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ELASTICITY SOLUTIONS FOR  
FIBER-REINFORCED, POLYMERIC  
COMPOSITE LAMINATES

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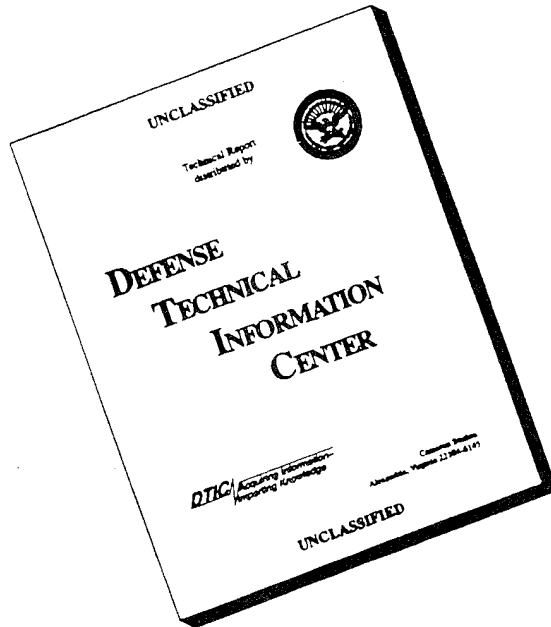
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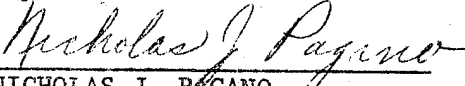
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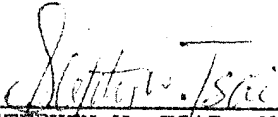
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COMPOSITE LAMINATES

Richard D. Schile

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## LIST OF SYMBOLS

$A_{ij}$	Stress function components
$A_{ij}^n$	Stress function components defined by $A_{ij} = \sum_{n=0}^1 A_{ij}^n z^n$
$a_{ij}$	Elastic constants of an individual ply referred to principal material axes
$b_{ij}$	Elastic constants of an individual ply
$d_{ij}$	Elastic constants of a laminated plate containing an infinite number of plies
$e_i$	Thermal expansion coefficients of a laminated plate containing an infinite number of plies
$2h$	Plate thickness
$i, j$	Indexes
$M_x, M_y, M_{xy}$	Bending moments per unit of length of edge
$N_x, N_y, N_{xy}$	In - plane forces per unit of length of edge
$n$	Index
$P(x, y)$	Lateral pressure distribution
$Q_x, Q_y$	Transverse shear forces per unit of length of edge
$T$	Temperature difference
$t$	Dummy variable of integration
$u_i$	Displacement components
$x, y, z$	(or $x_1, x_2, x_3$ ) Cartesian coordinates
$\alpha_i$	Thermal expansion coefficients of an individual ply referred to principal material axes
$\beta_i$	Thermal expansion coefficients of an individual ply
$\gamma_{ij}$	Strain components
$\phi_{ij}$	Stress function components
$\tau_{ij}$	Stress components
$\theta$	Angle between the x axis and the principal material axis of an individual ply

## INTRODUCTION

The problem to be considered is the analysis of rectangular, laminated, fiber-reinforced plates. The loading conditions are assumed to consist of normal pressure applied to one lateral face, arbitrary edge forces and moments applied to the edges and a uniform temperature distribution. Internal boundary conditions involve the continuity of the three components of the displacement vector and the components of the stress vector normal and tangent to the individual plies.

Assuming that the primary interaction is between plies, the plate is modeled as a laminated plate for which each ply is assumed to obey a stress-strain law with twelve independent elastic constants and three independent coefficients of thermal expansion. These local properties are assumed to vary discontinuously from ply to ply. Internal continuity conditions at ply interfaces can be identically satisfied by an integral formulation of three-dimensional stress function theory.

(References 1 and 2.)

## SECTION I

### FORMULATION OF THE PROBLEM

Let  $(x,y)$  be the in-plane coordinates and  $z$  the normal coordinate to the mid-plane of the laminated plate. It will be assumed that (1) individual laminations are homogeneous and elastic, (2) lamination interfaces are perfect, i.e., the displacement vector and the normal and tangential components of the stress vector are continuous across all interfaces and (3) all dependent variables are continuous in  $(x,y)$  and differentiable to any order required. The nature of the continuity of dependent variables with respect to the  $z$  coordinate will be specified by stating the appropriate function class  $C^n(z)$ ,  $n = 0,1$ . Functions which are discontinuous in  $z$  will be marked with an asterisk. Let  $a_{ij}^*(z)$ ,  $\alpha_i^*(z)$  be the elastic functions characterizing the kind and distribution of material within the plate,  $-h \leq z \leq h$ .

A sufficient condition for the continuity of the displacement vector within the plate ( $u_i \in C^0(z)$ ) is

$$u_i = \int_{-h}^z u_i^*(x,y,t) dt + u_i(x,y,-h) \quad (1)$$

The strain-displacement relations are

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2)$$

where a comma denotes partial differentiation with respect to  $x_1 = x$ ,  $x_2 = y$  or  $x_3 = z$ . Substituting equation (1) into equation (2), three of these equations can then be solved to give the displacement functions  $u_i^*$  in terms of the strains.

$$u_1^* = \gamma_{xz} - \int_{-h}^z \frac{\partial \gamma_{zz}}{\partial x} (x, y, t) dt - \frac{\partial}{\partial x} u_3(x, y, -h)$$

$$u_2^* = \gamma_{yz} - \int_{-h}^z \frac{\partial \gamma_{zz}}{\partial y} (x, y, t) dt - \frac{\partial}{\partial y} u_3(x, y, -h)$$

$$u_3^* = \gamma_{zz} \quad (3)$$

The remaining three equations can then be manipulated to produce three compatibility equations.

$$\begin{aligned} \gamma_{xx} = & 2 \int_{-h}^z \frac{\partial \gamma_{xz}}{\partial x} (x, y, t) dt - \int_{-h}^z (z-t) \frac{\partial^2 \gamma_{zz}}{\partial x^2} (x, y, t) dt \\ & - (z+h) \frac{\partial^2}{\partial x^2} u_3(x, y, -h) + \frac{\partial}{\partial x} u_1(x, y, -h) \end{aligned}$$

$$\begin{aligned} \gamma_{yy} = & 2 \int_{-h}^z \frac{\partial \gamma_{yz}}{\partial y} (x, y, t) dt - \int_{-h}^z (z-t) \frac{\partial^2 \gamma_{zz}}{\partial y^2} (x, y, t) dt \\ & - (z+h) \frac{\partial^2}{\partial y^2} u_3(x, y, -h) + \frac{\partial}{\partial y} u_2(x, y, -h) \end{aligned} \quad (4)$$

$$\begin{aligned} \gamma_{xy} = & \int_{-h}^z \left[ \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} \right]_{x, y, t} dt - \int_{-h}^z (z-t) \frac{\partial^2 \gamma_{zz}}{\partial x \partial y} (x, y, t) dt \\ & - (z+h) \frac{\partial^2}{\partial x \partial y} u_3(x, y, -h) + \frac{1}{2} \left[ \frac{\partial}{\partial x} u_2(x, y, -h) + \frac{\partial}{\partial y} u_1(x, y, -h) \right] \end{aligned}$$

Three other compatibility equations can be derived from equations (4) which are obviously not independent of equations (4) and which are satisfied when equations (4) are satisfied.

The equilibrium equations are,

$$\begin{aligned}
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= 0
\end{aligned} \tag{5}$$

These equations are identically satisfied by a representation in terms of six stress functions  $\phi_{ij}$ .

$$\begin{aligned}
\tau_{xx} &= \phi_{22,33} + \phi_{33,22} - 2\phi_{23,23} \\
\tau_{yy} &= \phi_{33,11} + \phi_{11,33} - 2\phi_{13,13} \\
\tau_{zz} &= \phi_{11,22} + \phi_{22,11} - 2\phi_{12,12} \\
\tau_{xy} &= -\phi_{33,12} + (\phi_{13,2} + \phi_{23,1} - \phi_{12,3}),_3 \\
\tau_{xz} &= -\phi_{22,13} + (\phi_{23,1} + \phi_{12,3} - \phi_{13,2}),_2 \\
\tau_{yz} &= -\phi_{11,23} + (\phi_{12,3} + \phi_{13,2} - \phi_{23,1}),_1
\end{aligned} \tag{6}$$

The stress components  $\tau_{xz}$  and  $\tau_{yz}$  must be continuous in  $z$ .

If  $\tau_{xz}, \tau_{yz} \in C^0(z)$  then since  $\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y}$ ,  $\tau_{zz} \in C^1(z)$ .

A sufficient condition for the satisfaction of these continuity conditions is

$$\begin{aligned}
\phi_{ij} &= \phi_{ij}^0(x,y) + (z+h)\phi_{ij}^1(x,y) + \int_{-h}^z (z-t)\phi_{ij}^*(x,y,t)dt \quad i,j=1,2 \\
\phi_{i3} &= \phi_{i3}^0(x,y) + \int_{-h}^z \phi_{i3}^*(x,y,t)dt \quad i=1,2 \\
\phi_{33} &= \phi_{33}^*
\end{aligned} \tag{7}$$

The stress function  $\phi_{33}$  may be discontinuous since it does not appear in  $\tau_{xz}$ ,  $\tau_{yz}$  or  $\tau_{zz}$ . Using equations (7), the stresses become

$$\tau_{xx} = \phi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*$$

$$\tau_{yy} = \phi_{33,11}^* + \phi_{11}^* - 2\phi_{13,1}^*$$

$$\begin{aligned} \tau_{zz} = & \int_{-h}^z (z-t) \left[ \phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^* \right]_{x,y,t} dt + \tau_{zz}(x,y,-h) \\ & - (z+h) \left[ \frac{\partial}{\partial x} \tau_{xz}(x,y,-h) + \frac{\partial}{\partial y} \tau_{yz}(x,y,-h) \right] \end{aligned}$$

$$\tau_{xy} = -\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \phi_{12}^*$$

$$\tau_{xz} = \int_{-h}^z \left[ \phi_{23,12}^* + \phi_{12,2}^* - \phi_{13,22}^* - \phi_{22,1}^* \right]_{x,y,t} dt + \tau_{xz}(x,y,-h)$$

$$\tau_{yz} = \int_{-h}^z \left[ \phi_{12,1}^* + \phi_{13,12}^* - \phi_{23,11}^* - \phi_{11,2}^* \right]_{x,y,t} dt + \tau_{yz}(x,y,-h) \quad (8)$$

The functions  $\phi_{ij}^0$  and  $\phi_{ij}^1$  have already been determined by evaluating the stresses at  $z = -h$ . Assuming that this face of the plate is free of applied stress, we can delete  $\tau_{xz}(x,y,-h)$ ,  $\tau_{yz}(x,y,-h)$  and  $\tau_{zz}(x,y,-h)$  from equations (8).

Hooke's law for an individual ply, referred to the principal material axes ( $y_1, y_2, y_3$ ) is

$$\gamma_{11} = a_{11}\tau_{11} + a_{12}\tau_{22} + a_{13}\tau_{33} + \alpha_1^T$$

$$\gamma_{22} = a_{21}\tau_{11} + a_{22}\tau_{22} + a_{23}\tau_{33} + \alpha_2^T$$

$$\gamma_{33} = a_{31}\tau_{11} + a_{32}\tau_{22} + a_{33}\tau_{33} + \alpha_3^T$$

$$\gamma_{12} = a_{44}\tau_{12}$$

$$\gamma_{13} = a_{55}\tau_{13}$$

$$\gamma_{23} = a_{66}\tau_{23} \tag{9}$$

In order to transform equations (9) to the  $(x,y,z)$  coordinate system, the coordinate system must be rotated through an angle  $\theta$  in the  $x,y$  plane. The transformation laws for the stress and strain tensors are

$$\gamma_{ij}^1 = k_{i\alpha} k_{j\beta} \gamma_{\alpha\beta}$$

$$\tau_{ij}^1 = k_{i\alpha} k_{j\beta} \tau_{\alpha\beta}$$

where  $k_{ij} = \bar{e}_i^1 \cdot \bar{e}_j$ . Let the primed components refer to the  $(x,y,z)$  coordinates and the unprimed components refer to the principal material axes  $(y_1, y_2, y_3)$ . The direction cosines of the transformation are

$$k_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By carrying out the transformation, substituting the results into equation (9) and solving for the strains, the stress-strain

law referred to the (x,y,z) axes is obtained.

$$\begin{aligned}
 \gamma_{xx} &= b_{11}\tau_{xx} + b_{12}\tau_{yy} + b_{13}\tau_{zz} + b_{14}\tau_{xy} + \beta_1 T \\
 \gamma_{yy} &= b_{21}\tau_{xx} + b_{22}\tau_{yy} + b_{23}\tau_{zz} + b_{24}\tau_{xy} + \beta_2 T \\
 \gamma_{zz} &= b_{31}\tau_{xx} + b_{32}\tau_{yy} + b_{33}\tau_{zz} + b_{34}\tau_{xy} + \beta_3 T \\
 \gamma_{xy} &= b_{41}\tau_{xx} + b_{42}\tau_{yy} + b_{43}\tau_{zz} + b_{44}\tau_{xy} + \beta_4 T \\
 \gamma_{xz} &= b_{55}\tau_{xz} + b_{56}\tau_{yz} \\
 \gamma_{yz} &= b_{65}\tau_{xz} + b_{66}\tau_{yz}
 \end{aligned} \tag{10}$$

The transformed material constants are given in terms of the constants referred to the principal material axes by the following relations:

$$\begin{aligned}
 b_{11} &= a_{11}\cos^4\theta + a_{22}\sin^4\theta + (a_{12} + a_{21} + 2a_{44})\sin^2\theta\cos^2\theta \\
 b_{12} &= a_{12}\cos^4\theta + a_{21}\sin^4\theta + (a_{11} + a_{22} - 2a_{44})\sin^2\theta\cos^2\theta \\
 b_{13} &= a_{13}\cos^2\theta + a_{23}\sin^2\theta \\
 b_{14} &= 2(a_{12} - a_{11} + a_{44})\sin\theta\cos^3\theta + 2(a_{22} - a_{21} - a_{44})\sin^3\theta\cos \\
 b_{21} &= a_{12}\sin^4\theta + a_{21}\cos^4\theta + (a_{11} + a_{22} - 2a_{44})\sin^2\theta\cos^2\theta \\
 b_{22} &= a_{11}\sin^4\theta + a_{22}\cos^4\theta + (a_{12} + a_{21} + 2a_{44})\sin^2\theta\cos^2\theta \\
 b_{23} &= a_{13}\sin^2\theta + a_{23}\cos^2\theta \\
 b_{24} &= 2(a_{22} - a_{21} - a_{44})\sin\theta\cos^3\theta + 2(a_{12} - a_{11} + a_{44})\sin^3\theta\cos \\
 b_{31} &= a_{31}\cos^2\theta + a_{32}\sin^2\theta
 \end{aligned} \tag{11}$$

more

$$b_{32} = a_{31}\sin^2\theta + a_{32}\cos^2\theta$$

$$b_{33} = a_{33}$$

$$b_{34} = 2(a_{32} - a_{31})\sin\theta\cos\theta$$

$$b_{41} = (a_{21} - a_{11})\sin\theta\cos\theta + (a_{22} - a_{12})\sin^3\theta\cos\theta \\ + a_{44}\sin\theta\cos\theta(1 - 2\sin^2\theta)$$

$$b_{42} = (a_{22} - a_{12})\sin\theta\cos^3\theta + (a_{21} - a_{11})\sin^3\theta\cos\theta \\ - a_{44}\sin\theta\cos\theta(1 - 2\sin^2\theta)$$

$$b_{43} = (a_{23} - a_{13})\sin\theta\cos\theta$$

$$b_{44} = 2(a_{11} + a_{22} - a_{12} - a_{21})\sin^2\theta\cos^2\theta + a_{44}(1 - 2\sin^2\theta)^2$$

$$b_{55} = a_{55}\cos^2\theta + a_{66}\sin^2\theta$$

$$b_{56} = (a_{66} - a_{55})\sin\theta\cos\theta = b_{65}$$

$$b_{66} = a_{55}\sin^2\theta + a_{66}\cos^2\theta$$

$$\beta_1 = \alpha_1\cos^2\theta + \alpha_2\sin^2\theta$$

$$\beta_2 = \alpha_1\sin^2\theta + \alpha_2\cos^2\theta$$

$$\beta_3 = \alpha_3$$

$$\beta_4 = (\alpha_2 - \alpha_1)\sin\theta\cos\theta \tag{11}$$

Equations (10) may be extended to apply to the entire laminated plate by allowing the material constants  $b_{ij}$ ,  $\beta_i$  to be discontinuous functions of the  $z$  coordinate. This discontinuous

dependence on  $z$  may be accomplished by allowing  $\theta$  to be a discontinuous function of  $z$  or by considering that the constants  $a_{ij}$ ,  $\alpha_i$  are discontinuous functions of  $z$  or allowing both possibilities simultaneously. The functional dependence of the material properties will be indicated by marking the constants  $b_{ij}$ ,  $\beta_i$  with an asterisk;  $b_{ij} = b_{ij}^*(z)$ ,  $\beta_i = \beta_i^*(z)$ .

By substituting equations (8) into equations (10), the strains may be obtained in terms of the stress functions  $\phi_{ij}^*$ . It is clear, however, from an examination of equations (4) that the strain components  $\gamma_{xx}$ ,  $\gamma_{yy}$ ,  $\gamma_{xy}$  are continuous functions of  $z$ . The remaining strain components may be discontinuous. Therefore, we shall examine the strain components  $\gamma_{xx}$ ,  $\gamma_{yy}$ ,  $\gamma_{xy}$ .

$$\begin{aligned} \gamma_{xx} = & b_{11}^* [\phi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{12}^* [\phi_{33,11}^* + \phi_{11}^* - 2\phi_{13,1}^*] \\ & + b_{13}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\ & + b_{14}^* [-\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \phi_{12}^*] + \beta_1^{*T} \\ \gamma_{yy} = & b_{21}^* [\phi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{22}^* [\phi_{33,11}^* + \phi_{11}^* - 2\phi_{13,1}^*] \\ & + b_{23}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\ & + b_{24}^* [-\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \phi_{12}^*] + \beta_2^{*T} \end{aligned} \quad (12)$$

more

$$\begin{aligned}
\gamma_{xy} = & b_{41}^* [\phi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{42}^* [\phi_{33,11}^* + \phi_{11}^* - 2\phi_{13,1}^*] \\
& + b_{43}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\
& + b_{44}^* [-\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \phi_{12}^*] + \beta_4^* T \quad (12)
\end{aligned}$$

If these strain components are to be continuous, as they must be in order to be consistent with equations (4), then we must seek some decomposition of the stress functions  $\phi_{ij}^*$  in terms of  $b_{ij}^*$ ,  $\beta_i^*$  and a new set of stress functions which are of class  $C^0(z)$ . Let us first eliminate the discontinuous temperature terms by making the transformation

$$\begin{aligned}
\phi_{11}^* &= \psi_{11}^* + F_{11}^{*T} \\
\phi_{22}^* &= \psi_{22}^* + F_{22}^{*T} \\
\phi_{12}^* &= \psi_{12}^* + F_{12}^{*T} \quad (13a)
\end{aligned}$$

and choosing  $F_{11}^{*T}$ ,  $F_{22}^{*T}$ ,  $F_{12}^{*T}$  so as to eliminate the temperature.

The result is

$$\begin{aligned}
D^*F_{11}^{*T} &= \begin{vmatrix} \beta_1^* & b_{11}^* & b_{14}^* \\ \beta_2^* & b_{21}^* & b_{24}^* \\ \beta_4^* & b_{41}^* & b_{44}^* \end{vmatrix} \\
D^*F_{22}^{*T} &= \begin{vmatrix} b_{12}^* & \beta_1^* & b_{14}^* \\ b_{22}^* & \beta_2^* & b_{24}^* \\ b_{42}^* & \beta_4^* & b_{44}^* \end{vmatrix} \quad (13b) \\
&\text{more}
\end{aligned}$$

$$D^*F_{12}^* = - \begin{vmatrix} b_{12}^* & b_{11}^* & \beta_1^* \\ b_{22}^* & b_{21}^* & \beta_2^* \\ b_{42}^* & b_{41}^* & \beta_4^* \end{vmatrix}$$

$$D^* = - \begin{vmatrix} b_{12}^* & b_{11}^* & b_{14}^* \\ b_{22}^* & b_{21}^* & b_{24}^* \\ b_{42}^* & b_{41}^* & b_{44}^* \end{vmatrix}$$

(13b)

The expressions for the strains then become

$$\begin{aligned} \gamma_{xx} = & b_{11}^* [\psi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{12}^* [\phi_{33,11}^* + \psi_{11}^* - 2\phi_{13,1}^*] \\ & + b_{13}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\ & + b_{14}^* [-\phi_{33,12}^* + \phi_{13,2}^* - \phi_{23,1}^* - \psi_{12}^*] \end{aligned}$$

$$\begin{aligned} \gamma_{yy} = & b_{21}^* [\psi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{22}^* [\phi_{33,11}^* + \psi_{11}^* - 2\phi_{13,1}^*] \\ & + b_{23}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\ & + b_{24}^* [-\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \psi_{12}^*] \end{aligned}$$

$$\begin{aligned} \gamma_{xy} = & b_{41}^* [\psi_{22}^* + \phi_{33,22}^* - 2\phi_{23,2}^*] + b_{42}^* [\phi_{33,11}^* + \psi_{11}^* - 2\phi_{13,1}^*] \\ & + b_{43}^* \int_{-h}^z (z-t) [\phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^*]_{x,y,t} dt \\ & + b_{44}^* [-\phi_{33,12}^* + \phi_{13,2}^* + \phi_{23,1}^* - \psi_{12}^*] \end{aligned}$$

Now, let

$$\begin{aligned}
 \psi_{11}^* &= C_{11}^* A_{11} + C_{12}^* A_{22} + C_{13}^* A_{12} + \Omega_{11}^* \\
 \psi_{22}^* &= C_{21}^* A_{11} + C_{22}^* A_{22} + C_{23}^* A_{12} + \Omega_{22}^* \\
 \psi_{12}^* &= -C_{31}^* A_{11} - C_{32}^* A_{22} - C_{33}^* A_{12} - \Omega_{12}^*
 \end{aligned} \tag{14}$$

where the new stress functions  $A_{ij}$  are assumed to be of class  $C^0(z)$  and the discontinuous functions  $C_{ij}^*$  are to be chosen so that

$$\begin{aligned}
 \gamma_{xx} &= A_{22} \\
 \gamma_{yy} &= A_{11} \\
 \gamma_{xy} &= A_{12}
 \end{aligned} \tag{15}$$

This leads to a set of nine equations for the discontinuous functions  $C_{ij}^*$ .

$$\begin{aligned}
 b_{11}^* C_{21}^* + b_{12}^* C_{11}^* + b_{14}^* C_{31}^* &= 0 \\
 b_{11}^* C_{22}^* + b_{12}^* C_{12}^* + b_{14}^* C_{32}^* &= 1 \\
 b_{11}^* C_{23}^* + b_{12}^* C_{13}^* + b_{14}^* C_{33}^* &= 0 \\
 b_{21}^* C_{21}^* + b_{22}^* C_{11}^* + b_{24}^* C_{31}^* &= 1 \\
 b_{21}^* C_{22}^* + b_{22}^* C_{12}^* + b_{24}^* C_{32}^* &= 0 \\
 b_{21}^* C_{23}^* + b_{22}^* C_{13}^* + b_{24}^* C_{33}^* &= 0 \\
 b_{41}^* C_{21}^* + b_{42}^* C_{11}^* + b_{44}^* C_{31}^* &= 0 \\
 b_{41}^* C_{22}^* + b_{42}^* C_{12}^* + b_{44}^* C_{32}^* &= 0 \\
 b_{41}^* C_{23}^* + b_{42}^* C_{13}^* + b_{44}^* C_{33}^* &= 1
 \end{aligned}$$

These equations can be broken down into three independent sets involving  $(C_{11}^*, C_{21}^*, C_{31}^*)$ ,  $(C_{12}^*, C_{22}^*, C_{32}^*)$ ,  $(C_{13}^*, C_{23}^*, C_{33}^*)$  respectively. All three sets have the same determinant

$$\Delta^* = \begin{vmatrix} b_{12}^* & b_{11}^* & b_{14}^* \\ b_{22}^* & b_{21}^* & b_{24}^* \\ b_{42}^* & b_{41}^* & b_{44}^* \end{vmatrix} \quad (16a)$$

and the functions  $C_{ij}^*$  are found to be

$$\begin{aligned} \Delta^* C_{11}^* &= b_{14}^* b_{41}^* - b_{11}^* b_{44}^* \\ \Delta^* C_{21}^* &= b_{12}^* b_{44}^* - b_{14}^* b_{42}^* \\ \Delta^* C_{31}^* &= b_{11}^* b_{42}^* - b_{12}^* b_{41}^* \\ \Delta^* C_{12}^* &= b_{21}^* b_{44}^* - b_{24}^* b_{41}^* \\ \Delta^* C_{22}^* &= b_{24}^* b_{42}^* - b_{22}^* b_{44}^* \\ \Delta^* C_{32}^* &= b_{22}^* b_{41}^* - b_{21}^* b_{42}^* \\ \Delta^* C_{13}^* &= b_{11}^* b_{24}^* - b_{14}^* b_{21}^* \\ \Delta^* C_{23}^* &= b_{22}^* b_{14}^* - b_{12}^* b_{24}^* \\ \Delta^* C_{33}^* &= b_{12}^* b_{21}^* - b_{11}^* b_{22}^* \end{aligned} \quad (16b)$$

The stress functions  $\Omega_{ij}$  are determined by eliminating all remaining discontinuous terms from the expressions for  $\gamma_{xx}$ ,  $\gamma_{yy}$ ,  $\gamma_{xy}$ . Before writing out the expressions for these stress functions however, it may be noted that there are only three compatibility equations to be satisfied, equations (4). There are already three continuous stress functions available;  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$ . In the analysis of the isotropic, laminated plate

(Ref. 1) it was found that the three stress functions  $\phi_{i3}^*$  disappeared from the final expressions for the stresses and could therefore be taken as identically zero. It will tentatively be assumed that these three stress functions will not be needed for the present analysis. By taking these three stress functions as zero, the expressions for  $\Omega_{ij}^*$  are greatly simplified.

$$\begin{aligned}
 \Omega_{11}^* &= - (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) A_{33} \\
 \Omega_{22}^* &= - (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) A_{33} \\
 \Omega_{12}^* &= - (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33}
 \end{aligned} \tag{16c}$$

For convenience, a new continuous stress function

$$\begin{aligned}
 A_{33} &= \int_{-h}^z (z-t) \left[ \phi_{11,22}^* + \phi_{22,11}^* - 2\phi_{12,12}^* \right]_{x,y,t} dt = \\
 &= \int_{-h}^z (z-t) \left[ C_{11}^* A_{11,22} + C_{12}^* A_{22,22} + C_{13}^* A_{12,22} \right. \\
 &\quad + C_{21}^* A_{11,11} + C_{22}^* A_{22,11} + C_{23}^* A_{12,11} + 2C_{31}^* A_{11,12} \\
 &\quad + 2C_{32}^* A_{22,12} + 2C_{33}^* A_{12,12} - (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) \\
 &\quad A_{33,22} - (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) A_{33,11} \\
 &\quad - 2(b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33,12} + F_{11}^{*T,22} \\
 &\quad \left. + F_{22}^{*T,11} - 2F_{12}^{*T,12} \right]_{x,y,t} dt
 \end{aligned} \tag{17}$$

has been defined. It should be noted that equation (17) is an integro-differential equation involving all four continuous

stress functions  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$ ,  $A_{33}$  and the temperature, through the terms  $F_{11}^*$ ,  $F_{22}^*$  from equation (13). The results of this decomposition may be summarized by writing out the final expressions for the stress functions  $\phi_{ij}^*$ , in terms of  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$ ,  $A_{33}$  and the temperature, and the corresponding expressions for the stresses and strains.

$$\begin{aligned}
 \phi_{11}^* &= C_{11}^* A_{11} + C_{12}^* A_{22} + C_{13}^* A_{12} \\
 &\quad - (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) A_{33} + F_{11}^* T \\
 \phi_{22}^* &= C_{21}^* A_{11} + C_{22}^* A_{22} + C_{23}^* A_{12} \\
 &\quad - (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) A_{33} + F_{22}^* T \\
 \phi_{12}^* &= - C_{31}^* A_{11} - C_{32}^* A_{22} - C_{33}^* A_{12} \\
 &\quad + (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33} + F_{12}^* T \\
 \phi_{i3}^* &\equiv 0 \quad i = 1, 2, 3
 \end{aligned} \tag{18a}$$

$$\begin{aligned}
 \tau_{xx} &= \phi_{22}^* \\
 \tau_{yy} &= \phi_{11}^* \\
 \tau_{zz} &= A_{33} \\
 \tau_{xy} &= - \phi_{12}^* \\
 \tau_{xz} &= \int_{-h}^z [\phi_{12,2}^* - \phi_{22,1}^*]_{x,y,t} dt \\
 \tau_{yz} &= \int_{-h}^z [\phi_{12,1}^* - \phi_{11,2}^*]_{x,y,t} dt
 \end{aligned} \tag{18b}$$

$$\gamma_{xx} = A_{22}$$

$$\gamma_{yy} = A_{11}$$

$$\begin{aligned} \gamma_{zz} = & (b_{31}^* C_{21}^* + b_{32}^* C_{11}^* + b_{34}^* C_{31}^*) A_{11} \\ & + (b_{31}^* C_{22}^* + b_{32}^* C_{12}^* + b_{34}^* C_{32}^*) A_{22} \\ & + (b_{31}^* C_{23}^* + b_{32}^* C_{13}^* + b_{34}^* C_{33}^*) A_{12} \\ & - [b_{31}^* (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) - b_{33}^* \\ & + b_{32}^* (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) \\ & + b_{34}^* (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*)] A_{33} \\ & + (b_{31}^* F_{22}^* + b_{32}^* F_{11}^* - b_{34}^* F_{12}^* + \beta_3^*) T \end{aligned}$$

$$\gamma_{xy} = A_{12}$$

$$\begin{aligned} \gamma_{xz} = & b_{55}^* \int_{-h}^z [-C_{31}^* A_{11,2} - C_{32}^* A_{22,2} - C_{33}^* A_{12,2} \\ & + (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33,2} \\ & - C_{21}^* A_{11,1} - C_{22}^* A_{22,1} - C_{23}^* A_{12,1} \\ & + (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) A_{33,1} + F_{12}^{*T,2} - F_{22}^{*T,1}]_{x,y,t} dt \\ & + b_{56}^* \int_{-h}^z [-C_{31}^* A_{11,1} - C_{32}^* A_{22,1} - C_{33}^* A_{12,1} \\ & + (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33,1} \\ & - C_{11}^* A_{11,2} - C_{12}^* A_{22,2} - C_{13}^* A_{12,2} \\ & + (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) A_{33,2} + F_{12}^{*T,1} - F_{11}^{*T,2}]_{x,y,t} dt \end{aligned}$$

$$\begin{aligned} \gamma_{yz} = & b_{65}^* \int_{-h}^z [-C_{31}^* A_{11,2} - C_{32}^* A_{22,2} - C_{33}^* A_{12,2} \\ & + (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33,2} \end{aligned}$$

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$$\begin{aligned}
& - C_{21}^* A_{11,1} - C_{22}^* A_{22,1} - C_{23}^* A_{12,1} \\
& + (b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*) A_{33,1} + F_{12}^{*T,2} - F_{22}^{*T,1} \Big]_{x,y,t} dt \\
& + b_{66}^* \int_{-h}^z \left[ - C_{31}^* A_{11,1} - C_{32}^* A_{22,1} - C_{33}^* A_{12,1} \right. \\
& + (b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*) A_{33,1} \\
& - C_{11}^* A_{11,2} - C_{12}^* A_{22,2} - C_{13}^* A_{12,2} \\
& \left. + (b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*) A_{33,2} + F_{12}^{*T,1} - F_{11}^{*T,2} \right]_{x,y,t} dt \\
& \qquad \qquad \qquad (19)
\end{aligned}$$

At this point, it may be proven without much difficulty, using equations (4), (17) and (19), that  $A_{11}, A_{22}, A_{12} \in C^0(z)$  and  $A_{33} \in C^1(z)$  and the formulation of the static problem is thus complete.

To obtain the governing equations, it is necessary to substitute the expressions for the strains, given by equation (19) into the compatibility equations, equations (4) and (17). The expressions for the strains can be simplified somewhat if several groups of terms which appear frequently are replaced by single symbols.

$$I_1^* = b_{13}^* C_{12}^* + b_{23}^* C_{11}^* + b_{43}^* C_{13}^*$$

$$I_2^* = b_{13}^* C_{22}^* + b_{23}^* C_{21}^* + b_{43}^* C_{23}^*$$

$$I_3^* = b_{13}^* C_{32}^* + b_{23}^* C_{31}^* + b_{43}^* C_{33}^*$$

$$I_4^* = b_{31}^* C_{21}^* + b_{32}^* C_{11}^* + b_{34}^* C_{31}^*$$

$$I_5^* = b_{31}^* C_{22}^* + b_{32}^* C_{12}^* + b_{34}^* C_{32}^*$$

$$I_6^* = b_{31}^* C_{23}^* + b_{32}^* C_{13}^* + b_{34}^* C_{33}^*$$

$$I_7^* = b_{33}^* - b_{31}^* I_2^* - b_{32}^* I_1^* - b_{34}^* I_3^*$$

$$I_8^* = b_{31}^* F_{22}^* + b_{32}^* F_{11}^* - b_{34}^* F_{12}^* + \beta_3^*$$

The strains can then be written in the form:

$$\gamma_{xx} = A_{22}$$

$$\gamma_{yy} = A_{11}$$

$$\gamma_{xy} = A_{12}$$

$$\gamma_{zz} = I_4^* A_{11} + I_5^* A_{22} + I_6^* A_{12} + I_7^* A_{33} + I_8^* T$$

$$\begin{aligned} \gamma_{xz} = & b_{55}^* \int_{-h}^z \left[ -C_{31}^* A_{11,2} - C_{32}^* A_{22,2} - C_{33}^* A_{12,2} - C_{21}^* A_{11,1} \right. \\ & - C_{22}^* A_{22,1} - C_{23}^* A_{12,1} + I_3^* A_{33,2} + I_2^* A_{33,1} + F_{12}^{*T},_2 \\ & \left. - F_{22}^{*T},_1 \right]_{x,y,t} dt \\ & + b_{56}^* \int_{-h}^z \left[ -C_{31}^* A_{11,1} - C_{32}^* A_{22,1} - C_{33}^* A_{12,1} - C_{11}^* A_{11,2} \right. \\ & - C_{12}^* A_{22,2} - C_{13}^* A_{12,2} + I_3^* A_{33,1} + I_1^* A_{33,2} + F_{12}^{*T},_1 \\ & \left. - F_{11}^{*T},_2 \right]_{x,y,t} dt \\ \gamma_{yz} = & b_{65}^* \int_{-h}^z \left[ -C_{31}^* A_{11,2} - C_{32}^* A_{22,2} - C_{33}^* A_{12,2} - C_{21}^* A_{11,1} \right. \\ & - C_{22}^* A_{22,1} - C_{23}^* A_{12,1} + I_3^* A_{33,2} + I_2^* A_{33,1} + F_{12}^{*T},_2 \\ & \left. - F_{22}^{*T},_1 \right]_{x,y,t} dt \\ & + b_{66}^* \int_{-h}^z \left[ -C_{31}^* A_{11,1} - C_{32}^* A_{22,1} - C_{33}^* A_{12,1} - C_{11}^* A_{11,2} \right. \end{aligned}$$

(20)  
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$$\begin{aligned}
& - C_{12}^* A_{22,2} - C_{13}^* A_{12,2} + I_3^* A_{33,1} + I_1^* A_{33,2} + F_{12}^{*T},_1 \\
& - F_{11}^{*T},_2 \Big]_{x,y,t} dt \quad (20)
\end{aligned}$$

Using equations (18) and (20), the inverted stress-strain law may be written in the form

$$\begin{aligned}
\tau_{xx} &= \left( C_{22}^* + \frac{I_2^* I_5^*}{I_7^*} \right) \gamma_{xx} + \left( C_{21}^* + \frac{I_2^* I_4^*}{I_7^*} \right) \gamma_{yy} \\
&+ \left( C_{23}^* + \frac{I_2^* I_6^*}{I_7^*} \right) \gamma_{xy} - \frac{I_2^*}{I_7^*} \gamma_{zz} + \left( F_{22}^* + \frac{I_2^* I_8^*}{I_7^*} \right) T \\
\tau_{yy} &= \left( C_{12}^* + \frac{I_1^* I_5^*}{I_7^*} \right) \gamma_{xx} + \left( C_{11}^* + \frac{I_1^* I_4^*}{I_7^*} \right) \gamma_{yy} \\
&+ \left( C_{13}^* + \frac{I_1^* I_6^*}{I_7^*} \right) \gamma_{xy} - \frac{I_1^*}{I_7^*} \gamma_{zz} + \left( F_{11}^* + \frac{I_1^* I_8^*}{I_7^*} \right) T \\
\tau_{xy} &= \left( C_{32}^* + \frac{I_3^* I_5^*}{I_7^*} \right) \gamma_{xx} + \left( C_{31}^* + \frac{I_3^* I_4^*}{I_7^*} \right) \gamma_{yy} \\
&+ \left( C_{33}^* + \frac{I_3^* I_6^*}{I_7^*} \right) \gamma_{xy} - \frac{I_3^*}{I_7^*} \gamma_{zz} - \left( F_{12}^* - \frac{I_3^* I_8^*}{I_7^*} \right) T \\
\tau_{zz} &= - \frac{1}{I_7^*} (I_5^* \gamma_{xx} + I_4^* \gamma_{yy} + I_6^* \gamma_{xy} - \gamma_{zz} + I_8^* T) \\
\tau_{xz} &= \frac{b_{66}^* \gamma_{xz} - b_{56}^* \gamma_{yz}}{b_{55}^* b_{66}^* - b_{56}^* b_{65}^*} \\
\tau_{yz} &= \frac{-b_{65}^* \gamma_{xz} + b_{55}^* \gamma_{yz}}{b_{55}^* b_{66}^* - b_{56}^* b_{65}^*} \quad (21)
\end{aligned}$$

The quantities  $C_{ij}^*$  are the moduli of the individual plies only under the classical plate theory assumption that the transverse normal stress is negligible. Otherwise, the correct expressions are those shown above.

The governing integro-differential equations may be obtained by substituting the expressions for the strains into equations (4) and (17). However, this will not be necessary at this point; it is only necessary to note that these equations involve integrals of discontinuous quantities with respect to the  $z$  coordinate. All of the terms on both sides of these equations are therefore continuous. The stress functions  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$  are continuous in  $z$  of class  $C^0$  and  $A_{33}$  is continuous in  $z$  of class  $C^1$ .

## SECTION II

### THE LIMITING CASE OF AN INFINITE NUMBER OF LAMINATIONS

The integrals involved in the compatibility equations are of two types:

$$\int_{-h}^z F^*(t) f(t) dt$$

$$\int_{-h}^z G^*(t) \int_{-h}^z H^*(s) f(s) ds dt$$

where  $F^*$ ,  $G^*$ ,  $H^*$  represent functions which are piece-wise constant in  $z$  and  $f$  represents functions which are continuous in  $z$  (of at least class  $C^0$ ) with a bounded derivative everywhere in the interval of integration. The function  $f$  is intended to represent any of the functions  $A_{ij}$ , and  $F^*$ ,  $G^*$ ,  $H^*$  represent combinations of the material property functions as they appear in the compatibility equations. If the laminated plate consists of an infinite number of laminations arranged periodically by repetition of a lamination subgroup containing a finite number of laminations, then it has been shown (Reference 2) that the integrals have the limiting values:

$$\lim_{n \rightarrow \infty} \int_{-h}^z F^*(t) f(t) dt = \bar{F} \int_{-h}^z f(t) dt$$

$$\lim_{n \rightarrow \infty} \int_{-h}^z G^*(t) \int_{-h}^z H^*(s) f(s) ds dt = \bar{G} \bar{H} \int_{-h}^z (z-t) f(t) dt \quad (22)$$

The average values are defined as arithmetic means taken over the thickness of the plate. By applying this theorem successively

to all terms contained in the governing integro-differential equations for the stress functions  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$ ,  $A_{33}$ , it can be shown that these equations are reduced to a form in which all material property functions are transformed into constants which are defined as mean values of those property functions. The case in which the number of laminations approaches infinity must therefore correspond to a homogeneous, anisotropic plate since the integro-differential equations for the homogeneous plate have constant coefficients. However, it is not known at this point what the material properties of the equivalent, homogeneous plate are. In order to determine this, it will be necessary to develop the integro-differential equations for a sufficiently general, homogeneous, anisotropic plate with undetermined material constants and then to evaluate these constants by matching coefficients between the two sets of integro-differential equations. The form of the stress-strain law for the equivalent, homogeneous plate must be determined by trial unless the most general form possible is assumed. The latter possibility is too tedious to evaluate but it seems reasonable to expect that a laminated plate in which the individual plies obey the stress-strain law given by equation (10) will not possess any higher degree of anisotropy than the individual plies. If all coefficients in the integro-differential equations for the two cases can be matched without any inconsistencies, then this assumption will have been proven correct. Accordingly, the stress-strain law for the equivalent, homogeneous plate will be taken in the form:

$$\begin{aligned}
\gamma_{xx} &= d_{11}\tau_{xx} + d_{12}\tau_{yy} + d_{13}\tau_{zz} + d_{14}\tau_{xy} + e_1^T \\
\gamma_{yy} &= d_{21}\tau_{xx} + d_{22}\tau_{yy} + d_{23}\tau_{zz} + d_{24}\tau_{xy} + e_2^T \\
\gamma_{zz} &= d_{31}\tau_{xx} + d_{32}\tau_{yy} + d_{33}\tau_{zz} + d_{34}\tau_{xy} + e_3^T \\
\gamma_{xy} &= d_{41}\tau_{xx} + d_{42}\tau_{yy} + d_{43}\tau_{zz} + d_{44}\tau_{xy} + e_4^T \\
\gamma_{xz} &= d_{55}\tau_{xz} + d_{56}\tau_{yz} \\
\gamma_{yz} &= d_{65}\tau_{xz} + d_{66}\tau_{yz}
\end{aligned} \tag{23}$$

The integro-differential equations for a plate composed of this material must now be derived by following formally the procedures carried out in the previous section: FORMULATION OF THE PROBLEM. It is necessary that this derivation be formal since the enforcement of internal continuity is not now required, and the motivation behind the development of these equations is thus obscured. It will be clear, however, that all of the equations thus obtained will be identical to those previously derived with  $b_{ij}^*$  replaced by  $d_{ij}$  and  $\beta_i^*$  replaced by  $e_i$ . The quantities  $C_{ij}^*$  which are defined by equations (16b) are replaced by  $C_{ij}^H$  which are defined by equations (16b) with  $b_{ij}^*$  replaced by  $d_{ij}$ . Also, the quantities  $I_i^*$  will be replaced by  $I_i^H$ , defined in a similar way. We are now in a position to compare the integro-differential equations for the homogeneous, anisotropic case and the limiting case of the laminated plate as the number of laminations approaches infinity. By equating all coefficients of the two sets of equations, the following set of relations is obtained. From equation (17):

$$C_{11}^H = \bar{C}_{11}$$

$$C_{12}^H = \bar{C}_{12}$$

$$C_{13}^H = \bar{C}_{13}$$

$$I_1^H = \bar{I}_1$$

$$F_{11}^H = \bar{F}_{11}$$

$$C_{21}^H = \bar{C}_{21}$$

$$C_{22}^H = \bar{C}_{22}$$

$$C_{23}^H = \bar{C}_{23}$$

$$I_2^H = \bar{I}_2$$

$$F_{22}^H = \bar{F}_{22}$$

$$C_{31}^H = \bar{C}_{31}$$

$$C_{32}^H = \bar{C}_{32}$$

$$C_{33}^H = \bar{C}_{33}$$

$$I_3^H = \bar{I}_3$$

$$F_{12}^H = \bar{F}_{12}$$

From equation (4a):

$$C_{31}^H d_{55} = \bar{C}_{31} \bar{b}_{55}$$

$$C_{32}^H d_{55} = \bar{C}_{32} \bar{b}_{55}$$

$$C_{33}^H d_{55} = \bar{C}_{33} \bar{b}_{55}$$

$$I_2^H d_{55} = \bar{I}_2 \bar{b}_{55}$$

$$F_{12}^H d_{55} = \bar{F}_{12} \bar{b}_{55}$$

$$F_{12}^H d_{56} = \bar{F}_{12} \bar{b}_{56}$$

$$C_{31}^H d_{56} = \bar{C}_{31} \bar{b}_{56}$$

$$C_{32}^H d_{56} = \bar{C}_{32} \bar{b}_{56}$$

$$C_{33}^H d_{56} = \bar{C}_{33} \bar{b}_{56}$$

$$I_3^H d_{56} = \bar{I}_3 \bar{b}_{56}$$

$$I_1^H d_{56} = \bar{I}_1 \bar{b}_{56}$$

$$C_{21}^H d_{55} = \bar{C}_{21} \bar{b}_{55}$$

$$C_{22}^H d_{55} = \bar{C}_{22} \bar{b}_{55}$$

$$C_{23}^H d_{55} = \bar{C}_{23} \bar{b}_{55}$$

$$I_3^H d_{55} = \bar{I}_3 \bar{b}_{55}$$

$$F_{22}^H d_{55} = \bar{F}_{22} \bar{b}_{55}$$

$$F_{11}^H d_{56} = \bar{F}_{11} \bar{b}_{56}$$

$$C_{11}^H d_{56} = \bar{C}_{11} \bar{b}_{56}$$

$$C_{12}^H d_{56} = \bar{C}_{12} \bar{b}_{56}$$

$$C_{13}^H d_{56} = \bar{C}_{13} \bar{b}_{56}$$

$$I_4^H = \bar{I}_4$$

$$I_5^H = \bar{I}_5$$

$$I_6^H = \bar{I}_6$$

$$I_7^H = \bar{I}_7$$

$$I_8^H = \bar{I}_8$$

From equation (4b):

$$C_{31}^H d_{65} = \bar{C}_{31} \bar{b}_{65}$$

$$C_{32}^H d_{65} = \bar{C}_{32} \bar{b}_{65}$$

$$C_{33}^H d_{65} = \bar{C}_{33} \bar{b}_{65}$$

$$I_3^H d_{65} = \bar{I}_3 \bar{b}_{65}$$

$$F_{12}^H d_{65} = \bar{F}_{12} \bar{b}_{65}$$

$$C_{31}^H d_{66} = \bar{C}_{31} \bar{b}_{66}$$

$$C_{32}^H d_{66} = \bar{C}_{32} \bar{b}_{66}$$

$$C_{33}^H d_{66} = \bar{C}_{33} \bar{b}_{66}$$

$$I_3^H d_{66} = \bar{I}_3 \bar{b}_{66}$$

$$F_{12}^H d_{66} = \bar{F}_{12} \bar{b}_{66}$$

$$C_{21}^H d_{65} = \bar{C}_{21} \bar{b}_{65}$$

$$C_{22}^H d_{65} = \bar{C}_{22} \bar{b}_{65}$$

$$C_{23}^H d_{65} = \bar{C}_{23} \bar{b}_{65}$$

$$I_2^H d_{65} = \bar{I}_2 \bar{b}_{65}$$

$$F_{22}^H d_{65} = \bar{F}_{22} \bar{b}_{65}$$

$$C_{11}^H d_{66} = \bar{C}_{11} \bar{b}_{66}$$

$$C_{12}^H d_{66} = \bar{C}_{12} \bar{b}_{66}$$

$$C_{13}^H d_{66} = \bar{C}_{13} \bar{b}_{66}$$

$$I_1^H d_{66} = \bar{I}_1 \bar{b}_{66}$$

$$F_{11}^H d_{66} = \bar{F}_{11} \bar{b}_{66}$$

The remaining relations for this equation are the same as the last five from equation (4a). Similar relations for equation (4c) only duplicate those already found. The distributive

property of the arithmetic mean has been used in some of the above relations. Asterisks have been omitted from the quantities  $b_{ij}$ ,  $\beta_i$ ,  $C_{ij}$  and  $I_i$ . All such quantities appear in terms under a bar which designates the arithmetic mean taken over the  $z$  coordinate.

The task now at hand is to solve the foregoing set of relations for the twenty equivalent elastic constants  $d_{ij}$  and the four equivalent coefficients of thermal expansion  $e_i$ . Considerable simplification may be obtained by noting that the foregoing set of equations can be represented by the more compact set:

$$\begin{aligned}
 C_{ij}^H &= \bar{C}_{ij} & i, j &= 1, \dots, 3 \\
 I_i^H &= \bar{I}_i & i &= 1, \dots, 8 \\
 d_{55} &= \bar{b}_{55} & d_{56} &= \bar{b}_{56} \\
 d_{65} &= \bar{b}_{65} & d_{66} &= \bar{b}_{66} \\
 F_{ij}^H &= \bar{F}_{ij} & i, j &= 1, 2
 \end{aligned} \tag{24}$$

There are thus twenty-four equations available for the determination of the twenty-four unknown coefficients. Consider first the solution of the first nine equations  $C_{ij}^H = \bar{C}_{ij}$  for the unknown coefficients  $d_{ij}$ . Remembering that the  $C_{ij}^H$  are defined by equation (16b) with  $b_{ij}^*$  replaced by  $d_{ij}$ , the  $C_{ij}^H$  then satisfy the nine equations following equation (15) with  $C_{ij}^*$  replaced by  $C_{ij}^H$  and  $b_{ij}^*$  replaced by  $d_{ij}$ .

$$d_{11}\bar{c}_{21} + d_{12}\bar{c}_{11} + d_{14}\bar{c}_{31} = 0$$

$$d_{11}\bar{c}_{22} + d_{12}\bar{c}_{12} + d_{14}\bar{c}_{32} = 1$$

$$d_{11}\bar{c}_{23} + d_{12}\bar{c}_{13} + d_{14}\bar{c}_{33} = 0$$

$$d_{21}\bar{c}_{21} + d_{22}\bar{c}_{11} + d_{24}\bar{c}_{31} = 1$$

$$d_{21}\bar{c}_{22} + d_{22}\bar{c}_{12} + d_{24}\bar{c}_{32} = 0$$

$$d_{21}\bar{c}_{23} + d_{22}\bar{c}_{13} + d_{24}\bar{c}_{33} = 0$$

$$d_{41}\bar{c}_{21} + d_{42}\bar{c}_{11} + d_{44}\bar{c}_{31} = 0$$

$$d_{41}\bar{c}_{22} + d_{42}\bar{c}_{12} + d_{44}\bar{c}_{32} = 0$$

$$d_{41}\bar{c}_{23} + d_{42}\bar{c}_{13} + d_{44}\bar{c}_{33} = 1$$

These are readily solved in sets of three with all three sets of equations having the same determinant.

$$\bar{\Delta} = \begin{vmatrix} \bar{c}_{21} & \bar{c}_{11} & \bar{c}_{31} \\ \bar{c}_{22} & \bar{c}_{12} & \bar{c}_{32} \\ \bar{c}_{23} & \bar{c}_{13} & \bar{c}_{33} \end{vmatrix}$$

The result is:

$$\bar{\Delta} d_{11} = \bar{c}_{31}\bar{c}_{13} - \bar{c}_{11}\bar{c}_{33}$$

$$\bar{\Delta} d_{12} = \bar{c}_{21}\bar{c}_{33} - \bar{c}_{31}\bar{c}_{23}$$

$$\bar{\Delta} d_{14} = \bar{c}_{11}\bar{c}_{23} - \bar{c}_{21}\bar{c}_{13}$$

$$\bar{\Delta} d_{21} = \bar{c}_{12}\bar{c}_{33} - \bar{c}_{32}\bar{c}_{13}$$

(25a)  
more

$$\begin{aligned}
\bar{\Delta} d_{22} &= \bar{c}_{23}\bar{c}_{32} - \bar{c}_{22}\bar{c}_{33} \\
\bar{\Delta} d_{24} &= \bar{c}_{22}\bar{c}_{13} - \bar{c}_{12}\bar{c}_{23} \\
\bar{\Delta} d_{41} &= \bar{c}_{11}\bar{c}_{32} - \bar{c}_{12}\bar{c}_{31} \\
\bar{\Delta} d_{42} &= \bar{c}_{22}\bar{c}_{31} - \bar{c}_{21}\bar{c}_{32} \\
\bar{\Delta} d_{44} &= \bar{c}_{21}\bar{c}_{12} - \bar{c}_{11}\bar{c}_{22}
\end{aligned} \tag{25a}$$

The next six equations  $I_i^H = I_i$  can now be solved for the six unknown coefficients  $d_{13}, d_{23}, d_{43}, d_{31}, d_{32}, d_{34}$ . Referring to the definitions preceding equation (20), and remembering that the  $I_i^H$  are defined by replacing the  $b_{ij}^*$  by  $d_{ij}$  and the  $c_{ij}^*$  by  $c_{ij}^H = \bar{c}_{ij}$ , we have:

$$\begin{aligned}
d_{13}\bar{c}_{12} + d_{23}\bar{c}_{11} + d_{43}\bar{c}_{13} &= \bar{I}_1 \\
d_{13}\bar{c}_{22} + d_{23}\bar{c}_{21} + d_{43}\bar{c}_{23} &= \bar{I}_2 \\
d_{13}\bar{c}_{32} + d_{23}\bar{c}_{31} + d_{43}\bar{c}_{33} &= \bar{I}_3 \\
d_{31}\bar{c}_{21} + d_{32}\bar{c}_{11} + d_{34}\bar{c}_{31} &= \bar{I}_4 \\
d_{31}\bar{c}_{22} + d_{32}\bar{c}_{12} + d_{34}\bar{c}_{32} &= \bar{I}_5 \\
d_{31}\bar{c}_{23} + d_{32}\bar{c}_{13} + d_{34}\bar{c}_{33} &= \bar{I}_6
\end{aligned}$$

These can also be solved in two sets of three equations, each having the same determinant  $\bar{\Delta}$  as the previous equations.

$$\begin{aligned}
d_{13} &= \bar{I}_1 d_{12} + \bar{I}_2 d_{11} + \bar{I}_3 d_{14} \\
d_{23} &= \bar{I}_1 d_{22} + \bar{I}_2 d_{21} + \bar{I}_3 d_{24}
\end{aligned} \tag{25b}$$

more

$$\begin{aligned}
d_{43} &= \bar{I}_1 d_{42} + \bar{I}_2 d_{41} + \bar{I}_3 d_{44} \\
d_{31} &= \bar{I}_4 d_{21} + \bar{I}_5 d_{11} + \bar{I}_6 d_{41} \\
d_{32} &= \bar{I}_4 d_{22} + \bar{I}_5 d_{12} + \bar{I}_6 d_{42} \\
d_{34} &= \bar{I}_4 d_{24} + \bar{I}_5 d_{14} + \bar{I}_6 d_{44}
\end{aligned} \tag{25b}$$

These results are in the most compact form when written in terms of the nine previously computed  $d_{ij}$ 's.

The transverse shearing constants are given directly as

$$\begin{aligned}
d_{55} &= \bar{b}_{55} & d_{66} &= \bar{b}_{66} \\
d_{56} &= d_{65} = \bar{b}_{56} = \bar{b}_{65}
\end{aligned}$$

Also, from  $I_7^H = \bar{I}_7$ ,  $I_8^H = \bar{I}_8$ ,

$$\begin{aligned}
d_{33} &= \bar{b}_{33} + d_{31}\bar{I}_2 + d_{32}\bar{I}_1 + d_{34}\bar{I}_3 - (\overline{b_{31}I_2}) - (\overline{b_{32}I_1}) - (\overline{b_{34}I_3}) \\
e_3 &= \bar{\beta}_3 + (\overline{b_{31}F_{22}}) + (\overline{b_{32}F_{11}}) - (\overline{b_{34}F_{12}}) - d_{31}\bar{F}_{22} - d_{32}\bar{F}_{11} + d_{34}\bar{F}_{12}
\end{aligned} \tag{25c}$$

The equivalent thermal expansion coefficients are then obtained in terms of the previously calculated material constants.

$$\begin{aligned}
e_1 &= \frac{D^H}{D} \left[ \bar{F}_{11}(D_5 D_9 - D_6 D_8) + \bar{F}_{22}(D_2 D_9 - D_3 D_8) + \bar{F}_{12}(D_3 D_5 - D_2 D_6) \right] \\
e_2 &= \frac{D^H}{D} \left[ \bar{F}_{11}(D_4 D_9 - D_6 D_7) + \bar{F}_{22}(D_1 D_9 - D_3 D_7) + \bar{F}_{12}(D_3 D_4 - D_1 D_6) \right] \\
e_4 &= \frac{D^H}{D} \left[ \bar{F}_{11}(D_4 D_8 - D_5 D_7) + \bar{F}_{22}(D_1 D_8 - D_2 D_7) + \bar{F}_{12}(D_2 D_4 - D_1 D_5) \right]
\end{aligned}$$

$$D^H = - \begin{vmatrix} d_{12} & d_{11} & d_{14} \\ d_{22} & d_{21} & d_{24} \\ d_{42} & d_{41} & d_{44} \end{vmatrix}$$

(25d)  
more

$$D = \begin{vmatrix} D_1 & -D_2 & D_3 \\ -D_4 & D_5 & -D_6 \\ D_7 & -D_8 & D_9 \end{vmatrix}$$

$$D_1 = d_{21}d_{44} - d_{24}d_{41}$$

$$D_2 = d_{11}d_{44} - d_{14}d_{41}$$

$$D_3 = d_{11}d_{24} - d_{14}d_{21}$$

$$D_4 = d_{22}d_{44} - d_{24}d_{42}$$

$$D_5 = d_{12}d_{44} - d_{14}d_{42}$$

$$D_6 = d_{12}d_{24} - d_{14}d_{22}$$

$$D_7 = d_{22}d_{41} - d_{21}d_{42}$$

$$D_8 = d_{12}d_{41} - d_{11}d_{42}$$

$$D_9 = d_{12}d_{21} - d_{11}d_{22} \tag{25d}$$

Since all of the equations have been satisfied without any inconsistencies, the original assumption concerning the form of the equivalent stress-strain law was therefore correct.

### SECTION III

#### THE CORRESPONDENCE PRINCIPLE

Since the stress functions  $A_{ij}$  are continuous in  $z$  of at least class  $C^0$ , it follows that

$$\lim_{n \rightarrow \infty} A_{ij} = A_{ij}^H$$

where the label "H" refers to the homogeneous, anisotropic plate. The temperature and displacement fields obey similar limiting laws.

$$\lim_{n \rightarrow \infty} T = T^H$$

$$\lim_{n \rightarrow \infty} u_i = u_i^H$$

Referring to equations (18) and (17), it can immediately be seen that the stress components  $\tau_{xz}$ ,  $\tau_{yz}$ ,  $\tau_{zz}$  are identical in the limit to the corresponding stress components for the homogeneous, anisotropic plate.

$$\lim_{n \rightarrow \infty} \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \end{pmatrix} = \begin{pmatrix} \tau_{xz}^H \\ \tau_{yz}^H \\ \tau_{zz}^H \end{pmatrix}$$

The remaining stress components in the laminated plate are:

$$\lim_{n \rightarrow \infty} \tau_{xx} = C_{21}^* A_{11}^H + C_{22}^* A_{22}^H + C_{23}^* A_{12}^H - I_2^* A_{33}^H + F_{22}^* T^H$$

$$\lim_{n \rightarrow \infty} \tau_{yy} = C_{11}^* A_{11}^H + C_{12}^* A_{22}^H + C_{13}^* A_{12}^H - I_1^* A_{33}^H + F_{11}^* T^H \quad (26)$$

more

$$\lim_{n \rightarrow \infty} \tau_{xy} = C_{31}^* A_{11}^H + C_{32}^* A_{22}^H + C_{33}^* A_{12}^H - I_3^* A_{33}^H - F_{12}^{*T^H} \quad (26)$$

The same stress components in the homogeneous plate are:

$$\begin{aligned} \tau_{xx}^H &= C_{21}^H A_{11}^H + C_{22}^H A_{22}^H + C_{23}^H A_{12}^H - I_2^H A_{33}^H + F_{22}^H T^H \\ \tau_{yy}^H &= C_{11}^H A_{11}^H + C_{12}^H A_{22}^H + C_{13}^H A_{12}^H - I_1^H A_{33}^H + F_{11}^H T^H \\ \tau_{xy}^H &= C_{31}^H A_{11}^H + C_{32}^H A_{22}^H + C_{33}^H A_{12}^H - I_3^H A_{33}^H - F_{12}^H T^H \end{aligned} \quad (27)$$

The elastic constants in equation (27) are, of course, known at this point, having been developed in the preceding section. It is now possible to obtain the stress components in the laminated plate, given by equation (26), in terms of the stress components in the homogeneous plate, given by equation (27), by eliminating the stress functions  $A_{ij}^H$  common to both sets of equations. The result of that calculation is

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau_{xx} &= (C_{21}^* d_{21} + C_{22}^* d_{11} + C_{23}^* d_{41}) \tau_{xx}^H \\ &+ (C_{21}^* d_{22} + C_{22}^* d_{12} + C_{23}^* d_{42}) \tau_{yy}^H \\ &+ (C_{21}^* d_{24} + C_{22}^* d_{14} + C_{23}^* d_{44}) \tau_{xy}^H \\ &+ (C_{21}^* d_{23} + C_{22}^* d_{13} + C_{23}^* d_{43} - I_2^*) \tau_{zz}^H \\ &+ [F_{22}^* - (C_{21}^* d_{21} + C_{22}^* d_{11} + C_{23}^* d_{41}) \bar{F}_{22} \\ &- (C_{21}^* d_{22} + C_{22}^* d_{12} + C_{23}^* d_{42}) \bar{F}_{11} \\ &+ (C_{21}^* d_{24} + C_{22}^* d_{14} + C_{23}^* d_{44}) \bar{F}_{12}] T^H \end{aligned} \quad (28)$$

more

$$\begin{aligned}
\lim_{n \rightarrow \infty} \tau_{yy} = & (C_{11}^{*d_{21}} + C_{12}^{*d_{11}} + C_{13}^{*d_{41}}) \tau_{xx}^H \\
& + (C_{11}^{*d_{22}} + C_{12}^{*d_{12}} + C_{13}^{*d_{42}}) \tau_{yy}^H \\
& + (C_{11}^{*d_{24}} + C_{12}^{*d_{14}} + C_{13}^{*d_{44}}) \tau_{xy}^H \\
& + (C_{11}^{*d_{23}} + C_{12}^{*d_{13}} + C_{13}^{*d_{43}} - I_1^*) \tau_{zz}^H \\
& + [F_{11}^* - (C_{11}^{*d_{21}} + C_{12}^{*d_{11}} + C_{13}^{*d_{41}}) \bar{F}_{22} \\
& - (C_{11}^{*d_{22}} + C_{12}^{*d_{12}} + C_{13}^{*d_{42}}) \bar{F}_{11} \\
& + (C_{11}^{*d_{24}} + C_{12}^{*d_{14}} + C_{13}^{*d_{44}}) \bar{F}_{12}] T^H \\
\lim_{n \rightarrow \infty} \tau_{xy} = & (C_{31}^{*d_{21}} + C_{32}^{*d_{11}} + C_{33}^{*d_{41}}) \tau_{xx}^H \\
& + (C_{31}^{*d_{22}} + C_{32}^{*d_{12}} + C_{33}^{*d_{42}}) \tau_{yy}^H \\
& + (C_{31}^{*d_{24}} + C_{32}^{*d_{14}} + C_{33}^{*d_{44}}) \tau_{xy}^H \\
& + (C_{31}^{*d_{23}} + C_{32}^{*d_{13}} + C_{33}^{*d_{43}} - I_3^*) \tau_{zz}^H \\
& - [F_{12}^* + (C_{31}^{*d_{21}} + C_{32}^{*d_{11}} + C_{33}^{*d_{41}}) \bar{F}_{22} \\
& + (C_{31}^{*d_{22}} + C_{32}^{*d_{12}} + C_{33}^{*d_{42}}) \bar{F}_{11} \\
& - (C_{31}^{*d_{24}} + C_{32}^{*d_{14}} + C_{33}^{*d_{44}}) \bar{F}_{12}] T^H \tag{28}
\end{aligned}$$

Some simplification has been accomplished by making use of equation (25b).

The Correspondence Principle for the stresses, as summarized by equation (28), gives explicit relations for the stresses in the laminated plate in terms of the stresses in

a corresponding, homogeneous, anisotropic plate subjected to the same boundary conditions. The equivalent elastic constants and thermal expansion coefficients for the laminated plate are also determined and are summarized by equations (25). The only assumption made in deriving these relations was that the number of laminations was indefinitely large. In that sense, this theory may be regarded as an asymptotic theory. The error involved in using these results for some finite number of laminations must be evaluated by the calculation of some specific cases. One caution to be emphasized is that the assumption of a very large number of laminations restricts these results to balanced laminates.

A computer program has been written to permit the calculation of the twenty equivalent elastic constants and the three equivalent coefficients of thermal expansion. This program was written in Dartmouth BASIC computer language and is listed in Appendix I together with the results of several sample runs using the Dartmouth Time Sharing Computer System.

SECTION IV  
NUMERICAL RESULTS

In order to evaluate the use of the Correspondence Principle for a finite number of laminations, a comparison solution is needed. The case of cylindrical bending of an isotropic, laminated plate is easily solved and will provide the required comparison solution. If the  $u_i(x, y, -h)$  are functions of  $x$  only and the  $A_{ij}$  are functions of  $x$  and  $z$  only, then equations (4) and (20) reduce to the two compatibility equations

$$\begin{aligned}
 A_{22} &= \frac{\partial}{\partial x} u_1(x, -h) - (z+h) \frac{\partial^2}{\partial x^2} u_3(x, -h) \\
 &\quad - 2 \int_{-h}^z \frac{1+\nu^*}{E^*} \int_{-h}^t \left[ \frac{E^*}{1-(\nu^*)^2} A_{22,11} + \frac{\nu^*}{1-\nu^*} A_{33,11} \right] ds dt \\
 &\quad + \int_{-h}^z (z-t) \left[ \frac{\nu^*}{1-\nu^*} A_{22,11} - \frac{1}{E^*} \left( 1 - \frac{2(\nu^*)^2}{1-\nu^*} \right) A_{33,11} \right] dt \\
 A_{12} &= \frac{1}{2} \frac{\partial}{\partial x} u_2(x, -h) - \int_{-h}^z \frac{1+\nu^*}{E^*} \int_{-h}^t \frac{E^*}{1+\nu^*} A_{12,11} ds dt
 \end{aligned}$$

As the number of laminations increases indefinitely, the strain distribution  $A_{22}$  approaches

$$\begin{aligned}
 L_{im} A_{22} &= A_{22}^o = \frac{\partial}{\partial x} u_1(x, -h) - (z+h) \frac{\partial^2}{\partial x^2} u_3(x, -h) \\
 &\quad + \left[ \left( \frac{\nu}{1-\nu} \right) - 2 \frac{b_{22} b_{66}}{\Delta} \right] \int_{-h}^z (z-t) A_{22,11}^o dt \\
 &\quad - \left[ \left[ \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \right] + 2 \left( \frac{\nu}{1-\nu} \right) b_{66} \right] \int_{-h}^z (z-t) A_{33,11}^o dt
 \end{aligned}$$

The integro-differential equations for  $A_{22}$  and  $A_{22}^{\circ}$  may be solved approximately, using Volterra iteration and retaining the same number of terms in each case.

$$\begin{aligned}
 A_{22} &\cong \left[ \frac{\partial}{\partial x} + \left[ \int_{-h}^z (z-t) \frac{v^*}{1-v^*} dt \right. \right. \\
 &\quad \left. \left. - 2 \int_{-h}^z \frac{1+v^*}{E^*} \int_{-h}^t \frac{E^*}{1-(v^*)^2} ds dt \right] \frac{\partial^3}{\partial x^3} \right] u_1(x_1, -h) \\
 &\quad - \left[ (z+h) \frac{\partial^2}{\partial x^2} + \left[ \int_{-h}^z (z-t) (t+h) \frac{v^*}{1-v^*} dt \right. \right. \\
 &\quad \left. \left. - 2 \int_{-h}^z \frac{1+v^*}{E^*} \int_{-h}^t \frac{E^*}{1-(v^*)^2} (s+h) ds dt \right] \frac{\partial^4}{\partial x^4} \right] u_3(x_1, -h) \\
 A_{22}^{\circ} &\cong \left[ \frac{\partial}{\partial x} + \left[ \left( \frac{v}{1-v} \right) - 2 \frac{b_{22}^b 66}{\Delta} \right] \frac{(z+h)^2}{2} \frac{\partial^3}{\partial x^3} \right] u_1(x, -h) \\
 &\quad - \left[ (z+h) \frac{\partial^2}{\partial x^2} + \left[ \left( \frac{v}{1-v} \right) - 2 \right] \frac{b_{22}^b 66}{\Delta} \frac{(z+h)^3}{6} \frac{\partial^4}{\partial x^4} \right] u_3(x, -h)
 \end{aligned}$$

The terms containing  $A_{33}$  are negligible.

The lower order terms in these expansions will be recognized from classical plate theory. The higher order terms are corrections which are of the order of the square of the plate thickness. The boundary displacements are determined from the differential equations which result from applying the boundary conditions at the upper face of the plate,  $z = h$ . In a plate with a finite number of laminations, the longitudinal strain distribution  $A_{22}$  is seen to be of class  $C^0(z)$  with a kink at every lamination interface. As the number of laminations increases, the kink locations approach one another and the strain distribution approaches a function having a continuous derivative.

The two cases may be compared by simply comparing the z-dependent coefficients in  $A_{22}$  with the constant coefficients in  $A_{22}^{\circ}$ . The coefficients of the lower order terms are identical so that, in terms of the assumptions of classical plate theory, the Correspondence Principle gives exact results. The most meaningful comparison is thus between the coefficients of the bending strain terms

$$C(z) = \int_{-h}^z (z-t)(t+h) \frac{\nu^*}{1-\nu^*} dt$$

$$- 2 \int_{-h}^z \frac{1+\nu^*}{E^*} \int_{-h}^t \frac{E^*}{1-(\nu^*)^2} (s+h) ds dt$$

and

$$C^{\circ}(z) = \left[ \left( \frac{\nu}{1-\nu} \right) - 2 \frac{b_{22} b_{66}}{\Delta} \right] \frac{(z+h)^3}{6}$$

which are "thick plate" corrections. Results are shown in Figures 1 - 3 for the case in which the plate is made up of an odd number of laminations of identical thickness. For the odd numbered plies,  $E=10 \times 10^6$ ,  $\nu = .25$  and for the even numbered plies,  $E = 1 \times 10^6$ ,  $\nu = .45$ . The rapid approach of the solutions for a finite number of plies to the Correspondence Principle solution may be interpreted as indicating that a condition of nearly complete mutual constraint exists between plies, even when the number of plies is relatively small. Taking into account the equality of the classical bending terms in  $A_{22}$  and  $A_{22}^{\circ}$ , the maximum error in using the Correspondence Principle is less than one percent for three plies and decreases rapidly for any greater number of plies.

SECTION V

SOME SIMPLE, EXACT SOLUTIONS FOR EDGE  
LOADED PLATES AT UNIFORM TEMPERATURE

If the lateral pressure is zero, then solutions of equations (4) and the boundary conditions

$$\tau_{xz}(x, y, \pm h) = \tau_{yz}(x, y, \pm h) - \tau_{zz}(x, y, \pm h) = 0$$

described by

$$A_{ij} = A_{ij}^0 + zA_{ij}^1, \quad A_{ij}^n = \text{constant}$$

are possible.

The edge forces and moments are all constant and the transverse shear forces are zero. If the edge forces and moments at the edges and the temperature are specified, then the six constant values  $A_{ij}^n$  are determined from the matrix equation

$$\bar{C} \bar{A} = \bar{N}$$

where

$$\bar{C} = \begin{pmatrix} C_{21}^0 & C_{22}^0 & C_{23}^0 & C_{21}^1 & C_{22}^1 & C_{23}^1 \\ C_{11}^0 & C_{12}^0 & C_{13}^0 & C_{11}^1 & C_{12}^1 & C_{13}^1 \\ C_{31}^0 & C_{32}^0 & C_{33}^0 & C_{31}^1 & C_{32}^1 & C_{33}^1 \\ C_{21}^1 & C_{22}^1 & C_{23}^1 & C_{21}^2 & C_{22}^2 & C_{23}^2 \\ C_{11}^1 & C_{12}^1 & C_{13}^1 & C_{11}^2 & C_{12}^2 & C_{13}^2 \\ C_{31}^1 & C_{32}^1 & C_{33}^1 & C_{31}^2 & C_{32}^2 & C_{33}^2 \end{pmatrix}$$

$$\bar{\bar{A}} = \begin{vmatrix} A_{11}^0 \\ A_{22}^0 \\ A_{12}^1 \\ A_{11}^1 \\ A_{22}^1 \\ A_{12}^1 \end{vmatrix} \quad \bar{\bar{N}} = \begin{vmatrix} N_x - F_{22}^0 T \\ N_y - F_{11}^0 T \\ N_{xy} + F_{12}^0 T \\ M_x - F_{22}^1 T \\ M_y - F_{11}^1 T \\ M_{xy} + F_{12}^1 T \end{vmatrix}$$

$$C_{ij}^n = \int_{-h}^h C_{ij}^* z^n dz$$

$$F_{ij}^n = \int_{-h}^h F_{ij}^* z^n dz$$

The displacement components at the lower lateral face may then be found by evaluating equations (4) at  $z=-h$  and integrating

$$u_1(x,y,-h) = \left( A_{22}^0 - hA_{22}^1 \right) x + \alpha y + \beta$$

$$u_2(x,y,-h) = \left( A_{11}^0 - hA_{11}^1 \right) y + 2 \left( A_{12}^0 - hA_{12}^1 \right) x - \alpha x + \gamma$$

$$u_3(x,y,-h) = -A_{22}^1 \frac{x^2}{2} - A_{12}^1 xy - A_{11}^1 \frac{y^2}{2} + k + \lambda x + \mu y$$

The six constants of integration  $\alpha, \beta, \gamma, k, \lambda, \mu$  are determined from the displacement boundary conditions. If the displacements referred to the mid-plane are desired, they can be calculated from equations (1), (3) and (20).

This set of solutions describes all possible cases of uniform edge loading and thermal warping. The transverse shear stresses

are, of course, zero. The solutions are exact only in regard to the matching of arbitrary edge forces and moments at the boundaries. At a free edge, for example, the edge forces and moments vanish but the stresses, in general, do not. The stress distribution near an edge is, in general, three-dimensional.

In the case of unrestrained thermal warping, if the edge displacement conditions are chosen so that the corner (0,0) of the plate is fixed,

$$\begin{aligned} u_1(0,0,-h) &= u_2(0,0,-h) = u_3(0,0,-h) = \\ &= \frac{\partial u_1}{\partial y}(0,0,-h) - \frac{\partial u_2}{\partial x}(0,0,-h) = \\ &= \frac{\partial u_3}{\partial x}(0,0,-h) = \frac{\partial u_3}{\partial y}(0,0,-h) = 0 \end{aligned}$$

then the displacement field is

$$\begin{aligned} u_1(x,y,-h) &= \left( A_{22}^0 - hA_{22}^1 \right) x + \left( A_{12}^0 - hA_{12}^1 \right) y \\ u_2(x,y,-h) &= \left( A_{11}^0 - hA_{11}^1 \right) y + \left( A_{12}^0 - hA_{12}^1 \right) x \\ u_3(x,y,-h) &= - A_{22}^1 \frac{x^2}{2} - A_{12}^1 xy - A_{11}^1 \frac{y^2}{2} \end{aligned}$$

For the case of simple tension in a balanced laminate, the elastic constants computed from this solution agree exactly with those found from equations (25a).

SECTION VI  
APPROXIMATE THEORY FOR THIN PLATES

The governing integro-differential equations to be solved are obtained by substituting equations (20) into the compatibility equations (4). The unknown quantities to be determined are the four stress functions  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$ ,  $A_{33}$ . The other equation needed is equation (17). By evaluating equations (4) at  $z = -h$ , the displacement components at the lower lateral face are obtained.

$$\begin{aligned}
 A_{22}(x,y,-h) &= \frac{\partial}{\partial x} u_1(x,y,-h) \\
 A_{11}(x,y,-h) &= \frac{\partial}{\partial y} u_2(x,y,-h) \\
 2A_{12}(x,y,-h) &= \frac{\partial}{\partial x} u_2(x,y,-h) + \frac{\partial}{\partial y} u_1(x,y,-h) \\
 \frac{\partial A_{22}}{\partial z}(x,y,-h) &= -\frac{\partial^2}{\partial x^2} u_3(x,y,-h) \\
 \frac{\partial A_{11}}{\partial z}(x,y,-h) &= -\frac{\partial^2}{\partial y^2} u_3(x,y,-h) \\
 \frac{\partial A_{12}}{\partial z}(x,y,-h) &= -\frac{\partial^2}{\partial x \partial y} u_3(x,y,-h)
 \end{aligned} \tag{29}$$

The displacement components are thus determined when the stress functions have been specified.

The boundary conditions on the lateral faces of the plate will be taken to be:

$$\begin{aligned}
 \tau_{xz}(x,y,\pm h) &= \tau_{yz}(x,y,\pm h) = 0 \\
 \tau_{zz}(x,y,-h) &= 0 \\
 \tau_{zz}(x,y,h) &= -p(x,y)
 \end{aligned} \tag{30}$$

The boundary conditions on the edges of the plate will be left arbitrary. The temperature is assumed to be uniform. The order of the various terms in the compatibility equations can be ascertained by examining equations (20). If  $A_{11}$ ,  $A_{22}$  and  $A_{12}$  are assumed to be of order  $h^0$  then  $\gamma_{xx}$ ,  $\gamma_{yy}$ ,  $\gamma_{xy}$ ,  $\gamma_{zz}$  are of order  $h^0$  and  $\gamma_{xz}$  and  $\gamma_{yz}$  are  $o(h)$ . The transverse shearing strains are, in terms of the displacements:

$$2 \gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}$$

$$2 \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}$$

If the stress functions are assumed to be given in terms of power series expansions in  $z$ , then the displacements are given by similar expansions.

$$u_1 \cong U + zU_1$$

$$u_2 \cong V + zV_1$$

$$u_3 \cong W$$

If the transverse shearing strains are to be of order  $h$  then,

$$2 \gamma_{xz} = U_1 + \frac{\partial W}{\partial x} = 0$$

$$2 \gamma_{yz} = V_1 + \frac{\partial W}{\partial y} = 0$$

Therefore, the displacements are

$$u_1 = U - z \frac{\partial W}{\partial x}$$

$$u_2 = V - z \frac{\partial W}{\partial y} \tag{31}$$

$$u_3 = W$$

and the stress function expansions in terms of U, V and W are

$$\begin{aligned}
 A_{22} &\cong \frac{\partial u}{\partial x} - z \frac{\partial^2 W}{\partial x^2} \\
 A_{11} &\cong \frac{\partial V}{\partial y} - z \frac{\partial^2 W}{\partial y^2} \\
 A_{12} &\cong \frac{1}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - z \frac{\partial^2 W}{\partial x \partial y}
 \end{aligned}
 \tag{32}$$

According to equation (31), the functions U, V and W are the displacement components referred to the middle surface. However, this is inconsistent with equation (29) which identifies W as the transverse displacement referred to the lower surface. This inconsistency should be regarded as resulting from the truncation of the power series expansion given by equation (31).

If equations (32) are substituted into the compatibility equations (4), it is found that these equations are identically satisfied to order  $h$ , that is, the error is proportional to  $h^2$ , which is consistent with equations (31) and (32). Satisfaction of the boundary conditions (30) then results in three differential equations to be solved for the three displacement components U, V and W. Equation (17) will supply the expression for the other stress function  $A_{33}$ . An examination of the integrand of this equation discloses that the terms containing  $A_{33}$  are small compared to the terms containing  $A_{11}$ ,  $A_{22}$  and  $A_{12}$  and may be neglected. A suitable expression for  $A_{33}$  is therefore

$$A_{33} \cong \int_{-h}^z (z-t) \left\{ \left[ C_{21}^* \frac{\partial^2}{\partial x^2} + 2C_{31}^* \frac{\partial^2}{\partial x \partial y} + C_{11}^* \frac{\partial^2}{\partial y^2} \right] A_{11} \right.$$

more

$$\begin{aligned}
& + \left[ C_{22}^* \frac{\partial^2}{\partial x^2} + 2C_{32}^* \frac{\partial^2}{\partial x \partial y} + C_{12}^* \frac{\partial^2}{\partial y^2} \right] A_{22} \\
& + \left[ C_{23}^* \frac{\partial^2}{\partial x^2} + 2C_{33}^* \frac{\partial^2}{\partial x \partial y} + C_{13}^* \frac{\partial^2}{\partial y^2} \right] A_{12} \left. \vphantom{\frac{\partial^2}{\partial x^2}} \right\} dt
\end{aligned} \tag{33}$$

continued

The governing differential equations are

$$L_1 U + L_2 V - L_3 W = P_1$$

$$L_4 U + L_5 V - L_6 W = P_2$$

$$L_7 U + L_8 V - L_9 W = P_3$$

$$L_1 = C_{22}^{\circ} \frac{\partial^2}{\partial x^2} + \left( C_{32}^{\circ} + \frac{1}{2} C_{23}^{\circ} \right) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} C_{33}^{\circ} \frac{\partial^2}{\partial y^2}$$

$$L_2 = \frac{1}{2} C_{23}^{\circ} \frac{\partial^2}{\partial x^2} + \left( C_{21}^{\circ} + \frac{1}{2} C_{33}^{\circ} \right) \frac{\partial^2}{\partial x \partial y} + C_{31}^{\circ} \frac{\partial^2}{\partial y^2}$$

$$\begin{aligned}
L_3 = & C_{22}^1 \frac{\partial^3}{\partial x^3} + \left( C_{32}^1 + C_{23}^1 \right) \frac{\partial^3}{\partial x^2 \partial y} + \left( C_{21}^1 + C_{33}^1 \right) \frac{\partial^3}{\partial x \partial y^2} \\
& + C_{31}^1 \frac{\partial^3}{\partial y^3}
\end{aligned}$$

$$L_4 = C_{32}^{\circ} \frac{\partial^2}{\partial x^2} + \left( C_{12}^{\circ} + \frac{1}{2} C_{33}^{\circ} \right) \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} C_{13}^{\circ} \frac{\partial^2}{\partial y^2}$$

$$L_5 = \frac{1}{2} C_{33}^{\circ} \frac{\partial^2}{\partial x^2} + \left( C_{31}^{\circ} + \frac{1}{2} C_{13}^{\circ} \right) \frac{\partial^2}{\partial x \partial y} + C_{11}^{\circ} \frac{\partial^2}{\partial y^2}$$

$$\begin{aligned}
L_6 = & C_{32}^1 \frac{\partial^3}{\partial x^3} + \left( C_{12}^1 + C_{33}^1 \right) \frac{\partial^3}{\partial x^2 \partial y} + \left( C_{31}^1 + C_{13}^1 \right) \frac{\partial^3}{\partial x \partial y^2} \\
& + C_{11}^1 \frac{\partial^3}{\partial y^3}
\end{aligned}$$

$$\begin{aligned}
L_7 = & \left( hc_{22}^{\circ} - C_{22}^1 \right) \frac{\partial^3}{\partial x^3} + \left( 2hc_{32}^{\circ} - 2C_{32}^1 + \frac{1}{2} hc_{23}^{\circ} - \frac{1}{2} C_{23}^1 \right) \frac{\partial^3}{\partial x^2 \partial y} \\
& + \left( hc_{12}^{\circ} - C_{12}^1 + hc_{33}^{\circ} - C_{33}^1 \right) \frac{\partial^3}{\partial x \partial y^2} + \frac{1}{2} \left( hc_{13}^{\circ} - C_{13}^1 \right) \frac{\partial^3}{\partial y^3}
\end{aligned}$$

$$L_8 = \frac{1}{2} \left( hc_{23}^{\circ} - c_{23}^1 \right) \frac{\partial^3}{\partial x^3} + \left( hc_{21}^{\circ} - c_{21}^1 + hc_{33}^{\circ} - c_{33}^1 \right) \frac{\partial^3}{\partial x^2 \partial y} \\ + \left( 2hc_{31}^{\circ} - 2c_{31}^1 + \frac{1}{2} hc_{13}^{\circ} - \frac{1}{2} c_{13}^1 \right) \frac{\partial^3}{\partial x \partial y^2} + \left( hc_{11}^{\circ} - c_{11}^1 \right) \frac{\partial^3}{\partial y^3}$$

$$L_9 = \left( hc_{22}^1 - c_{22}^2 \right) \frac{\partial^4}{\partial x^4} + \left( 2hc_{32}^1 - 2c_{32}^2 + hc_{23}^1 - c_{23}^2 \right) \frac{\partial^4}{\partial x^3 \partial y} \\ + \left( hc_{21}^1 - c_{21}^2 + hc_{12}^1 - c_{12}^2 + 2hc_{33}^1 - 2c_{33}^2 \right) \frac{\partial^4}{\partial x^2 \partial y^2} \\ + \left( 2hc_{31}^1 - 2c_{31}^2 + hc_{13}^1 - c_{13}^2 \right) \frac{\partial^4}{\partial x \partial y^3} + \left( hc_{11}^1 - c_{11}^2 \right) \frac{\partial^4}{\partial y^4}$$

$$P_1 = \int_{-h}^h \left( I_2^* \frac{\partial}{\partial x} + I_3^* \frac{\partial}{\partial y} \right) A_{33} dz$$

$$P_2 = \int_{-h}^h \left( I_3^* \frac{\partial}{\partial x} + I_1^* \frac{\partial}{\partial y} \right) A_{33} dz$$

$$P_3 \cong -P$$

Equations (34) are a set of three differential equations of eighth order in the three displacement components. Before considering the general case, it will be helpful to consider two special cases which are much simpler and which will provide some insight into the nature of the solutions.

## 1. Homogeneous, Isotropic Material

This is a single ply plate of isotropic material. The elastic constants of the material are

$$b_{11} = b_{22} = b_{33} = \frac{1}{E}$$

$$b_{12} = b_{21} = b_{13} = b_{31} = b_{23} = b_{32} = -\frac{\nu}{E}$$

$$b_{44} = b_{55} = b_{66} = \frac{1+\nu}{E}$$

$$b_{14} = b_{24} = b_{34} = b_{41} = b_{42} = b_{43} = b_{56} = b_{65} = 0$$

$$C_{11} = C_{22} = \frac{E}{1-\nu^2}$$

$$C_{12} = C_{21} = \frac{E\nu}{1-\nu^2}$$

$$C_{33} = \frac{E}{1+\nu}$$

$$C_{13} = C_{31} = C_{23} = C_{32} = 0$$

$$C_{ij}^0 = \int_{-h}^h C_{ij} dz = 2hC_{ij}$$

$$C_{ij}^1 = 0$$

$$C_{ij}^2 = \int_{-h}^h z^2 C_{ij} dz = \frac{2}{3} h^3 C_{ij}$$

$$I_1 = I_2 = I_3 = 0$$

The differential operators of equation (34) are

$$L_1 = \frac{2Eh}{1+\nu} \left[ \frac{1}{1-\nu} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \frac{\partial^2}{\partial y^2} \right]$$

$$L_2 = \frac{Eh}{1-\nu} \frac{\partial^2}{\partial x \partial y}$$

$$L_3 = 0$$

$$L_4 = \frac{Eh}{1-\nu} \frac{\partial^2}{\partial x \partial y}$$

$$L_5 = \frac{2Eh}{1+\nu} \left[ \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{1-\nu} \frac{\partial^2}{\partial y^2} \right]$$

$$L_6 = 0$$

$$L_7 = \frac{2Eh^2}{1-\nu^2} \left[ \frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x \partial y^2} \right]$$

$$L_8 = \frac{2Eh^2}{1-\nu^2} \left[ \frac{\partial^3}{\partial x^2 \partial y} + \frac{\partial^3}{\partial y^3} \right]$$

$$L_9 = \frac{-2Eh^3}{3(1-\nu^2)} \nabla^4$$

Since  $P_1$  and  $P_2$  are zero, the displacements  $U$  and  $V$  may be taken as zero provided that this is consistent with the boundary conditions. The third equation then reduces to

$$\nabla^4 W = - \frac{P}{\frac{2Eh^3}{3(1-\nu^2)}}$$

The edge forces and moments are

$$N_x = \frac{E}{1-\nu^2} \left[ \frac{\partial U}{\partial x} + \nu \frac{\partial V}{\partial y} \right]$$

$$N_y = \frac{E}{1-\nu^2} \left[ \frac{\partial V}{\partial y} + \nu \frac{\partial U}{\partial x} \right]$$

$$N_{xy} = \frac{E}{2(1+\nu)} \left[ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right]$$

$$M_x = -\frac{Eh^2}{3(1-\nu^2)} \left[ \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right]$$

$$M_y = -\frac{Eh^2}{3(1-\nu^2)} \left[ \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right]$$

$$M_{xy} = -\frac{Eh^2}{3(1+\nu)} \frac{\partial^2 W}{\partial x \partial y}$$

$$Q_x = -\frac{Eh^2}{3(1-\nu^2)} \frac{\partial}{\partial x} \nabla^2 W$$

$$Q_y = -\frac{Eh^2}{3(1-\nu^2)} \frac{\partial}{\partial y} \nabla^2 W$$

and the stresses are

$$\tau_{xx} = -\frac{EZ}{1-\nu^2} \left[ \frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{\partial y^2} \right] W$$

$$\tau_{yy} = -\frac{EZ}{1-\nu^2} \left[ \nu \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] W$$

$$\tau_{xy} = -\frac{EZ}{1-\nu^2} \frac{\partial^2 W}{\partial x \partial y}$$

$$\tau_{zz} = -\left( \frac{z^3}{6} - \frac{zh^2}{2} - \frac{h^3}{3} \right) \frac{E}{1-\nu^2} \nabla^4 W = \left( \frac{z^3}{6} - \frac{zh^2}{2} - \frac{h^3}{3} \right) \frac{3P}{2h^3}$$

$$\tau_{xz} = \frac{E(z^2 - h^2)}{2(1-\nu^2)} \frac{\partial}{\partial x} \nabla^2 W$$

$$\tau_{yz} = \frac{E(z^2 - h^2)}{2(1-\nu^2)} \frac{\partial}{\partial y} \nabla^2 W$$

If the plate is simply supported,

$$W = W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} ,$$

and the transverse shearing stresses are

$$\tau_{xz} = - \frac{E(z^2-h^2)}{2(1-\nu^2)} W_{mn} \left( \frac{m\pi}{a} \right) \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\tau_{yz} = - \frac{E(z^2-h^2)}{2(1-\nu^2)} W_{mn} \left( \frac{n\pi}{b} \right) \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

## 2. Balanced Laminate

A balanced laminate is defined as a laminated plate for which  $C_{ij}^1 = I_i^1 = 0$ . The material property distributions are even-symmetric functions with respect to the mid-plane of the plate. In this case, the operators  $L_3$  and  $L_6$  are zero and the in-plane displacements can thus be determined independently of the transverse displacement. By inspection, the particular integrals of  $U$  and  $V$  are of order  $h^0$ . Substituting these into the last of equations (34), the equation

$$L_9 W = -P (1 + O(h^2))$$

is obtained. Since the terms in this equation arising from  $U$  and  $V$  are of order  $h^2$  compared to unity, they may be discarded with the result that the transverse displacement is approximately uncoupled from the in-plane displacements. Since  $U$  and  $V$  are of order  $h^3 W$ , they may be ignored compared to  $W$ . However, this does not necessarily mean that the effect of  $U$  and  $V$  on the edge forces and moments is negligible; this remains to be evaluated. Contributions to various terms will be considered negligible only if they are of order  $h^2$  compared to unity. The edge forces and moments are

$$N_x = C_{21}^0 \frac{\partial V}{\partial y} + C_{22}^0 \frac{\partial U}{\partial x} + \frac{1}{2} C_{23}^0 \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \int_{-h}^h I_2^* \tau_{zz} dz + F_{22}^0 T$$

$$N_y = C_{11}^{\circ} \frac{\partial V}{\partial y} + C_{12}^{\circ} \frac{\partial U}{\partial x} + \frac{1}{2} C_{13}^{\circ} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \int_{-h}^h I_1^* \tau_{zz} dz + F_{11}^{\circ T}$$

$$N_{xy} = C_{31}^{\circ} \frac{\partial V}{\partial y} + C_{32}^{\circ} \frac{\partial U}{\partial x} + \frac{1}{2} C_{33}^{\circ} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \int_{-h}^h I_3^* \tau_{zz} dz - F_{12}^{\circ T}$$

$$-M_x = C_{21}^2 \frac{\partial^2 W}{\partial y^2} + C_{32}^2 \frac{\partial^2 W}{\partial x^2} + C_{23}^2 \frac{\partial^2 W}{\partial x \partial y}$$

$$-M_y = C_{11}^2 \frac{\partial^2 W}{\partial y^2} + C_{12}^2 \frac{\partial^2 W}{\partial x^2} + C_{13}^2 \frac{\partial^2 W}{\partial x \partial y}$$

$$-M_{xy} = C_{31}^2 \frac{\partial^2 W}{\partial y^2} + C_{32}^2 \frac{\partial^2 W}{\partial x^2} + C_{33}^2 \frac{\partial^2 W}{\partial x \partial y}$$

$$-Q_x = \frac{\partial}{\partial y} \left[ C_{31}^2 \frac{\partial^2 W}{\partial y^2} + C_{32}^2 \frac{\partial^2 W}{\partial x^2} + C_{33}^2 \frac{\partial^2 W}{\partial x \partial y} \right]$$

$$+ \frac{\partial}{\partial x} \left[ C_{21}^2 \frac{\partial^2 W}{\partial y^2} + C_{22}^2 \frac{\partial^2 W}{\partial x^2} + C_{23}^2 \frac{\partial^2 W}{\partial x \partial y} \right]$$

$$-Q_y = \frac{\partial}{\partial x} \left[ C_{31}^2 \frac{\partial^2 W}{\partial y^2} + C_{32}^2 \frac{\partial^2 W}{\partial x^2} + C_{33}^2 \frac{\partial^2 W}{\partial x \partial y} \right]$$

$$+ \frac{\partial}{\partial y} \left[ C_{11}^2 \frac{\partial^2 W}{\partial y^2} + C_{12}^2 \frac{\partial^2 W}{\partial x^2} + C_{13}^2 \frac{\partial^2 W}{\partial x \partial y} \right]$$

Terms have been discarded from the moment and transverse shear force expression which are of order  $h^2$ . Since the in-plane displacements do not appear in the moments and transverse shears and the transverse displacement does not appear in the edge forces, the problem is truly uncoupled at this level of approximation. The transverse pressure terms which appear in the expressions for the edge forces are due to the Poisson constraint effect. The presence of these terms means that the vanishing of the in-plane

displacement terms does not ensure that the edge forces vanish. This must be taken into account in the specification of the boundary conditions.

In the determination of the stresses, the dominance of the transverse displacement is clearly evident since the in-plane displacements contribute only to  $O(h^2)$  compared to the transverse displacement terms.

$$\tau_{xx} = - \left[ C_{21}^* \frac{\partial^2 W}{\partial y^2} + C_{22}^* \frac{\partial^2 W}{\partial x^2} + C_{23}^* \frac{\partial^2 W}{\partial x \partial y} \right] z + F_{22}^* T$$

$$\tau_{yy} = - \left[ C_{11}^* \frac{\partial^2 W}{\partial y^2} + C_{12}^* \frac{\partial^2 W}{\partial x^2} + C_{13}^* \frac{\partial^2 W}{\partial x \partial y} \right] z + F_{11}^* T$$

$$\tau_{xy} = - \left[ C_{31}^* \frac{\partial^2 W}{\partial y^2} + C_{32}^* \frac{\partial^2 W}{\partial x^2} + C_{33}^* \frac{\partial^2 W}{\partial x \partial y} \right] z - F_{12}^* T$$

$$\begin{aligned} \tau_{zz} \cong & \int_{-h}^z t(z-t) \left[ C_{22}^* \frac{\partial^4}{\partial x^4} + (2C_{32}^* + C_{23}^*) \frac{\partial^4}{\partial x^3 \partial y} \right. \\ & + (C_{12}^* + C_{21}^* + 2C_{33}^*) \frac{\partial^4}{\partial x^2 \partial y^2} + (2C_{31}^* + C_{13}^*) \frac{\partial^4}{\partial x \partial y^3} \\ & \left. + C_{11}^* \frac{\partial^4}{\partial y^4} \right] W dt \end{aligned}$$

$$\begin{aligned} \tau_{xz} = & \int_{-h}^z \left[ C_{22}^* \frac{\partial^3 W}{\partial x^3} + (C_{23}^* + C_{32}^*) \frac{\partial^3 W}{\partial x^2 \partial y} \right. \\ & \left. + (C_{21}^* + C_{33}^*) \frac{\partial^3 W}{\partial x \partial y^2} + C_{31}^* \frac{\partial^3 W}{\partial y^3} \right] t dt \end{aligned}$$

$$\begin{aligned} \tau_{yz} = & \int_{-h}^z \left[ C_{32}^* \frac{\partial^3 W}{\partial x^3} + (C_{12}^* + C_{33}^*) \frac{\partial^3 W}{\partial x^2 \partial y} \right. \\ & \left. + (C_{13}^* + C_{31}^*) \frac{\partial^3 W}{\partial x \partial y^2} + C_{11}^* \frac{\partial^3 W}{\partial y^3} \right] t dt \end{aligned}$$

The transverse shearing stresses automatically vanish at  $z=h$  due to the even-symmetric material property distributions.

### 3. The General Case

If the laminate is not balanced, none of the operators in equations (34) vanish and the in-plane displacements and the transverse displacement are coupled. The edge forces and moments will be needed in order to formulate boundary conditions.

$$N_x = C_{21}^{\circ} \frac{\partial V}{\partial Y} + C_{22}^{\circ} \frac{\partial U}{\partial X} + \frac{1}{2} C_{23}^{\circ} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h I_2^* \tau_{zz} dz + F_{22}^{\circ} T - C_{21}^1 \frac{\partial^2 W}{\partial Y^2} - C_{22}^1 \frac{\partial^2 W}{\partial X^2} - C_{23}^1 \frac{\partial^2 W}{\partial X \partial Y}$$

$$N_y = C_{11}^{\circ} \frac{\partial V}{\partial Y} + C_{12}^{\circ} \frac{\partial U}{\partial X} + \frac{1}{2} C_{13}^{\circ} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h I_1^* \tau_{zz} dz + F_{11}^{\circ} T - C_{11}^1 \frac{\partial^2 W}{\partial Y^2} - C_{12}^1 \frac{\partial^2 W}{\partial X^2} - C_{13}^1 \frac{\partial^2 W}{\partial X \partial Y}$$

$$N_{xy} = C_{31}^{\circ} \frac{\partial V}{\partial Y} + C_{32}^{\circ} \frac{\partial U}{\partial X} + \frac{1}{2} C_{33}^{\circ} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h I_3^* \tau_{zz} dz - F_{12}^{\circ} T - C_{31}^1 \frac{\partial^2 W}{\partial Y^2} - C_{32}^1 \frac{\partial^2 W}{\partial X^2} - C_{33}^1 \frac{\partial^2 W}{\partial X \partial Y}$$

$$M_x = C_{21}^1 \frac{\partial V}{\partial Y} + C_{22}^1 \frac{\partial U}{\partial X} + \frac{1}{2} C_{23}^1 \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h z I_2^* \tau_{zz} dz + F_{22}^1 T - C_{21}^2 \frac{\partial^2 W}{\partial Y^2} - C_{22}^2 \frac{\partial^2 W}{\partial X^2} - C_{23}^2 \frac{\partial^2 W}{\partial X \partial Y}$$

$$M_y = C_{11}^1 \frac{\partial V}{\partial Y} + C_{12}^1 \frac{\partial U}{\partial X} + \frac{1}{2} C_{13}^1 \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h z I_1^* \tau_{zz} dz + F_{11}^1 T - C_{11}^2 \frac{\partial^2 W}{\partial Y^2} - C_{12}^2 \frac{\partial^2 W}{\partial X^2} - C_{13}^2 \frac{\partial^2 W}{\partial X \partial Y}$$

$$M_{xy} = C_{31}^1 \frac{\partial V}{\partial Y} + C_{32}^1 \frac{\partial U}{\partial X} + \frac{1}{2} C_{33}^1 \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) - \int_{-h}^h z I_3^* \tau_{zz} dz - F_{12}^1 T - C_{31}^2 \frac{\partial^2 W}{\partial Y^2} - C_{32}^2 \frac{\partial^2 W}{\partial X^2} - C_{33}^2 \frac{\partial^2 W}{\partial X \partial Y}$$

$$\begin{aligned}
Q_x = & (c_{31}^1 - h c_{31}^{\circ}) \frac{\partial^2 V}{\partial y^2} + \frac{1}{2} (c_{33}^1 - h c_{33}^{\circ} + 2c_{21}^1 - 2hc_{21}^{\circ}) \frac{\partial^2 V}{\partial x \partial y} \\
& + \frac{1}{2} (c_{23}^1 - hc_{23}^{\circ}) \frac{\partial^2 V}{\partial x^2} + \frac{1}{2} (c_{33}^1 - hc_{33}^{\circ}) \frac{\partial^2 U}{\partial y^2} \\
& + \frac{1}{2} (c_{23}^1 - hc_{23}^{\circ} + 2c_{32}^1 - 2hc_{32}^{\circ}) \frac{\partial^2 U}{\partial x \partial y} + (c_{22}^1 - hc_{22}^{\circ}) \frac{\partial^2 U}{\partial x^2} \\
& + \int_{-h}^h \int_{-h}^z \left[ I_3^* \frac{\partial}{\partial y} + I_2^* \frac{\partial}{\partial x} \right] \tau_{zz} dt dz \\
& - (c_{31}^2 - hc_{31}^1) \frac{\partial^3 W}{\partial y^3} - (c_{33}^2 - hc_{33}^1 + c_{21}^2 - hc_{21}^1) \frac{\partial^3 W}{\partial x \partial y^2} \\
& - (c_{32}^2 - hc_{32}^1 + c_{23}^2 - hc_{23}^1) \frac{\partial^3 W}{\partial x^2 \partial y} - (c_{22}^2 - hc_{22}^1) \frac{\partial^3 W}{\partial x^3} \\
Q_y = & (c_{32}^1 - hc_{32}^{\circ}) \frac{\partial^2 U}{\partial x^2} + \frac{1}{2} (c_{33}^1 - hc_{33}^{\circ} + 2c_{12}^1 - 2hc_{12}^{\circ}) \frac{\partial^2 U}{\partial x \partial y} \\
& + \frac{1}{2} (c_{13}^1 - hc_{13}^{\circ}) \frac{\partial^2 U}{\partial y^2} + \frac{1}{2} (c_{33}^1 - hc_{33}^{\circ}) \frac{\partial^2 V}{\partial x^2} + \frac{1}{2} (c_{13}^1 - hc_{13}^{\circ}) \\
& + 2c_{31}^1 - 2hc_{31}^{\circ}) \frac{\partial^2 V}{\partial x \partial y} + (c_{11}^1 - hc_{11}^{\circ}) \frac{\partial^2 V}{\partial y^2} \\
& + \int_{-h}^h \int_{-h}^z \left[ I_3^* \frac{\partial}{\partial x} + I_1^* \frac{\partial}{\partial y} \right] \tau_{zz} dt dz \\
& - (c_{32}^2 - hc_{32}^1) \frac{\partial^3 W}{\partial x^3} - (c_{33}^2 - hc_{33}^1 + c_{12}^2 - hc_{12}^1) \frac{\partial^3 W}{\partial x^2 \partial y} \\
& - (c_{31}^2 - hc_{31}^1 + c_{13}^2 - hc_{13}^1) \frac{\partial^3 W}{\partial x \partial y^2} - (c_{11}^2 - hc_{11}^1) \frac{\partial^3 W}{\partial y^3}
\end{aligned}$$

The stresses are

$$\begin{aligned}
\tau_{xx} = & c_{21}^* \frac{\partial V}{\partial y} + c_{22}^* \frac{\partial U}{\partial x} + \frac{c_{23}^*}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\
& - \left( c_{21}^* \frac{\partial^2 W}{\partial y^2} + c_{22}^* \frac{\partial^2 W}{\partial x^2} + c_{23}^* \frac{\partial^2 W}{\partial x \partial y} \right) z - I_2^* \tau_{zz} + F_{22}^* T
\end{aligned}$$

$$\tau_{yy} = C_{11}^* \frac{\partial V}{\partial y} + C_{12}^* \frac{\partial U}{\partial x} + \frac{C_{13}^*}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\ - \left( C_{11}^* \frac{\partial^2 W}{\partial y^2} + C_{12}^* \frac{\partial^2 W}{\partial x^2} + C_{13}^* \frac{\partial^2 W}{\partial x \partial y} \right) z - I_1^* \tau_{zz} + F_{11}^* T$$

$$\tau_{xy} = C_{31}^* \frac{\partial V}{\partial y} + C_{32}^* \frac{\partial U}{\partial x} + \frac{C_{33}^*}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\ - \left( C_{31}^* \frac{\partial^2 W}{\partial y^2} + C_{32}^* \frac{\partial^2 W}{\partial x^2} + C_{33}^* \frac{\partial^2 W}{\partial x \partial y} \right) z - I_3^* \tau_{zz} - F_{12}^* T$$

$$\tau_{zz} = \int_{-h}^z (z-t) \left[ \left( C_{22}^* \frac{\partial^3}{\partial x^3} + \left( 2C_{32}^* + \frac{1}{2} C_{23}^* \right) \frac{\partial^3}{\partial x^2 \partial y} \right. \right. \\ + \left. \left( C_{12}^* + C_{33}^* \right) \frac{\partial^3}{\partial x \partial y^2} + \frac{1}{2} C_{13}^* \frac{\partial^3}{\partial y^3} \right) U \\ + \left. \left( \frac{1}{2} C_{23}^* \frac{\partial^3}{\partial x^3} + \left( C_{21}^* + C_{33}^* \right) \frac{\partial^3}{\partial x^2 \partial y} + \left( 2C_{31}^* + \frac{1}{2} C_{13}^* \right) \frac{\partial^3}{\partial x \partial y^2} \right. \right. \\ + \left. \left. C_{11}^* \frac{\partial^3}{\partial y^3} \right) V \right] dt - \int_{-h}^z t(z-t) \left[ C_{22}^* \frac{\partial^4}{\partial x^4} \right. \\ + \left. \left( 2C_{32}^* + C_{23}^* \right) \frac{\partial^4}{\partial x^3 \partial y} + \left( C_{21}^* + C_{12}^* + 2C_{33}^* \right) \frac{\partial^4}{\partial x^2 \partial y^2} \right. \\ + \left. \left. \left( 2C_{31}^* + C_{13}^* \right) \frac{\partial^4}{\partial x \partial y^3} + C_{11}^* \frac{\partial^4}{\partial y^4} \right] W dt$$

$$\tau_{xz} = - \int_{-h}^z \left[ C_{22}^* \frac{\partial^2 U}{\partial x^2} + \left( C_{33}^* + \frac{1}{2} C_{23}^* \right) \frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2} C_{33}^* \frac{\partial^2 U}{\partial y^2} \right. \\ + \frac{1}{2} C_{23}^* \frac{\partial^2 V}{\partial x^2} + \left( C_{21}^* + \frac{1}{2} C_{33}^* \right) \frac{\partial^2 V}{\partial x \partial y} + C_{31}^* \frac{\partial^2 V}{\partial y^2} \\ - \left. \left( C_{22}^* \frac{\partial^3 W}{\partial x^3} + \left( C_{23}^* + C_{32}^* \right) \frac{\partial^3 W}{\partial x^2 \partial y} + \left( C_{21}^* + C_{33}^* \right) \frac{\partial^3 W}{\partial x \partial y^2} \right. \right. \\ + \left. \left. C_{31}^* \frac{\partial^3 W}{\partial y^3} \right) t \right] dt + \int_{-h}^z \left[ I_3^* \frac{\partial}{\partial y} + I_2^* \frac{\partial}{\partial x} \right] \tau_{zz} dt$$

$$\begin{aligned}
\tau_{yz} = & - \int_{-h}^z \left[ C_{32}^* \frac{\partial^2 U}{\partial x^2} + (C_{12}^* + \frac{1}{2} C_{33}^*) \frac{\partial^2 U}{\partial x \partial y} + \frac{1}{2} C_{13}^* \frac{\partial^2 U}{\partial y^2} \right. \\
& + \frac{1}{2} C_{33}^* \frac{\partial^2 V}{\partial x^2} + (C_{31}^* + \frac{1}{2} C_{13}^*) \frac{\partial^2 V}{\partial x \partial y} + C_{11}^* \frac{\partial^2 V}{\partial y^2} \\
& - \left( C_{32}^* \frac{\partial^3 W}{\partial x^3} + (C_{33}^* + C_{12}^*) \frac{\partial^3 W}{\partial x^2 \partial y} + (C_{13}^* + C_{31}^*) \frac{\partial^3 W}{\partial x \partial y^2} \right. \\
& \left. \left. + C_{11}^* \frac{\partial^3 W}{\partial y^3} \right) t \right] dt + \int_{-h}^z \left[ I_3^* \frac{\partial}{\partial x} + I_1^* \frac{\partial}{\partial y} \right] \tau_{zz} dt
\end{aligned}$$

Let us consider first the case in which the laminate is severely unbalanced, that is  $C_{ij}^0$  and  $h^{-1}C_{ij}^1$  are of the same order of magnitude. Then,  $W = O(h^{-3})$  and, from the first two of equations (34),  $U, V = O(h^{-2})$ . Thus,  $P_1$  and  $P_2$  are of order  $h$  and are therefore negligibly small compared to  $L_3W$  and  $L_6W$  which are of order  $h^{-1}$ . In the edge force and moment expressions, the Poisson constraint terms which contain the quantities  $I_i^*$  are negligibly small. None of the other terms can be dropped.

Now, let us suppose that the laminate is only slightly unbalanced, so that  $C_{ij}^1 = O(h^4)$ . Then,  $W = O(h^{-3})$  as before and  $U, V$  are of order  $h^0$ . None of the terms in the first two of equations (34) can be dropped but the terms  $L_7U$  and  $L_8V$  in the last of equations (34) are negligibly small. No terms can be dropped from the edge force expressions but the  $U, V$  and  $I_i^n$  terms can be neglected in the expressions for the moments and the transverse shear forces.

If the equations for the general case are to be equally applicable regardless of the degree of unbalance of the laminate, then

no terms can be dropped from the differential equations (34). The only terms which can be dropped from the edge force and moments are the  $I_i^*$  terms in the bending moment and transverse shear forces. The terms containing  $I_i^*$  in the stresses can be dropped since they are negligible in both of the above cases.

The equations derived in this section, which are the equations of ordinary, laminated plate theory, have been derived from the equations of elasticity for laminated, anisotropic plates by making two assumptions:

- (1) The displacements are linear functions of the  $z$  coordinate.
- (2) Terms of order  $h^2$  are negligible compared to unity.

The assumptions customarily made in deriving the ordinary laminated plate equations are therefore dependent only on these two assumptions. The expressions for the transverse normal and shearing stresses which have been derived are those which would be obtained from the ordinary theory by using the equilibrium equations.

$$\begin{aligned} \tau_{xz} &= - \int_{-h}^z \left[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right] dt \\ \tau_{yz} &= - \int_{-h}^z \left[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \right] dt \\ \tau_{zz} &= \int_{-h}^z (z-t) \left[ \frac{\partial^2 \tau_{xx}}{\partial x^2} + 2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{yy}}{\partial y^2} \right] dt \end{aligned}$$

If the "first order theory" of laminated, anisotropic plates (the present simplified theory) proves inadequate, then the reason must be due to the above two assumptions and their modification must then lead to improvements in the accuracy of the theory. An obvious modification is to assume that the displacements are

quadratic functions of the  $z$  coordinate. It should be remembered, however, that any representation of the displacements in terms of continuously differentiable functions is inherently limited due to the fact that the displacements are, in reality, elements of class  $C^0(z)$ .

The expansion (32) is a formal one and the relative magnitude of each of the terms is unknown at the start. In fact, it has been shown that, in the general case, the first and second terms in the expansions for  $u_1$  and  $u_2$  are of the same order of magnitude. For a balanced laminate, the second terms are actually larger than the first terms. One can only hope that, in creating a higher order theory, the correction terms eventually turn out to be smaller than the first few terms in the expansion. This, of course, must be verified after the fact.

## SECTION VII

### A PLATE THEORY CORRECTED FOR TRANSVERSE NORMAL AND SHEARING DEFORMATION

As was pointed out in the previous section, the ordinary laminated plate equations are based on the assumption of a displacement field which is linear in the  $z$  coordinate and on the neglect of all terms of order  $h^2$ . Even with this limitation, however, the complete stress field, including the transverse normal and shearing stresses, can still be obtained, at least to an accuracy consistent with the stated assumptions. For moderately thick or highly anisotropic plates, the previous assumptions are open to question. In order to develop a theory which is corrected for transverse normal and shearing deformation, both of the previous assumptions will have to be discarded, since as will be shown, they are actually not independent of one another.

The linear displacement field given by equation (32) implies that the normal derivatives of the displacement components are continuous across ply interfaces, an implication which is not supported by equations (1) and (2). An examination of these equations discloses, however, that the discontinuities are proportional to the transverse shearing strains and therefore must be small if the shearing strains are small. It seems reasonable, therefore, to develop a higher order plate theory by assuming arbitrary but small additions to the displacement expressions given by equation (32) which are of class  $C^0(z)$ . The "smallness" of the corrections will be of value in simplifying the resulting analysis. Since the corrections are proportional to the trans-

verse shearing strains and since the transverse shearing stresses must vanish on the lower lateral surface, the correction terms must also vanish at  $z = -h$ .

In accordance with the previous remarks, the displacements will be taken in the form

$$\begin{aligned}
 u_1 &= U - z \frac{\partial W}{\partial X} + \int_{-h}^z U^* dt \\
 u_2 &= V - z \frac{\partial W}{\partial Y} + \int_{-h}^z V^* dt \\
 u_3 &= W + \int_{-h}^z W^* dt
 \end{aligned} \tag{35}$$

These modified expressions are thus consistent with equations (1) and (2) and with the boundary conditions. It should be noted, however, that the interpretation of the displacement components  $U$ ,  $V$  and  $W$  as the displacements referred to the middle surface of the plate is no longer possible.

$$\begin{aligned}
 u_1(x, y, 0) &= U + \int_{-h}^0 U^* dt \\
 u_2(x, y, 0) &= V + \int_{-h}^0 V^* dt \\
 u_3(x, y, 0) &= W + \int_{-h}^0 W^* dt
 \end{aligned}$$

If the corrected theory is to be easily reducible to the previous theory, this inconvenience will have to be tolerated.

This approach may be compared to an alternative method which consists of a Volterra expansion of equations (4) and (17) for the stress functions  $A_{11}$ ,  $A_{22}$ ,  $A_{12}$  and  $A_{33}$ . Equations (29) and (32) are, in fact, the first few terms of such an expansion. The displacements  $u_1(x, y, -h)$ ,  $u_2(x, y, -h)$  and  $u_3(x, y, -h)$

are carried along as unknowns and later determined by applying the boundary conditions at the upper lateral face,

$$\tau_{xz}(x,y,h) = \tau_{yz}(x,y,h) = 0$$

$$\tau_{zz}(x,y,h) = -P$$

These result in three differential equations for the determination of the displacements  $u_1(x,y,-h)$ ,  $u_2(x,y,-h)$ ,  $u_3(x,y,-h)$ . This procedure is thus directly comparable to that recommended at the beginning of this section.

The expressions for the strains which follow from equations (35) are

$$\gamma_{xx} = A_{22} = \frac{\partial U}{\partial x} - z \frac{\partial^2 W}{\partial x^2} + \int_{-h}^z \frac{\partial U^*}{\partial x} dt$$

$$\gamma_{yy} = A_{11} = \frac{\partial V}{\partial y} - z \frac{\partial^2 W}{\partial y^2} + \int_{-h}^z \frac{\partial V^*}{\partial y} dt$$

$$2 \gamma_{xy} = 2 A_{12} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} - 2 z \frac{\partial^2 W}{\partial x \partial y} + \int_{-h}^z \left[ \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right] dt$$

$$\epsilon_{zz} = W^*$$

$$2 \epsilon_{xz} = U^* + \int_{-h}^z \frac{\partial W^*}{\partial x} dt$$

$$2 \epsilon_{yz} = V^* + \int_{-h}^z \frac{\partial W^*}{\partial y} dt$$

Using the fourth of equations (20) gives an expression for  $W^*$ .

$$W^* \cong I_4^* \left[ \frac{\partial V}{\partial y} - z \frac{\partial^2 W}{\partial y^2} + \int_{-h}^z \frac{\partial V^*}{\partial y} dt \right]$$

$$\begin{aligned}
& + I_5^* \left[ \frac{\partial U}{\partial x} - z \frac{\partial^2 W}{\partial x^2} + \int_{-h}^z \frac{\partial U^*}{\partial x} dt \right] \\
& + I_6^* \left[ \frac{1}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - \frac{\partial^2 W}{\partial x \partial y} \right. \\
& \left. + \frac{1}{2} \int_{-h}^z \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) dt \right]
\end{aligned}$$

Terms of order  $h^2$  have been neglected.

The last two of the above equations can now be used to obtain expressions for  $U^*$  and  $V^*$ .

$$U^* = 2 \gamma_{xz} - \int_{-h}^z \frac{\partial W^*}{\partial x} dt$$

$$V^* = 2 \gamma_{yz} - \int_{-h}^z \frac{\partial W^*}{\partial y} dt$$

Assuming that  $U$ ,  $V$  and  $W$  are given, the last three equations, when written out in full (using equations 20)), together with equation (17), are a set of four integro-differential equations for the four unknown quantities  $U^*$ ,  $V^*$ ,  $W^*$  and  $A_{33}$  in terms of  $U$ ,  $V$  and  $W$ .

$$\begin{aligned}
U^* = 2 b_{55}^* \int_{-h}^z & - \left[ \left( C_{21}^* \frac{\partial}{\partial x} + C_{31}^* \frac{\partial}{\partial y} \right) \left( \frac{\partial V}{\partial y} - t \frac{\partial^2 W}{\partial y^2} \right. \right. \\
& + \left. \int_{-h}^t \frac{\partial V^*}{\partial y} ds \right) - \left( C_{22}^* \frac{\partial}{\partial x} + C_{32}^* \frac{\partial}{\partial y} \right) \left( \frac{\partial U}{\partial x} - t \frac{\partial^2 W}{\partial x^2} \right. \\
& + \left. \int_{-h}^t \frac{\partial U^*}{\partial x} ds \right) - \left( C_{23}^* \frac{\partial}{\partial x} + C_{33}^* \frac{\partial}{\partial y} \right) \left( \frac{1}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \right. \\
& \left. - t \frac{\partial^2 W}{\partial x \partial y} + \frac{1}{2} \int_{-h}^t \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) ds \right) + \left( I_2^* \frac{\partial}{\partial x} \right.
\end{aligned}$$

$$\begin{aligned}
& + I_3^* \left( \frac{\partial}{\partial Y} \right) A_{33} \Big] dt \\
& + 2 b_{56}^* \int_{-h}^z \left[ - \left( C_{31}^* \frac{\partial}{\partial X} + C_{11}^* \frac{\partial}{\partial Y} \right) \left( \frac{\partial V}{\partial Y} - t \frac{\partial^2 W}{\partial Y^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial V^*}{\partial Y} ds \right) - \left( C_{32}^* \frac{\partial}{\partial X} + C_{12}^* \frac{\partial}{\partial Y} \right) \left( \frac{\partial U}{\partial X} - t \frac{\partial^2 W}{\partial X^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial U^*}{\partial X} ds \right) - \left( C_{33}^* \frac{\partial}{\partial X} + C_{13}^* \frac{\partial}{\partial Y} \right) \left( \frac{1}{2} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) \right. \right. \quad (36) \\
& - \left. \left. t \frac{\partial^2 W}{\partial X \partial Y} + \frac{1}{2} \int_{-h}^t \left( \frac{\partial U^*}{\partial Y} + \frac{\partial V^*}{\partial X} \right) ds \right) + \left( I_3^* \frac{\partial}{\partial X} + I_1^* \frac{\partial}{\partial Y} \right) A_{33} \Big] dt \\
& - \int_{-h}^z \frac{\partial}{\partial X} \left[ I_4^* \left( \frac{\partial V}{\partial Y} - t \frac{\partial^2 W}{\partial Y^2} + \int_{-h}^t \frac{\partial V^*}{\partial Y} ds \right) \right. \\
& + \left. I_5^* \left( \frac{\partial U}{\partial X} - t \frac{\partial^2 W}{\partial X^2} + \int_{-h}^t \frac{\partial U^*}{\partial X} ds \right) \right. \\
& + \left. \frac{1}{2} I_6^* \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} - 2t \frac{\partial^2 W}{\partial X \partial Y} + \int_{-h}^t \left( \frac{\partial U^*}{\partial Y} + \frac{\partial V^*}{\partial X} \right) ds \right) + I_7^* A_{33} \Big] dt
\end{aligned}$$

$$\begin{aligned}
v^* = & 2 b_{65}^* \int_{-h}^z \left[ - \left( C_{31}^* \frac{\partial}{\partial Y} + C_{21}^* \frac{\partial}{\partial X} \right) \left( \frac{\partial V}{\partial Y} - t \frac{\partial^2 W}{\partial Y^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial V^*}{\partial Y} ds \right) - \left( C_{22}^* \frac{\partial}{\partial X} + C_{32}^* \frac{\partial}{\partial Y} \right) \left( \frac{\partial U}{\partial X} - t \frac{\partial^2 W}{\partial X^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial U^*}{\partial X} ds \right) - \left( C_{23}^* \frac{\partial}{\partial X} + C_{33}^* \frac{\partial}{\partial Y} \right) \left( \frac{1}{2} \left( \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \right) \right. \right. \quad (37) \\
& - \left. \left. t \frac{\partial^2 W}{\partial X \partial Y} + \frac{1}{2} \int_{-h}^t \left( \frac{\partial U^*}{\partial Y} + \frac{\partial V^*}{\partial X} \right) ds \right) + \left( I_2^* \frac{\partial}{\partial X} \right. \right. \\
& + \left. \left. I_3^* \frac{\partial}{\partial Y} \right) A_{33} \Big] dt
\end{aligned}$$

$$\begin{aligned}
& + 2 b_{66}^* \int_{-h}^z \left[ - \left( c_{31}^* \frac{\partial}{\partial x} + c_{11}^* \frac{\partial}{\partial y} \right) \left( \frac{\partial U}{\partial y} - t \frac{\partial^2 W}{\partial y^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial V^*}{\partial y} ds \right) - \left( c_{32}^* \frac{\partial}{\partial x} + c_{12}^* \frac{\partial}{\partial y} \right) \left( \frac{\partial U}{\partial x} - t \frac{\partial^2 W}{\partial x^2} \right. \right. \\
& + \left. \left. \int_{-h}^t \frac{\partial U^*}{\partial x} ds \right) - \left( c_{33}^* \frac{\partial}{\partial x} + c_{13}^* \frac{\partial}{\partial y} \right) \left( \frac{1}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \right. \right. \\
& - \left. \left. t \frac{\partial^2 W}{\partial x \partial y} + \frac{1}{2} \int_{-h}^t \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) ds \right) + \left( I_3^* \frac{\partial}{\partial x} \right. \right. \\
& + \left. \left. I_1^* \frac{\partial}{\partial y} \right) A_{33} \right] dt - \int_{-h}^z \frac{\partial}{\partial y} \left[ I_4^* \left( \frac{\partial V}{\partial y} - t \frac{\partial^2 W}{\partial y^2} + \int_{-h}^t \frac{\partial V^*}{\partial y} ds \right) \right. \\
& + \left. I_5^* \left( \frac{\partial U}{\partial x} - t \frac{\partial^2 W}{\partial x^2} + \int_{-h}^t \frac{\partial U^*}{\partial x} ds \right) + \frac{1}{2} I_6^* \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \right. \\
& \left. - 2t \frac{\partial^2 W}{\partial x \partial y} + \int_{-h}^t \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) ds + I_7^* A_{33} \right] dt
\end{aligned} \tag{37}$$

$$\begin{aligned}
W^* & = I_4^* \left( \frac{\partial V}{\partial y} - z \frac{\partial^2 W}{\partial y^2} + \int_{-h}^z \frac{\partial V^*}{\partial y} dt \right) \\
& + I_5^* \left( \frac{\partial U}{\partial x} - z \frac{\partial^2 W}{\partial x^2} + \int_{-h}^z \frac{\partial U^*}{\partial x} dt \right) \\
& + \frac{1}{2} I_6^* \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} - 2z \frac{\partial^2 W}{\partial x \partial y} + \int_{-h}^z \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) dt \right) \\
& + I_7^* A_{33}
\end{aligned} \tag{38}$$

$$\begin{aligned}
A_{33} = & \int_{-h}^z (z-t) \left[ \left( C_{21}^* \frac{\partial^2}{\partial x^2} + C_{11}^* \frac{\partial^2}{\partial y^2} + 2C_{31}^* \frac{\partial^2}{\partial x \partial y} \right) \right. \\
& \left( \frac{\partial V}{\partial y} - t \frac{\partial^2 W}{\partial y^2} + \int_{-h}^t \frac{\partial V^*}{\partial y} ds \right) + \left( C_{22}^* \frac{\partial^2}{\partial x^2} + C_{12}^* \frac{\partial^2}{\partial y^2} \right. \\
& \left. + 2C_{32}^* \frac{\partial^2}{\partial x \partial y} \right) \left( \frac{\partial U}{\partial x} - t \frac{\partial^2 W}{\partial x^2} + \int_{-h}^t \frac{\partial U^*}{\partial x} ds \right) \\
& + \frac{1}{2} \left( C_{23}^* \frac{\partial^2}{\partial x^2} + C_{13}^* \frac{\partial^2}{\partial y^2} + 2C_{33}^* \frac{\partial^2}{\partial x \partial y} \right) \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right. \\
& \left. - 2t \frac{\partial^2 W}{\partial x \partial y} + \int_{-h}^t \left( \frac{\partial U^*}{\partial y} + \frac{\partial V^*}{\partial x} \right) ds \right) \\
& \left. - \left( I_2^* \frac{\partial^2}{\partial x^2} + I_1^* \frac{\partial^2}{\partial y^2} + 2I_3^* \frac{\partial^2}{\partial x \partial y} \right) A_{33} \right] dt
\end{aligned} \tag{39}$$

Equations (36) through (39) are exact equations of elasticity.

The equilibrium equations, the internal continuity conditions, and the boundary conditions at  $z=-h$  have all been identically satisfied.

Equations (36) through (39) are integro-differential equations of Volterra type. Solutions of these equations obtained by Volterra iteration will be rapidly convergent since the range of integration ( $2h$ ) is small. The Volterra iteration solutions coincide with the asymptotic solutions of these equations in powers of the thickness parameter,  $h$ . The powerful existence and uniqueness theorems of Volterra equation theory guarantee that these solutions are regular for all values of  $z$ . Further, since  $U^*$  and  $V^*$  are of order  $p h^{-1}$  and  $W^*$  is of order  $p h^{-2}$ , the corrections to equations (35) are of order  $h^2$  compared to the terms

U, V and W. The one-term Volterra iteration solutions of equations (36) through (39) will therefore produce a plate theory which is correct to within terms of the order of  $h^2$ .

$$\begin{aligned}
 U^* \cong & -2b_{55}^* \int_{-h}^z \left[ L_1^* U + L_2^* V - L_3^* W \right] dt \\
 & -2b_{56}^* \int_{-h}^z \left[ L_4^* U + L_5^* V - L_6^* W \right] dt \\
 & - \int_{-h}^z \left[ \left( I_5^* \frac{\partial^2}{\partial x^2} + \frac{1}{2} I_6^* \frac{\partial^2}{\partial x \partial y} \right) U \right. \\
 & + \left. \left( \frac{1}{2} I_6^* \frac{\partial^2}{\partial x^2} + I_4^* \frac{\partial^2}{\partial x \partial y} \right) V \right. \\
 & \left. - t \left( I_5^* \frac{\partial^3}{\partial x^3} + I_6^* \frac{\partial^3}{\partial x^2 \partial y} + I_4^* \frac{\partial^3}{\partial x \partial y^2} \right) W \right] dt
 \end{aligned} \tag{40a}$$

$$\begin{aligned}
 V^* \cong & -2b_{65}^* \int_{-h}^z \left[ L_1^* U + L_2^* V - L_3^* W \right] dt \\
 & -2b_{66}^* \int_{-h}^z \left[ L_4^* U + L_5^* V - L_6^* W \right] dt \\
 & - \int_{-h}^z \left[ \left( I_5^* \frac{\partial^2}{\partial x \partial y} + \frac{1}{2} I_6^* \frac{\partial^2}{\partial y^2} \right) U + \frac{1}{2} \left( I_6^* \frac{\partial^2}{\partial x \partial y} + I_4^* \frac{\partial^2}{\partial y^2} \right) V \right. \\
 & \left. - t \left( I_5^* \frac{\partial^3}{\partial x^2 \partial y} + I_6^* \frac{\partial^3}{\partial x \partial y^2} + I_4^* \frac{\partial^3}{\partial y^3} \right) W \right] dt
 \end{aligned} \tag{40b}$$

$$\begin{aligned}
 W^* \cong & \left[ I_5^* \frac{\partial}{\partial x} + \frac{1}{2} I_6^* \frac{\partial}{\partial y} \right] U \\
 & + \left[ \frac{1}{2} I_6^* \frac{\partial}{\partial x} + I_4^* \frac{\partial}{\partial y} \right] V \\
 & - z \left[ I_5^* \frac{\partial^2}{\partial x^2} + I_6^* \frac{\partial^2}{\partial x \partial y} + I_4^* \frac{\partial^2}{\partial y^2} \right] W
 \end{aligned} \tag{40c}$$

$$A_{33} \cong \int_{-h}^z (z-t) \left[ L_7^* U + L_8^* V - L_9^* W \right] dt \quad (40d)$$

The operators  $L_i^*$  are defined by

$$L_i = \int_{-h}^h L_i^* dz$$

They are thus obtained from the operators following equations (34) by replacing the material constants  $C_{ij}^0$  with the material property functions  $C_{ij}^*$ ,  $C_{ij}^1$  with  $z C_{ij}^*$ , and  $C_{ij}^2$  with  $z^2 C_{ij}^*$ . It is now possible to write corrections ( $\delta A_{ij}$ ) for transverse normal and shearing deformation to the stress functions  $A_{ij}$ .

$$\delta A_{11} = \int_{-h}^z \frac{\partial V^*}{\partial Y} dt$$

$$\delta A_{22} = \int_{-h}^z \frac{\partial U^*}{\partial X} dt$$

$$\delta A_{12} = \frac{1}{2} \int_{-h}^z \left[ \frac{\partial U^*}{\partial Y} + \frac{\partial V^*}{\partial X} \right] dt$$

$$\delta A_{33} = \int_{-h}^z (z-t) \left[ L_7^* \int_{-h}^t U^* ds + L_8^* \int_{-h}^t V^* ds - \left( I_2^* \frac{\partial^2}{\partial X^2} + I_1^* \frac{\partial^2}{\partial Y^2} + 2I_3^* \frac{\partial^2}{\partial X \partial Y} \right) A_{33} \right] dt$$

These corrections will, in turn, lead to correction terms in the differential equations (34). Let the correction terms which are to be added to the left-hand sides of equations (34) be designated by  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  respectively.

$$\begin{aligned}
\delta_1 &= \int_{-h}^h \left[ L_1^* \int_{-h}^z U^* dt + L_2^* \int_{-h}^z V^* dt \right] dz \\
\delta_2 &= \int_{-h}^h \left[ L_4^* \int_{-h}^z U^* dt + L_5^* \int_{-h}^z V^* dt \right] dz \\
\delta_3 &= \int_{-h}^h (h-z) \left[ L_7^* \int_{-h}^z U^* dt + L_8^* \int_{-h}^z V^* dt \right. \\
&\quad \left. - \left( I_2^* \frac{\partial^2}{\partial x^2} + I_1^* \frac{\partial^2}{\partial y^2} + 2I_3^* \frac{\partial^2}{\partial x \partial y} \right) \int_{-h}^z (z-t) \right. \\
&\quad \left. \left[ L_7^* U + L_8^* V - L_9^* W \right] dt \right] dz
\end{aligned} \tag{41}$$

The full expressions for these corrections are, of course, obtained by inserting the expressions for  $U^*$  and  $V^*$  from equations (40). An examination of the results of the expansion shows that the coefficients of the correction operators are of the form

$$\begin{aligned}
K_{ijk\ell mn}^{pq} &= \int_{-h}^h z^p C_{ij}^* \int_{-h}^z b_{k\ell}^* \int_{-h}^t s^q C_{mn}^* ds dt dz \\
i, j &= 1, 2, 3 & k, \ell &= 5, 6 & m, n &= 1, 2, 3 & p, q &= 0, 1 \\
K_{ijk}^{pq} &= \int_{-h}^h z^p C_{ij}^* \int_{-h}^t s^q I_k^* ds dt dz \\
i, j &= 1, 2, 3 & k &= 4, 5, 6 & p, q &= 0, 1 \\
J_{ijk}^{pq} &= \int_{-h}^h z^p I_i^* \int_{-h}^z (z-t) t^q C_{jk}^* dt dz \\
p &= 0, 1 & q &= 0, 1, 2 & i &= 1, 2, 3 & j, k &= 1, 2, 3
\end{aligned} \tag{42}$$

There are 1296 possible coefficients of the type  $K_{ijk\ell mn}^{pq}$  and 108 coefficients of the type  $K_{ijk}^{pq}$ .

The expansion of equations (42) can be facilitated by using the notation

$$K_{ijk\ell mn}^{pq} = K_{ij}^p K_{k\ell} K_{mn}^q$$

for coefficients of the first type, and

$$K_{ijk}^{pq} = K_{ij}^p K_k^q$$

for coefficients of the second type, with the understanding that the indicated operations are not commutative. The corrections to the differential equations (34) may then be written, in operator notation, in terms of a set of coefficients  $l_i$ ,  $m_i$ ,  $n_i$  which must be calculated by a computer program.

$$\begin{aligned} \delta L_1 &= l_1 \frac{\partial^4}{\partial x^4} + l_2 \frac{\partial^4}{\partial x^3 \partial y} + l_3 \frac{\partial^4}{\partial x^2 \partial y^2} + l_4 \frac{\partial^4}{\partial x \partial y^3} + l_5 \frac{\partial^4}{\partial y^4} \\ \delta L_2 &= l_6 \frac{\partial^4}{\partial x^4} + l_7 \frac{\partial^4}{\partial x^3 \partial y} + l_8 \frac{\partial^4}{\partial x^2 \partial y^2} + l_9 \frac{\partial^4}{\partial x \partial y^3} + l_{10} \frac{\partial^4}{\partial y^4} \\ \delta L_3 &= l_{11} \frac{\partial^5}{\partial x^5} + l_{12} \frac{\partial^5}{\partial x^4 \partial y} + l_{13} \frac{\partial^5}{\partial x^3 \partial y^2} + l_{14} \frac{\partial^5}{\partial x^2 \partial y^3} \\ &\quad + l_{15} \frac{\partial^5}{\partial x \partial y^4} + l_{16} \frac{\partial^5}{\partial y^5} \\ \delta L_4 &= m_1 \frac{\partial^4}{\partial x^4} + m_2 \frac{\partial^4}{\partial x^3 \partial y} + m_3 \frac{\partial^4}{\partial x^2 \partial y^2} + m_4 \frac{\partial^4}{\partial x \partial y^3} + m_5 \frac{\partial^4}{\partial y^4} \\ \delta L_5 &= m_6 \frac{\partial^4}{\partial x^4} + m_7 \frac{\partial^4}{\partial x^3 \partial y} + m_8 \frac{\partial^4}{\partial x^2 \partial y^2} + m_9 \frac{\partial^4}{\partial x \partial y^3} + m_{10} \frac{\partial^4}{\partial y^4} \\ \delta L_6 &= m_{11} \frac{\partial^5}{\partial x^5} + m_{12} \frac{\partial^5}{\partial x^4 \partial y} + m_{13} \frac{\partial^5}{\partial x^3 \partial y^2} + m_{14} \frac{\partial^5}{\partial x^2 \partial y^3} \\ &\quad + m_{15} \frac{\partial^5}{\partial x \partial y^4} + m_{16} \frac{\partial^5}{\partial y^5} \end{aligned} \tag{43}$$

$$\delta L_7 = n_1 \frac{\partial^5}{\partial x^5} + n_2 \frac{\partial^5}{\partial x^4 \partial y} + n_3 \frac{\partial^5}{\partial x^3 \partial y^2} + n_4 \frac{\partial^5}{\partial x^2 \partial y^3} \\ + n_5 \frac{\partial^5}{\partial x \partial y^4} + n_6 \frac{\partial^5}{\partial y^5}$$

$$\delta L_8 = n_7 \frac{\partial^5}{\partial x^5} + n_8 \frac{\partial^5}{\partial x^4 \partial y} + n_9 \frac{\partial^5}{\partial x^3 \partial y^2} + n_{10} \frac{\partial^5}{\partial x^2 \partial y^3} \\ + n_{11} \frac{\partial^5}{\partial x \partial y^4} + n_{12} \frac{\partial^5}{\partial y^5}$$

$$\delta L_9 = n_{13} \frac{\partial^6}{\partial x^6} + n_{14} \frac{\partial^6}{\partial x^5 \partial y} + n_{15} \frac{\partial^6}{\partial x^4 \partial y^2} + n_{16} \frac{\partial^6}{\partial x^3 \partial y^3} \\ + n_{17} \frac{\partial^6}{\partial x^2 \partial y^4} + n_{18} \frac{\partial^6}{\partial x \partial y^5} + n_{19} \frac{\partial^6}{\partial y^6}$$

The computer program for calculating the constants  $K_{ijklmn}^{pq}$ ,  $K_{ijk}^{pq}$  and the coefficients of the above operators is described in Appendix II together with the results of a sample calculation for a five ply plate.

The coefficients of the uncorrected differential equations (34) for an "isotropic", five ply, 0-90 laminate are given below. These coefficients were calculated by the computer program PLY.

DIFF. EQ. COEFF. ARE (multiply by E6)  
(listed in decreasing order of X-derivatives)

FIRST EQUATION

1.57975	1.33807 E-6	0.06	1.39452 E-6	0.442785
3.56595 E-6	0	-1.48389 E-15	-2.54365 E-10	-1.84575 E-15

SECOND EQUATION

-5.64515 E-8	0.442785	5.3573 E-6	0.06	8.92325 E-6
1.57975	2.87856 E-17	-2.39822 E-10	-6.69603 E-15	-7.02698 E-10

THIRD EQUATION

9.47848 E-2	7.6897 E-8	3.01671 E-8	3.21438 E-7	8.36712 E-8
3.01671 E-2	7.49352 E-7	9.47848 E-2	-2.62481 E-3	-1.8733 E-9
-1.20668 E-3	-1.24926 E-8	-1.16658 E-3		

The coefficients of the correction operators given by equations (43) are shown in the following table. These were computed by the computer program DIFFCO.

COEFF. OF CORRECTION OPERATORS: DELTA L(1):  
(MULTIPLY BY E6)  
(LISTED IN DECREASING ORDER OF X-DERIVATIVES)

FIRST EQUATION

-0.190196	1.12672 E-2	-2.42698 E-2	-4.51758 E-4	-1.83759 E-4
4.37318 E-3	-3.07968 E-2	2.65778 E-3	-2.38774 E-2	-1.83943 E-7
-7.22392 E-3	6.70288 E-4	-2.2455 E-3	6.30183 E-5	-6.03174 E-4
-4.37178 E-9				

SECOND EQUATION

-1.1275 E-9	-5.60855 E-2	5.52693 E-3	-4.68619 E-2	-3.46724 E-3
-3.67503 E-4	4.14684 E-4	-1.08149 E-2	-1.10292 E-2	-0.078595
-1.20691 E-10	-2.19585 E-3	3.46082 E-4	-2.77002 E-3	-3.05428 E-4
-1.90015 E-3				

THIRD EQUATION

-4.22148 E-3	2.4875 E-4	-0.002003	-2.06286 E-5	-1.60055 E-3
-1.35644 E-4	1.51649 E-6	-1.32916 E-4	6.70463 E-5	-8.88607 E-4
-4.26205 E-4	-2.87753 E-3	-1.89911 E-4	1.52315 E-5	-3.4868 E-4
1.19751 E-5	-1.30294 E-4	-1.95585 E-5	-7.73107 E-5	

Some of these coefficients should be identically zero for a balanced laminate but are not computed as such due to roundoff errors. In order to make a valid comparison between the coefficients of equations (34) and the correction operators, the coefficients of the correction operators should be divided by the square of the plate length since the order of the correction operators is two higher than that of the coefficients of equations (34). It can be seen that, even for an "isotropic", balanced laminate, some of the corrections are not negligible.

The corrected plate theory leads to a set of governing partial differential equations of eighteenth order in the three displacement components U, V and W. Once these three functions have been determined, the stresses can be found from equations (18) and

(20). The transverse normal and shearing stresses are included and all of the stresses in the plate are determined to high accuracy.

From an asymptotic point of view, the solutions for  $U$ ,  $V$  and  $W$  fall into two classes; slowly-varying "interior" solutions and rapidly-varying "edge-effect" solutions. The interior solutions correspond to particular solutions of the differential equations in terms of the in-plane variables. The edge-effect solutions correspond to the complementary solutions which are rapidly-varying, a fact which may be determined by inspection. Equations (36)-(38) disclose that the rapidly-varying solutions for  $U$ ,  $V$ ,  $W$  result in  $U^*$ ,  $V^*$ ,  $W^*$  terms which are of order  $h^{-1}$ . In the edge zone, therefore, all of the stresses are of the same order of magnitude.

In order to apply the high order plate theory to practical plate problems, it will be necessary to develop the proper boundary conditions from a variational principle. Expressions for the strain energy and work done by the external forces are thus required.

SECTION VIII  
 WORK AND ENERGY EXPRESSIONS  
 FOR THICK PLATE THEORY

In accordance with the principle of virtual work, the variation of the quantity  $u - w$  must vanish where

$$\begin{aligned} \delta u &= \iiint \tau_{ij} \delta \gamma_{ij} dv \\ \delta w &= \iint_S^V -p \delta u_3(x, y, h) dS + \iint_B T_i^B \delta u_i^B dB \end{aligned} \quad (44)$$

and

$$T_i^B = \bar{\tau}_{ij} n_j^B, \quad \bar{n}^B = \bar{e}_\alpha n_\alpha^B$$

$S$  is the area of the upper lateral surface of the plate and  $B$  is the area over which the edge forces are applied. The dependent variables of the problem are the three "displacement" components,  $U, V, W$ . The first task is to develop the expressions for the strains which follow from equations (35). This can be expedited by writing the displacements  $U^*, V^*, W^*$  in the form

$$\begin{aligned} U^* &= \lambda_{11}^* U + \lambda_{12}^* V + \lambda_{13}^* W \\ V^* &= \lambda_{21}^* U + \lambda_{22}^* V + \lambda_{23}^* W \\ W^* &= \lambda_{31}^* U + \lambda_{32}^* V + \lambda_{33}^* W \end{aligned} \quad (45)$$

where the  $\lambda_{ij}^*$  are the appropriate differential operators in the variables  $x$  and  $y$  and integral operators in  $z$  with piecewise constant coefficients. The term  $I_7^* A_{33}$  in  $W^*$  is of higher order and has been dropped. The strains can now be written in

the form

$$\gamma_{xx} = \frac{\partial U}{\partial x} - z \frac{\partial^2 W}{\partial x^2} + \int_{-h}^z \frac{\partial}{\partial x} \left[ \lambda_{11}^* U + \lambda_{12}^* V + \lambda_{13}^* W \right] dt$$

$$\gamma_{yy} = \frac{\partial V}{\partial y} - z \frac{\partial^2 W}{\partial y^2} + \int_{-h}^z \frac{\partial}{\partial y} \left[ \lambda_{21}^* U + \lambda_{22}^* V + \lambda_{23}^* W \right] dt$$

$$\gamma_{xy} = \frac{1}{2} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) - z \frac{\partial^2 W}{\partial x \partial y} + \frac{1}{2} \int_{-h}^z \left\{ \frac{\partial}{\partial y} \left[ \lambda_{11}^* U + \lambda_{12}^* V + \lambda_{13}^* W \right] + \frac{\partial}{\partial x} \left[ \lambda_{21}^* U + \lambda_{22}^* V + \lambda_{23}^* W \right] \right\} dt$$

$$\gamma_{xz} = \frac{1}{2} \left[ \lambda_{11}^* U + \lambda_{12}^* V + \lambda_{13}^* W \right] + \frac{1}{2} \int_{-h}^z \frac{\partial}{\partial x} \left[ \lambda_{31}^* U + \lambda_{32}^* V + \lambda_{33}^* W \right] dt$$

$$\gamma_{yz} = \frac{1}{2} \left[ \lambda_{21}^* U + \lambda_{22}^* V + \lambda_{23}^* W \right] + \frac{1}{2} \int_{-h}^z \frac{\partial}{\partial y} \left[ \lambda_{31}^* U + \lambda_{32}^* V + \lambda_{33}^* W \right] dt$$

$$\gamma_{zz} = \lambda_{31}^* U + \lambda_{32}^* V + \lambda_{33}^* W$$

The strain energy per unit plate thickness is thus

$$\delta u_1 = \iint_S \left\{ \left[ \tau_{xx} \frac{\partial}{\partial x} + \tau_{xy} \frac{\partial}{\partial y} + \tau_{xx} \int_{-h}^z \lambda_{11}^* \frac{\partial}{\partial x} dt \right. \right. \\ \left. \left. + \tau_{yy} \int_{-h}^z \lambda_{21}^* \frac{\partial}{\partial y} dt + \tau_{xy} \int_{-h}^z \left( \lambda_{11}^* \frac{\partial}{\partial y} + \lambda_{21}^* \frac{\partial}{\partial x} \right) dt \right\}$$

$$\begin{aligned}
& + \tau_{xz} \lambda_{11}^* + \tau_{xz} \int_{-h}^z \lambda_{31}^* \frac{\partial}{\partial x} dt + \tau_{yz} \lambda_{21}^* \\
& + \tau_{yz} \int_{-h}^z \lambda_{31}^* \frac{\partial}{\partial y} dt + \tau_{zz} \lambda_{31}^* \left. \vphantom{\int_{-h}^z} \right] \delta U \\
& + \left[ \tau_{yy} \frac{\partial}{\partial y} + \tau_{xy} \frac{\partial}{\partial x} + \tau_{xx} \int_{-h}^z \lambda_{12}^* \frac{\partial}{\partial x} dt \right. \\
& + \tau_{yy} \int_{-h}^z \lambda_{22}^* \frac{\partial}{\partial y} dt + \tau_{xy} \int_{-h}^z \left( \lambda_{12}^* \frac{\partial}{\partial y} + \lambda_{22}^* \frac{\partial}{\partial x} \right) dt \\
& + \tau_{xz} \lambda_{12}^* + \tau_{xz} \int_{-h}^z \lambda_{32}^* \frac{\partial}{\partial x} dt + \tau_{yz} \lambda_{22}^* \\
& \left. + \tau_{yz} \int_{-h}^z \lambda_{32}^* \frac{\partial}{\partial y} dt + \tau_{zz} \lambda_{32}^* \right] \delta V \\
& + \left[ -z \left( \tau_{xx} \frac{\partial^2}{\partial x^2} + \tau_{xy} \frac{\partial^2}{\partial x \partial y} + \tau_{yy} \frac{\partial^2}{\partial y^2} \right) + \tau_{xx} \int_{-h}^z \lambda_{13}^* \frac{\partial}{\partial x} dt \right. \\
& + \tau_{yy} \int_{-h}^z \lambda_{23}^* \frac{\partial}{\partial y} dt + \tau_{xy} \int_{-h}^z \left( \lambda_{13}^* \frac{\partial}{\partial y} + \lambda_{23}^* \frac{\partial}{\partial x} \right) dt \\
& + \tau_{xz} \lambda_{13}^* + \tau_{xz} \int_{-h}^z \lambda_{33}^* \frac{\partial}{\partial x} dt + \tau_{yz} \lambda_{23}^* \\
& \left. + \tau_{yz} \int_{-h}^z \lambda_{33}^* \frac{\partial}{\partial y} dt + \tau_{zz} \lambda_{33}^* \right] \delta W \left. \vphantom{\int_{-h}^z} \right\} dS
\end{aligned} \tag{47a}$$

From equations (35), (44) and (45), the work done by the surface pressure is

$$\delta W_S = \iint_S -P \left[ \delta W + \int_{-h}^h \left( \lambda_{31}^* \delta U + \lambda_{32}^* \delta V + \lambda_{33}^* \delta W \right) dz \right] dS \tag{47b}$$

The components of the stress vector on the edge of the plate are

$$T_1^B = \bar{\tau}_{xx} \frac{\partial y}{\partial \ell} - \bar{\tau}_{xy} \frac{\partial x}{\partial \ell}$$

$$T_2^B = \bar{\tau}_{xy} \frac{\partial y}{\partial \ell} - \bar{\tau}_{yy} \frac{\partial x}{\partial \ell}$$

$$T_3^B = \bar{\tau}_{xz} \frac{\partial y}{\partial \ell} - \bar{\tau}_{yz} \frac{\partial x}{\partial \ell}$$

where  $\ell$  is the arc length measured along the edge of the plate in the middle plane and considered positive when the plate perimeter is traversed with the area on the left. A bar over a stress component denotes an applied stress. The work done by the edge force is now

$$\begin{aligned} \delta W_B = & \oint_L \int_{-h}^h \left\{ \left[ \bar{\tau}_{xx} \frac{\partial y}{\partial \ell} - \bar{\tau}_{xy} \frac{\partial x}{\partial \ell} \right] \left[ \delta U - z \frac{\partial}{\partial x} \delta W \right. \right. \\ & + \left. \int_{-h}^z (\lambda_{11}^* \delta U + \lambda_{12}^* \delta V + \lambda_{13}^* \delta W) dt \right] \\ & + \left. \left[ \bar{\tau}_{xy} \frac{\partial y}{\partial \ell} - \bar{\tau}_{yy} \frac{\partial x}{\partial \ell} \right] \left[ \delta V - z \frac{\partial}{\partial y} \delta W \right. \right. \\ & + \left. \left. \int_{-h}^z (\lambda_{21}^* \delta U + \lambda_{22}^* \delta V + \lambda_{23}^* \delta W) dt \right] \right. \\ & + \left. \left. \left[ \bar{\tau}_{xz} \frac{\partial y}{\partial \ell} - \bar{\tau}_{yz} \frac{\partial x}{\partial \ell} \right] \left[ \delta W + \int_{-h}^z (\lambda_{31}^* \delta U \right. \right. \right. \\ & \left. \left. \left. + \lambda_{32}^* \delta V + \lambda_{33}^* \delta W) dt \right] \right\} dz d\ell \end{aligned} \quad (47c)$$

where  $L$  is the perimeter of the area  $S$ .

In order to formulate the admissible boundary conditions and to check the previous derivation of the differential equations governing the thick plate theory, the integrals over the surface

S in equations (47a) and (47b) must be transformed by means of Gauss's theorem. After this has been done, the full expressions for the stresses in terms of U, V and W can be inserted into the variational principle.

## SECTION IX

## FORMULAS FOR INTEGRATION BY PARTS

Gauss's theorem is:

$$\iint_S \left( \frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y} \right) ds = \oint_L \left( F_1 \frac{\partial y}{\partial \ell} + F_2 \frac{\partial x}{\partial \ell} \right) d\ell$$

Various formulas for integration by parts can be obtained from this theorem by allowing  $F_1$  and  $F_2$  to be products of two functions. For example, if  $F_1 = fu$  and  $F_2 = 0$ , we have

$$\iint_S \frac{\partial}{\partial x} (fu) ds = \oint_L fu \frac{\partial y}{\partial \ell} d\ell. \text{ However, beginning with the mixed,}$$

second order derivative, it is found that two or more forms can be obtained for the surface integral. Taking  $F_1 = f \frac{\partial u}{\partial y}$  and  $F_2 = \frac{\partial f}{\partial x} u$  for example, leads to the expression

$$\iint_S \left( f \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} u \right) ds = \oint_L \left( f \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} + \frac{\partial f}{\partial x} u \frac{\partial x}{\partial \ell} \right) d\ell$$

Taking  $F_1 = \frac{\partial f}{\partial y} u$  and  $F_2 = -f \frac{\partial u}{\partial x}$  leads to the expression

$$\iint_S \left( f \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} u \right) ds = - \oint_L \left( f \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} + \frac{\partial f}{\partial y} u \frac{\partial y}{\partial \ell} \right) d\ell$$

In order to show that both forms are correct, it is necessary to use the fact that the integral of a perfect differential taken around a closed curve vanishes.

$$\oint_L d(fu) = \oint_L \left[ f \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} \right) + u \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial \ell} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \ell} \right) \right] d\ell = 0$$

hence

$$\oint_L \left( f \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} + \frac{\partial f}{\partial x} u \frac{\partial x}{\partial \ell} \right) d\ell = - \oint_L \left( f \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} + \frac{\partial f}{\partial y} u \frac{\partial y}{\partial \ell} \right) d\ell$$

The number of possible forms for each of the derivatives appearing in equations (60) is shown in the following table.

$(m,n)$ in $\iint_S f \frac{\partial^{m+n} u}{\partial x^m \partial y^n} ds$	Number of possible forms
(1,0)	1
(0,1)	1
(2,0)	1
(1,1)	2
(0,2)	1
(3,0)	1
(2,1)	3
(1,2)	3
(0,3)	1
(4,0)	1
(3,1)	4
(2,2)	4
(1,3)	4
(0,4)	1

From the empirical evidence presented in the table, the number of possible forms for the mixed derivative of order  $m+n$  is  $m+n$ . The following table lists all of the combinations of functions which can be used to evaluate the mixed derivatives through the fourth order.

Integrand	$F_1$	$F_2$
$f \frac{\partial^2 u}{\partial x \partial y}$	$f \frac{\partial u}{\partial y}$	$\frac{\partial f}{\partial x} u$
$- \frac{\partial^2 f}{\partial x \partial y} u$	$- \frac{\partial f}{\partial y} u$	$- f \frac{\partial u}{\partial x}$

Integrand	$F_1$	$F_2$
$f \frac{\partial^3 u}{\partial x^2 \partial y}$	$f \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial y} u$	$\frac{\partial f}{\partial x} \frac{\partial u}{\partial x}$
$+ \frac{\partial^3 f}{\partial x^2 \partial y} u$	$-\frac{\partial f}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial x \partial y} u$ $f \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial f}{\partial x} \frac{\partial u}{\partial y}$	$-f \frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 f}{\partial x^2} u$
$f \frac{\partial^3 u}{\partial x \partial y^2}$	$-\frac{\partial f}{\partial y} \frac{\partial u}{\partial y}$ $-\frac{\partial^2 f}{\partial y^2} u$	$-f \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} u$ $-f \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial f}{\partial y} \frac{\partial u}{\partial x}$
$+ \frac{\partial^3 f}{\partial x \partial y^2} u$	$-\frac{\partial f}{\partial y} \frac{\partial u}{\partial y}$	$-f \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} u$
$f \frac{\partial^4 u}{\partial x^3 \partial y}$	$f \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial x} - \frac{\partial^3 f}{\partial x^2 \partial y} u$ $-\frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial x} - \frac{\partial^3 f}{\partial x^2 \partial y} u$	$\frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial x^2}$ $-f \frac{\partial^3 u}{\partial x^3}$
$-\frac{\partial^4 f}{\partial x^3 \partial y} u$	$f \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y} u$ $f \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 f}{\partial x^2} \frac{u}{y}$	$-\frac{\partial^2 f}{\partial x^2} \frac{\partial u}{\partial x}$ $\frac{\partial^3 f}{\partial x^3} u$
$+ \frac{\partial^4 u}{\partial x \partial y^3}$	$-\frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial y^2}$ $\frac{\partial^2 f}{\partial y^2} \frac{\partial u}{\partial y}$	$-f \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial y} + \frac{\partial^3 f}{\partial x \partial y^2} u$ $-f \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^3 f}{\partial x \partial y^2} u$
$-\frac{\partial^4 f}{\partial x \partial y^3} u$	$-\frac{\partial^3 f}{\partial y^3} u$ $f \frac{\partial^3 u}{\partial y^3}$	$-f \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 f}{\partial y^2} \frac{\partial u}{\partial x}$ $\frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial y} + \frac{\partial^3 f}{\partial x \partial y^2} u$
$f \frac{\partial^4 u}{\partial x^2 \partial y^2}$	$f \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial y}$ $f \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial y^2}$	$\frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^3 f}{\partial x^2 \partial y} u$ $-\frac{\partial^2 f}{\partial x^2} \frac{\partial u}{\partial y} + \frac{\partial^3 f}{\partial x^2 \partial y} u$
$-\frac{\partial^4 f}{\partial x^2 \partial y^2} u$	$-\frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^3 f}{\partial x \partial y^2} u$ $\frac{\partial^2}{\partial y^2} \frac{\partial u}{\partial x} - \frac{\partial^3 f}{\partial x \partial y^2} u$	$-f \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial x}$ $-f \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x^2}$

Since the natural boundary conditions are to be obtained by setting equal to zero the coefficients of the various derivatives of  $\delta U$ ,  $\delta V$  and  $\delta W$ , the form taken by these coefficients is extremely important. It seems clear that, in order to obtain correct results, the various possible forms of the results of the application of Gauss's theorem should be averaged. This procedure leads to the following set of formulas for integration by parts.

$$\iint_S f \frac{\partial u}{\partial x} ds = - \iint_S \frac{\partial f}{\partial x} u ds + \oint_L f u \frac{\partial y}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial u}{\partial y} ds = - \iint_S \frac{\partial f}{\partial y} u ds - \oint_L f u \frac{\partial x}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^2 u}{\partial x^2} ds = \iint_S \frac{\partial^2 f}{\partial x^2} u ds + \oint_L \left( f \frac{\partial u}{\partial x} - \frac{\partial f}{\partial x} u \right) \frac{\partial y}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^2 u}{\partial y^2} ds = \iint_S \frac{\partial^2 f}{\partial y^2} u ds - \oint_L \left( f \frac{\partial u}{\partial y} - \frac{\partial f}{\partial y} u \right) \frac{\partial x}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^2 u}{\partial x \partial y} ds = \iint_S \frac{\partial^2 f}{\partial x \partial y} u ds + \frac{1}{2} \oint_L \left[ f \left( \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} - \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} \right) + u \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial \ell} - \frac{\partial f}{\partial y} \frac{\partial y}{\partial \ell} \right) \right] d\ell$$

$$\iint_S f \frac{\partial^3 u}{\partial x^3} ds = - \iint_S \frac{\partial^3 f}{\partial x^3} u ds + \oint_L \left( f \frac{\partial^2 u}{\partial x^2} - \frac{\partial f}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial x^2} u \right) \frac{\partial y}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^3 u}{\partial y^3} ds = - \iint_S \frac{\partial^3 f}{\partial y^3} u ds - \oint_L \left( f \frac{\partial^2 u}{\partial y^2} - \frac{\partial f}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial y^2} u \right) \frac{\partial x}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^3 u}{\partial x^2 \partial y} ds = - \iint_S \frac{\partial^3 f}{\partial x^2 \partial y} u ds + \frac{1}{3} \oint_L \left[ f \left( 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \ell} - \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \ell} \right) \right.$$

$$\left. + \frac{\partial f}{\partial x} \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} - \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} \right) - \frac{\partial f}{\partial y} \frac{\partial u}{\partial x} \frac{\partial y}{\partial \ell} \right.$$

$$\left. + \left( 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial y}{\partial \ell} - \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial \ell} \right) u \right] d\ell$$

$$\iint_S f \frac{\partial^3 u}{\partial x \partial y^2} ds = - \iint_S \frac{\partial^3 f}{\partial x \partial y^2} ds + \frac{1}{3} \oint_L \left[ f \left( \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \ell} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \ell} \right) \right. \\ \left. + \frac{\partial f}{\partial x} \frac{\partial u}{\partial y} \frac{\partial x}{\partial \ell} + \frac{\partial f}{\partial y} \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} - \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} \right) \right. \\ \left. + \left( \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial \ell} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial \ell} \right) u \right] d\ell$$

$$\iint_S f \frac{\partial^4 u}{\partial x^4} ds = \iint_S \frac{\partial^4 f}{\partial x^4} u ds + \oint_L \left( f \frac{\partial^3 u}{\partial x^3} - \frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial x^2} \right. \\ \left. + \frac{\partial^2 f}{\partial x^2} \frac{\partial u}{\partial x} - \frac{\partial^3 f}{\partial x^3} u \right) \frac{\partial y}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^4 u}{\partial y^4} ds = \iint_S \frac{\partial^4 f}{\partial y^4} u ds - \oint_L \left( f \frac{\partial^3 u}{\partial y^3} - \frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial y^2} \right. \\ \left. + \frac{\partial^2 f}{\partial y^2} \frac{\partial u}{\partial y} - \frac{\partial^3 f}{\partial y^3} u \right) \frac{\partial x}{\partial \ell} d\ell$$

$$\iint_S f \frac{\partial^4 u}{\partial x^3 \partial y} ds = \iint_S \frac{\partial^4 f}{\partial x^3 \partial y} u ds + \frac{1}{4} \oint_L \left[ f \left( 3 \frac{\partial^3 u}{\partial x^2 \partial y} \frac{\partial y}{\partial \ell} \right) \right. \\ \left. - \frac{\partial^3 u}{\partial x^3} \frac{\partial x}{\partial \ell} \right) + \frac{\partial f}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \ell} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \ell} \right) \\ \left. - \frac{\partial f}{\partial y} \frac{\partial^2 u}{\partial x^2} \frac{\partial y}{\partial \ell} + \frac{\partial^2 f}{\partial x^2} \left( \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} - \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} \right) \right. \\ \left. + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial x} \frac{\partial y}{\partial \ell} - 3 \frac{\partial^3 f}{\partial x^2 \partial y} u \frac{\partial y}{\partial \ell} + \frac{\partial^3 f}{\partial x^3} u \frac{\partial x}{\partial \ell} \right] d\ell$$

$$\iint_S f \frac{\partial^4 u}{\partial x \partial y^3} ds = \iint_S \frac{\partial^4 f}{\partial x \partial y^3} u ds + \frac{1}{4} \oint_L \left[ f \left( \frac{\partial^3 u}{\partial y^3} \frac{\partial y}{\partial \ell} \right) \right. \\ \left. - 3 \frac{\partial^3 u}{\partial x \partial y^2} \frac{\partial x}{\partial \ell} \right) + \frac{\partial f}{\partial x} \frac{\partial^2 u}{\partial y^2} \frac{\partial x}{\partial \ell} + \frac{\partial f}{\partial y} \left( 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \ell} \right. \\ \left. - \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \ell} \right) - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial u}{\partial y} \frac{\partial x}{\partial \ell} + \frac{\partial^2 f}{\partial y^2} \left( \frac{\partial u}{\partial y} \frac{\partial y}{\partial \ell} \right. \\ \left. - \frac{\partial u}{\partial x} \frac{\partial x}{\partial \ell} \right) + 3 \frac{\partial^3 f}{\partial x \partial y^2} u \frac{\partial x}{\partial \ell} - \frac{\partial^3 f}{\partial y^3} u \frac{\partial y}{\partial \ell} \right] d\ell$$

$$\begin{aligned}
\iint_S f \frac{\partial^4 u}{\partial x^2 \partial y^2} ds &= \iint_S \frac{\partial^4 f}{\partial x^2 \partial y^2} u ds + \frac{1}{4} \oint_L \left[ 2f \left( \frac{\partial^3 u}{\partial x \partial y^2} \frac{\partial y}{\partial l} \right. \right. \\
&\quad \left. \left. - \frac{\partial^3 u}{\partial x^2 \partial y} \frac{\partial x}{\partial l} \right) - \frac{\partial f}{\partial x} \left( \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial l} - \frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial l} \right) \right. \\
&\quad \left. - \frac{\partial f}{\partial y} \left( \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial l} - \frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial l} \right) - \frac{\partial^2 f}{\partial x^2} \frac{\partial u}{\partial y} \frac{\partial x}{\partial l} \right. \\
&\quad \left. + \frac{\partial^2 f}{\partial y^2} \frac{\partial u}{\partial x} \frac{\partial y}{\partial l} + \frac{\partial^2 f}{\partial x \partial y} \left( \frac{\partial u}{\partial y} \frac{\partial y}{\partial l} - \frac{\partial u}{\partial x} \frac{\partial x}{\partial l} \right) \right. \\
&\quad \left. - 2 \left( \frac{\partial^3 f}{\partial x \partial y^2} \frac{\partial y}{\partial l} - \frac{\partial^3 f}{\partial x^2 \partial y} \frac{\partial x}{\partial l} \right) u \right] dl
\end{aligned}$$

SECTION X  
BOUNDARY CONDITIONS

By applying the preceding formulas for integration by parts, the boundary integral part of the variational principle gives the boundary conditions on the edges of the plate. Results have been derived for a general form of curved boundary for which the direction cosines of the outward unit normal are  $\left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s}\right)$  where  $ds$  is the element of arc length of the boundary curve measured by traversing the boundary with the interior of the region on the left. The coefficients appearing in these expressions are defined by the following:

$$P_{ijkl}^n = \int_{-h}^z b_{ij}^* \int_{-h}^t s^n C_{kl}^* ds dt$$

$$P_{ijkl}^{*n} = b_{ij}^* \int_{-h}^z t^n C_{kl}^* dt = \frac{d}{dz} P_{ijkl}^n$$

$$P_i^n = \int_{-h}^z \int_{-h}^t s^n I_i^* ds dt$$

$$P_i^{*n} = \int_{-h}^z t^n I_i^* dt = \frac{d}{dz} P_i^n$$

\* Stresses written with a bar above are applied stresses on the edges of the plate.

$\delta U$ :

$$\int_{-h}^h \left[ \left( T_{xx} + T_{zz} I_5^* \right) \frac{\partial y}{\partial s} - \left( T_{xy} + \frac{1}{2} T_{zz} I_6^* \right) \frac{\partial x}{\partial s} \right]$$

\* Stresses are designated by the symbol  $T_{ij}$ .

$$\begin{aligned}
& + \frac{\partial T_{xz}}{\partial x} \left[ 2 \left( P_{5522}^{*o} + P_{5632}^{*o} \right) \frac{\partial y}{\partial s} - \left( P_{5532}^{*o} + P_{5612}^{*o} \right. \right. \\
& + \left. \left. \frac{1}{2} P_{5523}^{*o} + \frac{1}{2} P_{5633}^{*o} \right) \frac{\partial x}{\partial s} \right] + \frac{\partial T_{xz}}{\partial y} \left[ \left( P_{5532}^{*o} + P_{5612}^{*o} \right. \right. \\
& + \left. \left. \frac{1}{2} P_{5523}^{*o} + \frac{1}{2} P_{5633}^{*o} \right) \frac{\partial y}{\partial s} - \left( P_{5533}^{*o} + P_{5613}^{*o} \right) \frac{\partial x}{\partial s} \right] \\
& + \frac{\partial T_{yz}}{\partial x} \left[ 2 \left( P_{6522}^{*o} + P_{6632}^{*o} \right) \frac{\partial y}{\partial s} - \left( P_{6532}^{*o} + P_{6612}^{*o} \right. \right. \\
& + \left. \left. \frac{1}{2} P_{6523}^{*o} + \frac{1}{2} P_{6633}^{*o} \right) \frac{\partial x}{\partial s} \right] + \frac{\partial T_{yz}}{\partial y} \left[ \left( P_{6532}^{*o} + P_{6612}^{*o} \right. \right. \\
& + \left. \left. \frac{1}{2} P_{6523}^{*o} + \frac{1}{2} P_{6633}^{*o} \right) \frac{\partial y}{\partial s} - \left( P_{6533}^{*o} + P_{6613}^{*o} \right) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^2 T_{xx}}{\partial x^2} \left[ \left( 2 P_{5522}^o + 2 P_{5632}^o + P_5^o \right) \frac{\partial y}{\partial s} - \frac{1}{3} \left( 2 P_{5532}^o + 2 P_{5612}^o \right. \right. \\
& + \left. \left. P_{5523}^o + P_{5633}^o + \frac{1}{2} P_6^o \right) \frac{\partial x}{\partial s} \right] - \frac{\partial^2 T_{xx}}{\partial x \partial y} \left[ \frac{2}{3} \left( 2 P_{5532}^o + 2 P_{5612}^o \right. \right. \\
& + \left. \left. P_{5523}^o + P_{5633}^o + \frac{1}{2} P_6^o \right) \frac{\partial y}{\partial s} - \frac{2}{3} \left( P_{5533}^o + P_{5613}^o \right) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^2 T_{xx}}{\partial y^2} \left[ \frac{1}{3} \left( P_{5533}^o + P_{5613}^o \right) \frac{\partial y}{\partial s} \right] - \frac{\partial^2 T_{xy}}{\partial x^2} \left[ 2 \left( P_{6522}^o + P_{6632}^o \right) \frac{\partial y}{\partial s} \right. \\
& - \left. \frac{2}{3} \left( P_{5522}^o + P_{5632}^o + P_{6532}^o + P_{6612}^o + \frac{1}{2} P_{6523}^o + \frac{1}{2} P_{6633}^o + P_5^o \right) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^2 T_{xy}}{\partial x \partial y} \left[ \frac{4}{3} \left( P_{5522}^o + P_{5632}^o + P_{6532}^o + P_{6612}^o + \frac{1}{2} P_{6523}^o \right. \right. \\
& + \left. \left. \frac{1}{2} P_{6633}^o + P_5^o \right) \frac{\partial y}{\partial s} - \frac{4}{3} \left( P_{5532}^o + P_{5612}^o + \frac{1}{2} P_{5633}^o + \frac{1}{2} P_{5523}^o \right. \right. \\
& + \left. \left. \frac{1}{2} P_{6533}^o + \frac{1}{2} P_{6613}^o + \frac{1}{2} P_6^o \right) \frac{\partial x}{\partial s} \right] - \frac{\partial^2 T_{xy}}{\partial y^2} \left[ \frac{2}{3} \left( P_{5532}^o + P_{5612}^o \right. \right. \\
& + \left. \left. \frac{1}{2} P_{5523}^o + \frac{1}{2} P_{5633}^o + \frac{1}{2} P_{6533}^o + \frac{1}{2} P_{6613}^o + \frac{1}{2} P_6^o \right) \frac{\partial y}{\partial s} \right. \\
& + \left. \left( P_{5533}^o + P_{5613}^o \right) \frac{\partial x}{\partial s} \right] + \frac{\partial^2 T_{yy}}{\partial x^2} \left[ \frac{2}{3} \left( P_{6522}^o + P_{6632}^o \right) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^2 T_{yy}}{\partial x \partial y} \left[ \frac{4}{3} \left( P_{6522}^o + P_{6632}^o \right) \frac{\partial y}{\partial s} - \frac{2}{3} \left( 2 P_{6532}^o + 2 P_{6612}^o + P_{6523}^o \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + P_{6633}^0 + P_5^0) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{YY}}{\partial y^2} \left[ \frac{1}{3} (2 P_{6532}^0 + 2 P_{6612}^0 + P_{6523}^0 + P_{6633}^0 \right. \\
& + P_5^0) \frac{\partial y}{\partial s} - (P_{6533}^0 + P_{6613}^0 + \frac{1}{2} P_6^0) \frac{\partial x}{\partial s} \Big] dz \\
& = \int_{-h}^h \left[ \bar{T}_{xx} \frac{\partial y}{\partial s} - \bar{T}_{xy} \frac{\partial x}{\partial s} \right] dz
\end{aligned}$$

$\delta V$ :

$$\begin{aligned}
& \int_{-h}^h \left[ (T_{xy} + \frac{1}{2} T_{zz} I_6^*) \frac{\partial y}{\partial s} - (T_{yy} + T_{zz} I_4^*) \frac{\partial x}{\partial s} \right. \\
& + \frac{\partial T_{xz}}{\partial x} \left[ (P_{5523}^* + P_{5633}^*) \frac{\partial y}{\partial s} - (P_{5521}^* + P_{5631}^* + \frac{1}{2} P_{5533}^* \right. \\
& + \frac{1}{2} P_{5613}^*) \frac{\partial x}{\partial s} \Big] + \frac{\partial T_{xz}}{\partial y} \left[ (P_{5521}^* + P_{5631}^* + \frac{1}{2} P_{5533}^* + \frac{1}{2} P_{5613}^*) \frac{\partial y}{\partial s} \right. \\
& - 2 (P_{5531}^* + P_{5611}^*) \frac{\partial x}{\partial s} \Big] + \frac{\partial T_{yz}}{\partial x} \left[ (P_{6523}^* + P_{6633}^*) \frac{\partial y}{\partial s} - (P_{6521}^* \right. \\
& + P_{6631}^* + \frac{1}{2} P_{6533}^* + \frac{1}{2} P_{6613}^*) \frac{\partial x}{\partial s} \Big] + \frac{\partial T_{yz}}{\partial y} \left[ (P_{6521}^* + P_{6631}^* \right. \\
& + \frac{1}{2} P_{6533}^* + \frac{1}{2} P_{6613}^*) \frac{\partial y}{\partial s} - 2 (P_{6531}^* + P_{6611}^*) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{xx}}{\partial x^2} (P_{5523}^* \\
& + P_{5633}^* + \frac{1}{2} P_6^0) \frac{\partial y}{\partial s} - \frac{1}{3} (2 P_{5521}^0 + 2 P_{5631}^0 + P_{5533}^0 + P_{5613}^0 \\
& + P_4^0) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{xx}}{\partial x \partial y} \left[ \frac{2}{3} (2 P_{5521}^0 + P_{5533}^0 + 2 P_{5631}^0 + P_{5613}^0 + P_4^0) \frac{\partial y}{\partial s} \right. \\
& - \frac{4}{3} (P_{5531}^0 + P_{5611}^0) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{xx}}{\partial y^2} \left[ \frac{2}{3} (P_{5531}^0 + P_{5611}^0) \frac{\partial y}{\partial s} \right] \\
& - \frac{\partial^2 T_{xy}}{\partial x^2} \left[ (P_{6523}^0 + P_{6633}^0) \frac{\partial y}{\partial s} - \frac{1}{3} (2 P_{6521}^0 + 2 P_{6631}^0 + P_{6533}^0 \right. \\
& + P_{5523}^0 + P_{5633}^0 + P_{6613}^0 + P_6^0) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{xy}}{\partial x \partial y} \left[ \frac{2}{3} (2 P_{6521}^0 + 2 P_{6631}^0 \right. \\
& P_{6533}^0 + P_{6613}^0 + P_{5523}^0 + P_{5633}^0 + P_6^0) \frac{\partial y}{\partial s} - \frac{2}{3} (P_{5521}^0 + P_{5631}^0 \\
& + P_{6531}^0 + P_{6611}^0 + \frac{1}{2} P_{5533}^0 + \frac{1}{2} P_{5613}^0 + P_4^0) \frac{\partial x}{\partial s} \Big] - \frac{\partial^2 T_{xy}}{\partial y^2} \left[ \frac{2}{3} (P_{5521}^0 \right. \\
& + P_{5631}^0 + P_{6531}^0 + P_{6611}^0 + \frac{1}{2} P_{5533}^0 + \frac{1}{2} P_{5613}^0 + P_4^0) \frac{\partial y}{\partial s}
\end{aligned}$$

$$\begin{aligned}
& - 2 \left( P_{5531}^O + P_{5611}^O \right) \frac{\partial x}{\partial s} \Big] + \frac{\partial^2 T_{YY}}{\partial x^2} \left[ \frac{1}{3} \left( P_{6523}^O + P_{6633}^O \right) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^2 T_{YY}}{\partial x \partial y} \left[ \frac{2}{3} \left( P_{6523}^O + P_{6633}^O \right) \frac{\partial y}{\partial s} - \frac{2}{3} \left( 2 P_{6521}^O + 2 P_{6631}^O + P_{6533}^O \right. \right. \\
& + P_{6613}^O + \frac{1}{2} P_6^O \left. \right) \frac{\partial x}{\partial s} - \frac{\partial^2 T_{YY}}{\partial y^2} \left[ \frac{1}{3} \left( 2 P_{6521}^O + 2 P_{6631}^O + P_{6533}^O \right. \right. \\
& + P_{6613}^O + \frac{1}{2} P_6^O \left. \right) \frac{\partial y}{\partial s} - \left. \left. \left( 2 P_{6531}^O + 2 P_{6611}^O + P_4^O \right) \frac{\partial x}{\partial s} \right] \right] dz \\
& = \int_{-h}^h \left[ \bar{T}_{xy} \frac{\partial y}{\partial s} - \bar{T}_{yy} \frac{\partial x}{\partial s} \right] dz
\end{aligned}$$

δW:

$$\begin{aligned}
& \int_{-h}^h \left[ z \left( \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{zz}}{\partial x} I_5^* \right) \frac{\partial y}{\partial s} - z \left( \frac{\partial T_{xy}}{\partial x} + \frac{1}{2} \frac{\partial T_{zz}}{\partial x} I_6^* \right) \frac{\partial x}{\partial s} \right. \\
& + z \left( \frac{\partial T_{xy}}{\partial y} + \frac{1}{2} \frac{\partial T_{zz}}{\partial y} I_6^* \right) \frac{\partial y}{\partial s} - z \left( \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zz}}{\partial y} I_4^* \right) \frac{\partial x}{\partial s} + \frac{\partial^2 T_{xz}}{\partial x^2} \\
& \left. \left[ 2 \left( P_{5522}^{*1} + P_{5632}^{*1} \right) \frac{\partial y}{\partial s} - \frac{2}{3} \left( P_{5532}^{*1} + P_{5523}^{*1} + P_{5612}^{*1} + P_{5633}^{*1} \right) \frac{\partial x}{\partial s} \right] \right. \\
& + \frac{\partial^2 T_{xz}}{\partial x \partial y} \left[ \frac{4}{3} \left( P_{5532}^{*1} + P_{5523}^{*1} + P_{5612}^{*1} + P_{5633}^{*1} \right) \frac{\partial y}{\partial s} - \frac{4}{3} \left( P_{5521}^{*1} + P_{5533}^{*1} \right. \right. \\
& + P_{5631}^{*1} + P_{5613}^{*1} \left. \right) \frac{\partial x}{\partial s} \left. \right] + \frac{\partial^2 T_{xz}}{\partial y^2} \left[ \frac{2}{3} \left( P_{5521}^{*1} + P_{5533}^{*1} + P_{5631}^{*1} \right. \right. \\
& + P_{5613}^{*1} \left. \right) \frac{\partial y}{\partial s} - 2 \left( P_{5531}^{*1} + P_{5611}^{*1} \right) \frac{\partial x}{\partial s} \left. \right] + \frac{\partial^2 T_{yz}}{\partial x^2} \left[ 2 \left( P_{6522}^{*1} + P_{6632}^{*1} \right) \frac{\partial y}{\partial s} \right. \\
& - \frac{2}{3} \left( P_{6532}^{*1} + P_{6523}^{*1} + P_{6612}^{*1} + P_{6633}^{*1} \right) \frac{\partial x}{\partial s} \left. \right] + \frac{\partial^2 T_{yz}}{\partial x \partial y} \left[ \frac{4}{3} \left( P_{6532}^{*1} \right. \right. \\
& + P_{6523}^{*1} + P_{6612}^{*1} + P_{6633}^{*1} \left. \right) \frac{\partial y}{\partial s}
\end{aligned}$$

$$\begin{aligned}
& - \frac{4}{3} \left( P_{6521}^{*1} + P_{6533}^{*1} + P_{6631}^{*1} + P_{6613}^{*1} \right) \frac{\partial x}{\partial s} \Bigg] \\
& + \frac{\partial^2 T_{yz}}{\partial y^2} \left[ \frac{2}{3} \left( P_{6521}^{*1} + P_{6533}^{*1} + P_{6631}^{*1} + P_{6613}^{*1} \right) \frac{\partial y}{\partial s} \right. \\
& - 2 \left( P_{6531}^{*1} + P_{6611}^{*1} \right) \frac{\partial x}{\partial s} \Bigg] - \frac{\partial^3 T_{xx}}{\partial x^3} \left[ \left( 2 P_{5522}^1 + 2 P_{5632}^1 + P_5^1 \right) \frac{\partial y}{\partial s} \right. \\
& + \frac{1}{4} \left( 2 P_{5532}^1 + 2 P_{5523}^1 + 2 P_{5612}^1 + 2 P_{5633}^1 + P_6^1 \right) \frac{\partial x}{\partial s} \Bigg] \\
& - \frac{\partial^3 T_{xx}}{\partial x^2 \partial y} \left[ \frac{3}{4} \left( 2 P_{5532}^1 + 2 P_{5523}^1 + 2 P_{5612}^1 + 2 P_{5633}^1 + P_6^1 \right) \frac{\partial y}{\partial s} \right. \\
& - \frac{1}{2} \left( 2 P_{5521}^1 + 2 P_{5533}^1 + 2 P_{5631}^1 + 2 P_{5613}^1 + P_4^1 \right) \frac{\partial x}{\partial s} \\
& - \frac{\partial^3 T_{xx}}{\partial x \partial y^2} \left[ \frac{1}{2} \left( 2 P_{5521}^1 + 2 P_{5533}^1 + 2 P_{5631}^1 + 2 P_{5613}^1 + P_4^1 \right) \frac{\partial y}{\partial s} \right. \\
& - \frac{3}{2} \left( P_{5531}^1 + P_{5611}^1 \right) \frac{\partial x}{\partial s} \Bigg] - \frac{\partial^3 T_{xx}}{\partial y^3} \left[ \frac{1}{2} \left( P_{5531}^1 + P_{5611}^1 \right) \frac{\partial y}{\partial s} \right] \\
& - \frac{\partial^3 T_{xy}}{\partial x^3} \left[ 2 \left( P_{6522}^1 + P_{6632}^1 \right) \frac{\partial y}{\partial s} - \frac{1}{2} \left( P_{5522}^1 + P_{5632}^1 + P_{6532}^1 \right. \right. \\
& + P_{6523}^1 + P_{6612}^1 + P_{6633}^1 + P_5^1 \Bigg] \frac{\partial x}{\partial s} - \frac{\partial^3 T_{xy}}{\partial x^2 \partial y} \left[ \frac{3}{2} \left( P_{5522}^1 + P_{5632}^1 \right. \right. \\
& + P_{6532}^1 + P_{6523}^1 + P_{6612}^1 + P_{6633}^1 + P_5^1 \Bigg] \frac{\partial y}{\partial s} - \left( P_{5532}^1 + P_{5523}^1 \right. \\
& + P_{5612}^1 + P_{5633}^1 + P_{6521}^1 + P_{6533}^1 + P_{6631}^1 + P_{6613}^1 + P_6^1 \Bigg] \frac{\partial x}{\partial s} \Bigg] \\
& - \frac{\partial^3 T_{xy}}{\partial x \partial y^2} \left[ \left( P_{5532}^1 + P_{5523}^1 + P_{5612}^1 + P_{5633}^1 + P_{6521}^1 + P_{6533}^1 \right. \right. \\
& + P_{6631}^1 + P_{6613}^1 + P_6^1 \Bigg) \frac{\partial y}{\partial s} - \frac{3}{2} \left( P_{5521}^1 + P_{5533}^1 + P_{5631}^1 + P_{5613}^1 \right. \\
& + P_{6531}^1 + P_{6611}^1 + P_4^1 \Bigg) \frac{\partial x}{\partial s} \Bigg] - \frac{\partial^3 T_{xy}}{\partial y^3} \left[ \frac{1}{2} \left( P_{5521}^1 + P_{5533}^1 + P_{5631}^1 \right. \right. \\
& + P_{5613}^1 + P_{6531}^1 + P_{6611}^1 + P_4^1 \Bigg) \frac{\partial y}{\partial s} - 2 \left( P_{5531}^1 + P_{5611}^1 \right) \frac{\partial x}{\partial s} \Bigg]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\partial^3 T_{YY}}{\partial x^3} \left[ \frac{1}{2} (P_{6522}^1 + P_{6632}^1) \frac{\partial x}{\partial s} \right] - \frac{\partial^3 T_{YY}}{\partial x^2 \partial y} \left[ \frac{3}{2} (P_{6522}^1 + P_{6632}^1) \frac{\partial y}{\partial s} \right. \\
& - \left. \frac{1}{2} (2P_{6532}^1 + 2P_{6523}^1 + 2P_{6612}^1 + 2P_{6633}^1 + P_5^1) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^3 T_{YY}}{\partial x \partial y^2} \left[ \frac{1}{2} (2P_{6532}^1 + 2P_{6523}^1 + 2P_{6612}^1 + 2P_{6633}^1 + P_5^1) \frac{\partial y}{\partial s} \right. \\
& - \left. \frac{3}{4} (2P_{6521}^1 + 2P_{6533}^1 + 2P_{6631}^1 + 2P_{6613}^1 + P_6^1) \frac{\partial x}{\partial s} \right] \\
& - \frac{\partial^3 T_{YY}}{\partial y^3} \left[ \frac{1}{4} (2P_{6521}^1 + 2P_{6533}^1 + 2P_{6631}^1 + 2P_{6613}^1 + P_6^1) \frac{\partial y}{\partial s} \right. \\
& - \left. (2P_{6531}^1 + 2P_{6611}^1 + P_4^1) \frac{\partial x}{\partial s} \right] dz = \int_{-h}^h \left[ \bar{T}_{xz} \frac{\partial y}{\partial s} - \bar{T}_{yz} \frac{\partial x}{\partial s} \right] dz
\end{aligned}$$

Other boundary conditions can be written which involve the coefficients of the following variations:

$$\begin{aligned}
& \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y} \right] \delta U \\
& \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y} \right] \delta V \\
& \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^3}{\partial x^3}, \frac{\partial^3}{\partial x^2 \partial y}, \frac{\partial^3}{\partial x \partial y^2}, \frac{\partial^3}{\partial y^3} \right] \delta W
\end{aligned}$$

There are, in all, 22 natural boundary conditions. Further, all of the natural boundary conditions are inhomogeneous. These are all extremely complication expressions. Because of the length and complexity of this corrected theory, its practical application seems highly questionable and, for this reason, the remaining natural boundary conditions will not be written out in full.

## SECTION XI

### DISCUSSION

The Correspondence Principle results are exact within the limitations of the small deformation theory of elasticity. However, the application of these results will be for a finite number of laminations. It was shown in Section IV that good agreement between the Correspondence Principle and "exact" results can be obtained with a relatively small number of isotropic plies. In other cases, the error depends on the ply properties and the lamination scheme. In general, the error depends on functions of the type

$$\int_{-h}^z F^*(z) f(z) dz - \lim_{n \rightarrow \infty} \bar{F} \int_{-h}^z f(z) dz$$

where  $F^*(z)$  is a piece-wise constant function. If the mean value  $\bar{F}$  defined over the entire plate thickness is identical with the mean value defined over a comparatively small number of plies, then good agreement should be expected. Considerations of this nature limit the application of the Correspondence Principle to plates which are constructed using a lamination scheme which consists of a repeating subgroup containing a number of plies which is small compared to the total number of plies in the plate. This rules out the use of the Correspondence Principle for unbalanced laminates.

The attempt to derive a plate theory corrected for transverse normal and shearing deformation, described in Section VII, although technically successful, results in a set of partial differential

equations of high order and an extremely complex set of boundary conditions on the edges of the plate. The difficulty of solving this problem in any practical case renders this attempt a practical failure. Corrections to the classical plate theory are important only in the neighborhood of the edges of the plate where the stress distribution is three dimensional. The starred corrections to the displacement field in equation (35) represent the rapidly varying part of the displacement field near the edges of the plate. The high order corrections to the differential equations (34) which resulted from the inclusion of these additional terms contribute additional, rapidly varying, edge-effect solutions to the problem. In the edge zone, the displacement corrections to equations (35) in terms of  $U^*$ ,  $V^*$ ,  $W^*$  are of the same order as the respective displacements  $U$ ,  $V$ ,  $W$ .

An alternate approach to the development of asymptotic solutions to the stress field near the edges of a laminated plate is to employ the procedure recommended by Reiss and Locke (Ref. 3) for the solution of the generalized plane stress problem in homogeneous media. Briefly, the recommended procedure would be to perform a "stretching transformation" on the coordinate normal to the edge of the plate in equations (4) and then to develop regular asymptotic solutions in powers of the thickness parameter  $h$ . The greatest advantage of this procedure is that it would give the edge solutions directly without having to resort to the simultaneous solution of a high order set of differential equations.

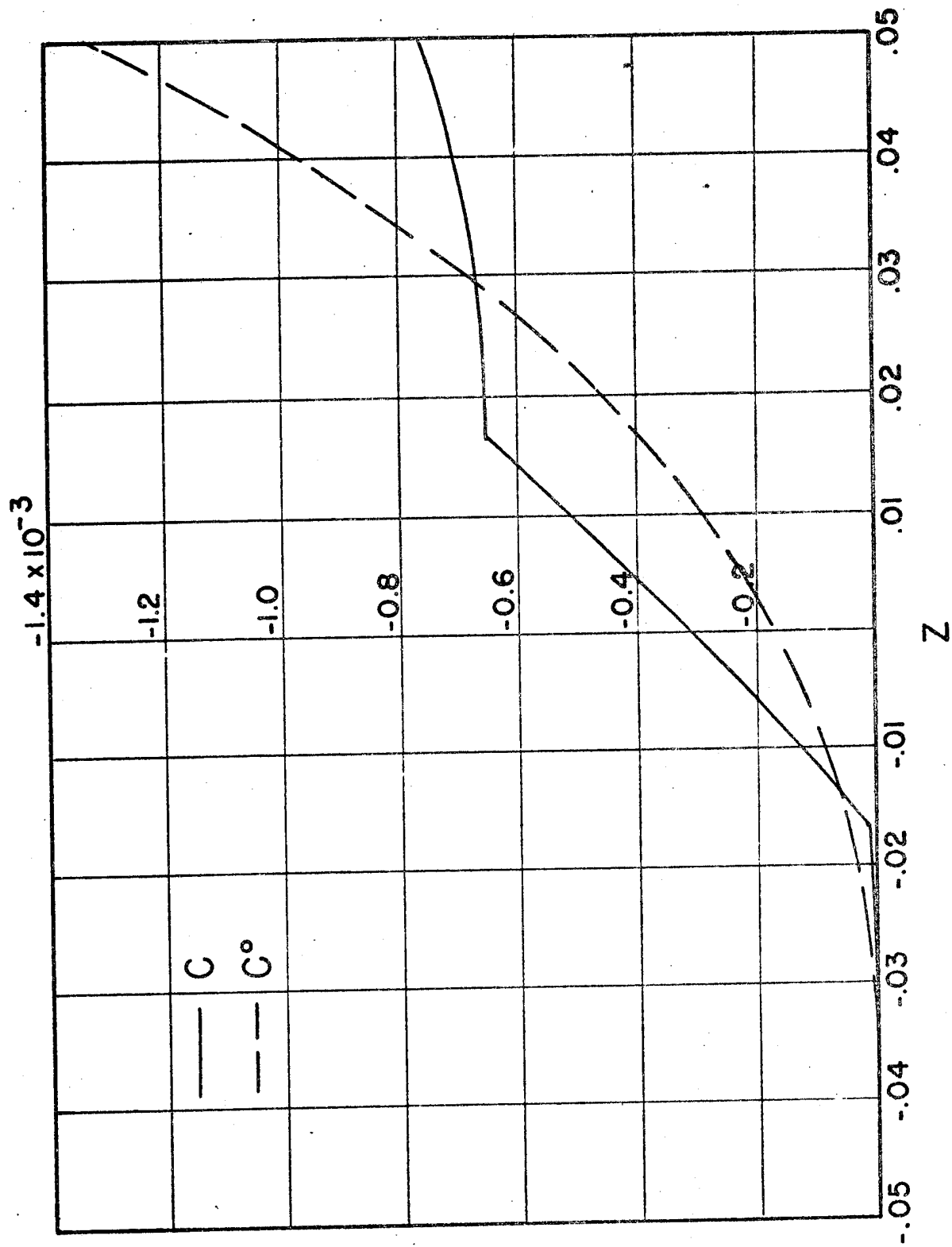


Fig. 1. Comparison of  $C$  and  $C^0$  for three plies.

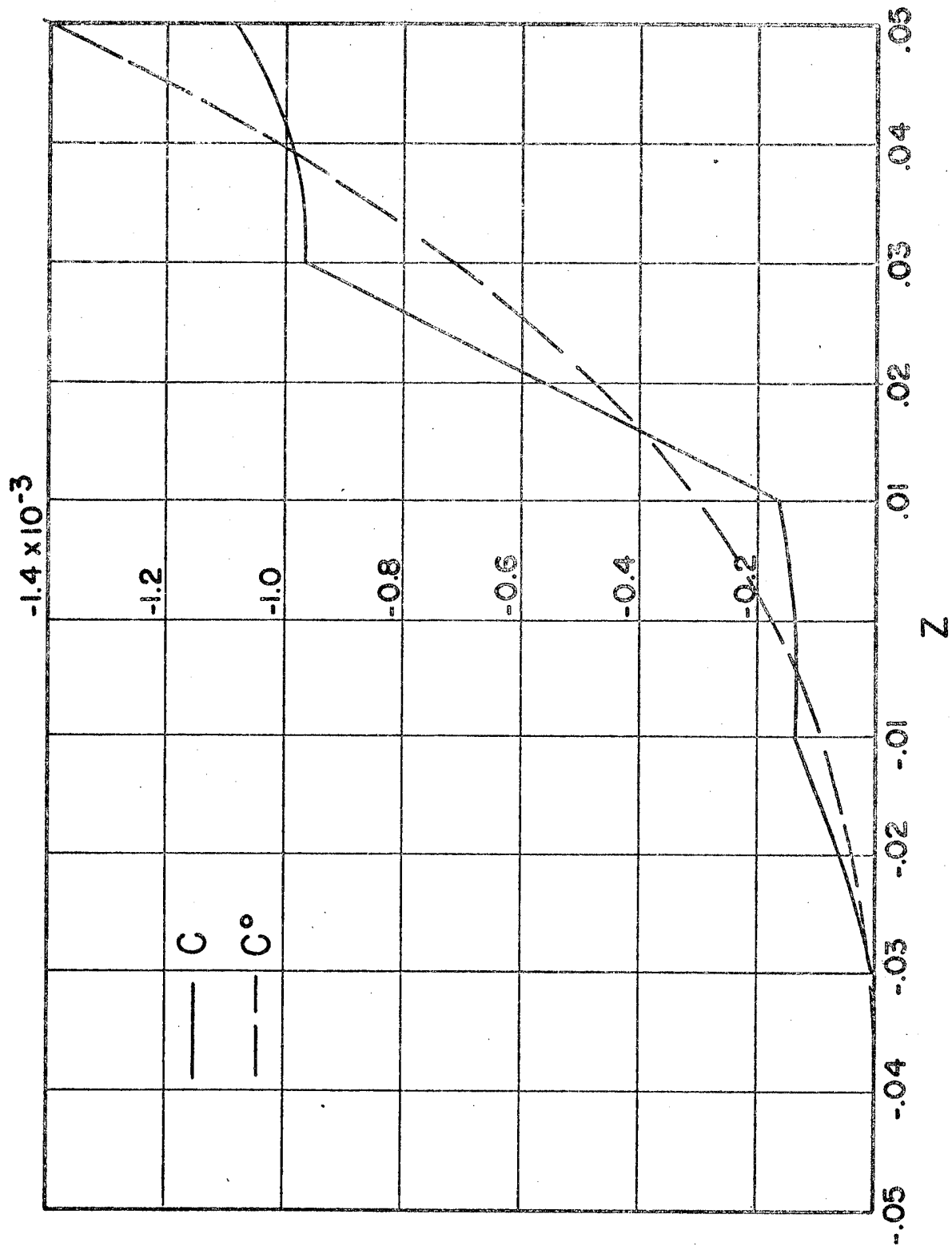


Fig. 2. Comparison of C and C<sup>0</sup> for five plies.

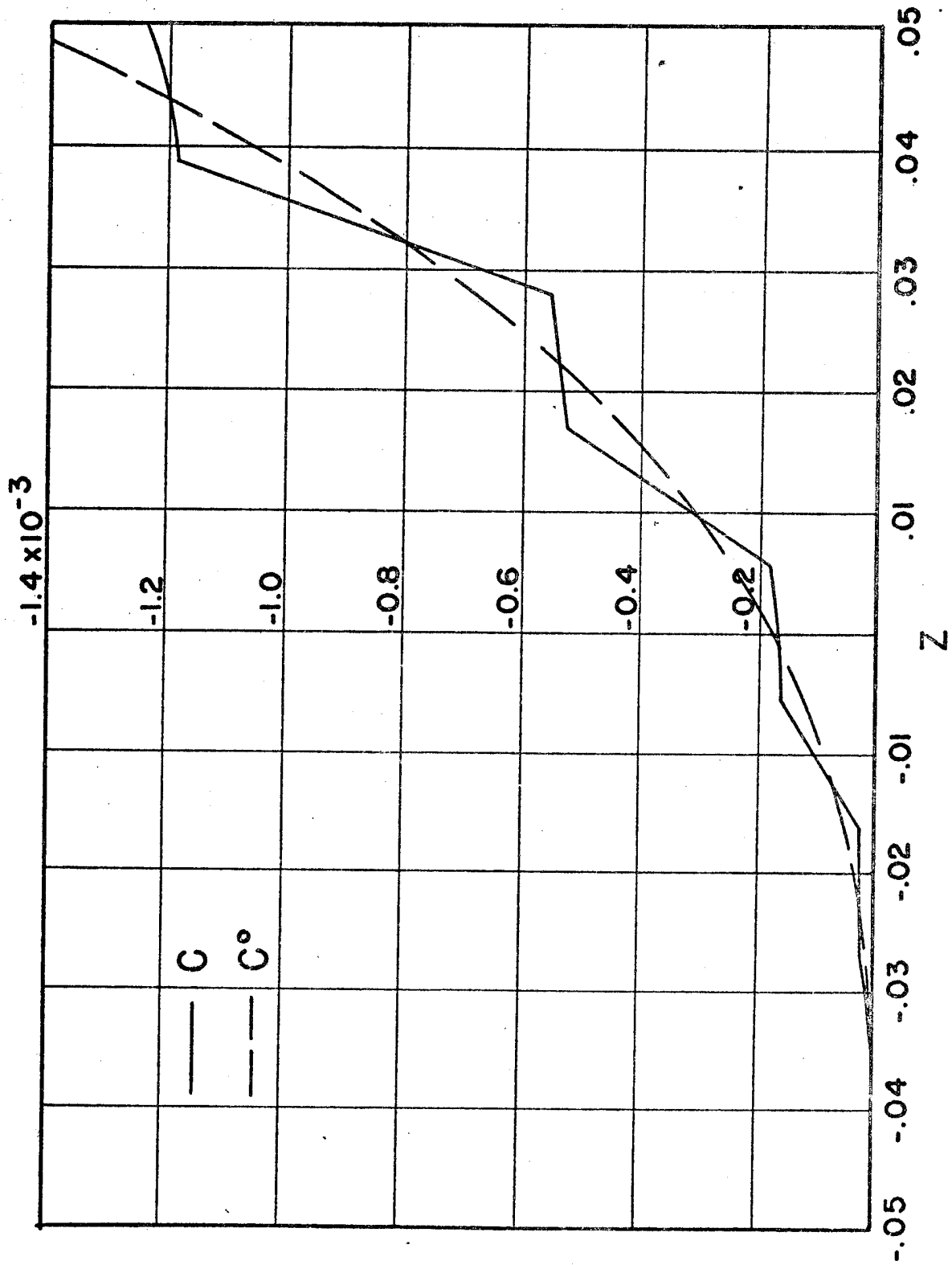


Fig. 3. Comparison of C and C° for nine plies.

APPENDIX I

"BASIC" COMPUTER PROGRAM FOR THE CALCULATION  
OF EQUIVALENT MATERIAL CONSTANTS

This program allows the rapid calculation of the equivalent material elastic compliances  $d_{ij}$  and the equivalent coefficients of thermal expansion  $e_j$  defined in Section II. It is written in "BASIC" language and compiled and run on the Dartmouth College Time-Sharing Computer System.

The input data is inserted sequentially in the DATA statements 898 through 910. The meaning of the symbols used in the program is given in the following chart.

<u>SYMBOL</u>	<u>MEANING</u>
E1, E2 .....	Longitudinal and transverse moduli of a single ply; $E_L, E_T$
G1, G2 .....	Shear moduli of a single ply; $G_{LT}, G_{TT}$
N1, N2, N3 .....	Poisson's ratios of a single ply; $\nu_{LT}, \nu_{TL}, \nu_{TT}$
T1, T2, T3 .....	Thermal expansion coefficients of a single ply; $\alpha_1, \alpha_2, \alpha_3$
M .....	Number of plies
Q(K) .....	Principal direction of each ply relative to the x axis, degrees. (K = 1 to M)
H(K) .....	Thickness of each ply, units arbitrary

( $\nu_{LT}$  represents the contraction in the longitudinal direction due to an applied stress in the transverse direction.)

The program operates in one of three modes, depending on the value of the integer control number  $G\emptyset$  in data statement 898. If  $G\emptyset$

is  $\emptyset$ , the program computes the compliances and thermal expansion coefficients, using the Correspondence Principle. It is up to the user of the program not to insert data defining an unbalanced laminate for this part of the program as the results will be meaningless. If  $G\emptyset$  is 1, the zeroth, first and second order moments of the  $C_{ij}$  are computed. These quantities are needed to define the edge force and moment relations in terms of the stress function components, and to evaluate the coefficients of the equations to be solved. If  $G\emptyset$  is 2, the program computes and prints the individual  $C_{ij}^*$ ,  $I_i^*$  and thermal expansion coefficients for each ply.

Several assumptions have been made in writing this program which should be pointed out. First, the program assumes that all plies are of the same material and that the ply orientation angle is the only variable which changes from one ply to the next. Second, it was assumed that the coefficients of equation (9) are constrained by the relations which appear in statements 35, 45, 55 and 64. This was done for expediency since data giving these coefficients was not available. All of these assumptions can be easily changed, if desired.

The results for a five ply plate are shown below. The data shown in the listed program in lines 898 - 910.

PLY

5 PLYS

PLY THICKNESSES ARE

0.02 0.03 0.02 0.03 0.02

PLY ANGLES ARE

0 90 0 90 0

COMPLIANCES ARE (MULTIPLY BY E-6 SO IN/LB)

8.06997 E-2	-1.95541 E-2	-1.00619 E-2	-1.29663 E-7
-1.95541 E-2	8.06997 E-2	-1.00619 E-2	-6.75107 E-6
-0.25	-0.25	1.	-4.12569 E-13
6.19339 E-7	-2.40729 E-6	8.08713 E-7	1.
2.5	0		
0	2.5		

THERMAL COEFFICIENTS ARE (UNITS ARE PER DEG)

1.06502 E-5	1.06502 E-5	0.00003	-6.52188 E-11
-------------	-------------	---------	---------------

ZEROth MOMENTS OF C(I,J) ARE (MULTIPLY BY E6)

1.57975	0.382785	1.07146 E-5
0.382785	1.57975	2.78904 E-6
3.56595 E-6	-5.64515 E-8	0.12

FIRST MOMENTS OF C(I,J) ARE (MULTIPLY BY E6)

-7.02698 E-10	-1.70300 E-10	-4.85029 E-15
-1.84846 E-10	-9.14042 E-10	-1.51268 E-15
-1.84575 E-15	2.87856 E-17	-6.95195 E-11

SECOND MOMENTS OF C(I,J) ARE (MULTIPLY BY E6)

1.16658 E-3	6.47696 E-4	7.50023 E-9
2.70987 E-4	2.62481 E-3	1.95233 E-9
2.49616 E-9	-3.95161 E-11	0.000144

ZEROth MOMENTS OF I(I) ARE (MULTIPLY BY E0)

-1.97468 E-2	-1.97468 E-2	6.17334 E-8
--------------	--------------	-------------

FIRST MOMENTS OF I(I) ARE (MULTIPLY BY E0)

8.3844 E-12	9.35785 E-12	-2.97613 E-17
-------------	--------------	---------------

SECOND MOMENTS OF I(I) ARE (MULTIPLY BY E0)

-2.97722 E-5	-1.76203 E-5	4.32133 E-11
--------------	--------------	--------------

THIRD MOMENTS OF I(I) ARE (MULTIPLY BY E0)

4.33265 E-14	2.39253 E-14	-5.94453 E-20
--------------	--------------	---------------

ZEROth MOMENTS OF F(I,J) ARE (MULTIPLY BY E6)

-2.09013 E-5	-2.09013 E-5	2.95504 E-11
--------------	--------------	--------------

FIRST MOMENTS OF F(I,J) ARE (MULTIPLY BY E6)

9.16281 E-15	1.22263 E-14	-1.33805 E-20
--------------	--------------	---------------

DIFF. EQ. COEFF. ARE (MULTIPLY BY E6)

(LISTED IN DECREASING ORDER OF X-DERIVATIVES)

FIRST EQUATION

1.57975	1.33807 E-6	0.06	1.39452 E-6	0.442785
3.56595 E-6	0	-1.48389 E-15	-2.54365 E-10	-1.84575 E-15

SECOND EQUATION

-5.64515 E-8	0.442785	5.3573 E-6	0.06	8.92325 E-6
1.57975	2.87856 E-17	-2.39822 E-10	-6.69603 E-15	-7.02698 E-10

THIRD EQUATION

9.47848 E-2	7.6897 E-8	3.01671 E-2	3.21438 E-7	8.36712 E-8
3.01671 E-2	7.49352 E-7	9.47848 E-2	-2.62481 E-3	-1.8733 E-9
-1.20668 E-3	-1.24926 E-8	-1.16658 E-3		

LOADING COEFF. ARE (DIMENSIONLESS)

(LISTED IN DECREASING ORDER OF X-DERIVATIVES)

FIRST EQUATION

6.16034 E-3	-1.84343 E-8
-------------	--------------

SECOND EQUATION

-1.84343 E-8	7.00422 E-3
--------------	-------------

THIRD EQUATION

-2.22785 E-4	5.96756 E-9	-1.68101 E-3	-1
--------------	-------------	--------------	----

PLY

```
2 READ G0
3 REM INSERT DATA IN LINES 898 THRU 910 . MODULI IN LINE 920 SHOULD BE
4 REM IN MILLIONS OF LB/SQ IN . ALL OTHER DATA IN NORMAL UNITS .
5 READ E1,E2,G1,G2,N1,N2,N3,T1,T2,T3
10 LET A(1,1) = 1./E1
15 LET A(2,2) = 1./E2
20 LET A(3,3) = A(2,2)
25 LET A(4,4) = 1./(2.*G1)
30 LET A(5,5) = 1./(2.*G2)
35 LET A(6,6) = A(5,5)
40 LET A(1,2) = -N1/E1
45 LET A(1,3) = A(1,2)
50 LET A(2,1) = -N2/E2
55 LET A(3,1) = A(2,1)
60 LET A(2,3) = -N3/E2
64 LET A(3,2) = A(2,3)
65 READ M
66 FOR K=1 TO M
67 READ Q(K)
68 LET P(K)=Q(K)
69 NEXT K
70 FOR K=1 TO M
71 READ H(K)
72 NEXT K
73 LET H=0.
74 FOR K=1 TO M
75 LET H=H+H(K)
76 NEXT K
77 FOR K=1 TO M
78 LET Q(K)=(6.2832/360.)*Q(K)
79 NEXT K
80 FOR K=1 TO M
81 LET Q=Q(K)
85 LET Q1=SIN(Q)
90 LET Q2=COS(Q)
95 LET Q3=Q1+2
100 LET Q4=Q2+2
105 LET Q5=Q1+3
110 LET Q6=Q2+3
115 LET Q7=Q1+4
120 LET Q8=Q2+4
125 LET A0=A(1,2)+A(2,1)+2.*A(4,4)
130 LET A1=A(1,1)+A(2,2)-2.*A(4,4)
135 LET A2=A(1,2)-A(1,1)+A(4,4)
140 LET A3=A(2,2)-A(2,1)-A(4,4)
145 LET A4=A(3,2)-A(3,1)
150 LET A5=A(2,1)-A(1,1)
155 LET A6=A(2,2)-A(1,2)
160 LET A7=A(2,3)-A(1,3)
165 LET A8=A(1,1)+A(2,2)-A(1,2)-A(2,1)
170 LET A9=A(6,6)-A(5,5)
175 LET B(1,1)=A(1,1)*Q8+A(2,2)*Q7+A0*Q3*Q4
180 LET B(1,2)=A(1,2)*Q8+A(2,1)*Q7+A1*Q3*Q4
185 LET B(1,3)=A(1,3)*Q4+A(2,3)*Q3
190 LET B(1,4)=2.*A2*Q1*Q6+2.*A3*Q5*Q2
191 LET B(1,5)=0.
192 LET B(1,6)=0.
```

```

195 LET B(2,1)=A(1,2)*Q7+A(2,1)*Q8+A1*Q3*Q4
200 LET B(2,2)=A(1,1)*Q7+A(2,2)*Q8+A0*Q3*Q4
205 LET B(2,3)=A(1,3)*Q3+A(2,3)*Q4
210 LET B(2,4)=2.*Q3*Q1+Q6*(2.*A2+Q5+Q2
211 LET B(2,5)=0.
212 LET B(2,6)=0.
215 LET B(3,1)=A(3,1)*Q4+A(3,2)*Q3
220 LET B(3,2)=A(3,1)*Q3+A(3,2)*Q4
225 LET B(3,3)=A(3,3)
230 LET B(3,4)=2.*A4*Q1*Q2
231 LET B(3,5)=0.
232 LET B(3,6)=0.
235 LET B(4,1)=A5*Q1*Q2+A6*Q5*Q2+A(4,4)*Q1*Q2*(1.-2.*Q3)
240 LET B(4,2)=A6*Q1*Q6+A5*Q5*Q2-A(4,4)*Q1*Q2*(1.-2.*Q3)
245 LET B(4,3)=A7*Q1*Q2
250 LET B(4,4)=2.*A8*Q4*Q3-A(4,4)*(1.-2.*Q3)*2
251 LET B(4,5)=0.
252 LET B(4,6)=0.
255 LET B(5,5)=A(5,5)+Q4+A(6,6)*Q3
260 LET B(5,6)=A9*Q1*Q2
261 LET B(5,1)=0.
262 LET B(5,2)=0.
263 LET B(5,3)=0.
264 LET B(5,4)=0.
265 LET B(6,5)=B(5,6)
270 LET B(6,6)=A(5,5)*Q5+A(6,6)*Q4
271 LET B(6,1)=0.
272 LET B(6,2)=3.
273 LET B(6,3)=0.
274 LET B(6,4)=0.
275 LET T(1)=T1*Q4+T2*Q3
276
280 LET T(2)=T1*Q3+T2*Q4
285 LET T(3)=T3
287 LET T(4)=(T2-T1)*Q1*Q2
290 LET D1=B(1,2)*(B(2,1)+B(4,4))-B(2,4)*B(4,1))
295 LET D2=B(1,1)*(B(2,2)+B(4,4))-B(2,4)*B(4,2))
300 LET D3=B(1,4)*(B(2,2)+B(4,1))-B(2,1)*B(4,2))
305 LET D=D1-D2+D3
310 LET C(1,1)=(B(1,4)*B(4,1)-B(1,1)*B(4,4))/D
315 LET C(2,1)=(B(1,2)*B(4,4)-B(1,4)*B(4,2))/D
320 LET C(3,1)=(B(1,1)*B(4,2)-B(1,2)*B(4,1))/D
325 LET C(1,2)=(B(2,1)*B(4,4)-B(2,4)*B(4,1))/D
330 LET C(2,2)=(B(2,4)*B(4,2)-B(2,2)*B(4,4))/D
335 LET C(3,2)=(B(2,2)*B(4,1)-B(2,1)*B(4,2))/D
340 LET C(1,3)=(B(1,1)*B(2,4)-B(1,4)*B(2,1))/D
345 LET C(2,3)=(B(2,2)*B(1,4)-B(1,2)*B(2,4))/D
350 LET C(3,3)=(B(1,2)*B(2,1)-B(1,1)*B(2,2))/D
355 LET I(1)=B(1,3)*C(1,2)+B(2,3)*C(1,1)+B(4,3)*C(1,3)
360 LET I(2)=B(1,3)*C(2,2)+B(2,3)*C(2,1)+B(4,3)*C(2,3)
365 LET I(3)=B(1,3)*C(3,2)+B(2,3)*C(3,1)+B(4,3)*C(3,3)
370 LET I(4)=B(3,1)*C(2,1)+B(3,2)*C(1,1)+B(3,4)*C(3,1)
375 LET I(5)=B(3,1)*C(2,2)+B(3,2)*C(1,2)+B(3,4)*C(3,2)
380 LET I(6)=B(3,1)*C(2,3)+B(3,2)*C(1,3)+B(3,4)*C(3,3)
385 LET I(7)=B(3,3)-B(3,1)*I(2)-B(3,2)*I(1)-B(3,4)*I(3)
390 LET D4=-B(1,2)*(B(2,1)+B(4,4))-B(2,4)*B(4,1))
391 LET D5=B(1,1)*(B(2,2)+B(4,4))-B(2,4)*B(4,2))

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392 LET D6=-B(1,4)*(B(2,2)*B(4,1)-B(2,1)*B(4,2))
393 LET D7=D4+D5+D6
394 LET F1=T(1)*(B(2,1)*B(4,4)-B(2,4)*B(4,1))/D7
395 LET F2=-T(2)*(B(1,1)*B(4,4)-B(1,4)*B(4,1))/D7
396 LET F3=T(4)*(B(1,1)*B(2,4)-B(1,4)*B(2,1))/D7
397 LET F1=F1+F2+F3
398 LET F2=-T(1)*(B(2,2)*B(4,4)-B(2,4)*B(4,2))/D7
399 LET F3=T(2)*(B(1,2)*B(4,4)-B(1,4)*B(4,2))/D7
400 LET F4=-T(4)*(B(1,2)*B(2,4)-B(1,4)*B(2,2))/D7
401 LET F2=F2+F3+F4
402 LET F3=-T(1)*(B(2,2)*B(4,1)-B(2,1)*B(4,2))/D7
403 LET F4=T(2)*(B(1,2)*B(4,1)-B(1,1)*B(4,2))/D7
404 LET F5=-T(4)*(B(1,2)*B(2,1)-B(1,1)*B(2,2))/D7
405 LET F3=F3+F4+F5
406 LET I(8)=B(3,2)*F1+B(3,1)*F2-B(3,4)*F3+T(3)
410 LET J(1)=B(3,1)*I(2)
412 LET J(2)=B(3,2)*I(1)
414 LET J(3)=B(3,4)*I(3)
416 LET J(4)=B(3,1)*F2
418 LET J(5)=B(3,2)*F1
420 LET J(6)=B(3,4)*F3
422 LET J(7)=F1
424 LET J(8)=F2
426 LET J(9)=F3
427 IF G0=2 THEN 920
428 IF G0=1 THEN 430
429 IF G0=0 THEN 600
430 LET F(0)=-H/2.
432 LET F(K)=F(K-1)+H(K)
434 FOR J=1 TO 3
436 FOR I=1 TO 3
438 LET X(I,J)=X(I,J)+C(I,J)*H(K)
439 LET Y(I,J)=Y(I,J)+(C(I,J)*(F(K)+2-F(K-1)+2))/2
440 LET Z(I,J)=Z(I,J)+(C(I,J)*(F(K)+3-F(K-1)+3))/3
441 NEXT I
442 NEXT J
443 FOR I=1 TO 3
444 LET E(I)=E(I)+I(I)*H(K)
445 LET G(I)=G(I)+(I(I)*(F(K)+2-F(K-1)+2))/2
446 LET K(I)=K(I)+(I(I)*(F(K)+3-F(K-1)+3))/3
447 LET N(I)=N(I)+(I(I)*(F(K)+4-F(K-1)+4))/4
448 NEXT I
449 FOR I=7 TO 9
450 LET L(I)=L(I)+J(I)*H(K)
451 LET M(I)=M(I)+(J(I)*(F(K)+2-F(K-1)+2))/2
452 NEXT I
454
456
457 IF M-K=0 THEN 460
458 GO TO 680
460 PRINT "ZEROTH MOMENTS OF C(I,J) ARE (MULTIPLY BY E6) "
462 PRINT X(1,1),X(1,2),X(1,3)
464 PRINT X(2,1),X(2,2),X(2,3)
466 PRINT X(3,1),X(3,2),X(3,3)
468 PRINT "FIRST MOMENTS OF C(I,J) ARE (MULTIPLY BY E6) "
470 PRINT Y(1,1),Y(1,2),Y(1,3)
472 PRINT Y(2,1),Y(2,2),Y(2,3)

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474 PRINT Y(3,1),Y(3,2),Y(3,3)
476 PRINT "SECOND MOMENTS OF C(I,J) ARE (MULTIPLY BY E6) "
478 PRINT Z(1,1),Z(1,2),Z(1,3)
480 PRINT Z(2,1),Z(2,2),Z(2,3)
482 PRINT Z(3,1),Z(3,2),Z(3,3)
484 PRINT "ZEROTH MOMENTS OF I(I) ARE (MULTIPLY BY E0)"
486 PRINT E(1),E(2),E(3)
488 PRINT "FIRST MOMENTS OF I(I) ARE (MULTIPLY BY E0)"
490 PRINT G(1),G(2),G(3)
492 PRINT "SECOND MOMENTS OF I(I) ARE (MULTIPLY BY E0)"
493 PRINT K(1),K(2),K(3)
494 PRINT "THIRD MOMENTS OF I(I) ARE (MULTIPLY BY E0)"
495 PRINT N(1),N(2),N(3)
496 PRINT "ZEROTH MOMENTS OF F(I,J) ARE (MULTIPLY BY E6) "
498 PRINT L(7),L(8),L(9)
500 PRINT "FIRST MOMENTS OF F(I,J) ARE (MULTIPLY BY E6) "
502 PRINT M(7),M(8),M(9)
503 DIM U(41)
504 LET H=H/2
505 LET U(1)=X(2,2)
506 LET U(2)=X(3,2)+X(2,3)/2
507 LET U(3)=X(3,3)/2
508 LET U(4)=X(2,3)/2
509 LET U(5)=X(2,1)+X(3,3)/2
510 LET U(6)=X(3,1)
511 LET J(7)=Y(2,2)
512 LET U(8)=Y(3,2)+Y(2,3)
513 LET U(9)=Y(2,1)+Y(3,3)
514 LET U(10)=Y(3,1)
515 LET U(11)=X(3,2)
516 LET U(12)=X(1,2)+X(3,3)/2
517 LET U(13)=X(1,3)/2
518 LET U(14)=X(3,3)/2
519 LET U(15)=X(3,1)+X(1,3)/2
520 LET U(16)=X(1,1)
521 LET U(17)=Y(3,2)
522 LET U(18)=Y(1,2)+Y(3,3)
523 LET U(19)=Y(3,1)+Y(1,3)
524 LET U(20)=Y(1,1)
525 LET U(21)=H*X(2,2)-Y(2,2)
526 LET U(22)=2*H*X(3,2)-2*Y(3,2)+H*X(2,3)/2-Y(2,3)/2
527 LET U(23)=H*X(1,2)-Y(1,2)+H*X(3,3)-Y(3,3)
528 LET U(24)=(H*X(1,3)-Y(1,3))/2
529 LET U(25)=(H*X(2,3)-Y(2,3))/2
530 LET U(26)=H*X(2,1)-Y(2,1)+H*X(3,3)-Y(3,3)
531 LET U(27)=2*H*X(3,1)-2*Y(3,1)+H*X(1,3)/2-Y(1,3)/2
532 LET U(28)=H*X(1,1)-Y(1,1)
533 LET U(29)=H*Y(2,2)-Z(2,2)
534 LET U(30)=2*H*Y(3,2)-2*Z(3,2)+H*Y(2,3)-Z(2,3)
535 LET U(31)=H*Y(2,1)-Z(2,1)+H*Y(1,2)-Z(1,2)+2*H*Y(3,3)-2*Z(3,3)
536 LET U(32)=2*H*Y(3,1)-2*Z(3,1)+H*Y(1,3)-Z(1,3)
537 LET U(33)=H*Y(1,1)-Z(1,1)
538 PRINT
539 PRINT "DIFF. EQ. COEFF. ARE (MULTIPLY BY E6)"
540 PRINT "(LISTED IN DECREASING ORDER OF X-DERIVATIVES)"
541 PRINT "FIRST EQUATION"

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542 FOR I=1 TO 10
543 PRINT U(I),
544 NEXT I
545 PRINT "SECOND EQUATION"
546 FOR I=11 TO 20
547 PRINT U(I),
548 NEXT I
549 PRINT "THIRD EQUATION"
550 FOR I=21 TO 33
551 PRINT U(I),
552 NEXT I
553 LET U(34)=- (E(2)/4+G(2)/(2*H)+K(2)/(4*H^2))
554 LET U(35)=- (E(3)/4+G(3)/(2*H)+K(3)/(4*H^2))
555 LET U(36)=- (E(3)/4+G(3)/(2*H)+K(3)/(4*H^2))
556 LET U(37)=- (E(1)/4+G(1)/(2*H)+K(1)/(4*H^2))
557 LET U(38)=(H*E(2)+G(2)-K(2)/H-N(2)/(H^2))/4
558 LET U(39)=2*(H*E(3)+G(3)-K(3)/H-N(3)/(H^2))
559 LET U(40)=H*E(1)+G(1)+K(1)/H-N(1)/(H^2)
560 LET U(41)=-1
561 PRINT
562 PRINT "LOADING COEFF. ARE (DIMENSIONLESS)"
563 PRINT "(LISTED IN DECREASING ORDER OF X-DERIVATIVES)"
564 PRINT "FIRST EQUATION"
565 PRINT U(34),U(35)
566 PRINT "SECOND EQUATION"
567 PRINT U(36),U(37)
568 PRINT "THIRD EQUATION "
569 PRINT U(38),U(39),U(40),U(41)
598 GO TO 999
600 FOR I=1 TO 3
605 FOR J=1 TO 3
610 LET R(I,J)=(C(I,J)*H(K))/H + R(I,J)
615 NEXT J
620 NEXT I
625 FOR I=1 TO 8
630 LET S(I)=(I(I)*H(K))/H + S(I)
635 NEXT I
640 FOR I=1 TO 9
645 LET W(I)=(J(I)*H(K))/H+W(I)
650 NEXT I
655 LET V(5,5)=(B(5,5)*H(K))/H + V(5,5)
660 LET V(6,6)=(B(6,6)*H(K))/H + V(6,6)
665 LET V(5,6)=(B(5,6)*H(K))/H + V(5,6)
670 LET V(3,3)=(B(3,3)*H(K))/H + V(3,3)
680 NEXT K
685 IF G0>0 THEN 999
690 LET C1=R(2,1)*(R(1,2)*R(3,3)-R(3,2)*R(1,3))
695 LET C2=R(1,1)*(R(2,2)*R(3,3)-R(3,2)*R(2,3))
700 LET C3=R(3,1)*(R(2,2)*R(1,3)-R(1,2)*R(2,3))
705 LET C=C1-C2+C3
706
710 LET D(1,1)=(R(3,1)*R(1,3)-R(1,1)*R(3,3))/C
715 LET D(1,2)=(R(2,1)*R(3,3)-R(3,1)*R(2,3))/C
720 LET D(1,4)=(R(1,1)*R(2,3)-R(2,1)*R(1,3))/C
725 LET D(2,1)=(R(1,2)*R(3,3)-R(3,2)*R(1,3))/C
730 LET D(2,2)=(R(2,3)*R(3,2)-R(2,2)*R(3,3))/C

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735 LET D(2,4)=(R(2,2)*R(1,3)-R(1,2)*R(2,3))/C
740 LET D(4,1)=(R(1,1)*R(3,2)-R(1,2)*R(3,1))/C
745 LET D(4,2)=(R(2,2)*R(3,1)-R(2,1)*R(3,2))/C
750 LET D(4,4)=(R(2,1)*R(1,2)-R(1,1)*R(2,2))/C
755 LET D(1,3)=S(1)*D(1,2)+S(2)*D(1,1)+S(3)*D(1,4)
760 LET D(2,3)=S(1)*D(2,2)+S(2)*D(2,1)+S(3)*D(2,4)
765 LET D(4,3)=S(1)*D(4,2)+S(2)*D(4,1)+S(3)*D(4,4)
770 LET D(3,1)=S(4)*D(2,1)+S(5)*D(1,1)+S(6)*D(4,1)
775 LET D(3,2)=S(4)*D(2,2)+S(5)*D(1,2)+S(6)*D(4,2)
780 LET D(3,4)=S(4)*D(2,4)+S(5)*D(1,4)+S(6)*D(4,4)
785 LET D(5,5)=V(5,5)
790 LET D(6,6)=V(6,6)
795 LET D(5,6)=V(5,6)
800 LET D(6,5)=D(5,6)
805 LET D(3,3)=V(3,3)+D(3,1)*S(2)+D(3,2)*S(1)+D(3,4)*S(3)-W(1)-W(2)-W(3)
806 LET E(3)=S(8)-D(3,1)*W(8)-D(3,2)*W(7)+D(3,4)*W(9)
807 LET D1=D(2,1)*D(4,4)-D(2,4)*D(4,1)
808 LET D2=D(1,1)*D(4,4)-D(1,4)*D(4,1)
809 LET D3=D(1,1)*D(2,4)-D(1,4)*D(2,1)
810 LET D4=D(2,2)*D(4,4)-D(2,4)*D(4,2)
811 LET D5=D(1,2)*D(4,4)-D(1,4)*D(4,2)
812 LET D6=D(1,2)*D(2,4)-D(1,4)*D(2,2)
813 LET D7=D(2,2)*D(4,1)-D(2,1)*D(4,2)
814 LET D8=D(1,2)*D(4,1)-D(1,1)*D(4,2)
815 LET D9=D(1,2)*D(2,1)-D(1,1)*D(2,2)
816 LET B1=D1*(D5*D9-D6*D8)
817 LET B2=D2*(D6*D7-D4*D9)
818 LET B3=D3*(D4*D8-D5*D7)
819 LET B1=B1+B2+B3
820 LET B2=-D(1,2)*D1
821 LET B3=D(1,1)*D4
822 LET B4=D(1,4)*D7
823 LET B2=B2+B3+B4
824 FOR I=7 TO 9
825 LET W(I)=W(I)*B2/B1
826 NEXT I
827 LET E(1)=W(7)*(D5*D9-D6*D8)+W(8)*(D2*D9-D3*D8)+W(9)*(D3*D5-D2*D6)
828 LET E(2)=W(7)*(D4*D9-D6*D7)+W(8)*(D1*D9-D3*D7)+W(9)*(D3*D4-D1*D6)
829 LET E(4)=W(7)*(D4*D8-D5*D7)+W(8)*(D1*D8-D2*D7)+W(9)*(D2*D4-D1*D5)
831 PRINT M;"PLIES"
832 PRINT "PLY THICKNESSES ARE"
833 FOR K=1 TO M
834 PRINT H(K);
835 NEXT K
836 PRINT
837 PRINT "PLY ANGLES ARE"
838 FOR K=1 TO M
839 PRINT P(K);
840 NEXT K
841 PRINT
850 PRINT "COMPLIANCES ARE (MULTIPLY BY E-6 SQ IN/LB)"
852 PRINT D(1,1),D(1,2),D(1,3),D(1,4)
854 PRINT D(2,1),D(2,2),D(2,3),D(2,4)
856 PRINT D(3,1),D(3,2),D(3,3),D(3,4)
858 PRINT D(4,1),D(4,2),D(4,3),D(4,4)
860 PRINT D(5,5),D(5,6)

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862 PRINT D(6,5),D(6,6)
864 PRINT "THERMAL COEFFICIENTS ARE (UNITS ARE PER DEG)"
866 PRINT E(1),E(2),E(3),E(4)
870 GO TO 999
898 DATA 1
900 DATA 25.,1.,.5,.2
902 DATA .05,.25,.25
904 DATA 10.E-6,30.E-6,30.E-6
906 DATA 5
908 DATA 0,90,0,90,0
910 DATA .02,.03,.02,.03,.02
920 PRINT "PLY C(I,J) (MULTIPLY BY E6), PLY";K
924 PRINT C(1,1),C(1,2),C(1,3)
926 PRINT C(2,1),C(2,2),C(2,3)
928 PRINT C(3,1),C(3,2),C(3,3)
930 PRINT "PLY F(I,J) (MULTIPLY BY E6), PLY";K
932 PRINT J(7),J(8),J(9)
934 PRINT "PLY I(I), PLY ";K
936 FOR I=1 TO 3
938 PRINT I(I);
940 NEXT I
942 GO TO 680
999 END
```

APPENDIX II

EVALUATION OF THE CORRECTION COEFFICIENTS FOR  
TRANSVERSE NORMAL AND SHEARING STRAIN

The coefficients  $K_{ijklmn}^{pq}$  and  $K_{ijk}^{pq}$  are of the form

$$I = \int_{-h}^h z^p F^*(z) \int_{-h}^z G^*(t) \int_{-h}^t s^q H^*(s) ds dt dz$$

To obtain  $K_{ijklmn}^{pq}$ , insert

$$\begin{aligned} F^* &= C_{ij}^* \\ G^* &= b_{kl}^* \\ H^* &= C_{mn}^* \end{aligned}$$

and to obtain  $K_{ijk}^{pq}$ , insert

$$\begin{aligned} F^* &= C_{ij}^* \\ G^* &= 1 \\ H^* &= I_k^* \end{aligned}$$

The functions  $F^*$ ,  $G^*$  and  $H^*$  are assumed to be piece-wise constant functions of  $z$ , defined within each ply by constants  $F_i$ ,  $G_i$ ,  $H_i$ . Let

$$d_i = -h + (h_1 + h_2 \dots + h_i)$$

be the distance, measured in the  $z$  direction, from the middle surface of the plate to the top of the  $i^{\text{th}}$  ply. Let  $h_i$  be the individual ply thicknesses. Then,

$$\begin{aligned} \int_{-h}^t s^q H^*(s) ds &= \sum_{i=1}^k \frac{H_i}{q+1} \left[ d_i^{q+1} - d_{i-1}^{q+1} \right] \\ &+ \frac{H_{k+1}}{q+1} \left[ t^{q+1} - d_k^{q+1} \right] \end{aligned}$$

where  $d_i \leq t \leq d_{k+1}$

Now,

$$\begin{aligned}
 & \int_{-h}^z G^*(t) \int_{-h}^t s^q H^*(s) ds dt = \\
 & = \sum_{j=1}^{k-1} \int_{d_{j-1}}^{d_j} G_j \left\{ \sum_{i=1}^{j-1} \frac{H_i}{q+1} \left[ d_i^{q+1} - d_{i-1}^{q+1} \right] + \frac{H_j}{q+1} \left[ t^{q+1} - d_{j-1}^{q+1} \right] \right\} dt \\
 & + \int_{d_{k-1}}^z G_k \left\{ \sum_{i=1}^{k-1} \frac{H_i}{q+1} \left[ d_i^{q+1} - d_{i-1}^{q+1} \right] \right. \\
 & \left. + \frac{H_k}{q+1} \left[ t^{q+1} - d_{k-1}^{q+1} \right] \right\} dt \quad d_{k-1} \leq z \leq d_k
 \end{aligned}$$

The third and final integration can now be performed.

$$\begin{aligned}
 I = & \sum_{k=1}^N \int_{d_{k-1}}^{d_k} z^p F_k \left\{ \sum_{j=1}^{k-1} \int_{d_{j-1}}^{d_j} \left[ \sum_{i=1}^{j-1} \frac{H_i}{q+1} \left[ d_i^{q+1} - d_{i-1}^{q+1} \right] \right. \right. \\
 & \left. \left. + \frac{H_j}{q+1} \left[ t^{q+1} - d_{j-1}^{q+1} \right] \right] dt \right. \\
 & \left. + \int_{d_{k-1}}^z G_k \left[ \sum_{j=1}^{k-1} \frac{H_j}{q+1} \left[ d_j^{q+1} - d_{j-1}^{q+1} \right] \right. \right. \\
 & \left. \left. + \frac{H_k}{q+1} \left[ t^{q+1} - d_{k-1}^{q+1} \right] \right] dt \right\} dz
 \end{aligned}$$

N is the total number of plies. The integrations indicated may now be carried out and the result written in the form

$$\begin{aligned}
I = & \sum_{k=3}^N F_k \left[ \frac{d_k^{p+1} - d_{k-1}^{p+1}}{p+1} \right] \sum_{j=2}^{k-1} h_j G_j \sum_{i=1}^{j-1} \frac{H_i}{q+1} \left[ d_i^{q+1} - d_{i-1}^{q+1} \right] \\
& + \sum_{k=2}^N F_k \left[ \frac{d_k^{p+1} - d_{k-1}^{p+1}}{p+1} \right] \sum_{j=1}^{k-1} \frac{G_j H_j}{q+1} \left[ \frac{d_j^{q+2} - d_{j-1}^{q+2}}{q+2} - d_j^{q+1} h_j \right] \\
& + \sum_{k=2}^N G_k G_k \sum_{j=1}^{k-1} \frac{H_j}{q+1} \left[ d_j^{q+1} - d_{j-1}^{q+1} \right] \left[ \frac{d_k^{p+2} - d_{k-1}^{p+2}}{p+2} \right] \\
& - d_{k-1} \left( \frac{d_k^{p+1} - d_{k-1}^{p+1}}{p+1} \right) \\
& + \sum_{k=1}^N \frac{F_k G_k H_k}{q+1} \left[ \frac{d_k^{p+q+3} - d_{k-1}^{p+q+3}}{(q+2)(p+q+3)} \right. \\
& - \frac{d_{k-1}^{q+2}}{q+2} \left( \frac{d_k^{p+1} - d_{k-1}^{p+1}}{p+1} \right) - d_{k-1}^{q+1} \left[ \frac{d_k^{p+2} - d_{k-1}^{p+2}}{p+2} \right. \\
& \left. \left. - d_{k-1} \left( \frac{d_k^{p+1} - d_{k-1}^{p+1}}{p+1} \right) \right] \right]
\end{aligned}$$

The initial indexes have been changed to 1, 2, or 3, as appropriate to avoid requiring the convention that a summation is not to be carried out unless the upper limit equals or exceeds the lower limit.

### APPENDIX III

#### COMPUTER PROGRAMS FOR THE EVALUATION OF THE CORRECTIONS FOR TRANSVERSE NORMAL AND SHEARING STRAIN

The integral which was evaluated in Appendix I has been programmed for computer evaluation, using BASIC language. However, since BASIC does not allow the use of a subscripted variable with eight levels of subscripting, it was necessary to regard the coefficients  $K_{ijk\ell mn}^{pq}$  and  $K_{ijk}^{pq}$  as functions of their indexes. The program is called DIFFCO and is listed on the following pages. The individual summations in the integral I (Appendix I) are written as subroutines FNA, FNB, FNC, and FND in lines 910-1000, 1010-1130, 1150-1270, and 1300-1500, respectively. These subroutines do double duty for the evaluation of the coefficients  $K_{ijk\ell mn}^{pq}$  and  $K_{ijk}^{pq}$  by replacing  $b_{kl}^*$  by 1 and  $C_{mn}^*$  by  $I_k^*$  for the latter calculation. The coefficients of the correction operators defined by equations (56) are also calculated by this program. In order to carry out the programming within the limitations of BASIC, it was necessary to label the eight-level and five-level subscripted coefficients as singly subscripted variables, using the re-labeling program MATCH, which is listed immediately following DIFFCO. The output of MATCH lists the subscripts  $i, j, k, \ell, m, n, p, q$  in the left-hand columns and the corresponding single subscript in the right-hand column. Following this table is a list of the indexes  $i, j, k, p, q$  of the five-level subscripted coefficients and the corresponding single subscript. The coefficients of the correction operators  $l_i, m_i$  and  $n_i$  are computed by the subroutines DIFFCO 1, DIFFCO 2, DIFFCO 3, and DIFFCO 4,

which are listed at the end of the program DIFFCO. In order to check the calculation of a specific coefficient  $\lambda_i$ ,  $m_i$  and  $n_i$  in DIFFCO, it is necessary to look up the single subscript of the constants  $K_{ijklmn}^{pq}$  and  $K_{ijk}^{pq}$  in MATCH and then locate these quantities in DIFFCO. For example,  $K_{335633}^{00}$  which is involved in the calculation of  $\lambda_8$  is referred to in DIFFCO as K(306).

Using the data listed in lines 1600-1620 of DIFFCO (an "isotropic", five-ply, symmetric laminate), all of the coefficients  $K_{ijklmn}^{pq}$  and  $K_{ijk}^{pq}$  have been computed and are listed following MATCH. The left-hand columns show the pertinent indexes and the right-hand columns, the value of the coefficient with those indexes.

DIFFCO

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1 READ M3
5 READ E1,E2,G1,G2,N1,N2,N3
10 LET A(1,1)=1/E1
15 LET A(2,2)=1/E2
20 LET A(3,3)=A(2,2)
25 LET A(4,4)=1/(2*G1)
30 LET A(5,5)=1/(2*G2)
35 LET A(6,6)=A(5,5)
40 LET A(1,2)=-N1/E1
45 LET A(1,3)=A(1,2)
50 LET A(2,1)=-N2/E2
55 LET A(3,1)=A(2,1)
60 LET A(2,3)=-N3/E2
65 LET A(3,2)=A(2,3)
70 READ Y
75 FOR X=1 TO Y
80 READ Q(X)
85 LET P(X)=Q(X)
90 NEXT X
95 FOR X=1 TO Y
100 READ H(X)
105 NEXT X
110 LET H=0
115 FOR X=1 TO Y
120 LET H=H+H(X)
122 NEXT X
124 FOR X=1 TO Y
126 LET Q(X)=(6.2832/360)*Q(X)
128 NEXT X
129 LET M2=0
130 FOR I=1 TO 3
132 FOR J=1 TO 3
134 FOR K=1 TO 2
136 FOR L=1 TO 2
138 FOR M=1 TO 3
140 FOR N=1 TO 3
145 FOR X=1 TO Y
150 LET Q=Q(X)
155 LET Q1=SIN(Q)
160 LET Q2=COS(Q)
165 LET Q3=Q1*2
170 LET Q4=Q2*2
175 LET Q5=Q1*3
180 LET Q6=Q2*3
185 LET Q7=Q1*4
190 LET Q8=Q2*4
195 LET A1=A(1,1)+A(2,2)-2*A(4,4)
200 LET A0=A(1,2)+A(2,1)+2*A(4,4)
205 LET A2=A(1,2)-A(1,1)+A(4,4)
210 LET A3=A(2,2)-A(2,1)-A(4,4)
215 LET A4=A(3,2)-A(3,1)
220 LET A5=A(2,1)-A(1,1)
225 LET A6=A(2,2)-A(1,2)
230 LET A7=A(2,3)-A(1,3)
235 LET A8=A(1,1)+A(2,2)-A(1,2)-A(2,1)
240 LET A9=A(6,6)-A(5,5)
245 LET B(1,1)=A(1,1)*Q8+A(2,2)*Q7+A0*Q3*Q4
250 LET B(1,2)=A(1,2)*Q8+A(2,1)*Q7+A1*Q3*Q4

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255 LET B(1,4)=2*A2*Q1+A6+2*A3*Q5*Q2
260 LET B(2,1)= A(1,2)*Q7+A(2,1)*Q8+A1*Q3*Q4
265 LET B(2,2)= A(1,1)*Q7+A(2,2)*Q8+A1*Q3*Q4
270 LET B(2,4)= 2*A3*Q1*Q6+2*A2*Q5*Q2:
275 LET B(3,1)= A(3,1)*Q4+A(3,2)*Q3:
280 LET B(3,2)= A(3,1)*Q3+A(3,2)*Q4
285 LET B(3,4)= 2*A4*Q1*Q2
290 LET B(4,1)= A5*Q1*Q2+A6*Q5*Q2+A(4,4)*Q1*Q2*(1-2*Q3,
295 LET B(4,2)= A6*Q1*Q6+A5*Q5*Q2-A(4,4)*Q1*Q2*(1-2*Q3)
300 LET B(4,4)= 2*A8*Q1*Q3+A(4,4)*(1-2*Q3)*2
305 LET B(5,5)= A(5,5)*Q4+A(6,6)*Q3
310 LET B(5,6)=A9*Q1*Q2
315 LET B(6,5)= B(5,6)
320 LET B(6,6)= A(5,5)*Q3+A(6,6)*Q4
325 LET D1=B(1,2)*(B(2,1)*B(4,4)-B(2,4)*B(4,1))
330 LET D2=B(1,1)*(B(2,2)*B(4,4)-B(2,4)*B(4,2))
335 LET D3=B(1,4)*(B(2,2)*B(4,1)-B(2,1)*B(4,2))
340 LET D=D1-D2+D3
345 LET C(1,1)= (B(1,4)*B(4,1)-B(1,1)*B(4,4))/D
350 LET C(2,1)= (B(1,2)*B(4,4)-B(1,4)*B(4,2))/D
355 LET C(3,1)= (B(1,1)*B(4,2)-B(1,2)*B(4,1))/D
360 LET C(1,2)= (B(2,1)*B(4,4)-B(2,4)*B(4,1))/D
365 LET C(2,2)= (B(2,4)*B(4,2)-B(2,2)*B(4,4))/D
370 LET C(3,2)= (B(2,2)*B(4,1)-B(2,1)*B(4,2))/D
375 LET C(1,3)= (B(1,1)*B(2,4)-B(1,4)*B(2,1))/D
380 LET C(2,3)= (B(2,2)*B(1,4)-B(1,2)*B(2,4))/D
385 LET C(3,3)= (B(1,2)*B(2,1)-B(1,1)*B(2,2))/D
386
405 LET B(1,1)= B(5,5)
410 LET B(1,2)= B(5,6)
415 LET B(2,1)= B(5,6)
420 LET B(2,2)= B(6,6)
500 LET D(0)= -H/2
510 LET D(X)= D(X-1)+H(X)
520 LET Z=FNA(I,J,K,L,M,N,R,S,X)+FNB(I,J,K,L,M,N,R,S,X)
530 LET Z=Z+FNC(I,J,K,L,M,N,R,S,X)+FND(I,J,K,L,M,N,R,S,X)
535
540 NEXT X
545 LET T=T+1
550 IF M3>0 THEN 564
555 IF T>1 THEN 562
560 PRINT "VALUES OF K(I,J,K,L,M,N,P,Q) ARE (MULTIPLY BY E6) "
561 PRINT
562 PRINT I,J,K+4,L+4,M,N,R,S,TAB(35);Z
563 IF M3=0 THEN 580
564 LET K(T)=Z
570
580 NEXT N
590 NEXT M
600 NEXT L
610 NEXT K
620 NEXT J
630 NEXT I
640 LET M1=M1+1
650 IF M1=1 THEN 680
660 IF M1=2 THEN 700
670 IF M1=3 THEN 720
675 IF M1>3 THEN 1650

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680 LET S=1
690 GO TO 130
700 LET R=1
705 LET S=0
710 GO TO 130
720 LET S=1
725 GO TO 130
910 DEF FNA(I,J,K,L,M,N,R,S,X)
912 IF M2=0 THEN 920
914 LET B1=C(I,J)*I(K)/(S+1)
916 GO TO 930
920 LET B1=C(I,J)*B(K,L)*C(M,N)/(S+1)
930 LET B2=(D(X)*(R+S+3)-D(X-1)*(R+S+3))/((S+2)*(R+S+3))
940 LET B3=(D(X-1)*(S+2)*(D(X)*(R+1)-D(X-1)*(R+1)))/((S+2)*(R+1))
950 LET B4=(D(X-1)*(S+1)*(D(X)*(R+2)-D(X-1)*(R+2)))/(R+2)
960 LET B5=(D(X-1)*(S+2)*(D(X)*(R+1)-D(X-1)*(R+1)))/(R+1)
970 LET B6=B1*(B2-B3-B4+B5)
972 IF X>1 THEN 976
974 LET C0=0
976 LET C0=B6+C0
980 LET FNA=C0
1000 FN END
1010 DEF FNB(I,J,K,L,M,N,R,S,X)
1012 IF X>1 THEN 1050
1014 LET F3=0
1016 LET F4=0
1050 LET B7=C(I,J)*(D(X)*(R+1)+D(X-1)*(R+1))/(R+1)
1066 IF M2=0 THEN 1070
1067 LET B8=I(K)*(D(X)*(S+2)-D(X-1)*(S+2))/((S+2)*(S+1))
1068 GO TO 1080
1070 LET B8=B(K,L)*C(M,N)*(D(X)*(S+2)-D(X-1)*(S+2))/((S+2)*(S+1))
1080 LET B9=(D(X)*(S+1))*H(X)/(S+1)
1082 IF M2=0 THEN 1085
1083 LET B9=B9+I(K)
1084 GO TO 1090
1085 LET B9=B9*B(K,L)*C(M,N)
1090 LET B8=B8-B9
1100 LET F3=B7+F4+F3
1105
1110 LET F4=B8+F4
1120 LET FNB=F3
1130 FNB END
1150 DEF FNC(I,J,K,L,M,N,R,S,X)
1155 IF X>1 THEN 1182
1160 LET C3=0
1165 LET C4=0
1182 IF M2=0 THEN 1190
1184 LET C1=C(I,J)*(D(X)*(R+2)+D(X-1)*(R+2))/(R+2)
1186 GO TO 1200
1190 LET C1=C1+B(K,L)*(D(X)*(R+2)-D(X-1)*(R+2))/(R+2)
1200 LET C2=(D(X-1)*(D(X)*(R+1)-D(X-1)*(R+1)))/(R+1)
1201 IF M2=0 THEN 1204
1202 LET C2=C2*C(I,J)
1203 GO TO 1205
1204 LET C2=C2*B(K,L)
1205 LET C1=C1-C2
1216 IF M2=0 THEN 1220
1217 LET C2=I(K)*(D(X)*(S+1)-D(X-1)*(S+1))/(S+1)

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1218 GO TO 1230
1220 LET C2=C(M,N)*(D(X)+(S+1)+D(X-1)+(S+1))/(S+1)
1230 LET C3=C1+C4+C3
1240 LET C4=C2+C4
1250 LET FNC=C3
1270 FNCND
1300 DEF FND(I,J,K,L,M,N,R,S,X)
1310 IF X=1 THEN 1370
1320 LET F0=0
1330 LET F1=0
1340 LET F2=0
1370 LET C5=C(I,J)*(D(X)+(R+1)+D(X-1)+(R+1))/(R+1)
1392 IF M=0 THEN 1400
1394 LET C6=I(K)*(D(X)+(S+1)-D(X-1)+(S+1))/(S+1)
1396 GO TO 1432
1400 LET C6=C(M,N)*(D(X)+(S+1)+D(X-1)+(S+1))/(S+1)
1432 IF M=0 THEN 1440
1434 LET C7=H(X)
1436 GO TO 1450
1440 LET C7=B(K,L)*H(X)
1450 LET F0=C5+F1+F0
1460 LET F1=C7+F2+F1
1470 LET F2=C6+F2
1480 LET FND=F0
1500 FNCND
1599 DATA 1
1600 DATA 25, 1, 5, 2
1605 DATA .05, .25, .25
1610 DATA 5
1615 DATA 0,90,0,90,0
1620 DATA .02,.03,.02,.03,.02
1650 LET M2=1
1652 LET T=0
1653 FOR R=0 TO 1
1654 FOR S=0 TO 3
1655 FOR J=1 TO 3
1660 FOR I=1 TO 3
1665 FOR K=1 TO 3
1670 FOR X=1 TO Y
1675 LET Q=Q(X)
1680 LET Q1=SIN(Q)
1685 LET Q2=COS(Q)
1690 LET Q3=Q1+2
1695 LET Q4=Q2+2
1700 LET Q5=Q1+3
1705 LET Q6=Q2+3
1710 LET Q7=Q1+4
1715 LET Q8=Q2+4
1720 LET A0=A(1,2)+A(2,1)+2*A(4,4)
1725 LET A1=A(1,1)+A(2,2)-2*A(4,4)
1730 LET A2=A(1,2)-A(1,1)+A(4,4)
1735 LET A3=A(2,2)-A(2,1)-A(4,4)
1740 LET A4=A(3,2)-A(3,1)
1745 LET A5=A(2,1)-A(1,1)
1750 LET A6=A(2,2)-A(1,2)
1755 LET A7=A(2,3)-A(1,3)
1760 LET A8=A(1,1)+A(2,2)-A(1,2)-A(2,1)
1765 LET A9=A(6,6)-A(5,5)

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1770 LET B(1,1)=A(1,1)*Q8+A(2,2)*Q7+A0*Q3*Q4
1775 LET B(1,2)=A(1,2)*Q8+A(2,1)*Q7+A1*Q3*Q4
1780 LET B(1,4)=2*A2*Q1*A6+2*A3*Q5*Q2
1785 LET B(2,1)=B(1,2)*Q7+A(2,1)*Q8+A1*Q3*Q4
1790 LET B(2,2)=A(1,1)*Q7+A(2,2)*Q9+A0*Q3*Q4
1795 LET B(2,4)=2*A3*Q1*Q6+2*A2*Q5*Q2
1800 LET D(3,1)=A(3,1)*Q4+A(3,2)*Q3
1805 LET B(3,2)=A(3,1)*Q3+A(3,2)*Q4
1810 LET B(3,4)=2*A4*Q1*Q2
1815 LET B(4,1)=A5*Q1*Q2+A5*Q5*Q2+A(4,4)*Q1*Q2*(1-2*Q3)
1820 LET B(4,2)=A6*Q1*Q6+A5*Q5*Q2-A(4,4)*Q1*Q2*(1-2*Q3)
1825 LET B(4,4)=2*A8*Q4*Q3+A(4,4)*(1-2*Q3)*2
1830 LET B(5,5)=A(5,5)*Q4+A(6,6)*Q3
1835 LET B(5,6)=A9*Q1*Q2
1840 LET B(6,5)=B(5,6)
1845 LET B(6,6)=B(5,5)*Q3+A(6,6)*Q4
1850 LET D1=B(1,2)*(B(2,1)*B(4,4)-3(2,4)*B(4,1))
1855 LET D2=B(1,1)*(B(2,2)*B(4,4)-3(2,4)*B(4,2))
1860 LET D3=B(1,4)*(B(2,2)*B(4,1)-3(2,1)*B(4,2))
1865 LET D=D1-D2+D3
1870 LET C(1,1)=(B(1,4)*9(4,1)-B(1,1)*3(4,4))/D
1875 LET C(2,1)=(B(1,2)*8(4,4)+B(1,4)*3(4,2))/D
1880 LET C(3,1)=(B(1,1)*8(4,2)+B(1,2)*3(4,1))/D
1885 LET C(1,2)=(B(2,1)*8(4,4)+B(2,4)*3(4,1))/D
1890 LET C(2,2)=(B(2,4)*8(4,2)+B(2,2)*3(4,4))/D
1895 LET C(3,2)=(B(2,2)*8(4,1)+B(2,1)*3(4,2))/D
1900 LET C(1,3)=(B(1,1)*8(2,4)-B(1,4)*3(2,1))/D
1905 LET C(2,3)=(B(2,2)*8(1,4)+B(1,2)*3(2,4))/D
1910 LET C(3,3)=(B(1,2)*8(2,1)-B(1,1)*3(2,2))/D
1915 LET I(1)=B(3,1)*C(2,1)+B(3,2)*C(1,1)+B(3,4)*C(3,1)
1920 LET I(2)=B(3,1)*C(2,2)+B(3,2)*C(1,2)+B(3,4)*C(3,2)
1925 LET I(3)=B(3,1)*C(2,3)+B(3,2)*C(1,3)+B(3,4)*C(3,3)
1930 LET D(0)=-H/2
1935 LET D(X)=D(X-1)+H(X)
1940 LET Z=FNA(I,J,K,L,M,N,R,S,X)+FNB(I,J,K,L,M,N,R,S,X)
1945 LET Z=Z+FNC(I,J,K,L,M,N,R,S,X)+FND(I,J,K,L,M,N,R,S,X)
1950 NEXT X
1955 LET T=T+1
1957 IF M3>0 THEN 1972
1960 IF T>1 THEN 1970
1962 PRINT
1965 PRINT "VALUES OF K(I,J,K,P,Q) ARE (MULTIPLY BY E6) "
1966 PRINT
1970 PRINT I;J;K+3;R;S;TAB(30);Z
1971 IF M3=0 THEN 1975
1972 LET J(T)=Z
1975 NEXT K
1980 NEXT J
1985 NEXT I
1990 NEXT S
1995 NEXT R
1998 DIM K(1300),J(110),L(16),M(16),N(19)
2000 IF M3=0 THEN 4000
2005 PRINT "COEFF. OF CORRECTION OPERATORS: DELTA L(I) "
2006 PRINT "(MULTIPLY BY E6)"
2007 PRINT "(LISTED ON DECREASING ORDER OF X-DERIVATIVES)"
2010 PRINT
2015 PRINT "FIRST EQUATION"

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2020 LET H=H/2
2025 CALL "DIFFCO1"K(),J(),L(),H
2030 FOR T=1 TO 16
2035 PRINT L(T),
2040 NEXT T
2045 PRINT
2050 PRINT "SECOND EQUATION"
2055 CALL "DIFFCO2"K(),J(),ZH(),H
2060 FOR T=1 TO 16
2065 PRINT M(T),
2070 NEXT T
2075 PRINT
2080 PRINT "THIRD EQUATION"
2085 CALL "DIFFCO3"K(),J(),N(),H
2090 CALL "DIFFCO4"K(),J(),N(),H
2095 FOR T=1 TO 19
3000 PRINT N(T),
3005 NEXT T
4000 END
5000 SUB "DIFFCO1":K(),J(),L():H
5001 LET L(1)=-2*(K(149)+K(161)+J(14)/2)
5002 LET L(2)=-2*(K(152)-K(150)+2*(K(155)-K(162)-J(15)/2-2*(K(257)
5003 LET L(2)=L(2)-2*(K(269)-J(23)-K(185)-K(197)-J(17)/2)
5004 LET L(2)=L(2)-K(206)-K(204)/2-K(209)-K(216)/2-2*(K(131)-2*(K(143)
5005 LET L(2)=L(2)-K(311)-K(323)-J(17)/2)
5006 LET L(3)=-K(153)-K(156)-2*(K(260)-K(258)-2*(K(263)-K(270)
5007 LET L(3)=L(3)-K(188)-K(186)/2-K(191)-K(198)/2-K(293)-K(305)
5008 LET L(3)=L(3)-J(24)/2-J(18)/4-J(27)/2-K(207)/2-K(210)/2-2*(K(134)
5009 LET L(3)=L(3)-K(132)-2*(K(137)-K(144)-K(314)-K(312)/2-K(317)
5010 LET L(3)=L(3)-K(324)/2-2*(K(239)+2*(K(251)-J(18)/4-J(11)-J(26)/2)
5011 LET L(4)=-K(261)-K(264)-K(189)/2-K(192)/2-K(296)-K(294)/2)
5012 LET L(4)=L(4)-K(299)-K(306)/2-J(27)/4-K(135)-K(138)-K(315)/2)
5013 LET L(4)=L(4)-K(318)/2-2*(K(242)+K(240)-2*(K(245)-K(252)-J(12)/2)
5014 LET L(4)=L(4)-J(27)/4-J(20)
5015 LET L(5)=-K(297)/2-K(300)/2-K(243)-K(246)-J(21)/2)
5016 LET L(5)=-K(150)-K(162)-J(15)/2-K(204)/2-K(216)/2)
5017 LET L(7)=-2*(K(148)-K(153)-2*(K(160)-K(156)-K(258)-K(270)-K(186)/2)
5018 LET L(7)=L(7)-K(198)/2-J(13)-J(24)/2-J(18)/4)
5019 LET L(7)=L(7)-K(202)-K(207)-K(214)-K(210)/2-K(132)-K(144)
5020 LET L(7)=L(7)-K(312)/2-K(324)/2-J(18)/4)
5021 LET L(8)=-2*(K(151)+2*(K(154)-2*(K(256)-K(261)-K(268)+2)
5022 LET L(8)=L(8)-K(264)-K(184)+K(199)/2-K(196)-K(192)/2-K(294)/2)
5023 LET L(8)=L(8)-K(306)/2-J(22)-J(16)/2-J(27)/4)
5024 LET L(8)=L(8)-K(205)-K(208)-2*(K(130)+2*(K(135)-2*(K(142)
5025 LET L(8)=L(8)-K(138)-K(310)-K(315)-K(322)-K(318)/2-K(240)
5026 LET L(8)=L(8)-K(252)-J(16)/2-J(12)/2-J(27)/4)
5027 LET L(9)=-2*(K(259)-2*(K(262)-K(187)-K(190)-K(292)-K(297)/2)
5028 LET L(9)=L(9)-K(304)-K(300)/2-J(25)/2-2*(K(133)-2*(K(136)-K(313)
5029 LET L(9)=L(9)-K(316)-2*(K(238)-2*(K(243)-2*(K(250)
5030 LET L(9)=L(9)-K(246)-J(10)-J(25)/2-J(21)/2)
5031 LET L(10)=-K(295)-K(296)-2*(K(241)-2*(K(244)-J(19)
5032 LET L(11)=2*(K(473)+2*(K(485)+J(41)+2*(K(527)+2*(K(539)
5033 LET L(12)=2*(K(476)+2*(K(474)+2*(K(479)+2*(K(486)+2*(K(581)+2*(K(593)
5034 LET L(12)=L(12)+K(509)+K(521)+J(42)+J(50)+J(44)/2)
5035 LET L(12)=L(12)+K(530)+K(528)+K(533)+K(540)+K(455)+K(467)+K(635)
5036 LET L(12)=L(12)+K(647)+J(44)/2)
5037 LET L(13)=2*(K(472)+2*(K(477)+2*(K(494)+2*(K(480)+2*(K(584)+2*(K(582)
5038 LET L(13)=L(13)+2*(K(587)+2*(K(594)+K(512)+K(510)+K(515)+K(522)

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5039 LET L(13)=L(13)+K(617)+K(629)+J(40)+J(51)+J(45)/2+J(53)/2
5040 LET L(13)=L(13)+K(526)+K(531)+K(538)+K(534)+2*K(458)+2*K(456)
5041 LET L(13)=L(13)+2*K(461)+2*K(469)+K(638)+K(636)+K(641)+K(648)
5042 LET L(13)=L(13)+2*K(563)+2*K(575)+J(45)/2+J(38)+J(53)/2
5043 LET L(14)=2*K(475)+2*K(478)+2*K(580)+2*K(585)+2*K(592)+2*K(588)
5044 LET L(14)=L(14)+K(508)+K(513)+K(520)+K(516)+K(620)+K(618)
5045 LET L(14)=L(14)+K(623)+K(630)+J(49)+J(43)/2+J(54)/2+K(529)+K(532)
5046 LET L(14)=L(14)+2*K(454)+2*K(459)+2*K(466)+2*K(462)+K(634)+K(639)
5047 LET L(14)=L(14)+K(646)+K(642)+2*K(566)+2*K(564)+2*K(569)+2*K(576)
5048 LET L(14)=L(14)+J(43)/2+J(39)+J(54)/2+J(47)
5049 LET L(15)=2*K(583)+2*K(586)+K(511)+K(514)+K(616)+K(621)+K(628)
5050 LET L(15)=L(15)+K(624)+J(52)/2
5051 LET L(15)=L(15)+2*K(457)+2*K(460)+K(637)+K(640)+2*K(562)
5052 LET L(15)=L(15)+2*K(567)+2*K(574)+2*K(570)+J(37)+J(52)/2+J(48)
5053 LET L(16)=K(619)+K(622)+2*K(565)+2*K(568)+J(46)
5054 SUBEND
5055 SUB "DIFFCO2":K(),J(),M(),H
5056 LET M(1)=-2*K(257)-2*K(269)-J(23)-K(311)-K(323)
5057 LET M(2)=-2*K(260)-K(258)-2*K(263)-K(270)-J(24)/2-2*K(41)-2*K(53)
5058 LET M(2)=M(2)-J(5)-K(293)+K(305)-J(26)/2-K(314)-K(312)/2-K(317)
5059 LET M(2)=M(2)-K(324)/2+2*K(239)+2*K(251)-K(95)-K(107)-J(26)
5060 LET M(3)=-K(261)-K(264)-2*K(44)+K(42)-2*K(47)-K(54)-J(6)/2-K(296)
5061 LET M(3)=M(3)-K(294)/2-K(299)-K(306)/2-J(27)/4-K(77)-K(89)-J(8)/2
5062 LET M(3)=M(3)-K(315)/2-K(318)-J(27)/2-2*K(242)-K(240)-2*K(245)
5063 LET M(3)=M(3)-K(98)-K(96)/2-K(101)-K(108)/2-J(8)/2-K(252)-J(20)
5064 LET M(3)=M(3)-2*K(23)-2*K(35)
5065 LET M(4)=-K(45)-K(48)-K(297)/2-K(300)/2-K(80)-K(78)/2-K(83)
5066 LET M(4)=M(4)-K(90)/2-J(9)/4-K(243)-K(246)-J(21)/2-K(99)/2
5067 LET M(4)=M(4)-K(102)/2+J(9)/4+2*K(267)-K(24)-2*K(29)-K(36)-J(2)
5068 LET M(5)=-K(81)/2-K(84)/2+K(27)+K(30)-J(3)/2
5069 LET M(6)=-K(258)-K(306)-J(24)/2+K(312)/2-K(324)
5070 LET M(7)=-2*K(256)-K(261)-2*K(268)-K(264)-J(22)-K(42)-K(54)-J(6)/2
5071 LET M(7)=M(7)-K(294)/2+K(306)/2+J(27)/4-K(310)-K(315)/2-K(322)
5072 LET M(7)=M(7)-K(318)/2-J(27)/4-K(240)-K(252)-K(96)/2-K(108)/2
5073 LET M(8)=-2*K(259)-2*K(262)-2*K(40)-K(45)-2*K(52)-K(48)-J(4)
5074 LET M(8)=M(8)-K(292)-K(297)/2-K(304)-K(300)/2-J(25)/2-J(78)/2
5075 LET M(8)=M(8)-K(90)/2+J(9)/4-K(313)-K(316)-J(25)/2-2*K(238)
5076 LET M(8)=M(8)-K(243)-2*K(290)-K(246)-J(21)/2-K(94)-K(99)/2-K(106)
5077 LET M(8)=M(8)-K(102)/2+J(9)/4-K(24)-K(36)
5078 LET M(9)=-2*K(43)-2*K(46)+K(295)-K(298)-K(76)-K(81)/2-K(88)
5079 LET M(9)=M(9)-K(84)/2-J(7)/2-2*K(241)-2*K(244)-J(19)-K(97)-K(100)
5080 LET M(9)=M(9)-J(7)/2-2*K(22)-K(27)-2*K(34)-K(30)-J(3)/2
5081 LET M(10)=-K(79)-K(82)-2*K(25)-2*K(28)-J(17)
5082 LET M(11)=2*K(581)+2*K(593)+J(50)+K(635)+K(647)
5083 LET M(12)=2*K(584)+2*K(582)+2*K(587)+2*K(594)+J(51)+2*K(365)
5084 LET M(12)=M(12)+2*K(377)+J(32)+K(617)+K(629)+J(53)/2+K(638)+K(636)
5085 LET M(12)=M(12)+K(641)+K(648)+J(53)+2*K(563)+2*K(575)+K(419)+K(431)
5086 LET M(13)=2*K(580)+2*K(585)+2*K(592)+2*K(588)+J(49)+2*K(368)
5087 LET M(13)=M(13)+2*K(366)+2*K(371)+2*K(378)+J(33)+K(620)+K(618)
5088 LET M(13)=M(13)+K(623)+K(630)+J(54)/2+K(401)+K(413)+J(35)/2+K(634)
5089 LET M(13)=M(13)+K(639)+K(646)+K(642)+J(54)/2+2*K(566)+2*K(564)
5090 LET M(13)=M(13)+2*K(569)+2*K(575)+J(47)+K(422)+K(420)+K(425)+K(432)
5091 LET M(13)=M(13)+J(35)/2+2*K(347)+2*K(359)
5092 LET M(14)=2*K(583)+2*K(586)+2*K(364)+2*K(369)+2*K(376)+2*K(372)
5093 LET M(14)=M(14)+J(31)+K(616)+K(621)+K(628)+K(624)+J(52)/2+K(404)
5094 LET M(14)=M(14)+K(402)+K(407)+K(414)+J(36)/2+K(537)+K(640)+J(52)/2
5095 LET M(14)=M(14)+2*K(562)+2*K(567)+2*K(574)+2*K(570)+J(48)+K(418)
5096 LET M(14)=M(14)+K(423)+K(430)+K(426)+J(36)/2+2*K(350)+2*K(348)

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5097 LET M(14)=M(14)+2*K(353)+2*K(360)+J(29)
5098 LET M(15)=2*K(367)+2*K(370)+K(619)+K(622)+K(400)+K(405)+K(412)
5099 LET M(15)=M(15)+K(408)+J(34)/2+2*K(565)+2*K(568)+J(46)+K(421)
5100 LET M(15)=M(15)+K(424)+J(34)/2+2*K(346)+2*K(351)+2*K(358)
5101 LET M(15)=M(15)+2*K(354)+J(30)
5102 LET M(16)=K(403)+K(406)+2*K(349)+2*K(352)+J(28)
5103 SUBEND
5104 SUB=DIFFCO3=K(),J(),N(),M
5105 LET N(1)=-2*H*K(149)-2*H*K(161)+H*J(14)+2*K(797)+2*K(809)+J(68)
5106 LET N(1)=N(1)-H*K(203)-H*K(215)+K(851)+K(863)
5107 LET N(2)=-2*H*K(152)-H*K(150)+2*H*K(155)-H*K(162)+H*J(15)/2
5108 LET N(2)=N(2)+2*K(800)+K(798)+2*K(803)+K(810)+J(69)/2-4*H*K(257)
5109 LET N(2)=N(2)-4*H*K(269)-2*H*J(23)+4*K(905)+4*K(917)+2*J(77)
5110 LET N(2)=N(2)-H*K(185)-H*K(197)+H*J(17)/2+K(833)+K(845)+J(71)/2
5111 LET N(2)=N(2)-H*K(206)-H*K(204)/2-H*K(209)-H*K(216)/2-H*J(17)/2
5112 LET N(2)=N(2)+K(854)+K(852)/2+K(857)+K(864)/2+J(71)/2-2*H*K(131)
5113 LET N(2)=N(2)-2*H*K(143)+2*K(779)+2*K(791)-2*H*K(311)-2*H*K(323)
5114 LET N(2)=N(2)+2*K(959)+2*K(971)
5115 LET N(3)=-H*K(153)-H*K(156)+K(801)+K(804)-4*H*K(260)-2*H*K(258)
5116 LET N(3)=N(3)-4*K(263)+2*H*J(270)-H*J(24)+4*K(908)+2*K(906)
5117 LET N(3)=N(3)+4*K(911)+2*K(918)+J(78)-H*K(188)-H*K(186)/2-H*K(191)
5118 LET N(3)=N(3)-H*K(198)/2-H*J(18)/4+K(836)+K(834)/2+K(839)+K(846)/2
5119 LET N(3)=N(3)+J(72)/4-2*H*K(41)+2*H*K(53)+2*K(689)+2*K(701)+J(59)
5120 LET N(3)=N(3)-2*H*K(293)-2*H*K(305)-H*J(267)+2*K(941)+2*K(953)+J(80)
5121 LET N(3)=N(3)-H*K(207)/2-H*K(210)/2-H*J(18)/4+K(855)/2+K(858)/2
5122 LET N(3)=N(3)+J(72)/4-2*H*K(134)-4*K(132)-2*H*K(137)-H*K(144)-H*J(11)
5123 LET N(3)=N(3)+2*K(762)+K(780)+2*K(785)+K(792)+J(65)-2*H*K(314)
5124 LET N(3)=N(3)-H*K(312)-2*H*K(317)-H*K(324)-H*J(26)+2*K(962)+K(960)
5125 LET N(3)=N(3)+2*K(965)+K(972)+J(80)-4*K(239)+H*4*H*K(251)+4*K(887)
5126 LET N(3)=N(3)+4*K(899)-H*K(895)-H*K(107)+K(743)+K(755)
5127 LET N(4)=-2*H*K(261)-2*H*K(264)+2*K(909)+2*K(912)+H*K(189)/2
5128 LET N(4)=N(4)-H*K(192)/2+K(837)/2+K(840)/2-2*H*K(44)-H*K(42)
5129 LET N(4)=N(4)-2*H*K(47)-H*K(54)+H*J(6)/2+2*K(692)+K(690)+2*K(695)
5130 LET N(4)=N(4)+K(702)+J(60)/2-2*H*K(296)-H*K(294)-2*H*K(299)-H*K(306)
5131 LET N(4)=N(4)-H*J(27)/2+2*K(894)+K(942)+2*K(947)+K(954)+J(81)/2
5132 LET N(4)=N(4)-H*K(77)-H*K(89)-H*J(8)/2+K(725)+K(737)+J(62)/2
5133 LET N(4)=N(4)-H*K(135)-H*K(135)+H*J(12)/2+K(783)+K(786)
5134 LET N(4)=N(4)+J(66)/2-H*K(315)-H*K(318)-H*J(27)/2+K(963)+K(966)
5135 LET N(4)=N(4)+J(81)/2-4*H*K(242)-2*H*K(240)-4*H*K(245)-2*H*K(252)
5136 LET N(4)=N(4)-2*H*J(20)+4*K(890)+2*K(888)+4*K(893)+2*K(900)
5137 LET N(4)=N(4)+2*J(74)-H*K(99)-H*K(96)/2-H*K(101)-H*K(108)/2
5138 LET N(4)=N(4)-H*K(8)/2+K(746)+K(744)/2+K(749)+K(756)/2+J(62)/2
5139 LET N(4)=N(4)-2*H*K(23)-2*H*K(35)+2*K(671)+2*K(683)
5140 LET N(5)=-H*K(45)-H*K(48)+K(693)+K(696)-H*K(297)-H*K(300)+K(945)
5141 LET N(5)=N(5)+K(948)-H*K(80)-4*K(78)/2-H*K(83)-H*K(90)/2-H*J(9)/4
5142 LET N(5)=N(5)+K(728)+K(726)/2+K(731)+K(738)/2+J(63)/4-2*H*K(243)
5143 LET N(5)=N(5)-2*H*K(246)-H*J(21)+2*K(891)+2*K(894)+J(75)-H*K(99)/2
5144 LET N(5)=N(5)-H*K(102)/2-H*J(9)/4+K(747)/2+K(750)/2+J(63)/4
5145 LET N(5)=N(5)-2*H*K(26)-H*K(24)+2*H*K(29)-H*K(36)-H*J(2)+2*K(674)
5146 LET N(5)=N(5)+K(672)+2*K(677)+K(684)+J(56)
5147 LET N(6)=-H*K(81)/2-H*K(84)/2+K(729)/2+K(732)/2-H*K(27)-H*K(30)
5148 LET N(6)=N(6)-H*J(3)/2+K(675)+K(678)+J(57)/2
5149 LET N(7)=-H*K(150)-H*K(162)-H*J(15)/2+K(798)+K(810)+J(69)/2
5150 LET N(7)=N(7)-H*K(204)/2-H*K(216)/2+K(852)+K(864)
5151 LET N(8)=-2*H*K(148)-H*K(153)-2*H*K(160)-H*K(156)-H*J(13)+2*K(796)
5152 LET N(8)=N(8)+K(801)+2*K(808)+K(804)+J(67)-2*H*K(258)-2*H*K(270)
5153 LET N(8)=N(8)-H*J(24)+2*K(906)+2*K(918)+J(78)-H*K(186)/2-H*K(198)/2
5154 LET N(8)=N(8)-H*J(18)/2+K(834)/2+K(846)/2+J(72)/4-H*K(202)

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5155 LET N(8)=N(8)-H*K(287)/2-H*K(224)-H*K(210)-H*J(18)/4*K(856)+K(855)/2
5156 LET N(8)=N(8)+K(862)+K(858)/2+J(72)/4-H*K(132)-H*K(144)+K(780)
5157 LET N(8)=N(8)+K(792)-H*K(312)-H*K(324)+K(960)+K(972)
5158 LET N(9)=-2+H*K(151)-2+H*K(154)+2*K(799)+2*K(802)-4+H*K(256)
5159 LET N(9)=N(9)-2+H*K(261)-4+H*K(268)-2+H*K(264)-2+H*J(22)+4*K(904)
5160 LET N(9)=N(9)+2*K(909)+4*K(916)+2*K(912)+2+J(76)-H*K(184)-H*K(189)/2
5161 LET N(9)=N(9)-H*K(196)-H*K(192)/2-H*J(16)/2+K(832)+K(837)+2+K(844)
5162 LET N(9)=N(9)+K(840)/2+J(70)/2-H*K(42)-H*K(54)-H*J(6)/2+2+K(690)+K(702)
5163 LET N(9)=N(9)+J(60)/2-H*K(294)-H*K(306)-H*J(27)/2+K(942)+J(81)/2
5164 LET N(9)=N(9)-H*K(205)-H*K(208)+H*J(16)/2+K(853)+K(856)+J(70)/2
5165 LET N(9)=N(9)-2+H*K(130)-H*K(135)-2+H*K(142)-H*K(138)-H*J(12)/2
5166 LET N(9)=N(9)+2+K(778)+K(783)+2+K(790)+K(786)+J(66)/2-2+H*K(310)
5167 LET N(9)=N(9)-H*K(315)-2+H*K(322)-H*K(318)-H*J(27)/2+2+K(958)
5168 LET N(9)=N(9)+K(963)+2+K(970)+K(966)+J(81)/2-2+H*K(240)
5169 LET N(9)=N(9)-2+H*K(252)+2+K(888)+2+K(900)+H*K(96)/2-H*K(808)/2
5170 LET N(9)=N(9)+K(744)/2+K(756)/2
5171 SUBEND
5172 SUB "DIFFCO4":K(),J(),N():M
5173 LET N(10)=-4+H*K(259)-4+H*K(262)+4*K(907)+4*K(910)-H*K(187)
5174 LET N(10)=N(10)-H*K(190)+K(835)+K(838)-2+H*K(40)-H*K(45)-2+H*K(52)
5175 LET N(10)=N(10)-H*K(48)-H*J(4)+2+K(688)+K(693)+2+K(700)+K(696)
5176 LET N(10)=N(10)+J(58)-2+H*K(292)-H*K(297)-2+H*K(304)+K(300)
5177 LET N(10)=N(10)-H*J(25)+2+K(940)+K(945)+2+K(952)+K(948)+J(79)
5178 LET N(10)=N(10)-H*K(78)/2+H*K(90)/2-H*J(9)/4+K(726)/2+K(738)/2
5179 LET N(10)=N(10)+J(63)/4-2+H*K(133)-2+H*K(136)-H*J(10)+2+K(781)
5180 LET N(10)=N(10)+2+K(784)+J(64)-2+H*K(313)-2+H*K(316)-H*J(25)
5181 LET N(10)=N(10)+2+K(961)+2+K(964)+J(79)-4+H*K(236)-2+H*K(243)
5182 LET N(10)=N(10)-4+H*K(250)+2+H*K(246)-H*J(21)+4+K(886)
5183 LET N(10)=N(10)+2+K(891)+4+K(898)+2+K(894)+J(75)-H*K(94)
5184 LET N(10)=N(10)-H*K(99)/2+H*K(106)-H*K(102)/2-H*J(9)/4+K(742)
5185 LET N(10)=N(10)+K(747)/2+K(754)+K(750)/2+J(63)/4-H*K(24)
5186 LET N(10)=N(10)-H*K(36)+K(672)+K(664)
5187 LET N(11)=-2+H*K(43)-2+H*K(46)+2+K(691)+2+K(694)-2+H*K(295)
5188 LET N(11)=N(11)-2+H*K(298)+2+K(943)+2+K(946)-H*K(76)-H*K(81)/2
5189 LET N(11)=N(11)-H*K(88)-H*K(84)/2-H*J(7)/2+K(724)+K(729)+2+K(736)
5190 LET N(11)=N(11)+K(732)/2+J(61)/2-4+H*K(241)-4+H*K(244)-2+H*J(19)
5191 LET N(11)=N(11)+4+K(889)+4+K(892)+2+J(73)-H*K(97)-H*K(100)
5192 LET N(11)=N(11)-H*J(7)/2+K(745)+K(748)+J(61)/2-2+H*K(22)-H*K(27)
5193 LET N(11)=N(11)-2+H*K(34)+H*K(30)-H*J(3)/2+2+K(670)+K(675)
5194 LET N(11)=N(11)+2+K(682)+K(678)+J(57)/2
5195 LET N(12)=-H*K(79)-H*K(82)+K(727)+K(730)-2+H*K(25)-2+H*K(28)-H*J(1)
5196 LET N(12)=N(12)+2+K(673)+2+K(676)+J(55)
5197 LET N(13)=2+H*K(473)+2+H*K(465)+H*J(41)-2+K(1121)-2+K(1133)-J(95)
5198 LET N(13)=N(13)+H*K(527)+H*K(539)-K(1175)-K(1187)
5199 LET N(14)=2+H*K(478)+2+H*K(474)+2+H*K(479)+2+H*K(486)+H*J(42)
5200 LET N(14)=N(14)-2+K(1124)+2+K(1122)-2+K(1127)-2+K(1134)
5201 LET N(14)=N(14)-J(96)+4+H*K(581)+4+H*K(593)+2+H*J(50)-4+K(1229)
5202 LET N(14)=N(14)-4+K(1241)+2+J(104)+H*K(509)+H*K(521)+H*J(44)/2
5203 LET N(14)=N(14)-K(1157)-K(1163)+J(98)/2+2+H*K(455)+2+H*K(467)
5204 LET N(14)=N(14)-2+K(1103)+2+K(1113)+2+H*K(635)+2+H*K(647)-2+K(1263)
5205 LET N(14)=N(14)-2+K(1295)+H*K(530)+H*K(528)+H*K(533)+H*K(540)
5206 LET N(14)=N(14)+H*J(44)/2+K(1178)-K(1176)-K(1181)-K(1188)-J(98)/2
5207 LET N(15)=2+H*K(472)+2+H*K(477)+2+H*K(484)+2+H*K(480)+J(40)
5208 LET N(15)=N(15)-2+K(1120)-2+K(1125)-2+K(1132)-2+K(1128)-J(94)
5209 LET N(15)=N(15)+4+H*K(584)+4+H*K(582)+4+H*K(587)+4+H*K(594)
5210 LET N(15)=N(15)+2+H*J(51)+4+K(1232)-4+K(1230)-4+K(1235)-4+K(1242)
5211 LET N(15)=N(15)+2+J(105)+H*K(512)+H*K(510)+H*K(515)+H*K(522)
5212 LET N(15)=N(15)+H*J(45)/2+K(1160)-K(1158)-K(1163)-K(1170)-J(99)/2

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5213 LET N(15)=N(15)+2\*HOK(365)+2\*HOK(377)+HOK(32)-2\*K(1013)-2\*K(1025)  
 5214 LET N(15)=N(15)-J(86)+2\*HOK(617)+2\*HOK(629)+HOK(53)-2\*K(1265)  
 5215 LET N(15)=N(15)-2\*K(1277)+J(107)+HOK(526)+HOK(531)+HOK(538)+HOK(534)  
 5216 LET N(15)=N(15)+HOK(45)+2\*K(1174)+K(1179)+K(1186)+K(1162)  
 5217 LET N(15)=N(15)-J(89)+2\*2\*HOK(438)+2\*HOK(456)+2\*HOK(461)+2\*HOK(468)  
 5218 LET N(15)=N(15)+HOK(38)-2\*K(1105)-2\*K(1104)-2\*K(1109)-2\*K(1116)  
 5219 LET N(15)=N(15)-J(92)+2\*HOK(638)+2\*HOK(636)+2\*HOK(641)+2\*HOK(648)  
 5220 LET N(15)=N(15)+HOK(53)-2\*K(1286)+2\*K(1284)-2\*K(1289)-2\*K(1296)  
 5221 LET N(15)=N(15)-J(107)+4\*HOK(563)+4\*HOK(575)+4\*K(1211)+4\*K(1223)  
 5222 LET N(15)=N(15)+HOK(419)+HOK(631)+K(1067)+K(1079)  
 5223 LET N(16)=2\*HOK(475)+2\*HOK(479)+2\*K(1123)+2\*K(1126)+4\*K(589)+H  
 5224 LET N(16)=N(16)+4\*HOK(585)+4\*HOK(592)+4\*HOK(588)+2\*HOK(49)+4\*K(1228)  
 5229 LET N(16)=N(16)+4\*K(1233)+4\*K(1240)+4\*K(1236)+2\*J(103)+HOK(508)  
 5226 LET N(16)=N(16)+HOK(513)+HOK(520)+HOK(516)+HOK(43)+2\*K(1156)+K(1161)  
 5227 LET N(16)=N(16)+K(1168)+K(1164)+J(97)+2\*2\*HOK(368)+2\*HOK(366)  
 5228 LET N(16)=N(16)+2\*HOK(371)+2\*HOK(378)+HOK(33)+2\*K(1016)+2\*K(1014)  
 5229 LET N(16)=N(16)+2\*K(1019)+2\*K(1025)+J(87)+2\*HOK(620)+2\*HOK(618)  
 5230 LET N(16)=N(16)+2\*HOK(623)+2\*HOK(630)+HOK(54)+2\*K(1268)+2\*K(1266)  
 5231 LET N(16)=N(16)+2\*K(1271)+2\*K(1276)+J(188)+HOK(401)+HOK(433)  
 5232 LET N(16)=N(16)+HOK(35)+2\*K(1049)+K(1061)+J(89)+2\*HOK(529)+HOK(532)  
 5233 LET N(16)=N(16)+HOK(43)+2\*K(1177)+K(1188)+J(97)+2\*2\*HOK(454)  
 5234 LET N(16)=N(16)+2\*HOK(459)+2\*HOK(466)+2\*HOK(462)+HOK(39)+2\*K(1102)  
 5235 LET N(16)=N(16)+2\*K(1107)+2\*K(1114)+2\*K(1110)+J(93)+2\*HOK(634)  
 5236 LET N(16)=N(16)+2\*HOK(639)+2\*HOK(646)+2\*HOK(642)+HOK(54)+2\*K(1282)  
 5237 LET N(16)=N(16)+2\*K(1287)+2\*K(1290)+J(108)+4\*HOK(566)+4\*HOK(564)  
 5238 LET N(16)=N(16)+4\*HOK(569)+4\*HOK(576)+2\*HOK(47)+4\*K(1214)  
 5239 LET N(16)=N(16)+4\*K(1212)+4\*K(1217)  
 5240 LET N(16)=N(16)+4\*K(1224)+2\*J(101)+HOK(422)+HOK(420)+HOK(425)  
 5241 LET N(16)=N(16)+HOK(432)+HOK(35)+2\*K(1070)+K(1068)+K(1073)+J(89)+2  
 5242 LET N(16)=N(16)+2\*HOK(347)+2\*HOK(359)+2\*K(995)+2\*K(1007)  
 5243 LET N(17)=4\*HOK(583)+4\*HOK(586)+4\*K(1231)+4\*K(1234)+HOK(511)  
 5244 LET N(17)=N(17)+HOK(514)+K(1159)+K(1162)+2\*HOK(364)+2\*HOK(369)  
 5245 LET N(17)=N(17)+2\*HOK(376)+2\*HOK(372)+HOK(31)+2\*K(1012)+2\*K(1017)  
 5246 LET N(17)=N(17)+2\*K(1024)+2\*K(1020)+J(85)+2\*HOK(626)+2\*HOK(621)  
 5247 LET N(17)=N(17)+2\*HOK(629)+2\*HOK(624)+HOK(52)+2\*K(1264)+2\*K(1269)  
 5248 LET N(17)=N(17)+2\*K(1276)+2\*K(1272)+J(106)+HOK(404)+HOK(402)  
 5249 LET N(17)=N(17)+HOK(407)+HOK(414)+HOK(36)+2\*K(1052)+K(1050)+K(1055)  
 5250 LET N(17)=N(17)+K(1062)+J(90)+2\*2\*HOK(457)+2\*HOK(460)+HOK(37)  
 5251 LET N(17)=N(17)+2\*K(1105)+2\*K(1108)+J(91)+2\*HOK(637)+2\*HOK(640)  
 5252 LET N(17)=N(17)+J(52)+2\*K(1285)+2\*K(1286)+J(106)+4\*HOK(562)  
 5253 LET N(17)=N(17)+4\*HOK(567)+4\*HOK(574)+4\*HOK(570)+2\*HOK(48)  
 5254 LET N(17)=N(17)+4\*K(1210)+4\*K(1215)+4\*K(1222)+4\*K(1218)+2\*J(102)  
 5255 LET N(17)=N(17)+HOK(418)+HOK(423)+HOK(430)+HOK(426)+HOK(36)+2  
 5256 LET N(17)=N(17)+K(1066)+K(1071)+K(1078)+K(1074)+J(90)+2\*2\*HOK(358)  
 5257 LET N(17)=N(17)+2\*HOK(348)+2\*HOK(353)+2\*HOK(360)+HOK(29)+2\*K(998)  
 5258 LET N(17)=N(17)+2\*K(996)+2\*K(1001)+2\*K(1008)+J(83)  
 5259 LET N(18)=2\*HOK(369)+2\*HOK(370)+2\*K(1015)+2\*K(1010)+2\*HOK(619)  
 5260 LET N(18)=N(18)+2\*HOK(622)+2\*K(1267)+2\*K(1270)+HOK(400)+HOK(405)  
 5261 LET N(18)=N(18)+HOK(412)+HOK(408)+HOK(34)+2\*K(1048)+K(1053)+K(1060)  
 5262 LET N(18)=N(18)+K(1056)+J(88)+2\*4\*HOK(565)+4\*HOK(568)+2\*HOK(46)  
 5263 LET N(18)=N(18)+4\*K(1213)+4\*K(1216)+2\*J(100)+HOK(621)+HOK(624)  
 5264 LET N(18)=N(18)+HOK(34)+2\*K(1069)+K(1072)+J(88)+2  
 5265 LET N(18)=N(18)+2\*HOK(346)+2\*HOK(351)+2\*HOK(358)+2\*HOK(354)+HOK(36)  
 5266 LET N(18)=N(18)+2\*K(994)+2\*K(999)+2\*K(1006)+2\*K(1002)+J(84)  
 5267 LET N(19)=HOK(403)+HOK(406)+K(1051)+K(1054)+2\*HOK(349)+2\*HOK(352)  
 5268 LET N(19)=N(19)+HOK(28)+2\*K(997)+2\*K(1000)+J(82)  
 5269 SUBEND

## MATCH

```
10 FOR P=0 TO 1
20 FOR Q=0 TO 1
30 FOR I=1 TO 3
40 FOR J=1 TO 3
50 FOR K=1 TO 2
60 FOR L=1 TO 2
70 FOR M=1 TO 3
80 FOR N=1 TO 3
90 LET T=T+1
100 PRINT I;J;K+4;L+4;MIN;P;Q;TAB(35);T
110 NEXT N
120 NEXT M
130 NEXT L
140 NEXT K
150 NEXT J
160 NEXT I
170 NEXT Q
180 NEXT P
190 PRINT
200 LET T=0
210 FOR P=0 TO 1
220 FOR Q=0 TO 1
230 FOR I=1 TO 3
240 FOR J=1 TO 3
250 FOR K=1 TO 3
260 LET T=T+1
270 PRINT I;J;K+3;P;Q;TAB(35);T
280 NEXT K
290 NEXT J
300 NEXT I
310 NEXT Q
320 NEXT P
330 END
```

MATCH

1	1	5	5	4	1	0	0	1
1	1	5	5	4	2	0	0	2
1	1	5	5	4	3	0	0	3
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1	1	5	5	4	7	0	0	7
1	1	5	5	4	8	0	0	8
1	1	5	5	4	9	0	0	9
1	1	5	5	4	10	0	0	10
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1	1	5	5	4	16	0	0	16
1	1	5	5	4	17	0	0	17
1	1	5	5	4	18	0	0	18
1	1	5	5	4	19	0	0	19
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1	1	5	5	4	21	0	0	21
1	1	5	5	4	22	0	0	22
1	1	5	5	4	23	0	0	23
1	1	5	5	4	24	0	0	24
1	1	5	5	4	25	0	0	25
1	1	5	5	4	26	0	0	26
1	1	5	5	4	27	0	0	27
1	1	5	5	4	28	0	0	28
1	1	5	5	4	29	0	0	29
1	1	5	5	4	30	0	0	30
1	1	5	5	4	31	0	0	31
1	1	5	5	4	32	0	0	32
1	1	5	5	4	33	0	0	33
1	1	5	5	4	34	0	0	34
1	1	5	5	4	35	0	0	35
1	1	5	5	4	36	0	0	36
1	1	5	5	4	37	0	0	37
1	1	5	5	4	38	0	0	38
1	1	5	5	4	39	0	0	39
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1	1	5	5	4	52	0	0	52
1	1	5	5	4	53	0	0	53
1	1	5	5	4	54	0	0	54





































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2	1	6	6	6	1	1	1125
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2	1	6	6	6	29	1	1153
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2	1	6	6	6	31	1	1155
2	1	6	6	6	32	1	1156









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2	3	5	1	1
2	3	6	1	1
3	1	4	1	1
3	1	5	1	1
3	1	6	1	1
3	2	4	1	1
3	2	5	1	1
3	2	6	1	1
3	3	4	1	1
3	3	5	1	1
3	3	6	1	1

92  
93  
94  
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96  
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100  
101  
102  
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104  
105  
106  
107  
108

DIFFCO

VALUES OF K(I,J,K,L,M,N,P,Q) ARE (MULTIPLY BY E6)

1	1	5	5	1	1	0	0	4.72616	E-2
1	1	5	5	1	2	0	0	1.56731	E-2
1	1	5	5	1	3	0	0	-1.70966	E-4
1	1	5	5	2	1	0	0	1.12845	E-2
1	1	5	5	2	2	0	0	6.81198	E-2
1	1	5	5	2	3	0	0	-3.42554	E-3
1	1	5	5	3	1	0	0	1.04722	E-7
1	1	5	5	3	2	0	0	-1.65782	E-9
1	1	5	5	3	3	0	0	4.38228	E-3
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1	1	6	6	1	3	0	0	-1.70966	E-4
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1	1	6	6	2	2	0	0	6.81198	E-2
1	1	6	6	2	3	0	0	-3.42554	E-3
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1	1	6	6	3	3	0	0	4.38228	E-3
1	1	2	5	5	1	1	0	9.46502	E-3
1	1	2	5	5	1	2	0	5.0569	E-3
1	1	2	5	5	1	3	0	-3.2572	E-5
1	1	2	5	5	2	1	0	2.17294	E-3
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2	2	6	1	1
2	3	4	1	1
2	3	5	1	1
2	3	6	1	1
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4	1	6	1	1
4	2	4	1	1
4	2	5	1	1
4	2	6	1	1
4	3	4	1	1
4	3	5	1	1
4	3	6	1	1

*2,54565	E-6
6,52612	E-6
1,24116	E-6
1,29711	E-6
*3,37576	E-6
*6,33181	E-7
*1,42203	E-5
1,12269	E-5
6,55275	E-6
*4,07301	E-7
1,04417	E-6
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*6,07091	E-12
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*8,30452	E-12
*1,57939	E-12
*6,89421	E-7
7,70891	E-7
3,21105	E-7

147,878 SEC, 98 I/O

#### REFERENCES

1. Schile, R.D., "A Correspondence Principle for Laminated Plates". Presented at the Ninth Annual Meeting of the Society of Engineering Science, Inc., November 10, 1971
2. Schile, R.D., "Laminate Elastostatics", Int'l. Journal Eng'g Sci., 9, 1, 1971
3. Reiss, E.L. and S. Locke, "On the Theory of Plane Stress", Qtly. of Appl. Math., XIX, 195, 1961.

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13. ABSTRACT  
The objective of this program is the determination of the state of stress in laminated, rectangular plates in which the individual plies are composed of fiber-reinforced, polymeric material. The starting point for the analysis is the three-dimensional theory of elasticity. The continuity conditions between plies are taken into account by means of an integral formulation of three-dimensional stress function theory. When the number of laminations is very large, a Correspondence Principle has been derived which relates the stress state in the laminated plate to that in a corresponding, homogeneous, anisotropic plate. For unsymmetrically laminated plates, equations governing the coupled bending and stretching deformation have been derived. Solutions of these equations are exhibited for the cases of loading by uniformly distributed edge forces and moments and a uniform temperature. An attempt was made to derive a theory describing the state of stress near the edges of a laminated plate where the stress distribution is three dimensional. Although technically successful, the resulting equations and boundary conditions are so complex that the corrected theory is not useful.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Composites Plates Elasticity Stress						

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