REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
2/2/96

3. REPORT TYPE AND DATES COVERED
Final 15 Oct 92 - 14 Oct 95

4. TITLE AND SUBTITLE
PDE, Differential and Algebraic Methods for Non-linear Filtering

5. FUNDING NUMBERS
DAAH04-93-G-0006

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 29674.11-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

19960621 144

13. ABSTRACT *(Maximum 200 words)*

We have found the best solution to Duncan-Mortensen-Zakai (DMZ) equation for linear filtering system and exact filtering system. We show that this equation can be solved explicitly with an arbitrary initial condition by solving a system of ordinary differentialequations and a Kolmogorov type equation. Let n be the dimension of state space. We show that we need only n sufficient statistics in order to solve the DMZ equation.

In the other direction, we prove that if the estimation algebra is finite dimensional and of maximal rank, then the $\Omega = \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right)$ matrix, where f denotes the drift term, is a linear matrix in the sense that all the entries in Ω are degree one polynomials. This theorem plays a fundamental role in the classification of finite dimensional estimate algebra of maximal rank.

14. SUBJECT TERMS
Duncan-Mortensen-Zakai equation, linear filtering, exact filtering, estimation algebra.

15. NUMBER IF PAGES
23

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

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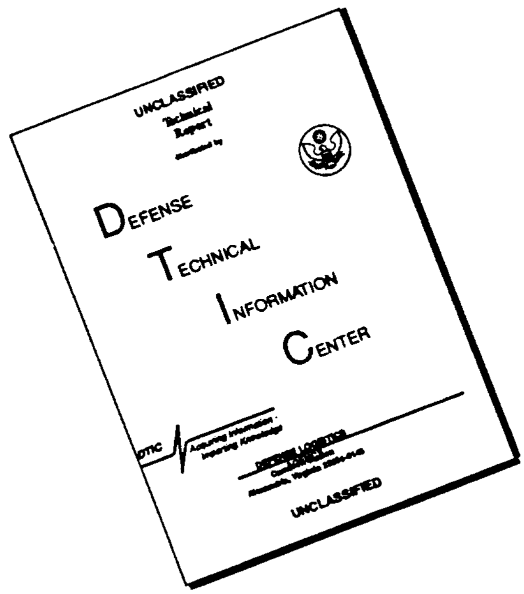

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PDE, DIFFERENTIAL GEOMETRIC AND ALGEBRAIC METHODS IN NONLINEAR FILTERING

Final Report

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December 15, 1995

U.S. Army Research Office

Army Grant DAAH04-93-G-0006

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

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A. Statement of problem studied.

The filtering problem considered here is based on the following signal observation model:

$$\begin{cases} dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dv(t) & x(0) = x_0 \\ dy(t) = h\big(x(t)\big)dt + dw(t) & y(0) = 0 \end{cases} \tag{1}$$

in which x, v, y and w, are, respectively, $\mathbf{R}^n$, $\mathbf{R}^p$, $\mathbf{R}^m$ and $\mathbf{R}^m$ valued processes, and v and w have components which are independent, standard Brownian process. We further assume that $n = p$, f, h are C^∞ smooth, and that g is an orthogonal matrix. We will refer to $x(t)$ as the state of the system at time t and $y(t)$ as the observation at time t.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s); 0 \leq s \leq t\}$. It is well-known that $\rho(t,x)$ is given by normalizing a function, $\sigma(t,x)$, which satisfies the following Duncan-Mortensen-Zakai equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy(t), \sigma(0,x) = \sigma_0 \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$$

and for $i = 1, \ldots, m$, L_i is the zero degree differential operator of multiplication by h_i. σ_0 is the probability density of the initial point, x_0.

Equation (2) is a stochastic partial differential equation. In real applications, we are interested in constructing robust state estimators from observed sample paths with some property of robustness. Davis studied this problem and proposed some robust algorithms. In our case, his basic idea reduces to refining a new unnormalized density

$$\xi(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t,x).$$

It is easy to show that $\xi(t,x)$ satisfies the following time varying partial differential equation

$$\begin{aligned} \frac{d\xi}{dt}(t,x) &= L_0\xi(t,x) + \sum_{i=1}^{m} y_i(t)[L_0, L_i]\xi(t,x) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\big[[L_0, L_i], L_j\big]\xi(t,x) \\ \xi(0,x) &= \sigma_0 \end{aligned} \tag{3}$$

where $[L_0, L_i]$ denotes the Lie bracket of L_0 and L_i.

Definition. The estimation algebra E of a filtering problem (1), is defined to be the Lie algebra generated by $\{L_0, L_1, \ldots, L_m\}$, or $E = \langle L_0, L_1, \ldots, L_m \rangle_{L.A.}$

If in addition there exists a potential function ψ such that $f_i = \frac{\partial \psi}{\partial x_i}$ for all $1 \leq i \leq n$, then the estimation algebra is called exact.

The estimation algebra is said to be with maximal rank if $x_i + c_i$ is in E for all $1 \leq i \leq n$ where c_i is a constant.

The problem is to solve explicitly equation (3) in real-time. In particular, we want to construct all possible finite dimensional filters via direct method or Wei-Norman approach. This includes solving the Brockett problem on classification of finite dimensional estimation algebras.

B. Summary of the most important results.

(I) In [Mi 1], Mitter pointed out that the innovation approach to nonlinear filtering theory is not, in general, explicitly computable. It was first proposed by Brockett and Clark [Br-Cl], Brockett [Br], and Mitter [Mi 1] to use estimation algebras to construct finite-dimensional filters. The idea is to use the Lie algebraic method to solve the Duncan-Mortensen-Zakai (DMZ) equation, which is a stochastic partial differential equation. By working on the robust form of the DMZ equation, we can reduce the complexity of the problem to that of solving a time-variant partial differential equation.

In the past decade, the Lie algebraic viewpoint has been remarkably successful, and recent works [T-W-Y], [D-T-W-Y], [Ya], [Ch-Ya], [C-L-Y] have given us a deeper understanding of the DMZ equation, which was essential for progress in nonlinear filtering as well as in stochastic control. Nevertheless, it is extremely desirable to treat the DMZ equation by a direct method. We shall use a direct method to solve DMZ equation for linear filtering system and exact filtering system, respectively. For the linear filtering system (i.e., f, g, and h are linear functions), the explicit recursive filter was previously derived in a closed form for arbitrary initial conditions only for those systems that are completely reachable and completely observable.

For linear filtering with an arbitrary initial condition, Makowski [Ma 1] observed that the filtering question is genuinely one of nonlinear filtering and few results have been obtained before. Notable exceptions are the works of Beněs and Karatzas [Be-Ka], Ocone [Oc], Makowski [Ma 1], and Haussmann and Pardoux [Ha-Pa]. Makowski [Ma 1] further remarked that it was shown in [Be-Ka] and [Oc] that there always exists a set of sufficient statistics that can be recursively computed as outputs of a finite-dimensional dynamic system. In contrast with previous results, the sufficient statistics generated in [Ma 1] can be termed "universal" in the sense that they are independent of the initial state distribution. Furthermore, no assumptions on the moments of this initial state distribution or its absolute continuity are made in [Ma 1], as was the case in [Be-Ka] and [Oc].

However, Makowski's method has two major disadvantages. First, let n be the dimension of the state space. The number of sufficient statistics in order to compute the conditional expectation $\pi_t(\phi(x_t))$ of $\phi(x_t)$ in Makowski's method is a polynomial of degree two in n, while in our method below (Theorem 1), the number of sufficient statistics is only n (see Remark 3.1). Second, the universal formula (3.13) of [Ma 1] for $\pi_t(\phi(x_t))$ is implicit and depends on ϕ. For a given ϕ, one has to do some computation before one can find the number of sufficient statistics.

Haussmann and Pardoux [Ha-Pa] considered a more general linear filtering problem than ours. However, even restricted to the classical Kalman-Bucy problem with a Guassian initial condition, the number of sufficient statistics in order to compute the conditional probability density is again a polynomial of degree two in n in their method. Since the complexity of our method for linear filtering with an arbitrary initial condition is exactly n, the same as the classical Kalman-Bucy filter, and since our formulas are universal in the sense that they are independent of the initial state distribution, we can apply these results for the recursive numerical solution of filtering problems. These results make the results of practical importance.

Despite its usefulness, the Kalman-Bucy filter is not perfect. One of its weaknesses is that it needs a Gaussian assumption on the initial data. The situation is more complex when the statistics of the initial condition are modeled by an arbitrary distribution. In the case where the linear filtering system (i.e., f, g, and h are linear functions in (1) is completely reachable and completely observable, Hazewinkel observed on p. 115 of [Ha] that the estimation algebra E (i.e., a Lie algebra generated by differential operators L_0, $h_1(x), \ldots, h_m(x)$) is the $2n+2$ dimensional Lie algebra with basis L_0, $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$, $x_1, \ldots, x_n$, 1. Even in this case, the Wei-Norman approach used to find the solution of (3) is more complicated than the procedure in Theorem 1 below because not only must one solve a finite system of ordinary differential equation and a Kolmogorov equation, but also one has to integrate n partial differential equations corresponding to the operators $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$. More important, if the linear system is not completely reachable or completely observable, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, there is an additional disadvantage of the Wei-Norman approach, namely, one cannot write down the finite system of ordinary differential equation explicitly. The novelty of theorem 1 is that our finite system of ordinary differential equations is explicitly written down and our procedure to get the solution of (3) is simpler than the Lie algebra approach.

Theorem 1. Consider the linear filtering system (1) with

$$h_i(x) = \sum_{j=1}^{n} c_{ij} x_j + c_i, \ 1 \leq i \leq m, \tag{4}$$

where c_{ij} and c_i are constants, and

$$f_i(x) = \sum_{j=1}^{n} d_{ij} x_j + d_i, \ 1 \leq i \leq n, \tag{5}$$

where d_{ij} and d_i are constants. Choose a homogeneous quadratic $F(x) = \frac{1}{2} \sum_{i,j=1}^{n} e_{ij} x_i x_j$ with $e_{ij} = e_{ji}$ such that

$$(E+D)^T(E+D) = C^T C + D^T D. \tag{6}$$

Here $E = (e_{ij})$, $D = (d_{ij})$ are an $n \times n$ matrix and $C = (c_{ij})$ is an $m \times n$ matrix. Then the solution $u(t,x)$ for the Duncan-Mortensen-Zakai equation (3) is reduced to the solution $\widetilde{u}(t,x)$ for the Kolmogorov equation

$$\begin{aligned}\frac{\partial \widetilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta\widetilde{u}(t,x) - \sum_{i=1}^{n}\left(f_i(x) + \frac{\partial F}{\partial x_i}(x)\right)\frac{\partial \widetilde{u}}{\partial x_i}(t,x) \\ - \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_i}(x) + \frac{\partial^2 F}{\partial x_i^2}(x)\right)\widetilde{u}(t,x),\end{aligned} \tag{7}$$

where

$$\widetilde{u}(t,x) = e^{F(x)+\Sigma_{i=1}^{n} a_i(t)x_i + c(t)} u\big(t, x + b(t)\big) \tag{8}$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy the following system of ODEs:

$$\begin{aligned}a_i'(t) + \sum_{j=1}^{n} d_{ji}a_i(t) - \sum_{\ell=1}^{m}\sum_{j=1}^{n} c_{\ell j}b_j(t)c_{\ell i} - \sum_{j=1}^{m}\sum_{k=1}^{n} y_j(t)d_{ki}c_{jk} \\ + \sum_{j=1}^{n} d_j e_{ji} - \sum_{j=1}^{m} c_j c_{ji} = 0, \ 1 \le i \le n\end{aligned} \tag{9}$$

$$\begin{aligned}&\sum_{i=1}^{n} d_i a_i(t) + c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) - \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{ij}b_j(t)\right)^2 - \sum_{i=1}^{m} c_i \sum_{j=1}^{n} c_{ij}b_j(t) \\ &- \sum_{j=1}^{m}\sum_{i=1}^{n} y_j(t)c_{ji}\left(\sum_{k=1}^{n} d_{ik}b_k(t) + d_i\right) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(t)y_j(t)c_{ik}c_{jk} \\ &+ \frac{1}{2}\sum_{k=1}^{n} e_{kk} - \frac{1}{2}\sum_{i=1}^{m} c_i^2 = 0\end{aligned} \tag{10}$$

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) - \sum_{j=1}^{n} d_{ij}b_j(t) = 0 \qquad 1 \le i \le n. \tag{11}$$

(II) The exact filtering problem is related to the linear filtering problem by the gauge transformation, as pointed out in [Mi 2]. The idea is also related to the concept of equivalence of parabolic equations, as discussed in [Ba]. However, no explicit solution has been written down. Notable exceptions are [Be] and [T-W-Y], in which the maximal rank condition of the estimation algebra is assumed. In [Be] a special case of the maximal-rank condition is treated, while in [T-W-Y] the general case of the maximal-rank condition is treated. Even in the maximal-rank case, the Wei-Norman approach to finding the solution of the robust DMZ equation is more complicated than our approach (Theorem 2). Not only must one solve a finite system of ODEs and a Kolmogorov equation, but also one must integrate n partial differential equations. More important, for the nonmaximal-rank case, since the basis of the estimation algebra is not explicitly known, the recursive algorithms cannot be written down explicitly. Haussmann and Pardoux [Ha-Pa] considered a more general Benés filtering problem than ours. However, even restricted to the classical Benés problem, the number of sufficient statistics in order to compute the conditional probability density is a polynomial of degree two in n in their method. The novelty of our Theorem 2 is that our finite system of ordinary differential equations is explicitly written down and our algorithms are universal in the sense that they are independent of the initial state distribution. Moreover, we need only n sufficient statistics in order to compute the conditional probability density of the exact filtering with an arbitrary initial condition. So the result is of practical importance.

Let us observe that the elliptic differential operator L_0 in (2) can be more compactly represented if one defines $D_i = \frac{\partial}{\partial x_i} - f_i$. Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta\right), \tag{12}$$

where

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{13}$$

This compact representation of L_0 was exploited in [Wo 1], [Wo 2], and [T-W-Y] to derive necessary conditions and sufficient conditions for estimation algebras to be finite dimensional. (The concept of an estimation algebra was introduced in [Br-Cl], [Br], and [Mi 1]. It is defined to be the Lie algebra of differential operators generated by $\{L_0, L_1, \ldots, L_m\}$.) Observe that if we let $f_i = F_i(x) = \displaystyle\int_0^x f_i(t)dt$, then

$$D_i = \frac{\partial}{\partial x_i} - f_i = e^{f_i}\frac{\partial}{\partial x_i}e^{-f_i}. \tag{14}$$

Hence

$$D_i^2 = e^{f_i}\frac{\partial^2}{\partial x_i^2}e^{-f_i}, \tag{15}$$

and

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{f_i}\frac{\partial^2}{\partial x_i^2}e^{f_i} - \eta\right). \tag{16}$$

If f is the vector field of a potential function ϕ, we have a simple expression

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} e^{\phi}\frac{\partial^2}{\partial x_i^2}e^{-\phi} - \eta\right). \tag{17}$$

By defining $\xi = e^{-\phi}\sigma$, Equation (2) can be reformulated as

$$d\xi(t,x) = \frac{1}{2}(\Delta - \eta)\xi(t,x)dt + \sum_{i=1}^{m} L_i\xi(t,x)dy_i(t),\ \xi(0,x) = e^{-\phi}\sigma_0, \tag{18}$$

where Δ denotes the Laplacian operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. The connection between this representation and the gauge transformation was pointed out in [Mi 2]. This idea is also related to the concept of equivalence of parabolic equations, as discussed in [Ba]. If η is a quadratic polynomial in x, then it was usually thought that the semigroup generated by the differential operator $\Delta - \eta$ is well known as can be used to write down an explicit solution to the equation when h_i's are linear in x. However, to our knowledge, no explicit solution has been written down. Notable exceptions are references [Be] and [T-W-Y], both of which assume the maximal-rank condition of the estimation algebra. Using the Rozovsky's transformation.

$$u(t,x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\xi(t,x),$$

we can reduce (18) to the following time varying partial differential equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= \widetilde{L}_0 u(t,x) + \sum_{i=1}^{m} y_i(t)[\widetilde{L}_0, L_i]u(t,x) + \frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)\left[[\widetilde{L}_0, L_i], L_j\right]u(t,x) \\ u(0,x) &= e^{-\phi}\sigma_0, \end{cases} \tag{19}$$

where

$$\widetilde{L}_0 = \frac{1}{2}\left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \eta\right). \tag{20}$$

It is easy to see that (19) is equivalent to the following equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(t,x) + \sum_{j=1}^{n}\sum_{i=1}^{m} y_i(t)\frac{\partial h_i}{\partial x_j}(x)\frac{\partial u}{\partial x_j}(t,x) \\ & \quad + \left(-\frac{1}{2}\eta(x) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t)\frac{\partial^2 h_i}{\partial x_j^2} \right. \\ & \quad \left. +\frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{n} y_i(t)y_j(t)\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)\right)u(t,x) \\ u(0,x) &= e^{-\phi}\sigma_0. \end{cases} \tag{21}$$

If η is a quadratic in x and h_i's are linear in x, then the estimation algebra generated by $\widetilde{L}_0$, $L_1, \ldots, L_m$ is finite dimensional. If in addition the matrix H, which is defined by $h(x) = Hx$, is of maximal rank and $m \geq n$, then the estimation algebra has a basis $\widetilde{L}_0$, $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, $x_1, \ldots, x_n$, 1. Even in this case, using the Wei-Norman approach the procedure to find the solution of (21) is more complicated than the procedure in Theorem 2 below because not only must one solve a finite system of ordinary differential equations and a Kolmogorov equation, but also one has to integrate n partial differential equations corresponding to the operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. More important, if H is not of maximal rank, then the basis of the estimation algebra is not explicitly known (although it can be computed). As a result, one cannot write down recursive algorithms explicitly. Haussmann and Pardoux [Ha-Pa] considered a more general exact filtering problem than ours. However, even restricted to the classical Benés problem, the number of sufficient statistics in order to compute the conditional probability density is a polynomial of degree two in n in their method. The novelty of Theorem 2 is that our finite system of ordinary differential equations is explicitly written down and our algorithms apply uniformly for any η, h_i's, and any initial condition σ_0. Moreover, we need only n sufficient statistics in order to compute the conditional probability density of the exact filtering with arbitrary initial condition.

Theorem 2. Consider the exact filtering system (1) with

$$h_i(x) = \sum_{j=1}^{n} c_{ij}x_j + c_i, \ 1 \leq i \leq m, \tag{22}$$

where c_{ij} and c_i are constants;

$$f_i(x) = \frac{\partial F}{\partial x_i}(x), \ 1 \leq i \leq n, \tag{23}$$

where F is a C^∞ function; and

$$\Delta F(x) + \left|\nabla F(x)\right|^2(x) + \sum_{i=1}^{m} h_i^2(x) = \sum_{i,j=1}^{n} e_{ij}x_ix_j + \sum_{i=1}^{n} e_ix_i + e. \tag{24}$$

Choose a C^∞ function $G(x)$ such that

$$\Delta G(x) + |\nabla G|^2(x) = \sum_{i,j=1}^{n} e_{ij}x_ix_j. \tag{25}$$

(In view of (24) and Theorem 12 of [D-T-W-Y], there exists a G satisfying (25).) Then the solution $u(t,x)$ for the Duncan-Mortensen-Zakai equation (3) is reduced to the solution $\widetilde{u}(t,x)$ for the Kolmogorov equation

$$\frac{\partial \widetilde{u}}{\partial t}(t,x) = \frac{1}{2}\Delta\widetilde{u}(t,x) - \sum_{i=1}^{n}\frac{\partial G}{\partial x_i}(x)\frac{\partial \widetilde{u}}{\partial x_i}(t,x) - \sum_{i=1}^{n}\frac{\partial^2 G}{\partial x_i^2}(x)\widetilde{u}(t,x) \tag{26}$$

where

$$\widetilde{u}(t,x) = e^{-F(x+b(t))+G(x)+\Sigma_{i=1}^n a_i(t)x_i + c(t)}u\big(t, x+b(t)\big) \tag{27}$$

and $a_i(t)$, $b_i(t)$, and $c(t)$ satisfy the following system of ODEs:

$$b_i'(t) - a_i(t) + \sum_{j=1}^{m} c_{ji}y_j(t) = 0 \qquad 1 \leq i \leq n \tag{28}$$

$$c'(t) - \frac{1}{2}\sum_{i=1}^{n} a_i^2(t) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ik}c_{jk}y_i(t)y_j(t) - \frac{1}{2}\sum_{i=1}^{n} e_ib_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} e_{ij}b_i(t)b_j(t) - \frac{1}{2}e = 0 \tag{29}$$

$$a_i'(t) - \frac{1}{2}\sum_{j=1}^{n}(e_{ij}+e_{ji}) - b_j(t) - \frac{1}{2}e_i = 0 \ 1 \leq i \leq n. \tag{30}$$

(III) Although the concept of estimation algebra has proven to be an invaluable tool in the study of nonlinear filtering problems [Ma], until recently very little was known about estimation algebras. Beginning in the late 1980s, however, the structure and classification of finite-dimensional exact estimation algebras were studied in detail [T-W-Y] [D-T-W-Y]. In [Wo], the concept of Ω was introduced, which is defined as the matrix whose (i,j) element is $\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, where f is the drift term of the state evolution equation. For the class of exact filtering systems, Ω is identically zero. More recently, Yau [Ya] has studied filtering systems such that all entries of Ω are constants. He was able to classify all finite-dimensional estimation algebras of maximal rank in such filtering systems. Chiou-Yau [Ch-Ya] and Chen-Leung-Yau [C-L-Y] have shown respectively that if the dimension of the state space is two or three, all entries of Ω are constants as long as the estimate algebra is of maximal rank and finite dimensional. Thus, finite-dimensional estimation algebra of maximal rank is completely classified if the dimension of state space is at most three. The novelty of their theorems is that there are no a priori assumptions on the drift term of the nonlinear filtering system.

Our approach for the complete classification of finite-dimensional estimation algebras of maximal rank consists of two steps. The first step is to prove that for such an estimation algebra, all the entries in the Ω-matrix are degree one polynomials. The second step is to prove that in fact all the entries in Ω are constants. Then we can apply the result of Yau [Ya] to give a complete classification of finite-dimensional estimation algebras of maximal rank. Recently we have completed the first step. The following is our theorem.

Theorem 3. If E is a finite-dimensional estimation algebra of maximal rank, then all the entries $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ of Ω are degree one polynomials. Let k be the maximal rank of quadratic forms in E. Then ω_{ij} are constants for $1 \leq i,\ j \leq k$; ω_{ij} are degree one polynomials in $x_1, \ldots, x_k$ for $1 \leq i \leq k$ or $1 \leq j \leq k$; and ω_{ij} are degree one polynomials in $x_{k+1}, \ldots, x_n$ for $k+1 \leq i,\ j \leq n$.

Let n be the dimension of the state space. In case $n = 3$, there are three unknowns: ω_{12}, ω_{13}, and ω_{23}. It is easy to see that they are all degree two polynomials in view of Ocone's theorem. In [Ya-Le], Leung and Yau showed that the coefficients of the quadratic parts of ω_{12}, ω_{13}, and ω_{23} have to satisfy 90 quadratic equations. It was also shown in that paper that this system of 90 quadratic equations has only a trivial solution. Later Chen, Leung, and Yau proved that Ω is a matrix of constants [C-L-Y]. The novelty of our main theorem is that it holds for arbitrary n. Thus, it is the fundamental step in the classification of finite-dimensional estimation algebras of maximal rank.

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C. List of all Publications and Technical Reports

1. Recent result on classification of finite dimensional maximal rank estimation algebras with state space dimension 3 (with Chi-Wah Leung), Proceedings of the 31st Conference on Decision and Control, Tucson, Arizona, Dec. (1992), 2247-2250.

2. Explicit fundamental solution to Kolmogorov equations (with S.T. Yau), Proceedings of the 31st Conference on Decision and Control, Tucson, Arizona, Dec. (1992), 1508-1511.

3. Finite dimensional estimation algebras of maximal rank with dimension of state space equal to 3 (with Jie Chen, Chi-Wah Leung), Tenth Army Conference on Applied Mathematics and Computing, (1993), 337-344.

4. Finite dimensional estimation algebras of maximal rank with dimension of state space equal to 4 (with Jie Chen and Chi-Wah Leung) European Control Conference, Groningen, The Netherland, June 28-July 1, (1993), 2125-2130.

5. Some remarks on wavelet transforms (with Tomasz Bielecki, Jie Chen, E. Bing Lin). IEEE Proceedings of the first Regional Conference on Aerospace Control Systems, May 25-27, (1993), 148-150.

6. Explicit construction of finite dimensonal nonlinear filters with state space dimension 2, (with Wen-Lin Chiou), Proceedings of the 32nd Conference on Decision and Control, San Antonio, Texas, Dec. (1993), 710-713.

7. Classification of low dimensional estimation algebras (with Jie Chen and Chi-Wah Leung). Proceedings of 32nd Conference on Decision and Control, San Antonio, Texas, Dec. (1993), 732-734.
8. Wavelet and Wavelet Stieltjes transforms (with T. Bielecki, J. chen and E. Lin) Proceedings of the 32nd Conference on Decision and Control, San Antonio, Texas, Dec. (1993), 3062-3063.
9. Finite dimensional filters with non-linear drifts I: A class of filters containing both Kalman filters and Benés filters, J. of Math. Systems, Estimation and Control., Vol. 4, (1994), pp. 181-203.
10. Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensional estimation algebra (with Wen-Lin Chiou), SIAM J. Control and Optimization, Vol. 32, No. 1 (1994) 297-310.
11. Explicit formal solution to generalized Kolmogorov equation (with Shing Tung Yau), Eleventh Army Conference on Applied Mathematics and Computing, (1994), 373-386.
12. Computing the exponential of matrices, (with Hon Wing Cheng) Proceedings of the American Control Conference, Baltimore, Maryland, June, (1994), 3543-3547.
13. New direct method for Kalman-Bucy filtering system with arbitrary initial condition, (with S.T. Yau) Proceedings of the 33rd IEEE Conference on Decision and Control, Lake Buena Vista, Florida, Dec. 14-16, (1994), 1221-1225.
14. Random Wavelet Transformation and its Properties SPIE Proceedings on Wavelet Applications in Signal and Image Processing II, Andrew Laine and Michael Unser, eds. Vol. 2303 July (1994), 345-353 (with Tomasz R. Bielecki and Jie Chen).
15. Finite dimensional filters with nonlinear drift IV: classification of finite dimensional maximal rank estimation algebra with dimension of state space equal to 3, (with Jie Chen, Chi-Wah Leung), 22 pp. in ms., (to appear) SIAM J. Control and Optimization.
16. The wavelet application to Kolmogorov equation, (with Zhigang Liang) Proceedings of International Conference on Control and Information, (1995), 271-276.
17. A report on explicit formulas for $\exp(tA)$, (with Hon-Wing Cheng) Proceedings of International Conference on Control and Information, (1995), 69-75.
18. Recent results on classification of 4-dimensional estimation algebras (with Amid Rasoulian) Proceedings of International Conference on Control and Information, (1995), 371-374.
19. Random wavelet transform, algebraic geometric coding and their applications in compression and de-noising of signals. (with Tomasz Bielecki, Man K. Kwong, Li M. Song) Proceedings of International Conference on Control and Information, (1995), 283-289.
20. Explicit construction of finite dimensional nonlinear filters with state space dimension 3, (with Jie Chen and Chi-Wah Leung) Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, Louisiana, Dec. 13-15, (1995), 4030-4034.

21. Construction of new finite dimensional nonlinear filters, (with Amid Rasoulian) Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, Louisiana, Dec. 13-15, (195), 4002-4005.
22. Explicit solution of a Kolmogorov equation (with S.T. Yau) 33 pp. in ms. (to appear) Applied Mathematics and Optimization, An International Journal.
23. Direct method without Riccati equation for Kalman-Bucy filtering system with arbitrary initial condition, 9 pp. in ms. (with Q. Hu), to appear IFAC, 1996.
24. Filtering system with finite dimensional estimation algebras (with Rui-Tao Dong and Wing-Shing Wong) 13 pp. in ms., submitted for publication.
25. Finite dimensional filters with nonlienar drift III: Duncan-Mortenson-Zakai equation with arbitrary initial condition for Kalman-Bucy filtering system and Benés filtering system (with Shing Tung Yau), 19 pp. in ms., submitted for publication.
26. Wavelet representations of general signals (with T. Bielecki, J. Chen and E. Lin) 16 pp. in ms., submitted for publicaiton.
27. More explicit formulas for the matrix exponential 25 pp. in ms. (with Hon Wing Cheng).
28. Finite dimensional filters with nonlinear drift VI: Linear structure of Ω, 19 pp. in ms. (with Jie Chen).
29. Finite dimensional filters with nonlinear drift VII: Mitter conjecture and structure of η, 25 pp. in ms. (submitted for publication) with Jie Chen.
30. Finite dimensional filters with nonlinear drift VIII: Classification of finite dimensional estimation algebra of maximal rank with state space dimension 4, 15 pp. in ms. (submitted for publication) with Jie Chen and Chi-Wah Leung.
31. Finite dimensional filters with non-linear drift IX: construction of finite dimensional estimation algebra of non-maximal rank (with Amid Rasoulion), 13 pp. in ms., submitted for publication.
32. Finite dimensional filter with nonlinear drift V: Solution to Kolmogorov equation arising from linear filtering with non-Gaussian initial condition (with Zhigang Liang and Shing-Tung Yau), 16 pp. in ms., submitted for publication.

D. List of all participating scientific personnel

Senior Personnel:

Professor Stephen S.T. Yau	University of Illinois at Chicago
Professor S.T. Yau	Harvard University
Professor W.-S. Wong	AT&T Bell Lab. and Chinese University of Hong Kong
Professor T. Bielecki	University of Illinois at Chicago
Professor R.T. Dong	Johns Hopkins University and University of California, Irvine

Junior Personnel:

Dr. Li Min Song	University of Illinois at Chicago
Dr. Lixing Jia	University of Illinois at Chicago
Dr. J. Chen	University of Illinois at Chicago
Professor C.W. Leung	National Central University

Graduate Students:

Wei Yu Xiong	University of Illinois at Chicago
Amid Rasoulian	University of Illinois at Chicago
Yue He	University of Illinois at Chicago
Hangiang Hu	University of Illinois at Chicago
Qinglong Zhang	University of Illinois at Chicago
Fang Wang	University of Illinois at Chicago
Xiaozhong Liu	University of Illinois at Chicago
Zhigang Liang	University of Illinois at Chicago
Hon-Wing Cheng	University of Illinois at Chicago
Qingwei Li	University of Illinois at Chicago

Degree Awarded:

Master Degree

1)Weiyu Xiong
2) Yue He
3) Hanqiang Hu
4) Fang Wang
5) Xiaozhong Liu

Ph.D. Degree

1) Lixing Jia
2) Jie Chen

Report of Inventions

The best algorithms for linear filtering and exact filtering problems with arbitrary initial conditions were found. These algorithms are of practical importance.

Curriculum Vitae

Stephen Shing-Toung Yau

Education

Ph.D	The state University of New York at Stony Brook	1976
M.A.	The State University of New York at Stony Brook	1974

Positions

Director, Laboratory of Control and Information	1993-
Managing Editor, Journal of Algebraic Geometry	1991-
Visiting Professor, University of Pisa,Italy	Spring 1990
Visiting Professor, Johns Hopkins University	1989 - 90
University Scholar, University of Illinois at Chicago	1987 -90
Visiting Professor, Institute Mittag-Leffler, Sweden	Winter 1987
Professor,University of Illinois at Chicago	1984-
Visiting Professor, Yale University	1984-85
Vistiting Associate Professor, University of Southern California	1983-84
Member, The Institute for Advanced Study	1981-82
Visiting Research Mathematician, Princeton University	Spring 1981
Associate Professor,University of Illinois at Chicago	1980-84
Alfred P.Sloan Research Fellow	1980-82
Benjamin Pierce Assistant Professor, Harvard University	1977-80
Member, The Institute for Advanced Study	1976-77

Prizes and Awards

Alfred P.Sloan Research Fellowship 1980-82
University Scholar,University of Illinois at Chicago 1987-90
C.M. Cha Fellow from Hong Kong Baptist University May-July, 1995
Biographical note in American Men And Women of Science
Biographical note in Who's Who in the World
Biographical note in Who's Who in Science and Engineering
Biographical note in Who's Who in the Midwest
Biographical note in Who's Who in American Education
Biographical note in Who's Who among Asian Americans

Membership

Senior Member of the Institute of Electrical and Electronic Engineers
Member of Society for Industrial and Applied Mathematics
Member of American Mathematical Society

Grants

Army Research Office	1989-98
National Science Foundation	1989-97
	1976-88
National Science Foundation Special Year Grant	1987-88
Research Board, University of Illinois at Chicago	1987, 1984
	1983, 1981
The Clark and the Topier Fund,Harvard University	1977-80

Honors

Open Invitation to spend 1984-86 as a Research Professor at Sonderforschungesbereigh "Theoretische Mathematic," University of Bonn, Germany

Invited one-hour address at the AMS Meeting, Worcester,Massachusetts, April 1985

Invited one-hour address at the Swedish Mathematical Society Meeting, Linkoping, January 1987

Invited to visit one week and give a colloquium lecture at Aarhus University,Denmark, February 1987

Invitation to spend a month at Fudan University, Shanghai, People's Republic of China,to give a series of lectures, November and December 1987

Invitation to spend one week at the Institute of Mathematics, Academia Sinica, Beijing, People's Republic of China to give several lectures, December 1987

Invitation to spend eight weeks at the University of Pisa, Italy to give a series of lectures, January - February 1990

Invitation to spend 9 days at Global Analysis Research Center,Seoul National University,Korea to give a series of lectures February 1992

Invitation to spend one month at Nanjing University, People's Republic of China to give a series of lectures,May 1993

Invitation to spend one month at National Taiwan University, Republic of China to give a series of lectures, May 1994

Ph.D Student Supervision

Yung Yu(1988), Craig Seeley(1988), Yi-Jing Xu(1990), Wen-Lin Chiou (1991), Chi-Wah Leung(1993), Tan Jiang (1993) , Li-Xing Jia (1994), Jie Chen (1994)

Selected Professional Activities

Managing Editor and Founder, Journal of Algebraic Geometry,1991

General Chairman, IEEE International Conference on Control and Information, The Chinese University of Hong Kong, June, 1995

Coorganizer, International Conference on Singularities and Complex Geometry, Beijing, China. June, 1994

Coorganizer, Wavelets and Large-Scale Image Processing, Chicago, Oct. 1994

Coorganizer, Wavelets and their applications in PDE, a minisymposium during SIAM Annual Meeting, San Diego, CA, July 25-29, 1994

Organizer, Minisymposium on wavelets at the Third SIAM Conference on Linear Algebra in Signals, Systems and Control, University of Washington, Seattle, August,1993

Organizer, Wavelets and its application at IEEE Regional Conference on Aerospace Control Systems, Rockwell Science Center, Thousand, CA, May,1993

Co-organizer, Emerging Computational Advances in Systems and Control 31st IEEE Conference on Decision and Control, Tucson, Arizona, December 1992

Organizer, Midwest Algebraic Geometry Conference at the University of Illinois at Chicago, March 1988

Organizer, National Science Foundation Special Year Algebraic Cycles Conference at the University of Illinois at Chicago, March 1988

Special Session Chairman on Singularities and Complex Geometry, Worcester, Massachusetts, April 1985

Special Session Chairman on Differential Geometry of Submanifolds, Worcester, Massachusetts, April 1985

Stephen Sing-Toung Yau

List of Publications

1. Two theorems on higher dimensional singularities, Math. Ann. 231 (1977), 55 -59.
2. On almost minimally elliptic singularities, Bull. Amer. Math. Soc. 83 (1977), 362-364.
3. The signature of smoothing of higher dimensional singularities, Bull. Amer. Math. Soc. 83 (1977), 1313-1315.
4. Normal singularities of surfaces, Proceedings of Symposia in Pure Mathematics 32 (1978), 195-198.
5. The signature of Milnor Fibers and duality theorems for strongly pseudoconvex manifolds, Invent. Math. 46 (1978), 81-97.
6. Hypersurface weighted dual graphs of normal singularities of surfaces, Amer. J. Math. 101 (1979), 761-812.
7. Gorenstein singularities with geometric genus equal to two, Amer. J. Math. 101 (1979), 813-854.
8. On strongly elliptic singularities, Amer. J. Math. 101 (1979), 855-884.
9. Normal two-dimensional elliptic singularities, Trans. Amer. Math. Soc. 254 (1979), 117-134.
10. On maximally elliptic singularities, Trans. Amer. Math. Soc. 257 (1980), 269-329.
11. Index theory for the boundaries of complex analytic varieties, Proc. Nat. Acad. U.S.A. 77 (1980), 1248-1249.
12. Deformations and equitopological deformations of strongly pseudoconvex manifolds, Nagoya Math. J. 82 (1981), 113-192.
13. Kohn-Rossi cohomology and its application to the complex Plateau problem 1, Ann. of Math. 113 (1981) 67-110.
14. Sheaf cohomology on 1-convex manifolds, Recent Developments in Several Complex Variables, Ann. of Math. Study. 100 (1981), 429-452.
15. Existence of L^2-integrable holomorphic forms and lower estimates of T^1V, Duke Math. J. 48 (1981), 537-547.
16. Criterion for biholomorphic equivalence of isolated hypersurface singularities, (with John Mather), Proc. Nat. Acad. Sci., U.S.A. 78 (1981), 5946-5947.
17. Milnor number and classification of isolated singularities of holomorphic maps, (with Bruce Bennett), Lecture Notes in Mathematics 949, Springer-Verlag (1982), 1-34.
18. $s^{(n-1)}$invariant for isolated n-dimensional singularities and its application to moduli problems, Amer. J. Math. 104 (1982), 829-841.
19. Classification of isolated hypersurface singularities by their moduli algebras, (with John Mather), Invent. Math. 69 (1982), 243-251.
20. Various numerical invariants for isolated singularities, Amer. J. Math. 104 (1982), 1063-1100.
21. On irregularity and geometric genus of isolated singularities, Proc. Symp. Pure Math. 40, Part 2 (1983), 653-662.
22. Milnor algebras and equivalent relations among holomorphic functions, Bull. Amer. Math. Soc. 9 (1983), 235-239.
23. Continuous family of finite dimensional representations of a solvable Lie algebra arising from singularities, Proc. Nat. Acad. Sci. U.S.A. 80 (1983), 7694-7696.
24. Criteria for right-left equivalence and right equivalence of holomorphic functions with isolated critical points, Complex Analysis Several Complex Variables, Proc. Symp. Pure Math. 41 (1984), 291-297.
25. Riemann-Roch theorem for strongly pseudoconvex manifolds of dimension three, (with Paul Yang), Several Complex Variables, Proc. of the 1981 Hangzhous Conf., Birkhauser, Boston, (1984), 257-267.
26. An estimate of the gap of the first two eigenvalues in the Schrodinger operator,(with I.M. Singer, Bun Wong and Shing-Tung Yau), Ann. Scuola Norm. Sup., Pisa, Classe di Scienze, Serie IV, Vol. XII, N.2 (1985), 319-333.
27. Solvable Lie algebras and generalized Cartan Matrix arising from isolated singularities, Math. Z. 191 (1986), 489-506.
28. Singularities defined by sl(2,C)invariant polynomials and solvability of Lie algebras arising from isolated singularities, Amer. J. Math. 108 (1986),1215-1240.
29. Lie algebras and their representations arising from isolated singularities: Computer method in calculating the Lie algebra and their cohomology, (with Max Benson), Adv. Stud. Pure Math. 8, Complex Analytic Singularities (1986), 3-58.
30. A necessary and sufficient condition for a local commutative algebra to be a moduli algebra:weighted homogeneous case, Adv. Stud.Pure Math. 8, Complex Analytic Singularities (1986), 687-697.

31. Some surfaces covered by the ball and a problem in finite groups, (with G.D.Mostow), Lecture Notes in Math.Vol.1271, Springer-Verlag, Proc. of a Symposium in Honor of T.A, Springer, (1987), 201-228.
32. Holomorphic symmetry, (with Blaine Lawson), Ann. Sci. I'Ecole Norm. Sup. 4e series, t. 20 (1987), 557-577
33. Classification of Jacobian ideals invariant by sl(2,C) actions, Mem. Amer. Math. Soc. 72 (1988), 1-180.
34. Topological types and multiplicities of isolated quasi-homogeneous surface singularities, Bull. Amer. Soc. 19 (1988), 447-454.
35. The inequality $\mu \geq 12p_g - 4$ for weakly elliptic hypersurface singularities, (with Yi-Jing Xu), Contemp. Math. 90 (1989), 317-344.
36. Topological types of isolated hypersurface singularities, Contemp. Math.101 (1989), 303-321.
37. Classification of topological types of isolated quasi-homogeneous two-dimensional hypersurface singularities, (with Yi-Jing Xu), Mancripta Math. 64 (1989), 445-469.
38. Multiplicity of isolated two-dimensional hypersurface singularities:Zariski problem, Amer. J. Math. 111 (1989), 763-767.
39. Recent results on finite dimensional exact estimation algebra, (with L.F.Tam and W.S.Wong), Proceedings of the 28th Conf. on Decision and Control at Tampa, Florida, Dec. (1989), 2574-2575.
40. On a necessary and sufficient condition for finite dimensionality of estimation algebras, (with L.F. Tam and W.S, Wong), SIAM J. Control Optim. 28 (1990), 173-185.
41. Variations of complex structures and variation of Lie algebras, (with Craig Seeley), Invent. Math. 99 (1990), 545-565.
42. Equivalence between isolated hypersurface singularities, (with Max Benson), Math. Ann. 287 (1990), 107-134.
43. An obstruction for smoothing of Gorenstein surface singularities, (with A.Libgober), Comment. Math. Helv. 65 (1990), 413-433.
44. Recent results on nonlinear filtering: New class of finite dimensional filters, Proceedings of the 29th Conf. on Decision and Control at Honolulu, Hawaii, Dec. (1990), 231-233.
45. A remark on moduli of complex hypersurface, Amer. J. Math., 113 (1990), 287-292.
46. Classification of finite dimensional estimation algebras and Schrodinger equation, (with R.T. Dong, L.F. Tam and W.S. Wong), SIAM J. Control Optim., Vol 29, No.4 (1991), 886-877.
47. Obstructions to embedding of real compact (2n-1)-dimensional CR-manifold in C^{n+1}, (with H.S. Luk), Proceedings of Symposia in Pure Mathematics Vol. 52 (1991), Part 3, 261-276.
48. Regularity to Harvey-Lawson's solution to complex plateau problem, J. Differential Geom., 34 (1991) 425-429.
49. Solvability of the Lie algebras arising from singularities and non-isolatedness of the singularities defined by the invariant polynomials of sl(2,C), Amer. J. Math. 113 (1991), 773-778.
50. Algebraic methods in the study of simple-elliptic singularities, (with Craig Seeley), US-USSR Algebraic Geometry Symposium, Springer-Verlag, Lecture Notes in Mathematics 1479 (1991), 216-237.
51. Recent results on classification of finite dimensional estimation algebras: Dimension of state space$\leq$2, (with Wen-Lin Chiou), Proceedings of the 30th Conf. on Decision and Control, Brighton, England, Dec. 11-13 (1991), 2758-2760.
52. Topological types of seven classes of isolated singularities with C^* -action, (with Yi-Jing Xu), Rocky Mountain J. Math.,Vol 22, (1992) 1147-1215.
53. Classification of gradient space as sl(2,C)-module I (with J. Sampson and Yung Yu), Amer. J. Math.114 (1992), 1147-1161.
54. Sharp estimate of number of integral points in tetrahedron, (with Yi-Jing Xu), Journal fur die reine und angewandte Mathematik 423 (1992), 199-219.
55. Recent result on classification of finite dimensional maximal rank estimation algebras with state space dimension 3 (with Chi-Wah Leung), Proceedings of the 31st Conference on Decision and Control, Tucson, Arizona, Dec. (1992), 2247-2250.
56. Explicit fundamental solution to Kolmogorov equation(with S.T.Yau), Proceedings of the 31st Conference on Decision and Control, Tucson, Arizona, Dec. (1992), 1508-1511.
57. Classification of finite dimensional filters from Lie algebraic point of view, Transaction of the Ninth Army Conference on Applied Mathematics and Computing, (1992), 459-466.

58. Complex Hypersurface Singularities with Application in Complex Geometry, Algebraic Geometry and Lie Algebra, Lecture Notes Series Number 5, 1992, Research Institute of Mathematics, Global Analysis Research Center, Seoul National University, Seoul Korea.
59. On the smoothness of solutions to the complex plateau problem: An addendum to the paper"Holomorphic symmetries," (with Blain Lawson), 6 pp in ms., (preprint).
60. Durfee conjecture and coordinate free characterization of homogeneous singularities, (with Yi-Jing Xu), Journal of Differential Geometry 37 (1993), 375-396.
61. Gorenstein quotient singularities in dimension three,(with Yung Yu), Mem. Amer. Math. Soc. Vol. 105 (1993), 1-88.
62. Topological and differentiable structures of the complement of an arrangement of hyperplanes, (with Tan Jiang) Proceedings of Symposia in Pure Mathematics, Vol.54 (1993), part 2, 337-357.
63. Topological invariance of intersection lattices of arrangement in CP^2 (with Tan Jiang), Bulletin A.M.S. Vol.-39, No.1 (1993) 88-93.
64. Cohomology and Splitting Criterion for holomorphic vector bundles on CP^n (with Hing-Sun Luk) Math. Nachrichten 161 (1993), 223-238.
65. Finite dimensional estimation algebras of maximal rank with dimension of state space equal to 3 (with Jie Chen, Chi-Wah Leung), Tenth Army Conference on Applied Mathematics and Computing, (1993), 337-344.
66. Finite dimensional estimation algebras of maximal rank with dimension of state space equal to 4 (with Jie Chen and Chi -Wah Leung) European Control Conference, Groningen, The Netherland, June 28-July 1, (1993), 2126-2130.
67. Some remarks on wavelet transforms (with Tomasz Bielecki, Jie Chen, E. Bing Lin). IEEE Proceeding of the first Regional Conference on Aerospace Control Systems, May 25-27, (1993), 148-150.
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70. Wavelet and Wavelet Stieltjes transforms (with T. Bielecki, J. Chen and E. Lin) Proceedings of the 32nd Conference on Decision and Control, San Antonio, Texas, Dec. (1993), 3062-3063.
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72. Finite dimensional filters with nonlinear drift II: Brockett's problem on classification of finite dimensional estimation algebra (with Wen-Lin Chiou), SIAM J. Control and Optimization, vol.32, No.1 (1994) 297-310.
73. Diffeomorphic type of the complement of arrangement of hyperplanes, (with Tan Jiang) Composito Math, 92, (1994) 133-155.
74. Explicit formal solution to generalized Kolmogorov equation (with Shing Tung Yau), Eleventh Army Conference on Applied Mathematics and Computing, (1994), 373-386
75. Algebraic classification and obstructions of embedding of compact 3 dimensional CR manifolds in C^3. (with Hing Sun Luk and Yung Yu) , Math. Nachrichten 170 (1994), 183-200.
76. Computing the exponential of matrices, (with Hon Wing Cheng) Proceedings of the American Control Conference, Baltimore, Maryland. June, (1994), 3543-3547.
77. New direct mehtod for Kalman-Bucy filtering system with arbitrary initial condition, (with S.T. Yau) Proceedings of the 33rd IEEE conference on Decision and Control, Lake Buena Vista, Florida, Dec. 14-16,(1994), 1221-1225.
78. Random Wavelet Transformation and its Properties, SPIE Proceedings on Wavelet Applications in Signal and Image Processing II, Andrew Laine and Michael Unser, eds,Vol. 2303 July, (1994), 345-353 (with Tomasz R.Bielecki and Jie Chen).
79. Non-existence of negatively graded derivation of weighted homogeneous moduli algebra (with Hao Chen and Yi-Jing Xu), Journal of Algebra, 172, (1995), 243-254.
80. Finite dimensional filters with nonlinear drift IV: classification of finite dimensional maximal rank estimation algebra with dimension of state space equal to 3, (with Jie Chen, Chi-Wah Leung). 22 pp. in ms., (to appear) SIAM J. Control and Optimization.

81. The wavelet application to Kolmogorov equation, (with Zhigang Liang) Proceeding of International Conference on Control and Information, (1995), 271-276.
82. A report on explicit formulas for exp(tA), (with Hon-wing Cheng) Proceeding of International Conference on Control and Information, (1995), 69-75.
83. Recent results on classification of 4-dimensional estimation algebras (with Amid Rasoulian) Proceeding of International Conference on Control and Information, (1995), 371-374.
84. Random wavelet transfrom, algebraic geometric coding and their applications in compression and de-noising of signals. (with Tomasz Bielecki; Man K. Kwong; Li M. Song) Proceeding of International Conference on Control and Information, (1995), 283-289.
85. Explicit construction of finite dimensional nonlinear filters with state space dimension 3, (with Jie Chen and Chi-Wah Leung) Proceedings of the 34th IEEE conference on Decision and Control,New Orleans, Louisiana, Dec. 13-15, (1995), 4030-4034.
86. Construction of new finite dimensional nonlinear filters, (with Amid Rasoulian) Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, Louisiana, Dec. 13-15, (1995), 4002-4005.
87. Explicit solution of a Kolmogorov equation (with S.T.Yau) 33 pp in ms. (to appear) Applied Mathematics and Optimization, An International Journal .
88. Invariant Kohn-Rossi cohomology and obstruction to embedding of compact real (2n-1)-dimensional CR manifolds in C^n, 29 pp. in ms., (with H.-S.Luk)(to appear) Journal of the Mathematical Society of Japan (1996).
89. Complete Algebraic CR Invariants of codimension 3 strongly pseudoconvex CR manifolds (with Hing Sun Luk), 10 pp. in ms. (to appear) Proceeding of US-China Conference on Complex Geometry and Singularities, International Press.
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