The Gamma Transform: A Local Time-Frequency Analysis Method

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In this paper we introduce the gamma transform, a local time-frequency analysis method which applies to causal signals. The gamma transform resolves the identity, has good time-frequency resolution, and adapts resolution windows according to the time delay. The discretized version of the gamma transform produces gamma frame and, in fact, a tight gamma frame for $L^2(0,T)$ provided the frequency sampling interval w_0 is less than or equal to 2pi/T. We demonstrate the gamma frame decomposition and reconstruction with several numerical experiments using tight gamma frames and dual gamma frames. Finally, we give the four varieties of the gamma transform corresponding roughly to the four Fourier transforms.

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Abstract

In this paper we introduce the gamma transform, a local time-frequency analysis method which applies to causal signals. The gamma transform resolves the identity, has good time-frequency resolution, and adapts resolution windows according to the time delay (while the wavelet transform adapts according to the frequency). The discretized version of the gamma transform produces gamma frame, and in fact a tight gamma frame for $L^2(0,T)$ provided the frequency sampling interval ω_0 is $\leq 2\pi/T$. We demonstrate the gamma frame decomposition and reconstruction with several numerical experiments, using tight gamma frames and dual gamma frames. Finally, we give the four varieties of the gamma transform corresponding, roughly to the four Fourier transforms.

1 Introduction

Time-frequency analysis usually involves resolving an analog signal into time-frequency coefficients associated with a set of frame (or even basis) vectors via some transform, and reconstructing the signal from the time-frequency coefficients.

The Fourier transform is an extreme case of time-frequency analysis. It involves integrating over the whole time-domain, even including the future, and producing pure frequency information (independent of time). Therefore, in many applications such as analysis of nonstationary signals and real-time signal processing, the Fourier transform is quite inadequate.

Owing to the deficiency of Fourier transform, we look for transforms which are localized, and therefore able to resolve the frequency content of the signal locally in time.

A natural way to mend the Fourier transform's deficiency is to first window the signal, and then take its Fourier transform. This leads to the so called "windowed Fourier transform"(WFT) which is a standard technique for time-frequency localization. Actually, this is exactly what D. Gabor did in 1946 [4]. He showed that Gaussian windows optimize time-frequency resolution.

A drawback of the WFT is that resolution cells are rigid. This makes it inadequate for analyzing signals with very high and low frequencies [1]. This drawback is due to the use of a single window function with shifting and modulation in the WFT mechanism. Obviously one way to get rid of this drawback is to introduce variable window functions into the transform mechanism "properly". This approach develops into the variable-windowed Fourier transform the author proposes in [5].

Another well known method of time-frequency analysis is wavelet analysis. The wavelet transform has orthonormal bases and fast algorithms. This makes it attractive in digital computing. However, most orthonormal mother wavelets are sort of "pure mathematical animals" so that it is usually impossible to make direct links between the time-frequency coefficients obtained via the transform and physical phenomena.

The WFT and the wavelet transform each employ a single window function (or a mother wavelet). By shifting and modulating (or shifting and dilating) the window function, the time-frequency plane is "covered" and thus the transforms are defined. A natural question is why use a single window? What happens if one uses variable windows instead?

Actually variable window functions (or mother wavelets) are not new in wavelet transforms but they are hardly considered in the WFT. A. Grossmann, R. Kronland-Martinet, and Morlet found that employing variable mother wavelets (or multiple voices in wavelet terminology) improves the tightness of frames [2]. Is this also possible in the WFT? or what is possible in the WFT if variable windows are used?

The idea is to employ a family of window functions and let them "properly" cover the time-frequency plane to produce a variable-windowed Fourier transform (VWFT).

Unique properties are expected depending on the family of window functions chosen. The gamma family, namely

$$\gamma(t;n,\alpha) = \frac{\alpha}{\Gamma(n)}(\alpha t)^{n-1}e^{-\alpha t}u(t)$$

$u(t)$ is the unit step function, $\quad n \in \mathbf{N}, \quad \alpha > 0$

is the family of window functions under investigation in this reasearch. The gamma family is chosen because of its matching in pulse shape with dispersive phenomenon and its elegant analytical properties.

2 Continuous Gamma Transform

The continuous gamma transform is defined as follows.

Definition 1
Let $f(t) \in L^2(\mathbf{R}^+)$ be a causal signal and $g_n(t) = \gamma^{1/2}(t;n,\alpha)$. Then the gamma transform of $f(t)$ written as $(\Gamma_\alpha f)(\omega, n)$ is

$$(\Gamma_\alpha f)(\omega,n) = \left\langle f(t), g_n(t)e^{jwt}\right\rangle \tag{1}$$

$$= \int_0^\infty dt f(t)\gamma^{1/2}(t;n,\alpha)e^{-jwt} \tag{2}$$

where $\omega \in \mathbf{R}$ and $n \in \mathbf{N}$. □

We proved that the inverse continuous gamma transform is

$$f(t) = \frac{1}{2\pi\alpha}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} d\omega(\Gamma_\alpha f)(\omega,n)\gamma^{1/2}(t;n,\alpha)e^{jwt} \tag{3}$$

3 Properties of the Continuous Gamma Transform

The continuous gamma transform we just defined turns out to have very good time-frequency localization. One way to see this is to view this transform as a variable windowed Fourier transform with $\{g_n(t)\}$ the family of window functions and $\{\hat{g}_n(\omega)\}$ the corresponding Fourier transform. Each window function g_n has centers t^* and ω^* and radii $\triangle_{g_n}$ and $\triangle_{\hat{g}_n}$ in the time domain and the frequency domain respectively [5]

$$t^* = nt_0; \qquad t_0 = 1/\alpha \tag{4}$$

$$\omega^* = 0 \tag{5}$$

$$\triangle_{g_n} = \sqrt{n}t_0 \tag{6}$$

$$\triangle_{\hat{g}_n} = \alpha/2\sqrt{n-2} \tag{7}$$

Therefore the resolution cell defined by the window function g_n centered at (nt_0, ω) is

$$\left[\frac{n}{\alpha} - \frac{\sqrt{n}}{\alpha}, \frac{n}{\alpha} + \frac{\sqrt{n}}{\alpha}\right] \times \left[\omega - \frac{\alpha}{2\sqrt{n-2}}, \omega + \frac{\alpha}{2\sqrt{n-2}}\right]$$

The size of a resolution cell is thus

$$\begin{aligned}(2\triangle_{g_n})(2\triangle_{\hat{g}_n}) &= 2\frac{\sqrt{n}}{\sqrt{n-2}}, \quad n > 2 \\ &\longrightarrow 2, \quad \text{as} \quad n \longrightarrow \infty\end{aligned}$$

Note that this size varies as a function of n. When n increases it approaches 2 which is the lower bound of the Heisenberg Uncertainty Principle.

The resolution cells are not rigid. They narrow and widen according to small and large n, or small and large distance from the time origin. This is exactly what one would hope to have to analyze transient signals which tend to disperse naturally when propagating down the time- or space-axis. Furthermore, the ratio of the "center-time" to the "time width" is

$$\frac{n/\alpha}{2\sqrt{n}/\alpha} = \frac{\sqrt{n}}{2},$$

which depends on the n associated with window function $g_n(t)$, but not on the actual center-time nt_0. This seems to be the counterpart to the *constant-Q* property in wavelet analysis. A plot of the tiling for the continuous gamma transform is shown in Figure 1.

4 Discretized Gamma Transform and Tight Gamma Frame

The discretized gamma transform is defined as follows.

Definition 2
Let $f(t) \in L^2(\mathbf{R}^+)$ be a causal signal and let $g_n(t) = \gamma^{1/2}(t;n,\alpha)$. Then the discretized gamma transform of $f(t)$ written as $(\Gamma_\alpha f)(m,n)$ is

$$(\Gamma_\alpha f)(m,n) = \left\langle f(t), g_n(t)e^{jmw_0t}\right\rangle \tag{8}$$

$$= \int_0^\infty dt f(t)\gamma^{1/2}(t;n,\alpha)e^{-jmw_0t} \tag{9}$$

where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$. □

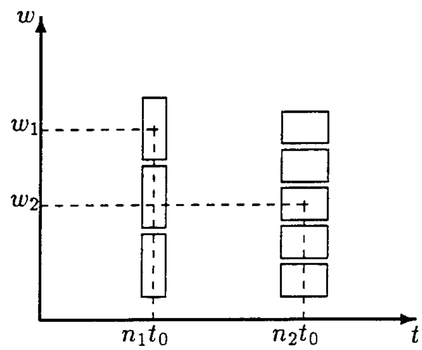

Figure 1: The tiling of the GT

The discretized gamma transform is invertible only when the $\{\phi_{mn} = g_n(t)e^{jmw_0t}\}$ constitute a frame, i. e. when there exist frame bounds $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{mn} |\langle f, \phi_{mn}\rangle|^2 \leq B||f||^2 \qquad (10)$$

for all f in the underlying Hilbert space [2, 6]. Then every function f in the underlying Hilbert space can be decomposed and reconstructed from the inverse transform

$$f = \sum_{mn} \langle f, \phi_{mn}\rangle \widetilde{\phi_{mn}} = \sum_{mn} \left\langle f, \widetilde{\phi_{mn}} \right\rangle \phi_{mn}. \qquad (11)$$

where $\widetilde{\phi_{mn}}$ is the dual frame of ϕ_{mn}.

Let $\phi_{mn}\chi[0, T]$ denote the function ϕ_{mn} restricted to $[0, T]$. These functions are a frame for $L^2(0, T)$ and a tight frame if $\omega_0 < 2\pi/T$ [5]. In this case equation (11) becomes

$$f = \frac{1}{A}\sum_{mn} \langle f, \phi_{mn}\rangle \phi_{mn} \qquad (12)$$

because the dual frame is just the frame itself scaled by the inversion of the frame bound so that the decomposition-reconstruction formula for tight frames is identical to that for orthonormal bases essentially. Daubechies called tight frame expansions "painless nonorthonormal expansions" or "quasiorthonormal expansions" and assigns great value to them [3].

5 Numerical Experiments

In this section we do some numerical experiments using gamma frames to decompose and reconstruct signals.

5.1 Tight Gamma Frame

For simplicity we only demonstrate the tight gamma frame decomposition and reconstruction with two cases: (1) $f(t) = \gamma(t; 3, 1)$ and (2) $f(t) = sin(t)$. We set ω_0 to be the "fundamental frequency" $2\pi/T$ so that the gamma frame is tight in $L^2(0, T)$ and therefore the experiment itself becomes simply the realization of equation (12) on a digital computer. For each case we present the experiment results with two plots. One is the plot of the original signal and its gamma reconstruction. The other is a plot of gamma coefficients in gray scales over the time-frequency plane, showing how the energy of the signal is distributed over the time-frequency plane in the gamma frame representation. The reconstruction is intentionally illustrated beyond $t = T$ to make the point that, with the choice of $\omega_0 = 2\pi/T$, the gamma frame is only tight on $L^2(0, T)$.

5.2 Dual Gamma Frame

If one does not insist tight gamma frames then the "frequency sampling interval " ω_0 needs not to be small than the fundamental frequency. It can be any finite positive real number. In this case, however, one needs dual gamma frames to do the decomposition and reconstruction. A theory and algorithm for computing dual frames can be found in [2]. We redo case (1) with gamma frame decomposition and dual gamma frame reconstruction. One can see "the tail" in Figure 2 does not appear in Figure 6 because the dual frame has been computed for the whole interval.

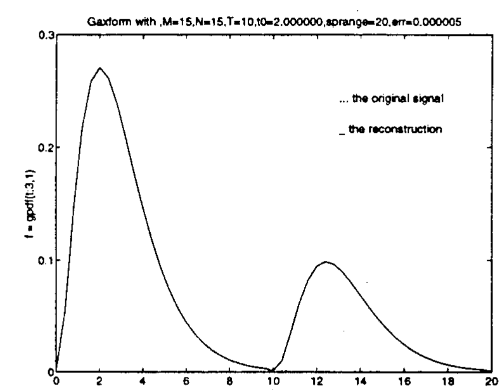

Figure 2: Tight frame reconstruction of $\gamma(t; 3, 1)$ with $m = -15 : 15; n = 1 : 15; t_0 = 2; T = 10; \omega_0 = 2\pi/T$

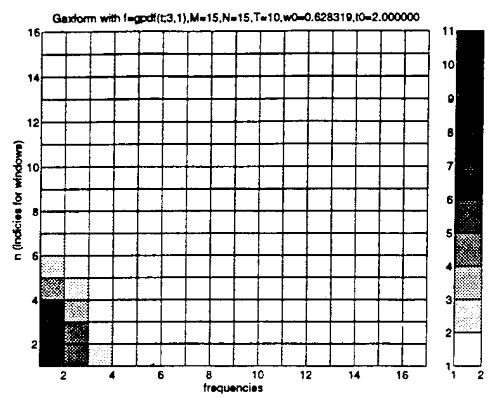

Figure 3: The gamma coefficients in gray scales of $\gamma(t;3,1)$ with $m = 0 : 15; n = 1 : 15;\ t_0 = 2; T = 10; \omega_0 = 2\pi/T$

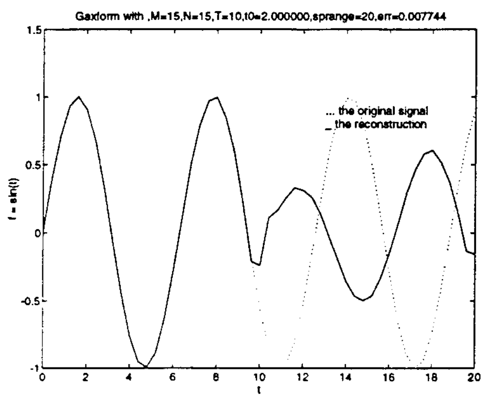

Figure 4: Tight frame reconstruction of $sin(t)$ with $m = -15 : 15; n = 1 : 15;\ t_0 = 2; T = 10; \omega_0 = 2\pi/T$

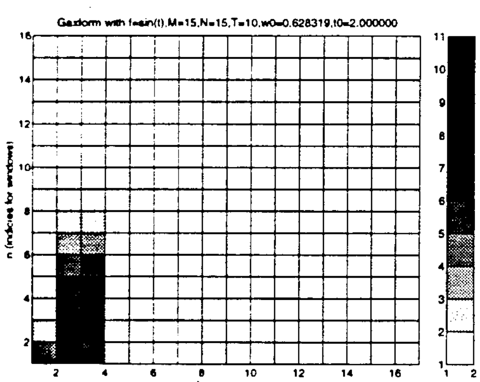

Figure 5: The gamma coefficients in gray scales of $sin(t)$ with $m = 0 : 15; n = 1 : 15;\ t_0 = 2; T = 10; \omega_0 = 2\pi/T$

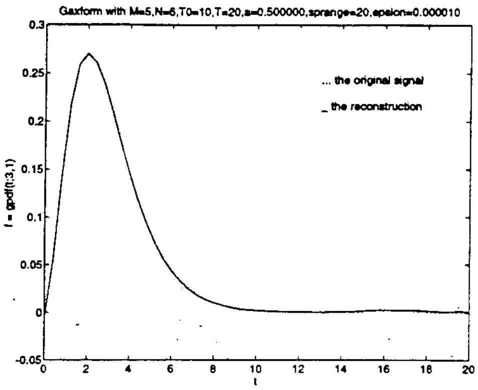

Figure 6: Dual frame reconstruction of $\gamma(t;3,1)$ with $m = -5 : 5; n = 1 : 6;\ t_0 = 2; T_0 = 10; \omega_0 = 2\pi/T_0; T = 20$

6 Four Varieties of Gamma Transforms

The gamma transform comes in four varieties, just as the Fourier transform does. Adopting the convention of Fourier analysis, we call

- gamma transforms with continuous parameters, taking continuous signals as input, *continuous gamma transforms* (CGTs).
- gamma transforms with discrete parameters, taking continuous signals as input, *gamma series* (GSs).
- gamma transforms with continuous parameters, taking discrete signals as input, *discrete-time gamma transforms* (DTGTs).
- gamma transforms with discrete parameters, taking discrete signals as input, *discrete gamma transforms* (DGTs).

In addition to the CGT, which has been defined in **Section** 2, we define the other three gamma transforms. We give their inverse transforms in three propositions, without proofs.

Definition 3

Let $f(t) \in L^2(0,T)$ supported functions and let $\omega_0 = \frac{2\pi}{T}$. Then the gamma series of $f(t)$ written as $(\Gamma_\alpha f)(m,n)$ is

$$(\Gamma_\alpha f)(m,n) \tag{13}$$

$$= \langle f(t), g_n(t) e^{jm\omega_0 t} \rangle \tag{14}$$

$$= \int_0^T dt f(t) g_n(t) e^{-jm\omega_0 t}$$

where

$$m \in \mathbf{Z}, \quad n \in \mathbf{N}, \quad \alpha > 0$$

□

Proposition 1

The inverse GS is

$$f(t) = \frac{1}{\alpha T} \sum_{mn} (\Gamma_\alpha f)(m,n) g_n(t) e^{jm\omega_0 t}$$

where

$$t \in [0, T]$$

□

Definition 4

Let $f(k) \in l^2(\mathbf{Z}^+)$ be a causal sequence and $g_n(k) = g_n(kt_d)$ for some $t_d > 0$. Then the discrete-time gamma transform of $f(k)$ written as $(\Gamma_\alpha f)(e^{j\theta}, n)$ is

$$(\Gamma_\alpha f)(e^{j\theta}, n) = \left\langle f(k), g_n(k) e^{jk\omega t_d} \right\rangle \quad (15)$$

$$= \sum_{k=0}^{\infty} f(k) \gamma^{1/2}(k; n, \alpha) e^{-jk\theta} \quad (16)$$

where

$$\theta = \omega t_d \in \mathbf{R}, \quad n \in \mathbf{N}, \quad \alpha > 0$$

□

Proposition 2

The inverse DTGT is

$$f(k) = \frac{1}{2\pi\alpha} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\theta (\Gamma_\alpha f)(e^{j\theta}, n) g_n(k) e^{jk\theta}$$

where

$$k \in \mathbf{Z}^+$$

□

Definition 5

Let $f(k)$ be a $l^2(0, K-1)$ supported sequence and $W_K = e^{j\frac{2\pi}{K}}$ be the principal K_{th} root of unity. Then the discrete gamma transform of $f(k)$ written as $(\Gamma_\alpha f)(m, n)$ is

$$(\Gamma_\alpha f)(m, n) = \left\langle f(k), g_n(k) e^{j\frac{2\pi}{K}mk} \right\rangle \quad (17)$$

$$= \sum_{k=0}^{K-1} f(k) \gamma^{1/2}(k; n, \alpha) W_K^{-mk} \quad (18)$$

where

$$m \in \mathbf{Z}, \quad n \in \mathbf{N}, \quad \alpha > 0$$

□

Proposition 3

The inverse DGT is

$$f(k) = \frac{1}{\alpha K} \sum_{m=0}^{K-1} \sum_{n=1}^{\infty} (\Gamma_\alpha f)(m,n) g_n(k) W_K^{mk}$$

where

$$k \in [0, K-1]$$

□

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