NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

THESIS

MATRIX ALGEBRA

by

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June, 1996

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REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE June, 1996	3. REPORT TYPE AND DATES COVERED Master's Thesis

4. TITLE AND SUBTITLE MATRIX ALGEBRA	5. FUNDING NUMBERS
6. AUTHOR(S) Susan A. Rife	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey CA 93943-5000	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.	12b. DISTRIBUTION CODE

13. ABSTRACT *(maximum 200 words)*

The purpose of this thesis is to develop a textbook which presents basic concepts of matrix algebra from a primarily computational perspective, for an introductory course in matrix algebra at the Naval Postgraduate School (NPS). The need for an introductory matrix algebra text is generated by the unique characteristics of the student body at NPS. Students at NPS are beginning graduate studies after several years away from the academic environment. As a result, most students benefit from a course which presents fundamental concepts and techniques in solving matrix algebra problems which are needed for advanced studies in mathematics, engineering, and operations research. Current publications in matrix algebra go into more detail on linear algebra than is needed for the introductory course and many texts do not cover complex numbers in sufficient detail to meet the needs of the students. This text presents techniques for solving systems of linear equations, the algebra of matrices, the connection between linear systems and algebraic operations on matrices, and an introduction to eigenvalues, eigenvectors, and complex numbers. The intent is to hone student skills in applying fundamental techniques in matrix algebra essential to success in future courses.

14.SUBJECT TERMS Introduction to Matrix Algebra.	15. NUMBER OF PAGES 228
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18 298-102

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MATRIX ALGEBRA

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Submitted in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

from the

NAVAL POSTGRADUATE SCHOOL
June 1996

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ABSTRACT

The purpose of this thesis is to develop a textbook which presents basic concepts of matrix algebra from a primarily computational perspective, for an introductory course in matrix algebra at the Naval Postgraduate School (NPS). The need for an introductory matrix algebra text is generated by the unique characteristics of the student body at NPS. Students at NPS are beginning graduate studies after several years away form the academic environment. As a result, most students benefit from a course which presents fundamental concepts and techniques in solving matrix algebra problems which are needed for advanced studies in mathematics, engineering, and operations research. Current publications in matrix algebra go into more detail on linear algebra than is needed for the introductory course and many texts do not cover complex numbers in sufficient detail to meet the needs of the students. This text presents techniques for solving systems of linear equations, the algebra of matrices, the connection between linear systems and algebraic operations on matrices, and an introduction to eigenvalues, eigenvectors, and complex numbers. The intent is to home student skills in applying fundamental techniques in matrix algebra essential to success in future courses.

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I. MATRIX ALGEBRA

Matrix algebra is the study of algebraic operations on matrices and their applications to solving systems of linear equations. Systems of linear equations occur in fields such as engineering, economics, sociology and the physical sciences. We need to understand the fundamentals of matrix algebra in order to solve systems of linear equations which make up a large percentage of mathematical problems in these fields. Because of the wide range of disciplines which require the understanding and ability to apply the basic concepts of matrix algebra, a need exists for a manual to accompany a course which presents the basic concepts from a computational perspective rather than the more in-depth theoretical approach of current publications. Students from different disciplines can use this text to learn the basic tools of matrix algebra and begin applying those tools immediately. Those students who require a more in-depth understanding of the subject of matrix algebra will benefit from a clear understanding of the fundamentals presented in this manual and will be prepared for follow-on courses. This thesis project provides a textbook which presents techniques for solving systems of linear equations, the algebra of matrices, the connection between linear systems and algebraic operations on matrices, and an introduction to eigenvalues, eigenvectors and complex numbers.

Chapter One of the text begins with methods for solving homogeneous and nonhomogeneous systems of linear equations. The methods presented are substitution, Gaussian elimination, and Gauss-Jordan elimination. In the process of learning to solve linear systems students learn the three types of solutions a linear system can have, the representation of a linear system as an augmented matrix, the row echelon and reduced row echelon forms of a matrix, and elementary row operations. The first chapter concludes with a look at the use of computers and the need to understand types of computer error in solving systems of linear equations.

In Chapter Two, the student is introduced to the algebra of vectors and matrices. Students learn to add, subtract, and multiply vectors and matrices. Using the dot product of vectors, the student can compute the magnitude of a vector and apply the Law of Cosines to compute the angle between two vectors. Additionally, some properties of matrices are given along with an explanation of what it means for a matrix to be square, symmetric, or a transpose of another matrix. The techniques learned in Chapters One and Two are used in Chapter Three.

The connection between linear systems and algebraic operations on matrices is made in Chapter Three. This chapter also includes the concept of a linear combination of vectors and how a linear combination applies to the definition of the span of a set of vectors. The next topic of discussion is linear dependence of vectors. The student learns how to determine the linear dependence relationship in a set of vectors using the solution to the homogeneous system of linear equations. The chapter concludes with the study of matrix transformations and the necessary conditions for a matrix transformation to be a linear transformation.

Elementary matrices are presented and applied in Chapter Four to motivate the concept of the inverse of a matrix and the factorization of a matrix into a lower and an upper triangular matrix, known as LU Decomposition. Using the inverse and the LU Decomposition of a matrix, the student learns two additional methods for solving a system of linear equations. The next topic is to characterize a matrix as singular or nonsingular by calculating the determinant of a matrix. The methods for calculating the determinant use elementary row operations and cofactor expansion. The chapter closes with the use of the determinant of a matrix in Cramer's Rule to find the solution to a linear system.

The final chapter of the text presents the idea of eigenvalues and eigenvectors of a matrix. The procedure of solving the characteristic equation is used to find the eigenvalues of a matrix, and the eigenvectors are found by solving the homogeneous system $(A - \lambda I)\vec{x} = \vec{0}$. Since the eigenvalues and eigenvectors can be real or complex, the text then defines the set of complex numbers and the rules of arithmetic for complex numbers.

The text is designed with examples worked out in detail for each new concept presented. Additionally, applications based on simplified models of real world situations are presented early in the text to show relevance of the material. Each chapter concludes with exercises to practice the techniques taught. Solutions to exercises are in the appendix.

This manual incorporates concepts from several different texts, published and unpublished. Specifically, network applications adapted from Leon [Ref. 1], illustrations of ill-conditioned systems from Rasmussen [Ref. 2], applications for eigenvalues and eigenvectors from Underwood [Ref. 3], exercises which clearly demonstrate key points about the span of a set of vectors and linear transformations adapted from Lay [Ref. 4], and important concepts involving elementary matrices from Anton [Ref. 5]. Reference to these sources may clarify points that are not obvious otherwise.

The study of matrix algebra is far more involved than the ideas presented in this manual. The intent of the text is specific: to provide students with the fundamental tools of matrix algebra essential to success in follow-on courses.

REFERENCES

[1] Leon, Steven J. *Linear Algebra with Applications.* 4th ed. New York: Macmillan College Publishing Company, Inc., page 21-22, 1994.

[2] Rasmussen, Craig W. *MAR142 Matrix Algebra Refresher: Lecture Notes.* California: Naval Postgraduate School, 1995.

[3] Underwood, Doug. *An Introduction to Matrix Algebra.* California: Naval Postgraduate School, 1991.

[4] Leon, David C. *Linear Algebra and Its Applications.* New York: Addison-Wesley Publishing Company, pages 46, 72-73, 1994.

[5] Anton, Howard. *Elementary Linear Algebra.* 3d ed. New York: John Wiley & Sons, Inc., page 41, 1981.

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PREFACE

The writing of this text is motivated by the need for a manual to accompany a course which presents basic concepts in linear algebra from a primarily computational perspective. The intent is to provide students with the building blocks and tools required for follow-on courses.

The audience is assumed to include a mix of math students, operations research students, and engineering students. Upon completion of the text, students are prepared to pursue courses of study which provide more advanced topics and theory such as linear algebra, and courses which expand on the concepts taught and present nontrivial applications such as linear programming, differential equations, and introductory engineering courses.

The author assumes students are familiar with algebra with respect to functions and first degree polynomial multiplication.

The text is designed with examples for each new concept which are worked out in detail so that the student should have no trouble reading and understanding the process followed to obtain the solution. Applications are presented in various sections to introduce the material to be studied and to show relevant application of the material. The applications tend to be based on simplified models of real world situations because no familiarity with calculus or differential equations is assumed. Each chapter concludes with exercises to practice the concepts taught. Most of the exercises are numerical rather than theoretical. Solutions for all exercises are at the end of the text.

We begin in *Chapter 1* with an introduction to methods for solving linear equations and systems of linear equations. The methods introduced are substitution, Gaussian elimination, and Gauss-Jordan elimination. In *Chapter 2* the student is introduced to the algebra of vectors and matrices. Additionally, some special matrices are presented. The connection between linear systems and algebraic operations on matrices is made in *Chapter 3*. We also consider the concepts of linear combination, linear independence, and linear transformation. In *Chapter 4* we introduce elementary matrices, and use them to motivate the concept of the inverse of a matrix and the standard matrix factorization in the form of LU decomposition. The chapter concludes with methods for finding the determinant of a matrix and some basic properties of the determinant. The final chapter, *Chapter 5*, provides a brief introduction to the concept of eigenvalues and eigenvectors and includes the study of complex numbers so that the student can explore complex eigenvalues and eigenvectors.

The material in this text can be covered in approximately 22 lecture hours.

Suggested Syllabus:

Chapter 1	Sections A-F	4 lectures
Chapter 2	Sections A-B	3 lectures
Chapter 3	Sections A-D	4 lectures
Chapter 4	Sections A-D	6 lectures
Chapter 5	Sections A-E	5 lectures

Total lecture hours: 22

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I. SYSTEMS OF LINEAR EQUATIONS

A large percentage of problems encountered in mathematics involve solving systems of linear equations. Systems of linear equations also have applications in economics, industry, physical sciences, sociology and many other disciplines. This section will introduce nonhomogeneous and homogeneous systems of linear equations and three methods for solving systems of linear equations. We can use systems of linear equations to model real world problems. We begin with a simplified application of linear systems to the personnel staffing problem.

Part of the management process in military organizations involves staffing. With the current reduction of forces, most military units are faced with a shortage of personnel. Commanders must organize their units to promote maximum productivity while operating under personnel constraints. The personnel shortage is compounded by unforecast requirements, such as the need for a Crisis Action Team.

Suppose a Crisis Action Team is needed to continually monitor and report on the tactical situation in an area of interest. The personnel who will make up this team must be taken out of existing departments; no new personnel are added to the organization. The commander wants to have the smallest team possible while still meeting the requirement that the team provide accurate and timely intelligence assessments of the situation in the area of interest. The commander decides to make a four-shift rotation schedule. The first shift is 2400 - 0600 hours. The second shift is 0600-1200 hours. The third shift is 1200-1800 hours. The fourth shift is 1800-2400 hours. Each team member will work two consecutive shifts and have the next two consecutive shifts off (twelve hours on, twelve hours off). Additionally, the commander decides each shift must have a least number of personnel working to complete all tasks. The minimal staffing numbers are shown in Table 1. We can develop a mathematical model to help the commander decide how many personnel should begin working each shift so that shift staffing requirements are met while using the

Shift	Number of Personnel Required
2400-0600 hours	4
0600-1200 hours	6
1200-1800 hours	8
1800-2400 hours	6

Table 1. Crisis Action Team Staffing Requirements

minimum number of personnel. The variables are defined below:

w is the number of personnel who begin working at 2400
x is the number of personnel who begin working at 0600
y is the number of personnel who begin working at 1200
z is the number of personnel who begin working at 1800.

Since we know the minimal shift staffing requirements, we can begin to formulate the equations in the model. The personnel working during the 2400-0600 shift will be those who begin working at 1800 and those who begin at 2400. There must be at least four personnel on duty from 2400-0600 hours, so the first equation will be:

$$\text{\# who begin working at 1800} + \text{\# who begin working at 2400} = \text{\# from 2400-0600}$$

$$z + w = 4.$$

The second equation will be:

$$\text{\# who begin working at 2400} + \text{\# who begin working at 0600} = \text{\# from 0600-1200}$$

$$w + x = 6.$$

The third equation will be:

$$\text{\# who begin working at 0600} + \text{\# who begin working at 1200} = \text{\# from 1200-1800}$$

$$x + y = 8.$$

The fourth equation will be:

$$\text{\# who begin working at 1200} + \text{\# who begin working at 1800} = \text{\# from 1800-2400}$$

$$y + z = 6.$$

We can solve these four equations simultaneously:

$$\begin{array}{ccccc} z & + & w & = & 4 \\ w & + & x & = & 6 \\ x & + & y & = & 8 \\ y & + & z & = & 6. \end{array}$$

Solve for y in terms of z in the fourth equation:

$$y = 6 - z.$$

Substitute for y in the third equation and solve for x in terms of z :

$$\begin{aligned} x + 6 - z &= 8 \\ x &= z + 2. \end{aligned}$$

Substitute for x in the second equation and solve for w in terms of z :

$$\begin{aligned} w + z + 2 &= 6 \\ w &= -z + 4. \end{aligned}$$

Substitute for w in the first equation and solve for z :

$$\begin{aligned} z + (-z + 4) &= 4 \\ 4 &= 4. \end{aligned}$$

This last equation is true for any value of z. We have defined each variable in terms of z:

$$\begin{array}{ccccc} w & = & -z & + & 4 \\ x & = & z & + & 2 \\ y & = & -z & + & 6 \\ z & = & z. & & \end{array}$$

Since z is the number of personnel who begin working at 1800, z can be $1, 2, 3, 4, 5$, or 6. We will not try any values larger than 6 because the requirement is for 6 personnel on duty from $1800 - 2400$. The commander will minimize personnel usage if the number of personnel beginning work at 1800,z, is 4,5,or 6. The negative values for w are disregarded because we cannot have a negative number of people. The commander can now make a

If z is	1	2	3	4	5	6
Then x is	3	4	5	6	7	8
y is	5	4	3	2	1	0
w is	3	2	1	0	-1	-2
Total personnel required	11	10	9	8	8	8

Table 2. Staffing Options

decision about how to schedule the rotation of personnel.

In reality, the commander would consider other factors, such as overlap of personnel on shifts to maintain operational continuity. This is a simplified example of how we can begin to solve real world issues by modeling the situation as a systems of linear equations. Let's find out what systems of linear equations are. We begin with linear equations.

A. LINEAR EQUATIONS

A *linear equation* is an equation which can be written in the form:

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,}$$

where $a_1, ..., a_n$ and b are real numbers, and $x_1, ..., x_n$ are variables. An equation of this form is linear if:

1. Every variable occurs to the first power.
2. No variable of the form x^c,where c is a constant not equal to one.
3. There are no products of variables.
4. No variables are arguments for radical, exponential, logarithmic, or trigonometric functions.

Example 1.1 The following are linear equations:

$$3x + 4y = 7, \qquad x_1 + 2x_2 - x_3 = \sqrt{8},$$

$$11x - 26y + z = e^c \quad (e^c \text{ is a constant}), \qquad 8x + 6y - 5 = 47z.$$

Example 1.2 The following are nonlinear equations:

$$2x^2 + 7y = 7, \quad 3x + 2y - xz = 4,$$

$$\sin x + y = 0, \quad \sqrt{x} + 2y = 1.$$

Consider linear equations in different dimensions. In one dimension, a linear equation has the form:

$$ax = b, \tag{I.1}$$

where a and b are real numbers. If $a = 0$ and $b \neq 0$, this implies that I.1 has no solution. If $a = 0$ and $b = 0$, this implies that I.1 has infinitely many solutions.

Example 1.3 Consider the equations:

$$0x = 7 \quad \text{and} \quad 0x = 0.$$

The equation $0x = 7$ has no solution because no matter what x we choose, $0 \neq 7$. The equation $0x = 0$ has infinitely many solutions because any value we choose for x will yield the result $0 = 0$.

In the plane, a linear equation has the form:

$$ax + by = c,$$

where $a, b,$and c are real numbers. In three dimensions, a linear equation has the form:

$$ax + by + cz = d,$$

where $a, b, c,$ and d are real numbers. In general, a linear equation has the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, ..., a_n$, and b are real numbers.

A *solution* to a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a sequence of numbers $(s_1, s_2, ..., s_n)$ such that the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is true when we substitute the values $s_1, s_2, ..., s_n$ for the variables $x_1, x_2, ..., x_n$, respectively. Solving a linear equation means finding all such solutions. The set of all solutions is the *solution set.*

Example 1.4 Find the solution set for the linear equation:

$$6x - 7y = 3.$$

Solution:

The general solution is represented by the following equations:

$$\begin{cases} x &= \frac{7}{6}t + \frac{1}{2} \\ y &= t. \end{cases}$$

For the solution we assign an arbitrary value to y and solve for x. The parameter t is an arbitrary real number. Each solution to the linear equation is uniquely determined as t varies over all possible real numbers.

Example 1.5 Find the general solution for the linear equation:

$$2x + 4y - 7z = 8.$$

Solution:

The general solution is represented by the following equations:

$$\begin{cases} x &= -2s + \frac{7}{2}t + 4 \\ y &= s \\ z &= t. \end{cases}$$

Assign parameters s and t to the variables y and z,respectively, and solve for x.

B. SYSTEMS OF LINEAR EQUATIONS

A *system of linear equations (linear system)* is a finite set of one or more linear equations with the same variable set. An $m \times n$ system of linear equations is a linear system of m equations in n unknowns of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The a_{ij}'s (the coefficients) and the b_i's are real numbers. The x_i's are the variables.

Example 1.6 An example of a 2×2 system of linear equations, 2 equations in 2 unknowns, is:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & + & 3x_2 & = & 8. \end{array}$$

Example 1.7 An example of a 2×3 system of linear equations, 2 equations in 3 unknowns, is:

$$\begin{array}{rcrcrcr} 5x_1 & - & x_2 & + & 10x_3 & = & 3 \\ -4x_1 & + & 2x_2 & + & 8x_3 & = & -1. \end{array}$$

A solution to a linear system is a sequence of numbers $(s_1, s_2, ..., s_n)$ which simultaneously satisfies each linear equation in the system of linear equations when we substitute the values $s_1, s_2, ..., s_n$ for the variables $x_1, x_2, ..., x_n$, in this order. Solving a system of linear equations means finding all such solutions. The solution set is the set of all possible solutions.

Example 1.8 Consider the linear system:

$$\begin{array}{rcrcrcr} -2x_1 & - & 3x_2 & - & 15x_3 & = & 7 \\ 6x_1 & + & 2x_2 & + & 18x_3 & = & -8. \end{array}$$

$(1, 2, -1)$ is a solution to this linear system because the values simultaneously satisfy each equation in the system. The values $(1, 7, -2)$ satisfy the first equation, but they do not satisfy the second equation; therefore, $(1, 7, -2)$ is not a solution to the linear system. The general solution is represented by the equations:

$$\left\{ \begin{array}{rcl} x_1 & = & -12/7s - 5/7 \\ x_2 & = & -27/7s - 13/7 \\ x_3 & = & s. \end{array} \right.$$

A system of linear equations is said to be *consistent* if there exists at least one solution to the system. If no solution exists, the system is inconsistent. There are three possibilities for solution sets to linear systems:

1. No solution.
2. A unique solution.
3. Infinitely many solutions.

To visualize these possibilities in the xy-plane, consider that a solution to a system of linear equations is a point of intersection of the lines represented by the linear equations in the system.

Example 1.9 A 2×2 system with no solution; see figure 1.

A linear system with two linear equations in two variables can have no solution if the two lines are parallel. Consider the system given by:

$$x_1 + 2x_2 = 5$$
$$x_1 + 2x_2 = 7.$$

Solution:

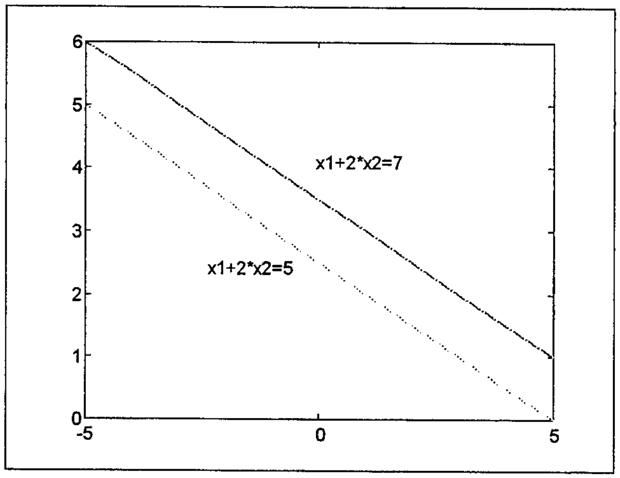

Figure 1. No Solution

Solve for x_1:

$$x_1 = 5 - 2x_2.$$

Substitute into the second equation:

$$\begin{aligned} 5 - 2x_2 + 2x_2 &= 7 \\ 5 &= 7. \end{aligned}$$

The last equation is not true; therefore, the system is inconsistent.

Example 1.10 A 2×2 system with one solution; see figure 2.

A linear system with two linear equations in two variables has exactly one solution when the lines in the system intersect at a single point. For instance, consider the system:

$$\begin{array}{rcrcr} 6x_1 & + & 2x_2 & = & 10 \\ 2x_1 & + & x_2 & = & 2. \end{array}$$

Solution:

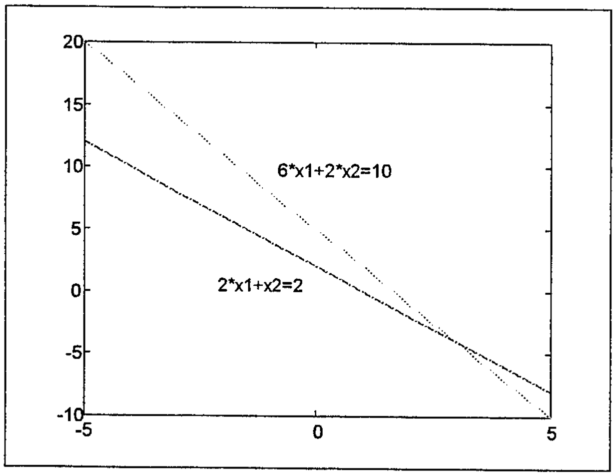

Figure 2. One Solution

Solve the first equation for x_1:

$$x_1 = -\frac{1}{3}x_2 + \frac{5}{3}.$$

Substitute the resulting expression into the second equation:

$$\begin{aligned} 2\left(-\frac{1}{3}x_2 + \frac{5}{3}\right) + x_2 &= 2 \\ \frac{1}{3}x_2 &= \frac{-4}{3} \\ x_2 &= -4. \end{aligned}$$

Solve for x_1:

$$\begin{aligned} x_1 &= -\frac{1}{3}(-4) + \frac{5}{3} \\ x_1 &= 3. \end{aligned}$$

This linear system is the intersection of two lines at a single point. The system has the unique solution $(x_1, x_2) = (3, -4)$.

Example 1.11 A 2×2 system with infinitely many solutions; see figure 3.

A linear system with two equations in two variables has infinitely many solutions when the lines coincide. For example, consider the system:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 5 \\ 2x_1 & + & 4x_2 & = & 10. \end{array}$$

Solution:

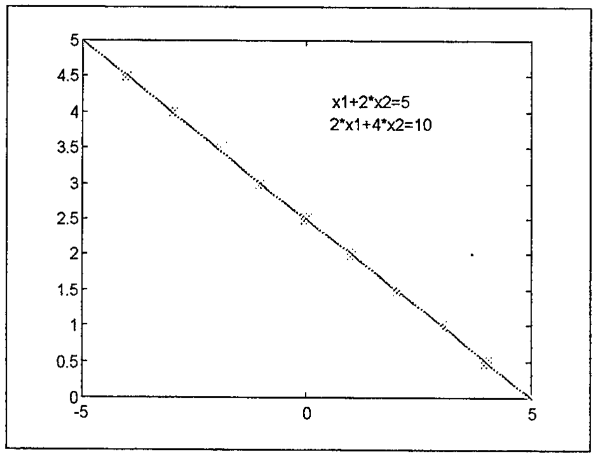

Figure 3. Infinitely Many Solutions

Solve for x_1 in the first equation:

$$x_1 = -2x_2 + 5.$$

Substitute into the second equation:

$$\begin{aligned} 2(-2x_2+5)+4x_2 &= 10 \\ -4x_2+10+4x_2 &= 10 \\ 10 &= 10. \end{aligned}$$

The last equation is true for any value assigned to x_2; the linear system is consistent.

We can visualize the three types of solutions to systems of linear equations as an intersection of hyperplanes. Each linear equation in our system represents a hyperplane. In R^2 a hyperplane is a line. In R^3 a hyperplane is a plane. In R^n we cannot visualize a hyperplane, but the concept is similar to that of a plane in R^3 and a line in R^2.

A system of linear equations is *overdetermined* if the number of equations is greater than the number of variables. Overdetermined linear systems are usually inconsistent; however, it is possible for an overdetermined linear system to be consistent.

Example 1.12 Consider the overdetermined linear system:

$$
\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & + & x_3 & = & 6 \\
3x_1 & - & x_2 & - & x_3 & = & 5 \\
5x_1 & + & 3x_2 & + & x_3 & = & 17 \\
-4x_1 & + & 6x_2 & + & 4x_3 & = & 2.
\end{array}
$$

The general solution for this linear system is:

$$
\begin{cases}
x_1 = 1/7s + 16/7 \\
x_2 = -4/7s + 13/7 \\
x_3 = s.
\end{cases}
$$

This overdetermined linear system is consistent because two of the linear equations in the system are multiples of the other two equations in the system.

Similarly, a system of linear equations is *underdetermined* when the number of equations is less than the number of variables. Underdetermined linear systems are usually consistent with infinitely many solutions. It is not possible for the underdetermined system to have a unique solution, because there will always be a free variable. It is possible for an underdetermined system to be inconsistent.

C. METHODS FOR SOLVING LINEAR SYSTEMS

Many methods for solving systems of linear equations exist. In this section we will study three methods: Method of Substitution, Gaussian Elimination with Back Substitution, and Gauss-Jordan Elimination.

1. Method of Substitution

Solving a system of linear equations by the method of substitution is the process of solving for any one variable in terms of the other variables and substituting into the remaining equations. The method of substitution is frequently used when solving relatively small systems of linear equations, but this method soon becomes cumbersome as the size of the linear system increases. The procedure is basic:

1. Eliminate variables by defining a variable in terms of other variables in the system.

2. Substitute into the remaining equations.

3. Repeat

Example 1.13 Find the general solution for the following linear equation using substitution:

$$x + 2y = 5.$$

Solution:

Solve for x in terms of y:

$$x = 5 - 2y.$$

For every value of y there is a different value for x. Let y be any real number. The general solution for the linear equation is $x = 5 - 2s$, $y = s$. There are infinitely many solutions as y varies through the real numbers.

Example 1.14 Use substitution to solve the linear system:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 5 \\ 4x & + & 8y & - & 6z & = & 2 \\ 3x & + & 6y & - & 5z & = & 0. \end{array}$$

Solution:

Solve for z in terms of x and y in the first equation:

$$z = 2x + 3y - 5.$$

Substitute for z in the second and third equations:

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 8y - 6(2x + 3y - 5) &= 2 \\ 3x + 6y - 5(2x + 3y - 5) &= 0. \end{aligned}$$

This simplifies to:

$$\begin{aligned} 2x + 3y - z &= \quad 5 \\ -8x - 10y &= -28 \\ -7x - 9y &= -25. \end{aligned}$$

Solve for y in the second equation.:

$$y = \frac{14}{5} - \frac{4}{5}x.$$

Substitute for y in the third equation:

$$\begin{aligned} 2x + 3y - z &= \quad 5 \\ -8x - 10y &= -28 \\ -7x - 9\left(\frac{14}{5} - \frac{4}{5}x\right) &= -25. \end{aligned}$$

This simplifies to:

$$\begin{aligned} 2x + 3y - z &= \quad 5 \\ -8x - 10y &= -28 \\ \frac{1}{5}x &= \quad \frac{1}{5}. \end{aligned}$$

Solving for x in the third equation we get $x = 1$. Substituting $x = 1$ into the second equation and solving for y we get $y = 2$. Substituting $x = 1, y = .2$ into the first equation we get $z = 3$. There is a unique solution for this system of linear equations: $(x, y, z) = (1, 2, 3)$. This solution simultaneously satisfies each equation in the linear system.

In the process of solving this system of linear equations we appear to have changed the linear equations in the system. The new equations are a result of making valid substitutions for variables into the old equations. Therefore, the systems of linear equations are actually equivalent, yet the new system is easier to solve. In solving systems of linear equations we want to try to replace the existing system with an equivalent system which has the same solution set, but is easier to solve. We say two linear systems are *equivalent* if they have the same solution set. There are three operations which maintain equivalence for linear systems.

1. *Scaling:* Multiply an equation in the system by a nonzero real number (scalar).
2. *Equation interchange:* Interchange two equations in the system.
3. *Equation replacement:* Add a multiple of one equation to another equation in the system.

Applying one, or any combination, of these operations to a system of linear equations will result in a new system of linear equations which is equivalent to the original system.

A *matrix* is a rectangular array of numbers. We will study the arithmetic of matrices in Chapter II. The *matrix of coefficients* is a matrix consisting of the coefficients of the variables of the linear equations. The *augmented matrix* for a system of linear equations is a shorthand notation with which we can represent the linear system. The augmented matrix consists of the matrix of coefficients and the right hand side constants of the linear equations. Given the system of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\ \ \cdot \\
&\ \ \cdot \\
&\ \ \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}$$

the matrix of coefficients is:

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}.$$

and the augmented matrix is:

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}},$$

Using the augmented matrix to represent a system of linear equations, we must keep track of the variables x_i, the $+$'s and the $=$'s.

A matrix with m rows and n columns is an $m \times n$ matrix; m and n are the *dimensions* of the matrix.

Example 1.15 Write the augmented matrix for the linear system:

$$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 5 \\ 4x & + & 8y & - & 6z & = & 2 \\ 3x & + & 6y & - & 5z & = & 0. \end{array}$$

Solution:

The augmented matrix for the above system of linear equations is:

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 8 & -6 & 2 \\ 3 & 6 & -5 & 0 \end{bmatrix}.$$

This is a matrix of dimension 3×4, because the matrix has 3 rows and 4 columns.

Just as we can create an equivalent system of linear equations using scaling, equation interchange, and equation replacement, we can produce an equivalent augmented matrix using elementary row operations. *Elementary row operations* are:

1. *Scaling:* Multiply all entries in a row of the augmented matrix by a nonzero constant (scalar).
2. *Row interchange:* Interchange two rows in the augmented matrix.
3. *Row replacement:* Add a multiple of one row to another row in the augmented matrix.

Elementary row operations can be performed on any matrix, not just the augmented matrix. However, we will use the augmented matrix and the elementary row operations to solve systems of linear equations.

2. Gaussian Elimination with Back Substitution

Gaussian elimination is a process which takes a system of linear equations and turns it into a triangular system of linear equations using elementary row operations. The solution is then easy to find using back substitution. So, the objective is to replace the system with an equivalent system which is easier to solve. Let's start with an example.

Example 1.16 Find the solution set for the following system of linear equations. We will work with the linear system on the left and the augmented matrix of the linear system on the right:

$$\begin{array}{c} \text{System of Linear Equations} \\ \begin{array}{rcrcrcr} 6x_1 & + & 4x_2 & - & 2_3 & = & -4 \\ 3x_1 & + & x_2 & - & 2x_3 & = & -5 \\ 9x_1 & + & 6x_2 & + & 3x_3 & = & 6, \end{array} \end{array} \qquad \begin{array}{c} \text{Augmented Matrix} \\ \left[\begin{array}{rrrr} 6 & 4 & -2 & -4 \\ 3 & 1 & -2 & -5 \\ 9 & 6 & 3 & 6 \end{array}\right]. \end{array}$$

Solution:

Add $\frac{-1}{2}$ times the first row to the second row.
Add $\frac{-3}{2}$ times the first row to the third row:

$$\begin{array}{rcrcrcr} 6x_1 & + & 4x_2 & - & 2x_3 & = & -4 \\ & - & x_2 & - & x_3 & = & -3 \\ & & & & 6x_3 & = & 12, \end{array} \qquad \left[\begin{array}{rrrr} 6 & 4 & -2 & -4 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 6 & 12 \end{array}\right].$$

Solve for x_3 in the third row:

$$\begin{aligned} 6x_3 &= 12 \\ x_3 &= 2. \end{aligned}$$

Using back substitution, solve for x_2 in the second equation:

$$\begin{aligned} -x_2 - x_3 &= -3 \\ -x_2 - 2 &= -3 \\ x_2 &= 1. \end{aligned}$$

Using back substitution, solve for x_1 in the first equation:

$$\begin{aligned} 6x_1 + 4x_2 - 2x_3 &= -4 \\ 6x_1 + 4(1) - 2(2) &= -4 \\ 6x_1 &= -4 \\ x_1 &= \frac{-2}{3}. \end{aligned}$$

The solution is $(x_1, x_2, x_3) = \left(\frac{-2}{3}, 1, 2\right)$. The solution is unique.

We stated that Gaussian elimination takes a system of linear equations and turns it into an equivalent triangular system of linear equations using elementary row operations. The solution is then easy to find using back substitution. Two questions arise:

1. What should a triangular system of linear equations look like?
2. How do I make the linear system triangular?

In order to answer these questions, we must first learn about two special forms for any matrix. These form are: row echelon form and reduced row echelon form.

a. Row Echelon Form/Reduced Row Echelon Form

A matrix is in *row echelon form* if the following three properties apply:

1. Rows which have zeros in all entries, are at the bottom of the matrix.

2. In rows which have nonzero entries, the *leading nonzero entry (pivot)* in the upper row is to the left of the leading nonzero entry (pivot) in the lower row.

3. All entries in a column below the pivot are zeros.

(Note: Some books will state that the pivot should be a 1; however, we will not make that additional requirement for row echelon form.)

A matrix is in *reduced row echelon form* if the matrix is in row echelon form and:

1. In rows which have nonzero entries, the leading nonzero entry (pivot) in the row is 1.

2. Each column containing a pivot has zeros in every position above and below the pivot.

Example 1.17

$$\overset{\text{Row Echelon Form}}{\begin{bmatrix} \underline{2} & 3 & 4 & -7 \\ 0 & \underline{5} & -1 & 3 \\ 0 & 0 & \underline{3} & 2 \end{bmatrix}}, \quad \overset{\text{Row Echelon Form}}{\begin{bmatrix} \underline{1} & -6 & 5 & 2 & -5 & 4 \\ 0 & 0 & \underline{-2} & 8 & -1 & 5 \\ 0 & 0 & 0 & 0 & \underline{3} & 4 \end{bmatrix}}.$$

Example 1.18

$$\overset{\text{Reduced Row Echelon Form}}{\begin{bmatrix} \underline{1} & -3 & 0 & 0 & 1 \\ 0 & 0 & \underline{1} & 0 & 9 \\ 0 & 0 & 0 & \underline{1} & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}, \quad \overset{\text{Reduced Row Echelon Form}}{\begin{bmatrix} \underline{1} & 7 & 0 & 9 & 0 & 0 \\ 0 & 0 & \underline{1} & -6 & 0 & 3 \\ 0 & 0 & 0 & 0 & \underline{1} & -7 \end{bmatrix}}.$$

Any matrix can be changed into row echelon form or reduced row echelon form by applying elementary row operations to the matrix using the following algorithm.

1. Start with the first nonzero column from the left; this is the *pivot column.*

2. The top position in the pivot column is the *pivot position*. Interchange the top row with another row, if necessary, to move a nonzero entry into the pivot position. This nonzero entry is the *pivot*. The row containing the pivot is the *pivot row.* The pivot is underlined in the examples above.

3. Use elementary row operations to get zeros in all the positions in the column below the pivot position.

4. Cover the row containing the pivot position. Start with step 1 and repeat the process on the remaining rows below. Continue this process until the entire matrix is in row echelon form.

To change the matrix to reduced row echelon form, put the matrix in row echelon form using the steps above and add one more step.

5. Start with the first pivot from the right. If a pivot is not 1, make it 1 by dividing each entry in the pivot row by the pivot. Use row replacement to get zeros above each pivot.

Row echelon form of a matrix is not unique. Changing the sequence in which the elementary row operations are performed will produce different coefficients. However, the systems are still equivalent. On the other hand, *reduced row echelon form is unique*.

The terms *leading nonzero entry, pivot,* and *pivot position* have been mentioned in developing the row echelon and reduced row echelon forms of a matrix. This brings up the concept of a pivot variable. What exactly is the pivot variable? Well, just as you would suspect, the *pivot variable* is the variable in the system of linear equations corresponding to the pivot position in a row of a matrix. The pivot is underlined in the following examples.

When we solve a system of linear equations we usually try to solve for the pivot variable in terms of the variables to the right in each linear equation. Since we solve for the pivot variables in terms of the other variables, we say the pivot variables are the *dependent variables.* The value of the pivot variables depend on the value of the other variables. You will also hear the term *basic variable* used to refer to the pivot variable.

If a variable is not a pivot variable, it is called a *free variable,* or an *independent variable.* We assign parameters to the independent variables to get the general solution. Each different choice of value for the independent variable determines a different solution for the linear system. A solution set can have more than one independent variable

Example 1.19 Consider the 4×5 augmented matrix:

Augmented Matrix of linear system with 4 equations and 4 variables

$$\begin{bmatrix} \underline{1} & 2 & -1 & 3 & 6 \\ 0 & \underline{-1} & -2 & 1 & 0 \\ 0 & 0 & \underline{-6} & -2 & -3 \\ 0 & 0 & 0 & \underline{-1} & 2 \end{bmatrix}.$$

There are four pivot positions which correspond to the four pivot variables, x_1, x_2, x_3, and x_4, in this augmented matrix. The corresponding linear system has a unique solution $\left(\frac{131}{6}, \frac{-13}{3}, \frac{7}{6}, -2\right)$.

Example 1.20 Consider the 3×6 augmented matrix:

Augmented Matrix of linear system with 3 equations and 5 variables.

$$\begin{bmatrix} \underline{1} & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \underline{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & \underline{1} & 1 \end{bmatrix}.$$

There are three pivot positions which correspond to the three pivot variables in the system of linear equations. The other two variables are independent variables. The system has the general solution:

$$\begin{cases} x_1 &= 1 + x_2 + x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= 3 \\ x_5 &= 1. \end{cases}$$

In this solution set, x_1, x_4, and x_5 are the pivot/dependent variables; x_2 and x_3 are the free/independent variables

The method of solving systems of linear equations by reducing the augmented matrix to row echelon form is called *Gaussian Elimination.*

Let's do another example. This time we will perform elementary row operations on the augmented matrix only.

Example 1.21 Given the system of linear equations:

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & + & 6x_6 & = & 2 \\ 2x_1 & + & 6x_2 & - & 4x_3 & & & + & 4x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & + & 9x_6 & = & 3 \\ 4x_1 & + & 12x_2 & - & 9x_3 & - & 2x_4 & + & 8x_5 & + & 21x_6 & = & 7 \end{array}$$

write the augmented matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 6 & 2. \\ 2 & 6 & -4 & 0 & 4 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & 9 & 3 \\ 4 & 12 & -9 & -2 & 8 & 21 & 7 \end{bmatrix}.$$

Solution:

Add -2 times the first row to the second and third rows,
add -4 times the first row to the fourth row:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & -12 & -4 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \end{bmatrix}.$$

Interchange the second and third rows:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 6 & 2 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -12 & -4 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \end{bmatrix}.$$

Add -1 times the second row to the fourth row:

$$\begin{bmatrix} \underline{1} & 3 & -2 & 0 & 2 & 6 & 2 \\ 0 & 0 & \underline{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & \underline{-12} & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is row echelon form. The pivots are underlined.
Write the corresponding system of linear equations using the new coefficients:

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & + & 6x_6 & = & 2 \\ & & & - & x_3 & - & 2x_4 & & & - & 3x_6 & = & -1 \\ & & & & & & & & & - & 12x_6 & = & -4. \end{array}$$

Solve the equations for the pivot variables. The pivot variables are the variables corresponding to the pivots of each linear equation:

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 - 6x_6 + 2 \\ x_3 &= -2x_4 - 3x_6 + 1 \\ x_6 &= \frac{1}{3}. \end{aligned}$$

Begin at the bottom and back substitute into the equations above:

$$\begin{aligned} x_3 &= -2x_4 - 3x_6 + 1 \\ x_3 &= -2x_4 - 3\left(\frac{1}{3}\right) + 1 \\ x_3 &= -2x_4. \end{aligned}$$

Substitute x_3 into the first equation:

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 - 6x_6 + 2 \\ x_1 &= -3x_2 + 2(-2x_4) - 2x_5 - 6\left(\frac{1}{3}\right) + 2 \\ x_1 &= -3x_2 - 4x_4 - 2x_5 \end{aligned}$$

Assign arbitrary values to the independent variables to get the general solution:

$$\begin{cases} x_1 = -3r - 4s - 2t \\ x_2 = r \\ x_3 = -2s \\ x_4 = s \\ x_5 = t \\ x_6 = \frac{1}{3}. \end{cases}$$

The general solution is represented by *parametric equations.* Each solution to the linear system is uniquely determined by the parameters r, s, and t, which are real numbers. The pivot variables are x_1,x_3, and x_6. The free variables are x_2,x_4, and x_5.

Remember, our objective is to replace the current system of linear equations with an equivalent system which is easier to solve. This method of solving systems of linear equations by reducing the augmented matrix to row echelon form then using back substitution to find the solution set is called Gaussian elimination with back substitution.

Notice we interchanged rows in one step of the reduction process. This interchange was necessary to obtain a nonzero entry in the pivot position. In general, interchanging rows may be a necessary operation to obtain a nonzero entry in the pivot position.

Another method for solving systems of linear equations is Gauss-Jordan elimination.

3. Gauss-Jordan Elimination

Gauss-Jordan elimination involves changing the augmented matrix of a linear system to reduced row echelon form. Since reduced row echelon form is a step beyond row echelon form, let's continue with the same example.

Example 1.22 We began Example 1.21 with the following system of equations:

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & + & 6x_6 & = & 2 \\
2x_1 & + & 6x_2 & - & 4x_3 & & & + & 4x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & + & 9x_6 & = & 3 \\
4x_1 & + & 12x_2 & - & 9x_3 & - & 2x_4 & + & 8x_5 & + & 21x_6 & = & 7
\end{array}$$

In Example 1.21, we changed the augmented matrix into row echelon form:

$$\begin{bmatrix}
\underline{1} & 3 & -2 & 0 & 2 & 6 & 2 \\
0 & 0 & \underline{-1} & -2 & 0 & -3 & -1 \\
0 & 0 & 0 & 0 & 0 & \underline{-12} & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

We now continue with our algorithm to change the augmented matrix to reduced row echelon form.
Multiply the second row by -1, and
multiply the third row by $\frac{-1}{12}$:

$$\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Add -3 times the third row to the second row:

$$\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Add 2 times the second row to the first row to produce the reduced row echelon form:

$$\begin{bmatrix}
1 & 3 & 0 & 4 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Rewrite the system of linear equations:

$$\begin{array}{rcrcrcrcrcrcl}
x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & & = & 0 \\
 & & & & x_3 & + & 2x_4 & & & & & = & 0 \\
 & & & & & & & & & & x_6 & = & \dfrac{1}{3}.
\end{array}$$

Solve for the pivot variables:

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3}. \end{aligned}$$

Assign arbitrary values to the independent variables to get the general solution:

$$\begin{cases} x_1 = -3r - 4s - 2t \\ x_2 = r \\ x_3 = -2s \\ x_4 = s \\ x_5 = t \\ x_6 = \tfrac{1}{3}. \end{cases}$$

Notice that the solution set is represented by parametric equations where the parameters r, s, and t are real numbers.

We have just solved the system of linear equations using Gauss-Jordan elimination. This method tends to involve more arithmetic than Gaussian elimination. Most computer algorithms perform Gaussian elimination to solve systems of linear equations.

D. EXISTENCE AND UNIQUENESS OF A SOLUTION

Now that we can solve systems of linear equations, let's consider how to determine whether a solution exists and, if a solution does exit, whether that solution is unique.

When the augmented matrix for a system of linear equations is in row echelon form or reduced row echelon form, there will sometimes be row of the form:

$$[0\ 0 \ldots 0\ b].$$

If $b \neq 0$ in any of the rows with this form, when we write the linear equation corresponding to this row we get:

$$\begin{aligned} 0x_1 + 0x_2 + \cdots + 0x_n &= b, \quad b \neq 0 \\ 0 &= b, \quad b \neq 0. \end{aligned}$$

There are no variables $x_1, x_2, \ldots, x_n$ which will make this linear equation true; therefore, the linear system is inconsistent.

If $b = 0$ in all of the rows with this form, the rows will result in linear equations:

$$0 = 0,$$

which is true for any variable $x_1, x_{2,}..., x_n$; therefore, the system is consistent.

This idea is summarized in the following theorem [Ref. 1]:

Theorem 1.1 *A system of linear equations is inconsistent if, and only if, in the (reduced) row echelon form, there is a row of the form:*

$$[0, 0...0\ b]\text{, } where\ b \neq 0.$$

A system of linear equations is consistent if it is not inconsistent. A consistent linear system will have:

1. A unique solution when every variable is a pivot variable.

2. Infinitely many solutions when at least one variable is an independent variable.

This theorem applies to *nonhomogeneous* systems of linear equations, which are sometimes consistent and sometimes inconsistent. The linear systems with which we have been working have been nonhomogeneous systems of linear equations. In the next section we will learn about a new type of linear system which is always consistent, the homogeneous system of linear equations. Before we study homogeneous linear systems, we'll look at the application of linear systems in solving network problems.

When we want to move some product from one location to another location, we design routes along which these products can move. We call these routes ***networks***. Electricity moves from power plants to our houses along power line networks. We communicate long distances using telephone networks. We travel to different places using traffic networks such as highways, railroads, airline routes, and sea routes. We want products to move along networks efficiently, so we plan the movement by modelling the network as a mathematical object. Network models consist of points called *nodes* which are connected by lines called *arcs*. Product movement through a network is called *flow*. For example, in a traffic network the roads are the arcs and the points of intersection of the roads are the nodes. Cars flow between nodes along arcs in directions indicated by arrows. A set of one-way streets would

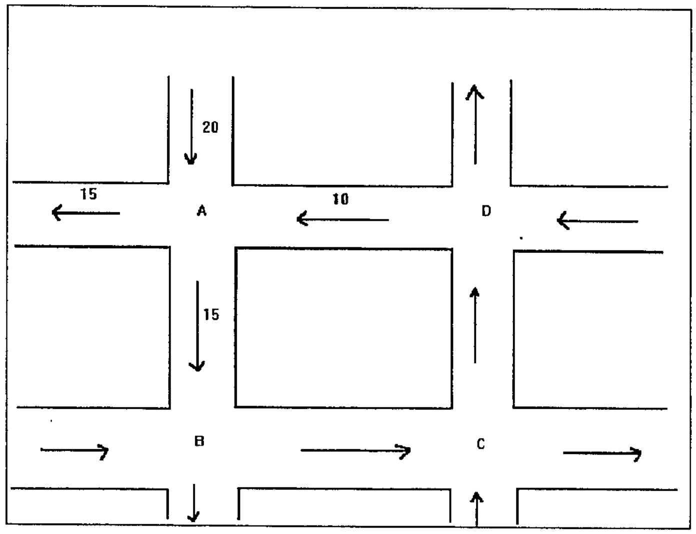

Figure 4. One-way street network. After Ref. [1].

be modeled as shown in figure 4. The basic idea behind solving a network problem is *balance of flow*. Balance of flow means what flows into a node must equal what flows out of a node. In figure 4 we see that 30 cars flow into node A and 30 cars flow out of node A.

Another network problem could deal with communication lines. Suppose that a signal company is establishing communication lines in a tactical area of responsibility. The company has received only partial information on the number of lines needed. The information received and the lines planned are shown in the network model in figure 5. Determine the number of lines which the signal company must establish along the arcs x_1, x_2, x_3, x_4, x_5, and x_6. Remember, in a network problem, balance of flow is the goal. The number of lines into a node must equal the number of lines out of the node. The equations for this communication network are:

$$
\begin{aligned}
20 + x_2 &= 15 + x_1 \quad \text{(node A)} \\
20 + x_4 &= 20 + x_3 \quad \text{(node B)} \\
20 + x_1 &= 25 + x_4 \quad \text{(node C)} \\
15 + x_5 &= 20 + x_2 \quad \text{(node D)}
\end{aligned}
$$

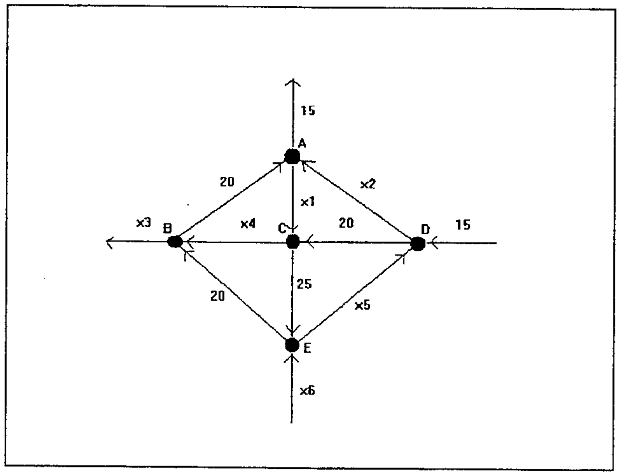

Figure 5. Communications Network

$$25 + x_6 = 20 + x_5 \quad \text{(node E)}.$$

We can rewrite these equations as:

$$\begin{aligned} x_1 - x_2 &= 5 \\ x_3 - x_4 &= 0 \\ x_1 - x_4 &= 0 \\ x_2 - x_5 &= -5 \\ x_5 - x_6 &= 5. \end{aligned}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

Interchange row 2 and row 4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 1 & 0 & 0 & -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

Interchange row 3 and row 4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

Add -1 times row 1 to row 4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

Add -1 times row 2 to row 4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

Multiply row 4 by -1:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5. \end{bmatrix}$$

When we solve for the pivot variables we get:

$$\begin{aligned} x_1 &= x_6 + 5 \\ x_2 &= x_6 \\ x_3 &= x_6 \\ x_4 &= x_6 \\ x_5 &= x_6 + 5 \\ x_6 &= x_6. \end{aligned}$$

The signal company needs to assign a fixed number of lines which will enter the network on arc x_6, after which, the remainder of the lines needed are determined by the equations above.

E. HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

A *homogeneous system of linear equations* is a system in which all the constant terms on the right hand side of the equations are zero:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Unlike nonhomogeneous systems of linear equations, homogeneous systems of linear equations are always consistent. Homogeneous systems have two possible solution sets:

1. One solution - the *trivial solution* in which each variable is equal to zero:

$$x_1 = 0, x_2 = 0, ..., x_n = 0.$$

2. Infinitely many solutions - *nontrivial solutions* as well as the trivial solution.

Let's solve a homogeneous system of linear equations.

Example 1.23 Given the following augmented matrix and equivalent reduced row echelon form of the matrix:

$$\overset{\text{Augmented Matrix}}{\begin{bmatrix} 2 & 4 & -1 & 0 & 2 & 0 \\ -2 & -3 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{\sim}{\text{row equivalent to}} \overset{\text{Reduced Row Echelon Form}}{\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{-1}{2} & 0 \end{bmatrix}},$$

Solution:

Write the corresponding system of linear equations:

$$\begin{array}{ccccccc} x_1 & & & & - & 2x_5 & = 0 \\ & x_2 & & & + & \frac{3}{2}x_5 & = 0 \\ & & x_3 & & & & = 0 \\ & & & x_4 & - & \frac{1}{2}x_5 & = 0. \end{array}$$

Solve for the pivot/dependent variables:

$$\begin{array}{rcl} x_1 & = & 2x_5 \\ x_2 & = & \frac{-3}{2}x_5 \\ x_3 & = & 0 \\ x_4 & = & \frac{1}{2}x_5. \end{array}$$

The general nontrivial solution is:

$$\left\{ \begin{array}{rcl} x_1 & = & 2s \\ x_2 & = & -3/2s \\ x_3 & = & 0 \\ x_4 & = & 1/2s \\ x_5 & = & s. \end{array} \right.$$

The trivial solution occurs when $s = 0$.

An important observation in solving homogeneous systems of linear equations is that performing elementary row operations will not change the zeros in the column of right hand side constants of the augmented matrix..

We have the theorem of existence and uniqueness for solutions to nonhomogeneous linear systems. Similarly, we have a theorem which addresses the existence of nontrivial solutions to homogeneous linear systems.

Theorem 1.2 *An underdetermined homogeneous system of linear equations has at least one independent variable and; therefore, has nontrivial solutions.*

Be careful! The theorem says nothing about a linear system in which the number of variables is equal to the number of equations. Below are two distinct examples in which the homogeneous linear system has the same number of variables as equations yet for the first example there are infinitely many nontrivial solutions and for the second example there is only one solution, the trivial solution.

Example 1.24 Determine if the following homogeneous system of linear equations has a nontrivial solution:

$$\begin{array}{rcrcrcl} 6x_1 & + & 10x_2 & + & 3x_3 & = & 0 \\ -6x_1 & - & 4x_2 & - & 3x_3 & = & 0 \\ 12x_1 & + & 2x_2 & + & 6x_3 & = & 0. \end{array}$$

Solution:

Write the augmented matrix:

$$\underset{\substack{\text{Add 1 times row 1 to row 2}\\ \text{Add -2 times row 1 to row 3}}}{\begin{bmatrix} 6 & 10 & 3 & 0 \\ -6 & -4 & -3 & 0 \\ 12 & 2 & 6 & 0 \end{bmatrix}} \sim \underset{\text{Add 3 times row 2 to row 3}}{\begin{bmatrix} 6 & 10 & 3 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -18 & 0 & 0 \end{bmatrix}} \sim \underset{\text{This is row echelon form}}{\begin{bmatrix} 6 & 10 & 3 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}.$$

Here we can see that x_3 is free; therefore, there are nontrivial solutions. Continue to reduced row echelon form:

$$\underset{\substack{\text{This is row echelon form}\\ \text{Multiply row 1 by } \frac{1}{6}\\ \text{Multiply row 2 by } \frac{1}{6}}}{\begin{bmatrix} 6 & 10 & 3 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \sim \underset{\text{Add } \frac{-5}{3} \text{ times row 2 to row 1}}{\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \sim \underset{\text{This is reduced row echelon form}}{\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}.$$

Write the corresponding system of linear equations:

$$\begin{array}{ccccccc} x_1 & & & + & \frac{1}{2}x_3 & = & 0 \\ & & x_2 & & & = & 0 \\ & & & & 0 & = & 0. \end{array}$$

Solve for the pivot variables:

$$\begin{array}{rcl} x_1 & = & \frac{-1}{2}x_3 \\ x_2 & = & 0 \\ x_3 & = & x_3. \end{array}$$

The general solution is:

$$\begin{cases} x_1 = \frac{-1}{2}s \\ x_2 = 0 \\ x_3 = s. \end{cases}$$

The trivial solution occurs when $s = 0$.

Example 1.25 Solve the following homogeneous system of linear equations:

$$\begin{array}{ccccccc} 2x_1 & + & 2x_2 & & & = & 0 \\ 4x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ 4x_1 & & & + & 6x_3 & = & 0. \end{array}$$

Solution:

Write the augmented matrix:

$$\underset{\substack{\text{Add -2 times row 1 to row 2}\\ \text{Add -2 times row 1 to row 3}}}{\begin{bmatrix} 2 & 2 & 0 & 0 \\ 4 & 2 & 4 & 0 \\ 4 & 0 & 6 & 0 \end{bmatrix}} \sim \underset{\text{Add -2 times row 2 to row 3}}{\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -4 & 6 & 0 \end{bmatrix}} \sim \underset{\text{Row echelon form}}{\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}}.$$

Write the corresponding system of linear equations:

$$\begin{array}{rcrcrcl} 2x_1 & + & 2x_2 & & & = & 0 \\ & - & 2x_2 & + & 4x_3 & = & 0 \\ & & & - & 2x_3 & = & 0. \end{array}$$

Solve for the pivot variables:

$$\begin{aligned} x_1 &= -x_2 \\ x_2 &= 2x_3 \\ x_3 &= 0. \end{aligned}$$

The only solution is the trivial solution

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0. \end{cases}$$

F. COMPUTERS AND SYSTEMS OF LINEAR EQUATIONS

Geometrically speaking, the solution to a linear system can be interpreted as the points of intersection of the linear equations. The problems we will study can be solved by hand and sometimes even graphed. In the real world, problems are much more extensive and require the use of calculators and computer software. In using automation, we must be aware of the probability of error. The errors we are talking about are not human error, but rather machine limitations and design.

Types of error:

- *Machine epsilon* - Machine epsilon is the smallest number ε such that $1 + \varepsilon > 1$ on the machine. ε is a measure of the relative error committed in storing numbers on the computer.
- *Truncation Error* - This error occurs when we approximate a number even though the number requires an infinitely long sequence of numbers.
- *Round-Off Error* - Digital computers generally use floating-point numbers of fixed word

length. The true value of a number is not expressed exactly due to the computer imperfection of round-off error.

Why are we concerned with error? Consider two lines which are nearly parallel. The two equations are not proportional, but are nearly proportional. A small error in the coefficient of a variable will result in turning, raising, or lowering a nearly parallel line slightly, thereby causing a large change in the point of intersection, relatively speaking. So, in the two equations, a small error in the coefficients can result in a large error in the solution. A system is said to be *ill-conditioned* when a small change in a coefficient or the right hand side constant results in a large change in the solution. The following is an example of an ill-conditioned system. [Ref. 2]

Example 1.26 Consider two nearly parallel lines,

$$\begin{aligned} 1.4142x - y &= 0 \\ \sqrt{2}x - y &= 0. \end{aligned}$$

This system has a unique solution $(x, y) = (0, 0)$. If we introduce a small change in the right hand side of the second equation we shift the line corresponding to the second equation slightly.

$$\begin{aligned} 1.4142x - y &= 0 \\ \sqrt{2}x - y &= 0.001. \end{aligned}$$

In this case, the y-intercept will change, but the slope remains the same. This system has the unique solution $(x, y) = (73.7, 104.3)$. So, the system of linear equations is ill-conditioned because a small change in a coefficient produced a large change in the solution.

Example 1.27 Machine computation also results in unexpected errors. Consider the system,

$$\begin{aligned} 0.0001x + y &= 1 \\ x + y &= 2. \end{aligned}$$

This system has the approximate solution $(x, y) = (1.0001, 0.9999)$. Using Gaussian elimination and rounding to three significant digits, we will get the solution $(x, y) = (0, 1)$. However, if we interchange the equations before eliminating variables, and still round to three significant digits, we will get the solution $(x, y) = (1, 1)$. This row interchange is an example of *partial pivoting*. Partial pivoting is a strategy for interchanging rows to obtain nonzero entries in the pivot position, which will reduce round-off error. The idea behind partial pivoting is that when a number has some error in it resulting from round-off, dividing that number by a value near zero will increase the error. For this reason, we want the pivot entry to be as far from zero as possible. In order to ensure we minimize error, we interchange rows as necessary to place the largest entry, in absolute value, in the pivot position. Partial pivoting is may be done for other reasons even when it is not necessary to obtain the largest pivot entry. This strategy and others are described in numerical analysis textbooks.

G. EXERCISES

1. Decide whether the following equations are linear or nonlinear in the variables x, y, z.

a) $2x_1 + 3x_1x_3 + x_2 = 2$

b) $-x_1 + 2x_2 + x_3 = \tan(k)$ (k is a constant)

c) $3x_1 - x_2^{\frac{1}{2}} + 2x_3 = 6$

d) $x_1 + \sqrt{5}x_2 = x_3 - 4$

e) $-x_1 - x_2^{-1} + 3 = x_3$

f) $x_2 = x_3$

g) $2x + 4yz = 6$

h) $x - 6y + z - \sin t = e^s$ (s,t are constants)

i) $x + 16y - 7z = 4$

j) $-3x^3 - 3 = 0$

2. Write the coefficient matrix and the augmented matrix for each of the following systems of linear equations.

a)
$$\begin{array}{rcrcr} -3x_1 & + & 5x_2 & = & 1 \\ x_1 & + & 2x_2 & = & -4 \\ 4x_1 & - & x_2 & = & -3 \end{array}$$

b)
$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & & & & & = & 1 \\ & & x_2 & - & x_3 & + & 3x_4 & & & = & -4 \\ x_1 & & & + & 5x_3 & & & - & 2x_5 & = & 6 \end{array}$$

3. Find the solution set for the following linear equations.

a) $-2x_1 + 5x_2 - x_3 + 3x_4 = 8$

b) $v - 4w + x + 5y + 3z = 0$

c) $x - y + 6z = 0$

4. Solve the following linear systems by the method of substitution.

a) $$\begin{array}{rcrcr} 2x & + & y & = & 5 \\ -3x & + & 2y & = & 3 \end{array}$$

b) $$\begin{array}{rcrcr} 4x & - & 2y & = & 0 \\ x & + & 3y & = & 7 \end{array}$$

c) $$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & 6 \\ x & + & 6y & - & 2z & = & 12 \\ -x & + & 4y & + & 6z & = & 24 \end{array}$$

d) $$\begin{array}{rcrcr} & & 6x_2 & = & 6 \\ 4x_1 & + & x_2 & = & -3 \end{array}$$

e) $$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 3 \end{array}$$

f) $$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 4 \end{array}$$

5. Solve the following linear systems using Gaussian Elimination.

a) $$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & 6 \\ x & + & 6y & - & 2z & = & 12 \\ -x & + & 4y & + & 6z & = & 24 \end{array}$$

b) $$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 3 \end{array}$$

c) $$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 4 \end{array}$$

d) $$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & + & 3x_4 & = & 5 \\ -x_1 & - & 2x_2 & & & - & 3x_4 & = & -9 \\ 2x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 18 \\ 2x_1 & + & x_2 & + & x_3 & + & 5x_4 & = & 12 \end{array}$$

$$\text{e)}\quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 9 \\ -2x_1 & - & x_2 & - & x_3 & = & -15 \\ -x_1 & + & x_2 & + & x_3 & = & -4 \end{array}$$

$$\text{f)}\quad \begin{array}{rcrcrcrcr} 1x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 7 \\ 2x_1 & - & 8x_2 & + & 2x_3 & + & 2x_4 & = & -10 \\ 2x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & = & 12 \\ 3x_1 & - & 2x_2 & + & 3x_2 & + & 3x_2 & = & 5 \end{array}.$$

6. Use Gauss-Jordan Elimination to solve the following systems.

$$\text{a)}\quad \begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & 5x_3 & = & 5 \\ 3x_1 & + & 4x_2 & + & 7x_3 & = & 8 \end{array}$$

$$\text{b)}\quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 7x_3 & = & 19 \\ 2x_1 & + & 4x_4 & + & 6x_3 & = & 16 \end{array}$$

$$\text{c)}\quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 6 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 7 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 15 \end{array}$$

$$\text{d)}\quad \begin{array}{rcrcrcr} -x_1 & - & 2x_2 & - & x_3 & = & -5 \\ x_1 & + & 3x_2 & + & 2x_3 & = & 7 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 10 \end{array}$$

7. A butcher sells regular and diet ground beef. Regular ground beef is 30% fat, while diet ground beef is 20% fat. One day the butcher finds that she has on hand 3 pounds of fat and 10 pounds of lean beef (assume lean beef has no fat in it). The butcher wants to use all the fat and lean beef she has on hand. How many pounds of regular ground beef and how many pounds of diet ground beef shall the butcher make if she wants to use all the fat and all the lean beef on hand?[Ref. 2]

8. Can the equation $ax = b$ always be solved for x?

9. Determine if the following systems are consistent?

a) The 2×4 coefficient matrix for the system has two pivot columns.

b) The 2×4 augmented matrix for the system has the fourth column as a pivot column.

10. State whether the following matrices are in row echelon form, reduced row echelon form or neither.

a) $\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -4 & 6 & 5 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) $\begin{bmatrix} 2 & -3 & 4 & 1 \\ 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

11. Solve the following systems of linear equations by the method you prefer.

a) $\begin{array}{rcrcr} -x_1 & - & 2x_2 & = & 6 \\ 2x_1 & + & 6x_2 & = & 3 \end{array}$

b) $\begin{array}{rcrcr} x_1 & - & 2x_2 & = & 3 \\ -3x_1 & + & 5x_2 & = & 1 \end{array}$

c) $\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & + & x_3 & = & 4 \\ -x_1 & - & 3x_2 & + & 2x_3 & = & 5 \\ x_1 & - & 3x_2 & - & 4x_3 & = & -17 \end{array}$

d) $\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 6 \\ -x_1 & + & 2x_2 & + & 3x_3 & = & -8 \\ & & 6x_2 & + & 4x_3 & = & -1 \end{array}$

e) $$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & 3 \\ x_1 & + & 4x_2 & + & 3x_3 & = & 8 \\ x_1 & + & 6x_2 & + & 4x_3 & = & 11 \end{array}$$

f) $$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & + & 2x_4 & = & 1 \\ -x_1 & + & 3x_2 & + & 5x_3 & - & x_4 & = & 6 \\ 3x_1 & - & 7x_2 & - & 4x_3 & + & 3x_4 & = & -5 \end{array}$$

g) $$\begin{array}{rcrcr} 2x_1 & + & 2x_2 & = & 8 \\ 2x_1 & - & 5x_2 & = & -6 \\ 6x_1 & - & 8x_2 & = & 3 \end{array}$$

h) $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & x_3 & = & -4 \\ -2x_1 & & & - & 2x_3 & = & 2 \end{array}$$

12. Solve the following homogeneous systems of linear equations.

a) $$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & 0 \\ -x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ & & 3x_2 & + & 8x_3 & = & 0 \end{array}$$

b) $$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 4x_3 & + & x_4 & = & 0 \\ 5x_1 & + & 3x_2 & + & x_3 & - & 2x_4 & = & 0 \end{array}$$

c) $$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & 2x_3 & + & 4x_4 & = & 0 \\ -3x_1 & - & x_2 & + & 3x_3 & & & = & 0 \\ & - & 3x_2 & - & x_3 & - & 2x_4 & = & 0 \\ x_1 & + & x_2 & & & + & x_4 & = & 0 \\ -x_1 & - & 2x_2 & - & 3x_3 & + & 3x_4 & = & 0 \end{array}$$

d) $$\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & + & x_3 & + & 2x_4 & = & 0 \\ & & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}$$

e) $$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 2x_3 & = & 0 \\ 2x_1 & - & 3x_2 & & & = & 0 \\ -3x_1 & + & 4x_2 & & & = & 0 \end{array}$$

II. VECTOR AND MATRIX EQUATIONS

In this section we will study the notation and algebra of vectors and matrices. Vectors and matrices give us a way to represent systems of linear equations in a "shorthand" notation which is algebraically easier to work with than the linear system itself. We began to manipulate linear systems in matrix form when we used the augmented matrix. This section will expand on basic algebraic concepts of vectors and matrices and their applications to systems of linear equations.

A. VECTOR OPERATIONS

A *vector*, for our purposes, is an ordered array of n scalars. We denote a vector with a lowercase letter which has an arrow over the top, for example, $\vec{v}$. We can represent a vector as a matrix of dimension $n \times 1$, called a *column vector,* whose components are scalars. We can also represent the same vector as a matrix of dimension $1 \times n$, called a *row vector,* whose components are the same scalars.

Example II.1 The vector $\vec{v}$ written as a column vector and as a row vector:

$$\vec{v} = \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}}_{\substack{\text{Column Vector} \\ (n \times 1)}}, \qquad \vec{v} = \underbrace{\begin{bmatrix} v_1 & v_2 & . & . & . & v_n \end{bmatrix}}_{\substack{\text{Row vector} \\ (1 \times n)}}.$$

The components $v_1, v_2, ...v_n$ are scalars.

Given an $n \times 1$ vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$, we associate with $\vec{v}$ the point $(v_1, v_2, ...v_n)$ in R^n. In our studies we will use column vectors unless otherwise required.

Example II.2 Below are three vectors with different numbers of components:

$$\vec{u} = \underset{\substack{\text{Vector in } R^2 \\ (2\times1)}}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}, \quad \vec{v} = \underset{\substack{\text{Vector in } R^3 \\ (3\times1)}}{\begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}}, \quad \vec{w} = \underset{\substack{\text{Vector in } R^n \\ (n\times1)}}{\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}}.$$

Two vectors are *equal* if and only if they have the same components in the same order.

Example II.3 For example, $\vec{u} = \vec{v}$, but $\vec{u} \neq \vec{w}$, and $\vec{v} \neq \vec{w}$:

$$\vec{u} = \underset{(3\times1)}{\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}}, \quad \vec{v} = \underset{(3\times1)}{\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}}, \quad \vec{w} = \underset{(3\times1)}{\begin{bmatrix} -6 \\ 3 \\ 4 \end{bmatrix}}.$$

We should note that a vector $\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \neq \begin{bmatrix} 3 & 4 & -6 \end{bmatrix}$ because the vectors have different dimensions, even though they have the same components.

Geometrically, we associate with $\vec{v}$ the directed line segment in R^n with initial point at the origin and terminal point at $(v_1, v_2, ...v_n)$. For example, we can associate a vector in R^2 with a point in the Cartesian plane. If we draw a directed line segment from the origin to the point (v_1, v_2), we get the conventional visualization of a vector.

Example II.4 The point (x, y) in the Cartesian plane can be associated with the vector $\begin{bmatrix} x \\ y \end{bmatrix}$, see figure 1.

1. Vector Addition and Subtraction

Vectors can be added and subtracted in the following manner. Given two vectors, if the vectors have the same number of components, then addition and subtraction is computed

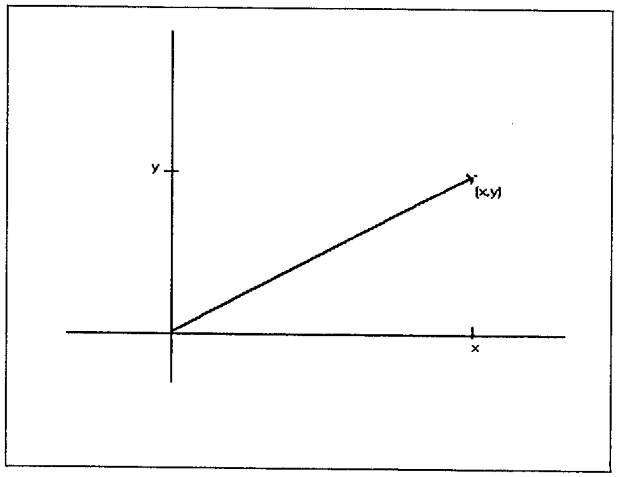

Figure 1. Plot of the Vector $[x\ y]$

component-wise.

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}.}$$

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \vec{x} - \vec{y} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}.}$$

If the vectors do not have the same number of components, the sum and difference are *undefined*.

Graphically, we can use the *parallelogram rule* to visualize the sum of two vectors. In the Cartesian plane, if the origin and the points associated with the vectors are three vertices of a parallelogram, the sum of the two vectors is a vector associated the point which would be the fourth vertex of the parallelogram, see figure 2.

Example II.5 Given $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 7 \\ 6 \\ 5 \end{bmatrix}$:

$$\vec{x} + \vec{y} = \begin{bmatrix} 2 \\ 6 \\ 11 \end{bmatrix}, \quad \vec{x} - \vec{y} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix},$$

$\vec{x} - \vec{z}$ and $\vec{y} + \vec{z}$ are undefined.

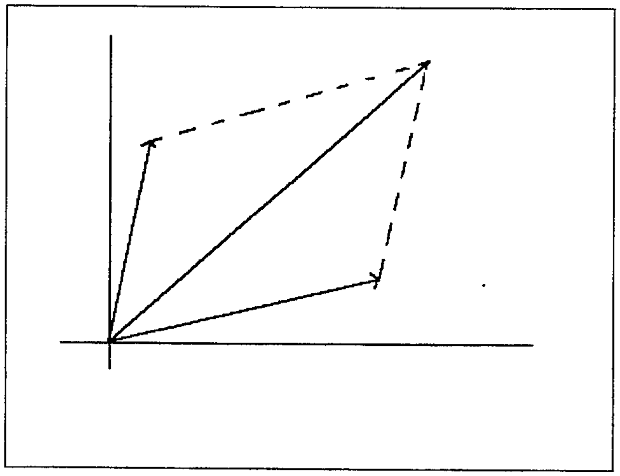

Figure 2. Illustration of the Parallelogram Rule

2. Vector Scalar Multiplication

Given a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$ and a scalar k, we can compute $k\vec{v}$ by multiplying each component of $\vec{v}$ by the scalar k:

$$\boxed{k\vec{v} = k \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \cdot \\ \cdot \\ \cdot \\ kv_n \end{bmatrix}.}$$

When we subtract a vector $\vec{y}$ from another vector $\vec{v}$, we are actually adding $(-1)\,\vec{y}$ to $\vec{v}$, that is,

$$\vec{v} - \vec{y} = \vec{v} + (-1)\vec{y}.$$

Graphically, $k\vec{v}$ transforms the vector $\vec{v}$ as follows. If $0 < k < 1$, then k contracts the vector $\vec{v}$. If $0 > k > -1$, then k contracts the vector $\vec{v}$ and points the associated directed line segment in the opposite direction. If $k > 1$, then k dilates the vector $\vec{v}$. If $k < -1$, then k dilates the vector $\vec{v}$ and points the associated directed line segment in the opposite direction, see figure 3.

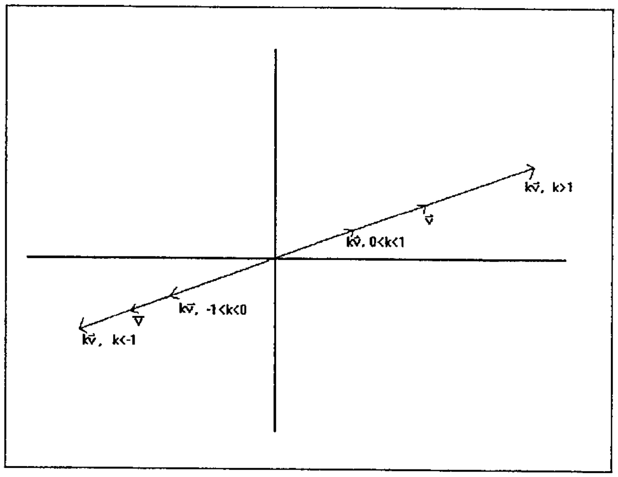

Figure 3. Vector Scalar Multiplication

Example II.6 Compute the scalar multiplication $k\vec{v}$:

$$\textbf{Solution:} \quad k = 6, \quad \vec{v} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}, \quad k\vec{v} = \begin{bmatrix} 6 \cdot 0 \\ 6 \cdot 6 \\ 6 \cdot 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 36 \\ 30 \end{bmatrix}.$$

3. Vector Multiplication

Vector multiplication is commonly called the *dot product* or the *scalar product*. The dot product of two vectors requires the vectors to have the same number of components. The result will be a scalar, not a vector, ergo the name scalar product. Given two vectors:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \text{and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}, \text{ we define the dot product as}$$

$$\boxed{\vec{x} \cdot \vec{y} = \sum_{k=1}^{n} x_k \cdot y_k = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.}$$

We know that multiplication of real numbers is commutative, so we can rewrite this equation as:

$$\begin{aligned}
\vec{x}\cdot\vec{y} &= x_1y_1 + x_2y_2 + \cdots + x_ny_n. \\
&= y_1x_1 + y_2x_2 + \cdots + y_nx_n \\
&= \sum_{k=1}^{n} y_k \cdot x_k \\
&= \vec{y}\cdot\vec{x}.
\end{aligned}$$

This demonstrates the property that vector multiplication is commutative, $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$.

Example II.7 Consider the vectors:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 7 \\ 6 \\ 5 \end{bmatrix},$$

$$\vec{x}\cdot\vec{y} = 2\cdot -2 + 3\cdot 1 + 4\cdot 3 = 11.$$

$\vec{y}\cdot\vec{z}$ is undefined, since the vectors do not have the same number of components.

The *norm,* or *magnitude,* of a vector is defined to be the square root of the dot product of the vector with itself. We denote the norm of a vector $\vec{x}$ by writing $|\vec{x}|$. We can use the dot product to calculate the norm of a vector, and to calculate the angle between the directed line segments associated with two vectors. The norm is defined by:

$$\boxed{|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}}.}$$

We can calculate the norm of any vector regardless of the dimensions of the vector.

Example II.8 Given the vectors:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 7 \\ 6 \\ 5 \end{bmatrix},$$

compute $|\vec{x}|$, $|\vec{y}|$, and $|\vec{z}|$.

Solution:

$$\begin{aligned}
|\vec{x}| &= \sqrt{2\cdot 2 + 3\cdot 3 + 4\cdot 4} = \sqrt{29} \approx 5.39, \\
|\vec{y}| &= \sqrt{-2\cdot -2 + 1\cdot 1 + 3\cdot 3} = \sqrt{14} \approx 3.74,
\end{aligned}$$

$$|\vec{z}| = \sqrt{0 \cdot 0 + 7 \cdot 7 + 6 \cdot 6 + 5 \cdot 5} = \sqrt{110} \approx 10.49.$$

We can use the dot product and the norm to compute the angle between the two directed line segments associated with the vectors $\vec{x}$ and $\vec{y}$ using the *Law of Cosines,* which is defined by:

$$\boxed{\cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|}}$$

θ is the smallest positive angle between the two directed line segments associated with the vectors $\vec{x}$ and $\vec{y}$, see figure 4.

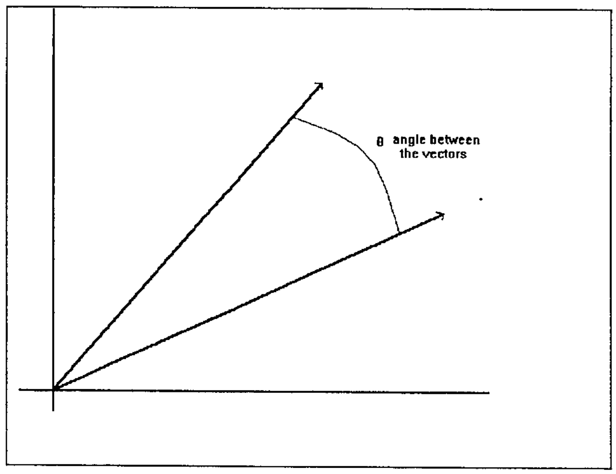

Figure 4. Law of Cosines, Angle Between Vectors

Example II.9 Given the vectors:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix},$$

find the angle between the associated directed line segments:

Solution:

$$\begin{aligned} \cos\theta &= \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|} \\ &= \frac{11}{(5.39)\,(3.74)} \\ &= \frac{11}{20.16} \end{aligned}$$

$$\approx \ .55.$$

$$\theta = \arccos(.55) \approx 56.93°.$$

We say two vectors are *orthogonal* if their dot product is equal to zero.

Example II.10 Determine which vectors are orthogonal.

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \qquad \vec{w} = \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}, \qquad \vec{u} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

$$\vec{x} \cdot \vec{w} = -8 + 15 - 7 = 0 \text{ (orthogonal)},$$
$$\vec{x} \cdot \vec{u} = -4 - 3 - 2 = -9 \text{ (not orthogonal)},$$
$$\vec{w} \cdot \vec{u} = 8 - 5 + 14 = 17 \text{ (not orthogonal)}.$$

Let's calculate the angle between the directed line segments associated with the orthogonal vectors $\vec{x}$ and $\vec{w}$.

Example II.11 Given $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$, use the law of cosines to calculate the angle between the associated directed line segments:

Solution:

$$\begin{aligned} \cos\theta &= \frac{\vec{x} \cdot \vec{w}}{|\vec{x}|\,|\vec{w}|} \\ &= \frac{0}{(3.74)\,(9.49)} \\ &= 0 \end{aligned}$$

$$\theta = \arccos(0) = 90°$$

From this example we see that orthogonal vectors form right angles.

4. Vector Form of the Solution to a Linear System

Now that we know a vector is an ordered array of scalars and the solution to a system of linear equations is an ordered set of values, we have a new way to write the solution to a system of linear equations.

Example II.12 Given the linear system,

$$\begin{array}{rcrcrcr} -2x_1 & - & 3x_2 & - & 15x_3 & = & 7 \\ 6x_1 & + & 2x_2 & + & 18x_3 & = & -8 \end{array},$$

the general solution is

$$\begin{cases} x_1 &= -12/7s - 5/7 \\ x_2 &= -27/7s - 13/7 \\ x_3 &= s. \end{cases}$$

We can also write the general solution in vector form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12/7s - 5/7 \\ -27/7s - 13/7 \\ s \end{bmatrix} = s \begin{bmatrix} -12/7 \\ -27/7 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/7 \\ -13/7 \\ 0 \end{bmatrix}$$

B. MATRIX OPERATIONS

A matrix is a rectangular array of numbers. We denote a matrix with a capital letter and the entries of a matrix are denoted with lowercase letters. If we have a matrix A with m rows and n columns we say that A is an $m \times n$ matrix. An entry in the matrix A is represented as a_{ij}, where i and j denote the position of the entry in the matrix (row i and column j).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ & & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

A is an $(m \times n)$ matrix with entries a_{ij}

Given two matrices A and B, we say the matrix A is *equal* to the matrix B if, and only if, $a_{ij} = b_{ij}$ for all i, j. In the remainder of this section we will treat matrices as algebraic objects and learn how to perform matrix addition, subtraction, multiplication, and scalar multiplication. We will learn some properties of matrix operations and discuss special types of matrices.

1. Matrix Addition and Subtraction

Two matrices can be added together provided they have the same dimensions. Addition is done entry-wise. Given two matrices A and B, both of which are $m \times n$ matrices, matrix addition is defined as follows:

$$\boxed{C = A + B, \text{ where } c_{ij} = a_{ij} + b_{ij} \text{ for each } i, j.}$$

C is also an $m \times n$ matrix.

Example II.13 Compute $A + B$:
Solution:

$$\underset{(2\times3)}{A = \begin{bmatrix} 4 & 1 & 6 \\ -1 & 3 & 2 \end{bmatrix}}, \quad \underset{(2\times3)}{B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \end{bmatrix}}, \quad \underset{(2\times3)}{A + B = \begin{bmatrix} 5 & 1 & 8 \\ 2 & 1 & 7 \end{bmatrix}},$$

Similarly, one matrix can be subtracted from another matrix provided they have the same dimensions. Subtraction is also done entry-wise. Given two matrices A and B, both of which are $m \times n$ matrices, matrix subtraction is defined as follows:

$$\boxed{C = A - B, \text{ where } c_{ij} = a_{ij} - b_{ij} \text{ for each } i, j.}$$

C is also an $m \times n$ matrix.

Example II.14 Compute $A - B$:
Solution:

$$\underset{(2\times3)}{A = \begin{bmatrix} 4 & 1 & 6 \\ -1 & 3 & 2 \end{bmatrix}}, \quad \underset{(2\times3)}{B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \end{bmatrix}}, \quad \underset{(2\times3)}{A - B = \begin{bmatrix} 3 & 1 & 4 \\ -4 & 5 & -3 \end{bmatrix}}.$$

If two matrices do not have the same dimensions, matrix addition and matrix subtraction are undefined.

2. Matrix Scalar Multiplication

We can multiply an $m \times n$ matrix A by a scalar k in the same manner we performed vector scalar multiplication. Given an $m \times n$ matrix A and a scalar k, kA is an $m \times n$ matrix with the entries ka_{ij} for each a_{ij} in the matrix A. Matrix scalar multiplication can be performed on any matrix regardless the dimensions of the matrix.

Example II.15 Consider the matrix A and the scalar k:

Solution

$$A = \underset{(2\times 3)}{\begin{bmatrix} 4 & 1 & 6 \\ -1 & 3 & 2 \end{bmatrix}}, \qquad k = 2.$$

$$kA = 2\begin{bmatrix} 4 & 1 & 6 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 12 \\ -2 & 6 & 4 \end{bmatrix}.$$

When we subtract a matrix B from a matrix A, we actually add $(-1)\,B$ to A.

$$A - B = A + (-1)\,B.$$

Remember, A and B must have the same dimensions to perform addition and subtraction.

3. Matrix Multiplication

Not only can we multiply a matrix by a scalar, but we can also multiply a matrix by a matrix. Unlike matrix addition and subtraction, matrix multiplication is not done entry-wise and the dimensions of the matrices may or may not be the same. Matrix multiplication is performed by taking the dot product of the row vector in the ith row of the first matrix with the column vector in the jth column of the second matrix to get the ij entry of the resulting matrix. Since the dot product of two vectors requires that the vectors have the same number of entries, the number of columns in the first matrix must equal the number of rows in the second matrix. The resulting matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix in the product. Let's take a closer look. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix, then

$$\underset{(m\times n)}{A}\;\underset{(n\times p)}{B} = \underset{(m\times p)}{AB}$$

The dimensions of the matrices must have the following relationship in order to perform matrix multiplication,

$$\text{dimensions of } AB = \underbrace{(m \times \underbrace{n)(n}_{\text{same}} \times p)}_{\text{dimensions of resulting matrix}} = (m \times p).$$

We define the product as follows:

$$\boxed{C = AB, \text{ where the entries of } C \text{ are } c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \text{ for each } i, j}.$$

Example II.16 Let $A = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \end{bmatrix}$. Since A is a (2×3) matrix and B is a (2×3) matrix,

$$\text{dimensions of } AB = (2 \times \underbrace{3)(2}_{\neq} \times 3).$$

Since the dimensions of A and B do not have the right relationship for matrix multiplication, AB is undefined.

Example II.17 Let $A = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 0 & 5 \end{bmatrix}$.

$$\text{dimensions of } AB = \underbrace{(2\times \underbrace{3)(3}_{\text{same}} \times 2)}_{\text{dimensions of resulting matrix}} = (2 \times 2).$$

$$\begin{aligned} AB &= \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot 6 + 1 \cdot 4 + 3 \cdot 0 & 4 \cdot 1 + 1 \cdot 2 + 3 \cdot 5 \\ -1 \cdot 6 + 3 \cdot 4 + 2 \cdot 0 & -1 \cdot 1 + 3 \cdot 2 + 2 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 28 & 21 \\ 6 & 15 \end{bmatrix}. \end{aligned}$$

Unlike vector multiplication, matrix multiplication is not a commutative operation. So in general,

$$\boxed{AB \neq BA.}$$

Example II.18 Let $A = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 0 & 5 \end{bmatrix}$. Then AB is a (2×2) matrix, while BA is a (3×3) matrix.

$$BA = \begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6\cdot 4+1\cdot -1 & 6\cdot 1+1\cdot 3 & 6\cdot 3+1\cdot 2 \\ 4\cdot 4+2\cdot -1 & 4\cdot 1+2\cdot 3 & 4\cdot 3+2\cdot 2 \\ 0\cdot 4+5\cdot -1 & 0\cdot 1+5\cdot 3 & 0\cdot 3+5\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 9 & 20 \\ 14 & 10 & 16 \\ -5 & 15 & 10 \end{bmatrix}$$

4. Properties of Matrix Operations

Given scalars a and b and matrices A, B, and C, with dimensions such that the operations below are defined, the following properties hold:

1. $A + B = B + A$ Commutative property of addition.
2. $A + (B + C) = (A + B) + C$ Associative property of addition.
3. $A(B + C) = AB + AC$ Left distributive property over addition.
4. $(A + B)C = AC + BC$ Right distributive property over addition.
5. $A(BC) = (AB)C$ Associative property of multiplication.
6. $a(B + C) = aB + aC$
7. $(a + b)C = aC + bC$
8. $(ab)C = a(bC)$
9. $a(BC) = (aB)C$ The order of B and C must be maintained.

5. Partitioned Matrices

At the beginning of this section we said an $m \times n$ matrix A has m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

We can think of an $m \times n$ matrix as a matrix which consists of n column vectors, each with

m components:

$$A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdot & \cdot & \cdots & \vec{c}_n \end{bmatrix}.$$

We can also think of an $m \times n$ matrix as a matrix which consists of m row vectors, each with n components:

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \cdot \\ \cdot \\ \cdot \\ \vec{r}_m \end{bmatrix}.$$

We have changed our view of a matrix from a rectangular array of numbers to an array of vectors, either column vectors or row vectors. Can we go one step further and consider a matrix to be an array of matrices? It turns out that we can. An $m \times n$ matrix can be *partitioned* into smaller matrices, which we shall call *submatrices* or *blocks*, by separating the entries in the matrix using horizontal and vertical lines between the rows and columns of the matrix. Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix},$$

we can rewrite the matrix A as a *partitioned matrix* consisting of submatrices

$$A = \left[\begin{array}{cc|ccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

where $A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_{12} = \begin{bmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{bmatrix}$, $A_{13} = \begin{bmatrix} a_{16} \\ a_{26} \end{bmatrix}$,

$A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}$, $A_{22} = \begin{bmatrix} a_{33} & a_{34} & a_{35} \end{bmatrix}$, and $A_{23} = [a_{36}]$.

The partitioning of A is not unique. We can rewrite the matrix A as a different partitioned matrix by choosing a different arrangement of the horizontal and vertical lines. When we write the matrix A as a matrix consisting of row vectors,

$$A = \left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{array}\right] = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix},$$

we are actually partitioning the matrix A into submatrices,

$$A = \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix}.$$

Similarly, we can partition A into submatrices which are column vectors.

$$A = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 & \vec{c}_5 & \vec{c}_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \end{bmatrix}.$$

Example II.19 Let $A = \begin{bmatrix} 1 & 4 & 2 & 6 \\ 3 & 5 & 4 & 7 \end{bmatrix}$.

a) Partition A into a matrix with two submatrices which are row vectors.

Solution:

$$A = \left[\begin{array}{cccc} 1 & 4 & 2 & 6 \\ \hline 3 & 5 & 4 & 7 \end{array}\right] = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}.$$

b) Partition A into a matrix with four submatrices which are column vectors.

Solution:

$$A = \left[\begin{array}{c|c|c|c} 1 & 4 & 2 & 6 \\ 3 & 5 & 4 & 7 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \end{bmatrix}.$$

c) Partition A into a matrix with at least three submatrices/blocks.

Solution:

$$A = \left[\begin{array}{c|ccc} 1 & 4 & 2 & 6 \\ \hline 3 & 5 & 4 & 7 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Not only can we partition matrices, but we can also treat partitioned matrices as algebraic objects just as we do matrices which are not partitioned. This is useful when we are dealing with large matrices on a computer. Instead of storing the entire matrix in the computer memory, we can store the partitions and only call the necessary partition into memory to complete a calculation.

a. Addition and Subtraction of Partitioned Matrices

Addition and subtraction of partitioned matrices is performed in exactly the same way as addition and subtraction of non-partitioned matrices. The submatrices must have dimensions such that addition and subtraction are defined.

Example II.20 Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 3 \\ 1 & 1 & 7 \\ 9 & 0 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 6 \\ 6 & 1 & 2 \\ 0 & 9 & 3 \end{bmatrix}$.

Let A and B be partitioned as follows:

$$A = \left[\begin{array}{c|cc} 1 & 2 & 5 \\ 2 & 6 & 3 \\ \hline 1 & 1 & 7 \\ 9 & 0 & 4 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$B = \left[\begin{array}{c|cc} 7 & 3 & 2 \\ 5 & 1 & 6 \\ \hline 6 & 1 & 2 \\ 0 & 9 & 3 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

a) Calculate $A + B$.

Solution:

$$\begin{aligned} A+B &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} 8 & 5 & 7 \\ 7 & 7 & 9 \\ \hline 7 & 2 & 9 \\ 9 & 9 & 7 \end{array}\right] \\ &= \begin{bmatrix} 8 & 5 & 7 \\ 7 & 7 & 9 \\ 7 & 2 & 9 \\ 9 & 9 & 7 \end{bmatrix} \end{aligned}$$

Compare this result with adding $A + B$ when A and B are not partitioned. Is the result the same? It should be.

b) Compute $A - B$.

Solution:

$$\begin{aligned} A - B &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - B_{11} & A_{12} - B_{12} \\ A_{21} - B_{21} & A_{22} - B_{22} \end{bmatrix} \\ &= \left[\begin{array}{r|rr} -6 & -1 & 3 \\ -3 & 5 & -3 \\ \hline -5 & 0 & 5 \\ 9 & -9 & 1 \end{array}\right]. \\ &= \begin{bmatrix} -6 & -1 & 3 \\ -3 & 5 & -3 \\ -5 & 0 & 5 \\ 9 & -9 & 1 \end{bmatrix}. \end{aligned}$$

Compare your solution with $A - B$ when A and B are not partitioned. Is the result the same? Again, it should be.

We see that the sum/difference of two partitioned matrices is equal to the sum/difference of the corresponding submatrices. The matrices must have the same dimensions with respect to the submatrices and the submatrices must have the same dimensions with respect to the elements. Using our example, A and B are both 2×2 partitioned matrices each with corresponding submatrices which have dimensions:

A_{11} and B_{11} are 2×1,

A_{12} and B_{12} are 2×2,

A_{21} and B_{21} are 2×1,

A_{22} and B_{22} are 2×2.

b. Scalar Multiplication of Partitioned Matrices

Multiplying a partitioned matrix by a scalar is the same as multiplying each submatrix by that same scalar. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and c is any scalar, then

$$\boxed{cA = \begin{bmatrix} cA_{11} & cA_{12} \\ cA_{21} & cA_{22} \end{bmatrix}.}$$

Example II.21 Let $A = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and let $c = 2$.

Compute cA.

Solution:

$$\begin{aligned} cA &= \begin{bmatrix} cA_{11} & cA_{12} \\ cA_{21} & cA_{22} \end{bmatrix} \\ &= \left[\begin{array}{c|cc} 2\cdot 1 & 2\cdot 2 & 2\cdot 3 \\ \hline 2\cdot 4 & 2\cdot 5 & 2\cdot 6 \end{array}\right] \\ &= \left[\begin{array}{c|cc} 2 & 4 & 6 \\ \hline 8 & 10 & 12 \end{array}\right]. \end{aligned}$$

c. Multiplication of Partitioned Matrices

Let A be an $m \times n$ matrix and let B be an $n \times r$ matrix. If the submatrices have dimensions such that multiplication is defined, the submatrices can be multiplied exactly as we perform matrix multiplication.

Example II.22 Let $A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$,

and let $B = \left[\begin{array}{ccc} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}$.

Compute AB using submatrix multiplication.

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \\ &= [A_{11}B_{11} + A_{12}B_{22}] \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 & 4 \\ 12 & 1 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -3 \\ -1 & -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{bmatrix}. \end{aligned}$$

In the example, we multiply the submatrices of A and B exactly as we perform matrix multiplication.

$$AB = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} = [A_{11}B_{11} + A_{12}B_{22}]\,.$$

But the multiplication of A and B is only defined if the submatrices have dimensions such that the submatrix multiplication is defined.

$$\text{The dimensions of } A_{11}B_{11} = (2 \times 3)\,(3 \times 3) = (2 \times 3)\,.$$

$$\text{The dimensions of } A_{12}B_{22} = (2 \times 1)\,(1 \times 3) = (2 \times 3)\,.$$

Finally, the resulting submatrix products must have dimensions such that addition is defined.

$$\text{The dimension of } A_{11}B_{11} + A_{12}B_{22} = (2 \times 3) + (2 \times 3) = (2 \times 3)\,.$$

6. Special Matrices

A *square matrix* is a matrix in which the number of rows is equal to the number of columns. The dimension of a square matrix A is denoted $n \times n$ and we say a square matrix of dimension $n \times n$ has *order n.*

$$A = \underset{(4\times4)}{\begin{bmatrix} \underline{a_{11}} & a_{12} & a_{13} & a_{14} \\ a_{21} & \underline{a_{22}} & a_{23} & a_{24} \\ a_{31} & a_{32} & \underline{a_{33}} & a_{34} \\ a_{41} & a_{42} & a_{43} & \underline{a_{44}} \end{bmatrix}}\,.$$

A is a 4×4 matrix of order 4. The *main diagonal* of A consists of the entries a_{11}, a_{22}, a_{33}, a_{44}, underlined in the matrix A above. Only square matrices have a main diagonal.

The *transpose of a matrix* A, denoted A^T, is a matrix where the columns of A are the rows of A^T and the rows of A are the columns of A^T. If A is an $m \times n$ matrix, then A^T will be an $n \times m$ matrix.

Example II.23 Let $A = \underset{(2\times3)}{\begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}}$, then $A^T = \underset{(3\times2)}{\begin{bmatrix} 4 & -1 \\ 1 & 3 \\ 3 & 2 \end{bmatrix}}$.

Since a vector is also a matrix, it is easy to see that the transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

Example II.24 Consider the column vector $\vec{x}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ and } \vec{x}^T = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}.$$

The transpose of a matrix has some interesting properties.

1. $(AB)^T = B^T A^T$.

Example II.25 Let $A = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ 4 & 2 \\ 0 & 5 \end{bmatrix}$, then $AB = \begin{bmatrix} 28 & 21 \\ 6 & 15 \end{bmatrix}$ as previously shown. So, $(AB)^T = \begin{bmatrix} 28 & 6 \\ 21 & 15 \end{bmatrix}$. Let's calculate $B^T A^T$.

Solution:

$$\begin{aligned} B^T A^T &= \underset{(2\times3)}{\begin{bmatrix} 6 & 4 & 0 \\ 1 & 2 & 5 \end{bmatrix}} \underset{(3\times2)}{\begin{bmatrix} 4 & -1 \\ 1 & 3 \\ 3 & 2 \end{bmatrix}} \\ &= \begin{bmatrix} 28 & 6 \\ 21 & 15 \end{bmatrix} \\ &= (AB)^T. \end{aligned}$$

2. $(A+B)^T = A^T + B^T$.
(Remember, to add two matrices, they must have the same dimension.)

Example II.26 Let $A = \begin{bmatrix} 4 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$, then $A+B = \begin{bmatrix} 6 & 4 & 7 \\ 0 & 8 & 9 \end{bmatrix}$, and $(A+B)^T = \begin{bmatrix} 6 & 0 \\ 4 & 8 \\ 7 & 9 \end{bmatrix}$. Let's calculate $A^T + B^T$.

Solution:

$$A^T + B^T = \begin{bmatrix} 4 & -1 \\ 1 & 3 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} 6 & 0 \\ 4 & 8 \\ 7 & 9 \end{bmatrix} \\ &= (A+B)^T. \end{aligned}$$

3. $(A-B)^T = A^T - B^T$.
4. $(A^T)^T = A$.
5. $(cA)^T = c(A)^T$, where c is a scalar.

Try to make up some simple examples on your own to see how properties 3, 4, and 5 work.

An $n \times n$ matrix is a *symmetric matrix* if $A = A^T$. The symmetry is across the main diagonal. A symmetric matrix is always a square matrix.

Example II.27 Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix}$ and $A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix}$. Since $A = A^T$, we say the matrix A is a symmetric matrix.

The *zero matrix*, denoted $\mathbf{0}$, is an $m \times n$ matrix in which all entries are 0's. The following are some properties of arithmetic with the zero matrix. Given matrices which have dimensions such that the operations are defined:

1. $A + \mathbf{0} = \mathbf{0} + A = A$.
2. $A - A = \mathbf{0}$.
3. $\mathbf{0} - A = (-1)A$.
4. $\mathbf{0}A = \mathbf{0}$.
5. $A\mathbf{0} = \mathbf{0}$.

Example II.28 The following are examples zero matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}.$$

By property 1,we can conclude the zero matrix is the *additive identity* for matrices.

Example II.29 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \end{bmatrix}$, and $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} A + \mathbf{0} &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 1 \end{bmatrix} \\ &= A. \end{aligned}$$

The *additive inverse* of an $m \times n$ matrix A is the matrix $(-1)\,A$ such that $A + (-1)\,A = A - A = \mathbf{0}$. This is property 2.

C. EXERCISES

1. Determine the dimensions of the resulting matrix if A is a 2×4 matrix, B is a 4×6 matrix, and C is a 6×2 matrix.
a) AB
b) CA
c) AC
d) BC
e) BA

2. Let $\vec{x} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} -1 \\ 3 \\ 4 \\ -2 \\ -5 \end{bmatrix}$, $\vec{t} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$.

Using the vectors above calculate the following:

a) $\vec{x} + \vec{y}$
b) $\vec{z} + \vec{t}$
c) $\vec{y} + \vec{t}$
d) $\vec{x} \cdot \vec{x}$
e) $\vec{y} \cdot \vec{t}$
f) $\vec{z} \cdot \vec{t}$
g) Calculate the magnitude of $\vec{z}$.
h) Calculate the angle between $\vec{x}$ and $\vec{y}$.

i) Calculate $5\vec{t}$.

3. Let $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & -1 \\ 6 & 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$,
$D = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} -2 & -1 \\ 1 & 3 \end{bmatrix}$, $F = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix}$,
$G = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

a) Calculate the matrix products AB, BC, BD, CA, CB, DC.
b) Write A^T.
c) Write C^T.
d) Write B^T.
e) Calculate $(EF)^T$.
f) Calculate $F^T E^T$.
g) Find the additive inverse for matrix A.
h) Find the additive inverse for matrix C.
i) What are the scalars on the main diagonal for matrix F?
j) What are the scalars on the main diagonal for matrix B?

4. Consider the matrices:
$A = \begin{bmatrix} -3 & 0 \\ -1 & 2 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 6 & 5 \end{bmatrix}$,
$D = \begin{bmatrix} 4 & 5 & 2 \\ -3 & 0 & 1 \\ -1 & 2 & 4 \end{bmatrix}$, $E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 5 & 2 \\ 4 & -2 & 3 \end{bmatrix}$.

Compute the following:

a) AB
b) $D + E$
c) $D - E$
d) DE
e) ED
f) $-7B$
g) $3C - D$
h) $(3E)D$
i) $A(BC)$
j) $(4B)C + 2B$

5. Let $A, B, \vec{x}$, and $\vec{y}$ be the matrices:

$$A = \begin{bmatrix} -3 & -1 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Find each of the following expressions, or state why it does not exist.

a) AB
b) $A\vec{x}$
c) $A^T\vec{y}$
d) $\vec{x} \cdot \vec{y}$

6. Let $A, B, C, \vec{x}$, and $\vec{y}$ be the matrices:

$$A = \begin{bmatrix} 4 & -2 & 3 \\ -1 & 5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \\ -1 & 3 \end{bmatrix},$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

Find the following expressions or state why the expression does not exist.

a) AC
b) $AB + C^T$
c) $\vec{x} \cdot \vec{y}$
d) $3\vec{x} + B$

7. True or False (Explain your answer.)

a) If matrices A and B have dimensions such that AB exists, then BA exists.
b) If A, B, and C have dimensions such that addition is defined, then $(A + B) + C = A + (B + C)$.
c) If A and B are square matrices, $AB = BA$.

8. Which pairs of vectors are orthogonal?

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}.$$

9. Let $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ -2 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$

For each pair of vectors, either show that they are orthogonal, or compute *exactly* the cosine of the angle between the associated directed line segment.

10. Let A, B, and C be the partitioned matrices:

$$A = \left[\begin{array}{ccc} 3 & 2 & 1 \\ 2 & -1 & 0 \\ \hline -3 & 1 & 2 \end{array}\right], \quad B = \left[\begin{array}{ccc} 1 & 2 & -2 \\ -1 & 0 & 3 \\ \hline -1 & -3 & 1 \end{array}\right], \quad C = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}.$$

Compute:

a) AB
b) $A + B$
c) BC

11. Perform each of the following partitioned matrix multiplications.

a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & 3 & -1 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & -1 \\ 2 & 3 \\ 0 & -1 \\ \hline 4 & -3 \end{array}\right].$

b) $\left[\begin{array}{cc} 4 & 2 \\ 1 & 0 \\ -1 & 3 \\ \hline 2 & 1 \end{array}\right] \left[\begin{array}{cc|c} 2 & 0 & -1 \\ 1 & 3 & 0 \end{array}\right].$

c) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & -1 \\ \hline 0 & -2 & 4 \end{array}\right] \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$

12. A student taking 4 classes will study for the classes as follows:

Class		Days to study class
I	:	Monday, Wednesday, Friday
II	:	Monday, Tuesday, Thursday
III	:	Tuesday, Wednesday, Thursday
IV	:	Monday, Wednesday, Friday

The student will study a fixed length of time for each class in each session.

Class		Hours Per Study Session
I	:	2 hours
II	:	3 hours
III	:	1 hour
IV	:	3 hours

How many hours will the student study each weekday? How many total hours will the student study each week? (Hint: Develop a matrix showing the days the student studies for each class by placing a 1 in the matrix if the student studies the class on that day and a 0 in the matrix if the student does not study the class on that day. Develop a vector showing the hours per session the student studies for each class. Complete the matrix vector multiplication to find the hours the student studies each day.)

13. A truck company performs three types of services: delivery of supplies, transport of personnel, and maintenance services. The number of trucks required of one performance of each service is:

Service		Number of Trucks Required for One Performance of Service
Supply Delivery	:	3 trucks
Troop Transport	:	8 trucks
Maintenance Services	:	2 trucks

The number of times each service is performed each day is:

Day/Service		Supply	Troop	Maintenance
Monday	:	3	1	1
Tuesday	:	2	2	1
Wednesday	:	3	2	1
Thursday	:	4	1	1
Friday	:	2	3	1

How many workers are needed each weekday if each service requires 2 workers every time the service is performed?

14. The division budget officer must make annual budget estimates. The division has 3 brigades, each of which will conduct training exercises. The number of exercises each brigade will conduct in a fiscal quarter is:

Brigade/Fiscal Quarter	1st Qtr	2nd Qtr	3rd Qtr	4th Qtr
1st Brigade	2	1	1	2
2nd Brigade	1	2	1	1
3rd Brigade	2	0	1	2

For each brigade an estimate is given for expenses (in \$1,000) incurred in a single training exercise for training equipment, fuel, repair parts, and meals.

Expense (\$1,000)/Brigade	1st Brigade	2nd Brigade	3rd Brigade
Equipment	50	45	60
Fuel	200	250	200
Repair Parts	350	400	300
Meals	50	45	60

The division budget officer wants to consolidate the estimated annual budget information in a single table showing the total expenses per quarter for each exercise expense (equipment, fuel, repair parts, meals) and the total annual cost for each exercise expense. Show how the budget officer would make this table?

III. MATRICES AND SYSTEMS OF LINEAR EQUATIONS

Now that we understand how to perform algebraic operations on matrices, we can begin to look at the relationships between systems of linear equations and matrices, and apply matrix operations to the linear systems. In this chapter we will discuss two new forms for a system of linear equations: the vector equation consisting of several vectors and the matrix equation consisting of a combination of matrices and vectors. Using these new forms for linear systems, we will then explore the concepts of linear combination, span, linear independence, and linear transformation.

A. MATRIX AND VECTOR EQUATIONS

Matrix multiplication is directly applicable to systems of linear equations. Given a system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\;\cdot \\ &\;\cdot \\ &\;\cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

this system can be written in the *matrix equation form:*

$$A\vec{x} = \vec{b},$$

where A is the matrix of coefficients , $\vec{x}$ is the vector of variables and $\vec{b}$ is the vector of right-hand side constants. Multiplying $A\vec{x}$ we get the result $\vec{b}$. Therefore, a system of linear equations can be replaced by the matrix equation:

$$\boxed{A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & \cdot & & \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = \vec{b}}.$$

If we think of the matrix A as a matrix consisting of a row of column vectors, this system of linear equations can also be written in the *vector equation form:*

$$\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}.}$$

In the vector equation x_i is the ith component of the vector $\vec{x}$ of variables, where $i = 1, 2, ..., n$. The term $\vec{a}_i$ is the ith column vector of the matrix of coefficients A above, where $i = 1, 2, ..., n$. $\vec{b}$ is the vector of right hand side constants. Given the matrix equation:

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdot & \cdot & \cdot & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix},$$

we can obtain the vector equation:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix},$$

which can be written:

$$\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}.} \qquad \text{(III.2)}$$

So we can view the linear system in three distinct ways:

1. a system of linear equations,
2. a matrix equation, and
3. a vector equation.

This means the system of linear equations has the same solution set as the matrix equation, which also has the same solution set as the vector equation.

Example 3.1 Given the system of linear equations:

$$\begin{array}{rcrcrcr} -2x_1 & + & 3x_2 & + & 4x_3 & = & 2 \\ x_1 & + & 2x_2 & & & = & 1 \\ & - & 2x_2 & + & x_3 & = & -3 \end{array},$$

the matrix equation is:

$$\begin{bmatrix} -2 & 3 & 4 \\ 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix},$$

and the vector equation is:

$$x_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

B. LINEAR COMBINATIONS OF VECTORS

Given k n×1 vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ and k scalars $c_1, c_2, ..., c_k$, a *linear combination* is a sum of the form

$$\boxed{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{y},}$$

where the c_i are scalars of the vectors $\vec{v}_i$. Notice, this is a vector equation. Referring to the vector equation form of our system of linear equations $(III.2)$, we see that $\vec{b}$ is written as a linear combination of the column vectors $\vec{a}_i$ with scalars x_i.

Example 3.2 Let $\vec{u}, \vec{v}, \vec{w}, \vec{z}$ be vectors in R^n such that:

$$4\vec{u} + 2\vec{v} = -\vec{w} + \vec{z}.$$

Express $\vec{u}$ as a linear combination of $\vec{v}, \vec{w}$, and $\vec{z}$.
Solution:

$$-2\vec{v} - \vec{w} + \vec{z} = 4\vec{u}$$
$$\frac{-1}{2}\vec{v} - \frac{1}{4}\vec{w} + \frac{1}{4}\vec{z} = \vec{u}.$$

$\vec{u}$ is a linear combination of $\vec{v}, \vec{w}$, and $\vec{z}$ with scalars $\frac{-1}{2}, \frac{-1}{4}$, and $\frac{1}{4}$.

If $\vec{u}$,$\vec{v}$,$\vec{w}$,$\vec{z}$ are vectors in R^n, then the *span* of $\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}$, denoted Span$\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}$, is the set of all possible linear combinations of $\vec{u}$,$\vec{v}$,$\vec{w}$,$\vec{z}$. We say a vector $\vec{x}$ is in Span$\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}$ if, and only if, $\vec{x}$ can be written as a linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$, *and* $\vec{z}$:

$$\vec{x} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w} + c_4\vec{z}. \tag{III.3}$$

If the system III.3 is inconsistent, then $\vec{x}$ is not in Span$\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}$.

It may be easier to understand the span if we can visualize it. If $\vec{v}$ is a nonzero vector in R^3, then Span$\{\vec{v}\}$ is the set of all multiples of $\vec{v}$, a line in R^3, see figure 10. Similarly, if

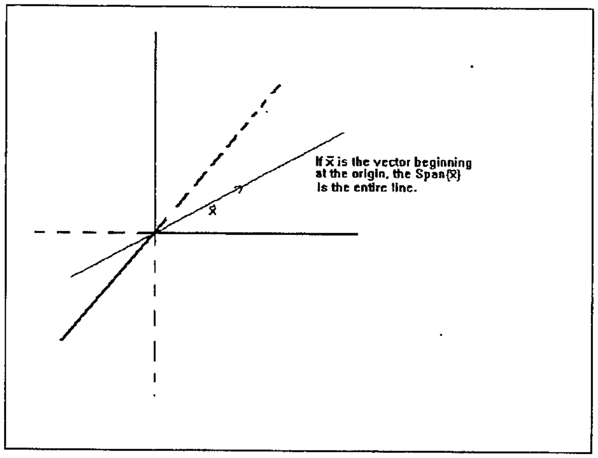

Figure 10. Span of a Vector

$\vec{u}$ and $\vec{v}$ are nonzero vectors in R^3, and $\vec{v}$ is not a multiple of $\vec{u}$, then Span $\{\vec{u}, \vec{v}\}$ is the set of linear combinations of $\vec{u}$ and $\vec{v}$, a plane in R^3, see figure 11

Example 3.3 Let $\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}$.

Show that $\vec{b}$ is in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$.

Solution:

We must show that $\vec{b}$ can be written as a linear combination of the vectors $\vec{a}_1$ and $\vec{a}_2$:

$$c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{b}.$$

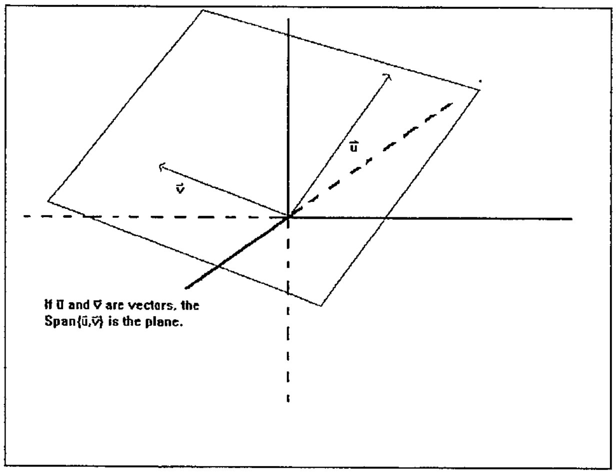

Figure 11. Span of Two Vectors

Now solve the linear system for scalars c_1 and c_2:

$$c_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

Using vector-scalar multiplication, we obtain:

$$\begin{bmatrix} 2c_1 \\ c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 2c_2 \\ -4c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

Using vector addition, we obtain:

$$\begin{bmatrix} 2c_1 & + & 3c_2 \\ c_1 & + & 2c_2 \\ -1c_1 & + & -4c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

We know two vectors are equivalent if they have identical components, so we set the components in the left side vector equal to the components in the right side vector:

$$\begin{array}{rcrcr} 2c_1 & + & 3c_2 & = & 0 \\ c_1 & + & 2c_2 & = & 1 \\ -c_1 & + & -4c_2 & = & -5. \end{array}$$

Solve the system of linear equations for c_1 and c_2:

$$\underbrace{\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 1 \\ -1 & -4 & -5 \end{bmatrix}}^{\text{Augmented matrix}} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The unique solution is $c_1 = 3, \; c_2 = -2$. Since the linear system is consistent, $\vec{b}$ is a linear combination of the vectors $\vec{a}_1$ and $\vec{a}_2$ with scalars $c_1 = 3$ and $c_2 = -2$, and $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2\}$.

C. LINEAR INDEPENDENCE OF VECTORS:

The vectors $\vec{u}$,$\vec{v}$,$\vec{w}$,$\vec{z}$ are *linearly independent* if the only solution to the homogeneous vector equation

$$\boxed{c_1\vec{u} + c_2\vec{v} + c_3\vec{w} + c_4\vec{z} = \vec{0},}$$

is the trivial solution. The vectors $\vec{u}$,$\vec{v}$,$\vec{w}$,$\vec{z}$ are *linearly dependent* if they are not linearly independent, in other words the vector equation

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} + c_4\vec{z} = \vec{0}$$

has nontrivial solutions. This means at least one c_i is nonzero. Some additional guidelines to determine linear dependence are :

1. Given a set of vectors in R^n, if the number of vectors is greater than the number of components in the vectors, then the set of vectors is linearly dependent. This statement is only intended to be applied to a set of vectors, not a system of linear equations.

2. If the zero vector, $\vec{0}$, is in a given set of vectors, then the set of vectors is linearly dependent.

Example 3.4 The following vectors are linearly dependent:

$$\underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -6 \end{bmatrix}}_{\text{4 vectors each with only 3 entries}}.$$

Example 3.5 The following vectors are linearly dependent:

$$\underbrace{\begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}}_{\text{The zero vector is a member of this set}}.$$

Example 3.6 Determine whether $\vec{x}, \vec{y}$, and $\vec{z}$ are linearly dependent or linearly independent:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution:

Start by solving the homogeneous equation:

$$c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}.$$

This gives us

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using vector scalar multiplication we obtain

$$\begin{bmatrix} c_1 \\ c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using vector addition gives us

$$\begin{bmatrix} c_1 & + & c_2 & + & c_3 \\ c_1 & + & c_2 & + & 0 \\ c_1 & + & 0 & + & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Setting the entries of the two vectors equal, we get the linear system

$$\begin{array}{ccccccc} c_1 & + & c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ c_1 & & & & & = & 0, \end{array}$$

which has only the trivial solution $c_1 = c_2 = c_3 = 0$. Therefore, the vectors are linearly independent.

Example 3.7 Determine whether the following vectors are linearly dependent or linearly independent:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ -13 \\ 17 \end{bmatrix}.$$

Solution:

Solve the homogeneous system of linear equations:

$$\begin{bmatrix} 1 & -2 & 4 & 0 \\ -1 & -4 & -13 & 0 \\ 2 & 2 & 17 & 0 \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & -6 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

x_1 and x_2 are pivot variables and x_3 is a free variable. Therefore, the linear system has nontrivial solutions and the three vectors are linearly dependent.

The *rank* of a matrix is the number of linearly independent rows or columns in the matrix. We can use what we have learned about linear independence to determine the rank of a matrix. To determine the rank of a matrix, use Gaussian elimination to reduce a matrix to row echelon form. The nonzero rows are linearly independent. Additionally, the columns with pivots are linearly independent.

Example 3.8 Let $A = \begin{bmatrix} 1 & -2 & 1 & 4 & -1 \\ 4 & -8 & -6 & 4 & -2 \\ 1 & -2 & 16 & -10 & -1 \\ 3 & -6 & 18 & -2 & -3 \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} \underline{1} & -2 & 1 & 4 & -1 \\ 0 & 0 & \underline{10} & 12 & -2 \\ 0 & 0 & 0 & \underline{32} & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

The rank of $A = 3$. The first three rows of A are linearly independent, the first, third, and fourth columns (columns with pivots underlined) are linearly independent. Notice, there is only one linearly dependent row vector, the fourth row; but, there are two linearly dependent column vectors, the second and the fifth columns.

The number of linearly independent rows is equal to the number of linearly independent columns in a matrix. The same is not necessarily true for linearly dependent rows and columns.

D. LINEAR TRANSFORMATIONS

In this section we will study vector-valued functions of the form

$$\boxed{T(\vec{x}) = y,}$$

where T takes a vector $\vec{x}$ in R^n and changes it into a vector $\vec{y}$ in R^m. We say that:

$$\boxed{T : R^n \to R^m}$$

is a *transformation* from R^n to R^m, with a rule T, which assigns to each independent *vector variable* $\vec{x}$ in the **domain** R^n, one and only one dependent vector variable $\vec{y}$ in R^m. $T(\vec{x})$ is the *image* of $\vec{x}$ under the transformation T. The **range** of T is the set of all images $T(\vec{x})$ in R^m. Let's look at an example.

Example 3.9 Define a transformation $T : R^2 \to R^2$ by the rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$. When $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$T(\vec{x}) = \vec{y}$$
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

We are concerned with transformations from R^n to R^m which are associated with matrices, called *matrix transformations*. Given an $m \times n$ matrix A and a column vector $\vec{x}$ in R^n, the product $A\vec{x}$ is an $m \times 1$ column vector $\vec{b}$ in R^m. We say $\vec{x}$ in R^n is transformed to $\vec{b}$ in R^m by the matrix A. Using matrix notation, we can define the transformation $T : R^n \to R^m$ by the notation:

$$T(\vec{x}) = A\vec{x}, \text{ where } A\vec{x} = \vec{b}.$$

We are used to working with the equation $A\vec{x} = \vec{b}$, so we have been working with matrix transformations every time we found a vector $\vec{x}$ for which the linear system $A\vec{x} = \vec{b}$ was consistent. The next example focuses on the rule, image and range of a transformation. [Ref. 4]

Example 3.10 Let $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix}$.

a) Define the rule for the transformation $T : R^2 \rightarrow R^3$ by $T(\vec{x}) = A\vec{x}$. This gives us:

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + 4x_2 \\ -1x_1 - 2x_2 \\ 3x_1 + 6x_2 \end{bmatrix} \end{aligned}$$

This is the rule T for the transformation.

b) Find $T(\vec{u})$, the image of $\vec{u}$ under T.

Solution:

$$\begin{aligned} T(\vec{u}) &= A\vec{u} \\ &= \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \end{aligned}$$

c) Find $\vec{x}$ in R^2, whose image under T is $\vec{b}$. In other words, solve the transformation $T(\vec{x}) = \vec{b}$, for $\vec{x}$. This is a linear system $A\vec{x} = \vec{b}$, which we can already solve:

Solution:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}. \end{aligned}$$

This is the matrix equation for a system of linear equations. We can solve this system by reducing the augmented matrix.

$$\begin{bmatrix} 2 & 4 & 2 \\ -1 & -2 & -1 \\ 3 & 6 & 3 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives us the solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 + 1 \\ x_2 \end{bmatrix}.$$

d) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?

Yes, x_2 is an independent variable. There are infinitely many vectors whose images under T are $\vec{b}$.

e) Is $\vec{c}$ in the range of the transformation T ? Recall, the range of T is the set of all images under T, so we need to determine if $\vec{c}$ is the image for some $\vec{x}$ in R^2. Again we will be solving a linear system:

Solution:

$$\begin{aligned} T(x) &= \vec{c} \\ A\vec{x} &= \vec{c} \\ \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 6 \\ 5 \end{bmatrix} \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 2 & 4 & 4 \\ -1 & -2 & 6 \\ 3 & 6 & 5 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 2 & 4 & 4 \\ 0 & 0 & 8 \\ 0 & 0 & -1 \end{bmatrix}.$$

This system is inconsistent; therefore, $\vec{c}$ is not an image for any $\vec{x}$ in R^2 and $\vec{c}$ is not in the range of T.

Example 3.11 Given $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 3 & -1 \end{bmatrix}$, if $T : R^2 \to R^3$, what is the rule T for the transformation?

Solution:

Since $T(\vec{x}) = A\vec{x}$, the rule is:

$$\begin{bmatrix} 1 & 2 \\ -2 & 0 \\ 3 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{vector in } R^2} = \overset{\text{Rule}}{\underbrace{\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 \\ 3x_1 - x_2 \end{bmatrix}}_{\text{vector in } R^3}}.$$

We should see a relationship between matrix transformations and our study of the linear system $A\vec{x} = \vec{b}$. In solving the system of equations $A\vec{x} = \vec{b}$, we have been finding all vectors $\vec{x}$ in R^n, which are transformed into the vector $\vec{b}$ in R^m, under the "action" of multiplication by A. It turns out that given a transformation $T(\vec{x}) = A\vec{x}$, this matrix transformation $T : R^n \to R^m$ is a *linear transformation* if:

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in the domain of the transformation T.
2. $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ and all scalars c

These two rules state that vector addition and scalar multiplication are preserved when a transformation is linear.

If a transformation satisfies both rules for all $\vec{u}$ and $\vec{v}$ in the domain of T and all scalars c and d, then the transformation must be linear. These rules can be generalized for more than two vectors:

$$\boxed{T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_pT(\vec{v}_p).}$$

This is called the *superposition principle* in engineering and physics.

Example 3.12 Determine if $T(\vec{x}) = \vec{y}$ is a linear transformation when $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$.

Solution:

A transformation is linear if the following rules hold

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$,$\vec{v}$ in the domain of T.
2. $T(c\vec{u}) = cT(\vec{u})$ for all vectors $\vec{u}$ in the domain of T and all scalars c.Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then:

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right)$$

$$
\begin{aligned}
&= \begin{bmatrix} 2(u_1+v_1) \\ u_2+v_2 \end{bmatrix} \\
&= \begin{bmatrix} 2u_1+2v_1 \\ u_2+v_2 \end{bmatrix} \\
&= \begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix} \\
&= T(\vec{u}) + T(\vec{v}).
\end{aligned}
$$

The first rule holds, but that is not enough. Both rules must hold! Check the second rule. Let c be a scalar:

$$
\begin{aligned}
T(c\vec{u}) &= T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2cu_1 \\ cu_2 \end{bmatrix} \\
&= c\begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix} \\
&= cT(\vec{u}).
\end{aligned}
$$

The second rule holds. Since both rules hold, the transformation is linear.

Let's think about this a second: in order to show that a transformation is linear, you must show that both vector addition and scalar multiplication are preserved. If one of the rules fails, you can conclude that the transformation is not linear without checking the other rule. Why? Because even if the second rule holds, the first rule already failed.

Let's look at a few more examples of linear transformations.

Example 3.13 Given an angle θ and a matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the transformation $T : R^2 \to R^2$, defined by the rule $T(\vec{x}) = A\vec{x}$, rotates the directed line segment associated with the vector $\vec{x}$ counterclockwise through the angle θ.

Let $\theta = 90°$ and let $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then:

$$
\begin{aligned}
T(\vec{x}) &= A\vec{x} \\
&= \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

If you plot the directed line segments associated with the vectors $\vec{x}$ and $A\vec{x}$, you will see that $A\vec{x}$ is the result of rotating the $\vec{x}$ counterclockwise by 90°.

Let $\theta = \frac{\pi}{4}$ and let $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then:

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \end{aligned}$$

Again, if you plot the directed line segments associated with the vectors $\vec{x}$ and $A\vec{x}$, you will see that $A\vec{x}$ is the result of rotating $\vec{x}$ counterclockwise through the angle $\frac{\pi}{4}$.

In the above example the linear transformation rotates a point in R^2 about the origin, through the angle θ. This linear transformation is called a *rotation*.

Example 3.14 We can define a transformation $T : R^n \to R^n$ by the rule $T(\vec{x}) = r\vec{x}$, where r is a scalar. Let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and let $r = \frac{1}{2}$:

$$\begin{aligned} T(\vec{x}) &= r\vec{x} \\ &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

The resulting directed line segment associated with the vector is shorter than the original associated directed line segment.

Using the same $\vec{x}$, let $r = 3$:

$$\begin{aligned} T(\vec{x}) &= r\vec{x} \\ &= 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}. \end{aligned}$$

This time the resulting directed line segment associated with the vector is longer than the original associated directed line segment.

When $0 \leq |r| < 1$, the transformation T compresses the directed line segment associated with the vector $\vec{x}$ into a shorter directed line segment by a factor of r; this is called a *contraction*. When $|r| > 1$, the transformation T stretches the associated directed line segment into a longer directed line segment by a factor of r; this is called a *dilation*. If $r < 0$, the associated directed line segment is also pointed in the opposite direction. Can you relate this transformation to a matrix transformation? What matrix would you use to define the transformation $T(\vec{x}) = A\vec{x}$ and get the same result as $T(\vec{x}) = r\vec{x}$? Try $\begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}$.

Let $r = 3, \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}\vec{x} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}. \end{aligned}$$

This is the same result as the transformation $T(\vec{x}) = r\vec{x}$. If you check to see whether the operations of vector addition and scalar multiplication are preserved, you will find that this is also a linear transformation. One final example:

Example 3.15 If we have a line l which passes through the origin, we can define a transformation $T(\vec{x}) = A\vec{x}$ which reflects each point in the plane across the line l. This transformation is called a *reflection.*

Let l be the y-axis. The standard matrix for reflection across the y-axis is $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then the transformation $T : R^2 \to R^2$ is:

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \end{bmatrix} \end{aligned}$$

If you plot the directed line segment associated with the vectors $\vec{x}$ and $A\vec{x}$, you will see that the new directed line segment is the mirror image of the original directed line segment across the y-axis.

How about a reflection across the x-axis? The standard matrix for reflection about the x-axis is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Using the same vector $\vec{x}$, the transformation $T : R^2 \to R^2$ is:

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

Plot the associated directed line segments to see the reflection.

Let's try one more reflection: Across the line $x = y$. The standard matrix for reflection about the line $x = y$ is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Using the same vector $\vec{x}$, the transformation $T : R^2 \to R^2$ is:

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

Plot the directed line segments to see the reflection across the line $x = y$.

E. EXERCISES

1. Write the following systems of linear equations in matrix equation form $A\vec{x} = \vec{b}$.

a) $\left[\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & 4x_3 & = & 0 \\ -2x_1 & - & 3x_2 & + & 3x_3 & = & -2 \\ x_1 & + & 2x_2 & - & x_3 & = & 2 \end{array}\right]$

b) $\left[\begin{array}{rcrcrcrcr} & & 4x_2 & - & x_3 & + & 2x_4 & = & 5 \\ 2x_1 & + & x_2 & + & 3x_3 & & & = & 6 \\ 4x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & = & 2 \end{array}\right]$

2. Write the matrix equation as a vector equation.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ -6 \end{bmatrix}$$

3. Write a system of equations which is equivalent to the given vector equation.

$$x_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4. Write a vector equation which is equivalent to the given system of equations.

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & = & 6 \\ x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & 3x_2 & + & x_3 & = & 4 \end{array}$$

5. Write the following system of linear equations as a matrix equation and as a vector equation.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 1 \\ -x_1 & - & 2x_2 & + & 3x_3 & - & x_4 & = & -1 \\ 2x_1 & - & x_2 & + & 5x_3 & - & x_4 & = & 4 \end{array}$$

6. Let $\vec{u}, \vec{v}, \vec{w}, \vec{x}$ be vectors in R^4 such that $-4\left(\vec{x} - \vec{u}\right) = 5\left(\vec{v} - 3\vec{w}\right)$. Express $\vec{x}$ as a linear combination of $\vec{u}, \vec{v}$, and $\vec{w}$.

7. Determine if $\vec{b}$ is a linear combination of the columns of the matrix A.

a) $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & -3 & -2 \\ -2 & 2 & 6 \\ 2 & -8 & -3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$

8. Do $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span R^4?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

9. Given $\vec{x}$ and A below, is $\vec{x}$ in the plane spanned by the columns of A?

$$\vec{x} = \begin{bmatrix} -5 \\ -3 \\ 4 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 \\ 4 & 5 \\ -3 & -2 \end{bmatrix}$$

10. Determine if $\vec{b}$ is in the Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

11. This exercise compares a set of vectors with the span of a set of vectors. [Ref. 4] Let $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & -1 \\ 6 & -2 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 8 \\ 6 \end{bmatrix}$. Denote the columns of A by $\vec{a}_1, \vec{a}_2$, and $\vec{a}_3$ and let $W =$Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

a) How many vectors are in $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?
b) Is $\vec{b}$ in $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$?
c) Is $\vec{b}$ in W?
d) How many vectors are in W?

e) Show that $\vec{a}_1$ is in W. (Hint: row operations are not necessary).

12. Given $\vec{x} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \vec{a} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 9 \\ -3 \\ -9 \end{bmatrix}$

a) Is $\vec{a} \in$Span$(\vec{x}, \vec{y})$?
b) Is $\vec{b} \in$Span$(\vec{x}, \vec{y})$?

13. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & -5 \\ 4 & 8 & 5 \end{bmatrix}$$

a) Determine the rank of A.

b) Given $\vec{x} = \begin{bmatrix} -1 \\ 8 \\ -14 \end{bmatrix}$, is $\vec{x}$ in the plane spanned by the columns of A?

14. Solve the equation $A\vec{x} = \vec{b}$ with A and $\vec{b}$ given below. Write the solution as a vector.

$$A = \begin{bmatrix} 7 & -2 & -5 \\ 3 & 4 & 1 \\ 0 & -6 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -6 \\ -14 \\ 20 \end{bmatrix}$$

15. Let $T(\vec{x}) = A\vec{x}$. Define the rule for the transformation. (Compute $A\vec{x}$).

a) $A = \begin{bmatrix} 4 & -2 & 5 \\ -1 & 3 & 8 \\ 2 & 1 & -2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} r \\ s \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

16. For A=$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, define $T : R^2 \to R^2$ by $T(\vec{x}) = A\vec{x}$.
Find the images under T of $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

17. Define $T(\vec{x}) = A\vec{x}$, find an $\vec{x}$ whose image under T is $\vec{b}$. Determine if $\vec{x}$ is unique.

$$A = \begin{bmatrix} 3 & -5 & -1 \\ 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -7 \\ 14 \\ 24 \end{bmatrix}$$

18. Find all $\vec{x}$ in R^4 that are mapped into the zero vector by the transformation $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 7 & 0 & 3 & 1 \\ -5 & 1 & -3 & 3 \\ 6 & 0 & 4 & -2 \end{bmatrix}$$

19. a) Let A be a 4×6 matrix. What values must a and b have to define a transformation $T : R^a \to R^b$ by $T(\vec{x}) = A\vec{x}$.

b) Let A be a 5×3 matrix. What values must a and b have to define a transformation $T : R^a \to R^b$ by $T(\vec{x}) = A\vec{x}$.

c) Let A be a 4×4 matrix. What values must a and b have to define a transformation $T : R^a \to R^b$ by $T(\vec{x}) = A\vec{x}$.

20. Let $A = \begin{bmatrix} 7 & 0 & 3 & 1 \\ -5 & 1 & -3 & 3 \\ 6 & 0 & 4 & -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 19 \\ -10 \\ 12 \end{bmatrix}$. Is $\vec{b}$ in the range of the linear transformation $T(\vec{x}) = A\vec{x}$?

21. Use a rectangular coordinate system to plot $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$, and their images under the transformation T. State what type of linear transformation T is.

$$T(\vec{x}) = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

22. Use a rectangular coordinate system to plot $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ -1 \end{bmatrix}$, and their images under the transformation T. State what type of linear transformation T is.

$$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

23. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$, $r = 2$.

a) Define $T(\vec{x}) = A\vec{x}$.
b) Compute $T(\vec{u} + \vec{v})$ and $T(\vec{u}) + T(\vec{v})$.
c) Compute $T(r\vec{u})$ and $rT(\vec{u})$.
d) Is the transformation defined by $T(\vec{x}) = A\vec{x}$ linear?

24. Suppose A is an $m \times n$ matrix, and its transformation $T : R^n \to R^m$ is defined by $T(\vec{x}) = A\vec{x}$. Find the rule of the transformation $T(\vec{x})$ and give the domain and range of $T(\vec{x})$. For example, if $A = \begin{bmatrix} 2 & -2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix}$, then the rule is:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 2 & -2 \\ -1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ -x_1 \\ 3x_1 + x_2 \end{bmatrix}$$

and the domain and range are R^2 and R^3, respectively.

a) Find the rule, domain and range for $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$.

b) Find the rule, domain and range for $A = \begin{bmatrix} 3 & -2 & 4 & 1 \end{bmatrix}$.

25. Determine if T is a linear transformation. Give the domain and range of T and if T is linear, find A such that $T(\vec{x}) = A\vec{x}$.

a) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y^2 \end{bmatrix}$

b) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 2y \\ x - 2y + 3z \end{bmatrix}$

26. For both transformations below:

a) Determine whether or not T is a linear transformation.
b) Give the domain and range of T.
c) For both transformations calculate $T(\vec{x})$ when $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.
d) If T is linear, find A such that $T(\vec{x}) = A\vec{x}$.
e) If T is linear, use A to find $T(\vec{x})$ when $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \\ x_2 \end{bmatrix}$ $\qquad$ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2 \\ x_2 \end{bmatrix}$

IV. MATRIX ALGEBRA

This chapter begins with the introduction of two new types of matrices, the identity matrix and elementary matrices. These matrices are essential in the development of the inverse of a matrix and matrix factorization. Next we learn two new methods for solving linear systems using inverses and LU decomposition. The chapter finishes with an introduction to the determinant of a matrix and another method for finding the solution to a linear system known as Cramer's rule.

A. ELEMENTARY MATRICES

We begin our study of elementary matrices by defining the identity matrix. The *identity matrix* is a square matrix with 1's in every position on the main diagonal. All other entries are 0's. An identity matrix of order n is denoted I_n.

The following matrices are examples of identity matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Given an $m \times n$ matrix A and an identity matrix of appropriate order, we have

$$\boxed{I_m A_{m\times n} = A_{m\times n} \text{ and } A_{m\times n} I_n = A_{m\times n}.}$$

Thus, when we multiply the matrix A on either side by the identity matrix, the result is the matrix A. For this reason, the identity matrix is called the multiplicative identity for the matrix A.

Example 4.1 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Then

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = A,$$

and

$$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = A.$$

An *elementary matrix* E_i is a matrix which is formed by performing a single elementary row operation on an $n \times n$ identity matrix. There are three types of elementary matrices, one for each of the three elementary row operations. An example of each type follows.

Example 4.2 Multiply a row by a nonzero constant:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_1 = \overset{\text{Multiply row 2 by 3.}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Example 4.3 Interchange two rows:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2 = \overset{\text{Interchange row 1 and row 2.}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Example 4.4 Add a multiple of one row to another row:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3 = \overset{\text{Add 4 times row 2 to row 3.}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}.$$

Theorem 4.1 *Let E be the $m \times m$ elementary matrix formed by performing a single elementary row operation e on an $m \times m$ identity matrix I_m. Then*

$$E = e(I_m).$$

Thus, for every $m \times n$ matrix A, the elementary row operation e can be performed on A by multiplying A on the left by the corresponding elementary matrix

$$e(A) = EA.$$

Example 4.5 Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix}$ and let $E = \overset{\text{Add -2 times row 1 to row 2.}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$. Then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix}.$$

Multiplying the matrix A by the matrix E adds -2 times row 1 of the matrix A to row 2 of the matrix A.

The process of multiplying a matrix by an elementary matrix can be repeated.

Example 4.6 Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix}$,

$$E_1 = \overset{\text{Add -2 times row 1 to row 2.}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \text{ and } E_2 = \overset{\text{Add -4 times row 1 to row 3.}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}.$$

We can compute:

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix}$$

and

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 0 & -7 & -5 & 9 \end{bmatrix}.$$

Compare the results of multiplying A on the left by E_1 and E_2 with the results of performing the corresponding row operations on A :

$$\underset{\text{Add -2 times row 1 to row 2.}}{\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix}} \sim \underset{\text{Add -4 times row 1 to row 3.}}{\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix}} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 0 & -7 & -5 & 9 \end{bmatrix}.$$

A nice feature of elementary row operations is that they can be reversed.

Example 4.7 Use the same matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix} \text{ and } E_1 = \overset{\text{Add -2 times row 1 to row 2.}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}. \text{ Then}$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix}.$$

Reverse the operation by adding 2 times row 1 to row 2. Let $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$E_2\left(E_1A\right) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -4 & -8 & 6 \\ 4 & 1 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & -2 & 4 \\ 4 & 1 & 7 & 5 \end{bmatrix} = A.$$

The result of the multiplication is the original matrix A.

Let's think about what this means for elementary matrices. We can take the identity matrix I and change it into an elementary matrix E by performing an elementary row operation. Then we can take that elementary matrix E and change it back into the identity matrix I by performing the reverse operation on E. The reverse operations are [Ref. 5]:

Row Operation	Reverse Row Operation
1. Multiply row i by nonzero constant c.	1. Multiply row i by nonzero constant $\frac{1}{c}$.
2. Interchange row i and row j.	2. Interchange row i and row j.
3. Add c times row i to row j.	3. Add $-c$ times row i to row j.

Let $e_1(I) = E_1$, an elementary matrix formed by performing a row operation on I

Let $e_2(I) = E_2$, the elementary matrix formed by performing the corresponding reverse row operation on I.

Then

$$E_2E_1 = I.$$

Since E_1 and E_2 perform reverse operations, we can also say

$$E_1E_2 = I.$$

B. INVERSE OF A MATRIX

In this section we will study the inverse of a matrix and the concept of singular and nonsingular matrices. Additionally, we will learn how to compute the inverse of a matrix.

We begin with the multiplicative inverse. The *multiplicative inverse* of an $n \times n$ matrix A is a matrix B such that

$$AB = BA = I.$$

If such a matrix B exists, we say that the matrix A is *invertible* and B is the *inverse* of A, denoted A^{-1}. Then the above equation can be written

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Example 4.8 Let $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Compute AB and BA.

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

AB is the identity matrix.

$$\begin{aligned} BA &= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

BA is also the identity matrix. Since $AB = I$, and $BA = I$, We say A is invertible and B is the inverse of A. We can also say B is invertible and A is the inverse of B.

Only square matrices will have inverses, but not all square matrices are invertible.

Example 4.9 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Now compute AB.

Solution:

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 \\ 0 & \underline{0} \end{bmatrix}. \end{aligned}$$

We will never find a matrix B which is the inverse of A, because no matter which values are in the matrix B, the multiplication AB will always result in zeros in the second row. There is no way to get a 1 in the second row, second column position (underlined above).

The matrix A does not have an inverse.

Theorem 4.2 *If a matrix A is invertible, its inverse is unique.*

How do we know the inverse is unique? Suppose the inverse is not unique. Then the matrix A has two distinct inverses B and C. Since B and C are inverses of A, we can conclude that $AB = BA = I$ and $AC = CA = I$. Substituting into the equations below,

$$B = BI = B(AC) = (BA)C = IC = C.$$

We see from these calculations that $B = C$. But, we started out saying that B and C were distinct inverses of A and came to a contradiction that $B = C$. All of our calculations are accurate; therefore, our assumption that B and C are distinct must be incorrect. Thus the matrix A has only one inverse.

We say the matrix A is *nonsingular* if the multiplicative inverse of A exists. If no multiplicative inverse of A exists, then A is *singular*. In our previous examples, the matrices $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$, and $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ are nonsingular matrices and the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is singular.

Three useful facts about invertible matrices are:

1. If A is a nonsingular matrix, then A^{-1} is also a nonsingular matrix:

$$\left(A^{-1}\right)^{-1} = A.$$

2. If A and B are $n \times n$ matrices which are both invertible, then the product AB is also invertible:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

We can generalize this fact. The product of k $n \times n$ invertible matrices is invertible:

$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}.$$

3. If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

By now you are probably asking: How do we know if a matrix has an inverse and how do we find the inverse of a matrix? There are simple formulas for the inverse of 1×1

and 2×2 matrices. Given a 1×1 matrix $A = [a]$, then

$$\boxed{A^{-1} = \left[\tfrac{1}{a}\right]}$$

For a 2×2 matrix, the formula is more involved. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then the matrix A is nonsingular and

$$\boxed{A^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.} \tag{IV.4}$$

If $ad - bc = 0$, clearly we cannot find a solution for A^{-1} and the matrix A is singular/not invertible. Let's look at our invertible matrices above.

Example 4.10 Given the matrix $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and using the formula in equation IV.4,

$$\begin{aligned} ad - bc &= (2 \cdot 3) - (-5 \cdot -1) \\ &= 6 - 5 \\ &= 1. \end{aligned}$$

Since this value is nonzero, we know the matrix A is invertible and

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}. \\ &= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

This is the inverse of the matrix A. Similarly, we could start with the matrix A^{-1}, apply the formula in equation IV.4 and the result would be the matrix A. Try this yourself.

What if our matrix is larger than a 2×2 matrix? We'll start with an example and follow the example with an explanation.

Example 4.11 Let $A = \begin{bmatrix} 4 & -2 & 6 \\ 2 & 0 & -4 \\ 2 & 0 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Use Gauss-Jordan elimination to reduce the matrix A to the matrix I. Simultaneously perform the same row operations on I as we perform on A:

$$A = \begin{bmatrix} 4 & -2 & 6 \\ 2 & 0 & -4 \\ 2 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Add $\frac{-1}{2}$ times row 1 to row2 and row 3:

$$A_1 = \begin{bmatrix} 4 & -2 & 6 \\ 0 & 1 & -7 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}.$$

Add -1 times row 2 to row 3:

$$A_2 = \begin{bmatrix} 4 & -2 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Multiply row 3 by $\frac{1}{6}$:

$$A_3 = \begin{bmatrix} 4 & -2 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -1/6 & 1/6 \end{bmatrix}.$$

Add 7 times row 3 to row 2 and add -6 times row 3 to row 1:

$$A_4 = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 1 & 1 & -1 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}.$$

Add 2 times row 2 to row 1:

$$A_5 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_5 = \begin{bmatrix} 0 & 2/3 & 8/6 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}.$$

Multiply row 1 by $\frac{1}{4}$:

$$A_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_6 = \begin{bmatrix} 0 & 1/6 & 2/6 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}$$

What does all of this mean? Using elementary matrices, we say A is row equivalent to I_n if there exists a sequence of row operations, $E_1, E_2, ...E_n$, such that

$$E_n E_{n-1} \cdots E_1 A = I_n.$$

Since the product of invertible matrices is invertible, we can multiply both sides of the equation by the inverse of the product:

$$(E_n E_{n-1} \cdots E_1)^{-1}(E_n E_{n-1} \cdots E_1)A = (E_n E_{n-1} \cdots E_1)^{-1} I_n.$$

This gives us

$$I_n A = (E_n E_{n-1} \cdots E_1)^{-1} I_n,$$

which is simply

$$A = (E_n E_{n-1} \cdots E_1)^{-1}.$$

This means that A is the inverse of the matrix $E_n E_{n-1} \cdots E_1$, in other words, A is an invertible matrix. Therefore,

$$A^{-1} = \left[(E_n E_{n-1} \cdots E_1)^{-1}\right]^{-1},$$

which simplifies to

$$A^{-1} = (E_n E_{n-1} \cdots E_1).$$

Thus

$$A^{-1} = (E_n E_{n-1} \cdots E_1) I_n,$$

and we see that the matrix A^{-1} is the result of performing the same row operations on I_n as we performed on the matrix A when we transformed A to I_n. This is the idea behind the next theorem.

Theorem 4.3 *An $n \times n$ matrix A is nonsingular if and only if A can be reduced to the identity matrix I_n with a sequence of row operations. The sequence of elementary row operations which reduces the matrix A to the matrix I_n also transforms the identity matrix I_n into A^{-1}, the inverse of A.*

The next example refers back to example 4.11,

Example 4.12 When we perform elementary row operations on $A = \begin{bmatrix} 4 & -2 & 6 \\ 2 & 0 & -4 \\ 2 & 0 & 2 \end{bmatrix}$

to show that A is row equivalent to $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we simultaneously perform the same row operations on $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and show that I is row equivalent to the matrix $B_6 = \begin{bmatrix} 0 & 1/6 & 2/6 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}$. By Theorem 4.3, B_6 must be the inverse of A. We can check to see if $I_6 = A^{-1}$ by computing $AB_6 = B_6A = I$. If the equality holds, then we have found A^{-1}.

Solution:

$$AB_6 = \begin{bmatrix} 4 & -2 & 6 \\ 2 & 0 & -4 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1/6 & 2/6 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now try:

$$B_6A = \begin{bmatrix} 0 & 1/6 & 2/6 \\ -1/2 & -1/6 & 7/6 \\ 0 & -1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 4 & -2 & 6 \\ 2 & 0 & -4 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It works! We have successfully found A^{-1}.

Using Theorem 4.3, we can develop a procedure for finding A^{-1}.

1. Place A and I side by side to form an augmented matrix

$$[A\ I]$$

2. Perform row operations on the augmented matrix $[A\ I]$. If A can be reduced to I, then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$ and we can read A^{-1} off of the augmented matrix. Otherwise, A does not have an inverse and we say A is singular.

Here is another example.

Example 4.13 Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 3 & -5 & 2 \end{bmatrix}$. Find A^{-1}.

Solution:

$$[A\,I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 3 & -5 & 2 & 0 & 0 & 1 \end{array}\right].$$

Add -3 times row 1 to row 3:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & -3 & 0 & 1 \end{array}\right].$$

Multiply row 2 by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & -2 & 2 & -3 & 0 & 1 \end{array}\right].$$

Add 1 times row 2 to row 1 and add 2 times row 2 to row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right].$$

Add $\frac{1}{2}$ times row 3 to row 1 and row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -3/2 & 1 & 1/2 \\ 0 & 0 & 1 & -3 & 1 & 1 \end{array}\right].$$

We have transformed A into I; therefore, our theorem tells us that we have transformed I into A^{-1}. Read A^{-1} off of the matrix, $A^{-1} = \begin{bmatrix} 1/2 & 1 & 1/2 \\ -3/2 & 1 & 1/2 \\ -3 & 1 & 1 \end{bmatrix}$. If it were not possible to transform A into I, then we would conclude that A is singular.

Computing the inverse of a matrix is an algebraically expensive operation which can lose accuracy quickly when using a computer program. However, inverses have theoretical importance which will be useful in future courses. For our purposes, when we have relatively small matrices, we can use the inverse to solve the matrix equation $A\vec{x} = \vec{b}$. If A is an invertible matrix, the unique solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ can found as follows:

$$A\vec{x} = \vec{b}$$

$$\begin{aligned}(A^{-1}A)\vec{x} &= \left(A^{-1}\right)\vec{b}\\ I\vec{x} &= A^{-1}\vec{b}\\ \vec{x} &= A^{-1}\vec{b}.\end{aligned}$$

Solving linear systems is almost always easier and faster using Gaussian elimination rather than computing an inverse.

Example 4.14 Solve the system of linear equations using the inverse.

$$\begin{array}{rcrcr} 3x_1 & + & 8x_2 & = & 18\\ 1x_1 & + & 3x_2 & = & 7\end{array}$$

Solution:

Write the matrix equation $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 3 & 8\\ 1 & 3\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix} = \begin{bmatrix} 18\\ 7\end{bmatrix}.$$

Find A^{-1}:

$$\begin{aligned}A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b\\ -c & a\end{bmatrix}\\ &= \frac{1}{9-8}\begin{bmatrix} 3 & -8\\ -1 & 3\end{bmatrix}\\ &= 1\begin{bmatrix} 3 & -8\\ -1 & 3\end{bmatrix}\\ &= \begin{bmatrix} 3 & -8\\ -1 & 3\end{bmatrix}.\end{aligned}$$

Solve the equation $\vec{x} = A^{-1}\vec{b}$:

$$\begin{aligned}\vec{x} &= \begin{bmatrix} 3 & -8\\ -1 & 3\end{bmatrix}\begin{bmatrix} 18\\ 7\end{bmatrix}\\ &= \begin{bmatrix} -2\\ 3\end{bmatrix}, \text{ a unique solution.}\end{aligned}$$

C. LU DECOMPOSITION

In this section we introduce another special type of matrix, the triangular matrix.

We will then learn how to solve the matrix equation $A\vec{x} = \vec{b}$ by factoring the matrix A into two triangular matrices.

An *upper triangular matrix* is an $n \times n$ matrix in which all entries below the main diagonal are 0's. The entries above and on the main diagonal may or may not be 0's.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \text{ is an upper triangular matrix.}$$

It is easy to show that the product of two upper triangular matrices is also an upper triangular matrix, as suggested by the following example.

Example 4.15 Let $A = \begin{bmatrix} -2 & 4 & 3 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 4 & -2 \\ 0 & 3 & 6 & -5 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & 12 & 22 & 35 \\ 0 & 3 & 10 & 34 \\ 0 & 0 & 14 & 49 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, a *lower triangular matrix* is an $n \times n$ matrix in which all entries above the main diagonal are 0's. The entries below and on the main diagonal may or may not be 0's.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ is a lower triangular matrix.}$$

The product of two lower triangular matrices is also lower triangular.

Example 4.16 $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 6 & 4 & 2 \end{bmatrix}$. Then $AB = \begin{bmatrix} 8 & 0 & 0 \\ 6 & 6 & 0 \\ 10 & 24 & 6 \end{bmatrix}$.

A *diagonal matrix* is an $n \times n$ matrix in which all entries above and below the main diagonal are zeros. A diagonal matrix is both upper and lower triangular.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \text{ is a diagonal matrix.}$$

Example 4.17 The following are diagonal matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A *unit triangular matrix* is an $n \times n$ triangular matrix with only 1's on the main diagonal.

Example 4.18 The following are unit triangular matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 6 & 8 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let's look at an example of a new way to solve a system of linear equations using triangular matrices.

Example 4.19 Let $A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & -2 & 19 \\ -4 & 6 & 4 \end{bmatrix}$. Then A can be written as the product of two different triangular matrices

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix}.$$

Let $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 30 \end{bmatrix}$. We can solve the equation,

$$A\vec{x} = \vec{b},$$

using the factorization

$$A = LU.$$

We first substitute the factorization $A = LU$ in $A\vec{x} = \vec{b}$ to get the new equation:

$$LU\vec{x} = \vec{b}.$$

If we let $\vec{y} = U\vec{x}$, then $LU\vec{x} = \vec{b}$ becomes

$$L\vec{y} = \vec{b}.$$

This gives us

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 30 \end{bmatrix}.$$

Write the corresponding linear equations. Then the solutions can be found by *forward substitution*:

$$\begin{array}{rcrcrcr} y_1 & & & & & = & -2 \\ 3y_1 & + & y_2 & & & = & 1 \\ -2y_1 & + & 4y_2 & + & y_3 & = & 30 \end{array}$$

Substituting in $y_1 = -2$ into the second equation, we get

$$\begin{aligned} -6 + y_2 &= 1 \\ y_2 &= 7. \end{aligned}$$

Now substitute into the third equation

$$\begin{aligned} 4 + 28 + y_3 &= 30 \\ y_3 &= -2. \end{aligned}$$

The solution to the equation $L\vec{y} = \vec{b}$ is $\vec{y} = \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix}$. Now use $\vec{y}$ to solve the equation

$$U\vec{x} = \vec{y}.$$

This gives us

$$\begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix}.$$

Write the corresponding linear equations. Then the solutions can be found by backward substitution:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 5x_3 & = & -2 \\ & & x_2 & + & 4x_3 & = & 7 \\ & & & & -2x_3 & = & -2 \end{array}$$

This will yield the solution $\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$. This is the solution to the matrix equation $A\vec{x} = \vec{b}$.

The example above uses matrix factorization of the matrix A. *Matrix factorization* is the process of decomposing a matrix A into a product of two or more matrices. We use matrix factorization to express a matrix in a form which is easier to use. A common factorization process used is *LU Decomposition* (Lower/Upper triangular matrix decomposition). An $m \times n$ matrix A has an LU factorization if it can be written in the form $A = LU$, where

L is an $m \times m$ lower triangular matrix and U is an $m \times n$ "modified" upper triangular matrix in the sense that all entries in U below the diagonal entries $u_{11}, u_{22}, ..., u_{nn}$ are zeros, even when U is not square. The format of the matrices is:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

When we factor A into LU we can solve the equation $A\vec{x} = \vec{b}$ more rapidly. Given:

$$A\vec{x} = \vec{b},$$

we substitute $A = LU$, obtaining

$$LU\vec{x} = \vec{b}.$$

Let $\vec{y} = U\vec{x}$ and solve the pair of equations,

$$\begin{aligned} L\vec{y} &= \vec{b} \\ U\vec{x} &= \vec{y}. \end{aligned}$$

We solve $L\vec{y} = \vec{b}$ using forward substitution and then plug the solution $\vec{y}$ into the equation $U\vec{x} = \vec{y}$ and solve for $\vec{x}$ using back substitution. This pair of equations is easy to solve because L and U are triangular and we can almost read the solution off of the matrix. Why would we want to solve a linear system using LU decomposition when we have already stated that Gaussian elimination is frequently the quickest and least computationally expensive method for solving linear systems? It turns out that solving a system of equations $A\vec{x} = \vec{b}$ once by either method, takes approximately the same amount of work. In both methods we use Gaussian elimination to reduce the matrix A to row echelon form, then we solve for the variables using substitution. What happens if we decide we want to solve a system of equations using the same matrix A but a different vector $\vec{b}$? This is a realistic possibility in real world problems and we should keep in mind that most real world problems will involve a matrix A that is significantly larger that those with which we have been working. It is not unreasonable to have a matrix A which is 100×100 in size. So, if we keep $A\vec{x}$ and change $\vec{b}$, using Gaussian elimination, we have to start at the beginning by reducing the augmented matrix to row echelon form and then solve for $\vec{x}$ using substitution. However, if we used LU decomposition we have already formed L and U, and do not have to use Gaussian elimination to reduce the matrix A to the matrix U again. All we have to do is solve $L\vec{y} = \vec{b}$ using forward substitution and then $U\vec{x} = \vec{y}$ using backward substitution; much less work!

Unfortunately, L and U will not always be given to us, so we need to learn how to find the LU factorization of a matrix A. Here is the procedure for a general LU decomposition

1. Use Gaussian Elimination to reduce the matrix A to the *row echelon* form U.

2. Construct L so that the same sequence of row operations which reduced A to U will reduce L to I.

Let's look at this a little closer. We want to transform the matrix A to the upper triangular matrix U using a sequence of elementary row operations $E_1, E_2, ...E_n$:

$$E_n E_{n-1} ... E_1 A = U. \tag{IV.5}$$

Then we can obtain the equation,

$$A = LU$$

by multiplying on both sides by $(E_n E_{n-1} ... E_1)^{-1}$:

$$(E_n E_{n-1} ... E_1)^{-1} (E_n E_{n-1} ... E_1) A = (E_n E_{n-1} ... E_1)^{-1} U,$$

which simplifies to:

$$A = (E_n E_{n-1} ... E_1)^{-1} U.$$

If we let $(E_n E_{n-1} ... E_1)^{-1} = L$, we get

$$A = LU.$$

Step 2 in the procedure is to construct L so that the same sequence of row operations which reduced A to U will reduce L to I. Let's see if this works on L:

$$(E_nE_{n-1}...E_1)\,L = (E_nE_{n-1}...E_1)\,(E_nE_{n-1}...E_1)^{-1} = I.$$

We have formed L as we said we would. This process is rather involved, but gives an idea of what is happening in theory. Just as we said that Gaussian elimination will not produce a unique triangular reduction of the matrix A, this procedure will not produce a unique LU factorization of the matrix A. If we want a unique LU Decomposition, we have to place additional requirements on the process. In this text we will focus on LU decomposition of square matrices where L is unit lower triangular and U is upper triangular. We will consider only those matrices A which can be factored using row replacement operations. This LU decomposition, if it exists, is unique.

Example 4.20 Find the LU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

Since the matrix A is 3×3, the matrices L and U will also be 3×3. We begin with A and a shell of a unit lower triangular matrix for L. Let m_{ij} be the multiplier of the pivot which will be added to row i to make the entry a_{ij} a zero. Reduce A to U using Gaussian elimination. Record $-m_{ij}$ in position l_{ij} of L.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ . & & 1 \end{bmatrix}$$

Add -2 times row 1 to row 2. $m_{21} = -2$.
Add 1 times row 1 to row 3. $m_{31} = 1$.

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 3 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & & 1 \end{bmatrix}$$

Add 3 times row 2 to row 3. $m_{32} = 3$.

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -6 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

This completes the factorization of A into L and U. We can check our work by multiplying

LU to see if the result is A. This gives us:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix} = A.$$

It works!

The procedure employed in the previous example can be summarized as follows:

1. Form U by reducing the matrix A to upper triangular form using row replacement (add a multiple of one row to another row) only. Do not use row interchanges or scaling.

2. Form L by starting with a unit lower triangular matrix shell and fill in the blank spaces by storing the negative of the multiplier of the pivot in the corresponding row in L.

3. Check your solution by multiplying LU to make sure the result is A.

An LU decomposition will not always exist. However, if the matrix A can be reduced to row echelon form without using row interchanges, an LU decomposition will exist.

If row interchanges are performed on a nonsingular matrix, then the product of L and U wil be a matrix which is a row permutation of A. For some permutation matrix P we have:

$$PA = LU.$$

A note about computers. Most computer programs will compute the LU decomposition where L is unit lower triangular. Instead of storing two new matrices L and U, the computer will overwrite the matrix A with the matrices L and U storing only the multipliers of L in the lower portion of U instead of storing all of L.

Example 4.21 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{bmatrix}$.

The LU decomposition for A is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -6 \end{bmatrix}$. The computer will replace the matrix A with the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & -6 \end{bmatrix}.$$

D. DETERMINANTS

Now we are going to study determinants. The *determinant* is a number assigned to a square matrix which may be used to characterize the matrix in some way, for example, singular or nonsingular. Calculating the determinant of a matrix is another concept which has more theoretical application than computational value when a matrix is larger than 4×4 in size. First, we will learn how to compute the determinant. Then we will use the determinant to solve the linear system $A\vec{x} = \vec{b}$. Common notations for the determinant of a matrix A are

$$\det(A), \text{ or } |A|.$$

If the matrix is 1×1 or 2×2 in size, the determinant can be calculated with little effort.

Example 4.22 Let $A = [3]$, and let $B = \begin{bmatrix} 12 & 7 \\ 3 & 1 \end{bmatrix}$. Find $\det(A)$ and $\det(B)$.

Solution:

$$\det(A) = 3.$$

$$\begin{aligned} \det(B) &= 12 \cdot 1 - 7 \cdot 3 \\ &= 12 - 21 \\ &= -9. \end{aligned}$$

When a matrix A is 1×1, the determinant is equal to the value of the single element in the matrix.

$$\boxed{\text{If } A = [a_{11}]\text{, then } \det(A) = a_{11}.}$$

When a matrix A is 2×2, the determinant is calculated by a simple formula.

$$\boxed{\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\text{, then } \det(A) = ad - bc.}$$

We used this formula when we were computing the inverse of a 2×2 matrix. Recall, we said if $ad - bc \neq 0$, then the 2×2 matrix is invertible/nonsingular. Now we know if the determinant of a 2×2 matrix is not equal to zero, then the matrix is nonsingular, and if the determinant of a 2×2 matrix is equal to zero, then the matrix is singular. We summarize this statement in the following theorem relating determinants to inverses of matrices..

Theorem 4.4 *An $n \times n$ matrix is invertible if and only if its determinant is nonzero.*

We need a way to compute the determinant of a matrix which is larger than 2×2. We will use cofactor expansion.

1. Cofactor Expansion

Cofactor expansion is a method for computing the determinant of a matrix. In order to calculate the determinant of matrices larger than 2×2 in size, we must first understand the concept of a submatrix. A *submatrix* of a matrix A is a matrix formed as the result of deleting certain rows and/or columns from the matrix A.

Example 4.23 Let $A = \begin{bmatrix} 1 & 3 & \underline{-5} & -3 \\ \underline{-1} & \underline{-5} & \underline{8} & \underline{4} \\ 4 & 2 & \underline{-5} & -7 \\ -2 & -4 & \underline{7} & 5 \end{bmatrix}$. The submatrix A_{23} is formed by deleting row 2 and column 3 (underlined above) from the matrix A.

$$A_{23} = \begin{bmatrix} 1 & 3 & -3 \\ 4 & 2 & -7 \\ -2 & -4 & 5 \end{bmatrix}.$$

In general, given the matrix A, the submatrix A_{ij} is formed by deleting row i and column j from the original matrix A. Using submatrices, we can develop a recursive definition of the determinant. *Recursive* means we calculate any later term in a formula from the terms that precede it. Let's look at an example.

Example 4.24 Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & -2 \\ 2 & -3 & 1 \end{bmatrix}$. Find $|A|$.

Solution:

$$|A| = 1 \cdot \underbrace{\begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} - 2 \cdot \underbrace{\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} + 5 \cdot \underbrace{\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}}$$
$$= 1(-2) + -2(7) + 5(-17)$$
$$= -101.$$

We found the determinant of the matrix A, a 3×3 matrix, using the determinants of three 2×2 submatrices.

In general, we can find the determinant of an $n \times n$ matrix using the determinants of n submatrices of dimension $(n-1) \times (n-1)$ whenever $n \geq 2$. Thus, we find the determinant of a 4×4 matrix by first finding the determinants of three 3×3 submatrices, and we find the determinants for each of the 3×3 submatrices by first finding the determinants of the two corresponding 2×2 submatrices. The formula we used in the example is

$$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

Notice the alternating signs. This formula can be generalized as

$$\det(A) = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j},$$

which means

$$\underbrace{\sum_{j=1}^{3}}_{\substack{\text{sum over} \\ \text{cols j}}} \underbrace{(-1)^{1+j}}_{\text{alternating sign}} \underbrace{a_{1j}}_{\substack{\uparrow \\ \text{row 1 col j} \\ \text{of } A}} \underbrace{\det A_{1j}}_{\text{det of submatrix}}.$$

where A_{1j} is the submatrix obtained by deleting row 1 and column j from A. The term $(-1)^{1+j} \det A_{1j}$ is called the $1j$ *cofactor of* A. Our definition of the determinant for an $n \times n$ matrix is:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j},$$

which is called the *cofactor expansion along the first row of* A. We can generalize even further using the term $(-1)^{i+j} \det A_{ij}$ which is the ij *cofactor of* A. Then we can compute the determinant of an $n \times n$ matrix A using the cofactor expansion along any row or column of A. Hence, the determinant can be defined as the *cofactor expansion along the* i*th row of*

A:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

where row i is fixed and we sum along the columns of A.

Example 4.25 Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & -2 \\ 2 & -3 & 1 \end{bmatrix}$, the same matrix as in our previous example. Compute $|A|$ using cofactor expansion along the 2nd row of A.

Solution:

We write the general formula with $i = 2$:

$$\det(A) = \sum_{j=1}^{3} (-1)^{2+j} a_{2j} \det A_{2j},$$

which gives us

$$\det(A) = (-1)^{2+1} \cdot 3 \cdot \underbrace{\begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} + (-1)^{2+2} \cdot 4 \cdot \underbrace{\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} + (-1)^{2+3} \cdot -2 \cdot \underbrace{\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}}$$

$$\det(A) = (-1)(3)(17) + (1)(4)(-9) + (-1)(-2)(-7)$$
$$\det(A) = -101.$$

This is the same solution we got when we computed the determinant using the cofactor expansion along the 1st row of A.

Similarly, the determinant can be defined as the *cofactor expansion along the* j*th column of* A:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

where column j is fixed and we sum along the rows of A.

Example 4.26 Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & -2 \\ 2 & -3 & 1 \end{bmatrix}$, the same matrix. Compute $|A|$ using cofactor expansion along the 2nd column of A.

Solution:

We write the general formula with $j = 2$:

$$\det(A) = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} \det A_{i2},$$

which gives us:

$$\det(A) = (-1)^{1+2} \cdot 2 \cdot \underbrace{\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} + (-1)^{2+2} \cdot 4 \cdot \underbrace{\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}} + (-1)^{3+2} \cdot -3 \cdot \underbrace{\begin{vmatrix} 1 & 5 \\ 3 & -2 \end{vmatrix}}_{\substack{\text{det of } 2\times 2 \\ \text{submatrix}}}$$

$$\det(A) = (-1)(2)(7) + (1)(4)(-9) + (-1)(-3)(-17)$$
$$\det(A) = -101.$$

This is the same solution. Good!

If the matrix A has a row or column which consists mostly of zeros, then choose the row/column which consists mostly of zeros as the row/column along which to expand. This will reduce the number of computations needed to calculate $\det(A)$ since, if $a_{ij} = 0$, then the associated term in the cofactor expansion will also be zero.

Example 4.27 Let $A = \begin{bmatrix} -2 & 5 & 2 \\ 4 & -2 & -1 \\ 0 & 3 & 0 \end{bmatrix}$. Compute $|A|$ using cofactor expansion along the 3rd row of A (because this row has two zeros).

Solution:

We write the general formula with $i = 3$:

$$\det(A) = \sum_{j=1}^{3} (-1)^{3+j} a_{3j} \det A_{3j},$$

which gives us:

$$\det(A) = (-1)^{3+1} \cdot 0 \cdot \det(A_{31}) + (-1)^{3+2} \cdot 3 \cdot \det(A_{32}) + (-1)^{3+3} \cdot 0 \cdot \det(A_{33})$$

The first and third terms drop out leaving us with:

$$\begin{aligned} \det(A) &= (-1)(3) \begin{vmatrix} -2 & 2 \\ 4 & -1 \end{vmatrix} \\ &= (-1)(3)(-6) \\ &= 18. \end{aligned}$$

This example illustrates the next theorem.

Theorem 4.5 *If an $n \times n$ matrix A has a row or column of all zeros, then* $\det(A) = 0$.

Let's try another example.

Example 4.28 Let $A = \begin{bmatrix} 2 & 3 & -2 & 3 \\ 0 & -1 & 1 & -4 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ -1 & -2 & -3 & 0 \\ 5 & 1 & 4 & 5 \end{bmatrix}$.

Find $|A|$ and $|B|$.

Solution:

To find $\det(A)$, use cofactor expansion along the first column since the first column consists mostly of zeros. Fix $j = 1$:

$$\det(A) = \sum_{i=1}^{4} (-1)^{i+1} a_{i1} \det A_{i1}$$

This gives us the equation:

$$\det(A) = (-1)^{1+1}(2) \left| \begin{bmatrix} -1 & 1 & -4 \\ 0 & -3 & -1 \\ 0 & 0 & -2 \end{bmatrix} \right|.$$

All other terms drop out because the elements a_{21}, a_{31}, and a_{41} are zeros. Continue by finding the determinant of the 3×3 submatrix using cofactor expansion along the first column because the first column consists mostly of zeros. Fix $j = 1$. We now have

$$\det(A) = \left[(-1)^{1+1}(2)\right](-1)^{1+1}(-1) \begin{vmatrix} -3 & -1 \\ 0 & -2 \end{vmatrix}.$$

All other terms drop out because the elements a_{21} and a_{31} of the 3×3 submatrix are zeros. Continue by finding the determinant of the 2×2 submatrix using the formula $ad - bc$, obtaining

$$\det(A) = \left[(-1)^{1+1}(2)\right]\left[(-1)^{1+1}(-1)\right]\left[(-3)(-2) - (-1)(0)\right].$$

Simplifying, we get

$$\det(A) = (2)(-1)(-3)(-2) = -12.$$

Notice that the matrix A is upper triangular and the determinant turned out to be the product of the entries on the main diagonal.

$$(2)(-1)(-3)(-2) = -12$$

To find $\det(B)$, use cofactor expansion along the first row since the first row consists mostly of zeros. Fix $i = 1$:

$$\det(B) = \sum_{j=1}^{4} (-1)^{1+j} a_{1j} \det A_{1j}$$
$$= (-1)^{1+1}(2) \left| \begin{bmatrix} -1 & 0 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 5 \end{bmatrix} \right|.$$

All other terms drop out because the elements a_{12}, a_{13}, and a_{14} are zeros. Continue by finding the determinant of the 3×3 submatrix using cofactor expansion along the first row because the first row consists mostly of zeros. Fix $i = 1$. Then

$$\det(B) = \left[(-1)^{1+1}(2)\right](-1)^{1+1}(-1) \begin{vmatrix} -3 & 0 \\ 4 & 5 \end{vmatrix}.$$

All other terms drop out because the elements a_{12} and a_{13} of the 3×3 submatrix are zeros. Continue by finding the determinant of the 2×2 submatrix using the formula $ad - bc$, obtaining

$$\det(B) = \left[(-1)^{1+1}(2)\right]\left[(-1)^{1+1}(-1)\right]\left[(-3)(5) - (0)(4)\right].$$

Simplifying, we get

$$\det(B) = (2)(-1)(-3)(4) = 30.$$

The matrix B is lower triangular and the determinant turned out to be the product of the entries on the main diagonal:

$$(2)(-1)(-3)(5) = 30.$$

From the above example, we can see a nice feature of triangular matrices.

Theorem 4.6 *If the matrix A is an $n \times n$ triangular matrix, the* $\det(A)$ *is equal to the product of the entries on the main diagonal.*

It would appear that given an $n \times n$ matrix A, the determinant is easier to compute if we can transform the matrix A to a triangular matrix. Clearly, an $n \times n$ matrix A is row equivalent to an upper triangular matrix which is obtained using Gaussian elimination. The question is, how do elementary row operations affect the determinant? When elementary row operations are performed on an $n \times n$ matrix A, the determinant of the new matrix will change in the following ways each time a row operation is performed:

Effect of row operations on the determinant of a matrix.

1. If a multiple of one row of the matrix A is added to another row to produce the matrix A_1, the determinant of the new matrix is the same as the determinant of the original matrix.

$$\det A_1 = \det A.$$

2. If two rows of the matrix A are interchanged to produce the matrix A_1, the determinant of the new matrix is equal to minus the determinant of the original matrix.

$$\det A_1 = -\det A.$$

3. If one row of the matrix A is multiplied by a scalar c to produce the matrix A_1, the determinant of the new matrix is equal to c times the determinant of the original matrix.

$$\det A_1 = (c)\det A.$$

Example 4.29 Let $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 4 & -1 \\ 0 & 5 & 0 \end{bmatrix}$. Compute $|A|$.

Solution:

Use Gaussian elimination to change the matrix A to row echelon form.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 4 & -1 \\ 0 & 5 & 0 \end{bmatrix}.$$

Add 2 times row 1 to row 2. The determinant is unchanged.

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 10 & -1 \\ 0 & 5 & 0 \end{bmatrix}.$$

Add $\frac{-1}{2}$ times row 2 to row 3. The determinant is unchanged.

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 10 & -1 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

This matrix is upper triangular. By Theorem 4.6, the determinant is equal to the product of the entries on the main diagonal. The determinant is unchanged by the row operations applied.

$$\det(A) = \det(U) = (1)(10)\left(\frac{1}{2}\right) = 5.$$

Example 4.30 Let $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -6 & 5 \\ -1 & 7 & 0 \end{bmatrix}$. Compute $|A|$.

Solution:

Use Gaussian elimination to change the matrix A to row echelon form.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -6 & 5 \\ -1 & 7 & 0 \end{bmatrix}.$$

Add 2 times row 1 to row 2. The determinant is unchanged.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 1 \\ -1 & 7 & 0 \end{bmatrix}.$$

Add 1 times row 1 to row 3. The determinant is unchanged.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 1 \\ 0 & 10 & -2 \end{bmatrix}.$$

Interchange rows 2 and 3. The determinant of the new matrix is equal to (-1) times the determinant of the previous matrix.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 10 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is an upper triangular matrix. The determinant is equal to the product of the entries on the main diagonal. The determinant has changed.

$$\det(U) = -1 \det(A).$$

Remember, we are trying to solve $|A|$; therefore,

$$\det(A) = -1 \det(U) = (-1)(1)(10)(1) = -10$$

Suppose, instead of calculating the determinant at this point, we had taken one more step by multiplying the second row by the scalar $\frac{1}{10}$. The determinant of the new matrix is equal to $\frac{1}{10}$ times the determinant of the previous matrix.

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1/5 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is upper triangular. The determinant is equal to the product of the entries on the main diagonal. The determinant has changed in two ways, once by interchanging rows and a second way by multiplying a row by a scalar.

$$\det(U) = (-1)\left(\frac{1}{10}\right)\det(A).$$

Again, stay focused, we are trying to find $|A|$; therefore,

$$\det(A) = -10\det(U) = (-10)(1)(1)(1) = -10.$$

Whew! The same answer.

Sometimes the determinant will be easier to compute using a combination of both elementary row operations and cofactor expansion.

Example 4.31 Let $A = \begin{bmatrix} 3 & 2 & 1 & -2 \\ 0 & 4 & 6 & 8 \\ 0 & 2 & 3 & 5 \\ -3 & -2 & 3 & 5 \end{bmatrix}$. Compute $|A|$.

Solution:

Begin by adding 1 times row 1 to row 4. The determinant is unchanged by the row operation.

$$\begin{bmatrix} 3 & 2 & 1 & -2 \\ 0 & 4 & 6 & 8 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

Use cofactor expansion along the first column because this column consists mostly of zeros and all but one term drops out of the summation. Fix $j = 1$,

$$(-1)^{1+1}(3)\left|\begin{bmatrix} 4 & 6 & 8 \\ 2 & 3 & 5 \\ 0 & 4 & 3 \end{bmatrix}\right|.$$

Compute the determinant of the 3×3 submatrix using elementary row operations. Add $\frac{-1}{2}$

times row 1 to row 2. The determinant is unchanged.

$$(-1)^{1+1}(3)\left|\begin{bmatrix} 4 & 6 & 8 \\ 0 & 0 & 1 \\ 0 & 4 & 3 \end{bmatrix}\right|.$$

At this point we have two options. We can interchange rows 2 and 3, or we can use cofactor expansion along the first column of the 3×3 submatrix. We'll interchange rows here. Try cofactor expansion on your own to see if you get the same solution.

Interchange row 2 with row 3. The determinant of the new matrix is equal to minus the determinant of the previous matrix.

$$(-1)^{1+1}(3)\left|\begin{bmatrix} 4 & 6 & 8 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}\right|.$$

This submatrix is upper triangular, so the determinant is equal to the product of the entries on the main diagonal. The determinant has changed because of the row interchange.

$$\det(U) = -\det(A).$$

We want the determinant of the matrix A.

$$\det(A) = -\det(U) = (-1)(1)^{1+1}(3)(4)(4)(1) = -48.$$

At this point we are ready to learn a few properties of determinants and how they relate to some of the concepts presented earlier in the book.

Properties of Determinants

1. The rows or columns of the matrix A are linearly dependent if and only if $\det(A) = 0$. (Recall, rows or columns of a matrix are linearly dependent when 2 or more rows or columns are the same, multiples of another row or column, or have a row of zeros.)

2. The rows or columns of the matrix A are linearly independent if and only if $\det(A) \neq 0$.

3. The determinant of the transpose of a matrix A is equal to the determinant of the matrix A.
$$\det\left(A^T\right) = \det(A).$$

This is consistent with calculating the determinant using cofactor expansion along a row or a column of A, because column operations on the matrix A are simply row operations on the matrix A^T.

4. Given $n \times n$ matrices A and B,

$$\det(AB) = \det(A)\det(B).$$

Notice, when we have $A = LU$, then:

$$\det(A) = \det(LU) = \det(L)\det(U).$$

Since L is unit lower triangular, $\det(L) = 1$, which gives us:

$$\det(A) = \det(U)$$

provided the only elementary row operation applied was row replacement.

5. If A is a nonsingular matrix, the determinant of the inverse to the matrix A is equal to the inverse of the determinant of the matrix A.

$$\det\left(A^{-1}\right) = [\det(A)]^{-1}$$

We can use the determinant to find the solution to a system of linear equations by applying Cramer's Rule.

2. Cramer's Rule

Cramer's rule is a formula which allows us to solve the system of equations $A\vec{x} = \vec{b}$ using what we just learned about determinants instead of using Gaussian elimination or LU decomposition. We discuss Cramer's rule because it will occasionally appear in science texts and is sometimes useful in theoretical calculations. However, the formula quickly becomes inefficient with matrices of size larger than 3×3. We should remember that Gaussian elimination is generally the fastest method for solving systems of equations.

Before we define Cramer's rule, we need to learn about a different type of matrix. Given any $n \times n$ matrix A and an $n \times 1$ column vector $\vec{b}$, we can form a new matrix $A_j(\vec{b})$ by replacing the jth column of the matrix A with the vector $\vec{b}$.

Example 4.32 Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & -1 \\ 3 & 4 & 0 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 6 \\ 3 \\ 18 \end{bmatrix}$.

Form the matrices $A_1(\vec{b})$, $A_2(\vec{b})$, and $A_3(\vec{b})$.

Solution:

$$A_1(\vec{b}) = \begin{bmatrix} 6 & 3 & 5 \\ 3 & 2 & -1 \\ 18 & 4 & 0 \end{bmatrix},$$
$$A_2(\vec{b}) = \begin{bmatrix} 1 & 6 & 5 \\ -2 & 3 & -1 \\ 3 & 19 & 0 \end{bmatrix},$$
$$A_3(\vec{b}) = \begin{bmatrix} 1 & 3 & 6 \\ -2 & 2 & 3 \\ 3 & 4 & 18 \end{bmatrix}.$$

We can use the determinants of the matrix A and the new matrices $A_j(\vec{b})$ to solve a system of linear equation.

Example 4.33 Given the linear system of equations

$$\begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 2x_2 & - & x_3 & = & 6 \\ -x_1 & + & 2x_2 & + & 3x_3 & = & 5 \end{array},$$

find the solution vector $\vec{x}$ using the determinants of the matrices A and $A_j(\vec{b})$.

Solution:

We can rewrite this system of equations as the matrix equation $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}.$$

Now compute the determinants of A, $A_1(\vec{b})$, $A_2(\vec{b})$, and $A_3(\vec{b})$:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 2 & -1 \\ -1 & 2 & 3 \end{bmatrix}, \quad |A| = 1\,(8) + 2\,(5) + 1\,(6) = 24,$$

$$A_1(\vec{b}) = \begin{bmatrix} 3 & -2 & 1 \\ 6 & 2 & -1 \\ 5 & 2 & 3 \end{bmatrix}, \quad \left|A_1(\vec{b})\right| = 3\,(8) + 2\,(23) + 1\,(2) = 72,$$

$$A_2(\vec{b}) = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & -1 \\ -1 & 5 & 3 \end{bmatrix}, \quad \left|A_2(\vec{b})\right| = 1\,(23) - 3\,(5) + 1\,(16) = 24,$$

$$A_3(\vec{b}) = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 2 & 6 \\ -1 & 2 & 5 \end{bmatrix}, \quad \left|A_3(\vec{b})\right| = 1\,(-2) + 2\,(16) + 3\,(6) = 48.$$

We can now solve for $\vec{x}$ as follows:

$$x_1 = \frac{\det[A_1(\vec{b})]}{\det(A)} = \frac{72}{24} = 3,$$

$$x_2 = \frac{\det[A_2(\vec{b})]}{\det(A)} = \frac{24}{24} = 1,$$

$$x_3 = \frac{\det[A_3(\vec{b})]}{\det(A)} = \frac{48}{24} = 2.$$

Hence, the solution vector is $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. This is a unique solution.

The method used to solve the system in the above example, called Cramer's rule, is generalized in the following theorem:

Theorem 4.7 *Cramer's Rule*
If A is an $n \times n$ nonsingular matrix and $\vec{b}$ is an $n \times 1$ vector, then the linear system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x}$, where the entries of $\vec{x}$ are:

$$x_j = \frac{\det[A_j(\vec{b})]}{\det(A)}, \quad j = 1, 2..., n.$$

We need to be careful not to read anything into the theorem.. If $\det(A) = 0$, then the matrix A is singular and Cramer's rule does not tell us whether there are infinitely many solutions or no solutions.

E. EXERCISES

1. Compute the inverses of the following matrices.

$$A = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

2. Which of the following are elementary matrices?

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

3. Consider the matrix $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$.

a) Find the elementary matrices E_1 and E_2 such that $E_2 E_1 A = I$.

b) Write A^{-1} as a product of two elementary matrices.

c) Write A as a product of two elementary matrices.

4. Solve the following linear system using the solution formula $\vec{x} = A^{-1}\vec{b}$

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 4 \\ 4x_1 & + & 5x_2 & + & 9x_3 & = & 5 \end{array}$$

5. Find the inverse of the following matrices, if it exists.

a) $A = \begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 7 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

c) $C = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$

d) $D = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -9 \\ 4 & -1 & 16 \end{bmatrix}$

e) $E = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ -1 & 0 & 8 \end{bmatrix}$

6. Solve the equation $A\vec{x} = \vec{b}$ by using the LU factorization given for A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 7 \\ -16 \end{bmatrix}$$

7. Solve the equation $A\vec{x} = \vec{b}$ by using the LU factorization given for A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}$$

8. Find the LU factorization of the following matrices.

a) $\begin{bmatrix} -2 & 4 & 3 \\ -4 & 10 & 3 \\ 2 & 2 & -11 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -1 & 3 \\ 15 & 1 & 8 \\ -20 & 12 & -16 \end{bmatrix}$

c) $\begin{bmatrix} 7 & -2 & 4 & -3 \\ 28 & -7 & 21 & -10 \\ -7 & 4 & 5 & 9 \\ 7 & 1 & 21 & -5 \end{bmatrix}$

d) $\begin{bmatrix} 2 & -1 & 6 \\ 12 & -1 & 39 \\ 8 & 11 & 29 \end{bmatrix}$

9. Compute the determinant of the following matrices using cofactor expansion.

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 2 & 3 \\ 5 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & -3 & -2 \\ -3 & 1 & 2 & 3 \end{bmatrix}$$

10. Find the determinant for the following matrices using Gaussian elimination.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 7 & -4 \\ 2 & 4 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 3 & -5 \\ -2 & -1 & 4 & 3 \\ 4 & 3 & 2 & -11 \\ -3 & 0 & 1 & -1 \end{bmatrix}$$

11. Combine the methods of Gaussian elimination and cofactor expansion to compute the determinant.

$$A = \begin{bmatrix} 4 & 10 & -6 & -2 \\ 3 & 0 & 1 & -3 \\ -4 & 0 & 2 & 5 \\ 2 & 5 & 0 & -1 \end{bmatrix}$$

12. Use determinants to decide if the following vectors are linearly dependent.

a) $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

c) $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ 1 \end{bmatrix}$

13. Use Cramer's rule to compute the solution for the following linear systems.

a) $\begin{aligned} 5x_1 + 6x_2 &= -9 \\ 2x_1 + 4x_2 &= -2 \end{aligned}$

b) $$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & = & -2 \\ 2x_1 & + & 6x_2 & + & 2x_3 & = & 10 \end{array}$$

c) $$\begin{array}{rcrcrcr} 5x_1 & + & 2x_2 & - & 6x_3 & = & 10 \\ 4x_1 & + & 2x_2 & - & 6x_3 & = & 1 \\ 3x_1 & - & x_2 & - & 4x_3 & = & 7 \end{array}$$

(d) $$\begin{array}{rcrcr} 2x_1 & - & 5x_2 & = & 1 \\ 3x_1 & + & 2x_2 & = & 11 \end{array}$$

e) $$\begin{array}{rcrcr} 4x_1 & + & 6x_2 & = & 2 \\ 5x_1 & + & 7x_2 & = & 3 \end{array}$$

V. EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are used in the study of discrete and continuous dynamical systems, differential equations, and many more areas of pure and applied mathematics. In Chapter IV we studied linear transformations. We looked at the effect that a matrix A might have on a vector $\vec{x}$. We said the linear transformation is the action of the matrix A on the vector $\vec{x}$. The study of eigenvalues and eigenvectors involves a linear transformation $A(\vec{x})$ in which the matrix A transforms a nonzero vector $\vec{x}$ into a scalar multiple λ of itself.

$$\boxed{A(\vec{x}) = \lambda\vec{x}.}$$

A. REAL EIGENVECTORS AND EIGENVALUES

Let A be an $n \times n$ matrix which is a linear transformation from $R^n \longrightarrow R^n$. Is there a nonzero vector $\vec{x}$ and a scalar λ for which $A\vec{x} = \lambda\vec{x}$? If such a vector and scalar exist, then we call λ an *eigenvalue* of A and $\vec{x}$ an *eigenvector* of A associated with the eigenvalue λ. The matrix A may have several eigenvalues associated with it. Each eigenvalue has infinitely many associated eigenvectors. So, we are looking for vectors which are transformed by the matrix A into scalar multiples of themselves.

Example 5.1 Let $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$, $\quad \vec{x} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$, $\quad \vec{y} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.
Are $\vec{x}$ and $\vec{y}$ eigenvectors of A?

$$A\vec{x} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} -28 \\ -14 \end{bmatrix} = 7\begin{bmatrix} -4 \\ -2 \end{bmatrix} = 7\vec{x}.$$

The vector $\vec{x}$ is transformed by the matrix A into a vector $7\vec{x}$. Therefore, $\vec{x}$ is an eigenvector of A associated with the eigenvalue $\lambda = 7$.

$$A\vec{y} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 47 \\ 23 \end{bmatrix} \neq \lambda\begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

We cannot find any λ such that $A\vec{y} = \lambda\vec{y}$; therefore, $\vec{y}$ is *not* an eigenvector of A.

Example 5.2 Let $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$.

Show that $\lambda = 1$ is an eigenvalue of A.

Solution:

1 is an eigenvalue of A if there exists a nonzero $\vec{x}$ which satisfies the equation:

$$A\vec{x} = 1\vec{x}$$

Substitute A into the equation:

$$\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute the matrix multiplication on the left side of the equation:

$$\begin{bmatrix} 5x_1 & + & 4x_2 \\ 2x_1 & + & 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Two vectors are equal if the corresponding components are equal:

$$\begin{array}{rcrcl} 5x_1 & + & 4x_2 & = & x_1 \\ 2x_1 & + & 3x_2 & = & x_2 \end{array}$$

Subtract x_1 and x_2 from the right hand sides of the equations:

$$\begin{array}{rcrcl} 4x_1 & + & 4x_2 & = & 0 \\ 2x_1 & + & 2x_2 & = & 0. \end{array}$$

Solve the homogeneous linear system.

$$\begin{bmatrix} 4 & 4 & 0 \\ 2 & 2 & 0 \end{bmatrix} \overset{\text{Add } \frac{-1}{2} \text{ row 1 to row 2}}{\sim} \begin{bmatrix} 4 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a row of zeros, there is a free variable. Let x_2 be the free variable. Then $x_1 = -x_2$. The general solution is $\vec{x} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s,\ s \neq 0$.
We found a vector $\vec{x}$ such that $A\vec{x} = 1\vec{x}$; therefore, 1 is an eigenvalue of the matrix A.

Once we find an eigenvector $\vec{x}$ of an $n \times n$ matrix A, we can find infinitely many eigenvectors, associated with the same eigenvalue λ, by multiplying the eigenvector $\vec{x}$ by a scalar $c, c \neq 0$. The vector $c\vec{x}$ is an eigenvector of A associated with λ, since:

$$A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x}).$$

Example 5.3 Let $A = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$. Then $\lambda = 7$ is an eigenvalue of A with associated eigenvector $\vec{x} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$.

We show that $c\vec{x}$ is also an eigenvector of A when $c \neq 0$.

$$\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -6c \\ -3c \end{bmatrix} = \begin{bmatrix} -42c \\ -21c \end{bmatrix} = 7 \begin{bmatrix} 6c \\ -3c \end{bmatrix}.$$

The above equation shows that $A\,(c\vec{x}) = \lambda\,(c\vec{x})$; therefore, $c\vec{x}$ is an eigenvector of the matrix A associated with the eigenvalue $\lambda = 7$.

1. Finding Eigenvalues and Eigenvectors

We will not always be given the eigenvalues or eigenvectors of a matrix A. Therefore, we need to learn how to find all eigenvalues of a square matrix and all eigenvectors associated with each eigenvalue. Eigenvalues and associated eigenvectors satisfy the equation:

$$A\vec{x} = \lambda\vec{x},$$

where A is an $n \times n$ matrix and $\vec{x}$ is an $n \times 1$ vector. We can rewrite this equation as:

$$A\vec{x} - \lambda\vec{x} = \vec{0},$$

where $\vec{0}$ is the $n \times 1$ zero vector. The vector $\vec{x}$ can also be represented as $I\vec{x}$, where I is the $n \times n$ identity matrix. We can rewrite the above equation as:

$$A\vec{x} - \lambda I\vec{x} = \vec{0}.$$

Then we can factor out the vector $\vec{x}$ to get the equation

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Notice this equation is simply a homogeneous system of linear equations. Recall, homogeneous linear systems are always consistent with either the trivial solution or infinitely many nontrivial solutions. We want the nontrivial solution to the homogeneous system. The homogeneous system has a nontrivial solution if and only if $(A - \lambda I)$ is a singular matrix. Recall when we say $(A - \lambda I)$ is a singular matrix, this means $(A - \lambda I)$ is not row equivalent to the identity matrix. Therefore, the solution vector will have a free variable and nontrivial solutions. We know $(A - \lambda I)$ is singular if and only if $\det(A - \lambda I) = 0$. The equation $\det(A - \lambda I) = 0$ is called the *characteristic equation*. We will use the characteristic equation to find the eigenvalues of the matrix A.

Theorem 5.1 *Let A be an $n \times n$ matrix. Then:*

1. λ is an eigenvalue of A if and only if λ satisfies the characteristic equation $det(A - \lambda I) = 0$

2. If λ is an eigenvalue of A, any nontrivial solution of $(A - \lambda I)\vec{x} = \vec{0}$ is an eigenvector of A associated with λ.

Thus, we find the eigenvalues of A by solving the characteristic equation

$$\det(A - \lambda I) = 0,$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdot & \cdot & a_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} - \lambda \end{vmatrix} = 0.$$

Using cofactor expansion on A to find the determinant of $(A - \lambda I)$ will produce the *characteristic polynomial*. If A is $n \times n$, the characteristic polynomial will have degree n in λ. We want to find the characteristic polynomial because the roots of the characteristic polynomial are the eigenvalues of A. Sometimes polynomials have repeated roots. If an eigenvalue λ_i appears as a root of the characteristic polynomial p times, we say λ_i has multiplicity p.

Example 5.4 Let $A = \begin{bmatrix} 5 & -2 & 6 \\ 0 & 3 & -8 \\ 0 & 0 & 5 \end{bmatrix}$. Then $\det(A - \lambda I) = \left| \begin{bmatrix} 5-\lambda & -2 & 6 \\ 0 & 3-\lambda & -8 \\ 0 & 0 & 5-\lambda \end{bmatrix} \right|$. This is a triangular matrix; therefore, the determinant of this matrix is the product of the main diagonal elements. The characteristic polynomial is $(5 - \lambda)^2(3 - \lambda)$. The characteristic polynomial has 5 as a root with multiplicity 2 and 3 as a root with multiplicity 1. We say eigenvalue 5 has multiplicity 2 and eigenvalue 3 has multiplicity 1.

Counting multiplicities, there are n roots of the characteristic polynomial, some of which are real and some of which might be nonreal. While it is possible to have repeated eigenvalues, the study of repeated eigenvalues is a bit more complicated and will be left for a future course. For now we will study only characteristic equations with distinct, real roots. Example 5.4illustrates the following theorem.

Theorem 5.2 *If A is an $n \times n$ triangular matrix, then the eigenvalues of A are the entries on the main diagonal.*

Be careful! If $A = LU$, then the eigenvalues of U are its diagonal entries, but there is no reason to expect A to have the same eigenvalues.

The idea behind the theorem is this. If we have a triangular matrix A, we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ. The matrix $A - \lambda I$ looks like

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

Recall that the determinant of a triangular matrix is the product of the entries on the main diagonal. We want to find the eigenvalues of A which satisfy the equation,

$$(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0.$$

This means that λ equals one of the entries on the main diagonal of A.

Once we know the eigenvalues of the matrix A, we solve the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x}$. For each eigenvalue λ of A, there will be an infinitely many solution vectors associated with λ.

Example 5.5 Find the eigenvalues and all associated eigenvectors of the matrix A.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{bmatrix}.$$

Solution:

We want to find the values λ and the vectors $\vec{x}$ such that

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Find the eigenvalues using the characteristic equation,

$$\det(A - \lambda I) = 0.$$

$$\begin{aligned}
\det(A - \lambda I) &= \left| \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| \\
&= \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & -2 & 4-\lambda \end{vmatrix}
\end{aligned}$$

Using cofactor expansion along the first row of $\det(A - \lambda I)$ we have:

$$(2-\lambda)\left|\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\right| - (1)\left|\begin{bmatrix} 1 & 1 \\ 1 & 4-\lambda \end{bmatrix}\right| + (3)\left|\begin{bmatrix} 1 & 1-\lambda \\ 1 & -2 \end{bmatrix}\right|$$

$$\begin{aligned}
&= (2-\lambda)\left[(1-\lambda)(4-\lambda)+2\right] - \left[4-\lambda-1\right] + (3)\left[-2-(1-\lambda)\right] \\
&= (2-\lambda)\left[\lambda^2 - 5\lambda + 6\right] - \left[3-\lambda\right] + 3\left[-3+\lambda\right] \\
&= -\lambda^3 + 7\lambda^2 - 12\lambda \\
&= (-\lambda)\left(\lambda^2 - 7\lambda + 12\right) \\
&= (-\lambda)(\lambda-4)(\lambda-3)
\end{aligned}$$

Setting $(-\lambda)(\lambda-4)(\lambda-3)$ equal to 0 and solving, we find roots

$$\lambda = 0,\ \lambda = 4,\ \lambda = 3.$$

Let $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = 3$. Find the eigenvectors for each eigenvalue by solving the homogeneous equation,

$$(A - \lambda I)\vec{x} = \vec{0}.$$

Remember, we are looking for a nonzero vector $\vec{x}$. Since we are looking for a nontrivial solution, the matrix $A - \lambda I$ must be singular, otherwise $A - \lambda I$ would be row equivalent to the identity matrix and the only solution would be the trivial solution. We know that the matrix $A - \lambda I$ is singular, because we found the eigenvalues λ by solving the characteristic equation $\det(A - \lambda I) = 0$. When we perform Gaussian elimination on the system, we will end up with at least one row of zeros. This means that there will be at least one free variable.

For $\lambda_1 = 0$:

$$(A - \lambda I) = \begin{bmatrix} (2-0) & 1 & 3 \\ 1 & (1-0) & 1 \\ 1 & -2 & (4-0) \end{bmatrix}$$

Solve the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Write the augmented matrix:

$$\begin{bmatrix} (2-0) & 1 & 3 & 0 \\ 1 & (1-0) & 1 & 0 \\ 1 & -2 & (4-0) & 0 \end{bmatrix} \sim \overset{\text{Add -1/2 times row 1 to row 2\&3}}{\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -2 & 4 & 0 \end{bmatrix}} \sim$$

$$\overset{\text{Add 5 times row 2 to row 3}}{\begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -5/2 & 5/2 & 0 \end{bmatrix}} \sim \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a row of zeros, there must be a free variable. Let x_3 be the free variable. Set $x_3 = r$. Then $x_2 = r$ and $x_1 = -2r$. The general solution is $\vec{x} = r \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ for any real scalar r.

For $\lambda_2 = 4$:

$$(A - \lambda I) = \begin{bmatrix} (2-4) & 1 & 3 \\ 1 & (1-4) & 1 \\ 1 & -2 & (4-4) \end{bmatrix}$$

Solve the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Write the augmented matrix:

$$\begin{bmatrix} (2-4) & 1 & 3 & 0 \\ 1 & (1-4) & 1 & 0 \\ 1 & -2 & (4-4) & 0 \end{bmatrix} \sim \overset{\text{Add 1/2 times row 1 to row 2\&3}}{\begin{bmatrix} -2 & 1 & 3 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}} \sim$$

$$\overset{\text{Add -3/5 times row 2 to row 3}}{\begin{bmatrix} -2 & 1 & 3 & 0 \\ 0 & -5/2 & 5/2 & 0 \\ 0 & -3/2 & 3/2 & 0 \end{bmatrix}} \sim \begin{bmatrix} -2 & 1 & 3 & 0 \\ 0 & -5/2 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Again, one row is entirely zeros; therefore, there must be a free variable. Let x_3 be the free variable. Set $x_3 = s$. Then $x_2 = s$ and $x_1 = 2s$. The general solution is $\vec{x} = s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ for any real scalar s.

For $\lambda_3 = 3$:

$$(A - \lambda I) = \begin{bmatrix} (2-3) & 1 & 3 \\ 1 & (1-3) & 1 \\ 1 & -2 & (4-3) \end{bmatrix}$$

Solve the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$. Write the augmented matrix:

$$\begin{bmatrix} (2-3) & 1 & 3 & 0 \\ 1 & (1-3) & 1 & 0 \\ 1 & -2 & (4-3) & 0 \end{bmatrix} \sim \overset{\text{Add 1 times row 1 to row 2\&3}}{\begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}} \sim$$

$$\overset{\text{Add -1 times row 2 to row 3}}{\begin{bmatrix} -1 & 1 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix}} \sim \begin{bmatrix} -1 & 1 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let x_3 be the free variable. Set $x_3 = t$. Then $x_2 = 4t$ and $x_1 = 7t$. The general solution is $\vec{x} = t \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$ for any real scalar t.

In this example, A has an eigenvalue $\lambda = 0$. This only happens if there exists $\vec{x} \neq \vec{0}$ such that

$$A\vec{x} = 0\vec{x} = \vec{0}$$

This equation has a nontrivial solution if and only if A is not invertible. This means 0 is an eigenvalue of A if and only if A is singular. Let's look at an application. [Ref. 3]

Example 5.6 Two companies A and B find they can mutually gain by cooperating. By cooperating, company A's worth will increase by 9% of company B's worth, and company B's worth will increase by 4% of company A's worth. However, the companies will only cooperate if their values increase at equal percentages. For example, if both companies have initial values of 100 units, after cooperating, company A will be worth 109 units (a gain of 9%) and company B will be worth 104 units (a gain of 4%). Clearly, company B would not consider this a fair partnership. Let A_0 and B_0 be the starting values of the companies before the partnership. Let A_1 and B_1 be the company values after developing a partnership. Find the initial values A_0 and B_0, so that companies A and B realize the same percentage gains by solving the linear system,

$$\begin{array}{rcrcl} A_0 & + & .09B_0 & = & A_1 \\ .04A_0 & + & B_0 & = & B_1, \end{array}$$

with corresponding matrix equation,

$$\begin{bmatrix} 1 & .09 \\ .04 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}.$$

In order for the companies to realize the same percentage gain we want to find scalars λ

such that

$$\lambda \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}.$$

We can set the two quantities equal to each other:

$$\begin{bmatrix} 1 & .09 \\ .04 & 1 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \lambda \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}.$$

If this equation can be solved, then λ is an eigenvalue of the matrix $\begin{bmatrix} 1 & .09 \\ .04 & 1 \end{bmatrix}$, and $\begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$ is an eigenvector. Solve the characteristic equation $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & .09 \\ .04 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda) - (.04)(.09)$$
$$= \lambda^2 - 2\lambda + .9964.$$

Solving $\lambda^2 - 2\lambda + .9964 = 0$ we have

$$\lambda = 1.06 \text{ or } .94.$$

Since the resulting values of the companies is equal to λ times the starting values of the companies, the companies' values would decrease if we let $\lambda = .94$, which is less than 1. We will disregard this value and try to find an eigenvector for $\lambda = 1.06$.

$$\begin{bmatrix} 1-1.06 & .09 & 0 \\ .04 & 1-1.06 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -.06 & .09 & 0 \\ .04 & -.06 & 0 \end{bmatrix}.$$

Add .04/.06 time the first row to the second row:

$$\begin{bmatrix} -.06 & .09 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let B_0 be the free variable and $A_0 = 1.5B_0$. The starting values of the companies must satisfy the relationship:

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = B_0 \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}.$$

The percentage gain of each company is 6%.

This next theorem relates our previous study of linear independence to eigenvectors of a matrix.

Theorem 5.3 *If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_r$ are eigenvectors of an $n \times n$ matrix A that correspond to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$ of A, then the vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_r$ are linearly independent.*

B. COMPLEX NUMBERS

In section A we found that the eigenvalues of a matrix are the roots of its characteristic polynomial. We know that the roots of polynomials are not always real; some roots may be complex. This suggests that some eigenvalues are complex. Ultimately, we want to learn how to find complex eigenvalues and eigenvectors. We begin with the study of complex numbers. Consider trying to take the square root of a negative number, such as $\sqrt{-4}$. We cannot do this with any of the tools we currently possess. However, if we introduce an imaginary unit i, we can find $\sqrt{-4}$. Let the *imaginary unit* $i = \sqrt{-1}$. We can develop this a little further:

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= (\sqrt{-1})^2 = -1 \end{aligned}$$

Knowing the imaginary unit will allow us to find the square root of a negative number. Let $k > 0$. Then

$$\sqrt{-k} = \sqrt{(-1)k} = \sqrt{-1}\sqrt{k} = i\sqrt{k}.$$

Example 5.7

$$\sqrt{-4} = \sqrt{(-1)4} = \sqrt{-1}\sqrt{4} = i2 = 2i.$$

The imaginary unit allows us to expand our current number system to include imaginary numbers. An *imaginary number* is any number of the form $ki, \;\; k \neq 0$. Imaginary numbers allow us to build a whole new set of numbers called the complex numbers. A *complex number* z is a number of the form $z = a + bi$, where a and b are real numbers and i is the imaginary unit defined above. The complex number z has a real part, denoted $re(z) = re(a + bi) = a$, and an imaginary part, denoted $im(z) = im(a + bi) = b$. We will denote the complex number system by the symbol C.

Let's consider the relationship of the complex numbers C to the real numbers R. Every complex number has two parts, a real part, $re(z) = a$, and an imaginary part, $im(z) = b$. If we assign the value of zero to the imaginary part, $im(z) = b = 0$, then we get a number $z = a + 0i = a$, which is real. So a real number is a complex number of the form $z = a + 0i$, and it follows that the real numbers are a special subset of the complex numbers, i.e.,

$$R = \{a + bi \mid b = 0\} \subset C.$$

A complex number can be associated with a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in R^2. Just as with vectors, complex numbers are equal if and only if the real parts are equal and the imaginary parts are equal. Given complex numbers $z = a + bi$ and $w = c + di$, $z = w$ if and only if $a = c$ and $b = d$. If we think of z as the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and w as the vector $\begin{bmatrix} c \\ d \end{bmatrix}$, these two vectors are equal if and only if their components are equal.

Example 5.8 If $z = 3 + 2i$ and $w = 3 - 2i$, we can see that $z \neq w$ because $im(z) = 2$, and $im(w) = -2$.

When working with real numbers, we can perform addition, subtraction, multiplication and division. We can also perform these operations with complex numbers. For addition and subtraction of complex numbers, we will associate the operations with vector addition and vector subtraction. For multiplication we will treat complex numbers as polynomials and for division we must first define the inverse of a complex number.

1. Addition and Subtraction of Complex Numbers

Recall that a complex number has two parts, the real part and the imaginary part. When we add and subtract complex numbers we want to add and subtract component-wise, just as we do with vectors. This means add the real part of the complex numbers to the real part and add the imaginary part to the imaginary part. Given two complex numbers $z = a + bi$ and $w = c + di$, we get:

$$z + w = (a + c) + (b + d)i,$$

and

$$z - w = (a - c) + (b - d)i.$$

Example 5.9 Let $z = 3 + 2i$ and $w = 4 - 7i$. Then

$$\begin{aligned} z + w &= 7 - 5i \\ w + z &= 7 - 5i \\ z - w &= -1 + 9i \\ w - z &= 1 - 9i. \end{aligned}$$

In the example we see an illustration of the fact that while addition of complex numbers is commutative, subtraction is not.

2. Multiplication of Complex Numbers

Multiplication of complex numbers is not done component-wise. Instead, we treat multiplication of complex numbers like multiplication of binomials. Each component of the first complex number is multiplied by each component of the second complex number. The important fact to remember when multiplying complex numbers is that $i^2 = -1$. Given two complex numbers $z = a + bi$ and $w = c + di$,

$$\begin{aligned} z \cdot w &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= \underbrace{(ac - bd)}_{\text{real part}} + \underbrace{(ad + bc)}_{\text{imaginary part}} i. \end{aligned}$$

Example 5.10 Let $z = 3 + 2i$ and $w = 4 - 7i$. Then

$$z \cdot w = (3 + 2i)(4 - 7i) = 12 - 21i + 8i - 14i^2 = 12 - 13i + 14 = 26 - 13i.$$

Multiplication of complex numbers, like addition, is commutative.

3. Division of Complex Numbers

Division of complex numbers requires that we first acquire new tools. Complex numbers come in pairs, in the sense that every complex number z has a *complex conjugate* associated with it. If $z = a + bi$, we denote the conjugate of z by putting a bar over the number $\bar{z}$ or $\overline{a+bi}$. The complex conjugate is obtained by reversing the sign of the imaginary part so that $im(\bar{z}) = -im(z)$. The real parts do not change: $re(\bar{z}) = re(z)$.

$$\bar{z} = \overline{a+bi} = a - bi.$$

Example 5.11 Find the complex conjugates of $3+4i$ and $-6-2i$.

$$\begin{aligned} \overline{3+4i} &= 3-4i, \\ \overline{-6-2i} &= -6+2i. \end{aligned}$$

If we multiply a nonzero complex number by its conjugate, we get a real, nonzero number, i.e.,

$$z\bar{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2.$$

Using this fact we can define the *magnitude* of a complex number z, denoted $|z|$, as $\sqrt{z\bar{z}}$

$$\boxed{|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}.}$$

This is sometimes called the *modulus of* z.

Example 5.12 Find the magnitude of $z = 3+2i$.

$$|3+2i| = \sqrt{(3+2i)(3-2i)} = \sqrt{9+4} = \sqrt{13}.$$

Let z be a real number $z = a + 0i$. Then, $|z| = \sqrt{a^2+0^2} = \sqrt{a^2} = |a|$. So the absolute value of a real number is a special case of the magnitude of z. Some other properties of complex numbers are:

1. $z = \bar{z}$ if and only $z \in R$.
2. $\overline{w+z} = \bar{w} + \bar{z}$.
3. $\overline{wz} = \bar{w} \cdot \bar{z}$.
4. $z\bar{z} = |z|^2$.
5. $|wz| = |w|\,|z|$.

6. $|w+z| \leq |w| + |z|$.

To define division of complex numbers we need the multiplicative inverse of a complex number. If $z \neq 0$, then z has a multiplicative inverse which we define as follows:

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

We can use this multiplicative inverse to perform division of complex numbers. Let w, z be two complex numbers, $z \neq 0$. Then

$$\frac{w}{z} = w \cdot z^{-1} = w\frac{\bar{z}}{z\bar{z}} = \frac{w\bar{z}}{z\bar{z}}$$

(We are multiplying $\frac{w}{z}$ by 1, where $1 = \frac{\bar{z}}{\bar{z}}$.)

Example 5.13 Let $w = 3+2i$ and $z = 4-2i$. Calculate $z\bar{z}$, $|z|$, and $\frac{w}{z}$.

$$z\bar{z} = (4-2i)(4+2i) = 16 - 4i^2 = 16 + 4 = 20.$$

$$|z| = \sqrt{a^2+b^2} = \sqrt{4^2+(-2)^2} = \sqrt{16+4} = \sqrt{20}.$$

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}} = \frac{(3+2i)(4+2i)}{(4-2i)(4+2i)} = \frac{12+14i+4i^2}{20} = \frac{8+14i}{20} = \frac{2}{5} + \frac{7}{10}i.$$

C. THE COMPLEX PLANE

We can plot a complex number as a point on the complex plane in the same way that we plot a vector in R^2 as a point on the cartesian plane, see figure 12. The complex plane looks like the cartesian plane except the horizontal axis is called the real axis, representing real numbers of the form $(a, 0)$, and the vertical axis is called the imaginary axis representing *imaginary numbers* of the form $(0, b)$. We then plot the complex number $a + bi$ as the point with cartesian coordinates (a, b) on the complex plane. The point (a, b) may then be labeled $a + bi$. Recall that we found that a real number is a special case of a complex number $z = a + bi$, where $b = 0$. If we plot the real numbers we see that real numbers are ordered. Given two real numbers $u, w \in R$, either $u < w$, $u > w$, or $u = w$. Complex numbers cannot be so ordered.

When we plot a complex number $z = a + bi$ and its conjugate $\bar{z} = a - bi$ on the

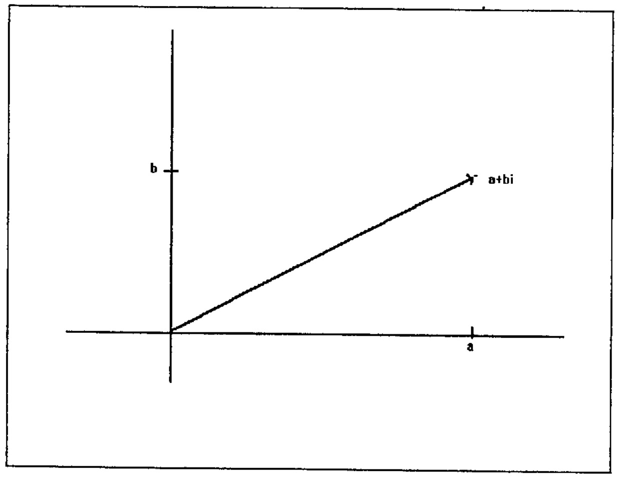

Figure 12. The Complex Plane

complex plane, we see that the complex conjugate is a reflection of the original complex number across the real axis (horizontal axis), see figure 13. We can calculate the distance from the origin to the point representing the complex number by computing the magnitude of the vector from the origin to the point representing the complex number. We already know the formula. Let $z = a + bi$. The distance from the origin to the point z is

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

The geometric interpretation of complex addition and subtraction in the complex plane is completed in the same way that we add vectors in the real plane using the parallelogram rule. Let $z = a + bi$ and $w = c + di$. Then $z + w$ is the point in the complex plane which is the 4th vertex of a parallelogram formed by the origin, z, w, and $z + w$, see figure 14. The geometric interpretation of complex multiplication in the complex plane requires that we take a side trip and learn how to represent a point in a plane using a new coordinate system.

In example 5.14 we locate the point z in the plane by beginning at the origin and projecting a ray from the origin along the positive real axis. This is called an *initial ray*. The initial ray has angle $\theta = 0°$. We next project a second ray from the origin in the direction of the vector. A point z with cartesian coordinates (a, b) can also be represented using *polar coordinates* (r, θ), where r is the distance from the origin to z and θ is the angle between the positive real axis and the vector representing z. By definition of modulus, we have $r = |z|$.

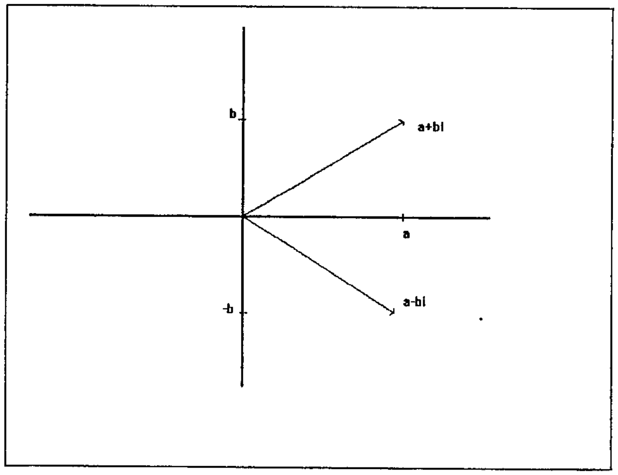

Figure 13. Complex Conjugate

We call θ the *argument* of z, denoted $\arg(z)$.

Example 5.14 Let $z = 1 + i$. In the complex plane, z is represented by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which has cartesian coordinates $(1, 1)$, see figure 15. We can also represent the point z by the coordinate $\left(\sqrt{2}, \frac{\pi}{4}\right)$ where $\sqrt{2}$ is the distance from the origin to the point z and $\frac{\pi}{4}$ is the angle from the positive real axis to the vector representing the complex number.

To find θ, we first need to define some basic concepts.

1. θ is positive when measured counterclockwise from the positive real axis, and negative when measured clockwise from the positive real axis.

2. θ is not unique. The angle $\frac{\pi}{4}$ is the same angle as $\frac{9\pi}{4}$ which is the same angle as $\frac{-7\pi}{4}$. In general, any angle θ is the same as the angle $\theta + 2k\pi$, where k is any integer. So, if $z \neq 0$, then $\arg(z)$ takes on infinitely many values..

We use these concepts to further define what we mean by $\arg(z)$. The argument of z can be viewed as the set of all arguments of z, a set containing infinitely many elements.

$$\arg(z) = \theta + 2k\pi, \qquad k \text{ any integer.}$$

The *principal argument* is the single argument of z that is in the range $-\pi < \theta \leq \pi$.

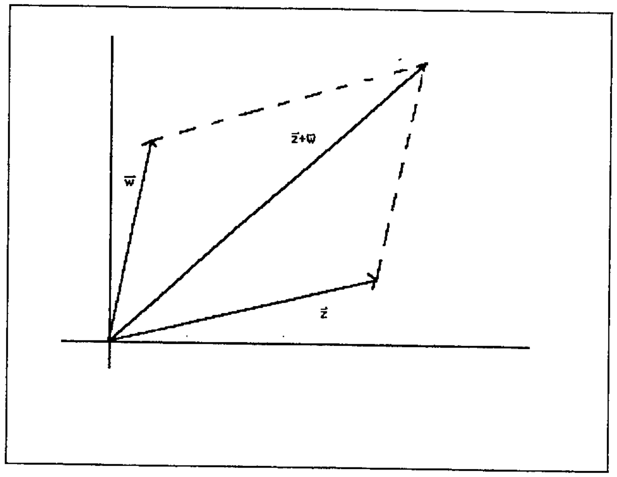

Figure 14. Addition of Complex Numbers

Just as the complex number z has cartesian coordinates and a representation in cartesian form, the same complex number has polar coordinates and a representation in polar form. Consider the complex number $z = a + bi$. In the complex plane, z is the point (a, b). z can also be written as the point (r, θ). Since θ is the angle between the positive real axis and the directed line segment from the origin to the point (a, b) and r is the magnitude of that directed line segment, $r = |z| = \sqrt{a^2 + b^2}$, from basic trigonometry we have $\cos\theta = \frac{a}{r}$, and $\sin\theta = \frac{b}{r}$. We can then write $a = r\cos\theta$, and $b = r\sin\theta$. If we substitute these values for a and b into $a + bi$, we obtain:

$$a + bi = (r\cos\theta) + (r\sin\theta)\, i.$$

Factor out an r and we obtain:

$$a + bi = r\left(\cos\theta + i\sin\theta\right)$$

So, we can change from cartesian form to polar form and vice versa for z using the following formulas:

$$a = r\cos\theta, \quad b = r\sin\theta, \quad r = \sqrt{a^2 + b^2}, \quad \tan\theta = \frac{b}{a},$$

which gives us:

cartesian coordinates: (a, b)	cartesian form: $a + bi$
polar coordinates: (r, θ)	polar form: $r\left(\cos\theta + i\sin\theta\right)$

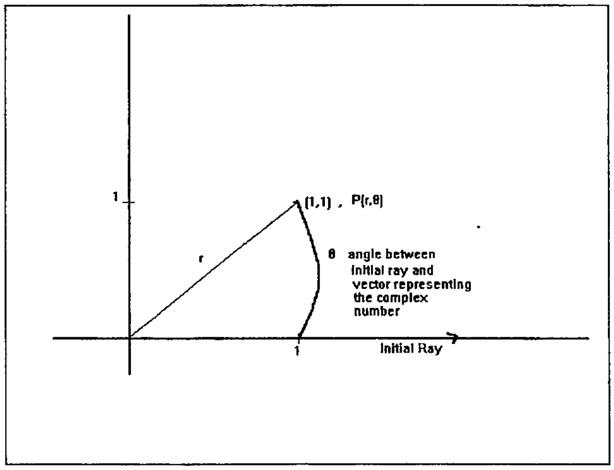

Figure 15. Polar Coordinates

Given a complex number $z = a + bi$, the polar form of the equation for z is

$$\begin{aligned} a + bi &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

The most common mistake made in converting from the cartesian to the polar form of the equation is the miscalculation of the argument θ. When calculating θ we must make sure that we locate the argument in the correct quadrant of the plane. A mistake can occur when we calculate θ using the formula $\tan\theta = \frac{b}{a}$, but we do not verify that the solution is consistent with the complex number z.

Example 5.15 Consider the two complex numbers, $z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, and $w = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, see figure 16. When we plot the points representing z and w, we find that z is in the fourth quadrant of the complex plane with argument $\theta = \frac{-\pi}{4}$, while w is in the second quadrant of the complex plane with argument $\phi = \frac{3\pi}{4}$. If we do not think about the physical placement of the angles in the plane, and use only the formula, we find that

$$\arg(z) = \tan^{-1}\frac{b}{a} = \tan^{-1}(-1)$$

and

$$\arg(w) = \tan^{-1}\frac{b}{a} = \tan^{-1}(-1).$$

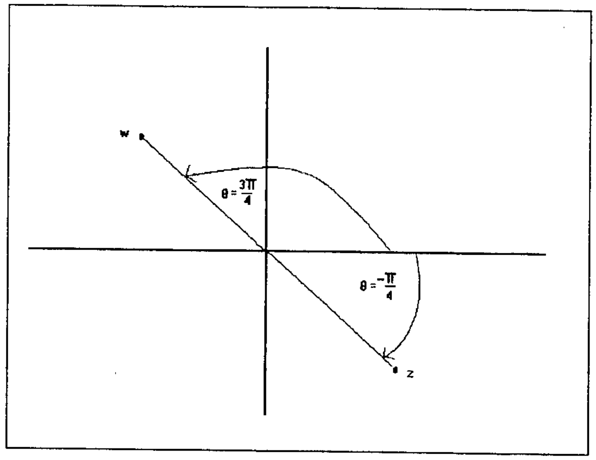

Figure 16. $z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ and $w = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

If we are not conscious of the quadrant we want, we could erroneously choose

$$\arg(z) = \arg(w) = \frac{-\pi}{4},$$

or

$$\arg(z) = \arg(w) = \frac{3\pi}{4},$$

or switch the arguments altogether.

Now we return to multiplication of complex numbers using polar coordinates. Given two complex numbers $z = a+bi = |z|(\cos\theta + i\sin\theta)$ and $w = c+di = |w|(\cos\phi + i\sin\phi)$ we calculate zw using the "foil" method as follows:

$$\begin{aligned}
zw &= [|z|(\cos\theta + i\sin\theta)] \cdot [|w|(\cos\phi + i\sin\phi)] \\
&= [|z|\cos\theta + |z|\, i\sin\theta] \cdot [|w|\cos\phi + |w|\, i\sin\phi] \\
&= |z||w|\cos\theta\cos\phi + |z||w|\cos\theta i\sin\phi + |z||w|\cos\phi i\sin\theta + |z||w|\, i\sin\theta i\sin\phi \\
&= |z||w|\cos\theta\cos\phi + |z||w|\cos\theta i\sin\phi + |z||w|\cos\phi i\sin\theta + |z||w|\, i^2\sin\theta\sin\phi
\end{aligned}$$

Factor out $|z||w|$ and regroup the equation into $re(zw)$ and $im(zw)$, remembering that $i^2 = -1$:

$$|z||w|\left[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \cos\phi\sin\theta)\right].$$

Using trigonometric identities we get the final form,

$$zw = |z|\,|w|\,[\cos(\theta + \phi) + i \sin(\theta + \phi)].$$

To put this in words, the magnitude of the product zw is equal to the product of the magnitude of z with the magnitude of w, and the argument of the product zw is equal to the sum of the argument of z with the argument of w.

Example 5.16 Let $z = 1 + i$ and let $w = 4i$.

Convert z and w to polar form. Then, compute zw in both cartesian and polar form.

Solution:
First we calculate the magnitudes of z and w

$$|z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$|w| = \sqrt{a^2 + b^2} = \sqrt{0^2 + 4^2} = \sqrt{16} = 4.$$

Now we compute the argument for z:

$$\begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{1}{1} \\ &= \tan^{-1} 1. \end{aligned}$$

Our choices are $\theta = \frac{-3\pi}{4}$ or $\theta = \frac{\pi}{4}$. Since z is in the first quadrant we want $\theta = \arg(z) = \frac{\pi}{4}$. Thus the polar equation for z is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right).$$

Now compute the argument for w:

$$\begin{aligned} \phi &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{4}{0}, \end{aligned}$$

which is undefined. Our choices are $\phi = \frac{\pi}{2}$ or $\phi = \frac{-\pi}{2}$. Since w is a positive pure imaginary number, we want $\phi = \arg(w) = \frac{\pi}{2}$. So the polar equation for w is

$$w = 4\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right).$$

Finally, we can compute zw. In cartesian form we get

$$\begin{aligned} zw &= (1+i)(0+4i) \\ &= 0+4i+0+4i^2 \\ &= -4+4i. \end{aligned}$$

In polar form we use the formula we developed for multiplication using polar coordinates, to obtain

$$\begin{aligned} zw &= \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\cdot 4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) \\ &= |z|\,|w|\,[\cos(\theta+\phi)+i\sin(\theta+\phi)] \\ &= \sqrt{2}\cdot 4\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right) \\ &= \sqrt{2}\cdot 4\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right). \end{aligned}$$

Note that if we evaluate the sine and cosine we obtain

$$\begin{aligned} zw &= \sqrt{2}\cdot 4\left(\frac{-\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) \\ &= 2(-2)+i2(2) \\ &= -4+4i, \end{aligned}$$

which is the cartesian form of $z+w$ as computed above.

Suppose we let $w=z$. Then when we compute zw, we are actually computing $zz=z^2$. If we multiply the result z^2 with z, we are actually computing z^3. We can continue in this manner to compute z^n. Once we know how to multiply complex numbers using their polar form, we can compute powers of complex numbers.

Let $z=r(\cos\theta+i\sin\theta)$. Then

$$\begin{aligned} zz &= z^2 = rr(\cos(\theta+\theta)+i\sin(\theta+\theta)) \\ &= r^2(\cos 2\theta+i\sin 2\theta). \end{aligned}$$

Let's try another step.

$$z^2 z = z^3 = r^2 r(\cos(2\theta+\theta)+i\sin(2\theta+\theta))$$

$$= \quad r^3 (\cos 3\theta + i \sin 3\theta).$$

This process continues and yields a general formula known as *DeMoivre's Theorem,* which states that if $z = r(\cos\theta + i\sin\theta)$, then

$$\boxed{z^n = r^n (\cos n\theta + i \sin n\theta) \quad n = 1, 2, ...}$$

If we define $z^0 = 1$ and $z^{-n} = \frac{1}{z^n}$, then, for all integers $n = 0, \pm1, \pm2, ...$, DeMoivre's Theorem will apply. Remember when we have the formula $\frac{1}{z^n}$, we are working with complex division.

Example 5.17 Let $z = 1 + i$. Compute z^4.

Solution:
Using DeMoivre's theorem, we get $z^4 = r^4 (\cos 4\theta + i \sin 4\theta)$. We already know the values for r and θ from the previous example:

$$\begin{aligned} z^4 &= \left(\sqrt{2}\right)^4 \left(\cos 4\left(\frac{\pi}{4}\right) + i \sin 4\left(\frac{\pi}{4}\right)\right) \\ &= 4(\cos \pi + i \sin \pi) \\ &= -4. \end{aligned}$$

If we can compute powers of complex numbers, it would seem logical that we can compute roots of complex numbers. Consider $z^{\frac{1}{n}}$ where n is a positive integer. We call $z^{\frac{1}{n}}$ the nth root of z. We will use DeMoivre's theorem to develop a formula for the nth roots of a complex number.

Let $w = z^{\frac{1}{n}}$. This means $z = w^n$. Now write z and w in polar form:

$$z = r(\cos\theta + i\sin\theta),$$

and

$$w = \rho(\cos\phi + i\sin\phi).$$

We also know how to write w^n using DeMoivre's Theorem:

$$w^n = \rho^n (\cos n\phi + i \sin n\phi).$$

We know that $z = w^n$. In polar form, we have

$$r(\cos\theta + i\sin\theta) = \rho^n(\cos n\phi + i\sin n\phi).$$

Now we equate the magnitudes, obtaining $r = \rho^n$, and it follows that $\rho = r^{\frac{1}{n}}$. Since r is the magnitude of z, $r \geq 0$. Therefore, ρ is the nonnegative nth root of r. Now equate the arguments, obtaining

$$\theta = n\phi.$$

Recall, that θ is not unique. Any angle θ is equal to the angle $\theta + 2k\pi$, for any integer k. So if k is an integer, we have

$$n\phi = \theta + 2k\pi$$

or

$$\phi = \frac{\theta + 2k\pi}{n}.$$

We now rewrite w using these values for ρ and ϕ :

$$\begin{aligned} w &= \rho(\cos\phi + i\sin\phi) \\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right). \end{aligned}$$

We began with the assumption

$$w = z^{\frac{1}{n}}.$$

This means

$$w = z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right).$$

Let's change the format of the above equation to:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{1}{n}(\theta + 2k\pi) + i\sin\frac{1}{n}(\theta + 2k\pi)\right).$$

When we let $k = 0, 1, ..., n - 1$, we get exactly n distinct values of $z^{\frac{1}{n}}$. These n distinct values are the nth roots of the complex number z. Notice that the n distinct roots are also complex numbers. What happens if we let $k \geq n$? The result will be a *repeated root.* Let's look a little closer at the equation,

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{1}{n}(\theta + 2k\pi) + i\sin\frac{1}{n}(\theta + 2k\pi)\right), \quad k = 0, 1, ..., n - 1$$

If we let $m = \frac{1}{n}$ and $\theta = \theta + 2k\pi$, then we can write

$$z^m = r^m \left(\cos m\theta + i \sin m\theta\right).$$

This is DeMoivre's formula. The formula for finding the roots of a complex number is summarized in the following theorem. [Ref. 2]

Theorem 5.4 *Every nonzero complex number $z = r(\cos\theta + i\sin\theta)$ has n distinct nth roots for any positive integer n. Each root has magnitude $r^{\frac{1}{n}}$ and the arguments are the angles $\frac{\theta+2k\pi}{n}$ for $k = 0, 1, ...n-1$.*

Theorem 5.4says that a complex number z has n distinct nth roots for any positive integer n. This means a complex number z has two square roots, three cube roots, four fourth roots, ..., n nth roots. Let's do an example.

Example 5.18 Let $z = 1 + 0i$. Find the formula for the nth roots of z.

We want the equation,

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left(\cos\frac{1}{n}(\theta + 2k\pi) + i\sin\frac{1}{n}(\theta + 2k\pi)\right).$$

Begin by finding r:

$$r = |z| = \sqrt{1^2 + 0^2} = 1.$$

Now find the principal argument θ:

$$\begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{0}{a} \\ &= \tan^{-1} 0. \end{aligned}$$

We have two choices for θ, $\theta = 0$ or $\theta = \pi$. Since z is a positive pure real number, we want $\theta = 0$. Substituting $1, 0$ for r, θ, respectively, we have

$$\begin{aligned} z^{\frac{1}{n}} &= r^{\frac{1}{n}} \left(\cos\frac{1}{n}(\theta + 2k\pi) + i\sin\frac{1}{n}(\theta + 2k\pi)\right) \\ &= 1^{\frac{1}{n}} \left(\cos\frac{1}{n}(0 + 2k\pi) + i\sin\frac{1}{n}(0 + 2k\pi)\right) \\ &= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right), \quad k = 0, 1, 2, ..., n-1. \end{aligned}$$

This will give us n nth roots of 1. These are also known as , the *nth roots of unity*. Let's use this formula to find the five fifth roots of z. Setting $n = 5$,

$$z^{\frac{1}{5}} = \left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right), \quad k = 0, 1, 2, 3, 4,$$

we obtain the formula for the five fifth roots of z. Evaluate for each k.

For $k = 0$, this takes on the value

$$\left(\cos\frac{2\,(0)\,\pi}{5} + i\sin\frac{2\,(0)\,\pi}{5}\right) = \cos 0 + i\sin 0 = 1.$$

For $k = 1$, this takes on the value

$$\left(\cos\frac{2\,(1)\,\pi}{5} + i\sin\frac{2\,(1)\,\pi}{5}\right) = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} \approx 0.309 + 0.951i.$$

For $k = 2$, this takes on the value

$$\left(\cos\frac{2\,(2)\,\pi}{5} + i\sin\frac{2\,(2)\,\pi}{5}\right) = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} \approx -0.809 + 0.588i.$$

For $k = 3$, this takes on the value

$$\left(\cos\frac{2\,(3)\,\pi}{5} + i\sin\frac{2\,(3)\,\pi}{5}\right) = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} \approx -0.809 - 0.588i.$$

For $k = 4$, this takes on the value

$$\left(\cos\frac{2\,(4)\,\pi}{5} + i\sin\frac{2\,(4)\,\pi}{5}\right) = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} \approx 0.309 - 0.951i.$$

Just for fun, let's compute the root for $k = 5$,

$$\begin{aligned}\left(\cos\frac{2\,(5)\,\pi}{5} + i\sin\frac{2\,(5)\,\pi}{5}\right) &= \cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5} \\ &= \cos 2\pi + i\sin 2\pi \\ &= 1.\end{aligned}$$

This is the same root that we got when we let $k = 0$, a repeated root as stated earlier.

If we plot each root we will produce figure 17. We see that the five roots are the ver-

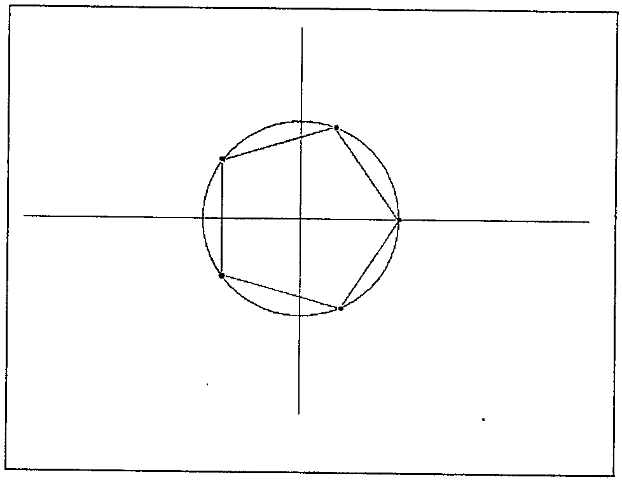

Figure 17. 5th Roots of the Unity

tices of a regular polygon of 5 sides inscribed in the unit circle around the origin, with one vertex located at $(1, 0)$.

It turns out that for any complex number z, the kth roots of z are equally spaced around a circle of radius $\sqrt[k]{|z|}$ centered about the origin.

Example 5.19 Let $z = 1 - i$. Find the fourth roots of z.

Begin by finding the magnitude of z

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Now find the principal argument θ for z.

$$\begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{-1}{1}\\ &= \tan^{-1}-1.\end{aligned}$$

The two choices for θ are $\theta = \frac{-\pi}{4}$ and $\theta = \frac{3\pi}{4}$. Since z is in the fourth quadrant, we want $\theta = \frac{-\pi}{4}$. Now substitute into the formula

$$z^{\frac{1}{4}} = r^{\frac{1}{4}}\left(\cos\frac{1}{4}\left(\theta + 2k\pi\right) + i\sin\frac{1}{4}\left(\theta + 2k\pi\right)\right)$$

$$= \sqrt{2}^{\frac{1}{4}} \left[\cos\left(\frac{1}{4}\right)\left(\frac{-\pi}{4} + 2k\pi\right) + i \sin\left(\frac{1}{4}\right)\left(\frac{-\pi}{4} + 2k\pi\right)\right], \quad k = 0, 1, 2, 3.$$

For $k = 0$, this takes on the value

$$2^{\frac{1}{8}} \left[\cos\left(\frac{-\pi/4}{4}\right) + i \sin\left(\frac{-\pi/4}{4}\right)\right]$$
$$= 2^{\frac{1}{8}} \left[\cos\frac{-\pi}{16} + i \sin\frac{-\pi}{16}\right].$$

For $k = 1$, this takes on the value

$$2^{\frac{1}{8}} \left[\cos\left(\frac{-\pi/4 + 2\pi}{4}\right) + i \sin\left(\frac{-\pi/4 + 2\pi}{4}\right)\right]$$
$$= 2^{\frac{1}{8}} \left[\cos\frac{7\pi}{16} + i \sin\frac{7\pi}{16}\right].$$

For $k = 2$, this takes on the value

$$2^{\frac{1}{8}} \left[\cos\left(\frac{-\pi/4 + 4\pi}{4}\right) + i \sin\left(\frac{-\pi/4 + 4\pi}{4}\right)\right]$$
$$= 2^{\frac{1}{8}} \left[\cos\frac{15\pi}{16} + i \sin\frac{15\pi}{16}\right].$$

For $k = 3$, this takes on the value

$$2^{\frac{1}{8}} \left[\cos\left(\frac{-\pi/4 + 6\pi}{4}\right) + i \sin\left(\frac{-\pi/4 + 6\pi}{4}\right)\right]$$
$$= 2^{\frac{1}{8}} \left[\cos\frac{23\pi}{16} + i \sin\frac{23\pi}{16}\right].$$

If we plot these values, they will be the vertices of a polygon with 4 sides centered around the origin, inscribed in a circle of radius $2^{\frac{1}{8}}$.

Before leaving our study of complex numbers, we want to consider one more use of complex numbers in the exponential function e^x.

D. COMPLEX EXPONENTIAL FUNCTION

A well known result in calculus is the ability to compute the exact value of the functions e^x, $\sin x$ and $\cos x$ using the formulas:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{V.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \tag{V.7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \tag{V.8}$$

We want to define e^x, where x is a complex number $a + bi$. We have the following rule of exponents:

$$e^{a+b} = e^a e^b.$$

This means we can write

$$e^{a+bi} = e^a e^{bi}.$$

We can compute e^a using the formula V.6:

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} \cdots.$$

But we do not have a rule for complex exponents, so how do we represent e^{bi}? Begin by using the formula V.6.

$$\begin{aligned}
e^{bi} &= 1 + bi + \frac{(bi)^2}{2!} + \frac{(bi)^3}{3!} + \frac{(bi)^4}{4!} + \frac{(bi)^5}{5!} + \frac{(bi)^6}{6!} + \cdots \\
&= 1 + \frac{(bi)^2}{2!} + \frac{(bi)^4}{4!} + \frac{(bi)^6}{6!} + \cdots + \left[bi + \frac{(bi)^3}{3!} + \frac{(bi)^5}{5!} + \cdots \right] \\
&= 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \cdots + i \left[b - \frac{b^3}{3!} + \frac{b^5}{5!} + \cdots \right] \\
&= \cos(b) + i \sin(b),
\end{aligned}$$

which means we can write

$$\boxed{e^{a+bi} = e^a e^{bi} = e^a \left[\cos(b) + i \sin(b)\right].}$$

This is known as *Euler's formula*. Notice that this is a polar form for e^{a+bi} where the magnitude of e^{a+bi} is e^a. *Euler's identity* is a result of Euler's formula when $a = 0$.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta.}$$

What happens if we have $e^{-i\theta}$? Using Euler's identity and the facts that:

$$\begin{aligned} \cos(-\theta) &= \cos\theta, \\ \sin(-\theta) &= -\sin\theta, \end{aligned}$$

and

$$\cos^2\theta + \sin^2\theta = 1,$$

we can write

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta. \end{aligned}$$

We also know that

$$\begin{aligned} e^{-i\theta} &= \frac{1}{e^{i\theta}} \\ &= \frac{1}{\cos\theta + i\sin\theta}. \end{aligned}$$

Remember, this is complex division:

$$\begin{aligned} \frac{1}{\cos\theta + i\sin\theta} &= \frac{1}{\cos\theta + i\sin\theta} \cdot \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\ &= \cos\theta - i\sin\theta. \end{aligned}$$

We can conclude that

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Example 5.20 Given $e^{\pi i}$, we write the cartesian form $(a+bi)$:
Solution:

$$\begin{aligned} e^{a+bi} &= e^{a}\left[\cos(b) + i\sin(b)\right] \\ e^{\pi i} &= e^{0}\left[\cos(\pi) + i\sin(\pi)\right] \\ &= 1\left[-1+0\right] \\ &= -1. \end{aligned}$$

Example 5.21 Given e^{2-3i}, write the cartesian form $(a+bi)$:
Solution:

$$\begin{aligned} e^{2-3i} &= e^{2}\left[\cos(-3) + i\sin(-3)\right] \\ &= e^{2}\left[\cos(3) - i\sin(3)\right]. \end{aligned}$$

Example 5.22 Given $e^{4+\frac{\pi}{2}i}$, write the cartesian form:
Solution:

$$\begin{aligned} e^{4+\frac{\pi}{2}i} &= e^{4}\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] \\ &= e^{4}\left[0+i\right] \\ &= ie^{4}. \end{aligned}$$

Example 5.23 Given $e^{\frac{\pi}{4}+\frac{\pi}{4}i}$, write the cartesian form:
Solution:

$$\begin{aligned} e^{a+bi} &= e^{a}\left[\cos(b) + i\sin(b)\right] \\ e^{\frac{\pi}{4}+\frac{\pi}{4}i} &= e^{\frac{\pi}{4}}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] \\ &= e^{\frac{\pi}{4}}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] \\ &= \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}} + i\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}}. \end{aligned}$$

Example 5.24 Given e^{2-2i}, write in cartesian form.
Solution:

$$\begin{aligned} e^{2-2i} &= e^{2}\left[\cos(-2) + i\sin(-2)\right] \\ &= e^{2}\left[\cos(2) - i\sin(2)\right] \\ &= e^{2}\cos(2) - ie^{2}\sin(2). \end{aligned}$$

This gives us a new way to write the polar form of z.

$$\begin{aligned} z &= a+bi \\ &= |z| \underbrace{[\cos\theta + i\sin\theta]}_{\text{Euler's idenity}} \\ &= |z| e^{i\theta} \\ &= re^{i\theta}. \end{aligned}$$

Example 5.25 Given $z = 3i$, write the polar form using the exponential function.

First we find $|z|$:

$$|z| = \sqrt{3^2} = 3.$$

$$\begin{aligned} z &= |z| [\cos\theta + i\sin\theta] \\ &= 3 [\cos\theta + i\sin\theta]. \end{aligned}$$

Then we find $\arg(z)$:

$$\tan^{-1}\theta = \frac{3}{0}, \quad \text{undefined}$$

$$\theta = \frac{\pi}{2}.$$

Finally we substitute into the polar equation:

$$\begin{aligned} z = 3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] \\ = 3e^{\frac{\pi}{2}i} \end{aligned}$$

Example 5.26 Given $z = 1 - i$, write the polar form using the exponential function.

Find $|z|$.

$$|z| = \sqrt{1^2 + (-i)^2} = \sqrt{2}.$$

$$\begin{aligned} 1 - i &= \sqrt{2} [\cos\theta + i\sin\theta] \\ &= \sqrt{2}\left[\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right] \\ &= \sqrt{2}e^{\frac{-\pi}{4}i} \end{aligned}$$

We conclude this section with properties of the complex exponential function.

Properties of the complex exponential function

Given complex number $z = a + bi$ and $w - c + di$.

1. $e^z e^w = e^{z+w}$.
2. $e^z \neq 0$, for all z.
3. $e^{-z} = \frac{1}{e^z}$, for all z.
4. $\frac{e^z}{e^w} = e^{z-w}$.
5. If b is real, $\left|e^{bi}\right| = 1$.

E. COMPLEX EIGENVALUES AND EIGENVECTORS

We study complex eigenvalues to uncover "hidden" information about matrices with real entries that arise in real life problems such as motion, vibration, and rotation in space. The methods we learned for finding real eigenvalues and eigenvectors apply to finding complex eigenvalues and eigenvectors. Let's begin with an example.

Example 5.27 Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Find all eigenvalues and associated eigenvectors for A.

Solution:
We first solve the characteristic equation:

$$\begin{aligned} \det(A - \lambda i) &= 0 \\ \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda + 5 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{4 \pm \sqrt{-4}}{2}. \\ \lambda &= 2 \pm i. \end{aligned}$$

The eigenvalues for the matrix A are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$.
Now we find the eigenvectors associated with each eigenvalue, by solving the corresponding homogeneous system.

For $\lambda_1 = 2 + i$, we have:

$$\begin{aligned}(A - \lambda_1 I)\,\vec{x} &= \vec{0} \\ \begin{bmatrix} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned}$$

Write the augmented matrix:

$$\begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \end{bmatrix}.$$

Add $-i$ times row 1 to row 2:

$$\begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives us the equation,

$$-ix_1 - x_2 = 0.$$

Let x_2 be the independent variable. Then we get

$$\begin{aligned} -ix_1 &= x_2, \text{ or} \\ x_1 &= ix_2. \end{aligned}$$

The eigenvector is

$$\begin{aligned} \vec{x} &= \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} \\ &= x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad x_2 \neq 0. \end{aligned}$$

The parametrized solution is

$$\vec{x} = s \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad s \neq 0.$$

We can check our solution by choosing $s = 1$:

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} &= (2+i) \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2i-1 \\ i+2 \end{bmatrix} &= \begin{bmatrix} 2i-1 \\ i+2 \end{bmatrix}. \end{aligned}$$

Therefore, $\lambda_1 = i$ is an eigenvalue for the matrix A with associated eigenvector $\vec{x} = s \begin{bmatrix} i \\ 1 \end{bmatrix}, s \neq 0.$

For $\lambda = 2 - i$,we have

$$(A - \lambda I)\,\vec{x} = \vec{0}$$

$$\begin{bmatrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Write the augmented matrix:

$$\begin{bmatrix} i & -1 & 0 \\ 1 & i & 0 \end{bmatrix}.$$

Add i times row 1 to row 2:

$$\begin{bmatrix} i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives us the equation,

$$ix_1 - x_2 = 0.$$

Let x_2 be the independent variable. Then we get:

$$\begin{aligned} ix_1 &= x_2, \text{ or} \\ x_1 &= -ix_2. \end{aligned}$$

The eigenvector is

$$\begin{aligned} \vec{x} &= \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} \\ &= x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad x_2 \neq 0. \end{aligned}$$

The parametrized solution is

$$\vec{x} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad t \neq 0.$$

We can check our solution by choosing $t = 1$:

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} &= 2-i \begin{bmatrix} -i \\ 1 \end{bmatrix} \\ \begin{bmatrix} -2i-1 \\ -i+2 \end{bmatrix} &= \begin{bmatrix} -2i-1 \\ -i+2 \end{bmatrix}. \end{aligned}$$

Therefore, $\lambda_2 = 2-i$ is an eigenvalue for the matrix A with associated eigenvector $\vec{x} = t \begin{bmatrix} -1 \\ -i \end{bmatrix}, \quad t \neq 0.$

We can see a special feature of complex eigenvalues in our example.

$$\begin{aligned} \lambda_1 &= 2+i, \\ \lambda_2 &= 2-i. = \overline{\lambda_1} \end{aligned}$$

When the matrix A is real, its complex eigenvalues and complex eigenvectors appear in conjugate pairs.

F EXERCISES

1. Find the characteristic polynomial and all eigenvalues and associated eigenvectors for the following matrices.

a) $\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 3 & -5 \\ 0 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} -2 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

f) $\begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

2. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$?

3. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$?

4. Calculate $(5-4i)(2+2i)$.

5. Calculate $(1+i)+(2+2i)$.

6. Calculate $\frac{2-2i}{3+i}$

7. Calculate $(\overline{2+4i})(3+6i)$.

8. Calculate$\frac{(6-i)(1+2i)}{(4-i)(1+i)}$.

9. Calculate i^3+3i^2-4.

10. Calculate$|3+4i|$.

11. Let $z=5-2i$. Find $|z|$.

12. Let $z=4+4i$,
a) Find the principal argument of z.
b) Write z in polar form.

13. Let $z=4i$, let $w=2+2i$.
a) Compute zw in rectangular form $(a+bi)$.
b) Write z and w in polar form.
c) Compute zw in polar form.
d) Compute z^3 in rectangular form.
e) Compute z^3 in polar form.

14. For each complex number z, compute $|z|$ and the principal argument of z, and write z in polar form (for some problems the principal argument may involve $\tan^{-1}\frac{b}{a}$).
a) $z=2+4i$
b) $z=4-i$
c) $z=6+3i$

15. Write z in rectangular form $(a+bi)$ where $z=2\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$.

16. Find $16^{\frac{1}{4}}$.

17. Write the exponential as a complex number in cartesian form $(a + bi)$.
a) $e^{\frac{\pi}{2}+i\frac{\pi}{2}}$
b) e^{3-5i}

18. Write z in polar form $re^{i\theta}$.
a) $z = 4i$
b) $z = 1 + i$

19. Find all square roots of $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$.

20. Plot the six sixth roots of 1.

21. Find all eigenvalues and eigenvectors of $\begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$.

22. Find all eigenvalues and eigenvectors of $\begin{bmatrix} -4 & -2 \\ 5 & 2 \end{bmatrix}$.

APPENDIX A. SOLUTIONS TO EXERCISES

CHAPTER ONE

1.a. Nonlinear, $(3x_1x_3)$

1.b. Linear

1.c. Nonlinear, $\left(-x_2^{\frac{1}{2}}\right)$

1.d. Linear

1.e. Nonlinear, $(-x_2^{-1})$

1.f. Linear

1.g. Nonlinear, $(4yz)$

1.h. Linear

1.i. Linear

1.j. Nonlinear, $(-3x^3)$

2.a.

$$\underbrace{\begin{bmatrix} -3 & 5 \\ 1 & 2 \\ 4 & -1 \end{bmatrix}}_{\text{Coefficient Matrix}}, \quad \underbrace{\begin{bmatrix} -3 & 5 & 1 \\ 1 & 2 & -4 \\ 4 & -1 & -3 \end{bmatrix}}_{\text{Augmented Matrix}}.$$

2.b.

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 1 & 0 & 5 & 0 & -2 \end{bmatrix}}_{\text{Coefficient Matrix}}, \quad \underbrace{\begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 1 \\ 0 & 1 & -1 & 3 & 0 & -4 \\ 1 & 0 & 5 & 0 & -2 & 6 \end{bmatrix}}_{\text{Augmented Matrix}}.$$

3.a.

$$\left\{ \begin{array}{rcl} x_1 & = & \frac{5}{2}s - \frac{1}{2}t + \frac{3}{2}u - 4 \\ x_2 & = & s \\ x_3 & = & t \\ x_4 & = & u. \end{array} \right. , \quad s, t, u \text{ are scalars.}$$

3.b.

$$\left\{\begin{array}{rcl} v &=& 4q-r-5s-3t \\ w &=& q \\ x &=& r \\ y &=& s \\ z &=& t. \end{array}\right. \quad , \quad q,r,s,t \text{ are scalars.}$$

3.c.

$$\left\{\begin{array}{rcl} x &=& s-6t \\ y &=& s \\ z &=& t \end{array}\right. \quad , \quad s,t \text{ are scalars.}$$

4.a.

$$\begin{array}{rcrcl} 2x &+& y &=& 5 \\ -3x &+& 2y &=& 3. \end{array}$$

Solve for y in the first equation:

$$y = 5 - 2x.$$

Substitute for y in the second equation:

$$\begin{aligned} -3x + 2(5-2x) &= 3 \\ -3x + 10 - 4x &= 3 \\ -7x &= -7 \\ x &= 1. \end{aligned}$$

Solve for y :

$$\begin{aligned} y &= 5 - 2(1) \\ &= 3. \end{aligned}$$

The linear system has a unique solution $\left\{\begin{array}{rcl} x &=& 1 \\ y &=& 3 \end{array}\right.$.

4.b

$$\left\{\begin{array}{rcl} x &=& 1 \\ y &=& 2 \end{array}\right. .$$

4.c

$$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & 6 \\ x & + & 6y & - & 2z & = & 12 \\ -x & + & 4y & + & 6z & = & 24. \end{array}$$

Solve for x in the first equation:

$$x = \frac{3}{2}y + \frac{1}{2}z + 3.$$

Substitute for x in the second and third equations:

$$\begin{aligned} \left(\frac{3}{2}y + \frac{1}{2}z + 3\right) + 6y - 2z &= 12 \\ -\left(\frac{3}{2}y + \frac{1}{2}z + 3\right) + 4y + 6z &= 24. \end{aligned}$$

These equations simplify to:

$$\begin{aligned} 15y - 3z &= 18 \\ 5y + 11z &= 54. \end{aligned}$$

Solve for y in the second equation:

$$y = \frac{1}{5}z + \frac{6}{5}.$$

Substitute into the third equation:

$$\begin{aligned} 5\left(\frac{1}{5}z + \frac{6}{5}\right) + 11z &= 54 \\ z &= 4. \end{aligned}$$

Back substitute to solve for y :

$$\begin{aligned} y &= \frac{1}{5}z + \frac{6}{5}. \\ y &= \frac{1}{5}(4) + \frac{6}{5} \\ y &= 2. \end{aligned}$$

Back substitute to solve for x :

$$\begin{aligned} x &= \frac{3}{2}y + \frac{1}{2}z + 3 \\ &= \frac{3}{2}(2) + \frac{1}{2}(4) + 3 \\ &= 8. \end{aligned}$$

This linear system has the unique solution $\left\{ \begin{aligned} x &= 8 \\ y &= 2 \\ z &= 4 \end{aligned} \right.$.

4.d

$$\begin{array}{rcrcr} & & 6x_2 & = & 6 \\ 4x_1 & + & x_2 & = & -3. \end{array}$$

Solve for x_2 in the first equation:

$$x_2 = 1.$$

Substitute for x_2 in the second equation:

$$\begin{aligned} 4x_1 + 1 &= -3 \\ 4x_1 &= -4 \\ x_1 &= -1. \end{aligned}$$

This linear system has the unique solution $\left\{ \begin{aligned} x_1 &= -1 \\ x_2 &= 1 \end{aligned} \right.$.

4.e.

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 3. \end{array}$$

Solve for x_1 in the first equation:

$$x_1 = x_2 + 3x_3 - 3.$$

Substitute into the second and third equation:

$$\begin{aligned} -2(x_2 + 3x_3 - 3) + x_2 + 5x_3 &= 0 \\ -3(x_2 + 3x_3 - 3) + 2x_2 + 8x_3 &= 3. \end{aligned}$$

These equations simplify to:

$$\begin{aligned} -x_2 - x_3 + 6 &= 0 \\ -x_2 - x_3 + 6 &= 0. \end{aligned}$$

Solve for x_2 in the second equation:

$$x_2 = -x_3 + 6.$$

Substitute into the third equation:

$$\begin{aligned} -(-x_3 + 6) - x_3 + 6 &= 0 \\ 0 &= 0. \end{aligned}$$

This equation is true for any value assigned to x_3; therefore, x_3 is an independent variable. Solve for x_1 in terms of x_3 :

$$\begin{aligned} x_1 &= -x_3 + 6 + 3x_3 - 3 \\ x_1 &= 2x_3 + 3. \end{aligned}$$

This linear system has infinitely many solutions. The general solution is:

$$\begin{cases} x_1 = 2s + 3 \\ x_2 = -s + 6 \\ x_3 = s \end{cases}, \quad s \text{ is a scalar.}$$

4.f.

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 0 \\ -3x_1 & + & 2x_2 & + & 8x_3 & = & 4. \end{array}$$

Solve for x_1 in the first equation:

$$x_1 = x_2 + 3x_3 - 3.$$

Substitute into the second and third equation:

$$-2(x_2 + 3x_3 - 3) + x_2 + 5x_3 = 0$$

$$-3(x_2+3x_3-3)+2x_2+8x_3 = 4.$$

These equations simplify to:

$$\begin{aligned} -x_2-x_3+6 &= 0 \\ -x_2-x_3+5 &= 0. \end{aligned}$$

Solve for x_2 in the second equation:

$$x_2 = -x_3+6.$$

Substitute into the third equation:

$$\begin{aligned} -(-x_3+6)-x_3+5 &= 0 \\ -1 &= 0. \end{aligned}$$

This equation is not true for any value assigned to x_3; therefore, this linear system is inconsistent.

5.a.

$$\underset{\substack{\text{Add -1/2 times row 1 to row 2} \\ \text{Add 1/2 times row 1 to row 3}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} 2 & -3 & -1 & 6 \\ 1 & 6 & -2 & 12 \\ -1 & 4 & 6 & 24 \end{bmatrix}}} \sim \underset{\text{Add -1/3 times row 2 to row 3}}{\begin{bmatrix} 2 & -3 & -1 & 6 \\ 0 & \frac{15}{2} & \frac{-3}{2} & 9 \\ 0 & \frac{5}{2} & \frac{11}{2} & 27 \end{bmatrix}} \sim \underset{\text{This is row echelon form.}}{\begin{bmatrix} 2 & -3 & -1 & 6 \\ 0 & \frac{15}{2} & \frac{-3}{2} & 9 \\ 0 & 0 & 6 & 24 \end{bmatrix}}.$$

Write the corresponding linear system:

$$\begin{array}{rcrcrcr} 2x & - & 3y & - & z & = & 6 \\ & & \frac{15}{2}y & - & \frac{3}{2}z & = & 9 \\ & & & & 6z & = & 24. \end{array}$$

Solve for z in the third equation:

$$z = 4.$$

Back substitute for z into the second equation:

$$\begin{aligned} \frac{15}{2}y-\frac{3}{2}z &= 9 \\ \frac{15}{2}y-\frac{3}{2}(4) &= 9 \end{aligned}$$

$$\begin{aligned} \frac{15}{2}y &= 15 \\ y &= 2. \end{aligned}$$

Back substitute for y and z into the first equations:

$$\begin{aligned} 2x - 3y - z &= 6 \\ 2x - 3(2) - 4 &= 6 \\ 2x &= 16 \\ x &= 8. \end{aligned}$$

This linear system has a unique solution:

$$\begin{cases} x = 8 \\ y = 2 \\ z = 4 \end{cases}.$$

5.b.

$$\underset{\substack{\text{Add -2 times row 1 to row 2} \\ \text{Add -3 times row 1 to row 3}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ -2 & 1 & 5 & 0 \\ -3 & 2 & 8 & 3 \end{bmatrix}}} \sim \underset{\text{Add -1 times row 2 to row 3}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ 0 & -1 & -1 & -6 \\ 0 & -1 & -1 & -6 \end{bmatrix}} \sim \underset{\text{This is row echelon form.}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}.$$

Write the corresponding linear system:

$$\begin{array}{rcrcrcr} -x_1 & + & x_2 & + & 3x_3 & = & 3 \\ & - & x_2 & - & x_3 & = & -6. \end{array}$$

There are two equations and three variables. There is one independent variable. Solve for x_2 in terms of x_3 in the second equation:

$$x_2 = -x_3 + 6.$$

Back substitute for x_2 in the first equation:

$$-x_1 + (-x_3 + 6) + 3x_3 = 3$$

$$\begin{aligned} -x_1 + 2x_3 &= -3 \\ x_1 &= 2x_3 + 3. \end{aligned}$$

This linear system has infinitely many solutions. The general solution is:

$$\begin{cases} x_1 = 2s+3 \\ x_2 = -s+6 \\ x_3 = s \end{cases}, \quad s \text{ is a scalar.}$$

5.c.

$$\underset{\substack{\text{Add -2 times row 1 to row 2} \\ \text{Add -3 times row 1 to row 3}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ -2 & 1 & 5 & 0 \\ -3 & 2 & 8 & 4 \end{bmatrix}}} \sim \underset{\text{Add -1 times row 2 to row 3}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ 0 & -1 & -1 & -6 \\ 0 & -1 & -1 & -5 \end{bmatrix}} \sim \underset{\text{This is row echelon form.}}{\begin{bmatrix} -1 & 1 & 3 & 3 \\ 0 & -1 & -1 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

The augmented matrix has a row $[0\ 0\ 0\ b]$, $b \neq 0$; therefore, the corresponding linear system is inconsistent.

5.d. $$\underset{\substack{\text{Add 1 times row 1 to row 2} \\ \text{Add -2 times row 1 to rows 3\&4}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} 1 & 0 & 1 & 3 & 5 \\ -1 & -2 & 0 & -3 & -9 \\ 2 & 2 & 1 & 2 & 18 \\ 2 & 1 & 1 & 5 & 12 \end{bmatrix}}} \sim \underset{\substack{\text{Add 1 times row 2 to row 3} \\ \text{Add 1/2 times row 2 to row 4}}}{\begin{bmatrix} 1 & 0 & 1 & 3 & 5 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 2 & -1 & -4 & 8 \\ 0 & 1 & -1 & -1 & 2 \end{bmatrix}} \sim \underset{\text{Interchange rows 3\&4}}{\begin{bmatrix} 1 & 0 & 1 & 3 & 5 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 0 & 0 & -4 & 4 \\ 0 & 0 & \frac{-1}{2} & -1 & 0 \end{bmatrix}} \sim$$

$$\underset{\text{This is row echelon form.}}{\begin{bmatrix} 1 & 0 & 1 & 3 & 5 \\ 0 & -2 & 1 & 0 & -4 \\ 0 & 0 & \frac{-1}{2} & -1 & 0 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}}, \qquad \begin{aligned} x_1 \qquad + x_3 + 3x_4 &= 5 \\ -2x_2 + x_3 \qquad &= -4 \\ \tfrac{-1}{2}x_3 - x_4 &= 0 \\ -4x_4 &= 4. \end{aligned}$$

This linear system has the unique solution:

$$\begin{cases} x_1 = 6 \\ x_2 = 3 \\ x_3 = 2 \\ x_4 = -1. \end{cases}$$

5.e. $\underset{\substack{\text{Add 2 times row 1 to row 2}\\ \text{Add 1 times row 1 to rows 3}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} 1 & 1 & 1 & 9 \\ -2 & -1 & -1 & -15 \\ -1 & 1 & 1 & -4 \end{bmatrix}}} \sim \underset{\text{Add -2 times row 2 to row 3}}{\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 5 \end{bmatrix}} \sim \underset{\text{Interchange rows 3\&4}}{\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$

This linear system is inconsistent.

5.f. $\underset{\substack{\text{Add -2 times row 1 to rows 2\&3}\\ \text{Add -3 times row 1 to row 4}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 2 & -8 & 2 & 2 & -10 \\ 2 & 3 & 2 & 2 & 12 \\ 3 & -2 & 3 & 3 & 5 \end{bmatrix}}} \sim \underset{\substack{\text{Add -1/12 times row 2 to row 3}\\ \text{Add -2/3 times row 2 to row 4}}}{\begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 0 & -12 & 0 & 0 & -24 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & -8 & 0 & 0 & -16 \end{bmatrix}} \sim \underset{\text{This is row echelon form.}}{\begin{bmatrix} 1 & 2 & 1 & 1 & 7 \\ 0 & -12 & 0 & 0 & -24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$

This linear system has the general solution $\begin{cases} x_1 &= 3-s-t \\ x_2 &= 2 \\ x_3 &= s \\ x_4 &= t. \end{cases}$

6.a. $\underset{\substack{\text{Add -2 times row 1 to row 2}\\ \text{Add -3 times row 1 to rows 3}}}{\overset{\text{Augmented Matrix}}{\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & 3 & 5 & 5 \\ 3 & 4 & 7 & 8 \end{bmatrix}}} \sim \underset{\text{Add -4/3 times row 2 to row 3}}{\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & -5 \\ 0 & 4 & 1 & -7 \end{bmatrix}} \sim \underset{\substack{\text{Multiply row 2 by 1/3}\\ \text{Multiply row 3 by -3}}}{\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & 1 & -5 \\ 0 & 0 & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}} \sim$

$\underset{\substack{\text{Add -1/3 times row 3 to row 2}\\ \text{Add -2 times row 3 to row 1}}}{\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & \frac{1}{3} & \frac{-5}{3} \\ 0 & 0 & 1 & 1 \end{bmatrix}} \sim \underset{\substack{\text{Multiply row 2 by 1/3}\\ \text{Multiply row 3 by -3}}}{\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}}$, This system has a unique solution.

$$\begin{cases} x_1 &= 3 \\ x_2 &= -2 \\ x_3 &= 1. \end{cases}$$

6.b. $\underset{\text{Add -2 times row 1 to row 2\&3}}{\begin{bmatrix} 1 & 1 & 2 & 5 \\ 2 & 5 & 7 & 19 \\ 2 & 4 & 6 & 16 \end{bmatrix}} \sim \underset{\text{Add -2/3 times row 2 to row 3}}{\begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 2 & 6 \end{bmatrix}} \sim \underset{\text{Multiply row 2 by 1/3}}{\begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 3 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \sim$

$$\begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ This system has infinitely many solutions.

Add -1 times row 2 to row 1

$$\begin{cases} x_1 &= 2-s \\ x_2 &= 3-s \\ x_3 &= s \end{cases}, \quad s \text{ is a scalar.}$$

Augmented Matrix

6.c. $$\begin{bmatrix} 1 & 2 & 1 & 6 \\ 1 & 2 & 2 & 7 \\ 2 & 4 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$ This linear system is inconsistent.

Add -1 times row 1 to row 2
Add -2 times row 1 to rows 3

Augmented Matrix

6.d. $$\begin{bmatrix} -1 & -2 & -1 & -5 \\ 1 & 3 & 2 & 7 \\ 2 & 4 & 2 & 10 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Add 1 times row 1 to row 2
Add 2 times row 1 to rows 3

Add 2 times row 2 to row 1

Multiply row 1 by -1

This linear system has infinitely many solutions:

$$\begin{cases} x_1 &= 1+s \\ x_2 &= 2-s \\ x_3 &= s \end{cases}, \quad s \text{ is a scalar.}$$

7. r = regular beef, d = diet beef:

$$\begin{aligned} .3r + .2d &= 3 \\ .7r + .8d &= 10 \end{aligned}$$

Solve for d in the first equation:

$$d = 15 - 1.5r$$

Substitute for d into the second equation:

$$\begin{aligned} .7r + .8(15 - 1.5r) &= 10 \\ -0.5r + 12 &= 10 \\ -0.5r &= -2 \end{aligned}$$

$$r = 4$$

Back substitute for r to solve for d :

$$\begin{aligned} d &= 15 - 1.5\,(4) \\ d &= 9. \end{aligned}$$

The butcher should make 4 pounds of regular beef and 9 pounds of diet beef.

8. No, if $a = 0$ and $b \neq 0$, the equation becomes $0 = b$, $b \neq 0$, which is not true.

9. Consistent, infinitely many solutions. The system has four variables, two variables are independent.

10.a. Reduced row echelon form.

10.b. Row echelon form.

10.c. Reduced row echelon form.

10.d. Reduced row echelon form.

10.e. Row echelon form

10.f. Neither. There is a row of zeros not at the bottom.

10.g. Row echelon form.

11.a. $\begin{cases} x_1 &= -21 \\ x_2 &= 15/2 \end{cases}$

11.b. $\begin{cases} x_1 &= 23 \\ x_2 &= 10 \end{cases}$

11.c. $\begin{cases} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 3 \end{cases}$

11.d. Inconsistent

11.e. $\left\{\begin{array}{lcl} x_1 & = & -1S+2 \\ x_2 & = & -1/2S+3/2 \\ x_3 & = & 1S+0 \end{array}\right.$, s is a scalar.

11.f. $\left\{\begin{array}{lcl} x_1 & = & \frac{-23}{2}s - t + \frac{27}{2} \\ x_2 & = & \frac{-11}{2}s + \frac{13}{2} \\ x_3 & = & s \\ x_4 & = & t \end{array}\right.$, s, t are scalars.

11.g. Inconsistent.

11.h. Inconsistent.

12.a. $\left\{\begin{array}{lcl} x_1 & = & -7/3s \\ x_2 & = & -8/3s \\ x3 & = & s \end{array}\right.$, s is a scalar.

12.b. $\left\{\begin{array}{lcl} x_1 & = & \frac{-14}{13}s + \frac{1}{13}t \\ x_2 & = & \frac{19}{13}s + \frac{7}{13}t \\ x_3 & = & s \\ x_4 & = & t \end{array}\right.$, s, t are scalars.

12.c. $\left\{\begin{array}{lcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \\ x_4 & = & 0 \end{array}\right.$.

12.d. $\left\{\begin{array}{lcl} x_1 & = & -7s-5t \\ x_2 & = & -2s-t \\ x_3 & = & s \\ x_4 & = & t \end{array}\right.$, s, t are scalars.

12.e. $\left\{\begin{array}{lcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \end{array}\right.$.

CHAPTER TWO

1.a. $(2 \times 4)\,(4 \times 6) = (2 \times 6)$

1.b. $(6 \times 2)\,(2 \times 4) = (6 \times 4)$

1.c. $(2 \times 4)(6 \times 2) =$ undefined.

1.d. $(4 \times 6)(6 \times 2) = (4 \times 2)$

1.e. $(4 \times 6)(2 \times 4) =$ undefined.

2.a. $\begin{bmatrix} -8 \\ 2 \\ -2 \end{bmatrix}$

2.b. $\begin{bmatrix} -4 \\ 3 \\ 5 \\ 0 \\ -9 \end{bmatrix}$

2.c. Undefined, $\vec{y}$, and $\vec{t}$ are not the same size.

2.d. $-3(-3) + 2(2) + -1(-1) = 14$

2.e. Undefined, $\vec{y}$, and $\vec{t}$ are not the same size.

2.f. $-1(-3) + 3(0) + 4(1) + -2(2) + -5(-4) = 23$

2.g. $\sqrt{-1(-1) + 3(3) + 4(4) + -2(-2) + -5(-5)} = \sqrt{55}$

2.h. $\cos\theta = \frac{16}{\sqrt{14}\sqrt{26}}, \quad \theta \approx 33.004°$

2.i. $\begin{bmatrix} -15 \\ 0 \\ 5 \\ 10 \\ -20 \end{bmatrix}$

3.a. $AB = \begin{bmatrix} 18 & 7 & 16 \\ 24 & 16 & 18 \end{bmatrix}, \quad BC = \begin{bmatrix} -3 & 1 \\ 27 & 8 \end{bmatrix}, \quad BD = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 15 & 18 \end{bmatrix},$

$$CA = \begin{bmatrix} -2 & 6 \\ 2 & 4 \\ -1 & 13 \end{bmatrix}, \quad CB = \begin{bmatrix} 0 & 4 & -2 \\ 6 & 3 & 5 \\ 6 & 9 & 2 \end{bmatrix}, \quad DC = \begin{bmatrix} 11 & 5 \\ -4 & 1 \\ 0 & 0 \end{bmatrix}$$

3.b. $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

3.c. $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

3.d. $\begin{bmatrix} 0 & 6 \\ 2 & 3 \\ -1 & 5 \end{bmatrix}$

3.e. $\begin{bmatrix} -2 & 11 \\ -5 & 5 \end{bmatrix}$

3.f. $\begin{bmatrix} -2 & 11 \\ -5 & 5 \end{bmatrix}$

3.g. $\begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix}$

3.h. $\begin{bmatrix} -2 & 0 \\ 0 & -1 \\ -3 & -1 \end{bmatrix}$

3.i. $-1, 1$

3.j. B does not have a main diagonal because B is not a square matrix.

4.a. $\begin{bmatrix} -9 & -3 \\ -3 & -5 \\ 12 & 2 \end{bmatrix}$

4.b. $\begin{bmatrix} 10 & 6 & 5 \\ -4 & 5 & 3 \\ 3 & 0 & 7 \end{bmatrix}$

4.c. $\begin{bmatrix} -2 & 4 & -1 \\ -2 & -5 & -1 \\ -5 & 4 & 1 \end{bmatrix}$

4.d. $\begin{bmatrix} 27 & 25 & 28 \\ -14 & -5 & -6 \\ 8 & 1 & 13 \end{bmatrix}$

4.e. $\begin{bmatrix} 18 & 36 & 25 \\ -21 & -1 & 11 \\ 91 & 26 & 18 \end{bmatrix}$

4.f. $\begin{bmatrix} -21 & -7 \\ 0 & 14 \end{bmatrix}$

4.g. Undefined, $3C$ is a 2×3 matrix and D is a 3×3 matrix.

4.h. $\begin{bmatrix} 54 & 108 & 75 \\ -63 & -3 & 33 \\ 57 & 78 & 54 \end{bmatrix}$

4.i. $\begin{bmatrix} -9 & 18 & -33 \\ -3 & -18 & -31 \\ 12 & -36 & 34 \end{bmatrix}$

4.j. Undefined, $(4B)\,C$ is a 2×3 matrix and $2B$ is a 2×2 matrix.

5.a. $\begin{bmatrix} -6 & -1 & -8 \\ 0 & 8 & -2 \\ 4 & -6 & 7 \end{bmatrix}$

5.b. Undefined, A is a 3×2 matrix and $\vec{x}$ is a 3×1 matrix.

5.c. $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$

5.d. 5

6.a. Undefined, A is a 2×3 matrix and C is a 2×2 matrix.

6.b. $\begin{bmatrix} -20 & -4 \\ 0 & 10 \end{bmatrix}$

6.c. -6

6.d. Undefined, $3\vec{x}$ is a 3×1 matrix and B is a 3×2 matrix

7.a. False, AB is a 3×1 matrix and BA is undefined.

7.b. True

7.c. False, matrix multiplication us not commutative.

8.

$$\begin{aligned}
\vec{x}\cdot\vec{y} &= 0 && \text{orthogonal}\\
\vec{x}\cdot\vec{z} &= 0 && \text{orthogonal}\\
\vec{x}\cdot\vec{w} &= -4 && \text{not orthogonal}\\
\vec{y}\cdot\vec{z} &= 65 && \text{not orthogonal}\\
\vec{y}\cdot\vec{w} &= 0 && \text{orthogonal}\\
\vec{z}\cdot\vec{w} &= 38 && \text{not orthogonal}
\end{aligned}$$

9.

$$\begin{aligned}
\vec{u}\cdot\vec{w} &= 0 && \text{orthogonal}\\
\vec{u}\cdot\vec{z} &= -1 && \cos\theta = \tfrac{-1}{3\sqrt{3}},\ \theta \approx 101.10^\circ\\
\vec{w}\cdot\vec{z} &= 0 && \text{orthogonal}
\end{aligned}$$

10.a. AB is undefined, A is a 2×1 partitioned matrix and B is a 2×1 partitioned matrix.

10.b. $A+B = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_1+B_1 \\ A_2+B_2 \end{bmatrix} = \left[\begin{array}{ccc} 4 & 4 & -1 \\ 1 & -1 & 3 \\ \hline -4 & -2 & 2 \end{array}\right]$

10.c. $BC = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}[C] = \begin{bmatrix} B_1C \\ B_2C \end{bmatrix}$. Multiplication is undefined, B_1 is a 2×3 and C is a 1×3 matrix. B_2 is a 1×3 matrix and C is a 1×3 matrix.

11.a. $\begin{bmatrix} A_1 & A_2 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = [A_1B_1 + A_2B_2] = \begin{bmatrix} 1 & 7 \\ 6 & 10 \end{bmatrix}$.

11.b. $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \end{bmatrix} = \left[\begin{array}{c|c} A_1B_1 & A_1B_2 \\ \hline A_2B_2 & A_2B_2 \end{array}\right] = \left[\begin{array}{cc|c} 10 & 6 & -4 \\ 2 & 0 & -1 \\ 1 & 9 & 1 \\ \hline 5 & 3 & -2 \end{array}\right]$

11.c. $\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] = \left[\begin{array}{c} A_{11}B_1 + A_{12}B_2 \\ \hline A_{21}B_1 + A_{22}B_2 \end{array}\right] = \left[\begin{array}{c} 7 \\ 4 \\ \hline 2 \end{array}\right]$

12.

Day/Class	Class I	Class II	Class III	Class IV
Monday	1	1	0	1
Tuesday	0	1	1	0
Wednesday	1	0	1	1
Thursday	0	1	1	0
Friday	1	0	0	1

Class I	:	2 hours / session
Class II	:	3 hours / session
Class III	:	1 hour / session
Class IV	:	3 hours / session

Each table can be represented by a matrix and multiplied.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 6 \\ 4 \\ 5 \end{bmatrix} \begin{array}{l} \text{hours on Monday} \\ \text{hours on Tuesday} \\ \text{hours on Wednesday} \\ \text{hours on Thursday} \\ \text{hours on Friday} \end{array}$$

13. The tables can be represented by matrices and multiplied together.

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 28 \\ 23 \\ 33 \end{bmatrix} \begin{array}{l} \text{missions on Monday} \\ \text{missions on Tuesday} \\ \text{missions on Wednesday} \\ \text{missions on Thursday} \\ \text{missions on Friday} \end{array}$$

Two workers are required for each mission.

$$2 \begin{bmatrix} 20 \\ 25 \\ 28 \\ 23 \\ 33 \end{bmatrix} = \begin{bmatrix} 40 \\ 50 \\ 56 \\ 46 \\ 66 \end{bmatrix} \begin{array}{l} \text{workers on Monday} \\ \text{workers on Tuesday} \\ \text{workers on Wednesday} \\ \text{workers on Thursday} \\ \text{workers on Friday} \end{array}$$

14. The tables can be represented by matrices and multiplied together.

$$\begin{bmatrix} 50 & 45 & 60 \\ 200 & 250 & 200 \\ 350 & 400 & 300 \\ 50 & 45 & 60 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 265 & 140 & 155 & 265 \\ 1050 & 700 & 650 & 1050 \\ 1700 & 1150 & 1050 & 1700 \\ 265 & 140 & 155 & 265 \end{bmatrix}$$

The budget estimate can be summarized in the following table:

Annual Training Budget Estimate ($1,000.00)

Expenses/Fiscal Quarter	1st Qtr	2nd Qtr	3rd Qtr	4th Qtr	Total Annual Expenses
Equipment	265	140	155	265	825
Fuel	1,050	700	650	1,050	3,450
Repair Parts	1,700	1,150	1,050	1,700	5,600
Meals	265	140	155	265	825
Total Quarterly Expenses	3,280	2,130	2,010	3,280	10,700

CHAPTER THREE

1.a.

$$\begin{bmatrix} 3 & 1 & -4 \\ -2 & -3 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

1.b.

$$\begin{bmatrix} 0 & 4 & -1 & 2 \\ 2 & 1 & 3 & 0 \\ 4 & -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 2 \end{bmatrix}$$

2.

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ -6 \end{bmatrix}$$

3.

$$\begin{array}{rcrcl} 3x_1 & + & 4x_2 & = & 1 \\ -x_1 & + & x_2 & = & 2 \\ 2x_1 & + & 5x_2 & = & 3 \end{array}$$

4.

$$x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

5. Matrix equation:

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ -1 & -2 & 3 & -1 \\ 2 & -1 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

Vector equation:

$$x_1\begin{bmatrix}1\\-1\\2\end{bmatrix}+x_2\begin{bmatrix}1\\-2\\-1\end{bmatrix}+x_3\begin{bmatrix}-2\\3\\5\end{bmatrix}+x_4\begin{bmatrix}1\\-1\\-1\end{bmatrix}=\begin{bmatrix}1\\-1\\4\end{bmatrix}$$

6.

$$\begin{aligned}-4\vec{x}+4\vec{u} &= 5\vec{v}-15\vec{w}\\ -4\vec{x} &= -4\vec{u}+5\vec{v}-15\vec{w}\\ \vec{x} &= \vec{u}-5/4\vec{v}+15/4\vec{w}\end{aligned}$$

7.a. Solve the linear system $A\vec{x}=\vec{b}$ for $\vec{x}$.

$$\begin{bmatrix}2 & 4 & 6\\ -1 & -2 & -3\\ 3 & 1 & 5\end{bmatrix}\underset{\sim}{\text{row equivalent to}}\begin{bmatrix}2 & 4 & 6\\ 0 & -5 & -4\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7/5\\4/5\end{bmatrix}$$

Yes, $\vec{b}$ is a linear combination of the columns of A because the corresponding linear system is consistent.

7.b. Solve the linear system $A\vec{x}=\vec{b}$ for $\vec{x}$.

$$\begin{bmatrix}1 & -3 & -2 & -4\\ -2 & 2 & 6 & -3\\ 2 & -8 & -3 & 2\end{bmatrix}\underset{\sim}{\text{row equivalent to}}\begin{bmatrix}1 & -3 & -2 & -4\\ 0 & -4 & 2 & -11\\ 0 & 0 & 0 & 31/2\end{bmatrix}$$

No, $\vec{b}$ is not a linear combination of the columns of A because the corresponding linear system is inconsistent.

8. Pick an arbitrary vector in R^4, say $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$, where $a, b, c,$ and d are scalars. If the vectors $\vec{v}_1, \vec{v}_2,$and $\vec{v}_3$ span R^4, then all vectors of the form $\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ will be in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. We

need to solve the system $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for c_i, where $i = 1, 2, 3$.

$$\begin{bmatrix} 1 & -1 & 1 & a \\ -1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & -1 & d \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 1 & -1 & 1 & a \\ 0 & -1 & 1 & a+b \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 0 & a+b+c+d \end{bmatrix}$$

No, the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ do not span R^4 because any vector of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, where $a+b+c+d \neq 0$, is not in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

9. Solve the system $A\vec{c} = \vec{x}$ for $\vec{c}$.

$$\begin{bmatrix} 2 & -1 & -5 \\ 4 & 5 & -3 \\ -3 & -2 & 4 \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 2 & -1 & -5 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Yes, $\vec{x}$ is in the plane spanned by the columns of A because the corresponding linear system is consistent.

10. Solve the linear system $c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = \vec{b}$ for $\vec{c}$.

$$\begin{bmatrix} 2 & 0 & 1 & 3 \\ -4 & 3 & 0 & -1 \\ 4 & 3 & 4 & 2 \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

No, $\vec{b}$ is not in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ because the corresponding linear system is inconsistent.

11.a. 3

11.b. No

11.c. Solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ -5 & 3 & -1 & 8 \\ 6 & -2 & 4 & 6 \end{bmatrix} \overset{\text{row equivalent to}}{\sim} \begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & 13 & 21 \\ 0 & 0 & -18 & -18 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$$

Yes, $\vec{b}$ is in W.

11.d. Infinitely many.

11.e. $1\vec{a}_1 + 0\vec{a}_2 + 0\vec{a}_3 = \vec{a}_1$

12.a. Solve $c_1\vec{x} + c_2\vec{y} = \vec{a}$ for $\vec{c}$.

$$\begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & -1 \\ -2 & 5 & 2 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 4 & -1 & 5 \\ 0 & 3 & -1 \\ 0 & 0 & 12 \end{bmatrix}$$

No, $\vec{a}$ is not in Span$\{\vec{x}, \vec{y}\}$ because the corresponding linear system is inconsistent.

12.b. Solve $c_1\vec{x} + c_2\vec{y} = \vec{b}$ for $\vec{c}$.

$$\begin{bmatrix} 4 & -1 & 9 \\ 0 & 3 & -3 \\ -2 & 5 & -9 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 4 & -1 & 9 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Yes, $\vec{b}$ is in Span$\{\vec{x}, \vec{y}\}$.

13.a. Rank is the number of linear independent rows/columns in the matrix. Using Gaussian elimination, the linearly independent rows are the nonzero rows when the matrix is in row echelon form. Rank$(A) = 3$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & -5 \\ 4 & 8 & 5 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

13.b. One way to answer this question is to note that A has 3 pivots; therefore, $A\vec{c} = \vec{x}$ has a unique solution. The other option is to solve $A\vec{c} = \vec{x}$ for $\vec{c}$.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 7 & -5 & 8 \\ 4 & 8 & 5 & -14 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 5 & -10 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

Yes, $\vec{x}$ is in the plane spanned by the columns of A.

14. Solve $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 7 & -2 & -5 & -6 \\ 3 & 4 & 1 & -14 \\ 0 & -6 & 2 & 20 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 7 & -2 & -5 & -6 \\ 0 & 34 & 22 & -80 \\ 0 & 0 & 100 & 100 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}.$$

15. a.

$$A\vec{x} = \begin{bmatrix} 4x_1 & - & 2x_2 & + & 5x_3 \\ -x_1 & + & 3x_2 & + & 8x_3 \\ 2x_1 & + & x_2 & - & 2x_3 \end{bmatrix}$$

15.b. $A\vec{x}$ is undefined.

15.c.

$$A\vec{x} = \begin{bmatrix} t \\ r \\ s \end{bmatrix}$$

16.

$$\begin{aligned} T(\vec{u}) &= A\vec{u} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 6 \end{bmatrix} \end{aligned} \quad \text{and} \quad \begin{aligned} T(\vec{v}) &= A\vec{v} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -6 \end{bmatrix} \end{aligned}$$

17.

$$\begin{bmatrix} 3 & -5 & -1 & -7 \\ 1 & 2 & 1 & 14 \\ 4 & 0 & 2 & 24 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 3 & -5 & -1 & -7 \\ 0 & 11 & 4 & 49 \\ 0 & 0 & 30 & 120 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$\vec{x}$ is unique.

18.

$$\begin{bmatrix} 7 & 0 & 3 & 1 & 0 \\ -5 & 1 & -3 & 3 & 0 \\ 6 & 0 & 4 & -2 & 0 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 7 & 0 & 3 & 1 & 0 \\ 0 & 7 & -6 & 26 & 0 \\ 0 & 0 & 10 & -20 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1s \\ -2s \\ 2s \\ s \end{bmatrix}$$

19.a. $a = 6, \quad b = 4$

19.b. $a = 3, \quad b = 5$

19.c. $a = 4, \quad b = 4$

20. Find an $\vec{x}$ whose image under T is $\vec{b}$.

$$\begin{bmatrix} 7 & 0 & 3 & 1 & 19 \\ -5 & 1 & -3 & 3 & -10 \\ 6 & 0 & 4 & -2 & 12 \end{bmatrix} \underset{\sim}{\text{row equivalent to}} \begin{bmatrix} 7 & 0 & 3 & 1 & 19 \\ 0 & 7 & -6 & 26 & 25 \\ 0 & 0 & 10 & -20 & -30 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s+4 \\ 1-2s \\ -3+2s \\ s \end{bmatrix}$$

Yes, $\vec{b}$ is in the range of the linear transformation.

21. This transformation is a contraction. Since $r < 0$, the transformation also points the directed line segment associated with the vector in the opposite direction.

$$\begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -3/2 \end{bmatrix}$$

$$\begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

22. This transformation is a reflection across the line $y = x$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

23.a.

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ 2x_1 + x_1 - 2x_3 \end{bmatrix}$$

23.b.

$$T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = T\left(\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 13 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

23.c.

$$T(r\vec{u}) = T\left(\begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

$$rT(\vec{u}) = 2\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

23.d. Yes, the transformation is linear because

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(r\vec{u}) = rT(\vec{u}).$$

24.a. Rule:

$$\begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 3x_2 \\ x_1 - x_2 \\ 4x_2 \end{bmatrix}$$

Domain: R^2. Range: R^4.

24.b. Rule

$$\begin{bmatrix} 3x_1 - 2x_2 + 4x_3 + x_4 \end{bmatrix}$$

Domain: R^4. Range: R^1.

25.a.

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ (u_2+v_2)^2 \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} u_1 \\ u_2^2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2^2+v_2^2 \end{bmatrix}$$

Since $T(\vec{u}+\vec{v}) \neq T(\vec{u}) + T(\vec{v})$, this is not a linear transformation. Domain: R^2. Range: R^2.

25.b.

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} 2(u_1+v_1) + 2(u_2+v_2) \\ (u_1+v_1) - 2(u_2+v_2) + 3(u_3+v_3) \end{bmatrix}$$

$$\begin{aligned} T(\vec{u}) + T(\vec{v}) &= \begin{bmatrix} 2u_1 + 2u_2 \\ u_1 - 2u_2 + 3u_3 \end{bmatrix} + \begin{bmatrix} 2v_1 + 2v_2 \\ v_1 - 2v_2 + 3v_3 \end{bmatrix} \\ &= \begin{bmatrix} 2u_1 + 2u_2 + 2v_1 + 2v_2 \\ u_1 - 2u_2 + 3u_3 + v_1 - 2v_2 + 3v_3 \end{bmatrix} \end{aligned}$$

$$T(r\vec{u}) = T\left(\begin{bmatrix} ru_1 \\ ru_2 \\ ru_3 \end{bmatrix}\right) = \begin{bmatrix} 2ru_1 + 2ru_2 \\ ru_1 - 2ru_2 + 3ru_3 \end{bmatrix}$$

$$rT(\vec{u}) = r\begin{bmatrix} 2u_1 + 2u_2 \\ u_1 - 2u_2 + 3u_3 \end{bmatrix} = \begin{bmatrix} 2ru_1 + 2ru_2 \\ ru_1 - 2ru_2 + 3ru_3 \end{bmatrix}$$

This is a linear transformation. Domain: R^3. Range: R^2.

26. a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \\ x_2 \end{bmatrix}$

Linear transformation

b) Domain: R^2. Range: R^3.

c) $T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$.

e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2 \\ x_3 \end{bmatrix}$

Not a linear transformation

b) Domain: R^2. Range: R^3

c) $T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

d) N/A e) N/A

CHAPTER FOUR

1. $A^{-1} = \frac{1}{(2)(3)-(-1)(-5)}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$B^{-1} = \frac{1}{(4)(1)-(2)(-3)}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{-1}{5} \\ \frac{3}{10} & \frac{2}{5} \end{bmatrix}$

$C^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

2. A: Add -2 times row 1 to row 2.

   B: Add 5 times row 1 to row 2.

   C: Not an elementary matrix.

   D: Interchange row 2 & 3.

   E: Not an elementary matrix.

F: Add 1 times row 2 to row 4.

3.a. $E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

3.b. $A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

3.c. $A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

4. $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 7 & -13 & 5 \\ -2 & 5 & -2 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 1 \end{bmatrix}.$

5.a.

$$\left[\begin{array}{rrr|rrr} 7 & -8 & 5 & 1 & 0 & 0 \\ -4 & 5 & -3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ -4 & 5 & -3 & 0 & 1 & 0 \\ 7 & -8 & 5 & 1 & 0 & 0 \end{array}\right] \sim$$

Interchange rows 1 and 3

Add 4 times row 1 to row 2
Add -7 times row 1 to row 3

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 4 \\ 0 & -1 & -2 & 1 & 0 & -7 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 4 \\ 0 & 0 & -1 & 1 & 1 & -3 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right]$$

Add 1 times row 2 to row 3

Multiply row 3 by -1

Add -1 times row 3 to rows 2 & 1

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array}\right], \quad A^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

Add 1 times row 2 to row 1

5.b. $B^{-1} = \begin{bmatrix} 23 & -7 & 3 \\ -16 & 5 & -2 \\ 10 & -3 & 1 \end{bmatrix}$

5.c. $C^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$

5.d.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ -2 & 1 & -9 & 0 & 1 & 0 \\ 4 & -1 & 16 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & -1 & 4 & -4 & 0 & 1 \end{array}\right] \sim$$

Add 2 times row 1 to row 2
Add -4 times row 1 to row 3

Add 1 times row 2 to row 3

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \quad D^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -4 & 4 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

Add 3 times row 3 to row 2
Add -3 times row 3 to row 1

6. Solve the equation:

$$L\vec{y} = \vec{b}$$

using forward substitution. This gives us:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -16 \end{bmatrix}.$$

We obtain:

$$\vec{y} = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}.$$

Solve the equation:

$$U\vec{x} = \vec{y}$$

using backward substitution. This gives us:

$$\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ -1 \end{bmatrix}.$$

We obtain:

$$\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

7. $\vec{y} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

8.a. Use Gaussian elimination to reduce the matrix to row echelon form to get U. Store the negative multiplier in L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

8.b. Use Gaussian elimination to reduce the matrix to row echelon form to get U. Store the negative multiplier in L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

8.c. Use Gaussian elimination to reduce the matrix to row echelon form to get U. Store the negative multiplier in L.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 3 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 7 & -2 & 4 & -3 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

8.d. Use Gaussian elimination to reduce the matrix to row echelon form to get U. Store the negative multiplier in L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 6 \\ 0 & 5 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

9.a. Expand along row 2.

$$\begin{aligned} |A| &= 2\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & -1 \\ 5 & 2 \end{vmatrix} \\ &= 2(-6) - 3(13) \\ &= -12 - 39 \\ &= -51 \end{aligned}$$

9.b. Expand along row 1.

$$\begin{aligned} |B| &= 2\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \\ &= 2(5) - 1(-3) + 1(-1) \\ &= 10 + 3 - 1 \\ &= 12 \end{aligned}$$

9.c. Determinant is undefined for matrices which are not square.

10.a. Reduce A to row echelon form, keeping track of the affect of row operations on the determinant of the resulting matrix.

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 7 & -4 \\ 2 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\det(A) = (1)(5)(1) = 5.$$

10.b. Reduce B to row echelon form, keeping track of the affect of row operations on the determinant of the resulting matrix.

$$\det(B) = 40$$

11.

$$\det(A) = 90$$

12.a. The determinant of the matrix $\begin{bmatrix} 1 & 4 & 3 \\ -2 & 6 & 2 \\ -3 & -2 & 1 \end{bmatrix}$ is equal to 60. Since the determinant is not equal to zero, the columns of the matrix are linearly independent.

12.b. The determinant of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$ is equal to -5. Since the determinant is not equal to zero, the columns of the matrix are linearly independent.

12.b. The determinant of the matrix $\begin{bmatrix} 2 & 4 & 8 \\ 2 & -6 & -7 \\ -2 & 2 & 1 \end{bmatrix}$ is equal to 0. Since the determinant is equal to zero, the columns of the matrix are linearly dependent.

13.a. Compute the determinants of the matrices A, $A_1\vec{b}$, and $A_2\vec{b}$.

$$\begin{aligned} |A| &= 8 \\ \left|A_1\vec{b}\right| &= -24 \\ \left|A_2\vec{b}\right| &= 8 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -24/8 \\ 8/8 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

13.b. Compute the determinants of the matrices A, $A_1\vec{b}$, $A_2\vec{b}$, and $A_3\vec{b}$.

$$\begin{aligned} |A| &= 10 \\ \left|A_1\vec{b}\right| &= -60 \\ \left|A_2\vec{b}\right| &= 30 \\ \left|A_3\vec{b}\right| &= 20 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix}$$

13.c. Compute the determinants of the matrices A, $A_1\vec{b}$, $A_2\vec{b}$, and $A_3\vec{b}$.

$$\begin{aligned} |A| &= -14 \\ \left|A_1\vec{b}\right| &= -126 \\ \left|A_2\vec{b}\right| &= 20 \\ \left|A_3\vec{b}\right| &= -75 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 9 \\ -10/7 \\ 75/14 \end{bmatrix}$$

13.d. Compute the determinants of the matrices A, $A_1\vec{b}$, and $A_2\vec{b}$.

$$\begin{aligned} |A| &= 19 \\ \left|A_1\vec{b}\right| &= 57 \\ \left|A_2\vec{b}\right| &= 19 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 57/19 \\ 19/19 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

13.e. Compute the determinants of the matrices A, $A_1\vec{b}$, and $A_2\vec{b}$.

$$\begin{aligned} |A| &= -2 \\ \left|A_1\vec{b}\right| &= -4 \\ \left|A_2\vec{b}\right| &= 2 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -4/-2 \\ 2/-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

CHAPTER FIVE

1.a. To find the characteristic polynomial, calculate

$$\det\left(A - \lambda I\right).$$

This gives us:

$$\begin{aligned} &\begin{vmatrix} 2-\lambda & 5 \\ 4 & 1-\lambda \end{vmatrix} \\ = \; &(2-\lambda)(1-\lambda) - 20 \\ = \; &\lambda^2 - 3\lambda - 18. \end{aligned}$$

To find the eigenvalues, compute the roots of the characteristic polynomial:

$$\lambda^2 - 3\lambda - 18 = 0$$

This equation factors into

$$(\lambda - 6)(\lambda + 3) = 0.$$

The eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -3$. Now find the associated eigenvectors for each eigenvalue by solving the homogeneous system

$$(A - \lambda I)\vec{x} = \vec{0}.$$

For $\lambda_1 = 6$ we obtain

$$\begin{bmatrix} -4 & 5 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We find that one variable is free. Let $x_2 = s$, and $x_1 = \frac{5}{4}s$. The eigenvector associated with $\lambda_1 = 6$ is $\vec{x} = s \begin{bmatrix} 5/4 \\ 1 \end{bmatrix}$, $s \neq 0$.

For $\lambda_1 = -3$ we obtain

$$\begin{bmatrix} 5 & 5 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find that one variable is free. Let $x_2 = t$, and $x_1 = -t$. The eigenvector associated with $\lambda_1 = 6$ is $\vec{x} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $t \neq 0$.

1.b. To find the characteristic polynomial, calculate

$$\det(A - \lambda I).$$

This gives us:

$$\begin{vmatrix} 3-\lambda & -5 \\ 0 & 2-\lambda \end{vmatrix}.$$

$$= (3-\lambda)(2-\lambda)$$
$$= \lambda^2 - 5\lambda + 6.$$

To find the eigenvalues, compute the roots of the characteristic polynomial:

$$\lambda^2 - 5\lambda + 6 = 0$$

This equation factors into

$$(\lambda - 3)(\lambda - 2) = 0.$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$. Now find the associated eigenvectors for each eigenvalue by solving the homogeneous system

$$(A - \lambda I)\vec{x} = \vec{0}.$$

For $\lambda_1 = 3$ we obtain

$$\begin{bmatrix} 0 & -5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find that $x_1 = 0$ and x_2 is free. Let $x_2 = s$. The eigenvector associated with $\lambda_1 = 3$ is $\vec{x} = s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $s \neq 0$.

For $\lambda_1 = 2$ we obtain

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We find that one variable is free. Let $x_2 = t$, and $x_1 = 5t$. The eigenvector associated with $\lambda_1 = 2$ is $\vec{x} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $t \neq 0$.

1.c. To find the characteristic polynomial, calculate

$$\det (A - \lambda I).$$

This gives us:

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & -\lambda & 4 \\ 2 & 0 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 4 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-\lambda)(1-\lambda)$$
$$= -\lambda^3 + 4\lambda^2 - 3\lambda \quad .$$

To find the eigenvalues, compute the roots of the characteristic polynomial:

$$-\lambda^3 + 4\lambda^2 - 3\lambda = 0$$

This equation factors into

$$-\lambda(\lambda-3)(\lambda-1) = 0.$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$, $\lambda_3 = 1$. Now find the associated eigenvectors for each eigenvalue by solving the homogeneous system

$$(A - \lambda I)\vec{x} = \vec{0}.$$

For $\lambda_1 = 0$ we obtain

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$$

We find that $x_1 = 0$, $x_3 = 0$ and x_2 is free. Let $x_2 = s$. The eigenvector associated with $\lambda_1 = 0$ is $\vec{x} = s\begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$, $s \neq 0$.

For $\lambda_2 = 3$ we obtain

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 4 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We find that one variable is free. Let $x_3 = t$. Then $x_1 = t$ and $x_2 = \frac{5}{3}t$ The eigenvector associated with $\lambda_2 = 3$ is $\vec{x} = t\begin{bmatrix} 1 \\ 5/3 \\ 1 \end{bmatrix}$, $t \neq 0$.

For $\lambda_3 = 1$ we obtain

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 4 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We find that $x_1 = 0$. One variable is free. Let $x_3 = r$. Then $x_2 = 4r$. The eigenvector associated with $\lambda_3 = 1$ is $\vec{x} = r\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, r \neq 0$

1.d. The characteristic polynomial is: $-\lambda^3 + 6\lambda^2 - 5\lambda$. The eigenvalues and associated eigenvectors are $\lambda_1 = 0$, $\vec{x}_1 = \begin{bmatrix} -1/5x_2 \\ x_2 \\ 2/5x_2 \end{bmatrix}$, $\lambda_2 = 5, \vec{x}_2 = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix}$, $\lambda_3 = 1$, $\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$.

1.e The characteristic polynomial is: $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$. The eigenvalues and associated eigenvectors are $\lambda_1 = 1$, $\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$, $\lambda_2 = -2, \vec{x}_2 = \begin{bmatrix} -3/2x_3 \\ 0 \\ x_3 \end{bmatrix}$, $\lambda_3 = 3$, $\vec{x}_3 = \begin{bmatrix} 2x_2 \\ x_2 \\ 2x_2 \end{bmatrix}$.

1.f. The characteristic polynomial is: $-\lambda^4 - 7\lambda^3 + 12\lambda^2$. The eigenvalues and associated eigenvectors are $\lambda_1 = 0$, $\vec{x}_1 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$, $\lambda_2 = 0, \vec{x}_2 = \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ 0 \end{bmatrix}$, $\lambda_3 = 3$, $\vec{x}_3 = \begin{bmatrix} -2x_4 \\ 0 \\ -4/3x_4 \\ x_4 \end{bmatrix}$, $\lambda_4 = 4, \vec{x}_4 = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix}$

2. $\lambda = 2$ is an eigenvalue of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ if there is a vector $\vec{x}$ which satisfies the equation

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives us

$$\begin{aligned} 4x_1 + x_2 &= 2x_1 \\ 2x_1 + 3x_2 &= 2x_2. \end{aligned}$$

which can be rewritten

$$2x_1 + x_2 = 0$$

$$2x_1 + x_2 \;=\; 0.$$

This is a system with one equation and two variables. Let $x_1 = t$. $\vec{x} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Since we found a vector which satisfies the equation, we say $\lambda = 2$ is an eigenvalue of $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ with associated eigenvector $= t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

3. $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ if there is a scalar λ which satisfies the equation

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

This gives us

$$\begin{bmatrix} 14 \\ 7 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$.

4. 18+2i

5. 3+3i

6. $\frac{(2-2i)(3-i)}{(3+i)(3-i)} = \frac{4-8i}{10} = \frac{2}{5} - \frac{4}{5}i$

7. $(2-4i)(3+6i) = 30$

8. $\frac{(6-i)(1+2i)}{(4-i)(1+i)} = \frac{(8+11i)(5-3i)}{(5+3i)(5-3i)} = \frac{73+31i}{34} = \frac{73}{34} + \frac{31}{34}i$

9. $i^3 + 3i^2 - 4 = -i - 3 - 4 = -7 - i$

10. $\sqrt{9+16} = \sqrt{25} = 5$

11. $\sqrt{25+4} = \sqrt{29}$

12.a. $\theta = \dfrac{\pi}{4}$

12.b. $|z| = \sqrt{32} = 4\sqrt{2}, \quad z = 4\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

13.a. $(0+4i)(2+2i) = -8+8i$

13.b. $|z| = 4, \quad |w| = 2\sqrt{2}, \quad \arg(z) = \dfrac{\pi}{2}, \quad \arg(w) = \dfrac{\pi}{4}.$

$$\begin{aligned} z &= 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \\ w &= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right). \end{aligned}$$

13.c.

$$zw = 8\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

13.d.

$$(4i)^3 = 64i^3 = -64i$$

13.e.

$$z^3 = 4^3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

14.a. $z = 2 + 4i, \quad |z| = \sqrt{20} = 2\sqrt{5}, \quad \arg(z) = \tan^{-1} 2,$

$$z = 2\sqrt{5}\left(\cos\left(\tan^{-1} 2\right) + i\sin\left(\tan^{-1} 2\right)\right)$$

14.b. $z = 4 - i, \quad |z| = \sqrt{17}, \quad \arg(z) = \tan^{-1}\left(\frac{-1}{4}\right),$

$$z = \sqrt{17}\left(\cos\left(\tan^{-1}\left(\frac{-1}{4}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{-1}{4}\right)\right)\right)$$

14.c. $z = 6 + 3i, \quad |z| = \sqrt{45} = 3\sqrt{5}, \quad \arg(z) = \tan^{-1}\left(\frac{1}{2}\right)$

$$z = 3\sqrt{5}\left(\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right)$$

15. $z = -\sqrt{2} + \sqrt{2}i$

16. $z = 16^{\frac{1}{4}}$. This means $z^4 = 16$. In polar form

$$z^4 = 16\left(\cos 0 + i\sin 0\right).$$

We want to find the fourth roots of z^4, or $(z^4)^{\frac{1}{4}}$.

$$z = 16^{\frac{1}{4}}\left(\cos\frac{1}{4}(0 + 2k\pi) + i\sin\frac{1}{4}(0 + 2k\pi)\right), \quad k = 0, 1, 2, 3.$$

For $k = 0$,this takes on the value

$$2\left(\cos 0 + i\sin 0\right) = 2.$$

For $k = 1$, this takes on the value

$$2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i.$$

For $k=2$, this takes on the value

$$2(\cos\pi + i\sin\pi) = -2.$$

For $k=3$, this takes on the value

$$2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i.$$

17.a. $e^{\frac{\pi}{2}+i\frac{\pi}{2}} = e^{\frac{\pi}{2}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}(0+i) = ie^{\frac{\pi}{2}}$

17.b. $e^{3-5i} = e^3(\cos(-5) + i\sin(-5)) = e^3\cos(-5) + ie^3\sin(-5)$

18.a. $4i = 4e^{i\frac{\pi}{2}}$

18.b. $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$

$z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \quad |z| = \sqrt{1} = i, \quad \theta = \tan^{-1}\sqrt{3} = \frac{2\pi}{3}$

$$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$z^{\frac{1}{2}} = \left(\cos\frac{1}{2}\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3} + 2k\pi\right)\right), \quad k = 0, 1$$

For $k=0$, this takes on the value

$$\left(\cos\frac{1}{2}\left(\frac{2\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3}\right)\right) = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

For $k=1$, this takes on the value

$$\begin{aligned}\left(\cos\frac{1}{2}\left(\frac{2\pi}{3} + 2\pi\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3} + 2\pi\right)\right) &= \left(\cos\frac{1}{2}\left(\frac{8\pi}{3}\right) + i\sin\frac{1}{2}\left(\frac{8\pi}{3}\right)\right) \\ &= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \\ &= \frac{-1}{2} - i\frac{\sqrt{3}}{2}\end{aligned}$$

20. $z = 1 = (\cos 0 + i\sin 0)$

$$z^{\frac{1}{6}} = \left(\cos\frac{1}{6}(2k\pi) + i\sin\frac{1}{6}(2k\pi)\right), \quad k = 0, 1, 2, 3, 4, 5.$$

For $k=0$, this takes on the value

$$\cos 0 + i\sin 0$$

For $k = 1$, this takes on the value

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

For $k = 2$, this takes on the value

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

For $k = 3$, this takes on the value

$$\cos\pi + i\sin\pi$$

For $k = 4$, this takes on the value

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

For $k = 5$, this takes on the value

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

21. $\lambda_1 = 2i, \quad \vec{x} = x_2\begin{bmatrix} -1-i \\ 1 \end{bmatrix}, \quad \lambda_2 = -2i, \quad \vec{x} = x_2\begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$

22. $\lambda_1 = -1+i, \quad \vec{x} = x_2\begin{bmatrix} \frac{-3+i}{5} \\ 1 \end{bmatrix}, \quad \lambda_2 = -1-i, \quad \vec{x} = x_2\begin{bmatrix} \frac{-3-i}{5} \\ 1 \end{bmatrix}.$

REFERENCES

[1] Leon, Steven J. *Linear Algebra with Applications.* 4th ed. New York: Macmillan College Publishing Company, Inc., page 21-22, 1994.

[2] Rasmussen, Craig W. *MAR142 Matrix Algebra Refresher: Lecture Notes.* California: Naval Postgraduate School, 1995.

[3] Underwood, Doug. *An Introduction to Matrix Algebra.* California: Naval Postgraduate School, 1991.

[4] Leon, David C. *Linear Algebra and Its Applications.* New York: Addison-Wesley Publishing Company, pages 46, 72-73, 1994.

[5] Anton, Howard. *Elementary Linear Algebra.* 3d ed. New York: John Wiley & Sons, Inc., page 41, 1981.