NONLINEAR PHYSICS
THEORY AND EXPERIMENT

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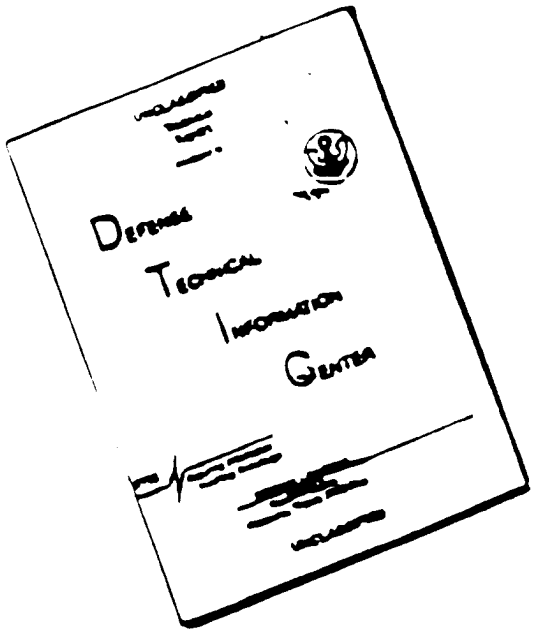

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PREFACE

This volume constitutes the proceedings of the Workshop "Nonlinear Physics. Theory and Experiment" held in Gallipoli (Lecce, Italy) from June 29 to July 7, 1995.

The purpose of the Workshop (tentatively the first of a series) was to bring together scientists whose common interest is the nature, structure and properties of nonlinear phenomena in various areas of physics and applied mathematics.

In some sense the meeting inherits the tradition of a successful series of Workshops titled "Nonlinear Evolution Equations and Dynamical Systems", NEEDS for short, the start of which was given in Lecce in 1979 by M. Boiti, F. Pempinelli and G. Soliani and then actively promoted and coordinated by F. Calogero. Significantly three of the NEEDS Workshops have been organized in Gallipoli (Lecce) in 1985, 1991 and 1993 by M. Boiti and F. Pempinelli and another one in Balaruc-les-Bains (Montpellier) in 1987 by J. Léon, organizers of this Workshop.

This time, as stressed by the title of the Workshop, an emphasis was made on both theory and experiments, the underlying objective being to offer to the nonlinear scientific community a truly interdisciplinary Workshop as a privileged place for a scientific interaction among theoreticians and experimentalists. Due to the actual increased relevance of solitons and other nonlinear effects in applications the organizers of the Workshop decided that it was just the right time for taking the scientific risk of trying this.

Crucial was the help of a large Scientific Advisory Committee with members from various areas. We warmly acknowledge its Chairman, M. Kruskal, and its members, M. Barthes, D. Campbell, A. Hasegawa, B. Konopelchenko, A. Osborne, R. Parmentier, P. Sabatier, G. Soliani, K. Spatschek.

There were 93 participants from 22 countries: Italy (28), Russia (16), USA (10), France (7), Germany (4), Belgium, Japan, United Kingdom (3), Australia, Canada, Israel, Ukraine (2), Brazil, Bulgaria, China, Moldavia, Poland, Portugal, Rumania, South Africa, Taiwan, Turkey (1). A special effort was made to assure a large attendance by researchers coming from the former Soviet Union. In several cases they took advantage of the existence of the Consortium EINSTEIN, and of its activity for developing scientific exchanges with those areas.

Remarkably, almost all the participants gave a lecture (45/30 minutes). Topics run from nonlinear optics, molecular dynamics, plasma waves,

hydrodynamics, quantum electronics and solid state physics to biology, and from inverse scattering transform methods, hamiltonian structures and geometrical aspects to turbulence and chaos, in nonlinear dynamical systems. Roughly speaking, about 40 talks are devoted in studying pure mathematical aspects of the theory of nonlinear equations, about 20 talks concern with nonlinear systems originated in specific theoretical physical contexts, and, finally, about 15 talks have a more phenomenological flavor, emphasizing the applications of well developed mathematical techniques in modelling and analyzing physical situations. Of course, this classification provides only a general scheme, which will also be used in the volume in order to give a guideline for the reader. However, not all papers fit exactly in a single section because of the multiplicity of subjects and methods covered by the authors.

In addition to the scheduled program of offical talks there were ample opportunities for informal discussions, a movie on multidimensional localized solitons was presented and there was the exposition of 8 posters.

Because of the large variety of topics and of the high quality of the contributions, these proceedings give a good up-to-date picture of the state of art in the field. However, they do not provide an exhaustive and self-contained description of the whole subject, but rather give an outline of the most recent results in such a way that they should stimulate the interested reader.

With great regret we inform the scientific community of the untimely death of S. Svinolupov, whose scientific activity participants to the Workshop had the opportunity to appreciate. A report in this volume is dedicated to a review of scientific results obtained by him on multi-component integrable systems.

The Workshop was mainly organized by F. Pempinelli and M. Boiti, who profited by the help and advice of J. Léon. The Scientific Secretary Mrs. M. C. Gerardi excellently managed all logistic and administrative problems, Mrs. H. Nagase helped at the reception of the Workshop during the first hot days and the logistic staff composed of Mr. R. De Donno and Mr. F. Spagna efficiently solved all usual and unusual practical problems which had to be faced during the Workshop.

Financial support from the University of Lecce, from Consortium EINSTEIN (European Institute of Nonlinear Studies via Transnationally Extended Interchanges), from IUPAP (International Union of Pure and Applied Physics), from the INFN (Istituto Nazionale di Fisica Nucleare) and from the University of Rome "La Sapienza" made the conference possible.

We wish also to thank the United States Air Force European Office of Aerospace Research and Development, and the United States Army European Research Office for their contribution to the success of this conference.

The editors of the present proceedings wish to thank the authors who sent their contributions. The original style of presentation has been preserved and only minor misprint have been corrected. Finally, the editors wish to acknowledge Mr. L. Pastore for the beautiful drawing of the cover.

Lecce, July, 1995

E. Alfinito
M. Boiti
J. Léon
L. Martina
F. Pempinelli

CONTENTS

Preface v

Part I Nonlinear Equations. General Methods

On the Rational Solutions of the Shabat Equation 3
V. E. Adler

Integration of Nonlinear Nonisospectral Difference-Differential Equations by Means of the Inverse Spectral Problem 11
Y. M. Berezansky

Analytical and Numerical Solutions of the Semiline Burgers Equation 21
G. Biondini and S. De Lillo

Continuous, Lattice and Q-Difference Integrable Systems and Their Transformation Properties via $\bar{\partial}$-Dressing Method 29
L. V. Bogdanov and B. G. Konopelchenko

Spectral Theory of Solitons on a Generic Background for the KPI Equation 37
M. Boiti, F. Pempinelli and A. K. Pogrebkov

Characterization and Solution of the Dispersionless Hirota Equations 53
R. Carrol and Y. Kodama

Some Thoughts on Integrating Non-Integrable Systems 60
P. J. Caudrey

Beyond the Two-Singular Manifold Methods 67
R. Conte and M. Musette

Nonlinear Initial-Boundary Evolutions with Singular Dispersion Laws Associated to the Quadratic Bundle 75
E. V. Doktorov and V. S. Shchesnovich

Geometrical Properties of Nonlinear Dynamical Systems with Regular and Chaotic Behaviour 83
V. Dryuma

The Application of the $\bar{\partial}$ Dressing Method to Some Integrable 2+1 Dimensional Nonlinear Equations 94
V. G. Dubrovsky

The Equations of the Associativity as Hydrodynamical Type System: Hamiltonian Representation and Integrability 104
E. V. Ferapontov and O. I. Mokhov

The Nonlinear Schrödinger Equation and N-Soliton Integrations. Generalization of the Karpman-Soloviev Approach 116
V. S. Gerdjikov, I. M. Uzunov and F. Lederer

Periodic Preserving Deformations of the Finite-Gap Solutions of the Soliton Equations 124
P. G. Grinevich and M. U. Schmidt

Integrable Boundary Conditions for Evolution Equations 131
T. B. Gürel, M. Gürses and I. Habibullin

Multi-Component Integrable Systems and Nonassociative Structures 139
I. T. Habibullin, V. V. Sokolov and R. I. Yamilov

How the Variational Method can Give Rise to False Instabilities 169
D. J. Kaup and T. I. Lakoba

P-Adic Description of Chaos 177
A. Khrennikov

Inverse Problem, Linearization and Related Topics of a Coupled Integrable, Dispersionless System 185
K. Konno and H. Kakuhata

Generating Equation for Constant Mean Curvature Surfaces as Hamiltonian System 193
B. G. Konopelchenko and I. A. Taimanov

Symmetry Analysis of Discrete Physical Systems 199
D. Levi and P. Winternitz

Binary Nonlinearization of Lax Pairs 217
W. X. Ma and B. Fuchssteiner

Non-Fuchsian Painlevé Test 225
M. Musette and R. Conte

Darboux Transformation for Integrable Lattice Systems 233
W. Oevel

N-Soliton Solution of Harry Dym Equation by Inverse Scattering Method 241
H. Oono

Algebraic and Geometrical Properties of Integrable Nonlinear Field Equations 249
M. Palese, E. Alfinito, M. Leo, R. A. Leo and G. Soliani

From Solitary Waves to Periodic Waves via Nonlinear Superposition: A Bilinear Approach 253
A. Parker

Bäcklund and Darboux Transformations for the Ablowitz-Ladik Spectral Problem 261
F. Pempinelli, M. Boiti and J. Leon

Lax Pairs and Exact Solutions from Painlevé Analysis 269
A. Pickering

On Generalized Ermakov Systems 278
C. Rogers and W. K. Schief

Noether Operator Relation and Conservation Laws for Partial Differential Equations 286
V. Rosenhaus and G. H. Katzin

The Method of the Multiple Scales and the Nonlinear Schrödinger Hierarchy 294
P. M. Santini

The Geometry of the LKR System. Application of a Laplace-Darboux-Type Transformation to Ernst-Type Equations 305
W. K. Schief and C. Rogers

On the Integrability of the Henon-Heiles Type Systems 313
G. Tondo

The nlBQ-Hierarchy as a pq=C Reduction of the KP-Hierarchy 321
R. Willox, I. Loris and J. Springael

Part II Nonlinear Models. Theory

Order and Chaos in Semiconductor Superlattices 333
K. N. Alekseev, G. P. Berman, D. K. Campbell, E. H. Cannon and M. C. Cargo

Nonlinear Waves Generated by Instabilities in Presence of a Conservation Law 341
D. E. Bar and A. A. Nepomnyashchy

Can Particle-Like Breathing Structures Occur in Superfluid Films? 349
I. V. Barashenkov

Soliton Analogs of Abrikosov-Nielsen-Olesen Vortices 357
I. L. Bogolubsky and A. A. Bogolubskaya

Statistics of the Particles and Quantum Groups 364
E. Celeghini

Breathers in a Nonlocal Sine-Gordon Model 372
M. D. Cunha, V. V. Konotop and L. V. Vàzquez

Forced Lattice Vibrations 377
P. Deift, T. Kriecherbauer and S. Venakides

Existence and Properties of Discrete Breathers 390
S. Flach

Reductive Perturbation Method, Multiple-Time Solutions and the KdV Hierarchy 398
R. A. Kraenkel, M. A. Manna, J. C. Montero and J. G. Pereira

Second Harmonic of Nonlinear Response of Magnets 406
A. V. Lazuta and V. A. Ryzhov

Integrable Models and Topological Field Theory 412
L. Martina, G. Soliani and O. K. Pashaev

On Dynamics of the Pulson Collisions 421
E. M. Maslov and A. G. Shagalov

On the Existence of Self-Similar Structures in the Resonance Domain 425
V. K. Mel'nikov

Nonlinear Waves in Covariant Shallow Waters 437
D. Müller and J. J. O'Brien

Chern-Simons Gauge Field Theory with Integrable Dynamics in 2+1 Dimensions 445
O. K. Pashaev

A Patchwork Approach to the Nonlinearity Inverse Problem 454
P. C. Sabatier

Part III Nonlinear Models. Applications

Proton Dynamics Along the Hydrogen Bond in Chains of the Peptide Groups: Polarons or Proton Transfer? 467
M. Barthes, M. Ribet and G. De Nunzio

Wavelet Analysis of the Electrocortical Activity 472
L. Battiston, C. Gabrieli, T. A. Minelli and A. Pascolini

The Quantum Easy Plane Ferro- and Antiferromagnet 477
C. Biagini, A. Cuccoli, V. Tognetti, R. Vaia and P. Verrucchi

Two Phases Behaviour of Porosity and Surface Width in Growth Phenomena 485
P. P. Delsanto, M. Scalerandi and S. Biancotto

Inverse Scattering Transform Analysis of the Stokes–Anti-Stokes Stimulated Raman Scattering 489
V. S. Gerdjikov and N. A. Kostov

The Geometry of Nonlinear Dynamics from Turing Patterns to Superconductors 493
R. E. Goldstein

Optical Soliton Transmission Through Fibers, Its Birth and Future Prospects 502
A. Hasegawa

Bose-Gas with Nontrivial Interparticle Interaction and Exotic Magnetic Solitons 508
A. S. Kovalev

Soliton Solution of a Non-Local Sine-Gordon Model 517
A. M. Kosevich, A. S. Kovalev and I. M. Babich

Families of Discrete Solitary Waves in Nonlinear Schrödinger Equations 521
E. W. Laedke, O. Kluth and K. H. Spatschek

The Spectral Transform on the Finite Interval and Interpretation of SRS Experiments 529
J. Leon

Anomalous Levy Processes and Sign-Singularity in Electro-Convective Turbulence 535
D. E. Lucchetta

Destruction of a Fundamental Soliton in a Periodically Modulated Waveguide 539
B. A. Malomed, R. Grimshaw and J. He

Superconducting and Spin-Glass Interaction in Coupled Layered Structures 547
L. V. Mercaldo, C. Attanasio, C. Coccorese, L. Maritato, S. L. Prischepa and M. Salvato

Nonlinear Simulation of the Electrocortical Activity 551
T. A. Minelli and L. Turicchia

Periodic Inverse Scattering Transform Analysis of Adriatic Sea Surface Waves 554
A. R. Osborne

Dynamics of Nonlinear Blood Pressure Waves in Arteries 563
J.-F. Paquerot, M. Remoissenet and L. Limat

Defect-like Solutions of 2D Sine-Gordon Equation as a Model of Inhomogeneous States in Large Area Josephson Junctions 567
A. G. Shagalov

Harmonic and Anharmonic Dynamics in Proteins and Molecular Crystals
J. C. Smith, J. Baudrey, S. Hery, A. Lamy, A. Micu and M. Souaille 575

Coupled Solitons in Continuous and Discrete Josephson Transmission Lines
A. V. Ustinov and R. D. Parmentier 582

Thermodynamic Functions of Nonlinear 1-D Systems
A. Visinescu and D. Grecu 590

List of Participants 599

Part I

Nonlinear Equations. General Methods

ON THE RATIONAL SOLUTIONS OF THE SHABAT EQUATION

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The Darboux transformation is found for the Shabat equation which is self-similar reduction of the dressing chain. This enables to construct a sequence of exact rational solutions of this equation and to prove some analytical properties of the general solution.

1 Introduction

The main object of this report is the ODE with deviating argument

$$q^2v'(qz) + v'(z) = (qv(qz) - v(z))^2 - 1, \quad q = \text{const}. \tag{1}$$

For the first time this equation was introduced by Shabat [1] in the form

$$q^2f'(qz) + f'(z) = q^2f^2(qz) - f^2(z) + 1 - q^2. \tag{2}$$

All analytical properties of the function f can be recovered from the properties of v in virtue of the substitution $f(z) = qv(qz) - v(z)$. Eq.(1) describes self-similar reduction

$$v_j(z) = q^jv(q^jz), \quad \beta_j = -q^{2j} \tag{3}$$

of the infinite chain

$$v'_{j+1} + v'_j = (v_{j+1} - v_j)^2 + \beta_j. \tag{4}$$

It is connected by formulae $f_j = v_{j+1} - v_j$, $u_j = 2v'_j$ with the *dressing chain*

$$f'_{j+1} + f'_j = f^2_{j+1} - f^2_j + \beta_{j+1} - \beta_j$$

which is equivalent to the sequence of the Darboux transformations (DT)

$$u_j = -f'_j + f^2_j + \beta_j, \quad u_{j+1} = u_j + 2f'_j$$

for the Schrödinger operators $L_j = -D^2 + u_j$. This chain was proved to be very powerful tool for constructing of exactly solvable potentials, see e.g.[2]. The

eq.(2) appeared within the same concept. Namely, it was shown in [1] that if $0 < q < 1$ and potential $u(z)$ corresponding to the real solution of (2) is regular and descends as z^{-2} as $z \to \pm\infty$ then its spectrum is $-q^{-2j}$, $j = 0, 1, 2, \ldots$.

From the other hand, eq.(1) is interesting in itself and standard questions arise, such as asymptotic behavior, character and distribution of singularities, exact solutions and so on. In this direction a few results are obtained till now.

In this paper a relation between different solutions of (1) is deduced. It originates from the nonlinear superposition principle for the DT (4) and can be considered as the DT for eq.(1). This observation yields the following results.

1) It is proved that almost all solutions of the eq.(1) are meromorphic in $\mathbb{C}$. (The proof fails for some initial values $v(0)$.)

2) A sequence of exact rational solutions is constructed. It is found by some specification of the method for constructing of the KdV rational solutions [3].

3) Moreover, the analysis of rational solutions prompts some formula connecting initial value and asymptotics of v. This formula is confirmed by numeric simulations in a rather wide domain of initial values. However, the finding of the general asymptotics and its justification is the open problem.

Everywhere below we assume $|q| < 1$. The case $|q| > 1$ is reduced to this one by obvious relation $v(z; 1/q) = qv(qz; q)$. The case $|q| = 1$ is not discussed, although some results remain valid. This case was partly considered in [4].

2 Preliminaries

It should be noted that the most general solution of the eq.(1) contains an arbitrary function. However, if we are interesting in the solutions which are analytical in the nearby of the origin then arbitrariness disappears and the Cauchy problem becomes uniquely solvable. In this paper we consider only such solutions. So, let $v(z)$ be the solution of (1) with initial condition

$$v(0) = \alpha \tag{5}$$

and let us find it as a series

$$v(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots \tag{6}$$

Using the recurrence relations

$$a_0 = \alpha, \quad a_1 = \frac{1}{1+q^2}(\alpha^2(1-q)^2 - 1), \quad a_n = \sum_{s=0}^{n-1} \frac{(1-q^{s+1})(1-q^{n-s})}{n(1+q^{n+1})} a_s a_{n-1-s}$$

one can easily prove by induction that $|a_n| \le A^{n+1}/\rho^n$, where $\rho = \frac{1-|q|}{1+|q|}$, $A = (1+|\alpha|^2)^{1/2}$, and therefore the series (6) converges in the disc $|z| < R_0 = \rho/A$.

Now one can rewrite (1) as a Riccati equation

$$v' - v^2 + av - b = 0 \tag{7}$$

with coefficients

$$a(z) = 2qv(qz),\ b(z) = q^2v^2(qz) - q^2v'(qz) - 1 \tag{8}$$

and use it for prolongation of $v(z)$ into the disc $|z| < R_1 = R_0/|q|$. Substitution $v = -\varphi'/\varphi$ maps (7) into the linear equation

$$\varphi'' + a\varphi' + b\varphi = 0. \tag{9}$$

Since $a(z)$ and $b(z)$ are holomorphic for $|z| < R_1$ hence φ is also holomorphic in this disc and its zeroes are simple, so that $v(z)$ is meromorphic and its poles are of the form $-\frac{1}{z-z_0}$. Then in the next disc $|z| < R_2 = R_0/|q|^2$ the coefficients (8) are meromorphic with Laurent expansions in the nearby of the pole $z_1 = z_0/q$

$$a(z) = -\frac{2}{z - z_1} + \ldots,\ b(z) = \frac{0}{(z - z_1)^2} + \ldots \tag{10}$$

One may hope that this prolongation procedure can be repeated infinitely, so that the solution will be constructed in the whole complex plane. In [4,5] this method was applied to the eq.(2) and the theorem was stated that the general solution is meromorphic in $\mathbb{C}$. This theorem is right, as we see later, but the proof was wrong, as one can see from the following theorem by Fuchs.

Theorem 1 (Fuchs) *Let $z = z_0$ be regular singular point of the eq.(9), that is Laurent expansions of its coefficients are of the form*

$$a(z) = \frac{1}{z - z_0}(a_0 + a_1(z - z_0) + \ldots),\ \ b(z) = \frac{1}{(z - z_0)^2}(b_0 + b_1(z - z_0) + \ldots)$$

and let λ_1, λ_2 are roots of the characteristic equation

$$\lambda(\lambda - 1) + a_0\lambda + b_0 = 0.$$

If $\lambda_1 - \lambda_2 \notin \mathbb{Z}$ then two linear independent solutions of the eq.(9) are

$$\varphi_1 = (z - z_0)^{\lambda_1}(1 + c_1(z - z_0) + \ldots),\ \ \varphi_2 = (z - z_0)^{\lambda_2}(1 + k_1(z - z_0) + \ldots)$$

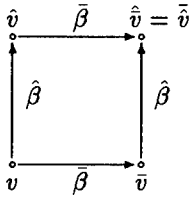
and if $\lambda_1 - \lambda_2 = n,\ n \in \mathbb{Z},\ n \geq 0$, then the linear independent solutions are

$$\begin{aligned} \varphi_1 &= (z - z_0)^{\lambda_1}(1 + c_1(z - z_0) + \ldots), \\ \varphi_2 &= (z - z_0)^{\lambda_2}(1 + k_1(z - z_0) + \ldots) + K\varphi_1 \log(z - z_0),\ \ K = const. \blacksquare \end{aligned}$$

In our case one see from (10) that eq.(9) singular points are regular in the disc $|z| < R_2$ and $\lambda_1 = 3$, $\lambda_2 = 0$. This means that $v(z)$ may contain logarithmic singularities in this disc and iterative procedure fails. Of course, logarithmic terms can be absent, either if $K = 0$ or if $\varphi = \varphi_1$ in virtue of initial condition, but to answer the question, which possibility is realized in fact, we need at least to know coefficients of the expansion (10). Nevertheless in the next section we shall prove meromorphy property by other method.

3 Darboux transformation

We shall use the well-known nonlinear superposition principle which means commutativity of two DT with different parameters. Let $\bar{}$ and $\hat{}$ denote the result of the DT with parameter $\bar{\beta}$ and $\hat{\beta}$ correspondingly. Then this property can be illustrated by the following diagram.



Moreover, the formula

$$(\hat{v} - \bar{v})(\hat{\bar{v}} - v) = \bar{\beta} - \hat{\beta}$$

is valid. Using these properties of DT we can establish some relations between functions v with different initial values $v(0)$. These relations can be regarded as the DT for the eq.(1) itself.

Theorem 2 *Let $\alpha, \tilde{\alpha}$ are related by formula*

$$(\tilde{\alpha} - q\alpha)(\alpha - q\tilde{\alpha}) = 1 \tag{11}$$

and $v, \tilde{v}$ are solutions of the eq.(1) with these initial data correspondingly. Then they are related by formulae

$$\tilde{v}' + v' = (\tilde{v} - v)^2 \tag{12}$$

$$(\tilde{v}(z) - qv(qz))(v(z) - q\tilde{v}(qz)) = 1 \tag{13}$$

Proof. Define function $y(z)$ as solution of the Cauchy problem

$$y' + v' = (y - v)^2, \; y(0) = \tilde{\alpha}. \tag{14}$$

Then $v(z)$ is connected with functions $y(z)$ and $qv(qz)$ via the DT with the parameters $\hat{\beta} = 0$ and $\bar{\beta} = -1$ correspondingly. Let

$$\bar{y}(z) = v(z) - 1/(y(z) - qv(qz)).$$

In virtue of the nonlinear superposition principle this function satisfies eqs.

$$\bar{y}' + y' = (\bar{y} - y)^2 - 1 \tag{15}$$

$$\bar{y}'(z) + q^2 v'(qz) = (\bar{y}(z) - qv(qz))^2 \tag{16}$$

Moreover, $\bar{y}(0) = q\tilde{\alpha}$ in virtue of special choice of initial value $y(0)$. Comparing eqs.(14) and (16) one see that $\bar{y}(z) = qy(qz)$ and then relation (15) means that $y(z)$ is a solution of the eq.(1), that is $y = \tilde{v}$. ■

For each α the eq.(11) associates two new values $\tilde{\alpha}$. On the next step one of the values $\tilde{\tilde{\alpha}}$ coincides with α, so that all values associated with α via iterations of the relation (11) generate some one-index sequence α_k. It is easy to prove that this sequence can be parametrized by formula

$$\alpha_k = \frac{1}{1-q^2}(cq^{k+1} + \frac{1}{cq^k}) \tag{17}$$

where c is arbitrary constant. When c is fixed we shall denote v_k the solution of equation (1) with initial value (17). (Pay attention on the different use of the subscripts in this notation and formula (3).) The following statement is almost obvious.

Lemma 3 *Let one of the functions v_j, say v_k, is meromorphic for $|z| < r$, then all v_j are meromorphic in this disc as well.*

Proof. Rewriting (13) in the form

$$v_{k+1}(z) = qv_k(qz) + 1/(v_k(z) - qv_{k+1}(qz))$$

one see that $v_{k+1}(z)$ is a rational function on $v_k(z), \ldots, v_k(q^m z)$ and $v_{k+1}(q^m z)$. Functions $v_k(z), \ldots, v_k(q^m z)$ are meromorphic by assumption and $v_{k+1}(q^m z)$ is holomorphic for m large enough, therefore $v_{k+1}(z)$ is meromorphic. Analogously one proves meromorphy of v_{k-1} and, by induction, all v_j. ■

Now we can prove the meromorphy property for almost all eq.(1) solutions.

Theorem 4 *Let $v(0) \neq \pm\frac{\sqrt{q}(q^k+q^{-k})}{1-q^2}$, $k \in \mathbb{Z}$. Then function $v(z)$ is meromorphic in $\mathbb{C}$ and its poles are of the form $-\frac{n(n+1)}{2(z-a)}$, $n = 1, 2, \ldots$.*

Proof. We have already proved that $v(z)$ is holomorphic in some finite disc $|z| < R_0$. Let us assume that statement of the theorem is true for $|z| < r$ and prove it for $|z| < r/|q|$. From the prolongation procedure (Section 1) it is clear that we need to check only the points $z = a/q$ where $z = a$ is a pole inside the disc $|z| < r$.

Consider the functions v_k related with $v = v_0$ via DT (11),(12). Let $z = a$ is a pole of the function v_k with the residue $-\frac{1}{2}n_k(n_k + 1)$. The neighbor functions v_{k-1}, v_{k+1} are solutions of the Riccati equation

$$y' + v_k' = (y - v_k)^2.$$

Linearization $y = -\varphi'/\varphi$ brings it to the form

$$\varphi'' + A\varphi' + B\varphi = 0 \tag{18}$$

where $A = 2v_k = -\frac{n_k(n_k+1)}{z-a} + \ldots$, $B = v_k^2 - v_k' = \frac{(n_k-1)n_k(n_k+1)(n_k+2)}{4(z-a)} + \ldots$ in the nearby of $z = a$. The roots of the characteristic equation

$$\lambda(\lambda-1) - n_k(n_k+1)\lambda + \frac{1}{4}(n_k-1)n_k(n_k+1)(n_k+2) = 0$$

are $\lambda_1 = \frac{1}{2}(n_k+1)(n_k+2)$, $\lambda_2 = \frac{1}{2}(n_k-1)n_k$, so that $n_{k+1} = n_k \pm 1$ and $n_{k-1} = n_k \pm 1$. Moreover, the both numbers n_{k+1} and n_{k-1} can be equal to n_k+1 only in the exceptional case $v_{k+1} = v_{k-1}$. Really, if $n_{k+1} = n_{k-1} = n_k+1$ then, in virtue of the Fuchs theorem, the corresponding solutions of the linear eq.(18) are given by the same series. This exceptional case corresponds to the omitted initial values $v(0) = \pm\frac{\sqrt{q}(q^k+q^{-k})}{1-q^2}$. For all the rest initial values at least one of two numbers n_{k+1}, n_{k-1} is less than n_k. Therefore the sequence $\ldots, n_{k-1}, n_k, n_{k+1}, \ldots$ terminates by zero at least in one direction. So a number s exists such that v_s is holomorphic in $z = a$ and therefore v_s is either holomorphic in $z = a/q$ or possesses a pole $-\frac{1}{z-a/q}$ (see first step of the prolongation procedure). Applying the Lemma 4 one obtains that all functions v_k, and v_0 in particular, are meromorphic in $z = a/q$. ∎

4 Rational solutions

In this section we construct the rational solutions of the eq.(1) by use of the method which was applied for the first time in [3] for constructing of the KdV rational solutions. (Of course, in our case there is no dependence on any t, since the eq.(1) is not consistent with KdV.)

Obviously eq.(1) admits stationary solutions $\pm\frac{1}{1-q}$. Let us construct the sequence of solutions $v_k(z)$ starting from $v_0 = \frac{1}{1-q}$ and applying DT (12). (It is clear that solutions associated with $-\frac{1}{1-q}$ are $-v_k(-z)$.) We have to solve the following relations

$$v_{k+1}' + v_k' = (v_{k+1} - v_k)^2, \quad v_{-1} = v_0 = \frac{1}{1-q}$$

$$v_k(0) = \alpha_k = \frac{1}{1-q^2}(q^{k+1} + q^{-k}) \tag{19}$$

(The constant c in (17) is equal to 1.) Substitution

$$v_k = \frac{1}{1-q} - \frac{w_k'}{w_k} \tag{20}$$

reduces this sequence of Riccati equations to the sequence of *linear* equations

$$w'_{k+1}w_{k-1} - w_{k+1}w'_{k-1} = w_k^2, \quad w_{-1} = w_0 = 1 \tag{21}$$

with initial values $w_k(0) = \omega_k$ determined by relations $\omega_{k+1} = \frac{q^{k+1}\omega_k^2}{(q^{2k+1}-1)\omega_{k-1}}$. It is easy to find by induction that

$$\omega_k = \frac{q^{k(k+1)(k+2)/6}}{(q^{2k-1}-1)(q^{2k-3}-1)^2\dots(q-1)^k}. \tag{22}$$

Linear eqs.(21) can be solved recursively and the solutions w_k turn out to be polynomials on z. There exists [3] elegant representation of these polynomials in terms of the Wronskians $W(\psi_1, \psi_2, \dots, \psi_k) = \det(\psi_j^{(i-1)})$. Namely, let

$$\psi_1 = z + \text{const}, \quad \psi_k'' = \psi_{k-1}, \ k = 2, 3, \dots \tag{23}$$

then polynomials $w_k = W(\psi_1, \psi_2, \dots, \psi_k)$ satisfy eqs. (21). It is obvious from the Jacobi's identity

$$w'_k(\chi)w_{k+1} - w_k(\chi)w'_{k+1} = w_{k+1}(\chi)w_k$$

where $w_k(\chi) = W(\psi_1, \psi_2, \dots, \psi_k, \chi)$ and relation $w_k(1) = (-1)^k w_{k-1}$ which follows from (23). So we have only to calculate integration constants in (23) to fit initial conditions (22). Several first polynomials are

$$w_1 = \zeta, \ \ 3w_2 = \zeta^3 + \frac{q^3}{1-q^3}, \ \ 45w_3 = \zeta^6 + \frac{5q^3}{1-q^3}\zeta^3 - \frac{9q^5}{1-q^5}\zeta - \frac{5q^6}{(1-q^3)^2}$$

where $\zeta = z - \frac{q}{1-q}$. It is easy to prove that

$$\deg w_k = k(k+1)/2. \tag{24}$$

We mention also an interesting difference analog of (21) which can be easily proved by induction

$$q^k w_{k+1}(z)w_{k-1}(qz) - q^{-k-1}w_{k+1}(qz)w_{k-1}(z) = w_k(z)w_k(qz).$$

5 Asymptotics

Investigation of the function v asymptotics is an open problem. Here we discuss the real solutions for $0 < q < 1$. (The case of negative q is more complicated.) Rational solutions give us some hint. Namely, comparing the

formulae (19),(20) and (24) one obtains the following relation connecting initial value $v(0)$ and asymptotics of $v(z)$:

$$\text{If} \qquad v(0) = -\frac{2q^{1/2}}{1-q^2}\cosh(\nu \log q) \tag{25}$$

$$\text{then} \qquad v(z) = -\frac{1}{1-q} - \frac{\nu^2 - \frac{1}{4}}{2z} + o(\frac{1}{z}),\ z \to +\infty. \tag{26}$$

This relation is certainly valid for half-integer ν corresponding to rational solutions. The natural conjecture is that it is valid for arbitrary ν as well.

Note that representation of initial values by formula (25) separates three domains on the real axis:

$$\begin{aligned} D_1 &: \ -\infty < v(0) \le -A, \quad && \nu \in \mathbb{R} \\ D_2 &: \ -A < v(0) < A, && \nu = i\mu,\ 0 < \mu < \pi/\log q \\ D_3 &: \ A \le v(0) < +\infty, && \nu = i\pi/\log q + \mu,\ \mu \in \mathbb{R} \end{aligned}$$

where $A = \frac{2q^{1/2}}{1-q^2}$. It is clear that relation (26) cannot be valid in the domain D_3. Numerical simulations substantiate our hypothesis at least in the domain D_1. In the domain D_2 the relation (26) remains valid for μ small enough, but then oscillations arise, so that the main part of asymptotics looks like

$$v(z) \sim -\frac{1}{1-q} - \frac{1}{z}(A + B\sin(\frac{2\pi}{\log q}\log z + C)).$$

Dependence of the parameters A, B, C on μ is not determined till now.

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INTEGRATION OF NONLINEAR NONISOSPECTRAL DIFFERERENCE-DIFFERENTIAL EQUATIONS BY MEANS OF THE INVERSE SPECTRAL PROBLEM

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Abstract

We prove that if the spectral measure of an infinite Jacobi matrix $L(t)$ changes appropriately, then $L(t)$ satisfies a generalized Lax equation, i.e., its elements are a solution of some difference-differential equation of Toda type. This result and the classical inverse problem for Jacobi matrices give the possibility of solving the Cauchy problem for a wide class of such equations.

1 Introduction

It is well known that, for many important evolutionary equations with spatial (possibly, discrete) variable x and time variable t, one can carry out the following procedure. Given such an equation, one constructs, for every t, some linear operator $L(t)$ with respect to x, whose coefficients depend on the solution $u(x,t)$ of this evolutionary equation (for example, one constructs the Sturm-Liouville equation on the axis with potential $u(x,t)$). For this linear operator, it is possible, in many cases, to set up and solve direct and inverse scattering problems, with scattering data $(\text{sc.d.})_t$ depending, of course, on t. The correspondence $u(x,t) \to (\text{sc.d.})_t$ is called a spectral transform, and it has, in many cases, the following remarkable property: a complicated law of change of $u(x,t)$ in t turns into a simple law of change of scattering data $(\text{sc.d.})_t$. As a result, this spectral transform gives the possibility of solving the Cauchy problem for our evolutionary equation: we transfer the initial data $u(x,0)$ into $(\text{sc.d.})_0$, find $(\text{sc.d.})_t$, and return to the solution $u(x,t)$ by means of the inverse scattering problem. This method is set forth in [10].

In 1984, we proposed in [2, 3, 4] another form of the spectral transform: as $L(t)$ one takes an infinite Jacobi matrix depending on t, and its spectral

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measure $d\rho(\lambda,t)$ is taken instead of $(\text{sc.d.})_t$. This gave us, in particular, the solution to the Cauchy problem for a semi-infinite Toda chain in the class of bounded sequences. It should be noted that, in the case of a finite Toda chain, such an approach was actually developed in the earlier work of J. Moser [16], and it has also some connections with the results of [12, 17, 11].

In case of a Toda chain, $\operatorname{supp} d\rho(\lambda,t) = \text{spectrum } L(t)$ does not change in time, i.e., such an equation is "isospectral." In this talk, we investigate a wide class of difference-differential equations for which the spectrum of $L(t)$ does change in time and $d\rho(\lambda,t)$ varies in some appropriate way (from the point of view of calculation). By solving the inverse spectral problem for a Jacobi matrix, we get a solution of the Cauchy problem for these equations.

Preliminary publications connected with the present paper are [7, 18, 8, 5]. Complete proofs can be found in [9]. The results of the paper were obtained jointly with M. E. Shmoish.

2 Some Facts of the Theory of Jacobi Matrices (e.g. [1])

A difference expression with bounded coefficients

$$(\mathcal{L}u)_j = a_{j-1}u_{j-1} + b_j u_j + a_j u_{j+1}, \quad a_j > 0, b_j \in \mathbf{R}^1, \qquad (1)$$
$$j \in \mathbf{Z}_+ = \{0,1,2,\ldots\}, a_{-1} = 0,$$

defines a bounded selfadjoint operator L acting in the space ℓ_2 of sequences $u = (u_j)_{j=0}^{\infty}$: $(Lu)_j = (\mathcal{L}u)_j$, $u \in \ell_2$, $j \in \mathbf{Z}_+$. This operator can be understood as an ordinary action of the bounded Jacobi infinite matrix

$$L = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \ldots \\ a_0 & b_1 & a_1 & 0 & 0 & \ldots \\ 0 & a_1 & b_2 & a_2 & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \end{pmatrix}, \qquad a_j > 0, b_j \in \mathbf{R}^1, \qquad (2)$$

in the space ℓ_2 (so the letter L denotes the Jacobi matrix as well as the corresponding operator).

Let $\phi = (\phi_j)_{j=0}^{\infty} \in \mathbf{C}^{\infty}$ be a generalized eigenvector of the operator L belonging to eigenvalue $\lambda \in \mathbf{C}^1$, i.e., a solution of the equation $(L\phi)_j = \lambda\phi_j$, $j \in \mathbf{Z}_+$. By (1), the last equation is a recursive relation for ϕ_j's, from which it is easy to deduce that, for each $n \in \mathbf{Z}_+$, $\phi_n = P_n(\lambda)$, where $P_n(\lambda)$ is a polynomial of power n. Suppose $\phi_0 > 0$ is fixed. Then, $P_n(\lambda)$'s are called the polynomials of the first kind connected with L. Define the Fourier transform with respect to the eigenvectors of L: $\ell_{2,0} \ni f \to \tilde{f}(\lambda) = \sum_{j=0}^{\infty} f_j P_j(\lambda)$, $\ell_{2,0}$ denoting the set of finite vectors from ℓ_2, $\lambda \in \mathbf{C}^1$. The spectral theorem for

Jacobi matrices asserts the existence of a finite measure ρ on $\mathbf{R}^1$ (the spectral measure of L, its support coinciding with the spectrum of L) for which the Parseval equality holds:

$$(f,g)_{\ell_2} = \int_{\mathbf{R}^1} \tilde{f}(\lambda)\overline{\tilde{g}(\lambda)}\, d\rho(\lambda), \qquad f,g \in \ell_{2,0}. \tag{3}$$

It is easy to see that $(P_n(\lambda))_{n=0}^{\infty}$ form an orthonormal basis in the space $L_2(\mathbf{R}^1, d\rho(\lambda))$ (the orthonormality follows from (3): as f and g it is necessary to take the "δ-sequences" $\delta_n = (\delta_{jn})_{j=0}^{\infty}$). The passage from L to ρ is called a direct spectral problem for a Jacobi matrix.

The set of all the spectral measures of bounded Jacobi matrices can be easily described: ρ is the spectral measure iff it is a measure on the Borel σ-algebra on $\mathbf{R}^1$ and its support is an infinite bounded set.

The inverse spectral problem in our case consists in finding a Jacobi matrix L from its spectral measure ρ. To this end, by using the classical Schmidt procedure, one has to construct from the powers $1, \lambda, \lambda^2, \ldots$ the orthonormal system of polynomials $P_0(\lambda) = (\rho(\mathbf{R}^1))^{-1/2}$, $P_1(\lambda), \ldots$ and then the elements of the Jacobi matrix are reconstructed by the formulas

$$a_n = \int_{\mathbf{R}^1} \lambda P_n(\lambda) P_{n+1}(\lambda)\, d\rho(\lambda), \quad b_n = \int_{\mathbf{R}^1} \lambda P_n^2(\lambda)\, d\rho(\lambda), \qquad n \in \mathbf{Z}_+. \tag{4}$$

The formulas (4) can be rewritten in another form, which is better from the point of view of calculation: let $s_n = \int_{\mathbf{R}^1} \lambda^n\, d\rho(\lambda)$, $n \in \mathbf{Z}_+$, be the momenta of the measure ρ, then

$$a_n = (D_{n-1}D_{n+1})^{1/2} D_n^{-1}, \quad b_n = \Delta_n D_n^{-1} - \Delta_{n-1} D_{n-1}^{-1}, \quad n \in \mathbf{Z}_+, \tag{5}$$

$$D_n = \begin{vmatrix} s_0 & s_1 & \ldots & s_n \\ s_1 & s_2 & \ldots & s_{n+1} \\ \cdot & \cdot & \ldots & \cdot \\ s_n & s_{n+1} & \ldots & s_{2n} \end{vmatrix}, \quad n \in \mathbf{Z}_+, D_{-1} = 1,$$

$$\Delta_n = \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} & s_{n+1} \\ s_1 & s_2 & \ldots & s_n & s_{n+2} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ s_n & s_{n+1} & \ldots & s_{2n-1} & s_{2n+1} \end{vmatrix}, \quad \begin{array}{l} n \in \mathbf{N} = \{1,2,\ldots\}, \\ \Delta_{-1} = 0, \Delta_0 = s_1. \end{array}$$

Remark 1. The above mentioned results take place also for unbounded sequences $a = (a_n)_{n=0}^{\infty}$, $b = (b_n)_{n=0}^{\infty}$ of elements of the matrix (2) when the closure of the operator $\ell_{2,0} \ni u \to Lu \in \ell_2$ is selfadjoint in ℓ_2 (for example, $\sum_{n=0}^{\infty} a_n^{-1} = \infty$). Just as before, $\operatorname{supp}\rho$ = spectrum L.

Remark 2. In case of a discrete spectrum, spectrum $L = \operatorname{supp}\rho = (\lambda_\nu)_{\nu=1}^{\infty}$, the spectral measure of a point λ_ν is equal to $\|\phi(\lambda_\nu)\|_{\ell_2}^{-1}$, where $\phi(\lambda_\nu) = (P_0(\lambda_\nu), P_1(\lambda_\nu), \ldots) \in \ell_2$ is the corresponding eigenvalue.

The formulas (4) show that the Jacobi matrix L is the matrix of the operator of multiplication by λ in the orthonormal basis $(P_n(\lambda))_{n=0}^{\infty}$ of the space $L_2(\mathbf{R}^1, d\rho(\lambda))$. Let $D_L = (d_{jk})_{j,k=0}^{\infty}$ be an analogous matrix of the operator of differentiation $' = d/d\lambda$, that is,

$$d_{jk} = \int_{\mathbf{R}^1} P_k'(\lambda) P_j(\lambda)\, d\rho(\lambda), \qquad j, k \in \mathbf{Z}_+. \tag{6}$$

It is clear that $d_{jk} = 0$ for $j \geq k$, i.e., D_L is a strictly upper triangular matrix. By differentiation in λ of the equality $(LP(\lambda))_j = \lambda P_j(\lambda)$, $j \in \mathbf{Z}_+$, $P(\lambda) = (P_0(\lambda), P_1(\lambda), \ldots)$, it is easy to get a recursion relation for d_{jk}'s. For example, one has

$$d_{k-1,k} = k a_{k-1}^{-1}, \quad d_{k-2,k} = a_{k-2}^{-1} a_{k-1}^{-1} \Big(\sum_{j=1}^{k} b_{j-1} - k b_{k-1} \Big), \quad k \in \mathbf{N}.$$

In any case, d_{jk}'s are expressed rationally via the elements a_n, b_n of L.

3 Main Results

We will investigate the Cauchy problem for a difference-differential equation of the form

$$\begin{gathered} \dot{a}_n(t) = F_n(a(t), b(t)), \quad \dot{b}_n(t) = G_n(a(t), b(t)), \\ \cdot = d/dt, n \in \mathbf{Z}_+, t \in [0, T], T \leq \infty. \end{gathered} \tag{7}$$

Here, $a(t) = (a_0(t), a_1(t), \ldots)$, $b(t) = (b_0(t), b_1(t), \ldots)$ are vectors of smooth unknowns

$$[0, T] \ni t \to a_n(t), b_n(t) \in \mathbf{R}^1,$$

and

$$\mathbf{R}^\infty \times \mathbf{R}^\infty \ni (a, b) \to F_n(a, b), G_n(a, b) \in \mathbf{R}^1$$

are some functions determining the difference expressions. For every $n \in \mathbf{Z}_+$, F_n, G_n depend only on a finite number of the coordinates of the vectors a, b

A classical representative of the equation (7) is a Toda chain:

$$\begin{gathered} \dot{a}_n(t) = 2^{-1} a_n(t) \big(b_{n+1}(t) - b_n(t) \big), \quad \dot{b}_n(t) = a_n^2(t) - a_{n-1}^2(t), \\ n \in \mathbf{Z}_+, \ a_{-1}(t) = 0, \ t \in [0, T]. \end{gathered} \tag{8}$$

We set up the Cauchy problem for (7) (particularly, for (8)): For arbitrary $(a(0), b(0))$, find, for each $t \in [0, T]$, a solution $(a(t), b(t))$ of (7) that is equal, at time $t = 0$, to the prescribed initial data $(a(0), b(0))$. This solution is supposed to belong to the class $K[0, T]$ of the real-valued functions $a_n(t)$, $b_n(t)$ satisfying the following conditions: for all t and n, $a_n(t) > 0$, for each t, $\|(a(t), b(t))\|_B := \sup_{n \in \mathbf{Z}_+}(a_n(t), |b_n(t)|) < \infty$, and the vector-function $[0, T] \ni t \to (a(t), b(t)) \in B$ is one time continuously differentiable (the Banach space B being defined by the last expression of the norm).

In other words, we set the correspondence between the solution $(a(t), b(t))$ of (7) and the Jacobi matrix $L(t) \equiv (a(t), b(t))$, $t \in [0, T]$, of the form (2) with $b(t)$ on its main diagonal and $a(t)$ on the two adjacent ones. Then, our conditions are equivalent to the operators $L(t)$'s being bounded (in ℓ_2) for each t and one time continuously differentiable. By $d\rho(\lambda, t)$ we denote the spectral measure of $L(t)$.

Let us first remind the corresponding results [2, 3, 4] for the Toda chain (8). The main assertion is the following: *if $(a(t), b(t))$ is a solution of (8), then the spectral measure $d\rho(\lambda, t)$ of the corresponding Jacobi matrix $L(t) = (a(t), b(t))$ is of the form*

$$d\rho(\lambda, t) = e^{\lambda t}\, d\rho(\lambda, 0), \quad t \in [0, \infty)\ (T = \infty). \tag{9}$$

So, under the spectral transform

$$(a(t), b(t)) = L(t) \to d\rho(\lambda, t), \quad t \in [0, T], \tag{10}$$

the complicated law of change in t of the solution of (8) turns into the very simple law (9).

Therefore, one can carry out the following procedure to find the solution of the Cauchy problem:

a) For the initial data $(a(0), b(0)) = L(0)$, find the spectral measure $d\rho(\lambda, 0)$.

b) By the rule (9), construct the spectral measure $d\rho(\lambda, t)$.

c) By means of the inverse spectral problem, i.e., by the formulas (4), (5) with $d\rho(\lambda, t)$ instead of $d\rho(\lambda)$, reconstruct the Jacobi matrix $L(t)$ corresponding to $d\rho(\lambda, t)$. Then, the elements $a(t)$, $b(t)$ of $L(t)$ give the solution to the Cauchy problem for (8).

Of course, a complete calculation of the solution by a)–c) can be carry out only in some special cases of initial data. These are connected with classical orthogonal polynomials for which the multiplication of the spectral measure by the factor $e^{\lambda t}$ gives a measure whose orthogonal polynomials can be calculated. Let us present two examples (for them, the initial data and solutions

are unbounded with respect to n, i.e., $L(t)$ is an unbounded operator for each t, but the above theory holds true, see Remark 1).

1. $(a(0), b(0)) = ((n+1)_{n=0}^{\infty}, (2n+1)_{n=0}^{\infty})$, $(a(t), b(t)) = ((1-t)^{-1}(n+1)_{n=0}^{\infty}, (1-t)^{-1}(2n+1)_{n=0}^{\infty})$, $t \in [0,1)$. The initial measure $d\rho(\lambda, 0)$ is equal to the Laguerre measure $e^{-\lambda}\, d\lambda$ supported on $[0, \infty)$. Here, $T < 1$ is because of the unboundedness of $L(0)$.

2. $(a(0), b(0)) = (((2^{-1}(n+1))^{1/2})_{n=0}^{\infty}, (0, 0, \ldots))$, $(a(t), b(t)) = (((2^{-1}(n+1))^{1/2})_{n=0}^{\infty}, 2^{-1}t(1, 1, \ldots))$, $t \in [0, \infty)$, $d\rho(\lambda, 0) = \pi^{-1/2} e^{-\lambda^2}\, d\lambda$ (the Gaussian measure and the Hermite polynomials for $t = 0$).

We would like also to remind the Lax form of a Toda chain. For an arbitrary matrix $A = (a_{jk})_{j,k=0}^{\infty}$, denote by $< A >$ the skew-symmetric matrix with zero main diagonal and elements $c_{jk} = a_{jk}$ if $j > k$ and $c_{jk} = -a_{kj}$ if $j < k$. Then, the Toda chain can be represented in the Lax form

$$\dot{L}(t) = [L(t), 2^{-1} < L(t) >], \qquad t \in [0, T], \tag{11}$$

where $L(t) = (a(t), b(t))$ is a Jacobi matrix and $[A, B] = AB - BA$ is the commutator.

The purpose of this paper is to introduce difference-differential equations of type (7), which are more general than (8) and for which the spectral transform (10) leads to a sufficiently simple rule of calculation of $d\rho(\lambda, t)$. Then, the procedure a)–c) is applicable and gives a solution of the corresponding Cauchy problem.

The formula (9) shows that, for each $t \in T$, $\operatorname{supp} d\rho(\lambda, t) = \operatorname{supp} d\rho(\lambda, 0)$ $(e^{\lambda t} > 0)$, i.e., spectrum $L(t) =$ spectrum $L(0)$. So, in a sense, a Toda chain is an isospectral equation. For the equations to be introduced, the spectrum of $L(t)$ changes in t, so they are nonisospectral. It will be clear that, in our case, the change of $d\rho(\lambda, t)$ in time has the form "mapping+multiplication," i.e., first, the initial measure $d\rho(\lambda, 0)$ is mapped by some mapping and then it is multiplied by a factor.

Let

$$\Phi(\lambda, t) = \sum_{i=0}^{p} \phi_i(t) \lambda^i, \quad \Psi(\lambda, t) = \sum_{i=0}^{q} \psi_i(t) \lambda^i, \quad \lambda \in \mathbf{C}^1, t \in [0, T], \tag{12}$$

be some fixed arbitrary polynomials with smooth coefficients $\phi_i(t)$, $\psi_i(t)$. We will investigate, instead of (11), a more general equation with respect to a Jacobi matrix $L(t) = (a(t), b(t))$:

$$\dot{L}(t) = \Phi(L(t), t) + [L(t), < \Phi(L(t), t) D_{L(t)} + 2^{-1} \Psi(L(t), t) >], \quad t \in [0, T]. \tag{13}$$

Here, $\Phi(L(t),t)$, $\Psi(L(t),t)$ are the polynomials (12) of $L(t)$, $D_{L(t)}$ is defined by (6) for every t. So, the equation (13) gives a nonlinear connection between the elements of the matrix $L(t)$ and their first derivatives, i.e., we have an equation of type (7).

More precisely, if we denote by $\{A\}_{jk}$ the (j,k)th element of the matrix A, $j,k \in \mathbf{Z}_+$, then it is easy to calculate that the equation (13) in the "coordinate" form can be rewritten as follows ($a_n = a_n(t)$, $b_n = b_n(t)$)

$$\begin{aligned}
\dot{a}_n &= \{\Phi(L(t),t)\}_{n+1,n} + 2^{-1}a_n\big(\{\Psi(L(t),t)\}_{n+1,n+1} - \{\Psi(L(t),t)\}_{n,n}\big) \\
&\quad + a_{n+1}\{\Phi(L(t),t)D_{L(t)}\}_{n+2,n} - a_{n-1}\{\Phi(L(t),t)D_{L(t)}\}_{n+1,n-1} \\
&\quad + (b_{n+1} - b_n)\{\Phi(L(t),t)D_{L(t)}\}_{n+1,n}, \\
\dot{b}_n &= \{\Phi(L(t),t)\}_{n,n} + a_n\{2\Phi(L(t),t)D_{L(t)} + \Psi(L(t),t)\}_{n+1,n} \\
&\quad - a_{n-1}\{2\Phi(L(t),t)D_{L(t)} + \Psi(L(t),t)\}_{n,n-1}, \\
&\quad n \in \mathbf{Z}_+, a_{-1} = 0, t \in [0,T].
\end{aligned} \tag{14}$$

In the case $\Phi = 0$, $\Psi = \lambda$, the equations (13), (14) turn into the Toda chain (11), (8).

Theorem 1 *Consider the Cauchy problem for the equations* (13) *and* (14) *in the* $K[0,T]$ *class: for arbitrary initial data* $(a(0),b(0)) = ((a_0(0),a_1(0),\ldots),(b_0(0)b_1(0),\ldots))$, *where* $a_n(0) > 0$, $b_n(0) \in \mathbf{R}^1$ *and are bounded with respect to* $n \in \mathbf{Z}_+$, *find a solution from* $K[0,T]$ *taking these initial data.*

Then, for sufficiently small $T > 0$, *such a solution exists, is unique, and can be found by the procedure a)–c) in which the stage b) is changed as follows.*

The knowledge of $d\rho(\lambda,t)$ *at moment* $t \in [0,T]$ *is equivalent to the knowledge of* $\int_{\mathbf{R}^1} f(\lambda)\,d\rho(\lambda,t)$ *for an arbitrary continuous finite function* $\mathbf{R}^1 \ni \lambda \to f(\lambda) \in \mathbf{R}^1$. *The rule of calculation of such an integral is the following. Let* $\lambda(t,\mu)$ *be a solution of the differential equation*

$$\frac{d\lambda(t)}{dt} = \Phi(\lambda(t),t), \quad t \in [0,T] \tag{15}$$

with initial data $\lambda(0) = \mu \in \mathbf{R}^1$, *and let* $s(\lambda,t)$ *be a solution of the partial differential equation*

$$\Phi(\lambda,t)\frac{\partial s(\lambda,t)}{\partial \lambda} + \frac{\partial s(\lambda,t)}{\partial t} = \Psi(\lambda,t)s(\lambda,t), \quad \lambda \in \mathbf{R}^1, t \in [0,T], \tag{16}$$

with initial data $s(\lambda,0) = 1$, $\lambda \in \mathbf{R}^1$. *Then,*

$$\int_{\mathbf{R}^1} f(\lambda)\,d\rho(\lambda,t) = \int_{\mathbf{R}^1} f(\lambda(t,\mu))s(\lambda(t,\mu),t)\,d\rho(0,\mu), \quad t \in [0,T]. \tag{17}$$

Remark 3. The solution $s(\lambda,t)$ of the above problem for (16) can be found by the method of characteristics, see [19].

4 Examples

There are some interesting examples of the equations (13), (14). Among them are, in particular, equations whose coefficients depend linearly on n (concerning similar equations, see [10]). In the following examples 1, 2, 4, $T > 0$ is sufficiently small.

1. *Inhomogeneous Toda chains (introduced by D. Levi, O. Ragmisco* [14, 15])
Equations have the form

$$\begin{aligned} &\dot{a}_n = 2^{-1}a_n\{b_n(1-\delta(2n-1)-b_{n+1}(1-\delta(2n+3))\},\\ &\dot{b}_n = a_{n-1}^2(1-2\delta(n-1))-a_n^2(1-2\delta(n+1))+\delta(b_n^2-4),\\ &n\in\mathbf{Z}_+, a_{-1}=0, \end{aligned} \tag{18}$$

where $\delta \in \mathbf{R}^1$ is a fixed parameter (in the case $\delta = 0$, (18) turns into the Toda chain (8) with $-t$ instead of$t$). The equations (18) have the form (14) with $\Phi = -4\delta + \delta\lambda^2$, $\Psi = (\delta - 1)\lambda$. The integral (17) takes the form

$$\begin{aligned} \int_{\mathbf{R}^1} f(\lambda)\, d\rho(\lambda,t) &= e^{2(\delta-1)t}\int_{\mathbf{R}^1} f\Big(2\,\frac{\mu+2+(\mu-2)e^{4\delta t}}{\mu+2-(\mu-2)e^{4\delta t}}\Big)\\ &\quad\times\big(4^{-1}(\mu+2-(\mu-2)e^{4\delta t}\big)^{(1-\delta)/\delta}\, d\rho(\mu,0). \end{aligned}$$

2. *Equations similar to* (18) *(and more simple)*

$$\begin{aligned} &\dot{a}_n = 2^{-1}a_n\big(b_{n+1}(3+2n)-b_n(2n-1)\big),\\ &\dot{b}_n = 2a_{n-1}^2(1-n)+2a_n^2(1+n)+b_n^2,\\ &n\in\mathbf{Z}_+, a_{-1}=0. \end{aligned}$$

Here, $\Phi = \lambda^2$, $\Psi = \lambda$, and the integral (17) is equal to

$$\int_{\mathbf{R}^1} f(\lambda)\, d\rho(\lambda,t) = \int_{\mathbf{R}^1} f\Big(\frac{\mu}{1-\mu t}\Big)\frac{1}{1-\mu t}\, d\rho(\mu,0).$$

3. *Nonisospectral M. Kac-P. van Moerbeke chain*
The classical M. Kac-P. van Moerbeke chain [13] has the form

$$\dot{a}_n = 2^{-1}a_n(a_{n+1}^2 - a_{n-1}^2), \quad n\in\mathbf{Z}_+, a_{-1}=0. \tag{19}$$

The equations (19) are isospectral ($\Phi = 0$, $\Psi = \lambda^2$) and $d\rho(\lambda,t) = e^{\lambda^2 t}\, d\rho(\lambda,0)$, $\lambda \in \mathbf{R}^1$, $t \in [0,\infty)$. Note that the initial data $b(0) = 0$ leads to the even measure $d\rho(\lambda,0)$, so the measure $d\rho(\lambda,t)$ is also even, and hence $b(t) = 0$ for each t. Thus, the b_n's are absent in (19).

A nonisospectral modification of (19) has the form

$$\dot{a}_n = -a_n + 2^{-1}a_n(a_{n+1}^2 - a_{n-1}^2), \quad n \in \mathbf{Z}_+, a_{-1} = 0.$$

Here, $\Phi = -\lambda$, $\Psi = \lambda^2$, and (17) has the form

$$\int_{\mathbf{R}^1} f(\lambda)\, d\rho(\lambda, t) = \int_{\mathbf{R}^1} f\big(\mu e^{-t}\big) e^{\frac{1}{2}\mu^2(1-e^{-2t})}\, d\rho(\mu, 0), \quad t \in [0, \infty).$$

4. *Another nonisospectral M. Kac-P. van Moerbeke chain*

$$\dot{a}_n = 2^{-1}a_n\big(a_{n-1}^2(3-2n) + 2a_n^2 + a_{n+1}^2(3+2n)\big), \quad n \in \mathbf{Z}_+, a_{-1} = 0.$$

Here, $\Phi = \lambda^3$, $\Psi = \lambda^2$, and (17) is of the form

$$\int_{\mathbf{R}^1} f(\lambda)\, d\rho(\lambda, t) = \int_{\mathbf{R}^1} f\Big(\frac{\mu}{\sqrt{1-2\mu^2 t}}\Big) \frac{1}{\sqrt{1-2\mu^2 t}}\, d\rho(\mu, 0).$$

5 Concluding Remarks

In the work [6] by M.I. Gekhtman and the author, there was developed a theory of integration of non-Abelian chains of Toda type. The "non-Abelian" means that, for each n, the unknowns $a_n(t)$, $b_n(t)$ in the equation of the type (8) are operator-valued functions of $t \in [0, T]$, whose values are bounded operators in some fixed Hilbert space. The equations of the work [6] were isospectral in the above mentioned sense. But it is possible also to extend the theory of Sections 2, 3, introduce non-Abelian chains of some type, and get their solutions.

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ANALYTICAL AND NUMERICAL SOLUTIONS OF THE SEMILINE BURGERS EQUATION

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We consider the semiline Burgers equation with oscillating boundary condition. Analytical solutions are compared to numerical simulations, relative to integrable and non-integrable discretizations. It is found that, unlike integrable discretizations, non-integrable discretizations fail at high frequencies.

1 Introduction

The Burgers equation is well known for its theoretical and applicative interest. In recent years the forced and semiline Burgers equations have been solved;[1,2,3] also, the asymptotic expansion for high frequency perturbations was introduced,[4] and two different kinds of discretization have been discussed,[5] called respectively "integrable" and "non-integrable".

In this contribution we will consider the semiline Burgers equation

$$u_t - u_{xx} - 2uu_x = 0\,, \qquad u \equiv u(x,t)\,, \qquad (x > 0,\, t > 0) \tag{1}$$

with null initial datum and oscillating boundary condition,

$$u(x,0) = 0\,, \tag{2}$$

$$u(0,t) = F(t) := k\sin(t/\varepsilon)\,. \tag{3}$$

Our goal is to obtain analytic approximations for the solution of Eq. 1 and to compare the two kinds of discretizations.

2 Analytical solutions

The appropriate linearizing transformation for Eq. 1 is the generalized Hopf-Cole transformation:[1,2,3]

$$v(x,t) = C(t)\,u(x,t)\exp\int_0^x dx' u(x',t)\,, \tag{4}$$

$$u(x,t) = \frac{v(x,t)}{C(t) + \int_0^x dx' v(x',t)} , \tag{5}$$

which maps the Burgers equation into the heat equation

$$v_t = v_{xx} , \tag{6}$$

with initial/boundary data

$$v(x,0) = 0 , \tag{7}$$

$$v(0,t) = C(t)F(t) , \tag{8}$$

together with the compatibility condition $\dot{C}(t) = v_x(0,t)$, which in turn yields the integral equation

$$C(t) = 1 - \frac{1}{\pi^{1/2}} \int_0^t dt' \frac{1}{(t-t')^{1/2}} F(t')C(t') . \tag{9}$$

Once $C(t)$ is known, $u(x,t)$ can be obtained by quadratures, since Eq.s 6–8 imply

$$v(x,t) = \int_0^t dt' (t-t')^{-3/2} \exp[-x^2/4(t-t')] C(t')F(t') , \tag{10}$$

$$\int_0^x dx' v(x',t) + C(t) = \int_0^t dt' (t-t')^{-1/2} \exp[-x^2/4(t-t')] C(t')F(t') . \tag{11}$$

In this way the solution of the semiline Burgers equation is reduced to the solution of the integral equation for $C(t)$ (Eq. 9).

To obtain an expression for $C(t)$ we proceed iteratively, introducing an appropriate expansion parameter λ and writing

$$C(t) = \sum_{n=0}^{\infty} \lambda^n C_n(t) . \tag{12}$$

We will consider two different cases for the boundary condition: small amplitudes and high frequencies.

2.1 Small amplitudes

When the amplitude k of the oscillations is small we can write a series expansion of $C(t)$ in powers of k. Then Eq. 9 becomes

$$C(t) \simeq 1 - \frac{kt^{1/2}}{\pi^{1/2}} f(t/\varepsilon) + \frac{k^2 t}{\pi} \int_0^1 dy\, y^{1/2} (1-y)^{-1/2} \sin(ty/\varepsilon) f(ty/\varepsilon) + O(k^3) , \tag{13}$$

where $f(z) = 2\,\mathrm{Im}[\,{}_1F_1(1,3/2,iz)\,]$ and

$$ {}_1F_1(\alpha,\beta,z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!} \tag{14}$$

is the degenerate hypergeometric function.

Substituting Eq. 13 in the RHS of Eq.s 10–11 and evaluating the resulting integrals we obtain a 2nd order approximation for $v(x,t)$ and $\int_0^x dx' v(x',t)$, which, in turn, gives an expansion of $u(x,t)$ up to $O(k^2)$ via the inverse Hopf-Cole transformation (Eq. 5).

2.2 High frequencies

In this case we solve the integral equation for $C(t)$ (Eq. 9) by performing an asymptotic expansion in ε of the RHS,[4,5] and iterating this procedure until all the subsequent terms in the expansion are of a given order in ε. The result is, up to $O(\varepsilon)$,

$$\begin{aligned} C(t) \simeq 1 &+ \varepsilon^{1/2}\left\{\frac{t^{1/2}}{(2\pi)^{1/2}} - \sin\left(\frac{t}{\varepsilon}-\frac{\pi}{4}\right)\right\} \\ &- \varepsilon\left\{\frac{1}{(\pi t)^{1/2}} + \frac{1}{8^{1/2}}\sin\frac{2t}{\varepsilon} + \frac{t^{1/2}}{(2\pi)^{1/2}}\sin\left(\frac{t}{\varepsilon}-\frac{\pi}{4}\right) + \frac{1}{8}t\right\} + O(\varepsilon^{3/2}). \end{aligned} \tag{15}$$

This gives, when inserted in Eq.s 10–11,

$$\begin{aligned} v(x,t) \simeq &-\frac{\varepsilon^{1/2}}{8^{1/2}}\,\mathrm{erfc}\frac{x}{2t^{1/2}} \\ &+ \frac{\varepsilon}{8}\left\{\frac{2}{\pi^{1/2}t^{3/2}}(2x-t^2)\,e^{-x^2/4t} + x\,\mathrm{erfc}\frac{x}{2t^{1/2}}\right\} + O(\varepsilon^{3/2}) \end{aligned} \tag{16}$$

and

$$\begin{aligned} \int_0^x dx' v(x',t) \simeq\ & \varepsilon^{1/2}\left\{\frac{t^{1/2}}{(2\pi)^{1/2}}(e^{-x^2/4t}-1) - \frac{1}{8^{1/2}}\,x\,\mathrm{erfc}\frac{x}{2t^{1/2}}\right\} \\ &+ \varepsilon\left\{\frac{1}{8}t - \frac{1}{8(\pi t)^{1/2}}(xt+8)\,e^{-x^2/4t} + \frac{1}{16}(x^2+2t)\,\mathrm{erfc}\frac{x}{2t^{1/2}}\right\} + O(\varepsilon^{3/2}). \end{aligned} \tag{17}$$

Eq.s 15–17 in turn give an expansion for $u(x,t)$ via the inverse Hopf-Cole transformation (Eq. 5):

$$\begin{aligned} u(x,t) \simeq &-\frac{\varepsilon^{1/2}}{8^{1/2}}\,\mathrm{erfc}\frac{x}{2t^{1/2}} + \frac{\varepsilon}{4}\left\{\frac{1}{\pi^{1/2}t^{3/2}}(2x-t^2)\,e^{-x^2/4t} + \left[\frac{t^{1/2}}{\pi^{1/2}}e^{-x^2/4t}\right.\right. \\ &\left.\left. + x - \frac{x}{2}\,\mathrm{erfc}\frac{x}{2t^{1/2}} - \sin\left(\frac{t}{\varepsilon}-\frac{\pi}{4}\right)\right]\mathrm{erfc}\frac{x}{2t^{1/2}}\right\} + O(\varepsilon^{3/2}). \end{aligned} \tag{18}$$

Unlike the series expansion, the validity of the above approximations is limited by the validity of the asymptotic expansions of the integrals in the RHS of Eq.s 9–11. Namely, it can be easily shown that the asymptotic expansion is valid only for $t \gg \varepsilon$, $x \gg \sqrt{\varepsilon}$.

It is also to be noted that, as for the Burgers equation under fast forcing,[4,5] the asymptotic approximation contains not only integer, but also semi-integer powers of the frequency.

3 Numerical schemes

In order to compute numerically the time evolution of the solution of the Burgers equation we discretize time and space, introducing a lattice of points $x_0, \ldots, x_N$ and points $t_0, \ldots, t_M$, with $x_i = x_0 + i\,\Delta x$ and $t_j = t_0 + j\,\Delta t$.

3.1 Non-Integrable Discretization (NID)

A direct approach to the discretization through a Forward Time Centered Space (FTCS) scheme yields

$$u_{i,j+1} = u_{i,j} + \alpha(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \beta u_{i,j}(u_{i+1,j} - u_{i-1,j})\,, \qquad (19)$$

where $u_{i,j} = u(x_i, t_j)$, $\alpha = \Delta t/(\Delta x)^2$ and $\beta = \Delta t/\Delta x$, with

$$u_{i,0} = 0\,, \qquad u_{0,j} = F_j\,. \qquad (20)$$

This method has the advantage of being very simple and to run very fast, but it has also some limitations, as we'll see later.

3.2 Integrable Discretization (ID)

A central feature of the Burgers equation is its exact integrability. An approach that preserves this characteristic consists in discretizing the heat equation (6) with the same FTCS scheme,

$$v_{i,j+1} = v_{i,j} + \alpha(v_{i+1,j} - 2v_{i,j} + v_{i-1,j})\,, \qquad (21)$$

with

$$v_{i,0} = 0\,, \qquad v_{0,j} = F_j C_j\,. \qquad (22)$$

The solution $u(x,t)$ is then recovered from the knowledge of $v(x,t)$ and $I(x,t) := \int_0^x dx' v(x',t)$, computing the values $I_{i,j}$ with a numerical quadrature

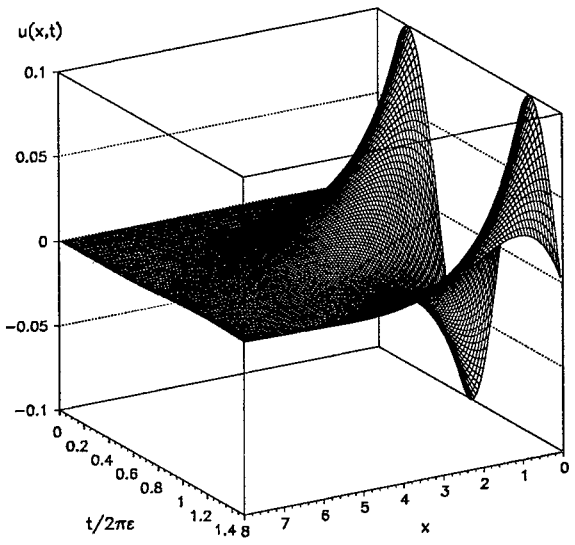

Figure 1: Three-dimensional plot of $u(x,t)$ for $k = \varepsilon = 1$, computed by the ID and the NID.

rule for $v(x,t_j)$ over the points $x_0,\ldots,x_N$, and using the inverse Hopf-Cole transformation (5):

$$u_{i,j} = v_{i,j}/(C_j + I_{i,j})\,. \tag{23}$$

A preliminary task for this method is the solution of the integral equation for $C(t)$, which can be achieved with the Nystrom method, i.e. a numerical quadrature of the integral in the RHS. In this case, however, a product rule must be used, because of the singular nature of the kernel $K(t,t') = (t - t')^{-1/2}\sin(t'/\varepsilon)$. The integral is then computed as

$$\int_0^{t_j} dt'\bar{K}(t_j,t')\sin(t'/\varepsilon)C(t') \simeq \sum_{m=1}^{j} w_{j,m}\sin(t_m/\varepsilon)C(t_m)\,, \tag{24}$$

where $\bar{K}(t,t') = (t-t')^{-1/2}$ and the $w_{j,m}$'s are appropriate integration weights, to be calculated explicitly by insisting that the rule is exact when $\sin(t/\varepsilon)C(t)$ is replaced by any polynomial of degree $\leq n$.

Two values have been chosen for the degree of the polynomials: $n = 1$ (the product analog of the trapezoid rule) and $n = 3$ (the product analog of the

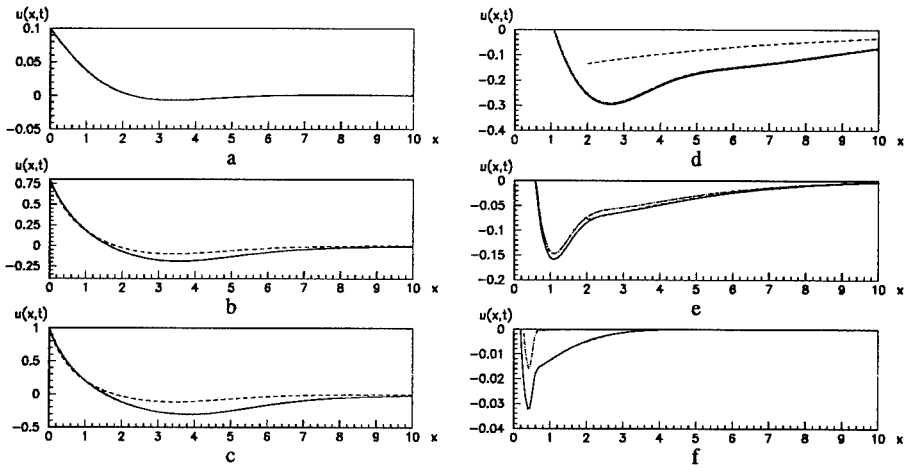

Figure 2: A comparison between different integration methods. Fig.s 2a–c are relative to small amplitudes (low k, with $\varepsilon = 1$), Fig.s 2d–f are relative to high frequencies (low ε, with $k = 1$). The full line refers to the ID, the dot-dashed line to the NID; the dashed line represents either the series expansion (Fig.s 2a–c) or the asymptotic expansion (Fig.s 2d–f). Fig. 2a: $k = 0.1$, $t/(2\pi) = 2.25$; Fig. 2b: $k = 0.8$, $t/(2\pi) = 2.25$; Fig. 2c: $k = 1$, $t/(2\pi) = 2.25$; Fig. 2d: $\varepsilon = 0.5$, $t/(2\pi\varepsilon) = 8.24$; Fig. 2e: $\varepsilon = 0.1$, $t/(2\pi\varepsilon) = 17.25$; Fig. 2f: $\varepsilon = 0.005$, $t/(2\pi\varepsilon) = 44.25$. Al low frequencies the NID coincides with the ID; at high frequencies the asymptotic approximation coincides with the ID.

Simpson rule). No appreciable difference was found between the corresponding results for $C(t)$.

4 Results and comments

The comparison between the various integration methods has been carried out with a number of tests performed on an uniform mesh of 100×100 points, for various combinations of the parameters.

It is important to note that both the analytic approximations are valid only for a limited period of time. This happens because, in both cases, higher order terms contain higher powers of t, that will dominate the expansion as $t \to \infty$. As an example, the difference between the asymptotic expansion for $C(t)$ with $\varepsilon = 0.01$ and the numerical result is less than 3% up to $t_{\max} = 12$, but is more than 15% when $t_{\max} = 100$. In a similar way, the difference between the series expansion for $C(t)$ with $k = 0.1$ and the numerical result is less than 0.2% up

to $t_{\max} = 20$, but is more than 8% when $t_{\max} = 2000$.

4.1 Small amplitudes

In this case we chose to test low frequencies ($\varepsilon = 1$), with values of k ranging from $k = 0.1$ to 1.0, for x from 0 to 20 and t from 0 up to a $t_{\max}$ varying from 20 to 2000.

The series expansion up to k^2 works very well for small values of k (for $k = 0.1$ the approximation is within 3% of the numerical value); predictably, for higher values of k the agreement is not completely satisfactory (for $k = 0.8$ the discrepancy is about 30%), reflecting the need to include higher order contributions.

About the numerical methods, at these frequencies both the ID and NID are stable (doubling the time step size produces discrepancies below 0.5% and 0.2%, respectively). They reproduce the same results with high accuracy for $k = 0.1$ (discrepancies lower than 0.1%), and the agreement is still very good for $k = 1.0$ (within 2%).

4.2 High frequencies

We considered values of ε from 0.005 to 0.5, with $k = 1$. The numerical methods have been compared for x from 0 to $x_{\max} = 30$ and t from 0 up to a $t_{\max} = 400$. The asymptotic expansion up to $\varepsilon^{3/2}$ has been tested for $x_{\min} < x < x_{\max}$ and $t_{\min} < t < t_{\max}$ (with $x_{\min}$ varying from 2 to 10, and $t_{\min}$ from 1 to 20).

At medium-high frequencies the asymptotic expansion agrees fairly well with the numerical data (for $\varepsilon < 0.1$ the approximation for $u(x,t)$ is within 6% of the numerical value), while for medium-low frequencies the agreement is unsatisfactory (for $\varepsilon = 0.5$ the difference is about 90%). This is probably due to the fact that semi-integer powers imply a lower convergence rate for the expansion.

About the numerical methods, the NID runs much more quickly than the ID (the run times on a HP 735/125 machine for $\varepsilon = 0.01$ and $t_{\max} = 400$ are less than 10^2s vs. about 10^5s, respectively). However, while the ID is always stable, (doubling the step size produces always differencies below 1%) the NID tends to be unstable at high frequencies (differencies go from 2% for $\varepsilon = 0.1$ to 36% for $\varepsilon = 0.005$).

Most importantly, the NID seems to have problems at medium frequencies, and to fail completely at high frequencies (the difference from the ID and the analytic approximation is about 20% for $\varepsilon = 0.1$, and is 100% for $\varepsilon \leq 0.01$).

This is to be compared with the results coming from previous tests,[5] in which no difference was found between the two integration methods.

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CONTINUOUS, LATTICE AND Q-DIFFERENCE INTEGRABLE SYSTEMS AND THEIR TRANSFORMATION PROPERTIES VIA $\bar{\partial}$-DRESSING METHOD

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A general scheme for introduction of lattice and q-difference variables to integrable hierarchies in frame of $\bar{\partial}$-dressing method is used to study some nonlinear integrable systems and their transformation properties.

1 Introduction

We will review here some aspects of application of the $\bar{\partial}$-dressing method [1] for construction and solving nonlinear systems and for study of their transformation properties.

We use the scheme based on $\bar{\partial}$-dressing method [2] (or, more generally, on Hirota bilinear identity [3]) that allows us to treat continuous, discrete and q-difference variables in the same way. This scheme provides a unifying framework to consider uniformly the following objects: the basic integrable system (it may be continuous, q-difference, lattice or mixed), Bäcklund transformations for this system, superposition principle for two Bäcklund transformations, and algebraic relations between three Bäcklund transformations.

From the symmetry point of view, this framework contains simultaneously the 'basic' system and its discrete, continuous and q-difference symmetries.

We also consider a special type of symmetry transformation – the Combescure transformation – which appears to be related to the fundamental question of connection between solutions of nonlocal $\bar{\partial}$ problem with different normalizations. We introduce the Combescure transformation for the modified KP equation.

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2 Lattice and q-difference variables in $\bar{\partial}$-dressing formalism

The scheme of the $\bar{\partial}$-dressing method uses the nonlocal $\bar{\partial}$-problem with the special dependence of the kernel on additional variables

$$\bar{\partial}(\chi(\mathbf{x},\lambda)-\eta(\mathbf{x},\lambda))=\iint_{\mathbf{C}} d\mu\wedge d\bar{\mu}\chi(\mu)g^{-1}(\mu)R(\mu,\lambda)g(\lambda)), \qquad (1)$$
$$(\chi(\mathbf{x},\lambda)-\eta(\mathbf{x},\lambda))_{|\lambda|\to\infty}\to 0.$$

where $\lambda\in\mathbf{C}$, $\bar{\partial}=\partial/\partial\bar{\lambda}$, $\eta(\mathbf{x},\lambda)$ is a rational function of λ (normalization). The function $\chi(\lambda)$ and the kernel $R(\lambda,\mu)$ in a general case are matrix-valued functions .

We will also often use the function ψ,

$$\psi=\chi g^{-1}.$$

This function plays a role of Baker-Akhiezer function in $\bar{\partial}$-dressing formalism.

A dependence of the solution $\chi(\lambda)$ of the problem (1) on extra variables is hidden in the function $g(\lambda)$. Usually these variables are continuous space and time variables, but it is possible also to introduce discrete (lattice) an q-difference variables into $\bar{\partial}$-dressing formalism. We will consider the following functions $g(\lambda)$

$$g_i^{-1} = \exp(K_i x_i);\quad \frac{\partial}{\partial x_i}g^{-1}=K_i g^{-1}, \qquad (2)$$

$$g_i^{-1} = (1+l_iK_i)^{n_i};\ \Delta_i g^{-1}=\frac{g^{-1}(n_i+1)-g^{-1}(n_i)}{l_i}=K_i g^{-1}, \qquad (3)$$

$$g_i^{-1} = \mathrm{e}_q(K_i y_i);\quad \delta_i^q g^{-1}=\frac{g^{-1}(qy_i)-g^{-1}(y_i)}{(q-1)y_i}=K_i g^{-1}. \qquad (4)$$

Here $K_i(\lambda)$ are meromorphic matrix functions commuting for different values of i. The function (2) introduces a dependence on continuous variable x_i, the function (3) - on discrete variable n_i and the function (4) defines a dependence of $\chi(\lambda)$ on the variable y_i (we will call it a q-difference variable). To introduce a dependence on several variables (may be of different type), one should consider a product of corresponding functions $g(\lambda)$ (all of them commute). Equations in the right part of (2-4) and the boundary condition $g(0)=1$ characterize the corresponding functions (and give a definition of $\mathrm{e_q(y)}$). These equations play a crucial role in the algebraic scheme of constructing integrable equations in frame of $\bar{\partial}$ dressing method. This scheme is based on the assumption of unique solvability of the problem (1) and on the existence of special operators, which

transform solutions of the problem (1) into the solutions of the same problem with other normalization.

We suppose that the kernel $R(\lambda,\mu)$ equals to zero in some open subset G of the complex plane with respect to λ and to μ. This subset should typically include all zeroes and poles of the considered class of functions $g(\lambda)$ and a neighborhood of infinity.

In this case the solution of the problem (1) normalized by η is the function

$$\chi(\lambda) = \eta(\mathbf{x},\lambda) + \varphi(\mathbf{x},\lambda),$$

where $\eta(\lambda)$ is a rational function of λ (normalization), all poles of $\eta(\lambda)$ belong to G, $\varphi(\lambda)$ decreases as $\lambda \to \infty$ and is *analytic* in G.

The solutions of the problem (1) with a rational normalization form a linear space, let us denote this space W. This space depends on corresponding extra variables (in fact it if a functional of the function g). It is easy to check that

$$W(g) = gW(1) \tag{5}$$

The $\bar\partial$-problem (1) implies the difference and q-difference extensions of the famous Hirota bilinear identity. Indeed, let us consider the problem (1) and its formally adjoint for the function normalized by $(\lambda-\mu)^{-1}$ with different functions g (i.e. with different values of coordinates)

$$\begin{aligned}
\frac{\partial}{\partial\bar\lambda}\chi(\lambda,\mu) &= 2\pi i\delta(\lambda-\mu) + \iint_{\mathbf{C}} d\nu\wedge d\bar\nu\chi(\nu,\mu)g_1(\nu)^{-1}R(\nu,\lambda)g_1(\lambda),\\
\frac{\partial}{\partial\bar\lambda}\chi^*(\lambda,\mu) &= -2\pi i\delta(\lambda-\mu) - \iint_{\mathbf{C}} d\nu\wedge d\bar\nu g_2(\mu)^{-1}R(\lambda,\nu)g_2(\nu)\chi^*(\nu,\mu).
\end{aligned} \tag{6}$$

After simple calculations (in the case of continuous variables see [4]) we obtain

$$\int_\gamma \chi(\nu,\lambda;g_1)g_1^{-1}(\nu)g_2(\nu)\chi^*(\nu,\mu;g_2)d\nu = 0, \tag{7}$$

where γ is the boundary of G. It follows from (7) that in $\bar G$ the function $\chi(\lambda,\mu)$ is equal to $\chi^*(\mu,\lambda)$, so in fact this identity should be written for one function. It is possible to take identity (7) instead of (1) as a starting point for the algebraic scheme of constructing equations.

The algebraic scheme of constructing equations is based on the following property of the problem (1) with the dressing functions (2-4): if $\chi(\mathbf{x},\mathbf{n},\mathbf{y},\lambda)\in W(\mathbf{x},\mathbf{n},\mathbf{y})$, then the functions

$$\begin{aligned}
D_i^c\chi &= \partial/\partial x_i\chi + \chi K_i(\lambda)\\
D_i^d\chi &= \Delta_i\chi + T_i\chi K_i(\lambda)\\
D_i^q\chi &= \delta_i^q\chi + T_i^q\chi K_i(\lambda)
\end{aligned} \tag{8}$$

also belong to W, where $Tf(n) = f(n+1)$, $T^q f(y) = f(qy)$. We can multiply the solution from the left by the arbitrary matrix function of additional variables, $u(\mathbf{x}, \mathbf{n}, \mathbf{y})\chi \in W$. So the operators (8) are the generators of Zakharov-Manakov ring of operators, that transform W into itself.

Combining this property with the unique solvability of the problem (1), one obtains the differential relations between the coefficients of expansion of functions $\chi(\mathbf{x}, \mathbf{n}, \mathbf{y}, \lambda)$ into powers of $(\lambda - \lambda_p)$ at the poles of $K_i(\lambda)$ [1].

3 Nonlinear superposition formulae

Let us consider the most simple case when functions K_i have a simple pole at the same point (let say, infinity). If all the variables are continuous, we obtain just trivial linear equations in this case. All continuous times corresponding to a simple pole at the same point are just proportional to each other. However, in the case of lattice and q-difference variables some nontrivial nonlinear equations arise. The lattice times in this case are not equivalent and they depend on continuous parameter - the step of the lattice.

So let us take $K_i = \lambda$, $i \in 1, 2, 3$ and canonically normalized solution χ of the problem (1) ($\eta = 1$). Unique solvability of the problem (1) implies the following linear equations for the function χ

$$(D_j^d - D_i^d)\chi = ((T_j - T_i)u)\chi \tag{9}$$

Taking a special combination of the first nontrivial term of expansion of these equations at infinity, one obtains a nonlinear equation for the function u

$$T_k(\Delta_j - \Delta_i)u + \text{c.p.} = T_k(u(T_j - T_i)u) + \text{c.p.} \tag{10}$$

Let us remind that the step of the lattice in our notations is hidden in the operator $\Delta_i = (T_i - 1)/l_i$.

If we introduce also the usual set of KP hierarchy times with $D_i = \frac{\partial}{\partial x_i} + \lambda^i$, the step of each lattice time can be interpreted as Bäcklund transformation. In our terms Bäcklund transformation can be considered as an equation containing one discrete time and two continuous times (in the case of KP hierarchy we will take first two times $x = x_1$, $y = x_2$. Using the standard technique, on easily obtains this equation

$$2\partial_x(\Delta u - u_x - lu\Delta u) = l\Delta(u_y - u_{xx} + 2u_x u). \tag{11}$$

The equation (10) can be interpreted as nonlinear superposition formulae for three Bäcklund transformations of the KP hierarchy. It is possible to consider also another form of superposition principle which contains two discrete

times and the first time of KP hierarchy x. In fact one can easily get it from the equation (10) by changing Δ_1 to ∂_x and T_1 to identity operator.

To get the superposition principle for the modified KP equation, we use the same functions K_i, but another function χ, normalized by $\frac{1}{\lambda}$. Using the similar procedure, one obtains a nonlinear equation for the first nontrivial coefficient of expansion of χ at infinity v

$$T_k((T_j v)^{-1}\Delta_j - (T_i v)^{-1}\Delta_i)v + \text{c.p.} = 0 \qquad (12)$$

This equation also can be interpreted as an algebraic relation between three Bäcklund transformations.

In fact the situation described hear for the KP and mKP is quite general. If we have a 'basic' continuous integrable system, we can introduce also a set of commuting discrete flows. The equation containing one lattice time and two continuous times of the 'basic' system is a Bäcklund transformation for the 'basic' system, the equation containing two lattice times and one continuous time is a superposition principle for two Bäcklund transformations and an equation with three discrete times is an algebraic relation between three Bäcklund transformations. In our scheme continuous and discrete times are included to a general framework and all these equations can be derived by the same procedure (see another approaches in [5]).

4 Darboux-Zakharov-Manakov system and its symmetry transformations

Another simple and fundamental case is the case when functions K_i have simple poles in *distinct* points [2]. This case corresponds to Darboux-Zakharov -Manakov system of equations. The derivation of equations in this case is given in [2], here we give only a brief sketch to introduce the notations.

First we choose three functions $K_i(\lambda)$, $K_j(\lambda)$, $K_k(\lambda)$ of the form $K_i(\lambda) = \frac{A_i}{\lambda-\lambda_i}$ where A_i, A_j, A_k are commuting matrices, $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i$. We introduce the solution of the problem (1) $\chi(\lambda)$ with the canonical normalization $(\eta(\lambda) = 1)$.

Then the linear equations and DZM system read

$$\delta_i^q\delta_j^q\psi = T_i^q((\delta_j^q H_i)H_i^{-1})\delta_i^q\psi + T_j^q((\delta_i^q H_j)H_j^{-1})\delta_j^q\psi, \qquad (13)$$

$$\delta_i^q\delta_j^q H_k = T_i^q((\delta_j^q H_i)H_i^{-1})\delta_i^q H_k + T_j^q((\delta_i^q H_j)H_j^{-1})\delta_j^q H_k \qquad (14)$$

where $\chi_i = \chi(\lambda_i)$, $\chi_k = H_k g_i(\lambda_k)g_j(\lambda_k)$; $\psi = \chi g_i(\lambda)g_j(\lambda)g_k(\lambda)$. The equations (14) represent a q-difference integrable deformation of Zakharov-Manakov system.

One may also consider the case of lattice or continuous variables. In fact each variable may be continuous, lattice or q-difference independently. To get

the difference variable one should just change in equation (14) δ_i^q for Δ_i and T_i for T_i^q, to get a continuous variable one should change δ_i^q for Δ_i and T_i^q for identity operator.

4.1 Bäcklund transformation

We can introduce three extra discrete variables with the poles at points λ_i. Using one extra variable and two 'basic' variables, one easily obtains a Bäcklund transformation for the DZM system just from the formula (14).

Taking two extra variables and a variable of the 'basic' DZM system with the pole at the third point, from the system (14) one obtains a superposition principle for the pair of Bäcklund transformations.

Introducing three extra discrete variables, we get an algebraic relation for the composition of three Bäcklund transformations.

So the formula (14) explicitly contains the following objects : the basic DZM system (it may be continuous, q-difference, lattice or mixed), three types of Bäcklund transformations for this system (characterized by the position of the pole), superposition principle for two different types of Bäcklund transformation, and algebraic relation between three different types of Bäcklund transformation.

From the symmetry point of view, the formula (14) contains the 'basic' system and its discrete, continuous and q-difference symmetries.

It is possible to introduce extra discrete variables with the poles at the same point and derive superposition principles for Bäcklund transformations of the same type. In fact some of the formulae for these superposition principles are given by simple modification of the equation (12).

5 Combescure transformation

The Combescure transformation is known for Darboux system for a long time and has a simple geometric sense in this case. But the derivation of this transformation in frame of $\bar{\partial}$ technique [2] implies that Combescure transformation should exist in every case when the function χ is normalized at the point not belonging to the divisor of poles of the functions K_i. This situation takes place, for example, for the modified KP equation and Ishimori equation.

In fact the Combescure transformation is connected with the fundamental question of connection between functions χ with different normalizations.

Here we introduce the Combescure transformation for the modified KP equation.

To derive Combescure transformation in frame of $\bar{\partial}$-dressing method, it is necessary to use freedom to choose a normalization of the problem (1) in quite

a nontrivial way.

Let us introduce solution of the problem (1) $\chi(\lambda,\mu)$ normalized by $(\lambda-\mu)^{-1}$, where μ is a parameter, $\mu \in G$.

We use modified D operators in frame of the standard dressing scheme for the mKP equation

$$D_i = \frac{\partial}{\partial x_i'} + (\lambda-\mu)^i$$

here the variables x_i' are connected with the initial variables by the linear transformation

$$\frac{\partial}{\partial x_i'} = \sum_{k=1}^{i} \left(\begin{array}{c} i \\ k \end{array} \right) (-\mu)^{i-k} \frac{\partial}{\partial x_k} \tag{15}$$

If we use the modified operators and the function $\chi(\lambda,\mu)$, we obtain exactly the linear equations corresponding to the mKP equation (in new coordinates). So the function $\chi(\lambda,\mu)$ also gives us a wave function and a solution for the mKP equation. Now we should find a connection between χ functions with different normalization points, this connection will give us a Combescure transformation for the mKP equation.

For the modified KP equation unique solvability of the problem (1) implies the equations

$$\begin{aligned}
(D_1 - c_1)\chi(\lambda,\mu) &= uD_1\chi(\lambda) \\
(D_2 - c_2)\chi(\lambda,\mu) &= v_1 D_1^2\chi(\lambda) + v_2 D_1\chi(\lambda) \\
(D_3 - c_3)\chi(\lambda,\mu) &= w_1 D_1^3\chi(\lambda) + w_2 D_1^2\chi(\lambda) + w_3 D_1\chi(\lambda)
\end{aligned} \tag{16}$$

here c_i depend only on μ. These equations connect the function $\chi(\lambda,\mu)$ with the function χ normalized by λ^{-1} (the standard choice). The compatibility conditions for these equations give the conditions to define the coefficients of this connection. Let us consider the compatibility conditions of the first two equations. Using the L-operator of the mKP equation

$$(D_2 - D_1^2)\chi(\lambda) = VD_1\chi(\lambda)$$

we get

$$\begin{aligned}
u &= v_1 \\
u_y + uV_x - v_x &= 0 \\
uV - u_x - v &= 0
\end{aligned} \tag{17}$$

These linear equations, containing as a potential the solution of mKP equation V, define the coefficients of the Combescure transformation for the mKP equation.

So the Combescure transformation in the case of mKP equation consists of two steps: first we express the function with the arbitrary normalization point through the canonical mKP wave function using (16) and (17), and then we make a transformation of coordinates (15) to get a new wave function (and new solution) for the mKP equation.

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SPECTRAL THEORY OF SOLITONS ON A GENERIC BACKGROUND FOR THE KPI EQUATION[a]

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The article reports on two lectures given at the Workshop. In the case of solutions of the KPI equation, non vanishing along a finite number of directions at large distances, a reformulation of the inverse scattering method in terms of a sort of complex extension of the resolvent of the related Lax operator is presented. Jost solutions for the continuum and discrete spectrum are constructed. The analytic properties and a bilinear representation of the resolvent are given. The study of scattering data and of their characterization properties is postponed to a following paper. The reader interested in more details on the resolvent approach proposed in this paper can consult ref.[1–7] and for previous works on the scattering transform for the KPI equation ref.[8–12].

1 The Image Space

Let a be a (matrix) differential operator given as a finite sum $a = \sum_n a_n(x)\,\partial_x^n$ where the coefficient $a_n(x)$ are (matrix) distributions in the Schwartz space $\mathcal{S}^d$ of d variables $x = (x_1, \ldots, x_d)$, $n = (n_1, \ldots, n_d)$ is a multi-index and $\partial_x^n = \partial_{x_1}^{n_1} \ldots \partial_{x_d}^{n_d}$. The set of these mathematical objects becomes a ring assuming the formal rules of differentiation.

To the (matrix) differential operator a we associate the (matrix) distribution $A(p;\mathbf{q})$ by the following rule

$$A(p;\mathbf{q}) = \sum_n A_n(p)(-i\mathbf{q})^n\,, \quad A_n(p) = \frac{1}{(2\pi)^d}\int dx\, e^{ipx}\, a_n(x) \tag{1.1}$$

where $px = \sum_{j=1}^d p_j x_j$ and $\mathbf{q} \in \mathbb{C}^d$. We call the distribution $A(p;\mathbf{q})$ the image of the differential operator a. The fact that $\mathbf{q}$ is complex shall play a relevant role in the following. Since below we have frequently to consider mathematical objects which depend both on complex and real variables, we stress this fact

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by writing $\mathbf{q}$ in boldface. Most naturally we can consider an extension of this set of images to the set $\mathcal{A}$ of Schwartz distributions $A(p;\mathbf{q})$ in the $p \in \mathbb{R}^d$ variables depending parametrically on the $\mathbf{q} \in \mathbb{C}^d$ variables. This dependence is supposed to be continuous with possible exceptions on a set S_A of surfaces, curves or points in $\mathbb{C}^d$ (possibly depending on p). The distributions $A(p;\mathbf{q})$, we consider in the following, admit a limit (in the distributional sense) for $\mathbf{q} \to \mathbf{q} \in S_A$ which in general depends on the specific chosen path in the complex space $\mathbb{C}^d$.

On the set of these images we introduce the following operations, the **transposition** $A^T(p;\mathbf{q}) = A(p;-p-\mathbf{q})^t$, where the index ^{t} indicates the standard matrix transposition, which gives the image of the operator dual to a, the **conjugation** $A^*(p;\mathbf{q}) = \overline{A(-p;-\overline{\mathbf{q}})}$, which gives the image of the operator with complex conjugate coefficients, the **hermitian conjugation** $A^\dagger = (A^T)^*$, or $A^\dagger(p;\mathbf{q}) = \overline{A(-p;p+\overline{\mathbf{q}})}^t$ and the **composition**

$$(AB)(p;\mathbf{q}) = \int dp'\, A(p-p';\mathbf{q}+p')\, B(p';\mathbf{q})\,, \tag{1.2}$$

which gives the image of the product of the differential operators a and b.

It is easy to verify that the standard properties

$$(AB)^T = B^T A^T, \quad (AB)^* = A^* B^*, \quad (AB)^\dagger = B^\dagger A^\dagger, \quad A(BC) = (AB)C \tag{1.3}$$

are satisfied. The distribution $\delta(p)\mathbf{1}$ (if considered as the image of 1) will be noted by $I(p;\mathbf{q})$. For some specific computations, it will be useful to consider the shifted value of the image $A(p;\mathbf{q})$ defined as $A^{(\mathbf{z})}(p;\mathbf{q}) = A(p;\mathbf{q}+\mathbf{z})$ where $\mathbf{z}$ is a complex vector.

We shall call the set $\mathcal{A}$ the extended image space and its elements the extended images or, for short, the image space and the images. In this space $\mathcal{A}$ it is convenient to consider the two subspaces

$$\mathcal{L} = \{A \in \mathcal{A} \mid A(p;\mathbf{q}) = A(p+\mathbf{q})\}, \quad \mathcal{R} = \{A \in \mathcal{A} \mid A(p;\mathbf{q}) = A(\mathbf{q})\}. \tag{1.4}$$

They are left and right ideals in $\mathcal{A}$, i.e.

$$\forall A \in \mathcal{A},\ B_{\mathcal{L}} \in \mathcal{L},\ B_{\mathcal{R}} \in \mathcal{R}, \qquad AB_{\mathcal{L}} \in \mathcal{L},\ B_{\mathcal{R}}A \in \mathcal{R}. \tag{1.5}$$

We can now define the direct product of two elements $A, B \in \mathcal{A}$

$$(A \otimes B)(p;\mathbf{q}) = A(p;\mathbf{q})\, B(p;\mathbf{q}). \tag{1.6}$$

$E(p;\mathbf{q}) \equiv 1$ is the unity of the direct product, i.e. $A \otimes E = A = E \otimes A$. Notice that there the associativity property of the direct product with respect to the

composition law in $\mathcal{A}$ is not satisfied, but there are some important special cases

$$\begin{array}{ll} \forall\, A_{\mathcal{L}} \in \mathcal{L},\, B, C \in A: & (A_{\mathcal{L}} \otimes B)C = A_{\mathcal{L}} \otimes (BC) \\ \forall\, A_{\mathcal{R}} \in \mathcal{R},\, B, C \in A: & B(C \otimes A_{\mathcal{R}}) = (BC) \otimes A_{\mathcal{R}} \end{array} \tag{1.7}$$

2 Extended Resolvent

GENERAL DEFINITION. Let us consider a bare differential operator, i.e. a differential operator with constant (matrix) coefficients. Its image has the form $L_0(p;\mathbf{q}) = \delta(p)\mathcal{L}(\mathbf{q})$ where $\mathcal{L}(\mathbf{q})$ is a polynomial in $\mathbf{q}$ with constant (matrix) coefficients. We dress L_0 with a (matrix) potential v

$$L = L_0 - v \tag{2.1}$$

and we want to define the spectral transform associated to the dressed operator L.

The extended resolvent M of L is defined as the inverse, with respect to the composition law (1.2), of the operator L. It can be considered an extension of the usual definition of the resolvent because L is obtained by considering an extension of the Fourier transform of a differential operator to complex variables.

Thanks to the associativity property (1.3) right and left inverse coincide

$$ML = I = LM \tag{2.2}$$

If we introduce the resolvent M_0 of the bare operator L_0, i.e. $M_0(p;\mathbf{q}) = \delta(p)\mathcal{L}(\mathbf{q})^{-1}$, the resolvent satisfies the two linear inhomogeneous integral equations

$$M = M_0 + M_0 v M, \qquad M = M_0 + M v M_0, \tag{2.3}$$

which can be considered mutually dual. If the potential v with respect to a certain norm is a "small perturbation" of L_0 we expect that these two integral equations have a unique solution.

By multiplying M from the right and from the left by L_0 we get two less singular quantities

$$\nu = ML_0\,, \quad \omega = L_0 M, \tag{2.4}$$

that again can be considered mutually dual. A convenient reduction of the first one will furnish the Jost solution and of the second one the dual Jost solution of the operator L. They satisfy the dual equations

$$L\nu = L_0\,, \qquad \omega L = L_0 \tag{2.5}$$

and the integral equations

$$\nu = I + M_0 v \nu, \qquad \omega = I + \omega v M_0. \tag{2.6}$$

A sort of truncated regular resolvent is obtained by considering

$$\rho = L_0(M - M_0)L_0. \tag{2.7}$$

It is related to ν and ω by the equations

$$\rho = v\nu = \omega v \tag{2.8}$$

and its reduction will furnish the spectral transform of v.

The integral equations for M, ν and ω can be written in terms of ρ as follows

$$M = M_0 + M_0 \rho M_0, \qquad \nu = I + M_0 \rho, \qquad \omega = I + \rho M_0. \tag{2.9}$$

The ρ satisfies the two dual integral equations $\rho = v + vM_0\rho$, $\rho = v + \rho M_0 v$.

HILBERT IDENTITIES. Let us consider another operator $L'(p;\mathbf{q})=L'_0(p;\mathbf{q}) - v'(p)$ and let M' be the corresponding resolvent. Then it is easy to get the so called Hilbert identity

$$M' - M = -M'(L' - L)M \tag{2.10}$$

If we choose $v' = v$ we obtain $M' - M = -M'\,(L'_0 - L_0)\,M$ that can also be written as $M' - M = \nu'\,(M'_0 - M_0)\,\omega$. Then due to definitions (2.4) and (2.7) we have

$$\nu' - \nu = \nu'\,(M'_0 - M_0)\,\rho \tag{2.11}$$

$$\omega' - \omega = \rho'\,(M'_0 - M_0)\,\omega \tag{2.12}$$

$$\rho' - \rho = \rho'\,(M'_0 - M_0)\,\rho \tag{2.13}$$

All these forms of the Hilbert identity will be used in what follows. In particular they can be used to get the $\overline{\partial}$ derivative with respect to $\mathbf{q}$ of the resolvent and of the Jost solutions and the characterization equations for the spectral data.

RESOLVENT OF THE NONSTATIONARY SCHRÖDINGER OPERATOR. We are specifically interested into the spectral transform theory for the Kadomtsev–Petviashvili equation in its version called KPI

$$(u_t - 6uu_x + u_{xxx})_x = 3u_{yy}, \qquad u = u(t,x,y) \tag{2.14}$$

for $u(t,x,y)$ real. Therefore we consider the image of the Lax operator $-i\partial_y + \partial_x^2 - u(x,y)$ which is associated to the KPI equation. In order to meet a notation

already established in literature it is convenient to define $px = p_1x - p_2y$ and, consequently, to define the Fourier Transform (image in our language), for instance of $u(x,y)$, as

$$v(p;\mathbf{q}) \equiv v(p) = \frac{1}{(2\pi)^2}\iint dx\,dy\,e^{ip_1x-ip_2y}\,u(x,y). \tag{2.15}$$

The image of the nonstationary Schrödinger operator is

$$L = Q_2 - Q_1^2 - v. \tag{2.16}$$

Due to the reality of the potential u we have $v^\dagger = v$, $L^\dagger = L$, $M^\dagger = M$, $\nu^\dagger = \omega$ and $\rho^\dagger = \rho$.

3 Jost Solutions

JOST SOLUTIONS IN THE IMAGE SPACE Note that the analytic properties of ν, ω can be studied by choosing in the Hilbert identities (2.11), (2.12) $\nu' = \nu^{(\mathbf{z})}$ and $\omega' = \omega^{(\mathbf{z})}$ and then by computing the $\overline{\partial}_j \equiv \partial/\partial\overline{\mathbf{q}_j}$ derivative with respect to $\mathbf{q}_j$. These properties are determined by those of the bare resolvent M_0. Recalling that $(\partial/\partial\overline{\mathbf{z}})1/\mathbf{z} = \pi\delta(\mathbf{z})$ we have $\overline{\partial_j}M_0 = \pi\overline{\partial_j L_0}\delta(L_0)$ where $\delta(L_0)(p;\mathbf{q}) = \delta(\mathbf{q}_2 - \mathbf{q}_1^2)\delta(p)$ and consequently only the reduced values of $\nu(p;\mathbf{q})$ and $\omega(p;\mathbf{q})$, respectively, at $\mathbf{q}_2 - \mathbf{q}_1^2 = 0$ and at $p_2 + \mathbf{q}_2 - (\mathbf{q}_1 + p_1)^2 = 0$ occur in the formulae for the $\overline{\partial}_j$ derivatives, which for potentials $u(x,y)$ vanishing sufficiently rapidly at large distances are well defined and analytic for $\mathbf{q}_{1\Im} \neq 0$ ($\mathbf{q}_{\Im} \equiv \Im\mathbf{q}$) and can be shown to be the Fourier transform, up to a gauge shift, of the usual Jost solutions.

However we are interested in potentials $u(x,y)$ going to a constant (at large distances) along some directions $x - 2\mu_n y = \text{const.}$ $(n = 1,2,...,N)$. Their Fourier transforms $v(p)$ are distributions singular on the lines $p_2 = 2\mu_n p_1$ of the form

$$v(p) = v^{(0)}(p) + \sum_{n=1}^{N}\left(\frac{v_n^{(+)}(p)}{p_2 - 2\mu_n p_1 + i0} - \frac{v_n^{(-)}(p)}{p_2 - 2\mu_n p_1 - i0}\right) \tag{3.1}$$

where $v^{(0)}(p)$, $v_n^{(+)}(p)$ and $v_n^{(-)}(p)$ are regular functions. In this case the reduced values of ν and ω are computed at a singular point and one has to choose the way along which to approach the singularity. This study has to be done on the integral equations (2.6) which can be rewritten explicitly

$$\nu(p;\mathbf{q}) = \delta(p) + \frac{(v\nu)(p;\mathbf{q})}{p_2 + \mathbf{q}_2 - (p_1 + \mathbf{q}_1)^2}, \qquad \omega(p;\mathbf{q}) = \delta(p) + \frac{(\omega v)(p;\mathbf{q})}{\mathbf{q}_2 - \mathbf{q}_1^2}. \tag{3.2}$$

We have to consider the reduction $\mathbf{q}_2 = \mathbf{q}_1^2$ for the first equation and $\mathbf{q}_2 = -p_2 + (p_1 + \mathbf{q}_1)^2$ for the second one on the lines $p_2 = 2\mu_n p_1$. For the first denominator we have

$$p_2 + \mathbf{q}_2 - (p_1 + \mathbf{q}_1)^2 \xrightarrow[\mathbf{q}_2 \to \mathbf{q}_1^2]{} \frac{p_2 - p_1(p_1 + 2\mathbf{q}_1)}{p_1}[p_1 + is_1 0] \tag{3.3}$$

where

$$s_1 = \operatorname{sgn}\left(\Im\left[\frac{p_1(\mathbf{q}_2 - \mathbf{q}_1^2)[p_2 - p_1(p_1 + 2\bar{\mathbf{q}}_1)]}{|p_2 - p_1(p_1 + 2\mathbf{q}_1)|^2}\right]\right) \tag{3.4}$$

Analogously for the second one

$$\mathbf{q}_2 - \mathbf{q}_1^2 \xrightarrow[\mathbf{q}_2 \to -p_2 + (p_1 + \mathbf{q}_1)^2]{} -\frac{p_2 - p_1(p_1 + 2\mathbf{q}_1)}{p_1}[p_1 + is_2 0] \tag{3.5}$$

where

$$s_2 = -\operatorname{sgn}\left(\Im\left[\frac{p_1(p_2 + \mathbf{q}_2 - (p_1 + \mathbf{q}_1)^2)[p_2 - p_1(p_1 + 2\bar{\mathbf{q}}_1)]}{|p_2 - p_1(p_1 + 2\mathbf{q}_1)|^2}\right]\right) \tag{3.6}$$

Therefore in the limits we consider the denominator in both integral equations in (3.2) can be rewritten as

$$\frac{1}{p_1 + is0}\frac{p_1}{p_2 - p_1(p_1 + 2\mathbf{q}_1)} \tag{3.7}$$

whose product by the distributions in (3.1) is well defined if the products of distributions are made in the order indicated. The signs s are obtained by computing formulae (3.4) and (3.6) at $p_2 = 2\mu_n p_1$. They depend in general on μ_n and consequently, in general, the integral equations (3.2) in the considered limit do not define functions with definite analytical properties. However the signs s_1 and s_2 do not depend on μ_n if $\mathbf{q}_2 - \mathbf{q}_1^2$ and $p_2 + \mathbf{q}_2 - (\mathbf{q}_1 + p_1)^2$ are approaching zero along real positive or negative values. We get therefore two types of Jost solutions

$$|\nu_\pm\rangle(p;\mathbf{q}) \equiv |\nu_\pm\rangle(p;\mathbf{q}_1) = \nu(p;\mathbf{q}_1, \mathbf{q}_1^2 \pm 0) \tag{3.8}$$

$$\langle\omega_\pm|(p;\mathbf{q}) \equiv \langle\omega_\pm|(p;\mathbf{q}_1) = \omega(p;\mathbf{q}_1, -p_2 + (p_1 + \mathbf{q}_1)^2 \pm 0) \tag{3.9}$$

that can be rewritten as

$$|\nu_\pm\rangle = \frac{i}{2}\int d\mathbf{z}_2 \wedge d\bar{\mathbf{z}}_2 \left(\nu\delta(Q_2 - Q_1^2 \mp 0I)\right)^{(\mathbf{z}_2)} \tag{3.10}$$

$$\langle\omega_\pm| = \frac{i}{2}\int d\mathbf{z}_2 \wedge d\bar{\mathbf{z}}_2 \left(\delta(Q_2 - Q_1^2 \mp 0I)\,\omega\right)^{(\mathbf{z}_2)} \tag{3.11}$$

The corresponding integral equations are

$$|\nu_\pm\rangle(p;\mathbf{q}) = \delta(p) + \frac{1}{p_1 \pm i0\mathbf{q}_{1\Im}}\left(\frac{p_1(v|\nu_\pm\rangle)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)}\right) \tag{3.12}$$

$$\langle\omega_\pm|(p;\mathbf{q}) = \delta(p) - \frac{1}{p_1 \mp i0\mathbf{q}_{1\Im}}\left(\frac{p_1(\langle\omega_\pm|v)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)}\right) \tag{3.13}$$

Notice once more that the products have to be performed in the order indicated. These dual Jost solutions are related by the equation $|\nu_\pm\rangle^\dagger = \langle\omega_\pm|$.

RELATIONS BETWEEN JOST SOLUTIONS. In order to get the relations connecting the two types of Jost solutions introduced in the previous section we can use the Hilbert identity (2.11). We have that

$$\nu(p;\mathbf{q}') - |\nu_\pm\rangle(p;\mathbf{q}) = \int dp'\nu(p-p';\mathbf{q}'+p')\left\{\frac{|\rho_\pm\rangle(p';\mathbf{q})}{p_2' + \mathbf{q}_2' - (p_1'+\mathbf{q}_1')^2}\right.$$
$$\left. -\frac{1}{p_1' \pm i0\mathbf{q}_{1\Im}}\left(\frac{p_1'(|\rho_\pm\rangle)(p';\mathbf{q})}{p_2' - p_1'(p_1'+2\mathbf{q}_1)}\right)\right\}$$

and then choosing $\mathbf{q}' = \mathbf{z}_2 + \mathbf{q}_1^2$ and taking the limit $\mathbf{z}_2 \to \mp 0$ the following relation between the Jost solutions

$$|\nu_\pm\rangle = |\nu_\mp\rangle T_\pm^{-1} \tag{3.14}$$

where

$$T_\pm^{-1}(p;\mathbf{q}) = \delta(p) \mp 2i\pi \operatorname{sgn}\mathbf{q}_{1\Im}\delta(p_1)\left(\frac{p_1(|\rho_\pm\rangle)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)}\right). \tag{3.15}$$

One can easily verify that $T_+^{-1}T_-^{-1} = I$. Introducing the simplified notations

$$|\nu\rangle = |\nu_+\rangle, \quad \langle\omega| = \langle\omega_+|, \qquad T = T_+ \tag{3.16}$$

we write

$$T_- = T^{-1}, \qquad |\nu_-\rangle = |\nu\rangle T^{-1} \tag{3.17}$$

Analogously for ω by using the Hilbert identity (2.12) we get $\langle\omega_\mp| = T_\pm^{-1}\langle\omega_\pm|$ with

$$T_\pm^{-1}(p;\mathbf{q}) = \delta(p) \mp 2i\pi \operatorname{sgn}\mathbf{q}_{1\Im}\delta(p_1)\left(\frac{p_1(\langle\rho_\pm|)(p;\mathbf{q})}{p_2 - p_1(p_1+2\mathbf{q}_1)}\right). \tag{3.18}$$

In the following we shall show that the two T^{-1}defined in (3.15) and (3.18) coincide.

SCALAR PRODUCTS OF JOST SOLUTIONS. From the definition of the resolvent (2.2) and the definition of ν and ω in (2.4) we have that $L\nu = L_0$, $\omega L = L_0$ and then, by taking the reductions as indicated in (3.10) and in (3.11),

$$L|\nu_\pm\rangle = |\nu_\pm\rangle L_0, \quad \langle\omega_\pm|L = L_0\langle\omega_\pm| \tag{3.19}$$

which can be considered to be, in the space of the images, the spectral problems satisfied by the Jost solutions. They can be rewritten as

$$M|\nu_\pm\rangle = |\nu_\pm\rangle M_0, \quad \langle\omega_\pm|M = M_0\langle\omega_\pm| \tag{3.20}$$

and by using (2.8)

$$\omega|\nu_\pm\rangle = |\nu_\pm\rangle + |\rho_\pm\rangle M_0, \quad \langle\omega_\pm|\nu = \langle\omega_\pm| + M_0\langle\rho_\pm|. \tag{3.21}$$

Performing the reduction $\mathbf{q}_2 = -p_2 + (p_1 + \mathbf{q}_1)^2 + 0s$ introduced in (3.13) for ω we obtain

$$\langle\omega_\mp|\nu_\pm\rangle = I, \qquad \langle\omega_\pm|\nu_\pm\rangle = T_\pm^{-1} = T^{\mp 1} \tag{3.22}$$

where $T_\pm^{-1}$ is defined as in (3.15) and where, in the second formula, we used (3.16) and (3.17). We have from (3.22) that $T^\dagger = T$ and, consequently, we recover for $T_\pm^{-1}(p;\mathbf{q})$ the formula obtained in (3.18) proving that the two T^{-1} obtained in relating $|\nu_+\rangle$ with $|\nu_-\rangle$ and $|\omega_+\rangle$ with $|\omega_-\rangle$ are equal.

Let us notice that due to (3.19) and (3.22)

$$L_0 T_\pm^{-1} = L_0\langle\omega_\pm|\nu_\pm\rangle = \langle\omega_\pm|L|\nu_\pm\rangle = \langle\omega_\pm|\nu_\pm\rangle L_0 = T_\pm^{-1} L_0 \tag{3.23}$$

and then $[L_0, T] = 0$. Therefore $T(p;\mathbf{q})$ must be of the form $T(p;\mathbf{q}) = \delta(p)t(\mathbf{q}_1)$ as expected.

Since the integral equations (3.12), (3.13) for the 'plus' Jost solutions $|\nu\rangle$ and $\langle\omega|$ are of Volterra type, as one can easily see by computing their Fourier transform, $|\nu\rangle$ and $\langle\omega|$ are analytic in the upper and lower half complex plane $\mathbf{q}_1$. Due to (3.22) T^{-1} has the same property and therefore we conclude that the function $t(\mathbf{q}_1)^{-1}$ is analytic in all the complex plane $\mathbf{q}_1$ excepted the real axis. Let us denote the zeroes (supposed simple) of this function and then the poles of $t(\mathbf{q}_1)$ by λ_j. For every pole at λ_j there is another one at $\overline{\lambda_j}$ and the corresponding residua are complex conjugate. If there are no real poles (as we suppose in the following) we can number the poles in such a way that positive numbers correspond to poles in the upper half–plane and negative numbers to poles in the lower half–plane. Therefore we have

$$t(\mathbf{q}_1) = \frac{t_j}{\mathbf{q}_1 - \lambda_j} + o(1), \qquad \mathbf{q}_1 \to \lambda_j \tag{3.24}$$

where $\lambda_{-j} = \overline{\lambda_j}$ and $t_{-j} = \overline{t_j}$. Notice that in the pure ray case T is self-transposed and $t_{-j} = -t_j$.

THE JOST SOLUTIONS IN THE (x, y) SPACE. Let us introduce two elements belonging, respectively, to the left and right ideals defined in section 1

$$\mathcal{L} \ni |x, y\rangle(p; \mathbf{q}) = \frac{1}{2\pi} e^{i(p_1+\mathbf{q}_{1\Re})x - i(p_2+\mathbf{q}_{2\Re})y} \tag{3.25}$$

$$\mathcal{R} \ni \langle x, y|(p; \mathbf{q}) = \frac{1}{2\pi} e^{-i\mathbf{q}_{1\Re}x + i\mathbf{q}_{2\Re}y} \tag{3.26}$$

that satisfy $|x, y\rangle^\dagger = \langle x, y|$ and

$$\langle x, y|x', y'\rangle = E\delta(x - x')\delta(y - y'), \qquad \iint dxdy\, |x, y\rangle \otimes \langle x, y| = I \tag{3.27}$$

They can used to get the Fourier transform of $v(p)$

$$\langle x, y|v|x', y'\rangle = Eu(x, y)\delta(x - x')\delta(y - y'), \tag{3.28}$$

and the Jost and dual Jost solutions in the (x, y) space

$$\Phi(x, y, \mathbf{q}_1) = 2\pi\, e^{\mathbf{q}_{1\Im}x + i(\mathbf{q}_1^2 - \mathbf{q}_{2\Re})y} \langle x, y|\nu\rangle(p; \mathbf{q}) \tag{3.29}$$

$$\Psi(x, y, \mathbf{q}_1) = 2\pi\, e^{-\mathbf{q}_{1\Im}x - i(\mathbf{q}_1^2 - \mathbf{q}_{2\Re})y} \langle\omega|x, y\rangle(p; \mathbf{q} - p). \tag{3.30}$$

Equations (3.19) transform in the (x, y) space into the standard spectral equations

$$(-i\partial_y + \partial_x^2 - u(x, y))\Phi(x, y, \mathbf{q}_1) = 0, \qquad (i\partial_y + \partial_x^2 - u(x, y))\Psi(x, y, \mathbf{q}_1) = 0. \tag{3.31}$$

Note that the relation between dual Jost solutions $|\nu_\pm\rangle^\dagger = \langle\omega_\pm|$ transforms into $\overline{\Phi(x, y, \mathbf{q}_1)} = \Psi(x, y, \overline{\mathbf{q}}_1)$.

4 Additional Discontinuities of the Resolvent

DISCONTINUITIES IN THE $\mathbf{q}_2$-PLANE. All singularities of the resolvent in the two–dimensional case are 'smoothed' if compared with the one–dimensional case. This is true in particular if solitons are present in the solution. In fact in one dimension the presence of a soliton reflects into the presence of a pole in the resolvent while in two dimensions the resolvent results to have a discontinuity along a line in the complex $\mathbf{q}_2$ plane at $\mathbf{q}_{1\Im}$ fixed.

The study of the special case $u(x, y) \equiv u(x - 2\mu y)$ considered as a special reduction of the two-dimensional case suggests that for potentials $u(x, y)$ going to a constant along the directions $x - 2\mu_n y = \text{const.}$ $(n = 1, 2, \ldots, N)$ the

resolvent has a discontinuity along the line $\mathbf{q}_{2\Im} = 2\mu_n \mathbf{q}_{1\Im}$. In the following, for any element $A(p;\mathbf{q}) \in \mathcal{A}$ we note by

$$A_{\Delta n}(p;\mathbf{q}) \equiv A_{\Delta n}(p;\mathbf{q}_1,\mathbf{q}_{2\Re}) \equiv A_{n+}(p;\mathbf{q}) - A_{n-}(p;\mathbf{q}) = A(p;\mathbf{q}_1,\mathbf{q}_{2\Re} + 2i\mu_n \mathbf{q}_{1\Im} + i0) - A(p;\mathbf{q}_1,\mathbf{q}_{2\Re} + 2i\mu_n \mathbf{q}_{1\Im} - i0)$$

its discontinuity across the line and by $A_n(p;\mathbf{q}) = A(p;\mathbf{q}_1,\mathbf{q}_{2\Re} + 2i\mu_n \mathbf{q}_{1\Im})$ its value on the line, if it is continuous.

Let $M_{\Delta n}$ be the discontinuity of M. $M_{\Delta n}$ satisfies the differential equations

$$L_n M_{\Delta n} = 0, \qquad M_{\Delta n} L_n = 0 \tag{4.1}$$

and the homogeneous integral equations

$$M_{\Delta n} = M_{0n} v M_{\Delta n}, \qquad M_{\Delta n} = M_{\Delta n} v M_{0n}. \tag{4.2}$$

SOME AUXILIARY OBJECTS. Let us introduce

$$\widehat{M}_{(n)}(p;\mathbf{q}) = \frac{1}{2i\pi} \int dp'_2 \frac{M_{\Delta n}(p;\mathbf{q}_1,p'_2)}{p'_2 + 2i\mu_n \mathbf{q}_{1\Im} - \mathbf{q}_2}. \tag{4.3}$$

Note that with the notation $A_{(n)}(p;\mathbf{q})$ we indicate an element of the image space that depends on the discrete variable n and that has not to be confused with $A_{\Delta n}(p;\mathbf{q})$, and $A_n(p;\mathbf{q})$ defined in the previous section. The quantity $\widehat{M}_{(n)}(p;\mathbf{q})$ shall play a relevant role in the following since the difference $M(p;\mathbf{q}) - \widehat{M}_{(n)}(p;\mathbf{q})$ has no discontinuity along the line $\mathbf{q}_{2\Im} = 2\mu_n \mathbf{q}_{1\Im}$. Let us also introduce the discontinuities of ω, ν, and ρ which are related to the discontinuity of M by the equations $\nu_{\Delta n} = M_{\Delta n} v$, $\omega_{\Delta n} = v M_{\Delta n}$, $\rho_{\Delta n} = v M_{\Delta n} v$ and the quantities $\widehat{\mathrm{N}}_{(n)} = \widehat{M}_{(n)} v$, $\widehat{\Omega}_{(n)} = v \widehat{M}_{(n)}$, $\widehat{\mathrm{P}}_{(n)} = v \widehat{M}_{(n)} v$. We will also use

$$m_{(n)}(p;\mathbf{q}) = \lim_{\mathbf{q}_2 \to \infty} \mathbf{q}_2 \widehat{M}_{(n)}(p;\mathbf{q}) \equiv -\frac{1}{2i\pi} \int dp'_2 \, M_{\Delta n}(p;\mathbf{q}_1,p'_2). \tag{4.4}$$

Due to (4.1) and (4.3) we have that $L\widehat{M}_{(n)}$ and $\widehat{M}_{(n)} L$ are no more discontinuous on the line and precisely we obtain that

$$\left(L\widehat{M}_{(n)}\right)_n = m_{(n)}, \qquad \left(\widehat{M}_{(n)} L\right)_n = m_{(n)} \tag{4.5}$$

and therefore, since $m_{(n)}$ does not depend on $\mathbf{q}_2$,

$$L\widehat{M}_{(n)} = m_{(n)}, \qquad \widehat{M}_{(n)} L = m_{(n)}, \tag{4.6}$$

which can be considered two differential equations satisfied by $\widehat{M}_{(n)}$ in the image space. They furnish two dual integral representations for $\widehat{M}_{(n)}$

$$\widehat{M}_{(n)} = m_{(n)}M, \qquad \widehat{M}_{(n)} = Mm_{(n)}. \tag{4.7}$$

SCALAR PRODUCTS. We start with the following Hilbert identities considered on both side of the discontinuity

$$M_{n\pm} - M^{(\mathbf{z})} = -M_{n\pm}(L_{(0)n} - L_0^{(\mathbf{z})})M^{(\mathbf{z})}.$$

We get

$$M_{\Delta n} = M_{\Delta n}(iQ_{2\Im} - 2i\mu_n Q_{1\Im} - 2\mathbf{z}_1 Q_1 + (\mathbf{z}_2 - \mathbf{z}_1^2)I)M^{(\mathbf{z})} \tag{4.8}$$

If $\mathbf{z}_1 = 0$ and only in this case we can extract the multiplier $(iQ_{2\Im} - 2i\mu_n Q_{1\Im} + \mathbf{z}_2 I)$ from inside of the r.h.s., since the imaginary parts of Q_j commute with all other operators. Thus by computing the discontinuity of the obtained formula across the line $\mathbf{q}_{2\Im} + \mathbf{z}_{2\Im} = 2\mu_k \mathbf{q}_{1\Im}$ we obtain the orthogonality relation for the $M_{\Delta n}$'s

$$M_{\Delta n} M_{\Delta k}^{(\mathbf{z}_{2\Re})} = -2i\pi\delta_{n,k}\delta(\mathbf{z}_{2\Re})M_{\Delta n} \tag{4.9}$$

This result can be used for computing $\widehat{M}_{(n)}\widehat{M}_{(k)}^{(\mathbf{z}_{2\Re})}$. In fact if we rewrite (4.3) as follows

$$\widehat{M}_{(n)}(p;\mathbf{q}) = \frac{1}{2i\pi}\int dp_2' \frac{M_{\Delta n}(p;\mathbf{q}_1, p_2' + \mathbf{q}_{2\Re})}{p_2' + 2i\mu_n \mathbf{q}_{1\Im} - i\mathbf{q}_{2\Im}}$$

we obtain that

$$\widehat{M}_{(n)}\widehat{M}_{(k)}^{(\mathbf{z}_{2\Re})} = -\frac{1}{2i\pi}\int dp_2' \frac{\delta_{n,k} M_{\Delta n}(p;\mathbf{q}_1, p_2')}{(p_2' + 2i\mu_n \mathbf{q}_{1\Im} - \mathbf{q}_2)\,(p_2' - \mathbf{z}_{2\Re} + 2i\mu_n \mathbf{q}_{1\Im} - \mathbf{q}_2)}$$

and, consequently, by continuing analytically in $\mathbf{z}_2$, the orthogonality relations for the singular parts of M

$$\widehat{M}_{(n)}\widehat{M}_{(k)}^{(\mathbf{z}_2)} = \frac{\delta_{n,k}}{\mathbf{z}_2}\left(\widehat{M}_{(n)} - \widehat{M}_{(n)}^{(\mathbf{z}_2)}\right). \tag{4.10}$$

From this equation and that one obtained by shifting $\mathbf{q}_2 \to \mathbf{q}_2 - \mathbf{z}_2$ we have, in the limit $\mathbf{z}_2 \to \infty$,

$$m_{(n)}\widehat{M}_{(k)} = \delta_{n,k}\widehat{M}_{(n)}, \qquad \widehat{M}_{(n)}m_{(k)} = \delta_{n,k}\widehat{M}_{(n)} \tag{4.11}$$

and, then, in the limit $\mathbf{q}_2 \to \infty$, the orthogonality relations and the normalization condition for the functions $m_{(n)}$

$$m_{(n)}m_{(k)} = \delta_{n,k}m_{(n)}. \tag{4.12}$$

Other useful relations can be obtained starting from

$$\frac{M_{\Delta n}(p;\mathbf{q})}{i\mathbf{q}_{2\Im}-2i\mu_n\mathbf{q}_{1\Im}+\mathbf{z}_2}=\int dp_2' M_{\Delta n}(p-p';\mathbf{q}_1+p_1',\mathbf{q}_{2\Re}+p_2')M(p';\mathbf{q}+\mathbf{z}_2).$$

that can be obtained directly from (4.8). Naming $\mathbf{q}_{2\Re}=p_2''$, shifting $\mathbf{z}_2\to\mathbf{z}_2+\mathbf{q}_{2\Re}-p_2''$ and then after integration with respect to p_2'' we have

$$\widehat{M}_{(n)}M^{(\mathbf{z}_2)}=\frac{1}{\mathbf{z}_2}\left(\widehat{M}_{(n)}-\widehat{M}_n^{(\mathbf{z}_2)}\right) \tag{4.13}$$

If we separate explicitly the part of M which is singular on the line $\mathbf{q}_{2\Im}=2\mu_n\mathbf{q}_{1\Im}$

$$M=\widehat{M}_{(n)}+M_{(n)\mathrm{reg}}. \tag{4.14}$$

from (4.13) and (4.10) we get

$$\widehat{M}_{(n)}M_{(n)\mathrm{reg}}^{(\mathbf{z}_2)}=0 \tag{4.15}$$

and on the discontinuity

$$M_{\Delta n}\left(M_{(n)\mathrm{reg}}^{(\mathbf{z}_2)}\right)_n=0 \tag{4.16}$$

Let us, now, turn back to (4.8) at $\mathbf{z}_1=0$. Multiplying it by $L_0^{(\mathbf{z}_2)}$ we get

$$M_{\Delta n}\left(iQ_{2\Im}-2i\mu_nQ_{1\Im}+\mathbf{z}_2I\right)^{-1}(Q_2-Q_1^2+\mathbf{z}_2I)=M_{\Delta n}\nu^{(\mathbf{z}_2)}. \tag{4.17}$$

Notice that the numerator and denominator in the l.h.s. have a common zero iff $\mathbf{q}_{2\Re}=\mathbf{q}_{1\Re}^2-\mathbf{q}_{1\Im}^2$, $\mathbf{q}_{1\Re}\mathbf{q}_{1\Im}=\mu_n\mathbf{q}_{1\Im}$, i.e., as we consider the case $\mathbf{q}_{1\Im}\neq 0$, at $\mathbf{q}=\mathbf{q}_n$, where $\mathbf{q}_{n,1}=\mu_n+i\mathbf{q}_{1\Im}$, $\mathbf{q}_{n,2}=(\mu_n+i\mathbf{q}_{1\Im})^2$. Thus, for $\mathbf{q}\neq\mathbf{q}_n$,

$$M_{\Delta n}|\nu_\pm\rangle=0,\qquad \langle\omega_\pm|M_{\Delta n}=0 \tag{4.18}$$

and from (4.3), since the Jost solutions do not depend on $\mathbf{q}_2$,

$$\widehat{M}_{(n)}|\nu_\pm\rangle=0,\qquad \langle\omega_\pm|\widehat{M}_{(n)}=0 \tag{4.19}$$

and by taking the limit $\mathbf{q}_2\to\infty$

$$m_{(n)}|\nu_\pm\rangle=0,\qquad \langle\omega_\pm|m_{(n)}=0. \tag{4.20}$$

$\overline{\partial}$–DERIVATIVES OF $M_{\Delta n}$ AND $\widehat{M}_{(n)}$. Let us insert (4.14) into the Hilbert identity for M

$$\begin{aligned}&\widehat{M}_{(n)}+M_{(n)\mathrm{reg}}-\widehat{M}_{(n)}^{(\mathbf{z})}-M_{(n)\mathrm{reg}}^{(\mathbf{z})}=\\&-\left[\widehat{M}_{(n)}+M_{(n)\mathrm{reg}}\right]\left[2\mathbf{z}_1Q_1-\left(\mathbf{z}_2-\mathbf{z}_1^2\right)I\right]\left[\widehat{M}_{(n)}^{(\mathbf{z})}+M_{(n)\mathrm{reg}}^{(\mathbf{z})}\right]\end{aligned}$$

We choose $\mathbf{q}_{2\Im} = 2\mu_n \mathbf{q}_{1\Im} + \varepsilon$, $\mathbf{q}_{2\Im} + \mathbf{z}_{2\Im} = 2\mu_n(\mathbf{q}_{1\Im} + \mathbf{z}_{1\Im}) + \varepsilon$, take the difference of the two limits $\varepsilon \to \pm 0$ and expand the formula obtained in terms of powers of $\mathbf{z}_1$ and $\overline{\mathbf{z}_1}$. The terms of order $\overline{\mathbf{z}}_1$, thanks to (4.16), give

$$\overline{\partial}_1 M_{\Delta n}^{(\mathbf{z}_{2\Re})} = \mu_n \left[\widehat{M}_{(n)n+} \widehat{M}_{(n)n+}^{(\mathbf{z}_{2\Re})} - \widehat{M}_{(n)n-} \widehat{M}_{(n)n-}^{(\mathbf{z}_{2\Re})} \right]$$

and using (4.13)

$$\overline{\partial}_1 M_{\Delta n}^{(\mathbf{z}_{2\Re})} = \frac{\mu_n}{\mathbf{z}_{2\Re}} \left[M_{\Delta n} - M_{\Delta n}^{(\mathbf{z}_{2\Re})} \right]$$

So in the limit $\mathbf{z}_{2\Re} \to 0$ we have $\overline{\partial}_1 M_{\Delta n} = -\mu_n M'_{\Delta n}$ where

$$M'_{\Delta n}(p; \mathbf{q}) = \frac{\partial}{\partial \mathbf{q}_{2\Re}} M_{\Delta n}(p; \mathbf{q}_1, \mathbf{q}_{2\Re}) \tag{4.21}$$

This means that $M_{\Delta n}(p; \mathbf{q}_1, \mathbf{q}_2 + 2\mu_n \mathbf{q}_1)$ is an analytic function of $\mathbf{q}_1$ and that, recalling the definition (4.4) of $m_{(n)}$, also $m_{(n)}$ is analytic in $\mathbf{q}_1$.

5 Bilinear Representation for the Resolvent

MODIFIED BILINEAR REPRESENTATION. If the resolvent M is discontinuous along some lines in the complex $\mathbf{q}_2$ plane the set of the Jost solutions is not complete, as follows from (4.19). Therefore, the standard bilinear representation of the resolvent must be modified adding a term for each singularity

$$M = |\nu\rangle M_0 T \langle\omega| + \textstyle\sum_n \widehat{M}_{(n)}. \tag{5.1}$$

These terms result to coincide with the $\widehat{M}_{(n)}$'s defined in section 4 as one can check computing the scalar products of the left and right and side of (5.1) with $|\nu_\pm\rangle$, $\langle\omega_\pm|$ and $\widehat{M}_{(k)}$ by using the equations (3.20), (3.22), (4.10), (4.13) and (4.19).

In the limit $\mathbf{q}_2 \to \infty$, we get the modified completeness relation

$$I = |\nu\rangle T \langle\omega| + \textstyle\sum_n m_{(n)} \tag{5.2}$$

and, multiplying (5.1) from the right and from the left by L_0, we obtain, thanks to the differential equations for $|\nu\rangle$ and $\langle\omega|$ in (3.19) and to the integral equations for $M_{(n)}$ in (4.6) and to (5.2),

$$\nu = I + |\nu\rangle M_0 T \langle\rho| + \textstyle\sum_n \widehat{\mathrm{N}}_{(n)}, \qquad \omega = I + |\rho\rangle M_0 T \langle\omega| + \textstyle\sum_n \widehat{\Omega}_{(n)} \tag{5.3}$$

CONNECTING SINGULARITIES OF THE TRANSMISSION COEFFICIENT WITH SINGULARITIES OF THE RESOLVENT. We showed in section 3 that the transmission

coefficient $t(\mathbf{q}_1)$ can have poles at $\mathbf{q}_1 = \lambda_j$ and we considered in section 4 possible discontinuities of the resolvent M along lines in the $\mathbf{q}_2$ plane. We expect, in fact, that the discontinuities of M are the reflex of poles in $t(\mathbf{q}_1)$. To show the connection it is convenient to start from the explicit form of (5.1)

$$M(p;\mathbf{q}) = \int dp' |\nu\rangle(p-p';\mathbf{q}+p') \frac{t(\mathbf{q}_1+p'_1)}{\mathbf{q}_2+p'_2-(\mathbf{q}_1+p'_1)^2} \langle\omega|(p';\mathbf{q}) + \textstyle\sum_n \widehat{M}_{(n)}(p;\mathbf{q}). \tag{5.4}$$

In the first term in the r.h.s. the pole of $t(\mathbf{q}_1)$ at $\mathbf{q}_1 = \lambda_j$ is smoothed by the double integral but there is a discontinuity at $\mathbf{q}_{1\Im} = \lambda_{j\Im}$ because as follows from (3.24)

$$t(\mathbf{q}_{1\Re} + i\lambda_{j\Im} + i0) - t(\mathbf{q}_{1\Re} + i\lambda_{j\Im} - i0) = -2i\pi t_j \delta(\mathbf{q}_{1\Re} - \lambda_{j\Re}). \tag{5.5}$$

This discontinuity must be compensated by the discontinuities of the terms $\widehat{M}_n$. It results that for each λ_j there exists a μ_n such that $\lambda_{j\Re} = \mu_n$ and viceversa that the resolvent M cannot have a discontinuity along the line $\mathbf{q}_{2\Im} = 2\mu_n \mathbf{q}_{1\Im}$ if the transmission coefficient $t(\mathbf{q}_1)$ has not a pole at λ_j with $\lambda_{j\Re} = \mu_n$. Consequently it is convenient to renumber the λ_j as $\lambda_{n,j} = \mu_n + i\kappa_{n,j}$ ($j = \pm 1, \pm 2, ..., \pm N_n$) where $\kappa_{n,j} > 0$ ($j > 0$) and $\kappa_{n,-j} = -\kappa_{n,j}$. Correspondingly we rename $t_{n,j}$ the residuum at the pole λ_j defined in (3.24). It obeys the condition $t_{n,-j} = \overline{t_{n,j}}$.

EIGENFUNCTIONS OF THE DISCRETE SPECTRUM AND BILINEAR REPRESENTATION OF $M_{\Delta n}$. Let us introduce special notations for special values of the Jost solutions

$$\begin{aligned} |\nu_{n,j}\rangle(p;\mathbf{q}) &= |\nu\rangle(p_1 + \mathbf{q}_{1\Re} - \mu_n, p_2 + \mathbf{q}_{2\Re} - \mu_n^2 + \kappa_{n,j}^2; \mu_n + i\kappa_{n,j}) \\ \langle\omega_{n,j}|(p;\mathbf{q}) &= \langle\omega|(\mu_n - \mathbf{q}_{1\Re}, \mu_n^2 - \kappa_{n,j}^2 - \mathbf{q}_{2\Re}; \mathbf{q}_{1\Re} + i\kappa_{n,j}). \end{aligned} \tag{5.6}$$

Note that $|\nu_{n,j}\rangle$ depends only on $p+\mathbf{q}$ and $\langle\omega_{n,j}|$ on $\mathbf{q}$, i.e. they belong respectively to the left $\mathcal{L}$ and right $\mathcal{R}$ ideal defined in section 1. In addition both of them are independent of $\mathbf{q}_{\Im}$. By using the analyticity properties of $M_{\Delta n}(p;\mathbf{q}_1, \mathbf{q}_2 + 2\mu_n\mathbf{q}_1)$ in $\mathbf{q}_1$ one can derive a bilinear representation for $M_{\Delta n}$

$$M_{\Delta n} = -2i\pi \sum_{j=\pm 1,...,\pm N_n} t_{n,j} \int dp' \frac{\delta(p'_2 - 2\mu_n p'_1)}{(p'_1 + i\kappa_{n,j})I - iQ_{1\Im}} (|\nu_{n,j}\rangle \otimes \langle\omega_{n,j}|)^{(p')} \tag{5.7}$$

The special reduced values of the Jost solutions $|\nu_{n,j}\rangle$ and $\langle\omega_{n,j}|$ obey the conjugation property $|\nu_{n,j}\rangle^\dagger = \langle\omega_{n,-j}|$. They obey integral equations which are homogeneous since $T^{-1}(p,\mathbf{q})$ at the values considered is identically zero

and the differential equations

$$(Q_{2\Re} - Q_{1\Re}^2 + \kappa_{n,j}^2 I + 2i\kappa_{n,j}(\mu_n I - Q_{1\Re}) - v)|\nu_{n,j}\rangle = 0 \tag{5.8}$$

$$\langle\omega_{n,j}|(Q_{2\Re} - Q_{1\Re}^2 + \kappa_{n,j}^2 I + 2i\kappa_{n,j}(\mu_n I - Q_{1\Re}) - v) = 0. \tag{5.9}$$

Their counterparts in the (x, y) space are given by

$$\Phi_{n,j}(x,y) = 2\pi\, e^{\kappa_{n,j}(x-2\mu_n y)+i\kappa_{n,j}^2 y}\langle x, y|\nu_{n,j}\rangle \tag{5.10}$$

$$\Psi_{n,j}(x,y) = 2\pi\, e^{-\kappa_{n,j}(x-2\mu_n y)-i\kappa_{n,j}^2 y}\langle\omega_{n,j}|x, y\rangle \tag{5.11}$$

They are connected by the formula $\overline{\Phi_{n,j}(x,y)} = \Psi_{n,-j}(x,y)$ and one can check easily by using (5.8) and (5.9) that they obey the differential equations

$$(-i\partial_y + \partial_x^2 - u(x,y))\Phi_{n,j}(x,y) = \kappa_{n,j}^2\Phi_{n,j}(x,y) \tag{5.12}$$

$$(i\partial_y + \partial_x^2 - u(x,y))\Psi_{n,j}(x,y) = \kappa_{n,j}^2\Psi_{n,j}(x,y) \tag{5.13}$$

and are therefore the mutually dual eigenfunctions of the discrete spectrum of the nonstationary Schrödinger equation.

USING THE BILINEAR REPRESENTATION FOR $M_{\Delta n}$. Let us consider the bilinear representation of $M_{\Delta n}$ in (5.7) and multiply it from the right by $(Q_{2\Re} + 2i\mu_n Q_{1\Im} - Q_1^2 - v)$. Recalling that $|\nu_{n,j}\rangle \in \mathcal{L}$ and using the second property listed in (1.7) and the differential equation (5.9) satisfied by $\langle\omega_{n,j}|$ we obtain that ($l = 0, 1, 2, ...$)

$$\sum_{j=\pm1,\ldots,\pm N_n} t_{n,j}\int ds\delta(s_2 - 2\mu_n s_1)(s_1 + i\kappa_{n,j})^l\,(|\nu_{n,j}\rangle\otimes\langle\omega_{n,j}|)^{(s)} = 0 \tag{5.14}$$

Up to some convenient shifts in the variables these expressions are just the coefficients of the $1/\mathbf{q}_1$–expansion of the function $M_{\Delta n}(p; \mathbf{q}_1, \mathbf{q}_2 + 2\mu_n\mathbf{q}_1)$. Since this function is analytic in the complex plane $\mathbf{q}_1$ with discontinuities along the lines $\mathbf{q}_{1\Im} = \kappa_{n,j}$ we can deduce that it is identically zero outside the widest strip $|\mathbf{q}_{1\Im}| \leq \kappa_{n,j}$. One can show that these properties of the discontinuity $M_{\Delta n}$ of the resolvent are equivalent to the following properties satisfied by the Jost solutions in the (x, y) space

$$\sum_{j=\pm1,\ldots,\pm N_n} t_{n,j}(\kappa_{n,j}^2)^m\Phi_{n,j}(x,y)\Psi_{n,j}(x',y') = 0, \quad m = 0, 1, \ldots \tag{5.15}$$

SCALAR PRODUCTS OF THE EIGENFUNCTIONS OF THE DISCRETE SPECTRUM. The only possibility to fulfill the conditions (5.15) is that the terms with

opposite j cancel out. This means that $\Phi_{n,j}$ must be proportional to $\Phi_{n,-j}$ and $\Psi_{n,j}$ to $\Psi_{n,-j}$ or more precisely that there exist constants $b_{n,j}$ such that

$$\Phi_{n,j}(x,y) = b_{n,j}\Phi_{n,-j}(x,y), \quad \Psi_{n,j}(x,y) = \frac{1}{\overline{b_{n,j}}}\Psi_{n,-j}(x,y). \tag{5.16}$$

To cancel the terms with opposite j in (5.15) the constants $b_{n,j}$ must satisfy the relation $b_{n,j} = -(\overline{t_{n,j}}/t_{n,j})\overline{b_{n,j}}$. Then from (4.9) one can derive the scalar products

$$\int dx \Psi_{n,j}(x,y)\Phi_{n,j'}(x,y) = -\frac{i}{t_{n,j}}\delta_{j,j'}, \quad j,j' > 0. \tag{5.17}$$

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CHARACTERIZATION AND SOLUTION OF THE DISPERSIONLESS HIROTA EQUATIONS

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The dispersionless differential Fay identity is shown to be equivalent to a kernel expansion providing a universal algebraic characterization and solution of the dispersionless Hirota equations. Some calculations based on D-bar data of the action are also indicated.

1 Introduction

We describe the results of ref.4. The dKP hierarchy (= dispersionless limit of the KP hierarchy) and various reductions thereof (such as dNKdV) play an increasingly important role in topological field theory and its connections to strings and 2-D gravity (cf. [1,2,3,5,6,8,10,13,14,16,17,18]). One can begin with two pseudodifferential operators ($\partial = \partial/\partial x$), $L = \partial + \sum_1^\infty u_{n+1}\partial^{-n}$ and $W = 1 + \sum_1^\infty w_n\partial^{-n}$ ($L = W\partial W^{-1}$) called the Lax operator and gauge operator respectively. The KP hierarchy then is determined by the Lax equations ($\partial_n = \partial/\partial t_n$), $\partial_n L = [B_n, L] = B_nL - LB_n$, where $B_n = L_+^n$. The wave function is defined via $\psi = W e^\xi = w(t,\lambda)e^\xi$; $\xi = \sum_1^\infty t_n\lambda^n$; $w(t,\lambda) = 1 + \sum_1^\infty w_n(t)\lambda^{-n}$ where $t_1 = x$. There is also an adjoint wave function $\psi^* = W^{*-1}\exp(-\xi) = w^*(t,\lambda)\exp(-\xi)$, $w^*(t,\lambda) = 1 + \sum_1^\infty w_i^*(t)\lambda^{-i}$, and equations $L\psi = \lambda\psi$; $\partial_n\psi = B_n\psi$; $L^*\psi^* = \lambda\psi^*$; $\partial_n\psi^* = -B_n^*\psi^*$. Note that the KP hierarchy is then given by the compatibility conditions among these equations with an isospectral property. Next one has the fundamental tau function $\tau(t)$ and vertex operators $G_\pm(\lambda)$ satisfying (**V**):

$$\psi(t,\lambda) = (exp(\xi)G_-(\lambda)\tau(t))/\tau(t) = (exp(\xi)\tau(t - [\lambda^{-1}]))/\tau(t);$$

$$\psi^*(t,\lambda) = (exp(-\xi)G_+(\lambda)\tau(t))/\tau(t) = (exp(-\xi)\tau(t + [\lambda^{-1}]))/\tau(t)$$

(here $G_\pm(\lambda) = \exp(\pm\xi(\tilde\partial,\lambda^{-1}))$ with $\tilde\partial = (\partial_1, (1/2)\partial_2, (1/3)\partial_3, \cdots)$ and $t \pm [\lambda^{-1}] = (t_1 \pm \lambda^{-1}, t_2 \pm (1/2)\lambda^{-2}, \cdots)$). Also (*) $exp(\xi) = \exp\left(\sum_1^\infty t_n\lambda^n\right) =$

$\sum_0^\infty \chi_j(t_1, t_2, \cdots, t_j)\lambda^j$ where the χ_j are the elementary Schur polynomials, which arise in many important formulas (cf. below).

We mention next the famous Hirota bilinear identity which generates the entire KP hierarchy. This has the form $\oint_\infty \psi(t,\lambda)\psi^*(t',\lambda)d\lambda = 0$, where $\oint_\infty(\cdot)d\lambda$ is the residue integral about ∞, which we also denote by $Res_\lambda[(\cdot)d\lambda]$. Using (**V**) this can also be written in terms of tau functions as $\oint_\infty \tau(t - [\lambda^{-1}])\tau(t'+[\lambda^{-1}])e^{\xi(t,\lambda)-\xi(t',\lambda)}d\lambda = 0$ leading to the characterization of the tau function in bilinear form and to the Hirota bilinear equations. Next one has the Fay identity (cf. [1,7] - c.p. means cyclic permutations) $\sum_{c.p.}(s_0 - s_1)(s_2 - s_3)\tau(t + [s_0] + [s_1])\tau(t + [s_2] + [s_3]) = 0$ which can be derived fr om the Hirota bilinear identity above. Some elementary manipulation (cf. [1]) then leads to the differential Fay identity $\tau(t)\partial\tau(t + [s_1] - [s_2]) - \tau(t + [s_1] - [s_2])\partial\tau(t) = (s_1^{-1} - s_2^{-1})\,[\tau(t + [s_1] - [s_2])\tau(t) - \tau(t + [s_1])\tau(t - [s_2])]$.

Now for the dispersionless theory (dKP) one takes $t_n \to \epsilon t_n = T_n$ ($t_1 = x \to \epsilon x = X$) and writes $L_\epsilon = \epsilon\partial + \sum_1^\infty u_{n+1}(T/\epsilon)(\epsilon\partial)^{-n}$ (think of $u_n(T/\epsilon) = U_n(T) + O(\epsilon)$, etc.). Then take a WKB form for the wave function with the action S, namely $\psi = \exp[\frac{1}{\epsilon}S(T,\lambda)]$. Replacing now ∂_n by $\epsilon\partial_n$, where $\partial_n = \partial/\partial T_n$ now, we define $P = \partial S = S_X$. Then $\epsilon^i\partial^i\psi \to P^i\psi$ as $\epsilon \to 0$ and the equation $L\psi = \lambda\psi$ becomes

$$\lambda = P + \sum_1^\infty U_{n+1}P^{-n}; \quad P = \lambda - \sum_1^\infty P_{i+1}\lambda^{-i} \tag{1.1}$$

where the second equation is simply the inversion of the first. We also note from $\partial_n\psi = B_n\psi = \sum_0^n b_{nm}(\epsilon\partial)^m\psi$ that one obtains $\partial_n S = \mathcal{B}_n(P) = \lambda_+^n$ where the subscript (+) refers now to powers of P (note $\epsilon\partial_n\psi/\psi \to \partial_n S$). Thus $B_n = L_+^n \to \mathcal{B}_n(P) = \lambda_+^n = \sum_0^n b_{nm}P^m$ and the KP hierarchy goes to (**) $\partial_n P = \partial\mathcal{B}_n$ which is the dKP hierarchy (note $\partial_n S = \mathcal{B}_n \Rightarrow \partial_n P = \partial\mathcal{B}_n$). The action S above can be computed from (**V**) in the limit $\epsilon \to 0$ as

$$S = \sum_1^\infty T_n\lambda^n - \sum_1^\infty \frac{\partial_m F}{m}\lambda^{-m}; \; \mathcal{B}_n = \partial_n S = \lambda^n - \sum_1^\infty \frac{F_{nm}}{m}\lambda^{-m} \tag{1.2}$$

where the function $F = F(T)$ (free energy) is defined by the formula $\tau = \exp[(1/\epsilon^2)F(T)]$ ($F_{nm} = \partial_n\partial_m F$ and $P = \mathcal{B}_1 = \partial S$). This S (or P) then solves the dKP hierarchy (**).

2 Results

Now following [18] we write the differential Fay identity with $\epsilon\partial_n$ replacing ∂_n, $\epsilon t_n = T_n$, etc. (cf. [4]). In passing this to limits ($\epsilon \to 0$) only the second order derivatives survive (cf. [5]), and one gets the dispersionless differential Fay identity

$$\sum_{m,n=1}^{\infty} \mu^{-m}\lambda^{-n}\frac{F_{mn}}{mn} = \log\left(1 - \sum_{1}^{\infty}\frac{\mu^{-n}-\lambda^{-n}}{\mu-\lambda}\frac{F_{1n}}{n}\right) \tag{2.1}$$

Although (2.1) only uses a subset of the Plücker relations defining the KP hierarchy it was shown in [18] that this subset is sufficient to determine KP; hence (2.1) characterizes the function F for dKP. Now note from (1.1) and (1.2) that $F_{1n} = nP_{n+1}$ so $\sum_1^\infty \lambda^{-n}(F_{1n}/n) = \sum_1^\infty P_{n+1}\lambda^{-n} = \lambda - P(\lambda)$. Consequently the right side of (2.1) becomes $\log[\frac{P(\mu)-P(\lambda)}{\mu-\lambda}]$ and for $\mu \to \lambda$ with $\dot{P} = \partial_\lambda P$ we have

$$\log\dot{P}(\lambda) = \sum_{m,n=1}^{\infty}\lambda^{-m-n}\frac{F_{mn}}{mn} = \sum_{j=1}^{\infty}\left(\sum_{n+m=j}\frac{F_{mn}}{mn}\right)\lambda^{-j} \tag{2.2}$$

Then using the elementary Schur polynomials defined in (*) and (1.1), we obtain

$$\dot{P}(\lambda) = \sum_{0}^{\infty}\chi_j(Z_2,\cdots,Z_j)\lambda^{-j} = 1 + \sum_{1}^{\infty} jP_{j+1}\lambda^{-j-1} = 1 + \sum_{1}^{\infty} F_{1j}\lambda^{-j-1}$$

where Z_i, $i \geq 2$, are defined by $Z_i = \sum_{m+n=i}(F_{mn}/mn)$. Note that $Z_1 = 0$. Thus we obtain the dispersionless Hirota equations

$$\textbf{(H)} \quad F_{1j} = \chi_{j+1}(Z_1 = 0, Z_2, \cdots, Z_{j+1})$$

These can be also derived in an equivalent form by expanding (2.2) in powers of λ^{-n} or by using the Hirota bilinear equations directly (cf. [4,6]). It is also interesting to note that the dispersionless Hirota equations can be regarded as algebraic equations for "symbols" F_{mn}, which are defined via (1.2). In fact one proves in [4]

THEOREM 2.1. The symbols F_{nm} of (1.2) satisfy the dispersionless Hirota equations **(H)** and $F_{nm} = F_{mn} = Res_P[\lambda^m d\lambda_+^n]$. Thus for λ, P given algebraically as in (1.1), with no a priori connection to dKP, and for $\mathcal{B}_n$ defined

as in (1.2) via a formal collection of symbols with two indices F_{mn}, it follows that the dispersionless Hirota equations are nothing but polynomial identities among the F_{mn}.

Now one very natural way of developing dKP begins with (1.2) and (**), since eventually the P_{j+1} can serve as universal coordinates (cf. here [3] for a discussion of this in connection with topological field theory = TFT). This point of view is also natural in terms of developing a Hamilton-Jacobi theory involving ideas from the hodograph − Riemann invariant approach (cf. [5,12,13,14] and remarks below). It is natural here to work with $Q_n = (1/n)\mathcal{B}_n$ and note that $\partial_n S = \mathcal{B}_n$ corresponds to $\partial_n P = \partial \mathcal{B}_n = n\partial Q_n$. Actually, given (1.1) and (**) the equation $\partial_n P = n\partial Q_n$ corresponds to Benney's moment equations and is equivalent to a system of Hamiltonian equations defining the dKP hierarchy (cf. [5,14] and remarks above). We recall here also an important formula for the functions Q_n (cf. [4,5,14]). Namely, a generating function of $\partial_P Q_n(\lambda)$ is given by

$$\frac{1}{P(\mu) - P(\lambda)} = \sum_1^\infty \partial_P Q_n(\lambda)\mu^{-n} \tag{2.3}$$

In particular we note that $\oint_\infty [\mu^n/(P(\mu) - P(\lambda))]d\mu = \partial_P Q_{n+1}(\lambda)$, which gives a key formula in the Hamilton-Jacobi method for dKP. Thus (2.3) represents a Cauchy kernel and has a version on Riemann surfaces related to a dispersionless limit of the Fay prime form. Further from (2.3) there follows

$$\partial_P Q_n = \chi_{n-1}(Q_1, \cdots, Q_{n-1})$$

and one proves (cf. [4])

THEOREM 2.2. The kernel formula (2.3) is equivalent to the dispersionless differential Fay identity (2.1).

Theorem 2.2 implies that the dispersionless Hirota equations (**H**) can be derived from the kernel formula (2.3) which is a direct consequence of the definition of F_{mn} in (1.2). Thus Theorem 2.2 gives a direct proof of Theorem 2.1. The following facts are also proved in [4] along with further explicit formulas. They indicate in particular that dKP theory can be characterized using only elementary Schur polynomials since the proofs provide all the information necessary for the kernel (2.3) or equivalently for the dispersionless differential Fay identity.

THEOREM 2.3. One can write $\chi_n = \partial_P Q_{n+1}$ and the F_{mn} can be ex-

pressed as polynomials in $P_{j+1} = F_{1j}/j$. Then the dispersionless Hirota equations can be solved totally algebraically via $F_{mn} = \Phi_{mn}(P_2, P_3, \cdots, P_{m+n})$ where Φ_{mn} is a polynomial in the P_{j+1} and the $F_{1n} = nP_{n+1}$ are generating elements for the F_{mn}. Finally for any $n \geq 2$, $\chi_n(-Q_1, \cdots, -Q_n) = -P_n$.

REMARK 2.4. We indicate here some connections to D-bar methods. It was shown in [4,6,12] how inverse scattering information is connected to the dispersionless theory for KdV and some other situations. We will see here that although such connections seem generally not to be expected, nevertheless, one can isolate D-bar data for S and P in the dispersionless NKdV situations leading to an expression for the generating elements $F_{1n} = nP_{n+1}$ which can be useful in computation. The technique also indicates another role for the Cauchy type kernel $1/(P(\lambda) - P(\mu))$ in (2.3). We only mention a few facts here and refer to [4,6,9] for more details. Thus consider the situation of [4,11,12] with a reduction of dKP to $\Lambda = \lambda^N = P^N + a_0 P^{N-2} + ... + a_{N-2}$ which is called dNKdV reduction. Assume $\Lambda(P)$ has N distinct real zeros $P_1 > P_2 > ... > P_N$ with $N-1$ interwoven turning points $\Lambda_k = \Lambda(\tilde{P}_k)$, $P_{k+1} > \tilde{P}_k > P_k$. Assume, as $X \to -\infty$, these Riemann invariants $\Lambda_k \to 0$ monotonically so $\Lambda \to P^N$ with $(P - \lambda) \to 0$. Such situations are considered in [12] and techniques from [14] are adapted (cf. also [5,6]). Here one has Riemann invariants $\lambda_k = \lambda(\tilde{P}_k)$ where $\partial_P \Lambda = 0$ and there is a collection L of finite cuts through the origin of angles $k\pi/N$ $(1 \leq k \leq N-1)$ in the λ plane with branch points λ_k. One takes Γ to be a suitable contour encircling the cuts clockwise (not containing μ) and sets $\Gamma = \Gamma_- - \Gamma_+$ where $+$ refers to the upper half plane. It turns out that $S|_{\Gamma_+} = \bar{S}|_{\Gamma_-}$ and the contours can be collapsed onto the cuts to yield integrals involving the Cauchy type kernel of (2.3) (see [9,12] for pictures and details). By reorganizing the terms in the integrals one can express such integrals in terms of D-bar data of P on the cuts. Thus P and S are analytic in λ for finite λ except on the cuts L where there is a jump discontinuity ΔP (yielding ΔS by integration in X ($S_X = P$) (e.g. for dKdV $\Delta P = 2\sqrt{U - k^2}$ on $(-\sqrt{U}, \sqrt{U})$ for suitable k). The techniques apply equally well to dKP provided the D-bar data $\bar{\partial}P$ for dKP lie in a bounded set Ω. Let us assume that $\bar{\partial}P$ (and hence $\bar{\partial}S$) is nontrivial only for a region Ω where say $|\lambda| \leq M < \infty$, and let Γ be a contour enclosing $|\lambda| \leq M$. Then without regard for the nature of such data (Riemann- Hilbert data, poles, simple nonanalyticity, etc.) one can in fact derive formulas following [5,12,14] (see [9] for details). We cannot collapse on cuts but we can think of integrals over D-bar data $\bar{\partial}P$ or $\bar{\partial}S$. It remains open to describe D-bar data for the dKP situations however. In summary (cf. [4,9] for

more detail) we can write for dNKdV (or dKP with given bounded D-bar data)

$$\Xi = -\frac{1}{2\pi i}\oint_\Gamma \frac{S(X,\lambda)\partial_\lambda P(X,\lambda)}{P(X,\lambda)-P(X,\mu)}d\lambda \tag{2.4}$$

where Γ encircles the cuts (or D-bar data) clockwise and $\Xi = S(X,\mu) - P(\mu)X - \sum_2^\infty \mathcal{B}_n(P(\mu),X)T_n$. The determination of D-bar data here is to be made via analysis of the polynomial λ^N for dNKdV and in this situation one can collapse (2.4) to the cuts, in which case Tsarev type generalized hodograph formulas arise (cf. [4,12]).

Perhaps the main point of this has been to show that S can be characterized via D-bar data of S or P (and to show again the emergence of the kernel (2.3)). There is also another interesting way in which D-bar data for P or S can be exploited. Thus, given that S has D-bar data as indicated above in a bounded region, one can say that for $|\lambda|$ large (†) $\partial_j F = (j/2\pi i)\int\int \zeta^{j-1}\bar\partial_\zeta S\ d\zeta\wedge d\bar\zeta$ (cf. [4,5]). This is very useful information about F_j, since from F_j one could in principle compute all of the functions F_{ij} for example. Also for dKP where $\partial_j F$ is known via residue calculations this serves as a moment problem for $\bar\partial S$. In fact, for dKdV, and possibly some other dNKdV situations, one can obtain a direct useful formula for the F_{1j}, which we know to generate all the F_{ij}. Thus for dKdV we know that in suitable variables $\Delta S = \int_{X_0}^X \Delta P dX' = 2\int_{X_0}^X \sqrt{U(X',T)-k^2}dX'$ on the cut $L = (-\sqrt{U},\sqrt{U})$ in the k-plane, and adjusting variables, ($\zeta^2 = -k^2$ etc.) one can write then

$$\partial F_j = -[(j(i)^{j-1})/\pi]\partial\int_{-\sqrt{U}}^{\sqrt{U}} k^{j-1}\int_{X_0}^X \sqrt{U(X',T)-k^2}\ dX'dk$$

Since all $F_{1\,2m} = 0$ by the residue formula in Theorem 2.1, we take $j = 2n-1$ to obtain (‡) $F_{1\,2n-1} = (-1)^n((2n-1)/\pi)\int_{-\sqrt{U}}^{\sqrt{U}} k^{2n-2}\sqrt{U-k^2}dk = (-1)^n(U/2)^n\prod_1^n(2l-1)/l)$ (here $k = a\sin\theta$ and $a^2 = U$). In summary (cf. [4] for more details)

THEOREM 2.5. For any situation with bounded D-bar data for S one can use (†) to compute F_j or as a moment problem for $\bar\partial S$. In the case of dKdV with ΔS as indicated one can determine directly the $F_{1\,2n-1}$ from (‡) which will generate all the functions F_{ij}. Generally $\bar\partial S = \int_{X_0}^X \bar\partial P(X',T,\zeta,\bar\zeta)dX'$ implies $F_{1j} = (j/2\pi i)\int\int\zeta^{j-1}\bar\partial P d\zeta\wedge d\bar\zeta$, and formally $\partial_n\partial_j F = (j/2\pi i)\int\int\zeta^{j-1}\bar\partial\mathcal{B}_n d\zeta\wedge d\bar\zeta$ (since $\partial_n\bar\partial S = \bar\partial\partial_n S = \bar\partial\mathcal{B}_n$).

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SOME THOUGHTS ON INTEGRATING NON-INTEGRABLE SYSTEMS

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S-integrable systems in dimensions higher than $2+1$ tend to be overdetermined and permit only constant solutions. In this talk I show that other systems *are* S-integrable but only if they are also C-integrable.

The real title of this talk should be *Some Unproductive Thoughts on Integrating Non-Integrable Systems.* In the book *Through the Looking Glass* by Lewis Carroll, also known as the English mathematician Charles Lutwidge Dodgson, the heroine, Alice, having passed through the looking glass eventually decides to go into the garden. She sees a hill in the distance and decides to walk to it. However, no matter which path she takes, she always ends up back at the house. This talk is a bit like that. The house is integrable systems, S-integrable in Calogero's terminology and the hill is non-integrable systems.

The spectral transform, by which S-integrable systems are solved was first discovered By Gardner et al. [1-5]. in 1967. Since then it has been greatly developed, and been extended from $1+1$ to higher dimensions [6,7]. S-integrable systems can all be expressed in terms of Lax pairs $\hat{A}$ and $\hat{B}$. In $1+1$ dimensions we have the spectral equation

$$\psi_x = \hat{A}(x,t,\lambda)\psi \tag{1}$$

together with an auxiliary spectral equation

$$\psi_t = \hat{B}(x,t,\lambda)\psi \tag{2}$$

where $\hat{A}$ and $\hat{B}$ are square matrices depending on x, t and a spectral parameter λ.

For $n+1$ dimensions this becomes

$$\psi_x = \hat{A}(x,\mathbf{y},t,\nabla)\psi \tag{3}$$

together with an auxiliary spectral equation

$$\psi_t = \hat{B}(x,\mathbf{y},t,\nabla)\psi \tag{4}$$

where $\hat{A}$ and $\hat{B}$ now depend on other space variables given by the vector $\mathbf{y} = (y_1, y_2, \cdots, y_{n-1})$. The spectral parameter is replaced by the del-operator

$\nabla = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_{n-1}})$. In both cases the integrability condition gives us the system under consideration.

$$\hat{A}_t - \hat{B}_x + \left[\hat{A}, \hat{B}\right] = 0. \tag{5}$$

An archetypical case is

$$\hat{A} = \frac{1}{4}\left\{\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (U_x + U_t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} \cos U & \sin U \\ \sin U & -\cos U \end{pmatrix}\right\} \tag{6}$$

and

$$\hat{B} = \frac{1}{4}\left\{\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (U_x + U_t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{\lambda}\begin{pmatrix} \cos U & \sin U \\ \sin U & -\cos U \end{pmatrix}\right\} \tag{7}$$

which gives the sine-Gordon equation in $1+1$ dimensions,

$$(U_{xx} + U_{tt} - \sin U)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{8}$$

If we try and raise this to $3+1$ dimensions we up the matrix dimensions to 4×4 and make the substitution

$$\frac{\partial}{\partial x} \to \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \tag{9}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \text{and} \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

We also introduce the matrices

$$\Sigma_x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \Sigma_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \text{and} \Sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

After making the substitutions and rearranging to get things into the proper form we have

$$\hat{A} = -\frac{1}{4} i\sigma_x \left\{\sin U \Sigma_x + (U_x\sigma_x + U_y\sigma_y + U_z\sigma_z)\Sigma_y - (1 - \cos U)\Sigma_z\right\} - i\sigma_z \frac{\partial}{\partial y} + i\sigma_y \frac{\partial}{\partial z}, \tag{12}$$

and

$$\hat{B} = \frac{1}{4} i \left\{ \sin U \Sigma_x + \left(U_x \sigma_x + U_y \sigma_y + U_z \sigma_z \right) \Sigma_y + \left(1 - \cos U \right) \Sigma_z \right\}. \tag{13}$$

The integrability condition now is

$$\frac{1}{4} i \left(U_{xx} + U_{yy} + U_{zz} - U_{tt} - \sin U \right) \sigma_x \Sigma_y + \hat{C} = 0 \tag{14}$$

where

$$\hat{C} = -\frac{1}{2} \sigma_x \Sigma_y \left\{ \left(U_y \sigma_z - U_z \sigma_y \right) \hat{A} + \left(U_y \sigma_z - U_z \sigma_y \right) \frac{\partial}{\partial y} + \left(U_y \sigma_z - U_z \sigma_y \right) \frac{\partial}{\partial z} \right\}. \tag{15}$$

The $3+1$ dimensional sine-Gordon equation is there, but so is

$$\hat{C} = 0 \tag{16}$$

and satisfying this requires that

$$U_x = U_y = U_z = 0, \tag{17}$$

which makes for some rather uninteresting solutions.

This is characteristic of what happens in $n+1$ dimensions when $n > 2$. The system becomes overdetermined and solutions become trivial. We need to loosen things up a bit. Assuming that the sine-Gordon equation (or whatever system we are examining) is satisfied with no further restrictions, the wave function ψ must evolve somehow. We keep the spectral equation

$$\psi_x = \hat{A} \psi \tag{18}$$

but generalize the auxiliary spectral equation to

$$\psi_t = \hat{B} \psi + \phi \tag{19}$$

where we have introduced the supplementary wave function ϕ. We close the system with the supplementary wave equation

$$\phi_x = \hat{A} \phi + \hat{C} \psi. \tag{20}$$

The integrability condition now is

$$\hat{A} - \hat{B} + \left[\hat{A}, \hat{B} \right] = \hat{C}, \tag{21}$$

which if we choose $\hat{A}$, $\hat{B}$ and $\hat{C}$ as above is just the sine-Gordon equation.

In order to integrate this system we need to know how ψ evolves in time. We put

$$\phi_t = \hat{B}\phi + \phi^{(2)}. \tag{22}$$

It follows that $\phi^{(2)}$ satisfies

$$\begin{aligned} \phi^{(2)}_x &= \phi_{xt} - \hat{B}_x\phi - \hat{B}\phi_x \\ &= \hat{A}\phi^{(2)} + \left(\hat{A}_t - \hat{B}_x + \left[\hat{A}, \hat{B}\right] + \hat{C}\right)\phi + \left(\hat{C}_t + \left[\hat{C}, \hat{B}\right]\right)\psi \\ &= \hat{A}\phi^{(2)} + 2\hat{C}\phi + \hat{C}^{(2)}\psi \end{aligned} \tag{23}$$

where

$$\hat{C}^{(2)} = \hat{C}_t + \left[\hat{C}, \hat{B}\right] = \left[\frac{\partial}{\partial t} - \hat{B}, \hat{C}\right]. \tag{24}$$

This argument repeats itself.

$$\phi^{(2)}_t = \hat{B}\phi^{(2)} + \phi^{(3)}. \tag{25}$$

It follows that $\phi^{(3)}$ satisfies

$$\phi^{(3)} = \hat{A}\phi^{(3)} + 3\hat{C}\phi^{(2)} + 3\hat{C}^{(2)}\phi + \hat{C}^{(3)}\psi \tag{26}$$

where

$$\hat{C}^{(3)} = \hat{C}^{(2)}_t + \left[\hat{C}^{(2)}, \hat{B}\right] = \left[\frac{\partial}{\partial t} - \hat{B}, \hat{C}^{(2)}\right]. \tag{27}$$

A pattern is now emerging. The system now consists of the original spectral equation

$$\psi_x = \hat{A}\psi, \tag{28}$$

the generalized auxiliary spectral equation

$$\psi_t = \hat{B}\psi + \phi^{(1)}, \tag{29}$$

the spectral equations which define the $\phi^{(n)}$'s for $n > 1$

$$\phi^{(n)}_x = \hat{A}\phi^{(n)} + \sum_{m=1}^{n-1} \binom{n}{m} \hat{C}^{(m)}\phi^{(n-m)} + \hat{C}^{(n)}\psi, \tag{30}$$

and the auxiliary spectral equations for the $\phi^{(n)}$'s

$$\phi^{(n)}_t = \hat{B}\phi^{(n)} + \phi^{(n+1)}. \tag{31}$$

The integrability conditions now are the non-linear evolution equation

$$\hat{A} - \hat{B} + \left[\hat{A}, \hat{B}\right] = \hat{C}, \tag{32}$$

and the iterative definitions of $\hat{C}^{(n+1)}$, $n \geq 1$

$$\hat{C}^{(n+1)} = \hat{C}_t^{(n)} + \left[\hat{C}^{(n)}, \hat{B}\right] = \left[\frac{\partial}{\partial t} - \hat{B}, \hat{C}^{(n)}\right]. \tag{33}$$

This is where we realize that we are returning to the house. The time derivative $\hat{C}_t^{(n)}$ at each stage of the iteration can be turned into space derivatives by using the non-linear evolution equation, but if we do this we might just as well integrate directly this way and not use the spectral transform at all.

Since we are approaching the house, we may as well go inside. We can do this by introducing a new variable s and writing ϕ as a power series in s,

$$\phi = \psi + \sum_{n=1}^{\infty} \frac{s^n}{n!} \phi^{(n)}. \tag{34}$$

Similarly we introduce the operator

$$\hat{C} = \hat{A} + \sum_{n=1}^{\infty} \frac{s^n}{n!} \hat{C}^{(n)}. \tag{35}$$

Then we find that

$$\begin{aligned} \phi_x &= \hat{A}\phi + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left(\hat{A}\phi^{(n)} + \sum_{m=1}^{n-1} \hat{C}^{(m)} \phi^{(n-m)} + \hat{C}^{(n)} \psi \right) \\ &= \left(\hat{A} + \sum_{n=1}^{\infty} \frac{s^n}{n!} \hat{C}^{(n)} \right) \left(\psi + \sum_{n=1}^{\infty} \frac{s^n}{n!} \phi^{(n)} \right) = \hat{C}\phi \end{aligned} \tag{36}$$

and

$$\phi_t = \hat{B}\psi + \phi^{(1)} + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left(\hat{B}\phi^{(n)} + \phi^{(n+1)} \right) = \hat{B}\phi + \phi_s. \tag{37}$$

Thus we are now back in the house with an integrable system with the variable $(t - s)$ instead of t.

Unfortunately it does not mean that the sine-Gordon equation in $3 + 1$ dimensions is properly S-integrable. *If* we look at the spectral data, which

consist of the coefficients when solutions of the spectral equation, subject to various boundary conditions, become linearly dependent,

$$\phi_j = \sum_k Q_{jk}\phi_k, \tag{38}$$

then we find

$$\frac{\partial}{\partial t}Q_{jk} = i\left(\mu_j - \mu_k\right)Q_{jk} + \frac{\partial}{\partial s}Q_{jk}, w, \tag{39}$$

where μ_j and μ_k are constants. Thus

$$Q_{jk}\left(t,s\right) = \exp\left\{i\left(\mu_j - \mu_k\right)t\right\}Q_{jk}\left(0,s-t\right). \tag{40}$$

We need only $Q_{jk}\left(t,0\right)$ to solve the inverse problem. This is given by

$$Q_{jk}\left(t,0\right) = \exp\left\{i\left(\mu_j - \mu_k\right)t\right\}Q_{jk}\left(0,-t\right), \tag{41}$$

and so we need $Q_{jk}\left(0,s\right)$, which comes from knowing ϕ_j and ϕ_k, for all values of s. Thus we have to solve for all values of s. This requires the t-derivatives of all orders and so the system is only S-integrable if all these are known, i.e. if it is already C-integrable.

It is clear that a similar argument can be followed through by generalizing the spectral equation instead of the auxiliary spectral equation. In this case it is x-derivatives of all orders which are needed. However, since x and t-derivatives are closely related by derivatives of the evolution equation, there is really no difference.

By the way, Alice solved her problem by walking a little way and then trying to get back to the house. This ploy got her to the hill. In real life, it seems, the gods of fate are not that easily fooled.

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BEYOND THE TWO-SINGULAR MANIFOLD METHOD[a]

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For its coherence, the Painlevé approach must find the Bäcklund transformation (BT) of a partial differential equation (PDE) when there is one. For the AKNS system, both the one-singular manifold method of Weiss and the two-singular manifold method fail to retrieve the BT. We find it by combining the point symmetries of the PDE and the one-singular manifold method.

1 Introduction

When a PDE satisfies some necessary conditions ("passes the Painlevé test" [24,15]) for the absence of movable critical singularities in its general solution (the "Painlevé property" (PP)), the next step is to find a BT [2,20], in order to prove the sufficiency of these conditions. A BT between two given PDEs

$$E_1(u,x,t)=0,\ E_2(U,X,T)=0 \tag{1}$$

is by definition ([8] vol. III chap. XII, [13]) a pair of relations

$$F_j(u,x,t,U,X,T)=0,\ j=1,2 \tag{2}$$

in which F_j depends on the derivatives of $u(x,t)$ and $U(X,T)$, such that the elimination of u (resp. U) between (F_1,F_2) implies $E_2=0$ (resp. $E_1=0$). In case the PDEs are the same, the BT is called the auto-BT.

For instance, the AKNS system [26,1]

$$E^{(1)}\equiv iu_t+p_ru_{xx}+q_ru^2v=0,\ E^{(2)}\equiv -iv_t+p_rv_{xx}+q_ruv^2=0 \tag{3}$$

has the BT [10,3,9,12] ($a^2=-2p_r/q_r,\ R=\pm\sqrt{(u+U)(v+V)/a^2-(\lambda-\mu)^2}$)

$$\begin{aligned}(u+U)_x &= -(u-U)R-i(\lambda+\mu)(u+U)\\ (v+V)_x &= -(v-V)R+i(\lambda+\mu)(v+V)\end{aligned}$$

$$\begin{aligned} +ip_r^{-1}(u+U)_t &= (u-U)_x R + (u+U)M + i(\lambda+\mu)(u+U)_x \\ -ip_r^{-1}(v+V)_t &= (v-V)_x R + (v+V)M - i(\lambda+\mu)(v+V)_x \\ M &= (uv+UV)/a^2 \end{aligned} \tag{4}$$

λ and μ being arbitrary complex constants. Galilean invariance (x,t,u,v) $\to (x-2p_rct, t, e^{i(cx-p_rc^2t)}u, e^{-i(cx-p_rc^2t)}v)$ allows to choose $c=\lambda+\mu=0$ [9].

Up to now, the above BT has been retrieved neither by the one–singular manifold method [23], nor by the two–singular manifold method [16]. This is a challenge to the Painlevé approach, which for most PDEs succeeds to find the BT by singularity analysis *only*. We do so in Section 6.

2 The ideas. Darboux, Lax and Bäcklund

For the six ordinary differential equations which bear his name, Painlevé proved the PP by showing [17,18] the existence of one (case of (P1)) or two ((P2)–(P6)) function(s) ψ linked to the general solution u by logarithmic derivatives

$$(\mathrm{P1}): \quad u = \mathcal{D}_1 \operatorname{Log}\psi \tag{5}$$

$$(\mathrm{P}n),\ n=2,\dots,6: \quad u = \mathcal{D}_n(\operatorname{Log}\psi_1 - \operatorname{Log}\psi_2) \tag{6}$$

where the operators $\mathcal{D}_n$ are linear (α is a constant):

$$\begin{aligned} &\mathcal{D}_1 = -\partial_x^2,\ \mathcal{D}_2 = \mathcal{D}_4 = \pm\partial_x, \\ &\mathcal{D}_3 = \pm e^{-x}\partial_x,\ \mathcal{D}_5 = \pm x e^{-x}(2\alpha)^{-1/2}\partial_x,\ \mathcal{D}_6 = \pm x(x-1)e^{-x}(2\alpha)^{-1/2}\partial_x. \end{aligned} \tag{7}$$

These functions ψ_1, ψ_2 have the same kind of singularities than solutions of *linear* ODEs, namely: they are entire functions for (P1)–(P5), and their only singularities for (P6) are three fixed critical points.

For PDEs, the analog of (5)–(6) is the *Darboux transformation* [7]

$$\mathrm{DT}: \ u = \sum_f \mathcal{D}_f \operatorname{Log}\psi_f + U \tag{8}$$

linking two solutions (u,U) of the PDE *via* logarithmic derivatives of scalars ψ_f attached to families f of movable singularities. Operators $\mathcal{D}_f$ can be derived by the Painlevé test. The ψ_f's satisfy a system of linear PDEs, the *Lax pair* [11]

$$\text{Lax pair}: \ L_1(U,\lambda)\Psi = 0,\ L_2(U,\lambda)\Psi = 0,\ \Psi = \operatorname{col}(\psi_f) \tag{9}$$

with coefficients depending on the second solution U and possibly some arbitrary constant λ, and the property that the vanishing of the commutator $[L_1, L_2]$ is equivalent to the PDE $E(U)=0$. The *Bäcklund transformation*

$$\mathrm{BT}: \ F_1(u,U,\lambda) = 0,\ F_2(u,U,\lambda) = 0 \tag{10}$$

results from the elimination [22] of Ψ between the DT (8) and the Lax pair (9).

3 Weiss method and its limitations

Let $\chi = 0$ be the equation for a given family of movable singularities, $-p$ and $-q$ the two positive integers equal to the singularity orders of u and $E(u)$. The assumption of the existence of a finite expression [24]

$$u_T = \sum_{j=0}^{-p} u_j \chi^{j+p} \tag{11}$$

implies

$$E_T = E(u_T) = \sum_{j=0}^{-q} E_j \chi^{j+q}, \tag{12}$$

where coefficients (u_j, E_j) only depend on the homographic invariants (S, C) occurring in the gradient of χ [4]

$$\chi_x = 1 + \frac{S}{2}\chi^2, \ \chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2 \tag{13}$$

and the arbitrary coefficients u_i of the Fuchs indices i in the interval $[0, -p]$. This "truncation $(p:0)$" (11) only exists if there is a nonempty solution to the set of overdetermined equations, called *Painlevé-Bäcklund equations* (PB),

$$\forall j \in]-p, -q] : \ E_j(S, C, \{u_i, i \text{ Fuchs index } \in [0, -p]\}) = 0. \tag{14}$$

In the case of Korteweg-de Vries (KdV) equation $(p = -2, q = -5)$

$$(\text{KdV}) : \ E \equiv u_t + \left(u_{xx} - (3/a)u^2\right)_x = 0, \ a \text{ constant}, \tag{15}$$

the method is successful (see details ref. [16] section 2): the second solution U and the scalar solution ψ of a linear system are

$$U = -a(C + 2S)/6, \ \text{Log}\,\psi = \int \chi^{-1} \mathrm{d}x, \tag{16}$$

resulting in the Darboux transformation

$$u_T = -2a\partial_x^2 \,\text{Log}\,\psi + U; \tag{17}$$

the spectral parameter λ is $(C-S)/6$, and the Lax pair is the linearized version of the Riccati system for χ

$$(\chi^{-1})_x = -\chi^{-2} + U/a + \lambda, \ (\chi^{-1})_t = ((2U/a - 4\lambda)\chi^{-1} - U_x/a)_x. \tag{18}$$

The BT then results from the elimination of χ between (17) and (18):

$$\chi^{-1} = -(w-W)/(2a),\ u_T = u = w_x,\ U = W_x \tag{19}$$

$$a(w+W)_x = -2a^2\lambda + (w-W)^2/2 \tag{20}$$

$$a(w+W)_t = (w-W)(W-w)_{xx} + 2(W_x^2 + w_x W_x + w_x^2). \tag{21}$$

As far as we know, this one–singular manifold method always succeeds for PDEs admitting exactly one family of movable singularities. It usually fails for PDEs with more than one family, as summarized in Table 1. Its success for Sawada-Kotera, which has two families, has no explanation yet.

Table 1: Ability of the one–singular manifold method to find the DT, the Lax pair and/or the BT, according to the number of families of the PDE. "Opposite" is short for "families with opposite leading terms $\pm u_0\chi^p$". For sine-Gordon, what is found is not exactly the DT, but the second solution U (we thank W. Schief for this precision).

PDE	Families	DT	Lax pair	BT
AKNS system	4	no	yes	no
Sine-Gordon	2 opposite	yes	no	no
MKdV	2 opposite	no	no	no
KdV	1	yes	yes	yes
Sawada-Kotera	2 non-opposite	yes	yes	yes [exception]
Kaup-Kupershmidt	2 non-opposite	no	no	no
Tzitzéica [21]	2 non-opposite	no	no	no

4 First attempt: the two–singular manifold method

While the one–singular manifold method extrapolates (5), the two–singular manifold method [16] extrapolates (6). The two basic assumptions (with some minor variations) are: the existence of a DT (8) with $\mathcal{D}_1 = -\mathcal{D}_2$ like in (6)

$$\text{DT}:\ u = \mathcal{D}(\text{Log}\,\psi_1 - \text{Log}\,\psi_2) + U \tag{22}$$

and the existence of a Riccati system satisfied by $\psi_1/\psi_2 = Y$. This results in a truncation $(p:-p)$ instead of $(p:0)$

$$u = u_T = \sum_{j=0}^{-2p} u_j Y^{j+p},\ E_T = E(u_T) = \sum_{j=0}^{-2q} E_j Y^{j+q}. \tag{23}$$

As far as we know, it succeeds [16,5] for all PDEs admitting exactly two opposite families: sine-Gordon, MKdV, Broer-Kaup, but fails for AKNS.

5 Second attempt: Weiss plus homography

This method uses the freedom in the choice of the expansion variable.

Given a family $\varphi - \varphi_0 = 0$ (φ is a function, φ_0 an arbitrary constant), the expansion variable, which must vanish as $\varphi - \varphi_0$, is chosen as [4]

$$\chi = \left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^{-1} = \frac{\psi}{\psi_x},\ \psi = (\varphi - \varphi_0)\varphi_x^{-\frac{1}{2}}. \tag{24}$$

Reciprocally, the most general expansion variable with the same property of generating homographically invariant coefficients (u_j, E_j) is [14]

$$Y = (A\chi^{-1} + B)^{-1}, \tag{25}$$

where A and B are two arbitrary homographically invariant functions.

Truncations $(p : 0)$ in χ and Y are equivalent (polynomials in χ^{-1} are polynomials in Y^{-1}), but a truncation $(p : -p)$ in Y may have solutions $B \neq 0$, i.e. more solutions than a truncation $(p : -p)$ in χ, see Ref. [19] for details. For instance, the Kaup-Kupershmidt equation admits a truncation $(-2 : 2)$ with $A = 1, B \neq 0$ and an arbitrary parameter λ [6], but this does not seem to provide the BT. For the AKNS system, this method also fails.

6 Present method: Weiss plus involutions

In case the one–singular manifold method fails to provide a BT but only provides some partial result $T(\chi, u, \lambda)$ for the truncation, one then considers all transformations on u conserving the equation $E(u) = 0$ in order to uncover a second solution U, see Table 2.

Table 2: Transformations of the dependent variable(s) conserving the equation(s), for the four PDEs of the AKNS group (complex conjugation, phase shift, parity).

PDE	Transformation(s)
AKNS system	$(u, v, i) \to (v, u, -i);\ \forall k :\ (u, v) \to (ku, v/k)$
Sine-Gordon	$u \to -u$
MKdV	$u \to -u$
KdV	none

For the AKNS system (3), the one–family truncation

$$u = u_0\chi^{-1} + u_1,\ v = v_0\chi^{-1} + v_1 \tag{26}$$

which has the general solution [23,16] (λ arbitrary complex constant)

$$
\begin{aligned}
u &= a(\chi^{-1} - f_x/(2f) - i\lambda)f && (27)\\
v &= a(\chi^{-1} + f_x/(2f) + i\lambda)/f && (28)\\
f_x/f &= -2i\lambda - (u/a)f^{-1} + (v/a)f && (29)\\
ip_r^{-1}f_t/f &= 2uv/a^2 + 4\lambda^2 + (u_x - 2i\lambda u)/(af) + (v_x + 2i\lambda v)f/a && (30)\\
(f_{xt} - f_{tx})/f &= (f^{-1}E^{(1)} + fE^{(2)})/a && (31)
\end{aligned}
$$

fails to introduce a second solution (U, V), see details in appendix C of Ref. [16]. This is done by applying the two point transformations of Table 2 to the above truncation T_1 (27)–(30):

$$
\left.
\begin{array}{llllllll}
T_1: & \chi_1 & u & v & i & f & \lambda & \text{(identity)}\\
T_2: & \chi_2 & v & u & -i & g & \mu & \text{(conjugation)}\\
T_3: & \chi_3 & kU & k^{-1}V & i & f & \lambda' & \text{(phase shift)}\\
T_4: & \chi_4 & k^{-1}V & kU & -i & g & \mu' & \text{(both)}
\end{array}
\right\} \qquad (32)
$$

These transformations act on (u, v, f, λ) like in Chen [3], not on (x, t) like in Yang and Schmid [25]. This is equivalent to successively process the four families of the AKNS system by the one–singular manifold method. In order that (u, v) and $(kU, V/k)$ be distinct, one must have $\lambda' = \mu, \mu' = \lambda$.

The four sets (27)–(28) define a system of eight equations in the eight unknowns $(\chi_1^{-1}, \chi_2^{-1}, \chi_3^{-1}, \chi_4^{-1}, u, v, kU, V/k)$. This system is linear with determinant $fg - 1/(fg)$ and it provides the DT straightforwardly (with the nonrestrictive choice $k = -1$):

$$
\begin{aligned}
u - U &= 2a[\partial_x \operatorname{Log}(g - 1/f) - i(\lambda + \mu)]/(g + 1/f) && (33)\\
v - V &= 2a[\partial_x \operatorname{Log}(f - 1/g) + i(\lambda + \mu)]/(f + 1/g) && (34)\\
u + U &= 2ia(\lambda - \mu)/(g - 1/f),\ v + V = 2ia(\lambda - \mu)/(f - 1/g) && (35)
\end{aligned}
$$

(to stick to our definition, the DT is made of two equations, either (33)–(34) or (35)). The nonconstant factor of the logarithmic derivatives is similar to that of (P3), (P5), (P6), see eq. (7). See Section 1 about $\lambda + \mu$.

The Lax pair in its Riccati form is made of the four equations resulting from the action of T_3 and T_4 on (29)–(30). The BT is made of the four equations resulting from the elimination of the two pseudopotentials (f, g) between the six equations defining the DT and the Lax pair, and these are precisely (4). This elimination is quite easy since equations (35) are algebraic in (f, g):

$$
f = ia(\lambda - \mu + R)/(v + V),\ g = ia(\lambda - \mu + R)/(u + U). \qquad (36)
$$

Remark. The "modified system" [3] of two equations for (f, g), obtained by eliminating (u, v, U, V) between (33)–(35) and the PDE, is invariant under exchange of (λ, μ). The elimination of g between this system provides the Broer-Kaup equation for $w = -i \operatorname{Log} f$, a result immediately obtainable by the Weiss truncation [16]

$$p_r^{-1} w_{tt} + 4 w_x w_{xt} + 2 w_t w_{xx} + p_r (6 w_x^2 w_{xx} + w_{xxxx}) = 0. \qquad (37)$$

7 Reductions of the DT of AKNS system

The "x–part" of the AKNS spectral problem admits the three reductions $v = \bar{u}, v = \pm u, v = 1$, and the DT, obtained only from the x–part, must admit them. This is indeed the case: equations (33)–(35) admit the two reductions $(v, V, g, \mu) = (\bar{u}, \bar{U}, \bar{f}, \bar{\lambda}), (\varepsilon u, \varepsilon U, \varepsilon f, -\lambda), \varepsilon^2 = 1$, and one must add the case of vanishing of the determinant $g = \varepsilon / f$. Table 3 summarizes these reductions and the homographic link between f and the χ of the invariant analysis.

Table 3: Reductions of the Darboux transformation of the AKNS system.

PDE	v, g, μ	χ^{-1}	$u - U$	$u + U$
NLS	$\bar{u}, \bar{f}, \bar{\lambda}$		(33)	$4a(\mathrm{Im}\ \lambda)/(1/f - \bar{f})$
SG MKdV	$\varepsilon u, \varepsilon f, -\lambda$ $e^2 = \varepsilon$	$\frac{\lambda}{Y} - \frac{eU}{4a}$ $Y = \frac{ef - 1}{ef + 1}$	$(4a/e)\frac{Y_x}{Y}$ $= \frac{2af_x}{\varepsilon f^2 - 1}$	$\frac{ia\lambda}{e}(\frac{1}{Y} - Y)$ $= 4ia\lambda \frac{f}{\varepsilon f^2 - 1}$
KdV	$1, \varepsilon/f, -\lambda$	$f - i\lambda$	$-2af_x$ $= -2a(\chi^{-1})_x$	$2a(f^2 - 2i\lambda f)$ $= 2a(\chi^{-2} + \lambda^2)$

8 Conclusion

Using Painlevé analysis *only*, one can find the Darboux transformation and the Lax pair, hence the Bäcklund transformation also for some PDEs which escape Weiss one–singular manifold method: one applies, first, Weiss method, secondly the discrete point symmetries. As compared to our previous "two-singular manifold method", the present method is simpler and, more important, it now succeeds for the ANKS system (NLS).

The next main difficulty to overcome is to handle PDEs with two non-opposite families, such as Kaup-Kupershmidt equation and Tzitzéica equation.

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NONLINEAR INITIAL-BOUNDARY EVOLUTIONS WITH SINGULAR DISPERSION LAWS ASSOCIATED TO THE QUADRATIC BUNDLE

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We obtain an hierarchy of nonlinear evolution equations with singular dispersion relations integrable by the spectral problem of the Wadati-Konno-Ichikawa type. The modified NLS equation with a self-consistent source is shown to be a gauge-transformed member of the hierarchy. The soliton solution for this equation is found.

1 Introduction

Recently Leon[1] constructed an extension of the spectral transform method that admits more general space and time dependence of the spectral data. It was shown on an example of nonlinear equations integrable by the Zakharov-Shabat spectral problem that such an approach leads to nonlinear systems for coupled waves which are integrable for arbitrary boundary values. A singularity (non-analyticity) of the dispersion relations is a special feature of these models. Physical situations described by the equations with the singular dispersion relations (SDR) involve, in particular, the interaction of radiation with matter. In the framework of this approach, Claude, Ginovart and Leon[2] proposed a nonlinear theory of stimulated Raman scattering of high energy laser pulses in two-level media.

Until quite recently, examples of nonlinear equations with SDR have been exhausted with those integrable by means of the AKNS problem. Not long ago, Zeng[3] obtained a hierarchy of nonlinear equations with SDR associated with the Kaup-Newell problem. In the first part of this paper we derive the integrable evolutions with SDR admitting a general quadratic bundle as the spectral problem. As an example, we obtain an integrable variant and one-soliton solution of the modified NLS equation with SDR. In the second part

we extend the above results to initial-boundary problems.

2 Spectral transform and hierarchy

We shall consider the 2×2-matrix $\overline{\partial}$-problem in the complex plane:

$$\frac{\partial}{\partial \overline{k}}\psi(k) \equiv \overline{\partial}\psi(k) = \psi(k)R(k), \quad \psi(k) = 1 + O(1/k),\ |k| \to \infty. \tag{1}$$

Here $R(k)$ is the spectral transform matrix, which we consider being off-diagonal and even in k, $R(-k) = R(k)$. The spatial and temporal dependencies of $R(k)$ are given by

$$R_x = \frac{i}{\alpha}(k^2 + \beta)\,[R, \sigma_3]\,, \quad R_t = (\omega_r + \omega_s)\,[R, \sigma_3]\,, \tag{2}$$

where α and β are real parameters, the dispersion law $\Omega(k) = (\omega_r + \omega_s)\sigma_3$ includes both the regular ω_r and singular ω_s parts :

$$\omega_r(k) = \sum_{j=0}^{J} \gamma_{2j} k^{2j}, \quad \omega_s(k) = \frac{1}{2i\pi} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l^2 - k^2}\, l^2 \rho(l^2). \tag{3}$$

The solution of the $\overline{\partial}$-problem (1) is given by the Cauchy-Green integral formula

$$\psi(k) = 1 + \frac{1}{2i\pi} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l - k} \psi(l)R(l) \equiv 1 + \psi(k)R(k)C_k. \tag{4}$$

Here the operator C_k acts on the left by the rule (4). The representation (4) enables us to write formally a solution of the $\overline{\partial}$-problem (1) in terms of the matrix R[4]:

$$\psi(k) = 1 \cdot (1 - RC_k)^{-1}. \tag{5}$$

with the diagonal ψ^d and off-diagonal ψ^a parts of ψ obeying the following properties with respect to k :

$$\psi^d(-k) = \psi^d(k), \quad \psi^a(-k) = -\psi^a(k). \tag{6}$$

In view of (2) and (6), differentiation of (5) with respect to x gives a linear spectral problem of the Wadati-Konno-Ichikawa type[5]:

$$\partial_x \psi = -\frac{i}{\alpha}(k^2 + \beta)\,[\sigma_3, \psi] + kQ\psi - \frac{i\alpha}{2} Q^2 \sigma_3 \psi. \tag{7}$$

Here a potential Q is defined by the formula

$$Q \equiv \begin{pmatrix} 0 & q(x,t) \\ -\overline{q}(x,t) & 0 \end{pmatrix} = -\frac{i}{\alpha}\,[\sigma_3, \langle \psi R \rangle] \tag{8}$$

and angle brackets stand for the bilinear form (tilde denotes transposition)

$$\langle\psi,\varphi\rangle = (2i\pi)^{-1}\iint \mathrm{d}k\wedge\mathrm{d}\overline{k}\psi(k)\tilde{\varphi}(k),\quad \langle\psi,1\rangle\equiv\langle\psi\rangle$$

Existence of the representation (8) for the potential Q allows us to obtain easily the hierarchy of evolutions $\partial_t Q$. Indeed, we get $\partial_t Q = -\frac{i}{\alpha}\left[\sigma_3,\partial_t\langle\psi R\rangle\right]$. The right hand side can be transformed as follows :

$$\begin{aligned}\partial_t(\psi R) &= \overline{\partial}\partial_t\psi = \overline{\partial}\partial_t\left[1\cdot(1-RC_k)^{-1}\right] = \overline{\partial}\left[\psi\partial_t RC_k(1-RC_k)^{-1}\right]\\ &= \overline{\partial}\left[\psi\partial_t R(1-C_kR)^{-1}\right]C_k = \psi\partial_t R(1-C_kR)^{-1}.\end{aligned}$$

Here we have used the property $\overline{\partial}f(k)C_k = f(k)$. Then

$$\partial_t Q = -\frac{i}{\alpha}\left[\sigma_3,\langle\psi\partial_t R(1-C_kR)^{-1},1\rangle\right] = -\frac{i}{\alpha}\left[\sigma_3,\langle\psi\partial_t R,1\cdot(1+\tilde{R}C_k)^{-1}\rangle\right].$$

It is easy to verify that $1\cdot(1+\tilde{R}C_k)^{-1} = \tilde{\psi}^{-1}(k)$ and $\overline{\partial}\tilde{\psi}^{-1} = -\tilde{\psi}^{-1}\tilde{R}$. Consequently,

$$\partial_t Q = -\frac{i}{\alpha}\left[\sigma_3,\sum_{j=0}^{J}\gamma_{2j}\langle\overline{\partial}(k^{2j}\mathcal{M})\rangle\right] + \frac{i}{\alpha}\left[\sigma_3,\langle k\rho(k^2)\mathcal{M}\rangle\right], \tag{9}$$

where $\mathcal{M}(k) = \psi(k)\sigma_3\psi^{-1}(k)$ satisfies the adjoint spectral equation

$$\partial_x\mathcal{M} + \frac{i}{\alpha}(k^2+\beta+\frac{1}{2}\alpha^2Q^2)\left[\sigma_3,\mathcal{M}\right] - k\left[Q,\mathcal{M}\right] = 0 \tag{10}$$

and can be expanded in the inverse power series $\mathcal{M} = \sigma_3 + \sum_{j=1}^{\infty}k^{-j}\mathcal{M}^{(j)}$, with $\mathcal{M}^{(2j+1)}$ being off-diagonal and $\mathcal{M}^{(2j)}$ being diagonal matrices.

After calculation we have $\mathcal{M}^{(2j+1)} = i\alpha\Lambda^jQ$, $\mathcal{M}^a = -i\alpha k\left(\Lambda-k^2\right)^{-1}Q$. The recursion operator Λ is given by

$$\begin{aligned}\Lambda &= (1+L)\left(\frac{i}{2}\alpha\sigma_3\partial_x - \beta - \frac{1}{2}\alpha^2Q^2\right),\\ L\cdot &= -\frac{i\alpha}{2}\sigma_3\left[Q,\int^x \mathrm{d}x\left[Q,\cdot\right]\right].\end{aligned}$$

The hierarchy of equations with SDR associated to the spectral problem (7) is represented by the system of the nonlinear evolution equation

$$Q_t = -2\sigma_3\sum_{j=0}^{J}\gamma_{2j}\Lambda^jQ + \frac{2i}{\alpha}\sigma_3\langle k\rho(k^2)\mathcal{M}^a(k)\rangle \tag{11}$$

and the spectral equation (9). Putting $J=2$, $\gamma_0 = \beta^2\gamma_4$, $\gamma_2 = 2\beta\gamma_4$, and $\gamma_4 = 2i/\alpha^2$, we obtain the evolution constituent of the system (9,10) in the form resembling the modified NLS equation :

$$i\partial_tQ + \sigma_3\partial_x^2Q + i\alpha Q(\partial_xQ)Q + 2\beta\sigma_3Q^3 + \frac{1}{2}\alpha^2\sigma_3Q^5 = -\frac{2}{\alpha}\sigma_3\langle k\rho(k^2)\mathcal{M}^a\rangle. \tag{12}$$

It should be noted that the generating operator for the WKI problem was derived by Boiti, Gerdjikov and Pempinelli[6]. To find the soliton solution for Eq. (1) we take the spectral transform matrix R in the form

$$R(k) = E^{-1} 2i\pi \begin{pmatrix} 0 & c_1[\delta(k+k_1)+\delta(k-k_1)] \\ \bar{c}_1[\delta(k+\overline{k}_1)+\delta(k-\overline{k}_1)] & 0 \end{pmatrix} E, \quad (13)$$

where $E = \exp[\frac{i}{\alpha}(k^2+\beta)\sigma_3 x]$, $c_1 = c_1(t)$. Substituting (2) into the integral equation (4) we find the matrix valued function $\psi(k)$ whose asymptotic expansion $\psi(k) = 1+(1/k)\psi^{(1)}(x,t)+...$ leads to the reconstruction of the soliton solution:

$$Q_s(x,t) = \frac{2i}{\alpha}\sigma_3\psi^{(1)}(x,t), \quad q_s(x,t) = \frac{2i}{\alpha}\psi_{12}^{(1)}(x,t).$$

Taking into account the time dependence $c_{1t} = -2(\omega_r+\omega_s)c_1$, we obtain the soliton solution in the form

$$\begin{aligned} q_s(x,t) &= -\frac{8}{\alpha}c_o\Delta^{-1}\exp\left[-4i\left(\frac{\alpha\nu_p'}{8\xi\eta}x+(\nu_r''+\nu_s'')t\right)\right] \\ &\quad \times\exp\left[4\left(\frac{\xi\eta}{\alpha}x+(\nu_r'+\nu_s')t\right)\right]. \end{aligned} \quad (14)$$

Here

$$\begin{aligned} &c_1 = c_o\exp\left[4(\nu_r'+\nu_s'-i(\nu_r''+\nu_s''))\right], \quad c_o = \text{const.}, \\ &\nu_r' = \frac{4\xi\eta}{\alpha^2}(\xi^2-\eta^2+\beta), \quad \nu_r'' = \frac{1}{\alpha^2}\left[(\xi^2-\eta^2+\beta)^2-4\xi^2\eta^2\right], \\ &\frac{1}{4\pi}\iint\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l^2-k^2}l^2\rho(l^2) = \nu_s'-i\nu_s'', \\ &\Delta(x,t) = 1+\frac{|c_o|^2\overline{k}_1^2}{\xi^2\eta^2}\exp\left[8\left(\frac{\xi\eta}{\alpha}x+(\nu_r'+\nu_s')t\right)\right]. \end{aligned}$$

As seen from (13), the singular part of the dispersion law modulates both the phase and the amplitude of the soliton.

3 Gauge transformation and modified NLS equation

The system (9) and (11) is canonical with respect to the normalization condition in (1), but has no direct physical applications. To obtain more relevant equations let us perform a gauge transformation $\psi = g\psi'$ with a function g obeying the equations :

$$g_x = -\frac{i}{2}\alpha\sigma_3 Q^2 g, \quad g_t = \left(-\frac{\alpha}{2}[Q,Q_x]+\frac{i}{4}\alpha^2\sigma_3 Q^4-\langle\rho(k^2)\mathcal{M}\rangle\right)g. \quad (15)$$

¿From the point of view of the $\overline{\partial}$-problem a gauge transformation with the function $g(x,t)$ is nothing but a transition from the canonical normalization

gauge (4) to new normalization gauge $g^{-1}(x,t)$, with the spectral transform matrix $R(k)$ being the same. The system of equations (9) and (11) transforms to

$$iQ'_t+\sigma_3 Q'_{xx}-i\alpha {Q'}^3_x+2\beta\sigma_3 {Q'}^3=-\langle\rho(k^2)\left(\frac{k}{\alpha}[\sigma_3,\mathcal{M}'^a(k)]+i[Q',\mathcal{M}'^d(k)]\right)\rangle, \tag{16}$$

$$\mathcal{M}'_x+\left[\frac{i}{\alpha}(k^2+\beta)\sigma_3-kQ',\mathcal{M}'\right]=0.$$

Here $Q'=g^{-1}Qg$ and $\mathcal{M}'=g^{-1}\mathcal{M}g$. The explicit expressions for the matrices U' and V' constituting the Lax pair are

$$\begin{aligned} U' &= -\frac{i}{\alpha}(k^2+\beta)\sigma_3+kQ', \\ V' &= -\frac{2i}{\alpha^2}(k^2+\beta)^2\sigma_3+\frac{2}{\alpha}k^3Q'-ik^2{Q'}^2\sigma_3+\frac{2\beta}{\alpha}kQ'+ik\sigma_3Q'_x+\alpha k{Q'}^3 \\ &\quad -\frac{k}{2i\pi}\iint\frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}\rho(l^2)\mathcal{M}'(l). \end{aligned}$$

The reduction

$$Q'=\begin{pmatrix}0 & \mathcal{E}\\ -\bar{\mathcal{E}} & 0\end{pmatrix},\quad \rho=ia\pi\delta\,(\mathrm{Im}k)\,\delta\left((\mathrm{Re}k)^2-\gamma^2\right),\ a,\gamma\in\mathrm{R},$$

$$\mathcal{M}'(\gamma)=\begin{pmatrix}n_1 & p_{12}\\ p_{21} & n_2\end{pmatrix}(\gamma),\quad (\mathcal{M}')^\dagger=\mathcal{M}',$$

leads to the modified NLS equation with SDR :

$$i\mathcal{E}_t+\mathcal{E}_{xx}+i\alpha(|\mathcal{E}|^2\mathcal{E})_x-2\beta|\mathcal{E}|^2\mathcal{E}=\frac{ia}{\alpha}p_{12}(\gamma)-\frac{a}{2\gamma}\mathcal{E}(n_2-n_1), \tag{17}$$

$$p_{12x}+2i\omega p_{12}=\gamma\mathcal{E}(n_2-n_1),\quad n_{1x}=\gamma(\mathcal{E}p_{21}+\bar{\mathcal{E}}p_{12}),\quad \omega=\frac{1}{\alpha}(\gamma^2+\beta).$$

These equations describe, in particular, a propagation of ultrashort light pulses in a nonlinear single-mode optical fiber with account of the group velocity dispersion nonlinearity and the two-level impurity atoms whose transition frequency is close to the carrying frequency of the pulse. Equations similar to (16) were proposed by Zabolotskii[7], but the Lax representation was obtained under some restricted choice of the parameters. The soliton solution $\mathcal{E}_s$ follows immediately from the soliton solution q_s (13) and yields

$$\mathcal{E}_s(x,t)=\left(\frac{\Delta(x,t)}{\bar{\Delta}(x,t)}\right)^2 q_s(x,t). \tag{18}$$

The N-soliton solution can be found in the same way with the use of an evident generalization of the spectral transform matrix R.

4 Initial-boundary evolutions

Here we show that the system of coupled equations for three fields $q(x,t)$, $a_1(x,t)$ and $a_2(x,t)$,

$$i\partial_t q+\partial_x^2 q+i\alpha\partial_x(|q|^2q)-2\beta|q|^2q=\int_{-\infty}^{\infty}dkG(k)\Big[ia_1\bar{a}_2+\frac{\alpha}{2k}(|a_1|^2-|a_2|^2)q\Big], \tag{19}$$

$$\partial_x a_1 = kqa_2, \qquad \partial_x a_2-\frac{2i}{\alpha}(k^2+\beta)=-k\bar{q}a_1,$$

is integrable for arbitrary boundary values $a_j(k,\pm\infty,t)$, $j=1,2$. Here $G(k,t)$ is an arbitrary function in L^2. We consider here only one type of the boundary values,

$$a_1\to I_1(k,t). \qquad a_2\to I_2\exp\left[\frac{2i}{\alpha}(k^2+\beta)x\right], \qquad x\to-\infty,$$

though any choice of boundaries ($+\infty$ or $\pm\infty$) is admissible.

The system (18) generalizes to the quadratic bundle the equations investigated by Leon[1]. Following his method, we take the evolution of R as $\partial_t R=[R,\Omega]+N$, where N is an even off-diagonal matrix. The simple generalization of the above hierarchy is written in the form

$$\partial_t Q=-2\sigma_3\sum_{j=0}^{J}\gamma_{2j}\Lambda^j Q-\frac{i}{\alpha}\left[\sigma_3,\langle\psi(N-\overline{\partial}\Omega)\psi^{-1}\rangle\right]$$

and after the gauge transformation (14) we obtain the modified NLS equation with SDR which we are interesting in (with primes omitted):

$$\begin{aligned}&i\partial_t Q+\sigma_3\partial_x^2 Q-i\alpha\partial_x(Q^3)+2\beta\sigma_3Q^3\\&=\frac{2}{\alpha}\sigma_3\langle\psi(N-\overline{\partial}\omega_s\sigma_3)\psi^{-1}\rangle-2i\langle\frac{1}{k}\psi(N-\overline{\partial}\omega_s\sigma_3)\psi^{-1}\rangle Q.\end{aligned}$$

In order to find the representations for N and $\overline{\partial}\omega_s$, we obtain the explicit expression for the matrix R in the case of the continuous spectrum. The curve of analyticity breaking is given by $\mathrm{Im}k^2=0$. It implies that the regions of analyticity on the complex k-plane coincide with quadrants of this plane with $\psi^+(\psi^-)$ belonging to first and third (second and fourth) quadrants. Then the solutions of (7) which analytic in the relevant regions are given by

$$\psi^+=\begin{pmatrix}e^{i\mu(x)} & 0\\ 0 & e^{-i\mu(x)}\end{pmatrix}$$

$$\times\left[\mathbf{1}+k\begin{pmatrix}\int_{-\infty}^{x}\mathrm{d}x'q\psi_{21}^+e^{-i\mu(x)} & -\int_{x}^{\infty}\mathrm{d}x'\psi_{22}^+e^{-i\chi(x-x')}e^{-i\mu(x')}\\ -\int_{-\infty}^{x}\mathrm{d}x'\bar{q}\psi_{11}^+e^{i\chi(x-x')}e^{i\mu(x')} & -\int_{-\infty}^{x}\mathrm{d}x'\bar{q}\psi_{12}ke^{i\mu(x')}\end{pmatrix}\right],$$

$$\psi^- = \begin{pmatrix} e^{i\mu(x)} & 0 \\ 0 & e^{-i\mu(x)} \end{pmatrix}$$

$$\times \left[1 + k \begin{pmatrix} \int_{-\infty}^{x} \mathrm{d}x' q\psi_{21}^-(x') e^{-i\mu(x')} & \int_{-\infty}^{x} \mathrm{d}x' q\psi_{22}^- e^{-i\chi(x-x')} e^{-i\mu(x')} \\ \int_{x}^{\infty} \mathrm{d}x' \overline{q}\psi_{11}^- e^{i\chi(x-x')} e^{i\mu(x')} & -\int_{-\infty}^{x} \mathrm{d}x' \overline{q}\psi_{12}^-(x') e^{i\mu(x')} \end{pmatrix}\right].$$

Here

$$\chi = \frac{2i}{\alpha}(k^2 + \beta), \qquad \mu(x) = \frac{i\alpha}{2}\int_{-\infty}^{x} |q|^2 \mathrm{d}x'.$$

To find R we should evaluate the jumps of the piecewise analytic function

$$\Psi(k) = \begin{cases} \psi^+, & \mathrm{Im}k^2 > 0, \\ \psi^-, & \mathrm{Im}k^2 < 0, \end{cases}$$

on the real and imaginary axes. We get ($k = k_R + ik_I$): $R(k) =$

$$\frac{i}{2}E^{-1}\left\{\mathrm{sgn}(k_R)\begin{pmatrix} 0 & \delta(\mathrm{sgn}(k_R)k_I - 0)\alpha^+(k_R) \\ -\delta(\mathrm{sgn}(k_R)k_I + 0)\alpha^-(k_R) & 0 \end{pmatrix}\right.$$

$$\left. -i\mathrm{sgn}(k_I)\begin{pmatrix} 0 & \delta(\mathrm{sgn}(k_I)k_R - 0)\alpha^+(ik_I) \\ -\delta(\mathrm{sgn}(k_I)k_R + 0)\alpha^-(ik_I) & 0 \end{pmatrix}\right\}E.$$

Here

$$\alpha^+(k,t) = -k\int_{-\infty}^{\infty} \mathrm{d}x q(x)\psi_{22}^+(k,x)e^{i\chi x - i\mu(x)},$$

$$\alpha^-(k,t) = k\int_{-\infty}^{\infty} \mathrm{d}x \overline{q}(x)\psi_{11}^-(k,x)e^{-i\chi x + i\mu(x)}.$$

The self-consistency of the evolution equation for R implies we should choose the matrix N in this form:

$$N(k,x,t) = \begin{pmatrix} 0 & n(k,t) \\ -\overline{n}(k,t) & 0 \end{pmatrix}E,$$

with a real function

$$n(k,t) = \frac{i}{2}n_0(k,t)\left[\mathrm{sgn}(k_R)k_R\delta(\mathrm{sgn}(k_R)k_I - 0) + \mathrm{sgn}(k_I)k_I\delta(\mathrm{sgn}(k_I)k_R - 0)\right].$$

As regards $\overline{\partial}\omega_s(k)$, we take the following representation for it:

$$\overline{\partial}\omega_s(k) = \frac{i}{2}p(k)\mathrm{sgn}(k_R)\left[\delta(\mathrm{sgn}(k_R)k_I - 0) + \delta(\mathrm{sgn}(k_R)k_I + 0)\right]$$

with real and even function $p(k)$. Taking into account the above structure of N and $\overline{\partial}\omega_s$, we get the scalar variant of the equation (19):

$$i\partial_t q + \partial_x^2 q + i\alpha\partial_x(|q|^2 q) - 2\beta|q|^2 q = \frac{2}{\alpha}\langle n\psi_{11}^2 e^{-i\chi x} + \overline{n}\psi_{12}^2 e^{i\chi x} + 2\overline{\partial}\omega_s\psi_{11}\psi_{12}\rangle$$

$$+2i\langle\frac{1}{k}\left(n\psi_{11}\psi_{21}e^{-i\chi x}+\overline{n}\psi_{12}\psi_{22}e^{i\chi x}+\overline{\partial}\omega_s(\psi_{11}\psi_{22}+\psi_{12}\psi_{21})\right)\rangle q.$$

Now our aim is to show that there exists a linear transformation between the vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and the vectors $\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{21} \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} \psi_{12} \\ \psi_{22} \end{pmatrix}$ which brings one system of equation to the other. This linear transformation has the form

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = I_1 \begin{pmatrix} \psi_{11}^+ \\ \psi_{21}^+ \end{pmatrix} + I_2 \begin{pmatrix} \psi_{12}^- \\ \psi_{22}^- \end{pmatrix} e^{\frac{2i}{\alpha}(k^2+\beta)x},$$

Then by means of the Riemann-Hilbert-type relations for the relevant jumps we may prove the equivalence of both systems of equations provided the functions $p(k)$ and $n_0(k)$ are connected with the boundary values as follows:

$$p(k) = \frac{\pi\alpha}{4}\mathrm{sgn}(k)G(k)\left(|I_1|^2 - |I_2|^2\right),$$

$$n_0(k) = -\frac{\pi\alpha}{4k}\mathrm{sgn}(k)\alpha^+(k)G(k)\left(|I_1|^2 - |I_2|^2\right).$$

Finally, we give here the equations of evolution of the spectral data:

$$\partial_t\alpha^+(k,t) = -2\left[\frac{2i}{\alpha}(k^2+\beta)^2 + \frac{1}{\pi i}\mathrm{V.p.}\int_{-\infty}^{\infty} dl\,\mathrm{sgn}(l)\frac{lp(l)}{l^2-k^2}\right]\alpha^+(k,t) + kn_0(k),$$

$$k_{nt} = 0, \quad c_{nt} = -2\left[\frac{2i}{\alpha}(k_n^2+\beta)^2 + \frac{1}{\pi i}\int_{-\infty}^{\infty} dl\,\mathrm{sgn}(l)\frac{lp(l)}{l^2-k_n^2}\right]c_n.$$

Hence, we see that the model system (19) is indeed integrable for arbitrary boundary values I_1 and I_2.

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GEOMETRICAL PROPERTIES OF NONLINEAR DYNAMICAL SYSTEMS WITH REGULAR AND CHAOTIC BEHAVIOUR

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Geometrical properties of the second order differential equations $y'' = f(x, y, y')$, which are equivalent to the nonlinear dynamical systems in the form:

$$dx/dt = P(x, y, z),\ \ dy/dt = Q(x, y, z),\ \ dz/dt = R(x, y, z)\,,$$

where P, Q, R – functions with parameters are considered.

1 Introduction

Many nonlinear phenomena in modern physics can be described by the systems of nonlinear equations in form:

$$\frac{dx}{dt} = P(x, y, z), \quad \frac{dy}{dt} = Q(x, y, z), \quad \frac{dz}{dt} = R(x, y, z)\,, \tag{1}$$

where P, Q, R – some functions containing parameters.The Lorenz and Ressler dynamical systems are the most famous examples of such type. An important feature of these systems is that the behaviour of their integral curves my be change radically with the change of parameters of the system – from regular up to chaotic at the some ranges of parameters. We suggest to apply the Cartan's geometric theory of the second order differential equation $y'' = f(x, y, y')$, which is equivalent to the initial system (1), for study the such properties of the systems as

1. existence of the first and partial integrals,
2. presence of the Painlevé property at the some meaning of parameters,
3. description with help of dual equation the set of integral curves of equations which determines the splitting of the phase space of the system on trajectories,
4. the Finsler's type metrics and gonometric properties for integral curves of equations and it dependence on parameters of equation.

2 Projectiv duality in theory of the second order differential equations

The fundamental object in theory of the second order differential equation $y'' = f(x, y, y')$ is its General Integral:

$$F(x, y, a, b) = 0\,. \tag{2}$$

All properties of equation with help of them are described. We can consider relation (2) as the 3-dim hyperspace of E^4 Euclidean space and on the its basic to introduce a dual equation $b'' = g(a, b, b')$ for given $y'' = f(x, y, y')$. With this aim it is necessary to except from (2) the variables x and y, using twice differentiation. So we obtain the following fundamental diagram:

$$\begin{array}{ccc} & F(x,y,a,b)=0 & \\ \swarrow\nearrow & & \searrow\nwarrow \\ y''=f(x,y,y') & & b''=g(a,b,b') \\ \Updownarrow & & \Updownarrow \\ M^3(x,y,y') & \Longleftrightarrow & N^3(a,b,b') \end{array}$$

which is presented the General Integral $F(x, y, a, b) = 0$ of equations (as some 3-dim multifold) in form of the twice nontrivial fibre bundles on circles over corresponding surface:

$$M^3(x, y, y') = U^2(x, y) \times S^1 \quad \text{and} \quad N^3(a, b, b') = V^2(a, b) \times S^1\,.$$

The properties of surfaces $U^2(x, y)$ and $V^2(a, b)$ with help of the functions $f(x, y, y')$, $g(a, b, b')$ and their derivations are determined. We can construct the connection and curvature of the spaces M^3, N^3 and to obtain the full list of Invariantes of equations under arbitrary transformations of variables (x, y) for initial and (a, b) for dual equation. Both set of Invariantes completely determines the General Integral of equation as some 3-dim multifold. The relation between equations and subspaces can be established by

Theorem 1 *For the equations in form:*

$$y'' + a_1(x, y){y'}^3 + 3a_2(x, y){y'}^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \tag{3}$$

with arbitrary coefficients $a_i(x, y)$, dual equations are $b'' = g(a, b, b')$, where function $g(a, b, b')$ from the relation is determined ($b' = c$):

$$g_{aacc} + 2cg_{abcc} + 2gg_{bccc} + c^2 g_{bbcc} + 2cgg_{bccc}$$

$$+g^2 g_{cccc} + (g_a + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bbc} \qquad (4)$$
$$-3gg_{bcc} - g_c g_{acc} + 4g_c g_{cc} - 3g_b g_{cc} + 6g_{bb} = 0\,.$$

So, both equations (3) and (4) some class of hyperspaces $F(x, y, a, b) = 0$ are determined.

Some simplest solutions of equations (4) was obtained in [1], but as exceptional significance of this equation for the theory of the second ODE it is necessary more detail investigation its solutions.

We present this equation in form of the system of two second order partial equations.

Proposition 1 *Equation (4) is equivalent to the following system of equations:*

$$\begin{aligned} g_{ac} &+ gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = h(a, b, c)\,, \\ h_{ac} &+ gh_{cc} - g_c h_c + ch_{bc} - 3h_b = 0\,. \end{aligned} \qquad (5)$$

This representation gives the possibility to get the large classes of solution of dual equation (4) and allow us to understand what type of multifolds $F(x, y, a, b) = 0$ can be realised for the equations (3) and how they are distributed in E^4 space according with values of parameters. The system (5) can be satisfy at condition $h = 0$ and so we get one equation:

$$g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = 0\,, \qquad (6)$$

which is more simplest than (4) and we can use Legendre-transformation for it integration. Another way of integration of equations (5) and (6) is connected with representation of the function $g(a, b, c)$ in the form:

$$g(a, b, \varphi) = \frac{\sum[A_n(a, b)\cos(n\varphi) + B_n(a, b)\sin(n\varphi) + C_n(a, b)]}{\cos^3\varphi}\,, \qquad (7)$$

where $c = b' = \tan\varphi$.

Let $g(a, b, c)$ be some function in the following form:

$$g(a, b, c) = \frac{A[b/a, b - ac]}{a^3}\,,$$

then from (6) one obtains the equation:

$$qA_{pq} + AA_{qq} - A_q^2/2 - 2A_p = 0\,,$$

where $(p = b/a, q = b - ac)$.

We can integrate this equation with by the method of division of variables and obtain partial solutions which depends of parameters. The values of these parameters the properties of initial equation are determined. Let us consider more general example.

Proposition 2 *There is exist some equations in form (3) which have the dual equations of the following type:*

$$b'' = a^{-2+\beta/\alpha} A\left(\frac{b^\alpha}{a^\beta}, \frac{c^\alpha}{a^{\beta-\alpha}}\right) , \tag{8}$$

where the function A is the solution of the equation:

$$[\alpha^2 p^{1-1/\alpha} q^{1/\alpha} - \alpha\beta p] A_{pq} + [\alpha^2 q^{1-1/\alpha} A + \alpha(\alpha-\beta) q] A_{qq} - \frac{1}{2}\alpha^2 q^{1-1/\alpha} A_q^2$$

$$+\alpha(\alpha-1) q^{-1/\alpha} A A_q - 2\alpha p^{1-1/\alpha} q^{-1+1/\alpha} A_p - [\alpha - (\alpha-1)(\beta-\alpha)] A_q = 0 ,$$

and $p = b^\alpha/a^\beta$, $q = c^\alpha/a^{\beta-\alpha}$.

For the such pairs of equation one obtains some two parametrically family of the 3-dim hyperspaces $F(x, y, a, b; \alpha, \beta) = 0$, which have the large discrete group symmetries in the space of parameters. This is followed from the properties of equation (8). At the change of parameters the structure of the space of integral curves may be change radically because the solutions of dual equation are dependent from them.

So, the conception of equations which dual pair are formed gives the basis of Poincare approach to the qualitative theory of the second order ODE.

3 The Finsler's metrics and gonometrical properties of the second order nonlinear ODE

Two parametrical set of the curves $F(x, y, a, b) = 0$ of the some second order ODE $y'' = f(x, y, y')$ is gonometric when the angle metric between integral curves has the Riemannian form:

$$ds^2 = g_{11}(a, b) da^2 + 2g_{12}(a, b) da db + g_{22}(a, b) db^2 . \tag{9}$$

In the general family of curves the angle metric has the Finsler form. In fact, let $F(x, y, a, b) = 0$ be some two parametrical family of the plane curves. Then the point of intersection of the some two curves which corresponds to the values of parameters a, b and $a_1 = a + da$, $b_1 = b + db$ is determined from the system of equations :

$$F(x, y, a, b) = 0, \qquad F_a da + F_b db = 0$$

and has the coordinates:

$$x = U(a, b, b_a), \quad y = V(a, b, b_a) . \tag{10}$$

Using the conditions:

$$F_x dx + F_y dy = 0, \quad y' = \tan\varphi = -F_x/F_y \,,$$

one obtains the relation:

$$d\varphi^2 = \frac{\left[(F_x F_{ya} - F_y F_{xa})\, da + (F_x F_{yb} - F_y F_{xb})\, db\right]^2}{\left[F_x^2 + F_y^2\right]^2 F_y^2}.$$

After substitution in this the expressions (10) instead of x, y one obtains some relation between parameters a, b and da, db, which can be consider as the some metric $ds = A(a, b, b')\, db$ in parametric space.

This metric is called the Finsler type and it is the generalization of the Riemannian form (9).

Theorem 2 (P.Rashevsky, 1937). *Let*

$$y'' = (1 + y'^2)^{\frac{3}{2}} K(x, y, \arctan y')$$

be some arbitrary second order ODE. Then it meets the angle metric of the Riemannian type in parametrical space when the function $K(x, y, \varphi = \arctan y')$ *satisfy to the such type of equation:*

$$[4K + 2\sin\varphi\,\partial_x - 2\cos\varphi\,\partial_y + \cos\varphi\,\partial^2_{x\varphi} + \sin\varphi\,\partial^2_{y\varphi} + K\partial^2_{\varphi\varphi}]AK = 0\,, \quad (11)$$

where

$$AK = \cos\varphi\,\partial_x K + \sin\varphi\,\partial_y K + K\partial_\varphi K\,.$$

So, for the equations with gonometrical properties the corresponding 3-dim hyperspaces
$F(x, y, a, b) = 0$ have the some special characteristics and it is interestingly to investigate the dual equation in this case.

Proposition 3 *The solutions of the Rashevsky equation (11) are determined the dual equations for the equations of geodesical lines of the Riemannian metrics (9).*

This means that the solutions of Rashevsky equation is only part of the full dual equation (5). We can use this fact for investigation of the properties of ODE which describe the geodesical lines on some surfaces with Riemannian metric type (9). The solutions of Rashevsky equation are determined the splitting of surface on integral curves with similar behaviour and can be apply in the theory of the hamiltonian dynamical systems.

Equation (11) has the particular solutions in the form

$$K = U(xK - \sin\varphi\,,\; yK + \cos\varphi\,)\,,$$

where $U(p,q)$ – arbitrary function. It is corresponded to the condition $AK=0$ at which the equation (11) is carrying out:

$$AK = \cos\varphi\,\partial_x K + \sin\varphi\,\partial_y K + K\partial_\varphi K = 0.$$

In the usual means this is equivalent to the equation

$$\partial_x f + y'\partial_y f + f\partial_{y'} f = \frac{3y'f^2}{1+y'^2}\,.$$

More general solutions can be obtain with the help of the type (7) representation.

Let consider the Finsler's properties of the some second order ODE with applications to the nonlinear dynamical systems. According to the E. Cartan theory the hyperspace of the E^4 space $F(x,y,a,b)=0$ which is the General Integral for the simplest form-invariant equation (3) is divided on circles over the some surfaces $V^2(x,y)$ equipping by the projective connection. The integral curves of equation are geodesical lines of the such type of connection. We can introduce some metric for this connection, which is generalization of the Riemannian quadratic form (9) and corresponding connection:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\partial_{x_i}g_{jl} + \partial_{x_j}g_{il} - \partial_{x_l}g_{ij}\right)\,.$$

The coefficients of equations (3) are arbitrary for the projectiv connection and have the form:

$$a_1 = -\Gamma^2_{11},\ 3a_2 = \Gamma^1_{11} - 2\Gamma^2_{12},\ 3a_3 = 2\Gamma^1_{12} - \Gamma^2_{22},\ a_4 = \Gamma^1_{22}\,.$$

Remark 1 *For the equations in form (3) the components of projective connection and curvature tensor are:*

$$\begin{aligned}
\Pi_{11} &= 2(a_3^2 - a_2a_4) + a_{3x} - a_{4y},\\
\Pi_{22} &= 2(a_2^2 - a_1a_3) + a_{1x} - a_{2y},\\
\Pi_{12} &= \Pi_{21} = a_2a_3 - a_1a_4 + a_{2x} - a_{3y},
\end{aligned}$$

$$\begin{aligned}
L_1 &= \Pi_{12x} - \Pi_{11y} + 2a_3\Pi_{12} - a_2\Pi_{11} - a_4\Pi_{22},\\
L_2 &= \Pi_{22x} - \Pi_{12y} + 2a_2\Pi_{12} - a_1\Pi_{11} - a_3\Pi_{22}\,.
\end{aligned}$$

Using components L_1, L_2 and coefficients $a_i(x,y)$ it is possible to obtain the following series of invariants of the equations (3) under arbitrary transformations of variables x,y:

$$[5t_m - (m-2)t_7t_{m-2}]\nu_5^{2/5} = 5[L_1\frac{\partial}{\partial y}t_{m-2} - L_2\frac{\partial}{\partial x}t_{m-2}]\,, \qquad (12)$$

where

$$\begin{aligned}\nu_5 &= L_2(L_1L_{2x} - L_2L_{1x}) - L_1(L_2L_{1y} - L_1L_{2y}) \\ &- a_1(x,y)L_1^3 + 3a_2(x,y)L_1^2L_2 - 3a_3(x,y)L_2^2L_1 + a_4(x,y)L_2^3,\end{aligned}$$

$$t_m = \nu_m\nu_5^{m/5}, \quad \nu_{m+2} = L_1\frac{\partial\nu_m}{\partial y} - L_2\frac{\partial\nu_m}{\partial x} + m\nu_m(L_{2x} - L_{1y}),$$

and similar series at condition $\nu_5 = 0$.

The question about the particular integrals for Eqs. (3) with help of the invariantes is solved. So, at the condition $\nu_5 = 0$ *the Eqs. (3) have particular integrals in form* $y' = -L_1/L_2$ (Liouville, 1887) *and the Painlevé properties at conditions* $\nu_5 = 0$, $w_1 = 0$ (V. Dryuma, L. Bordag,1994). *In this context the following class of hyperspaces* $F(x,y,a,b) = 0$

$$\begin{aligned}&(F_{xx} + F_{yy})(F_a^2 + F_b^2) + (F_{aa} + F_{bb})(F_x^2 + F_y^2) \\ - \;&2(F_{xa} + F_{yb})(F_xF_a + F_yF_b) + 2(F_{ya} - F_{xb})(F_xF_b + F_yF_a) = 0,\end{aligned}$$

which are the Levi-flat submanifolds of complex space C^2 *is interesting for the study these problems.*

The applications of the Finsler metrics in theory of the nonlinear dynamical systems (1) it possible because they equivalent to the some second order ODE: $y'' = f(x,y,y')$. For example let us consider, the Lorenz system of equations:

$$\frac{dx}{dt} = \sigma(x-y), \quad \frac{dy}{dt} = rx - y - zx, \quad \frac{dz}{dt} = xy - bz, \tag{13}$$

where σ, r and b – parameters is equivalent to the following system of equations:

$$\frac{dy}{dx} = \frac{rx - y - zx}{\sigma(y-x)}, \qquad \frac{dz}{dx} = \frac{xy - bz}{\sigma(y-x)}.$$

This system is equivalent after excluding the variable z to the one second order ODE in form (after some simplification)

$$y'' - \frac{3}{y}y'^2 + \left(\alpha y - \frac{1}{x}\right)y' + \left(\epsilon x - \beta x^3\right)y^4 - \left(\gamma + \beta x^2\right)y^3 + \frac{\delta}{x}y^2 = 0, \tag{14}$$

where:

$$\alpha = \frac{b+\sigma+1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma+1)}{\sigma^2}, \quad \delta = \frac{\sigma+1}{\sigma}, \quad \epsilon = \frac{b(r-1)}{\sigma^2}.$$

Theorem 3 *The equations in form*

$$y'' + \frac{A_y}{A}y'^2 + \left(\frac{n+1}{nA}B_y + \frac{n-1}{nA}A_x\right)y' + \frac{1}{A}B_x + \frac{B}{nA^2}(B_y - A_x) = 0$$

with arbitrary functions $A(x,y)$, $B(x,y)$ and parameter n are equations of geodesical lines on surfaces with Finsler metrics

$$ds = dx^n[B(x,y)dx + A(x,y)dy]^{1-n}\,. \tag{15}$$

We can investigate the question, at what values of parameters r, b, σ the integral curves of the equation (14) are geodesical lines of the Finsler's metric (15) ?

For this purposes we can write down the corresponding system of equations for the coefficients A, B and parameter n:

$$\frac{1}{A}A_y + \frac{3}{y} = 0\,,$$

$$(n+1)B_y + (n-1)A_x + nA\left(\frac{1}{x} - \alpha y\right) = 0\,,$$

$$nAB_x + B(B_y - A_x) + nA^2\left[(\beta x^3 - \epsilon x)y^4 + (\gamma + \beta x^2)y^3 - \frac{\delta}{x}y^2\right] = 0\,.$$

Using the solution of this system of equation one obtains the

Theorem 4 (V. Dryuma, M. Matsumoto, 1994).*Integral curves of the equation (14) are geodesical of the metric (15) at the following values of coefficients A, B and parameter n:*

1. $\sigma = 1/(2n-1)$, $\quad b = 2/(2n-1)$, $\quad r$ - arbitrary ,

$$A(x,y) = \frac{cx^{n/(1-n)}}{y^3}, \quad c = \text{const} \neq 0\,,$$

$$B(x,y) = cx^{1/(1-n)}\left[(2n-1)(r-1) - \frac{1}{2}(2n-1)^2x^2 - \frac{2n}{xy}\right]\,.$$

2. $\sigma = 1/(3-4n)$, $\quad b = 8n/(3-4n)$, $\quad r = (4n-1)/(3-4n)$,

$$A(x,y) = \frac{cx^{(n+2)/(n-1)}}{y^3}, \quad c = \text{const} \neq 0\,,$$

$$B(x,y) = cx^{3/(n-1)}\left[\frac{1}{y^2} - \frac{4nx}{y} + 8n(2n-1)x^2 - \frac{(3-4n)^2}{4}x^4\right]\,.$$

Here $n \neq 0, \pm 1, 1/2$.

3. $n = 1/2$ (Riemannian case), $\sigma = 1$, $b = 4$, r – arbitrary ,

$$A(x,y) = \frac{cx}{Ey^3}, \quad c = \text{const} \neq 0, \quad E = x^2 - 4r + 4 ,$$
$$B(x,y) = c\left[\frac{x^2}{y^2E^4} - \frac{2x}{yE^3} - \frac{x^2}{4E^2}\right] .$$

The construction of the Finsler metrics allow us to investigate analytically the properties of the dynamical systems with chaotic and regular behaviour. However, the main problem in this theory is how to choice the metrics of Finsler type for the all values of parameters of the equation. Some results in this approach gives the

Theorem 5 *The condition*

$$\det \begin{vmatrix} S' & 1 & 0 & 0 & P \\ S'' & S' & 1 & 0 & P' \\ S''' & 2S'' & S' & 1 & P'' \\ S^{IV} & 3S''' & 3S'' & S' & P''' \\ S^{V} & 4S^{IV} & 6S''' & 4S'' & P^{IV} \end{vmatrix} = 0 ,$$

where $P = -y'S_y - S_x$, $S' = S'_y \ldots P' = P_{y'} \ldots$, *is some differential equation for the function* $S = \log A_{y'y'}$ *which is determined all the invarianten types of the Finsler metrics* $ds = A(x,y,y')\,dx$ *for the Eqs. (3).*

The proof of this theorem is based on the condition $f_{cccc} = 0$ which is fulfilment for the Eqs. (3).

4 Geometry of surfaces in projective spaces and theory of the second order ODE

Some connection between the theory of the surfaces in projective spaces, nonlinear second ODE and integrable nonlinear partial differential equations is exist.

Proposition 4 (W. Denton, 1916, V. Dryuma, 1986). *The integrable system of equations :*

$$\begin{aligned} \psi_{xx} &= D(x,y)\psi_x + E(x,y)\psi , \\ \psi_{xy} &= A(x,y)\psi_x + B(x,y)\psi_y + C(x,y)\psi \\ \psi_{yyy} &= F(x,y)\psi_{yy} + G(x,y)\psi_y + H(x,y)\psi \end{aligned} \qquad (16)$$

has only four linearly independent solutions $\{\psi^i(x,y)\}$ *and the theory of this system is identical with: 1) the projective differential geometry of developable*

surfaces in three dimensional space RP^3; 2) the geometry of the second order ODE type (3) with condition $\nu_5 = 0$ on its coefficients. So, the equations which have the integrals and Painlevé properties can be described with help solutions of the system (16).

For the integration this system and corresponding nonlinear equations of compatibility we can apply theory of the Laplace -transformations. The various types of these solutions will be correspondent to the conditions equality to zero an invariantes of such type transformations.

Proposition 5 (E. Cech, G. Fubini, 1927). *Integrable system of equations :*

$$\begin{aligned} \psi_{xy} &= A(x,y)\psi_x + B(x,y)\psi_y + C(x,y)\psi\,, \\ \psi_{xxx} &= D(x,y)\psi_{xx} + E(x,y)\psi_{yy} + F(x,y)\psi_x + G(x,y)\psi_y + H(x,y)\psi \quad (17) \\ \psi_{yyy} &= K(x,y)\psi_{xx} + L(x,y)\psi_{yy} + M(x,y)\psi_x + N(x,y)\psi_y + P(x,y)\psi \end{aligned}$$

has only fife linearly independent solutions $\{\psi^i(x,y)\}$ and the theory of this system is identical with: 1) the projective differential geometry of surfaces in four dimensional space RP^4; 2) the geometry of the second order ODE type (3) with condition $\nu_5 \neq 0$ on coefficients. In particular, is existed the surfaces of (17) which equivalent to the equation of (14) with parameters $\sigma = 10$, $b = 8/3$, $r > 24,74$ when the Lorenz system has the strange attractor. This surfaces must possess unusual properties as compared with surfaces at parameters at which the behaviour of the system is regular. From the fact that the all of geometrical objects by their properties of Invariants are determined one can formulate the fundamental.

Proposition 6 *The relations between Invariantes*

$$I_{n+1} = U(I_{n-1},\, I_n,\, \alpha) \qquad (18)$$

constructed by the curvature and projective connection of surfaces are dependent on parameter α in such a way,that the sequence (18) is regular at the ones values of parameters or stochastic at the others values.The second ODE corresponding to these Invariantes the regular chaotic behaviour is described.

The sequence of Invariantes (12) and the sequence of the Laplace-Invariantes

$$h_{n+1} = 2h_n - h_{n-1} - \partial^2_{xy} \ln h_n \qquad (19)$$

of the first equation of the system (17, where $h_0 = A_x + AB - C$ and $h_{-1} = B_y + AB - C$ - the initial point of sequence is suitable object for the investigation of this question.

Some of the surfaces (17) with integrable nonlinear PDE are connected. Let us consider the such type of the system (17):

$$\begin{aligned} \psi_{xy} &= U_x\psi_y - 2U_{xy}\psi\,, \\ \psi_{yyy} &= -3U_{yy}\psi_y\,, \\ \psi_{xxx} &= K(x,y)\psi_{xx} + L(x,y)\psi_{yy} + M(x,y)\psi_x + N(x,y)\psi_y + P(x,y)\psi\,. \end{aligned} \tag{20}$$

From the conditions of compatibility of this system one obtains the relations between coefficients and the function $U(x,y)$ is the solution of the equation

$$U_{xyyy} = -3U_{yy}U_{xy}\,. \tag{21}$$

Condition for the determination of the coefficient $L(x,y)$ is

$$U_{xxxy} = -10U_x^2U_{xy} - 5U_xU_{xxy} - 7U_{xy}U_{xx} - L_yU_{yy} - LU_{yyy} - L_{yyy}$$

and lead to the some 5-th order ODE on variable y

$$\frac{d^5L}{dy^5} = \Phi\left[\frac{d^4L}{dy^4}\,,\frac{d^3L}{dy^3}\,,\frac{d^2L}{dy^2}\,,\frac{dL}{dy}\,,L,\,U(x,y)\right]$$

and parametrical on x. Its solutions are corresponded to the fife classes of surfaces in RP^4 space.

Equation (21) is the Hirota-Satsuma equation, integrable by the IST-method (1976, see also R. Conte and M. Musette, 1993). Note, that the soliton solutions of equation (21) can be obtained with the help the linear system (20) and their theory of the Laplace-invariantes (19).

In accordance with the connection between equations and surfaces from this it is followed that in the space of equations $y'' = f(x,y,y')$ and corresponding hypersurfaces $F(x,y,a,b) = 0$ some soliton like structure is existed.

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THE APPLICATION OF THE $\overline{\partial}$-DRESSING METHOD TO SOME INTEGRABLE 2+1 DIMENSIONAL NONLINEAR EQUATIONS

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Some integrable 2+1-dimensional nonlinear equations which are the generalizations of dispersive long wave,nonlinear Schroedinger, sinh-Gordon and heat equations are studied by the inverse spectral transform method.The solutions with functional parameters,line solitons and rational solutions of these equations are constructed via $\overline{\partial}$–dressing method.

1 Introduction

In this report we consider some well known integrable 2+1-dimensional generalizations of dispersive long wave,nonlinear Schroedinger, sinh-Gordon and heat equations.We apply $\overline{\partial}$–dressing method for the construction of broad classes of exact solutions of these equations.

Let us start from the following two linear auxiliary problems [1]:

$$L_1\Psi = \Psi_{\xi\eta} + V\Psi_\eta + U\Psi = 0,$$
$$L_2\Psi = \Psi_t + \alpha\Psi_{\xi\xi} + \beta\Psi_{\eta\eta} + W_1\Psi_\eta + W_2\Psi = 0 \qquad (1)$$

where α,β - constants;$\xi := x - \sigma y$,$\eta := x + \sigma y$, $\sigma^2 = 1$. The compatibility condition for the system (1) is the triad operator representation of the form [1]:

$$[L_1, L_2] = (W_{1\eta} - 2\alpha V_\xi)L_1. \qquad (2)$$

Here $W_1(\xi,\eta,t) = 2\beta\partial_\eta^{-1}V_\eta$, $W_2(\xi,\eta,t) = 2\alpha\partial_\eta^{-1}U_\xi$ and the system of nonlinear equations which is integrable by (1) is the following:

$$U_t - \alpha U_{\xi\xi} + \beta U_{\eta\eta} - 2\alpha(UV)_\xi + 2\beta(U\partial_\xi^{-1}V_\eta)_\eta = 0,$$
$$V_t + \alpha V_{\xi\xi} - \beta V_{\eta\eta} + 2\beta U_\eta - \alpha(V^2)_\xi - 2\alpha\partial_\eta^{-1}U_{\xi\xi} + 2\beta V_\eta\partial_\xi^{-1}V_\eta = 0. \qquad (3)$$

For $\alpha = 1, \beta = 0$ the system (3) has the form

$$U_t - U_{\xi\xi} - 2(UV)_\xi = 0,$$
$$V_{t\eta} + V_{\xi\xi\eta} - 2U_{\xi\xi} - (V^2)_{\xi\eta} = 0, \tag{4}$$

which is known as integrable 2+1-dimensional generalization of dispersive long wave system [2].On introduction of the new dependent variable $q = ln4U$ and by appropriate elimination of another variable V by the formula $V = (1/2)(e^{-q}\partial_\xi^{-1}(e^q q_t) - q_\xi)$ the system (4) reduces to single equation for q [2] - some 2+1-dimensional generalization (nonsymmetrical in ξ and η) of sinh-Gordon equation $q_{\xi t} + sinhq = 0$.

Under the change of dependent variables

$$V = -q_\xi/q \ , \ U = -pq \tag{5}$$

the system (3) reduces to the Davey-Stewartson (DS) system of equations:

$$q_t + \alpha q_{\xi\xi} - \beta q_{\eta\eta} - 2\alpha q\partial_\eta^{-1}(pq)_\xi + 2\beta q\partial_\xi^{-1}(pq)_\eta = 0,$$
$$p_t - \alpha p_{\xi\xi} + \beta p_{\eta\eta} + 2\alpha p\partial_\eta^{-1}(pq)_\xi - 2\beta p\partial_\xi^{-1}(pq)_\eta = 0. \tag{6}$$

When α and β are pure imaginary constants the system (6) admits the reduction $p = \kappa\overline{q}$ to the single DS equation :

$$q_t + \alpha q_{\xi\xi} - \beta q_{\eta\eta} - 2\alpha\kappa q\partial_\eta^{-1}\left(|q|^2\right)_\xi + 2\beta\kappa q\partial_\xi^{-1}\left(|q|^2\right)_\eta = 0. \tag{7}$$

Different choices of α and β correspond to DS-1,DS-2,... equations. It may be interesting also particular case $\beta = 0$ of the system (6):

$$q_t + \alpha q_{\xi\xi} - 2\alpha q\partial_\eta^{-1}(pq)_\xi = 0,$$
$$p_t - \alpha p_{\xi\xi} + 2\alpha p\partial_\eta^{-1}(pq)_\xi = 0 \tag{8}$$

with corresponding reduction $p = \kappa\overline{q}$ in the case $\overline{\alpha} = -\alpha$ [3].

2 Basic ingredients of $\overline{\partial}$-dressing method

Let us apply the $\overline{\partial}$-dressing method [4-8] for the system (1) in the case when $U(\xi, \eta, t)$ has generically non-zero asymptotic value $U_\infty = -\epsilon$ at infinity:

$$U(\xi, \eta, t) = \tilde{U}(\xi, \eta, t) + U_\infty = \tilde{U}(\xi, \eta, t) - \epsilon \tag{9}$$

where $\tilde{U}(\xi, \eta, t) \to 0$ as $\xi^2 + \eta^2 \to \infty$. At first one postulates nonlocal $\overline{\partial}$-problem:

$$\frac{\partial\chi(\lambda, \overline{\lambda})}{\partial\overline{\lambda}} = (\chi * R)(\lambda, \overline{\lambda}) = \int\int_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i}\chi(\lambda'.\overline{\lambda'})R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}). \tag{10}$$

The functions χ and R in our case are the scalar complex-valued functions.For the function χ we choose the canonical normalization ($\chi \to 1$,as $\lambda \to \infty$).We assume also that the problem (10) is uniquely solvable.

Then one introduces the dependence of kernel R on space and time variables ξ,η,t :

$$\frac{\partial R}{\partial \xi} = i\lambda' R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) i\lambda,$$

$$\frac{\partial R}{\partial \eta} = -\frac{i\epsilon}{\lambda'} R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) + R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) \frac{i\epsilon}{\lambda},$$

$$\frac{\partial R}{\partial t} = \left(\alpha\lambda'^2 + \frac{\beta\epsilon^2}{\lambda'^2}\right) R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) \left(\alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2}\right), \quad (11)$$

i.e.

$$R(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}; \xi, \eta, t) = R_0(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}) exp\left(F(\lambda') - F(\lambda)\right) \quad (12)$$

where

$$F(\lambda) := i\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) + \left(\alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2}\right) t. \quad (13)$$

With the use of "long" derivatives

$$D_\xi = \partial_\xi + i\lambda, \quad D_\eta = \partial_\eta - \frac{i\epsilon}{\lambda}, \quad D_t = \partial_t + \alpha\lambda^2 + \frac{\beta\epsilon^2}{\lambda^2} \quad (14)$$

one constructs then linear operators

$$L = \sum u_{lmn}(\xi, \eta, t) D_\xi^l D_\eta^m D_t^n \quad (15)$$

which satisfy to the condition

$$[\frac{\partial}{\partial\overline{\lambda}}, L] = 0 \quad (16)$$

of absence of singularities on λ.For such operators L the function $L\chi$ obeys the same $\overline{\partial}$-equation as the function χ.If there are several operators L_i of this type then by virtue of the unique solvability of (11) one has : $L_i\chi = 0$. In our case one can construct two such operators:

$$L_1\chi = \left(D_\xi D_\eta + \tilde{V} D_\xi + V D_\eta + U\right)\chi = 0,$$

$$L_2\chi = \left(D_t + \alpha D_\xi^2 + \beta D_\eta^2 + \tilde{W}_1 D_\xi + W_1 D_\eta + W_2\right)\chi = 0. \quad (17)$$

Then from (16),(17) we obtain for $\tilde{V}$,$\tilde{W}_1$ and V,U,W_1 , W_2 following reconstruction formulae :

$$\tilde{V} = 0, \qquad \tilde{W}_1 = 0,$$
$$V = -\tilde{\chi}_{0\xi}/\tilde{\chi}_0, \quad U = -\epsilon - i\chi_{-1\eta}, \quad W_1 = -2\beta\tilde{\chi}_{0\eta}/\tilde{\chi}_0, \quad W_2 = -2i\alpha\chi_{-1\xi} \quad (18)$$

where $\tilde{\chi}_0$,χ_{-1} are the coefficients of the series expansion of χ near the points $\lambda = 0$ and $\lambda = \infty$: $\chi = \tilde{\chi}_0 + \lambda\chi_1 + \lambda\chi_2 + ...$, $\chi = 1 + \chi_{-1}/\lambda + \chi_{-2}/(\lambda^2) + ...$

The solution of $\overline{\partial}$-problem (10) with the canonical normalization is equivalent to the solution of the following singular integral equation :

$$\chi(\lambda) = 1 + \int\int_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)} \int\int_C \frac{d\mu \wedge d\overline{\mu}}{2\pi i} \chi(\mu, \overline{\mu}) R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'}) e^{(F(\mu) - F(\lambda'))} \quad (19)$$

where $F(\lambda)$ is given by the formula (13).

In conclusion of this section let us consider the conditions of reality of U and V.One must to distinguish two different cases.For real values of α and β the condition of reality of U,V leads from (19) in the limit of weak fields to the following restriction on the kernel R of $\overline{\partial}$-problem:

$$\overline{R(\mu, \overline{\mu}; \lambda, \overline{\lambda})} = -R_0(-\overline{\mu}, -\mu; -\overline{\lambda}, -\lambda). \quad (20)$$

In the case of pure imaginary values of α and β the condition of reality of U leads from (19) in the limit of weak fields to the another restriction on the kernel R of $\overline{\partial}$-problem:

$$\overline{R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})} = -R_0(\overline{\lambda}, \lambda; \overline{\mu}, \mu). \quad (21)$$

In terms of variables p and q (5) the condition of reality of $U = -pq$ means that $p = \kappa\overline{q}$.

3 The solutions with functional parameters

At first let us consider general class of exact solutions of equations (3),(4) and (7)-(9) which corresponds to the following degenerate kernel R_0 of $\overline{\partial}$-problem :

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = i\sum_{k=1}^{N} f_k(\mu, \overline{\mu}) g_k(\lambda, \overline{\lambda}). \quad (22)$$

For the kernel of this type from equation (19) and reconstruction formulae (18) one obtains for the exact solutions V,U of systems (3),(4) the following

expressions:

$$V = -\frac{\partial}{\partial \xi} ln det(1+M), \qquad U = -\epsilon det\left[(1+M)(1+\tilde{M})\right] \tag{23}$$

where

$$M_{kn} := \sum_{l=1}^{N} \eta_k A^{-1}_{nl} \xi_l, \quad \tilde{M}_{kn} := \frac{1}{\epsilon} \sum_{l=1}^{N} \eta_{k\xi} A^{-1}_{nl} \xi_{l\eta}, \quad A_{kl} = \delta_{kl} - \partial^{-1}_{\eta}(\xi_{k\eta}\eta_l),$$

$$\xi_k(\xi,\eta,t) := \int\int_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} f_k(\lambda,\overline{\lambda}) e^{F(\lambda)}, \quad \eta_k(\xi,\eta,t) := \int\int_C \frac{d\lambda \wedge d\overline{\lambda}}{2\pi i} g_k(\lambda,\overline{\lambda}) e^{-F(\lambda)}. \tag{24}$$

The solutions q,p of the systems (7),(9) according (6),(38) have the form :

$$q = det(1+M), \quad p = \epsilon det(1+\tilde{M}). \tag{25}$$

The reality conditions (20) and (21) imply certain constraints on the functions f_k and g_k (k = 1,..,N).They are satisfied,in particular, if

$$\overline{f_k(\mu,\overline{\mu})} = f_k(-\overline{\mu},-\mu), \quad \overline{g_k(\mu,\overline{\mu})} = g_k(-\overline{\mu},-\mu) \tag{26}$$

for the case of real values of α,β,and

$$\overline{f_k(\mu,\overline{\mu})} = R_k G_k(\overline{\mu},\mu), \quad (R_k = \overline{R_k}) \tag{27}$$

for the pure imaginary values of α and β .The conditions (26) imply

$$\overline{\eta_k} = -\eta_k, \quad \overline{\xi_k} = \xi_k. \tag{28}$$

In this case matrices M,$\tilde{M}$ are real and as consequence the solutions V,U and q,p of the systems (3),(4) and (7),(9) are real.The conditions (27) lead to the relation

$$\overline{\xi_k} = R_k \eta_{k\xi} \tag{29}$$

from which follows $\tilde{M} = \overline{M}$, $p = \epsilon\overline{q}$ and from the formulae (24),(25),(29) one can obtain for the solution q of equation (8) the expression :

$$q = det(1+C) \tag{30}$$

where

$$C_{kn} = \sum_{l=1}^{N} \eta_k A^{-1}_{nl} R_l \overline{\eta}_{l\xi}, \quad A_{nl} = \delta_{nl} - \epsilon R_n \partial^{-1}_{\eta}(\overline{\eta}_n \eta_l). \tag{31}$$

4 Line solitons

Real valued line solitons of systems (3),(4) and (7),(9) in the case of real α and β correspond to the choice

$$f_k(\lambda,\overline{\lambda}) = \pi R_k \delta(\lambda - i\beta_k), \quad g_k(\lambda,\overline{\lambda}) = \pi\delta(\lambda - i\alpha_k) \tag{32}$$

where R_k,α_k and β_k are arbitrary real constants. In this case the solutions of systems (3),(4) and (7),(9) are given by the formulae (23)-(25) with the matrix A of the form :

$$A_{nl} = \delta_{nl} - \partial_\eta^{-1}(\xi_n \eta_l) = \delta_{nl} - \frac{R_n}{\alpha_l - \beta_n} exp\left[F(i\beta_n) - F(i\alpha_l)\right]. \tag{33}$$

The simplest solutions of such type corresponding to one term in the sum (22) are

$$q = \frac{1 - \frac{\beta_0}{\alpha_0}\varphi}{1-\varphi}, \qquad p = \epsilon\frac{1 - \frac{\alpha_0}{\beta_0}\varphi}{1-\varphi},$$

$$V = -\frac{q_\xi}{q} = -\frac{(\alpha_0 - \beta_0)^2\varphi}{\alpha_0(1-\varphi)\left(1 - \frac{\beta_0}{\alpha_0}\varphi\right)}, \quad U = -pq = -\epsilon\frac{\left(1 - \frac{\beta_0}{\alpha_0}\varphi\right)\left(1 - \frac{\alpha_0}{\beta_0}\varphi\right)}{(1-\varphi)^2} \tag{34}$$

where $\varphi = \frac{R_0 e^{\Delta F}}{(\alpha_0 - \beta_0)}$ and

$$\Delta F := F(i\beta_0) - F(i\alpha_0) = (\alpha_0 - \beta_0)\left(\xi - \frac{\epsilon}{\alpha_0\beta_0}\eta\right) + (\alpha_0^2 - \beta_0^2)\left(\alpha - \frac{\beta\epsilon^2}{\alpha_0^2\beta_0^2}\right)t$$

Under the restriction $R_0/(\alpha_0 - \beta_0) < 0$ the solutions q,p of (7),(9) and under the restrictions $R_0/(\alpha_0 - \beta_0) < 0$, $R_0\beta_0/(\alpha_0(\alpha_0 - \beta_0)) < 0$ the solutions V,U of (3),(4) are nonsingular and bounded line solitons .

Complex-valued line solitons of systems (3),(4) and (7),(9) in the case of pure imaginary α and β correspond to the choice

$$f_k(\lambda,\overline{\lambda}) = \pi R_k \delta(\lambda - \lambda_k), \qquad g_k(\lambda,\overline{\lambda}) = \pi\delta(\lambda - \overline{\lambda_k}) \tag{35}$$

where R_k are arbitrary real constants and $\lambda_k = \lambda_{kR} + i\lambda_{kI}$. The solutions of (3),(4) and (7),(9) corresponding to such choice of of kernel R_o are

$$V = -\frac{\partial}{\partial\xi} ln det(1+C), \quad U = -\epsilon det|1+C|^2; \quad q = det(1+C), \quad p = \epsilon\overline{q} \tag{36}$$

where the matrix C is given by (31) and the matrix A is

$$A_{kl} = \delta_{kl} - \epsilon R_k \partial_\eta^{-1}(\overline{\eta_k}\eta_l) = \delta_{kl} + \frac{iR_k e^{F(\lambda_k) - F(\overline{\lambda_l})}}{\lambda_k - \overline{\lambda_l}}. \tag{37}$$

The simplest solutions of (3),(4) and (7)-(9) of such type corresponding to one term $i\pi^2\delta(\mu-\lambda_0)\delta(\lambda-\overline{\lambda_0})$ in the sum (22) are given by the formulae :

$$V=\frac{4i\lambda_{0I}^2\phi}{(1+\phi)\left(1+\frac{\lambda_0}{\overline{\lambda_0}}\phi\right)},\quad U=-\epsilon\frac{\left|1+\frac{\lambda_0}{\lambda_0}\phi\right|^2}{(1+\phi)^2};\quad q=\frac{1+\frac{\lambda_0}{\overline{\lambda_0}}\phi}{1+\phi},\quad p=\epsilon\overline{q}\quad(38)$$

where $\lambda_0=\lambda_{0R}+i\lambda_{0I}$, $\phi:=\frac{R_0}{2\lambda_{0I}}e^{\Delta F}$ and

$$\Delta F=F(\lambda_0)-F(\overline{\lambda_0})=i(\lambda_0-\overline{\lambda_0})\left(\xi+\frac{\epsilon}{\lambda_0\overline{\lambda_0}}\eta\right)+(\lambda_0^2-\overline{\lambda_0}^2)\left(\alpha-\beta\frac{\epsilon^2}{\lambda_0^2\overline{\lambda_0}^2}\right)t$$

Under the restriction $\frac{R_0}{2\lambda_{0I}}>0$ the complex-valued solutions (56) are non-singular bounded line solitons of corresponding equations

5 Rational solutions

Via $\overline{\partial}$-dressing method can be easily constructed also rational solutions of integrable equations [8].Let us consider at first the case of real values of α and β in (1).To the reality condition (26) satisfies for example the following simple choice of kernel R of $\overline{\partial}$-problem :

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})=i\pi^2\sum_{k=1}^{N}S_k(\mu,\lambda)\delta(\mu-i\alpha_k)\delta(\lambda-i\alpha_k)\qquad(39)$$

where $\alpha_k=\overline{\alpha_k}$ and $\overline{S_k(\mu,\lambda)}=S_k(-\overline{\mu},-\overline{\lambda})$,$(k=1,2,...,N)$.

With the use of (18),(19) and (39) one obtains for the solutions q,p and V,U of the systems (7),(9) and (3),(4) correspondingly the following expressions :

$$q=det(1-BA^{-1}),\qquad p=\epsilon det(1+A^{-1}B^T),$$
$$V=-\frac{\partial}{\partial\xi}lndet(1-BA^{-1}),\quad U=-pq=-\epsilon det[(1-BA^{-1})(1+A^{-1}B^T)]\quad(40)$$

where $A_{kl}=d_k\delta_{kl}-\frac{1-\delta_{kl}}{\alpha_k-\alpha_l}$ and $d_k:=\xi-\frac{\epsilon}{\alpha_k^2}\eta+2\left(\alpha\alpha_k-\frac{\beta\epsilon^2}{\alpha_k^3}\right)t+\gamma_k$ and $\gamma_k=\overline{\gamma_k}$ and matrix B is: $B_{kl}:=1/\alpha_l$. The simplest solutions of such type which correspond to one term in the sum (39) have the form :

$$q=1-\frac{1/\alpha_0}{d_0},p=\epsilon\left(1+\frac{1/\alpha_0}{d_0}\right);\quad V=-\frac{1}{\alpha_0 d_0(d_0-1/\alpha_0)},\quad U=-\epsilon\left(1-\frac{1/\alpha_0^2}{d_0^2}\right).\qquad(41)$$

Analogous calculations one can made in the more complicated choice of the kernel R (α and β real constants):

$$R(\mu,\overline{\mu};\lambda,\overline{\lambda}) = i\pi^2\sum_{k=1}^{N}[S_k\delta(\mu-\lambda_k)\delta(\lambda-\lambda_k)+\overline{S_k}\delta(\mu+\overline{\lambda_k})\delta(\lambda+\overline{\lambda_k})] \quad (42)$$

The calculations in this case lead to the following expressions for the solutions p,q and V,U of equations(7),(9) and (3),(4) correspondingly :

$$q = det(1-iBA^{-1}), \quad p = \epsilon det(1+iA^{-1}B^T); \quad V = -q_\xi/q = \overline{V}, \quad U = -pq = \overline{U} \quad (43)$$

where the matrices A and B have now dimension $2N \times 2N$ and are given by the formulae :

$$A_{kl} = d_k\delta_{kl} - \frac{i}{\Lambda_k-\Lambda_l}, B_{kl} = 1/\Lambda_l, \quad (k,l = 1,2,...,2N) \quad (44)$$

where $d_k = \xi - \frac{\epsilon}{\Lambda_k}\eta + 2\left(\alpha\Lambda_k - \frac{\beta\epsilon^2}{\Lambda_k^3}\right)t + \gamma_k$. The set Λ of constants Λ_k is defined here as follows: $\Lambda := (\Lambda_1 = \lambda_1,...,\Lambda_N = \lambda_N; \Lambda_{N+1} = -\overline{\lambda_1},...,\Lambda_{2N} = -\overline{\lambda_N})$. The simplest solutions which correspond to one term in the sum (42) have the form :

$$q = 1 - i\left(\frac{d_0}{\lambda_0} - \frac{\overline{d_0}}{\overline{\lambda_0}}\right)\frac{1}{\Delta} - \frac{1}{|\lambda_0|^2\Delta}, p = \epsilon\left(1 + i\left(\frac{d_0}{\lambda_0} - \frac{\overline{d_0}}{\overline{\lambda_0}}\right)\frac{1}{\Delta} - \frac{1}{|\lambda_0|^2\Delta}\right),$$
$$V = -q_\xi/q = \overline{V}, \qquad U = -pq = \overline{U} \quad (45)$$

where $\Delta = |d_0|^2 - \frac{1}{4\lambda_{0R}^2}$. It is easy to see that the obtained rational solutions (40),(41),(43),(44) of the systems (3),(4) and (7),(9) are singular.

Completely analogous calculations of rational solutions one can made in the case of pure imaginary constants α and β in (1) : $\alpha := i\tilde{\alpha}, \beta := i\tilde{\beta}$ .We formulate in this case only final results.To the reality condition (27) of U (or to the condition $p = \epsilon\overline{q}$) corresponds for example the following simple choice of the kernel R of $\overline{\partial}$-problem :

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = i\pi^2\sum_{k=1}^{N} S_k\delta(\mu-\alpha_k)\delta(\lambda-\alpha_k) \quad (46)$$

with $S_k = \overline{S_k}$ and $\alpha_k = \overline{\alpha_k}$. For the exact solutions of of equations (3), (4) and (7)-(9) one obtains after some calculations:

$$q = det(1+iBA^{-1}), \quad p = \epsilon det(1-iA^{-1}B^T) = \epsilon\overline{q}; \quad V = -q_\xi/q, \quad U = -pq \quad (47)$$

$$B_{kl} := 1/\alpha_l, A_{kl} = d_k\delta_{kl} - i\frac{1-\delta_{kl}}{\alpha_k - \alpha_l}, d_k := \xi + \frac{\epsilon}{\alpha_k^2}\eta + 2\left(\tilde{\alpha}\alpha_k - \frac{\tilde{\beta}\epsilon^2}{\alpha_k^3}\right)t + \gamma_k \tag{48}$$

with some real constants γ_k. The simplest solutions of such type which correspond to the one term in the sum (46) have the form:

$$q = 1 - \frac{i/\alpha_0}{d_0}, p = \epsilon\left(1 + \frac{i/\alpha_0}{d_0}\right); \quad V = -\frac{i}{\alpha_0 d_0(d_0 - i/\alpha_0)}, U = -\epsilon\left(1 + \frac{1/\alpha_0^2}{d_0^2}\right). \tag{49}$$

The solutions (47),(49) as it is easy to see are singular.

To the reality condition (27) one can satisfy by the more complicated choice of kernel R of $\overline{\partial}$-problem :

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = i\pi^2 \sum_{k=1}^{N} [S_k\delta(\mu - \lambda_k)\delta(\lambda - \lambda_k) + \overline{S_k}\delta(\mu - \overline{\lambda_k})\delta(\lambda - \overline{\lambda_k})] \tag{50}$$

The solutions of equations (3),(4) and (7)-(9) which correspond to such type of the kernel R have the form :

$$q = det(1 - iBA^{-1}), \quad p = \epsilon det(1 + iA^{-1}B^T) = V = -q_\xi/q, \quad U = -pq = -\epsilon|q|^2 \tag{51}$$

where the matrices A and B have dimension $2N \times 2N$ and are given by the formulae

$$A_{kl} = d_k\delta_{kl} - \frac{i}{\Lambda_k - \Lambda_l}, \qquad B_{kl} = 1/\Lambda_l, \quad (k, l = 1, ..., 2N) \tag{52}$$

with $d_k := \xi - \frac{\epsilon}{\Lambda_k}\eta + 2\left(\tilde{\alpha}\Lambda_k - \frac{\tilde{\beta}\epsilon^2}{\Lambda_k^3}\right)t + \gamma_k$,(k=1,...2N),and the set Λ of constants Λ_k is defined as follows:$\Lambda := (\Lambda_1 = \lambda_1, ..., \Lambda_N = \lambda_N; \Lambda_{N+1} = \overline{\lambda_1}, ..., \Lambda_{2N} = \overline{\lambda_N})$. The simplest solutions of equations (3),(4) and (7),(9) of such type which correspond to the one term in the sum (50) have the form :

$$q = 1 - \frac{i}{\Delta}\left(\frac{d_0}{\overline{\lambda_0}} + \frac{\overline{d_0}}{\lambda_0} - \frac{i}{|\lambda_0|^2}\right), \quad p = \epsilon\overline{q}; \quad V = -q_\xi/q, \quad U = -pq = -\epsilon|q|^2 \tag{53}$$

where $\Delta := |d_0|^2 + \frac{1}{4\lambda_{0I}^2}$. As one can see from (60),(62) these rational solutions are nonsingular , bounded and in some sence localized.For example from (62) it is evident that all solutions q,p and V,U tend to their constant asymptotic values : $q \to 1, \quad p \to \epsilon, V \to 0, U \to -\epsilon$ as $\xi^2 + \eta^2 \to \infty$ rationaly in all directions of the plane ξ, η.

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THE EQUATIONS OF THE ASSOCIATIVITY AS HYDRODYNAMICAL TYPE SYSTEM: HAMILTONIAN REPRESENTATION AND INTEGRABILITY [a]

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1 Introduction

Let us consider a function of n independent variables $F(t^1, ..., t^n)$ satisfying the following two conditions:
1. The matrix

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} \qquad (\alpha, \beta = 1, ..., n)$$

is constant and nondegenerate.
2. For all $t = (t^1, ..., t^n)$ *the functions*

$$c^\alpha_{\beta\gamma}(t) = \eta^{\alpha\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\beta \partial t^\gamma} \qquad (here \quad \eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha_\beta)$$

are the structure constants of the associative algebra $A(t)$ *of dimension* n *with the basis* $e_1, ..., e_n$ *and the multiplication*

$$e_\beta \circ e_\gamma = c^\alpha_{\beta\gamma}(t) e_\alpha .$$

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Conditions 1 and 2 lead to a complicated overdetermined system of nonlinear partial differential equations of the third order on the function F. This system is known in two-dimensional topological field theory as the equations of the associativity or the Witten-Dijkgraaf-H.Verlinde-E.Verlinde (WDVV) system [10–13] (all necessary physical motivations and the theory of integrability of the equations of the associativity can be found in the survey of B.Dubrovin [1]).

For $n = 3$ Dubrovin considered two essentially different types of dependence of the function F on the fixed variable t^1:

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 + f(t^2, t^3)$$

and

$$F = \frac{1}{6}(t^1)^3 + t^1 t^2 t^3 + f(t^2, t^3).$$

For these cases the equations of the associativity reduce to the following two nonlinear equations of the third order on the function $f = f(x,t)$ of two independent variables ($x = t^2,\ t = t^3$):

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt} \tag{1.1}$$

and

$$f_{xxx}f_{ttt} - f_{xxt}f_{xtt} = 1, \tag{1.2}$$

respectively.

Following [2,3], we introduce the new variables

$$a = f_{xxx},\ b = f_{xxt},\ c = f_{xtt}.$$

As it was shown in the papers [2,3], in the new variables the equations (1.1) and (1.2) assume the form of the 3×3 systems of hydrodynamic type:

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (b^2 - ac)_x \end{cases} \tag{1.3}$$

and

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = \big((1 + bc)/a\big)_x \end{cases}, \tag{1.4}$$

respectively.

The main advantage of representation of the equations of the associativity in the form (1.3), (1.4) is the existence of the efficient and elaborate theory of integrability of systems of hydrodynamic type — see, for example, the surveys of Tsarev [4], Dubrovin and Novikov [5], and also the papers [6–9] devoted to systems of hydrodynamic type, which do not possess Riemann invariants.

In § 2 the Hamiltonian property of the system (1.3) is established. For this system a local nondegenerate Hamiltonian structure of hydrodynamic type (a Poisson bracket of Dubrovin-Novikov type [5]) is found. In contrast to (1.3), the integrable system of hydrodynamic type (1.4) possesses only nonlocal Hamiltonian structure of hydrodynamic type (see [14–16]). Investigation, which was made in [2,3], showed that both systems (1.3) and (1.4) are nondiagonalizable (i.e., do not possess Riemann invariants).

In § 3 we construct an explicit chain of differential substitutions, transforming the system (1.3) into the well-known 3-wave system.

In § 4 an explicit Bäcklund type transformation connecting solutions of the systems (1.3) and (1.4) is found.

In § 5 we apply a similar approach to the Witten equation $u_{ttt}u_{xxx} - u_{xxt}u_{xtt} = 0$, arising in topological field theory with two primary fields without the normalization constraint [10].

2 Hamiltonian representation of the system (1.3)

As it was noticed by Dubrovin [1], the equation (1.1) is connected with the spectral problem, which has the following form in the variables a, b, c:

$$\Psi_x = zA\Psi = z\begin{pmatrix} 0 & 1 & 0 \\ b & a & 1 \\ c & b & 0 \end{pmatrix}\Psi, \quad \Psi_t = zB\Psi = z\begin{pmatrix} 0 & 0 & 1 \\ c & b & 0 \\ b^2-ac & c & 0 \end{pmatrix}\Psi. \quad (2.1)$$

The compatibility conditions of the spectral problem (2.1) are equivalent to the following two relations between the matrices A and B:

$$A_t = B_x, \quad [A, B] = 0, \quad (2.2)$$

which are satisfied identically by virtue of the equations (1.3) (here $[\ ,\ ]$ denotes the commutator).

Let us demonstrate that the eigenvalues of the matrix A are densities of conservation laws of the system (1.3).

As far as the matrices A and B commute and have simple spectrum, they can be diagonalized simultaneously

$$A = PUP^{-1}, \quad B = PVP^{-1}.$$

Here $U = \mathrm{diag}\,(\mathrm{u}^1, \mathrm{u}^2, \mathrm{u}^3)$, $\mathrm{V} = \mathrm{diag}\,(\mathrm{v}^1, \mathrm{v}^2, \mathrm{v}^3)$. Substitution in the equation (2.2) gives

$$[P^{-1}P_t, U] + U_t = [P^{-1}P_x, V] + V_x.$$

It remains to note that the matrices $[P^{-1}P_t, U]$ and $[P^{-1}P_x, V]$ are off-diagonal. Hence,

$$U_t = V_x.$$

Thus, besides the three evident conservation laws with the densities a, b, c the system (1.3) has also three conservation laws with the densities u^1, u^2, u^3, which are the roots of the characteristic equation

$$\det(\lambda E - A) = \lambda^3 - a\lambda^2 - 2b\lambda - c = 0.$$

By virtue of the obvious linear relation $a = u^1 + u^2 + u^3$ only five conservation laws among them with the densities u^1, u^2, u^3, b, c are linearly independent. One can show that the system (1.3) has no other conservation laws of hydrodynamic type, i.e., with the densities of the form $h(a, b, c)$.

Let us change in the equations (1.3) from the variables a, b, c to the new field variables u^1, u^2, u^3, connected with a, b, c by the Viète formulas

$$a = u^1 + u^2 + u^3, \quad b = -\frac{1}{2}(u^1u^2 + u^2u^3 + u^3u^1), \quad c = u^1u^2u^3.$$

A direct calculation gives

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}_t = \frac{1}{2}\begin{pmatrix} u^2u^3 - u^1u^2 - u^1u^3 \\ u^1u^3 - u^2u^1 - u^2u^3 \\ u^1u^2 - u^3u^1 - u^3u^2 \end{pmatrix}_x = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\frac{d}{dx}\begin{pmatrix} \partial h/\partial u^1 \\ \partial h/\partial u^2 \\ \partial h/\partial u^3 \end{pmatrix}, \tag{2.3}$$

where $h = c = u^1u^2u^3$. Hence, the system under consideration is Hamiltonian with the Hamiltonian operator

$$M = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\frac{d}{dx} \tag{2.4}$$

and the Hamiltonian $H = \int c dx = \int u^1u^2u^3 dx$. The density of momentum and Casimirs of the corresponding Poisson bracket have the following form:

$2b = -u^1u^2 - u^2u^3 - u^3u^1$ (the density of momentum),
u^1, u^2, u^3 (Casimirs).

In the initial variables a, b, c the Hamiltonian operator (2.4) is expressed in the form

$$M = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2}a & b \\ \frac{1}{2}a & b & \frac{3}{2}c \\ b & \frac{3}{2}c & 2(b^2-ac) \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \frac{1}{2}a_x & b_x \\ 0 & \frac{1}{2}b_x & c_x \\ 0 & \frac{1}{2}c_x & (b^2-ac)_x \end{pmatrix}.$$

There exist two essentially different types of integrable Hamiltonian systems of hydrodynamic type: diagonalizable systems, which are integrable by the generalized hodograph transform [4], and nondiagonalizable systems, discussed in [6-9]. It was shown in [2-3], that the system (1.3) is nondiagonalizable, and hence the approach of [6-9] can be applied.

Theorem [7,8].*Nondiagonalizable Hamiltonian (with nondegenerate Poisson bracket of hydrodynamic type)* 3×3 *system of hydrodynamic type is integrable if and only if it is weakly nonlinear.*

We recall that a system of hydrodynamic type

$$u^i_t = v^i_j(u)u^j_x, \quad i,j = 1,...,n, \tag{2.5}$$

is called weakly nonlinear if for eigenvalues $\lambda^i(u)$ of the matrix $v^i_j(u)$ the following relations are satisfied for any $i = 1,...,n$:

$$L_{\vec{X}^i}(\lambda^i) = 0,$$

where $L_{\vec{X}^i}$ is the Lie derivative along the eigenvector $\vec{X}^i$ corresponding to the eigenvalue λ^i.

There exists a simple and efficient criterion of weak nonlinearity, which does not appeal to eigenvalues and eigenvectors.

Proposition [7]. *A system of hydrodynamic type (2.5) is weakly nonlinear if and only if*

$$(\mathrm{grad\ f_1})\mathrm{v^{n-1}} + (\mathrm{grad\ f_2})\mathrm{v^{n-2}} + ... + (\mathrm{grad\ f_n})\mathrm{E} = 0,$$

where f_i are the coefficients of the characteristic polynomial

$$\det(\lambda\delta^i_j - v^i_j(u)) = \lambda^n + f_1(u)\lambda^{n-1} + f_2(u)\lambda^{n-2} + ... + f_n(u),$$

and v^n denotes the n-th power of the matrix v^i_j.

As it was shown in [2,3], both systems (1.3) and (1.4) are weakly nonlinear.

3 Transformation into the 3-wave system

The eigenvalues λ^i and the corresponding left eigenvectors $\vec{l}^{\,i}$ of the system (2.3) are of the form:

$$\lambda^1 = -u^1, \quad \vec{l}^{\,1} = (k^1)^{-1}(u^2 - u^3, u^1 - u^3, u^2 - u^1),$$

$$\lambda^2 = -u^2, \quad \vec{l}^{\,2} = (k^2)^{-1}(u^2 - u^3, u^1 - u^3, u^1 - u^2),$$

$$\lambda^3 = -u^3, \quad \vec{l}^{\,3} = (k^3)^{-1}(u^2 - u^3, u^3 - u^1, u^2 - u^1),$$

where it is convenient to choose the normalizing coefficients k^i in the form

$$k^1 = 2(u^2 - u^3)\sqrt{(u^2 - u^1)(u^3 - u^1)},$$

$$k^2 = 2(u^3 - u^1)\sqrt{(u^2 - u^1)(u^2 - u^3)},$$

$$k^3 = 2(u^2 - u^1)\sqrt{(u^3 - u^1)(u^2 - u^3)}.$$

Then the equations (2.3) can be expressed as

$$\omega^i \wedge (dx - u^i dt) = 0, \quad i = 1, 2, 3, \tag{3.1}$$

where

$$\omega^1 = \frac{(u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^2 - u^1)du^3}{2(u^2 - u^3)\sqrt{(u^2 - u^1)(u^3 - u^1)}},$$

$$\omega^2 = \frac{(u^2 - u^3)du^1 + (u^1 - u^3)du^2 + (u^1 - u^2)du^3}{2(u^3 - u^1)\sqrt{(u^2 - u^1)(u^2 - u^3)}},$$

$$\omega^3 = \frac{(u^2 - u^3)du^1 + (u^3 - u^1)du^2 + (u^2 - u^1)du^3}{2(u^2 - u^1)\sqrt{(u^3 - u^1)(u^2 - u^3)}}. \tag{3.2}$$

The normalizing coefficients k^i are chosen in such a way, that the 1-forms ω^i satisfy the structure equations of the $SO(2,1)$ group:

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2 \tag{3.3}$$

Let us introduce the reciprocal transformation

$$d\tilde{x} = Bdx + Adt = (u^1 - u^2)dx + u^3(u^2 - u^1)dt,$$

$$d\tilde{t} = Ndx + Mdt = (2u^3 - u^1 - u^2)dx + (2u^1u^2 - u^1u^3 - u^2u^3)dt. \tag{3.4}$$

In the new independent variables $\tilde{x}, \tilde{t}$, the eigenvalues $\lambda^i = -u^i$ transform into 1, -1, 0, respectively. Hence, the exterior equations (3.1) become

$$\omega^1 \wedge (d\tilde{x} + d\tilde{t}) = 0, \quad \omega^2 \wedge (d\tilde{x} - d\tilde{t}) = 0, \quad \omega^3 \wedge d\tilde{x} = 0. \tag{3.5}$$

Introducing in the equations (3.5) the variables p^1, p^2, p^3 by the formulas (see [6,7])

$$\omega^1 = p^1(d\tilde{x} + d\tilde{t}), \quad \omega^2 = p^2(d\tilde{x} - d\tilde{t}), \quad \omega^3 = p^3 d\tilde{x}. \tag{3.6}$$

and substituting (3.6) in the structure equations (3.3) we obtain the integrable 3-wave system

$$p^1_{\tilde{t}} - p^1_{\tilde{x}} = -p^2 p^3,$$

$$p^2_{\tilde{t}} + p^2_{\tilde{x}} = -p^1 p^3,$$

$$p^3_{\tilde{t}} = -2p^1 p^2.$$

Remark. Using the explicit coordinate representation of the 1-forms $\omega^i = l^i_k(u)du^k$, we obtain for p^i the expressions of the form $p^i = l^i_k(u)u^k_{\tilde{x}}$. Hence, the change from u^i to p^i is a differential substitution of the first order.

Thus the transition from the equation (2.3) to the 3-wave system (3.7) can be decomposed in two steps:

1. The change from x, t to the new independent variables $\tilde{x}, \tilde{t}$ in accordance with the formulas (3.4).

2. The change of the field variables from u^1, u^2, u^3 to p^1, p^2, p^3 in accordance with the following formulas (compare with (3.2)):

$$p^1 = \frac{(u^2-u^3)u^1_{\tilde{x}} + (u^1-u^3)u^2_{\tilde{x}} + (u^2-u^1)u^3_{\tilde{x}}}{2(u^2-u^3)\sqrt{(u^2-u^1)(u^3-u^1)}},$$

$$p^2 = \frac{(u^2-u^3)u^1_{\tilde{x}} + (u^1-u^3)u^2_{\tilde{x}} + (u^1-u^2)u^3_{\tilde{x}}}{2(u^3-u^1)\sqrt{(u^2-u^1)(u^2-u^3)}},$$

$$p^3 = \frac{(u^2-u^3)u^1_{\tilde{x}} + (u^3-u^1)u^2_{\tilde{x}} + (u^2-u^1)u^3_{\tilde{x}}}{2(u^2-u^1)\sqrt{(u^3-u^1)(u^2-u^3)}}. \tag{3.8}$$

4 Relation between the systems (1.3) and (1.4)

The spectral problem corresponding to the system (1.4) has the form

$$\Psi_x = zA\Psi = z\begin{pmatrix} 0 & 1 & 0 \\ 0 & b & a \\ 1 & c & b \end{pmatrix}\Psi, \quad \Psi_t = zB\Psi = z\begin{pmatrix} 0 & 0 & 1 \\ 1 & c & b \\ 0 & (1+bc)/a & c \end{pmatrix}\Psi. \tag{4.1}$$

Let w^i be the eigenvalues of the matrix A, i.e., the roots of the characteristic equation

$$\det(\lambda E - A) = \lambda^3 - 2b\lambda^2 + (b^2 - ac)\lambda - a = 0.$$

In these coordinates the system (1.4) assumes the form:

$$\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}_t = \frac{1}{2}\begin{pmatrix} (w^1 - w^2 - w^3)/w^2w^3 \\ (w^2 - w^1 - w^3)/w^1w^3 \\ (w^3 - w^1 - w^2)/w^1w^2 \end{pmatrix}_x. \tag{4.2}$$

Note that the integrable systems of hydrodynamic type (1.4) and (4.2) do not possess local Hamiltonian structures of hydrodynamic type (the Poisson brackets of Dubrovin-Novikov type [5]). Hamiltonian structures of hydrodynamic type corresponding to them are strictly nonlocal ([14–16]).

We exhibit now the explicit relation between the systems (2.3) and (4.2). For this reason we change in the equations (2.3) from x, t to the new independent variables $\tilde{x}$, $\tilde{t}$ according to the formulas

$$d\tilde{x} = -\frac{1}{2}(u^1u^2 + u^1u^3 + u^2u^3)dx + u^1u^2u^3dt, \quad d\tilde{t} = dx. \tag{4.3}$$

After the transformation (4.3) the system (2.3) assumes the form

$$\begin{pmatrix} 1/u^1 \\ 1/u^2 \\ 1/u^3 \end{pmatrix}_{\tilde{t}} = \frac{1}{2}\begin{pmatrix} (u^2u^3)/u^1 - u^2 - u^3 \\ (u^1u^3)/u^2 - u^1 - u^3 \\ (u^1u^2)/u^3 - u^1 - u^2 \end{pmatrix}_{\tilde{x}},$$

which coincides with (4.2) after the transformation

$$w^i = \frac{1}{u^i}. \tag{4.4}$$

In terms of the initial equations (1.1) and (1.2) we can represent the transformations (4.3) and (4.4) in the following way: the equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt}$$

transforms into the equation

$$\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{f}_{\tilde{t}\tilde{t}\tilde{t}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{t}}\tilde{f}_{\tilde{x}\tilde{t}\tilde{t}} = 1$$

after the transformation

$$\tilde{x} = f_{xt}, \quad \tilde{t} = x, \qquad \tilde{f}_{\tilde{x}\tilde{x}} = t, \qquad \tilde{f}_{\tilde{x}\tilde{t}} = -f_{xx}, \quad \tilde{f}_{\tilde{t}\tilde{t}} = f_{tt}. \tag{4.6}$$

Note that transformation (4.6), connecting solutions of the associativity equations (1.1) and (1.2), is not contact.

5 On the Witten equation $f_{ttt}f_{xxx} - f_{xxt}f_{xtt} = 0$

The same approach applies to the equation

$$f_{ttt}f_{xxx} - f_{xxt}f_{xtt} = 0, \tag{5.1}$$

which was discussed by Witten [10] in the framework of 2-dimensional topological field theory for $n = 2$.

Introducing

$$a = f_{xxx}, \quad b = f_{xxt}, \quad c = f_{xtt},$$

we can rewrite (5.1) as the system of hydrodynamic type [2],[3]

$$\begin{cases} a_t = b_x, \\ b_t = c_x, \\ c_t = (bc/a)_x \end{cases} \tag{5.2}$$

Although very similar to (1.4), the system (5.2) is a much simpler object: as we are going to show below, it can be linearized by a sequence of variable transformations.

First of all we introduce the new dependent variables

$$R^1 = \sqrt{c/a}, \quad R^2 = b - \sqrt{ac}, \quad R^3 = b + \sqrt{ac}.$$

In the variables R^i the system (5.2) assumes the form

$$\begin{cases} R^1_t = R^1 \frac{(R^3+R^2)}{(R^3-R^2)} R^1_x, \\ R^2_t = -(R^1R^2)_x, \\ R^3_t = (R^1R^3)_x, \end{cases} \tag{5.3}$$

which shows that R^1 is a Riemann invariant (we emphasize that in contrast with (5.2) the system (1.4) does not possess a single Riemann invariant).

Let us introduce the reciprocal transformation

$$d\tilde{x} = (R^3 - R^2)dx + R^1(R^3 + R^2)dt,$$
$$d\tilde{t} = (R^3 + R^2)dx + R^1(R^3 - R^2)dt. \tag{5.4}$$

In the variables $\tilde{x}, \tilde{t}$ the equations (5.3) assume the form

$$\begin{cases} R^1_{\tilde{t}} = 0, \\ R^2_{\tilde{t}} = -R^2_{\tilde{x}} - \frac{R^2(R^3-R^2)}{2R^1R^3} R^1_{\tilde{x}}, \\ R^3_{\tilde{t}} = R^3_{\tilde{x}} - \frac{R^3(R^3-R^2)}{2R^1R^2} R^1_{\tilde{x}}, \end{cases} \tag{5.5}$$

We point out that after the transformation (5.4) the eigenvalues of the system (5.3), which are equal to

$$\lambda^1 = R^1\frac{(R^3 + R^2)}{(R^3 - R^2)}, \quad \lambda^2 = R^1, \quad \lambda^3 = -R^1,$$

become constants: $\lambda^1 = 0$, $\lambda^2 = 1$, $\lambda^3 = -1$, respectively.

Introducing the 1-forms

$$\begin{cases} \omega^1 = \frac{dR^1}{R^1}, \\ \omega^2 = (\frac{1}{R^3} - \frac{1}{R^2})\frac{dR^1}{R^1} + 2\frac{dR^3}{(R^3)^2}, \\ \omega^3 = (\frac{1}{R^2} - \frac{1}{R^3})\frac{dR^1}{R^1} + 2\frac{dR^2}{(R^2)^2}, \end{cases} \tag{5.6}$$

we can rewrite (5.5) in the exterior differential form (see e.g. [7],[8]):

$$\begin{cases} \omega^1 \wedge d\tilde{x} = 0, \\ \omega^2 \wedge (d\tilde{x} + d\tilde{t}) = 0, \\ \omega^3 \wedge (d\tilde{x} - d\tilde{t}) = 0, \end{cases} \tag{5.7}$$

Here the 1-forms ω^i satisfy the structure equations

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \frac{1}{2}(\omega^3 - \omega^2) \wedge \omega^1, \\ d\omega^3 = \frac{1}{2}(\omega^2 - \omega^3) \wedge \omega^1. \end{cases} \tag{5.8}$$

Introducing the variables p^1, p^2, p^3 by the formulas

$$\omega^1 = p^1 d\tilde{x}, \quad \omega^2 = p^2(d\tilde{x} + d\tilde{t}), \quad \omega^3 = p^3(d\tilde{x} - d\tilde{t})$$

(this representation for ω^i is a consequence of (5.7)) and using the structure equations (5.8), we arrive at the following equations for p^1, p^2, p^3:

$$\begin{cases} p^1_{\tilde{t}} = 0, \\ p^2_{\tilde{t}} = p^2_{\tilde{x}} - \frac{1}{2}p^1(p^2+p^3), \\ p^3_{\tilde{t}} = -p^3_{\tilde{x}} + \frac{1}{2}p^1(p^2+p^3). \end{cases} \tag{5.9}$$

Note, that the variables p^i and R^i are related by the formulae

$$p^1 = \frac{R^1_{\tilde{x}}}{R^1}, \quad p^2 = \left(\frac{1}{R^3} - \frac{1}{R^2}\right)\frac{R^1_{\tilde{x}}}{R^1} + 2\frac{R^3_{\tilde{x}}}{(R^3)^2}, \quad p^3 = \left(\frac{1}{R^2} - \frac{1}{R^3}\right)\frac{R^1_{\tilde{x}}}{R^1} + 2\frac{R^2_{\tilde{x}}}{(R^2)^2},$$

i.e. the transformation from R^i to p^i is just the differential substitution of the first order.

Finally we point out, that the equations (5.9) are in fact linear with the general solution given by

$$p^1 = \phi(\tilde{x}), \quad p^2 = \frac{1}{2}(f_{\tilde{x}} + f_{\tilde{t}}), \quad p^3 = \frac{1}{2}(f_{\tilde{x}} - f_{\tilde{t}}),$$

where $\phi(\tilde{x})$ is an arbitrary function of the variable $\tilde{x}$, while $f(\tilde{x},\tilde{t})$ is an arbitrary solution of the linear wave equation

$$f_{\tilde{t}\tilde{t}} - f_{\tilde{x}\tilde{x}} + \phi(\tilde{x})f_{\tilde{x}} = 0.$$

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THE NONLINEAR SCHRÖDINGER EQUATION AND N–SOLITON INTERACTIONS. GENERALIZATION OF THE KARPMAN–SOLOVIEV APPROACH

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A method for the description of the N–soliton interaction, which generalizes in a natural way the Karpman–Soloviev one for the nonlinear Schrödinger equation is proposed. Using this method we derive a nonlinear system of equations describing the dynamics of the parameters of N well separated solitons with nearly equal amplitudes and velocities. Next we study an exhaustive list of perturbations, relevant for nonlinear optics, which include linear and nonlinear dispersive and dissipative terms, effects of sliding filters, amplitude and phase modulation etc. We prove, that the linear perturbations affect each of the solitons separately, while the nonlinear ones lead also to additional interactive terms between the neighboring solitons.

1 Introduction

The interaction of the solitons of the nonlinear Schrödinger equation (NLSE) in the generic case when all solitons have different velocities is well known[1,2]. However, the cases, when two or several solitons move with the same velocity, or when the perturbed NLSE:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u], \tag{1}$$

is considered still contain open problems. At the same time such soliton trains moving in real media are of great interest for a number of different

physical applications. Typical examples of such applications are optical soliton transmission lines[3,4] and nonlinear fiber lasers[4].

To solve such type of problems one may use one of the following three methods.

The first one is the analytical approach based on the inverse scattering method[1,2] and the expansions over the 'squared' solutions of the Zakharov–Shabat system[5,6,7]. It allows one to relate the variations $\Delta C_k^{\pm}$, $\Delta\lambda_k^{\pm}$ of the soliton parameters to certain integrals containing the perturbation $R[u]$ itself and the corresponding 'squared' solution $\Phi_k^{\pm}$. This approach, known as the adiabatic approximation, is applicable for generic perturbations.

However, the corresponding formulae beyond $N = 1$ quickly become so involved, that one is not able to analyze them. Another problem arises with the necessity to determine the exact scattering data, corresponding to an initial pulse train of the form:

$$u(x,0) = \sum_{k=1}^{N} \frac{2\nu_k e^{i\phi_{0k}}}{\cosh 2\nu_k(x - x_{0k})} \tag{2}$$

This method has been used also for numeric simulations in several papers[8–12], where mostly 2–soliton interactions have been studied. With the growth of the number of pulses the difficulties of such investigations grow enormously (see[9,13] where some results for $N = 3$ have been obtained). Applying numerical methods in such approach means multiple execution of the following procedures. First, starting from the initial condition (2) one has to determine the corresponding scattering data ($C_k^{\pm}$, $\lambda_k^{\pm}$ and possible presence of radiation) and the 'squared' solutions $\Phi_k^{\pm}(x,\lambda)$ which enter in $\Delta C_k^{\pm}$ and $\Delta\lambda_k^{\pm}$. This allows one to determine $\Delta C_k^{\pm}$ and $\Delta\lambda_k^{\pm}$, i.e. the evolution of the scattering data. Finally one needs to determine the shape of the pulse, corresponding to the new data. This procedure must be repeated on each step of the calculation; it can be used effectively in the cases when the distances between the pulses are large and the number of solitons is comparatively low, see[10,11].

The second approach has been initiated by the pioneering paper of Karpman and Soloviev(KS)[14]; it is also known as the quasi–particle approach. With it one is able to study the interaction of soliton trains for some restricted class of initial conditions, namely: a) the solitons have nearly or exactly equal amplitudes and velocities; b) the separation between them is large as compare to their width (more precisely these conditions will be stated in Sec. 1 below). Under these approximations the N–soliton solution of the NLSE and the corresponding 'squared' solutions of the Zakharov–Shabat system are very well approximated by linear combinations of their one–soliton counterparts and so

one is able to derive a dynamical system of equations for the soliton parameters. This was performed effectively in[14] for $N = 2$ solitons, where moreover the corresponding dynamical system was solved explicitly. Later this approach was used in a number of papers for analyzing the 2–soliton interactions in the presence of various perturbations, for review see[3]. Although the region of soliton parameters, to which the KS method is applicable is comparatively small, it represents a substantial physical interest since a great part of the experimentally studied soliton–like pulses satisfy these conditions.

The third approach, which can be used for analyzing the soliton interactions is the so–called variational approach proposed by Anderson[15,16]. It is based on the Lagrangian formulation of the corresponding perturbed NLSE. To use it one needs an ansatz for the pulse solution, thus fixing up the parametrization of the pulse. Then, inserting it into the Lagrangian one is able to derive a set of dynamical equations for the evolution of the parameters. This method is more flexible than the KS approach in the sense that a larger class of initial pulses (e.g., chirped solitons[17]) can be considered. On the other hand it is limited by the requirement, that the perturbed NLSE has to be Hamiltonian, which is not necessary for the KS method.

Our aim in the present paper is to generalize the KS method to the case of N well separated interacting solitons with nearly or exactly equal amplitudes and velocities. In Sec. 2 we derive the generalization of the KS system for the N–soliton solution of the NLSE without perturbation. We prove, as it was conjectured in[18,19] that the interaction is of the nearest–neighbour type.

In Sec. 3 we obtain explicitly the effect of several classes of perturbations to the generalized KS system. We prove that the linear perturbations in u modify only the self–interaction terms for each of the solitons. The cubic perturbations in u give rise also to additional self–interaction terms between the neighboring solitons. We end up with some conclusions and open problems.

2 The N–soliton Karpman–Soloviev system

Here we consider the NLS equation, i.e. (1) with $R[u] = 0$. It is a completely integrable Hamiltonian system and can be solved with the help of the ISM applied to the so–called Zakharov–Shabat – AKNS system:

$$Lf(x,t,\lambda) \equiv \left(i\sigma_3 \frac{d}{dx} + Q(x,t)\right) f(x,t,\lambda) = \lambda f(x,t,\lambda), \tag{3}$$

where the potential $Q(x,t)$ is expressed in terms of $u(x,t)$ by:

$$Q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -u^*(x,t) & 0 \end{pmatrix}. \tag{4}$$

In what follows below we shall need a convenient parametrization for the one soliton solution and the corresponding eigenfunction of L:

$$u_{1s}(z,t) = \frac{2\nu e^{i\phi}}{\cosh z}, \qquad f(z,\lambda) = \frac{e^{i\lambda x}}{2\cosh z}\begin{pmatrix} -ie^{-i\phi} \\ e^{z} \end{pmatrix}, \qquad z = 2\nu(x-\xi(t)), \tag{5}$$

$$\xi(t) = 2\mu t + \xi_0, \qquad \delta(t) = 2(\mu^2+\nu^2)t + \delta_0, \qquad \phi(z,t) = \frac{\mu}{\nu}z + \delta(t). \tag{6}$$

By $\delta(t)$, $\xi(t)$ we have denoted the soliton phase and position respectively, and δ_0, ξ_0 determine their initial values for $t=0$; ν is the soliton amplitude and μ is its frequency. Let us also remind, that two eigenvalues of operator L given by $\lambda_k^{\pm} = \tilde{\mu}_k \pm i\tilde{\nu}_k$ correspond to the k–th soliton (with velocity $\tilde{\mu}_k = \mathrm{Re}\,\lambda_k$ and amplitude $\tilde{\nu}_k = \mathrm{Im}\,\lambda_k$).

Physically the most interesting initial configurations are those, which represent a superposition of equidistant pulses with nearly equal amplitudes and velocities $\nu_k - \nu_m \ll 1$ and velocities $\mu_k - \mu_m \ll 1$, that is:

$$u_0(x) = u_{\mathrm{Ns}}(x, t=0), \qquad u_{\mathrm{Ns}}(x,t) = \sum_{k=1}^{N} u_k(z_k,t) \tag{7}$$

where $u_k(z_k,t)$ is given by (5) with z, ϕ, ξ and δ replaced by z_k, ϕ_k, ξ_k and δ_k respectively.

We stress here, that generically the Zakharov–Shabat system with a potential fixed by the initial condition (7) possesses N pairs of eigenvalues $\lambda_k^{\pm}$ and some nontrivial scattering data on the continuous spectrum. It is a problem to determine the location of the eigenvalues corresponding to (7). But for well separated out of phase solitons we have $\nu_k \simeq \tilde{\nu}_k$ and $\mu_k \simeq \tilde{\mu}_k$, and only small percent of the energy comes from the continuous spectrum[9,13].

To our assumptions above we add one more: that the solitons are well separated, so that their overlap plays the role of the small parameter ϵ. Then the N–soliton solution can be well approximated by the sum of N one soliton terms as in (7). Mathematically these restrictions can be expressed as:

$$|\mu_k - \mu_n| \ll \mu, \qquad |\nu_k - \nu_n| \ll \nu, \qquad \mu = \frac{1}{N}\sum_{k=1}^{N}\mu_k,$$

$$\nu|\xi_{0k} - \xi_{0n}| \gg 1, \qquad |\nu_k - \nu_n||\xi_{0k} - \xi_{0n}| \ll 1, \qquad \nu = \frac{1}{N}\sum_{k=1}^{N}\nu_k. \tag{8}$$

Next we insert (7) into (1) with $R[u] = 0$. Due to the nonlinearity the k–th soliton will be influenced by the others. This influence will contain terms of first and second order with respect to the overlap ϵ. Thus we get:

$$iu_{k,t} + \frac{1}{2}u_{k,xx} + |u_k|^2 u_k = iR_k[u] + \tilde{R}_k[u] \tag{9}$$

where $R_k[u] = \sum_{n\neq k} \mathcal{R}_{kn}[u]$, $\tilde{R}_k[u] = \sum_{n\neq m\neq k} \tilde{\mathcal{R}}_{knm}[u]$, and

$$\mathcal{R}_{kn}[u] = 2|u_k|^2 u_n + u_k^2 u_n^*, \qquad \tilde{\mathcal{R}}_{knm}[u] = 2u_k u_m^* u_n + u_n u_m u_k^*, \tag{10}$$

It can be proved that all the terms in $\tilde{R}_k[u]$ are of order ϵ^2 and can be neglected.

Note that now we have no real perturbation; the terms $R_k[u]$ and $\tilde{R}_k[u]$ in the right hand side of (9) just take into account the fact that we deal with an approximation to the N–soliton solution.

Our first aim here will be to evaluate the effect of each of the summands $R_{k,n}$ in the r. h. side of (10) on the parameters of the k–th soliton. Next we will take into account additive perturbations. As a result we get:

$$\frac{d\nu_k}{dt} = N_k[u] + N_k'[u] + N_k''[u], \quad \frac{d\xi_k}{dt} = 2\mu_k + \Xi_k[u] + \Xi_k'[u] + \Xi_k''[u], \tag{11}$$

$$\frac{d\mu_k}{dt} = M_k[u] + M_k'[u] + M_k''[u], \quad \frac{d\delta_k}{dt} = p_k^2 + X_k[u] + X_k'[u] + X_k''[u], \tag{12}$$

where $p_k^2 = 2(\mu_k^2 + \nu_k^2)$, $X_k[u] = 2\mu_k \Xi_k[u] + D_k[u]$. The r.h.sides in (11)–(12) are determined by $R_k[u]$ (10) through:

$$N_k[u] = \mathrm{Re}\,\mathcal{F}\left(R_k[u]\right), \quad M_k[u] = \mathrm{Im}\,\mathcal{F}(f_1\left(R_k[u]\right), \tag{13}$$

$$\Xi_k[u] = \frac{1}{2\nu_k^2}\mathrm{Re}\,\mathcal{F}\left(f_2 R_k[u]\right), \quad D_k[u] = \frac{1}{\nu_k}\mathrm{Im}\,\mathcal{F}\left(f_3 R_k[u]\right), \tag{14}$$

$$\mathcal{F}(f_s R_k[u]) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k f_s(z_k)}{\cosh z_k} R_k[u] e^{-i\phi_k}, \tag{15}$$

with $f_1(z) = \tanh z$, $f_2(z) = z$, $f_3(z) = 1 - z\tanh z$. The terms $N'[u]$, $M'[u]$, $\Xi'[u]$, $D'[u]$ and $X'[u]$ (respectively $N''[u]$, $M''[u]$, $\Xi''[u]$, $D''[u]$ and $X''[u]$) are obtained by (13)–(14) by replacing $R[u]$ with the perturbation terms $R'[u]$ (respectively $R''[u]$) (see Section 3 below).

We insert (5) into (10) and calculate the contribution of each summand to the r.h.sides of (11) – (12). Thus we derive the following set of $4N$ equations, generalizing the Karpman–Soloviev system; for $R'[u] = R''[u] = 0$ we get:

$$\frac{d\nu_k}{dt} = 16\nu_k^2(S_{k,k-1} - S_{k,k+1}), \qquad \frac{d\mu_k}{dt} = -16\nu_k^2(C_{k,k-1} - C_{k,k+1}), \tag{16}$$

$$\frac{d\xi_k}{dt} = 2\mu_k - 4(S_{k,k-1} + S_{k,k+1}), \qquad \frac{d\delta_k}{dt} = p_k^2 + F_{k,k-1} + F_{k,k+1}, \tag{17}$$

where $F_{k,n} = 24\nu_k C_{k,n} - 8\mu_k S_{k,n}$,

$$S_{k,n} = e^{-|\beta_{kn}|}\nu_n \sin s_{kn}\phi_{0;kn}, \qquad C_{k,n} = e^{-|\beta_{kn}|}\nu_n \cos\phi_{0;kn}, \tag{18}$$

and $\beta_{kn} = 2\nu_n(\xi_k - \xi_n)$, $\phi_{0;kn} = \delta_k - \delta_n - 2\mu_n(\xi_k - \xi_n)$, $\beta_{10} = \beta_{N,N+1} = \infty$. Here we have assumed, that the solitons form a chain–like configuration of nearly equidistant solitons and that $\xi_k < \xi_{k+1}$; then the k–th soliton has as its nearest neighbours the $k-1$–st and $k+1$–st and $s_{k,k-1} = 1$ and $s_{k,k+1} = -1$.

3 The Perturbed Karpman–Soloviev System

Here we shall consider first the cases with:

$$R'[u]=\sum_{s=0}^{3}c_s\frac{d^s u}{dx^s},\qquad R''[u]=d_0|u|^2u+\frac{d_1}{4}u(|u|^2)_x+\frac{d_2}{4}(|u|^2u_x-u_x^*u^2),\ (19)$$

where $c_k = c_{k0} + ic_{k1}$ and $d_k = d_{k0} + id_{k1}$ are complex constants. Fixing up in a convenient way their values we can describe a number of physically important perturbations. Most of these terms have physical meaning and have already been analyzed in[22,24–30] for the $N = 1$ and $N = 2$ cases. The results are given by:

$$N_k'[u]=2\nu_k\left(c_{00}-2c_{11}\mu_k-4c_{20}\left(\tfrac{\nu_k^2}{3}+\mu_k^2\right)+8c_{31}\mu_k\left(\mu_k^2+\nu_k^2\right)\right),\ (20)$$

$$M_k'[u]=-\tfrac{4}{3}c_{11}\nu_k^2-\tfrac{16}{3}c_{20}\mu_k\nu_k^2+16c_{31}\nu_k^2\left(\mu_k^2+\tfrac{7}{15}\nu_k^2\right),\ (21)$$

$$\Xi_k'[u]=-c_{10}+4c_{21}\mu_k+4c_{30}\left(3\mu_k^2+\nu_k^2\right),\ (22)$$

$$X_k'[u]=c_{01}+4c_{21}\left(\mu_k^2-\nu_k^2\right)+16c_{30}\mu_k\left(\mu_k^2-\nu_k^2\right).\ (23)$$

As it is obvious from the above system, the linear perturbations in u provide only terms, which are 'local' in k. For the cubic perturbations in u we get:

$$N_k''[u]=\tfrac{16\nu_k^2}{3}\Big\{(d_{00}-d_{21}\mu_k)\nu_k+\sum_{n=k\pm1}\big[\left(3d_{00}-\tfrac{\nu_k}{3}d_{10}s_{kn}+d_{21}\mu_{kn}^+\right)C_{kn}+$$
$$+\left(d_{01}s_{kn}-d_{10}m_{kn}s_{kn}+d_{20}\mu_k s_{kn}+\tfrac{\nu_k}{3}(d_{11}-4d_{21})\right)S_{kn}\big]\Big\},\ (24)$$

$$M_k''[u]=-\tfrac{16\nu_k^2}{3}\Big\{\tfrac{\nu_k^2}{5}d_{11}+\sum_{n=k\pm1}\big[\left(3d_{01}+d_{11}\nu_k s_{kn}+d_{20}\mu_{kn}^+\right)s_{kn}C_{kn}-$$
$$-\left(d_{00}+(d_{10}-2d_{20})\nu_k s_{kn}+d_{11}m_{kn}-d_{21}\mu_k\right)S_{kn}\big]\Big\},\ (25)$$

$$\Xi_k''[u]=-\tfrac{2\nu_k^2}{3}d_{10}-4\sum_{n=k\pm1}\big\{\left[3d_{00}+d_{10}\nu_k s_{kn}-d_{21}\mu_{kn}^+\right]s_{kn}C_{kn}+$$
$$+\left[d_{01}+(d_{11}-2d_{21})\nu_k s_{kn}-d_{10}m_{kn}+d_{20}\mu_k\right]S_{kn}\big\},\ (26)$$

$$X_k''[u]=\tfrac{4\nu_k^2}{3}\left(3d_{01}+\mu_k(3d_{20}-d_{10})\right)+8\sum_{n=k\pm1}\left\{\delta_{kn}''C_{kn}-\zeta_{kn}''S_{kn}\right\},\ (27)$$

$$\delta_{kn}''=\left(3d_{01}-d_{10}\mu_k-d_{11}\nu_k s_{kn}+d_{20}\mu_{kn}^+\right)\nu_k+\left(d_{21}\mu_{kn}^+-3d_{00}\right)\mu_k s_{kn},$$
$$\zeta_{kn}''=d_{01}-d_{10}m_{kn}+d_{20}(\nu_k^2-\mu_k^2)+\left(d_{00}+d_{11}\mu_{kn}^-+3d_{21}\mu_k\right)\nu_k s_{kn},$$

with $\mu_{kn}^{\pm}=2\mu_k\pm\mu_n$, $m_{kn}=\mu_k-\mu_n$. Here, beside the self–interacting terms we get also nearest–neighbour interaction terms.

Finally we present the results for several perturbations, which are of particular physical interest. These include: $R_3[u]=(\tilde{c}x+\delta)u(x,t)+\beta u_{xx}$ which describes

bandwidth–limited amplification with sliding filters[11], the driving force perturbation: $R_4 = f_0 e^{i\Omega x} + f_1 e^{-i\Omega x}$, where we consider Ω to be a real and f_0 and f_1 – complex constants, and $R_5[u] = \left(f_0 e^{i\Omega x} + f_1 e^{-i\Omega x}\right) u(x,t)$, which describes the phase and amplitude modulation effects.

Skipping the calculations we collect the results in the following table:

	$R_3[u]$	$R_4[u]$	$R_5[u]$
$N_k[\cdot]$	$2\tilde{c}_0\nu_k\xi_k - \beta\sigma_k + 2\delta_k$	$\frac{\pi}{2\cosh\omega_k} H_+(\Omega\xi_k)$	$\frac{2\nu_k\omega_k}{\sinh\omega_k} H_+(\Omega\xi_k)$
$M_k[\cdot]$	$\frac{\tilde{c}_1}{2} - \frac{16}{3}\beta\mu_k\nu_k^2$	$\frac{\omega_k}{\cosh\omega_k} H_-(\Omega\xi_k)$	$\frac{2\nu_k\omega_k^2}{\pi\sinh\omega_k} H_-(\Omega\xi_k)$
$\Xi_k[\cdot]$	$\frac{\pi^2}{8\nu_k^2}\tilde{c}_0$	$-\frac{\pi^2\sinh\omega_k}{8\nu_k^2\cosh^2\omega_k} G_-(\Omega\xi_k)$	$\frac{\pi h(\omega_k)}{2\nu_k} G_-(\Omega\xi_k)$
$X_k[\cdot]$	$-\frac{\pi^2}{4\nu_k^2}\tilde{c}_0 + \xi_k\tilde{c}_1$	$\frac{\pi^2\sinh\omega_k}{8\nu_k^2\cosh^2\omega_k} G(\Omega\xi_k)$	$\frac{\pi\mu_k}{2\nu_k} + \frac{\pi g(\omega_k)}{4\nu_k} G(\Omega\xi_k)$

where $G(x) = \Omega G_+(x) - 2\mu_k G_-(x)$ and

$$H_\pm(x) = \mathrm{Re}\,(f_0 e^{ix} \pm f_1 e^{-ix}), \qquad G_\pm(x) = \mathrm{Im}\,(f_0 e^{ix} \pm f_1 e^{-ix}), \quad (28)$$

$$h(x) = \tfrac{1-x\coth x}{\sinh x}, \qquad g(x) = \tfrac{x\cosh x}{\sinh x}, \qquad \sigma_k = 8\nu_k\left(\tfrac{\nu_k^2}{3} + \mu_k^2\right), \quad (29)$$

A substantial difference between the linear and nonlinear interactions is that the latter change the character of nearest neighbour interaction.

In conclusion we have derived the generalized KS system in the presence of rather general perturbation terms. The system itself (even without perturbations), unlike the 2–soliton case, is rather complicated and can not be solved analytically. This calls for applying other methods, which would allow us to calculate its large t asymptotics.

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PERIODIC PRESERVING DEFORMATIONS OF THE FINITE-GAP SOLUTIONS OF THE SOLITON EQUATIONS [a]

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Ordinary differential equations on the moduli space of finite-gap Riemann surfaces are studied. These flows do not change the frequences of the corresponding finite-gap solutions therefore they leave invariant the subspace of Riemann surfaces generating periodic solutions. Rational form of these differential equations is suggested. We show that these flows are gradient in some flat Riemann metric. Coordinates on the space of all Riemann surfaces generating periodic solutions are introduced.

In the periodic theory of the soliton equations we always meet the following problem. The direct scattering transform is well-defined for the periodic data and no natural extension of this transform to the quasiperiodic potentials is known. Considering the inverse scattering transform we see that the general finite-gap data result in quasiperiodic potentials. Riemann surfaces corresponding to the periodic potentials may be characterized in the moduli space of all Riemann surfaces of given genus by a set of transcendental equations (4), (5) (the ratios of all periods of the quasimomentum differential are rational). To study this transcendental submanifold we use the following approach. We consider ordinary differential equations (ODE's) on the space of Riemann surfaces such that all frequences of the corresponding finite-gap solutions are integrals of motion for these ODE's (see formula (6) below. Such flows arose earlier in [1] where connections between topological quantum field theory and the

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Whitham equations were studied and in [2] in the theory of periodic nonintegrable perturbations of the soliton equations. For $1+1$ system it is natural to parameterize the Riemann surfaces by the branch points. Differential equations on the branch points have transcendental right hand side. We prove the following properties of these flows:

1) These equations can be imbedded to a bigger system of ODE's with rational right-hand side. Rational representation allows authors to write an effective program for numerical experiments.

2) These flows are gradient in the flat Riemann metric, playing an important role in the hamiltonian theory of the Whitham equations [3]. (This fact was pointed out to the authors by S.P.Novikov). It means that they can be written in the following form:

$$\frac{\partial u^k}{\partial t^j} = g^{kl}\frac{\partial H^j}{\partial u^l},\ g^{kl} = g^{lk} \tag{1}$$

where g^{kl} is the inverse matrix to a flat pseudo-Riemann metric, H^j are proportional to the periods of the quasimomentum differential. Representation (1) is rather similar to the standard Hamilton representation, but the matrix g^{kl} in (1) is symmetric instead of a skew-symmetric one. It is very important that the metric g^{kl} is not positive definite. For a positive definite metric the function generating the flow can not be a conservation law, but in our situations all the differentials dH^j, lie in the "light cone" of the metric and they are integrals of motion. It is possible to rewrite these equations in the Hamilton form with the same hamiltonians H^j, but the corresponding Poisson brackets looks less natural then the symmetric structure.

This technique is used by the authors to give a parameterization of the space of the all Riemann surfaces corresponding to the periodic potentials.

This approach is rather general. But for the sake of concreteness we will consider only one example which was the most important for us - the Nonlinear Schrödinger equations (NLS)

The auxiliary scattering problem for the (NLS) reads as:

$$L\Psi = 0,\ L = \partial_x - \begin{bmatrix} i\lambda & q \\ r & -i\lambda \end{bmatrix},\ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \tag{2}$$

Reduction $r = -\bar{q}$ corresponds to the self-focusing case, $r = \bar{q}$ to the defocusing one.

Let $q(x)$ be periodic with the period 1 $q(x+1) = q(x)$. The Bloch-Floquet eigenfunction $\Psi(x,\lambda)$ is defined as the common eigenfunction of L and the

monodromy operator:

$$L\Psi = 0,\ \Psi(x+1,\lambda) = \exp[ip(\lambda)]\Psi(x,\lambda),\ \Psi_1(0,\lambda) + \Psi_2(0,\lambda) = 1 \quad (3)$$

and is meromorphic on a two-sheeted Riemann surface Γ over the λ - plane with two points over ∞ - ∞_1 and ∞_2. The branch points of Γ and the poles of Ψ in λ are the scattering data. Detailed description of the scattering data for general periodic potentials can be found in [4]. The function $p(\lambda)$ is a locally holomorphic multifunction on the finite part of Γ. Finite-gap potentials correspond to algebraic Riemann surfaces (i.e. surfaces with finite number of branch points). Only such potentials will be discussed later.

The differential dp is uniquely defined by the following properties:

1) $dp(\lambda)$ is holomorphic on $\Gamma \setminus (\infty_1 \cup \infty_2)$.

2) $dp(\lambda) = \pm d\lambda + o(1)$ as $\lambda \to \infty_{1,2}$.

3) $\mathrm{Im} \oint_c dp = 0$ for any close cycle c.

For the real reductions of NLS condition 3) is equivalent to

3') $\oint_{a_k} dp = 0$ where $a_1, \ldots, a_g$ are the basic a - cycles.

Periodicity properties of $q(x)$ in x do not depend on the position of the poles of $\Psi(x,\lambda)$ and are completely determined by Γ. Let Γ be an arbitrary hyperelliptic Riemann surface of genus g with $2g+2$ finite pairwise different simple branch points $u^1, \ldots, u^{2g+2}$. The corresponding finite-gap potentials $q(x)$, $r(x)$ are periodic with the period P_* if and only if

$$P_* K^j \in 2\pi \mathbf{Z} \quad (4)$$

where

$$K^j = \oint_{a_j} dp,\ K^{g+j} = \oint_{b_j} dp,\ K^{2g+1} = \lim_{\substack{\gamma_1 \to \infty_1 \\ \gamma_2 \to \infty_2}} \left[\int_{\gamma_1}^{\gamma_2} dp - \lambda(\gamma_1) - \lambda(\gamma_2) \right]. \quad (5)$$

where the integral is taken along some path connecting the neighborhoods of the points ∞_1 and ∞_2,

$$dp = \frac{Q(\lambda)d\lambda}{\sqrt{P(\lambda)}},\ P(\lambda) = (\lambda - u^1)\ldots(\lambda - u^{2g+2}),\ Q(\lambda) = \lambda_{g+1} + \sum_{j=1}^{g} c_j \lambda^j,$$

the coefficients c_j are defined by 3). It is convenient to write $Q(\lambda)$ in terms of its roots $Q(\lambda) = (\lambda - \alpha_1)\ldots(\lambda - \alpha_{g+1})$.

Our approach is based on the following Lemma (see [4]):

Lemma 1 *Let $q(x)$, $r(x)$ be finite-gap periodic potentials, corresponding to a Riemann surface Γ, $\delta q(x)$, $\delta r(x)$ be arbitrary perturbations with the same period. Consider the corresponding perturbation of Γ (it may lie outside the finite-gap class). Then the differential*

$$\omega = \frac{\partial p}{\partial \tau} d\lambda - \frac{\partial \lambda}{\partial \tau} dp$$

is holomorphic on $\Gamma \setminus (\infty_1 \cup \infty_2)$ and has at most first order poles in the points ∞_1, ∞_2, where τ is the parameter of the deformation. We shall call such differentials weakly meromorphic.

It is rather natural to assume that $d\lambda/d\tau = 0$. Then

$$\frac{\partial dp}{\partial \tau} = \frac{\partial \omega}{\partial \lambda}, \text{ where } \omega = \nu d\lambda, \ \frac{\partial \omega}{\partial \lambda} = \frac{\partial \nu}{\partial \lambda} d\lambda, \ \nu = \frac{o(\lambda)}{\sqrt{P(\lambda)}} d\lambda, \tag{6}$$

$o(\lambda)$ is some polynomial of degree g. Formula (6) recalls the Flaschka-Forest-McLaughlin representation for the Whitham equations [5].

Lemma 2 *Let ω be an arbitrary weakly meromorphic differential on Γ. Then (6) generates an isoperiodic flow. In terms of the branch points it reads as*

$$\frac{\partial u^k}{\partial \tau} = -\frac{o(u^k)}{Q(u^k)}. \tag{7}$$

Proof of the Lemma 2. ν is a single valued function on Γ vanishing in the points ∞_1, ∞_2 thus all the periods of $\partial\omega/\partial\lambda$ are zero and the derivatives of all periods of dp are zero too.

Equations (7) has transcendental right-hand side because the coefficients c_i of $q(\lambda)$ are expressed via $u^1 \dots u^{2g+2}$ using the hyperelliptic integrals 3).

Theorem 1 *System (7) is equivalent to the following system*

$$\frac{\partial u^k}{\partial t} = -\frac{o(u^k)}{Q(u^k)},$$

$$\frac{\partial \alpha_k}{\partial t} = \frac{1}{\prod\limits_{j \neq k} (\alpha_k - \alpha_j)} \left[o'(\alpha_k) - \frac{1}{2} \left(\sum_{j=0}^{2g+2} \frac{1}{\alpha_k - \lambda_j} \right) o(\alpha_k) \right]. \tag{8}$$

if in the starting point $\tau = 0$ the data u^k and α_l are connected by 3).

The basis in the space of weakly meromorphic differentials can be chosen in the following natural way:

$$\omega_k = \frac{c_k}{\lambda - \alpha_k} dp, \ k = 1, \dots, g+1, \ c_k^2 = \frac{P(\alpha_k)}{\prod\limits_{j \neq k} (\alpha_k - \alpha_j)^2}.$$

Then

$$\frac{\partial u_j}{\partial \tau^k} = -\frac{c_k}{u^j - \alpha_k}, \quad \frac{\partial \alpha_j}{\partial \tau^k} = -\frac{c_k}{\alpha_j - \alpha_k} \text{ for } j \neq k,$$

$$\frac{\partial \alpha_k}{\partial \tau^k} = \sum_{j \neq k} \frac{c_j}{\alpha_j - \alpha_k} - \frac{1}{2} \sum_{j=1}^{2g+2} \frac{c_k}{u^j - \alpha_k}, \tag{9}$$

$$\frac{\partial c_l}{\partial \tau^k} = \frac{1}{c_l} \left[\frac{1}{2} \sum_j \frac{\frac{\partial \alpha_l}{\partial \tau^k} - \frac{\partial u^j}{\partial \tau^k}}{\alpha_l - u^j} - \sum_{j \neq l} \frac{\frac{\partial \alpha_l}{\partial \tau^k} - \frac{\partial \alpha_j}{\partial \tau^k}}{\alpha_l - \alpha_j} \right]$$

Flows corresponding to different k commute thus we have the action of $\mathbf{C}^{g+1}$ on our moduli space but with singular points. Using these flows we can introduce the following local coordinates on the space of the Riemann surfaces with fixed set of quasimomentum periods.

$$x_k = p(\tilde{\alpha_k}), \ k = 1, \ldots, g+1, \text{ where} \tag{10}$$

$\tilde{\alpha_k}$ in one of 2 points in Γ such that projection $\tilde{\alpha_k}$ to the λ plane coincides with $\tilde{\alpha_k}$ Similar coordinates were used in [6] for real KdV and by Krichever for general situation (private communication). From (9) it follows that

$$\frac{\partial x_k}{\partial \tau^j} = \delta_{kj}$$

thus the flows (9) describe coordinate shifts in (10).

Assume now that dp is normalized by 3'), $\omega_1 \ldots \omega_g$ are basic holomorphic differentials

$$\oint_{a_k} \omega_j = 2\pi\sqrt{-1}\delta_{jk} \tag{11}$$

ω_{g+1} is meromorphic on Γ with zero a-periods and first-order poles in ∞_1, ∞_2 such that

$$\text{res}_{\infty_2} \omega_j = 1.$$

Theorem 2 *The flows, corresponding to the differentials ω_j can be written in the form (1) where $g^{kl} = g^{kk}\delta^{kl}$ is the inverse matrix to the diagonal flat metric arising in the hamiltonian theory of the Whitham equations* [3,7]

$$g_{ii} = -res \Big|_{\lambda = u_i} \frac{(dp)^2}{d\lambda}, \ H^j = K^{j+g}. \tag{12}$$

$$g^{kl} \frac{\partial K^\alpha}{\partial u^k} \frac{\partial K^\beta}{\partial u^l} = 0 \text{ for all } \alpha, \beta,$$

thus all K^j are conservation laws for the system (1).

If $q(x)$, $r(x)$ are periodic with the period 1 the coordinates x_k are defined up to adding $2\pi n$, $n \in \mathbf{Z}$ and up to multiplication to -1 corresponding to 2 possible choices of $\tilde{\alpha_k}$ in Γ. Thus the functions $\Delta_k = 2\cos x_k$ are single-valued but they do not determine the Riemann surface uniquely and we have to add extra discrete data. One way to do it is the following.

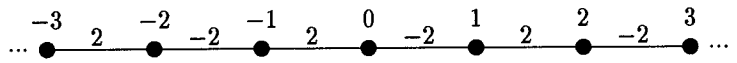
The function $\Delta(\lambda) = 2\cos p(\lambda)$ maps $\mathbf{C}^1$ to $\mathbf{C}^1$. The inverse function $\lambda(\Delta)$ is defined on a Riemann surface Y with infinitely many sheets and the branch points $-2, 2, \Delta_1, \ldots, \Delta_{g+1}$ (in fact there are infinitely many branch points over 2 and -2). Consider the situation of general position when all the branch points are of order 2 and the real part of the branch points $\Delta_1, \ldots, \Delta_{g+1}$ are pairwise distinct and not equal to $2, -2$. We can draw a system of cuts from the branch points to $i\infty$ parallel to the imaginary axes. For $|\lambda|$ sufficiently large $p(\lambda) = \lambda + o(\lambda)$ thus the sheets may be uniquely enumerated from the following condition: $Re\Delta(\lambda) \to \frac{\pi}{2} + \pi n$ as $\lambda \to -i\infty$ along the imaginary axes on the sheet n.

Consider the following graph. The vertices of the graph correspond to the sheets of Y, the edges to the branch points connecting these sheets, the numbers of the sheets are written on the vertices, the positions of the branch points are written on the edges.

This graph has the following properties:

1) It is a tree (connected graph without loops)

2) For $|n|$ sufficiently large the graph coincides with the standard graph corresponding to the zero potential.



In fact if we know the enumeration of vertices at one end we can uniquely reconstruct the whole enumeration.

Theorem 3 *In the points of general position we have one-to-one correspondence between the graphs with properties* 1), 2) *and the finite-gap Riemann surfaces generating periodic NLS solutions.*

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INTEGRABLE BOUNDARY CONDITIONS FOR EVOLUTION EQUATIONS

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Boundary value problems for integrable nonlinear partial differential equations are considered from the symmetry point of view. Families of boundary conditions compatible with an infinite number of generalized symmetries of the Burgers, Harry-Dym, KdV, mKdV equations are found. We also discuss the nonlinear Schrödinger and Boussinesq equations.

1 Introduction

In this work [1], we developed a method to construct the boundary value problems of the form

$$u_t = f(u, u_1, u_2, ..., u_n) \tag{1}$$

$$\mathbf{p}(u, u_1, u_2, ..., u_{n-1}) = 0 \ \text{ at } \ x = x_0 \tag{2}$$

completely compatible with the integrability property of equation (1). Here $u = u(x,t)$, $u_t = \partial u/\partial t$, $u_i = \partial^i u/\partial x^i$ and f is a scalar or vector field. It is well-known [2,3] that for some classes of completely integrable nonlinear evolution equations (1), there exist boundary conditions of the form (2) compatible with the inverse scattering transform or any other attribute of integrability. The following remarks can be extended to the vector field case of f.

Let the equation

$$u_\tau = g(u, u_1, ..., u_m) \ , \tag{3}$$

where $u_\tau = \partial u/\partial \tau$, for a fixed value of m, be a symmetry [4,5,6] of the equation (1). Now introducing new dynamical variables $\mathbf{v} = (u, u_1, u_2, ..., u_{n-1})$ we can

pass to a system of n-equations. All higher x-derivatives of u can be determined from the original equation (1) in terms of the new dynamical variables and their t-derivatives. So the symmetry (3) takes the form

$$\mathbf{v}_\tau = \mathbf{G}(\mathbf{v}, \mathbf{v}_t, \mathbf{v}_{tt}, ..., \mathbf{v}_{tt...t}) \ . \tag{4}$$

We call the boundary value problem defined by equations (1) and (2) as compatible with the symmetry (3) if the constraint $\mathbf{p}(\mathbf{v}) = 0$ (or the constraints $p^\alpha = 0$, $\alpha = 1, 2, ..., N$, N is the number of the constraints), is consistent with the τ-evolution

$$\frac{\partial \mathbf{p}}{\partial \tau} = 0 \ , (\text{mod } \mathbf{p} = 0) \tag{5}$$

Equation (5), by virtue of the equations in (4) must be automatically satisfied. In fact (5) means that the constraint $\mathbf{p} = 0$ defines an invariant surface in the manifold with local coordinates $\mathbf{v}$, or a compatible boundary condition with (4). This definition of compatibility is very close to the one introduced in Ref.7 but not identical.

We call the boundary condition (2) as compatible with the equation (1) if it is compatible with at least one of its higher symmetries. Here comes our main observation saying that if a boundary condition is compatible with one higher symmetry then it is compatible with an infinite number of symmetries. Also note that, all known boundary conditions of the form (2) consistent with the inverse scattering method are compatible with an infinite series of generalized (higher) symmetries. We propose a method to construct boundary conditions of the form (2), compatible with the time-independent generalized local symmetries of integrable nonlinear partial differential equations.

2 Boundary Conditions Compatible with Symmetries

In the sequel we suppose that equation (1) admits a recursion operator of the form [5,8],

$$R = \sum_{i=0}^{m} \alpha_i D^i + \sum_{i=0}^{k} \alpha_{-1_i} D^{-1} \alpha_{-2_i} \ , \ m \geq 0 \ , \ k \geq 0 \tag{6}$$

where $\alpha_i, \alpha_{-1_i}, \alpha_{-2_i}$ are functions of the dynamical variables, D is the total derivative with respect to x. Passing to the new dynamical variables $\mathbf{v}$, $\mathbf{v}_t$, $\mathbf{v}_{tt}$, ... we can obtain, by using (6), the recursion operator of the system of equations (4) takes the form [9]:

$$\mathbf{R} = \sum_{i=0}^{M} \mathbf{a}_i (\partial_t)^i + \sum_{i=0}^{K} \mathbf{a}_{-\mathbf{1}_i} (\partial_t)^{-1} \mathbf{a}_{-\mathbf{2}_i} \ , \ M \geq 0 \ , \ K \geq 0 \tag{7}$$

where the coefficient matrices $\mathbf{a}_i$, $\mathbf{a}_{-1_i}$ and $\mathbf{a}_{-2_i}$ depend on $\mathbf{v}$ and on a finite number of its t-derivatives, ∂_t is the operator of total derivative with respect to t. If (1) is a scalar equation, R is a scalar operator, then $\mathbf{R}$ is an $n \times n$ matrix valued operator. To find boundary conditions compatible with a higher symmetry $\mathbf{v}_\tau = \mathbf{R}^{n_0}\,\mathbf{v}_t$ of the equation (4), we need the coefficient matrix $\mathbf{b_N}$ in the expression

$$\mathbf{R}^{n_0} = \mathbf{b_N}(\partial_t)^N + \mathbf{b_{N-1}}(\partial_t)^{N-1} + ... \tag{8}$$

to be a scalar multiple of the identity matrix. The rest of this work depends highly on the following proposition:

Proposition .1 *Suppose* $\mathbf{p(v)} = 0$ *is a constraint of rank* $n-1$ *for the equation (4), and* $\mathbf{R}$ *be the recursion operator (7). Assume that* $\mathbf{p(v)} = 0$ *is compatible with a higher symmetry* $\mathbf{v}_\tau = \mathbf{R}^{n_0}\,\mathbf{v}_t$. *Then it is compatible with* $\mathbf{v}_\tau = H(\mathbf{R}^{n_0})\,\mathbf{v}_t$ *where* H *is a scalar polynomial with constant coefficients.*

Proof See Ref.9 .

2.1 Boundary Conditions for the Burgers Equation

First of all let us give the Burgers equation as an example since it is the easiest one. It is

$$u_t = u_{xx} + 2uu_x \tag{9}$$

which possesses the recursion operator [6,10]

$$R = D + u + u_x\,D^{-1}\,, \tag{10}$$

It may easily be proved that any symmetry of odd order of (9), i.e. $u_\tau = u_{2m+1} + h(u_{2m}, ..., u)$, does not possess an invariant surface because the correspondent system of equations has different orders in the highest-order t-derivatives

$$u_\tau = \partial_t^m\,u_1 + ...\ ,\quad u_{1\tau} = \partial_t^{m+1}\,u + ...\ . \tag{11}$$

But unlike the symmetries of odd order, for the symmetries of even order the correspondent system of equations has the same orders in the highest t-derivatives. This leads us to show that the symmetries of even order admit an invariant surface $p(u, u_1) = 0$, depending upon two arbitrary parameters.

Proposition .2 *If the boundary condition* $p(u, u_1) = 0$ *is compatible with an even order higher symmetry of the Burgers equation, then it is of the form* $c(u_1 + u^2) + c_1 u + c_2 = 0$ *and is compatible with every symmetry of the form* $u_\tau = P(\mathbf{R}^2)u_t$ *where* P *denotes any polynomial with constant coefficients.*

Proof See Ref.9 .

3 Applications to Other Partial Differential Equations

3.1 The Nonlinear Schrödinger Equation

Our first equation is the following system

$$\begin{aligned} u_t &= u_2 + 2u^2 v \\ -v_t &= v_2 + 2uv^2 \end{aligned} \tag{12}$$

which is, under the substitution $v \cdot\!\!\rightarrow u^*$ and $t \rightarrow it$, equivalent to the well-known nonlinear Schrödinger equation, where $*$ is the complex conjugation.

Since we are dealing with a system of two equations, the initial recursion operator will be the following 2×2 operator matrix

$$R = \begin{pmatrix} D + 2uD^{-1}v & 2uD^{-1}u \\ -2vD^{-1}v & -D - 2vD^{-1}u \end{pmatrix} \tag{13}$$

with respect to the column vector $(u_\tau, v_\tau)^T$, where T is the transposition operation. Our dynamical variables are u, u_1, v, v_1. Here u_1, v_1 denote u_x, v_x respectively. Trivially higher derivatives of u and v can be represented in terms these dynamical variables and their t-derivatives, with the aid of the system (12). Now after transforming R into t dependent form with respect to the above dynamical variables, we get the following 4×4 matrix

$$\mathbf{R} = \begin{pmatrix} -2u\partial_t^{-1}v_1 & 1 + 2u\partial_t^{-1}v & 2u\partial_t^{-1}u_1 & -2u\partial_t^{-1}u \\ \xi - 2u_1\partial_t^{-1}v_1 & 2u_1\partial_t^{-1}v & 2u_1\partial_t^{-1}u_1 & -2u_1\partial_t^{-1}u \\ 2v\partial_t^{-1}v_1 & -2v\partial_t^{-1}v & -2v\partial_t^{-1}u_1 & -1 + 2v\partial_t^{-1}u \\ 2v_1\partial_t^{-1}v_1 & -2v_1\partial_t^{-1}v & \eta - 2v_1\partial_t^{-1}u_1 & 2v_1\partial_t^{-1}u \end{pmatrix}$$

where $\xi = \partial_t - 2uv$ and $\eta = \partial_t + 2uv$. To obtain the coefficient matrix of the highest power of ∂_t as a scalar multiple of the identity matrix, we shall square $\mathbf{R}$, then apply to the column matrix $(u, u_1, v, v_1)^T$. Then we get a system of four equations which is supposed to admit a boundary condition of the form $u_1 = p^1(u, v)$, $v_1 = p^2(u, v)$ at $x = x_0$. So for nonlinear Schrödinger equation we get the compatible boundary conditions to be

$$u_1 = p^1 = cu \text{ and } v_1 = p^2 = cv \text{ at } x = x_0 \,. \tag{14}$$

The analytical properties of this boundary value problem are studied previously by means of the inverse scattering technique [2,11,12].

3.2 The Harry-Dym Equation

Among the integrable nonlinear partial differential equations, the Harry-Dym equation

$$u_t + u^3 u_3 = 0 \tag{15}$$

is of special interest. It is not semi-linear and so its analytical properties are not typical. It has the recursion operator

$$R = u^3 D^3 u D^{-1} \frac{1}{u^2} \ , \tag{16}$$

given in Ref.13 . For the Harry-Dym equation, the new dynamical variables are u, u_1, u_2 but passing to this set of variables is not regular since $u_3 = -u_t/u^3$ has a singular surface given by $u = 0$ which should be examined separately. The reflection symmetry $x \to -x$, $u \to -u$, $t \to t$ exists in the Harry-Dym equation itself and its all higher symmetries so the trivial boundary condition $u(0,t) = 0$ is consistent with the integrability.

The transformed recursion operator $\mathbf{R}$ is given by the matrix

$$\mathbf{R} = \begin{pmatrix} uw + u_t\,\partial_t^{-1}\,w & -uv - u_t\,\partial_t^{-1}\,v & u^2 + u_t\,\partial_t^{-1}\,u \\ \xi + vw + v_t\,\partial_t^{-1}\,w & -v^2 - v_t\,\partial_t^{-1}\,v & uv + v_t\,\partial_t^{-1}\,u \\ w^2 + w_t\,\partial_t^{-1}\,w & \xi - vw - w_t\,\partial_t^{-1}\,v & uw + w_t\,\partial_t^{-1}\,u \end{pmatrix}$$

where $v = u_x$, $w = u_{xx}$ and $\xi = \frac{1}{u}\,\partial_t - \frac{u_t}{u^2}$. In this case to have the coefficient matrix of highest order ∂_t as a scalar multiple of the identity matrix, we should cube $\mathbf{R}$. Now we can assume a boundary condition $p(u,v,w) = 0$ compatible with the ninth order symmetry given by

$$(u,v,w)_\tau^T = \mathbf{R}^3\,(u,v,w)_t^T \ . \tag{17}$$

We shall note that for the constraint $p = 0$ we have two choices of its rank; either one or two. If it is one; we don't have any regular invariant surface. The second choice leads to the invariant surfaces

$$u_x|_{x=x_0} = cu \ , \ u_{xx}|_{x=x_0} = \frac{c^2 u}{2} \tag{18}$$

which is compatible with an infinite number of symmetries.

3.3 The Korteweg de Vries and the Modified Korteweg de Vries Equations

Now we will consider the well-known Korteweg de Vries (KdV) equation. It is the following equation

$$u_t = u_3 + 6uu_1 \tag{19}$$

possessing the recursion operator

$$R = D^2 + 4u + 2u_1 D^{-1} \,. \tag{20}$$

This recursion operator R may be represented in the form

$$\mathbf{R} = \begin{pmatrix} 4u + 12v\,\partial_t^{-1}\,u & 0 & 1 + 2v\,\partial_t^{-1} \\ \partial_t + 12w\,\partial_t^{-1}\,u & -2u & 2w\,\partial_t^{-1} \\ 2w + 12(u_t - 6uv)\,\partial_t^{-1}\,u & \partial_t - 2v & -2u + 2(u_t - 6uv)\,\partial_t^{-1} \end{pmatrix}$$

To apply the technique we shall take the cube of $\mathbf{R}$, i.e. we are looking for the invariant surface of the system

$$(u, v, w)_\tau^T = \mathbf{R}^3\,(u, v, w)_t^T \,, \tag{21}$$

where $v = u_1$ and $w = u_2$ which are our new dynamical variables with u itself. It is straightforward to determine the boundary condition compatible with the symmetries is

$$u|_{x=x_0} = 0,\ w|_{x=x_0} = 0 \tag{22}$$

and of course v is any function of t at $x = x_0$.

The modified KdV equation

$$u_t = u_3 + 6u^2 u_1 \tag{23}$$

can be handled very similar. Its recursion operator is the following operator

$$R = D^2 + 4u^2 + 4u_1 D^{-1}\,u \,. \tag{24}$$

We can write R in the following matrix form with same new dynamical variables of KdV

$$\mathbf{R} = \begin{pmatrix} 4u^2 + 24v\,\partial_t^{-1}\,u^3 & -4v\,\partial_t^{-1}\,v & 1 + 4v\,\partial_t^{-1}\,u \\ \partial_t + 24w\,\partial_t^{-1}\,u^3 + 4w\,\partial_t^{-1}\,w & -2u^2 - 4w\,\partial_t^{-1}\,v & 4w\,\partial_t^{-1}\,u \\ 4\xi\,\partial_t^{-1}\,w + 24\xi\,\partial_t^{-1}\,u^3 + 4uw & \partial_t - 4uv - 4\xi\,\partial_t^{-1}\,v & -2u^2 + 4\xi\,\partial_t^{-1}\,u \end{pmatrix}$$

where $\xi = u_t - 6u^2 v$. Also in the modified KdV equation we will work with $\mathbf{R}^3$. Here the consistent boundary conditions turn out to be

$$u|_{x=x_0} = 0,\ u_x|_{x=x_0} = 0 \tag{25}$$

and $u_{xx}|_{x=x_0}$ is any function of t.

3.4 The Boussinesq Equation

The Boussinesq equation

$$u_{tt} = \frac{1}{3}u_{xxxx} + \frac{4}{3}(u^2)_{xx} \tag{26}$$

can be converted into an equivalent evolutionary system [6],

$$\begin{aligned} u_t &= v_x \\ v_t &= \tfrac{1}{3}u_{xxx} + \tfrac{8}{3}uu_x \,. \end{aligned} \tag{27}$$

This is a very typical system since the orders of the x-derivatives of the equations are not equal. It has two Hamiltonian operators given in Ref.6 which lead us to the determine the recursion operator R as the following

$$R = \begin{pmatrix} 3v + 2v_1 D^{-1} & D^2 + 2u + u_1 D^{-1} \\ \frac{1}{3}D^4 + \frac{10}{3}uD^2 + 5u_1 D + 3u_2 + \frac{16}{3}u^2 + \xi D^{-1} & 3v + v_1 D^{-1} \end{pmatrix}$$

where u_i and v_i denote corresponding i-th order x-derivatives as usual and $\xi = \frac{2}{3}u_3 + \frac{16}{3}uu_1$. Defining new dynamical variables u, z, w, v such that $z = u_1$ and $w = u_2$ we can find the transformed recursion operator $\mathbf{R}$, which is a 4×4 matrix operator. It is

$$\mathbf{R} = \begin{pmatrix} 3v + 8L_1 u & \partial_t & L_1 & 6K + 2u \\ \zeta + 8Su & 3v & S + \partial_t & 3z + 6L_2 \\ \eta & \mu & \kappa & \gamma \\ \delta + 8K & -\frac{1}{3}z & \frac{2}{3}u + K & 2v_t\partial_t^{-1} + 3v \end{pmatrix}$$

where

$$\begin{aligned} \eta &= 7z_t + 8(v_t - \tfrac{8}{3}uz)\partial_t^{-1}\, u - 8\partial_t z + 5z\partial_t & S &= \tfrac{1}{3}w\partial_t^{-1} \\ \mu &= 8u_t + 2u\partial_t - 8\partial_t\, u & L_2 &= \tfrac{1}{3}z_t\partial_t^{-1} \\ \gamma &= 4w + 3\partial_t^2 + 2w_t\partial_t^{-1} & \zeta &= 5u_t + 2u\partial_t \\ \delta &= \partial_t^2 + \tfrac{1}{3}w + \tfrac{16}{3}u^2 & L_1 &= \tfrac{1}{3}z\partial_t^{-1} \\ \kappa &= 3v + (v_t - \tfrac{8}{3}uz)\partial_t^{-1} & K &= \tfrac{1}{3}u_t\partial_t^{-1} \end{aligned}$$

In this case the $\mathbf{R}^3$ works, i.e. the invariant surface for system

$$(u, z, w, v)_\tau^T = \mathbf{R}^3\,(z, w, 3v_t - 8uz, u_t)^T \tag{28}$$

will be searched. Here the first symmetry used for the method is different from the previous ones since it is the suitable symmetry. We found the compatible boundary conditions in the form $u = c_1$, $z = c_2$, $w = c_3$, and $v = c_4$ where

c_i are arbitrary constants $\forall i = 1,2,3,4$, subject to satisfy some constraints. With respect to these constraints we have three distinct boundary conditions

$$\begin{array}{ll} i) & u = c_1,\ z = c_2,\ w = c_3,\ v = 0 \\ ii) & u = c_1,\ z = 0,\ w = 0,\ v = c_4 \\ iii) & u = c_1,\ z = c_2,\ w = c_3,\ v = c_4, \text{ where} \\ & c_4^2 = \frac{1}{7}c_2^2 - \frac{16}{21}c_1^3 - \frac{18}{35}c_3c_1 \text{ when } c_4 \neq 0,\ c_2 \neq 0 \text{ or } c_4 \neq 0,\ c_3 \neq 0 \end{array} \tag{29}$$

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MULTI-COMPONENT INTEGRABLE SYSTEMS AND NONASSOCIATIVE STRUCTURES

Dedicated to the memory of S.I. Svinolupov

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Some classification results concerning integrable multi-component evolution equations are presented. They are naturally formulated in terms of nonassociative algebraic structures and their deformations. A number of new examples of integrable evolution, hyperbolic and differential-difference multi-component equations and integrable boundary conditions for them are given.

1 Introduction

This survey contains all main results of Sergey Svinolupov concerning the classification of integrable multi-component systems. He intended to write a review article on the subject for the Proceedings, but his tragic death destroyed the plan in embryo. We, being coauthors of some of his works, are trying to do something instead of him.

Sometime, we formulate less general results than in original papers. Some results of Sections 3 and 5 are published for the first time.

The symmetry approach to the classification of integrable equations is based on observation (see [62,63,68]) that both linearizable equations and equations, integrable by the inverse scattering method, possess higher symmetries. The approach turns out to be most efficient for evolution equations of the form

$$\vec{u}_t = A\vec{u}_n + \vec{F}(\vec{u}, \vec{u}_x, \dots, \vec{u}_{n-1}), \tag{1}$$

where A is a constant matrix, $\vec{u} = (u^1, \dots, u^N)^t$, $\vec{u}_i = \partial^i \vec{u}/\partial x^i$.

The simplest case $N = 1$ was intensively investigated in 1979–1985 t [64,67,1,65,3,4,5,71]. It turns out that the existence of the higher symmetry

$$u_\tau = G(u, u_x, \dots, u_m), \qquad m > n,$$

implies the solvability of a triangular chain of equations of the form

$$D_x(X_{i+1}) = f_i(F, X_1, \dots, X_i), \qquad i = 1, \dots, m-n, \tag{2}$$

where D_x is the total derivative operator for equation (1 and X_i are local functions (i.e. functions depending on a finite number of variables $u, u_1, u_2, \ldots$). As the equation $D_x(X) = f$ has a local solution only if $\delta f/\delta u = 0$, the chain (2) leads to a set of conditions to which the rhs of (1) must satisfy. Using these integrability conditions, Svinolupov and Sokolov (see [4,12,6,3,66]) have found all integrable equations of the form $u_t = F(t, x, u, u_x, u_{xx})$ and scalar integrable equations (1) for $n = 3, 4, 5$.

R.I.Yamilov generalized the symmetry approach to the case of evolution differential-difference equations and classified integrable Volterra and Toda type lattices [48,49].

In the paper [50], higher symmetries were involved by Habibullin to investigate initial boundary value problems for integrable equations. An effective algorithm for describing integrable boundary conditions was presented. A number of new such conditions for nonlinear equations of mathematical physics were obtained in the paper [53].

Let us discuss multi-component equations (1). It is clear that the matrix A in (1) can be reduced to the Jordan form by a linear transformation of $\vec{u}$. The case with A being degenerate or nondiagonalizable has not been investigated yet. Let A be a diagonal nondegenerate matrix. Using a diagonalization procedure (see [72,70]), one can prove that it is sufficient to consider two opposite cases. First of them, when A has different eigenvalues, has been considered by Mikhailov, Shabat and Yamilov [69,70]. Svinolupov [11] investigated the second one in which A is the unity matrix. It turns out that this case is much more similar to the scalar case than the first one. The system (2) is replaced by a system of the form

$$D_x(X_{i+1}) + [F_{n-1}, X_{i+1}] = f_i(\vec{F}, X_1, \ldots, X_i), \qquad i = 1, \ldots, m-n, \tag{3}$$

where X_i are local matrices, F_i denotes the Jacobi matrix of $\vec{F}$ with respect to $\vec{u}_i$. For example, if we restrict ourselves to the equations

$$\vec{u}_t = \vec{u}_{xxx} + \vec{F}(\vec{u}, \vec{u}_x, \vec{u}_{xx}), \qquad i = 1, \ldots, N, \tag{4}$$

then first two equations of (3) take the form

$$D_x(X_1) + [F_2, X_1] = D_t(F_2), \tag{5}$$

$$D_x(X_2) + [F_2, X_2] = [X_1, F_1 - F_2^2] + D_t(F_1 - F_2^2). \tag{6}$$

Using classifying conditions (3), Svinolupov investigated second order systems and some of third order ones. The results obtained will be formulated in Section 2.

The main technical problem in the classification of multi-component integrable equations is that any straightforward calculation leads to enormous expressions quite impossible to deal with. The second point is that, in the case of polynomial equations, integrability conditions yield an overdetermined system of algebraic equations for coefficients of the rhs. As a rule, it is very difficult to understand how many solutions such a system has. The most essential problem is the third one. One should expect that the classification problem for equations with arbitrary many unknown variables contains, as a subproblem, a classical "unsolvable" classification problem of algebra, such as, for instance, the description of all finite dimensional Lie algebras.

In order to illustrate all above points, let us consider Svinolupov's result [13] concerning the multi-component generalizations

$$u^i_t = u^i_{xxx} + C^i_{jk} u^j u^k_x \tag{7}$$

of the Korteweg-de Vries equation. Here and below we assume that the summation is carried out over repeated indices. Since any linear transformation of $\vec{u}$ preserves the class (7), the description of integrable cases has to be invariant under these transformations.

To solve the problem of complication of computations, Svinolupov interpreted C^i_{jk} as the structure constants of an (noncommutative and nonassociative) algebra J and rewrote (7) in the form

$$U_t = U_{xxx} + U \circ U_x, \tag{8}$$

where $U(x,t)$ is a J-valued function. It is easy to see that equations related by linear transformations correspond to isomorphic algebras.

It turns out that if J is commutative, then (8) is integrable iff J is the Jordan algebra (see Appendix). Although there is no description of all the Jordan algebras this result allows one:

i) to check the integrability of a given system (7);

ii) to classify all integrable cases for small dimensions;

iii) to construct the most interesting examples of an arbitrary high dimension.

Let us explain what the term "most interesting" means. A system of equations (7) is called irreducible if it cannot be reduced to the block-triangular form by an appropriate linear transformation (in the case of the block-triangular system, the functions $u^1, \ldots, u^M$ $(M < N)$ satisfy an autonomous system of the form (7), and remaining equations are linear in $u^{M+1}, \ldots, u^N$). It turns out that irreducible systems are associated with the simple algebras. Thus, one can use a well-known algebraic result [36], namely, the exhaustive description of all the simple Jordan algebras to construct all irreducible systems. They are nothing but so-called vector and matrix Korteweg-de Vries equations [43,24].

The most interesting example here is the following vector KdV equation [24]

$$u_t = u_{xxx} + < C, u > u_x + < C, u_x > u - < u, u_x > C, \tag{9}$$

where $< , >$ is the standard scalar product, C is a given constant vector. Usually (see [43]), one refers to the system

$$u_t = u_{xxx} + < C, u > u_x + < C, u_x > u \tag{10}$$

as the vector KdV equation (cf. (9) and (10) with (96) and (97)). However, we recommend (9) as a claimant upon this role, since the system (10) is reducible unlike (9).

Section 3 is devoted to multi-component systems for which classification parameters are not constants, as above, but functions of $\vec{u} = (u^1, \ldots, u^N)^t$. One of the most interesting classes of such systems are hyperbolic systems of the form

$$u^i_{xy} = \alpha^i_{jk}(\vec{u}) u^j_x u^k_y. \tag{11}$$

This class is important for applications in the string theory. For instance, the chiral field models belong to it. The papers [24,30] contain new examples of integrable systems (11).

The class (11) is invariant under point transformations: $\vec{v} = \vec{\Psi}(\vec{u})$. The vector functions $\alpha^i_{jk}(\vec{u})$ are transformed as components of an affine connection Γ under these transformations. It is clear that an invariant description of integrable cases must be reduced to some conditions on the connection.

There exist the "geometric" classes of equations in the evolution case too. The simplest of them are

$$u^i_t = \Lambda u^i_{xx} + \alpha^i_{jk}(\vec{u}) u^j_x u^k_x, \tag{12}$$

with $\Lambda^2 = 1$, and

$$u^i_t = u^i_{xxx} + \alpha^i_{jk}(\vec{u}) u^j_x u^k_{xx} + \beta^i_{jks}(\vec{u}) u^j_x u^k_x u^s_x. \tag{13}$$

Subsection 3.2 contains some examples of integrable equations (12) and (13)

The classification of integrable equations (13) has been given by Svinolupov and Sokolov (see Subsection 3.3). In the process of classification, a new class of affinely connected spaces associated with integrable equations (13) has been found.

Apparently for any integrable system of the geometric type one can find a preferred system of coordinates which is singled out by the fact that the functions $\alpha^i_{jk}(\vec{u})$ are the structure constants of an N-parameter family $J(u)$

of nonassociative algebras like the Jordan or left-symmetric ones. One can regard such a family of nonassociative algebras, with the structure constants depending on parameters, as a deformation of an algebra with the structure constants $\alpha^i_{jk}(0)$. Deformations of nonassociative algebras defined by the overdetermined consistent system

$$\frac{\partial \alpha^i_{jk}}{\partial u^m} = \alpha^i_{rk}\alpha^r_{mj} + \alpha^i_{jr}\alpha^r_{mk} - \alpha^i_{mr}\alpha^r_{jk} \tag{14}$$

were investigated in [19,30]. All the examples of integrable systems of the geometric type known to the authors are associated with the deformation (14).

There are several papers of Svinolupov, Yamilov and Adler devoted to multi-component integrable generalizations of evolution differential-difference equations (chains) [14,23,47,32]. Unlike first Svinolupov's papers, the purpose of these ones was not to classify integrable cases, but only to construct integrable multi-component examples and to show that the algebraic approach gives results in the differential-difference case as well.

Results of Svinolupov and Yamilov obtained in the papers [14], [23] are discussed in Section 4. It is well-known [57] that Bäcklund transformations of integrable partial differential equations generate integrable differential-difference equations, and using some special Bäcklund transformations, we can obtain *evolution* differential-difference equations [44,45,46]. For example, the nonreduced Schrödinger equation (or ZS-AKNS equation)

$$u_t = u_{xx} + 2u^2v, \qquad -v_t = v_{xx} + 2v^2u \tag{15}$$

admits two special Bäcklund transformations. One of them is of the form

$$\tilde{u}_x = u + \tilde{u}^2 v, \qquad -v_x = \tilde{v} + v^2\tilde{u}, \tag{16}$$

the second one is the following explicit invertible auto-transformation:

$$\tilde{u} = u_{xx} - u_x^2/u + u^2 v, \qquad \tilde{v} = 1/u. \tag{17}$$

In the first case, if we consider a chain of Bäcklund transformations which links together solutions $(u, v) = (u_{n+1}, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_n, v_{n-1})$ of (15), we come to the system of differential-difference equations

$$u_{nx} = u_{n+1} + u_n^2 v_n, \qquad -v_{nx} = v_{n-1} + v_n^2 u_n, \tag{18}$$

where n is discrete integer variable. By carrying out the continuous limit, as one does for the Volterra equation to obtain the Korteweg-de Vries equation,

we get (15). In the second case, a chain of Bäcklund transformations which links together solutions $(u, v) = (u_n, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_{n+1}, v_{n+1})$ of (15) is a system of the form:

$$u_{nxx} = u_{nx}^2/u_n + u_{n+1} - u_n^2/u_{n-1}. \tag{19}$$

Introducing $u_n = \exp q_n$, we obtain the classical Toda model

$$q_{nxx} = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}). \tag{20}$$

According to [23], conservation laws and higher symmetries of (18) and (20) can be constructed using ones of (15).

In the same manner as above, multi-component generalizations of the differential-difference Schrödinger equation (18) and the Toda chain (20) possessing higher order symmetries and conservation laws will be constructed in Section 4. We will use the fact that there is a multi-component integrable analog of the system (15) (see [16] and Section 2).

In the next Section, we present results obtained by Svinolupov, Yamilov and Adler [32]. The paper [32] contains not only generalizations of the Volterra equation but also examples of multi-component local master symmetries (both differential-difference and partial differential ones). Such master symmetries exemplify local evolution equations explicitly depending on the spatial variable and integrable by the inverse scattering method.

The concept of the master symmetry was introduced in [60,61]. As it is known, for the first time the master symmetries have arisen and were investigated as integrable equations with the spectral problem in which the spectral parameter depends on the time (see e.g. [59,74]). Later it was observed that these equations generate higher symmetries for usual integrable equations, and a possibility appeared to give an algebraic definition for them.

The well-known Volterra equation

$$v_{nx} = v_n(v_{n+1} - v_{n-1}) \tag{21}$$

can be rewritten in the form

$$u_{nx} = u_n^2(u_{n+1} - u_{n-1}) \tag{22}$$

by introducing u_n such that $u_{n+1}u_n = v_n$. The form (22) is more convenient to construct multi-component generalizations. The chain (22) possesses the following local master symmetry

$$u_{n\tau} = u_n^2[(n+1)u_{n+1} - (n-1)u_{n-1}] \tag{23}$$

obtained for the first time in [58] as an integrable chain. We see that (23) is the local evolution equation, the rhs of which does not contain any integrals or their difference analogs unlike, for example, the master symmetry of the KdV equation

$$u_\tau = x(u_{xxx} + 6uu_x) + 4(u_{xx} + 2u^2) + 2u_x\partial_x^{-1}(u).$$

This is the reason why the master symmetry (23) is called local. This master symmetry admits an $L-A$ pair with the spectral parameter depending on the time [58].

Following [32], we point out in Section 5 the multi-component generalizations of (22) and (23) corresponding to an arbitrary Jordan triple system. Moreover, multi-component "continuous" examples of local master symmetries will arise there in a natural way.

In the last Section 6, we generalize results of [51,52,50] concerning the phenomena of integrable boundary conditions, to the case of the multi-component Burgers and nonlinear Schrödinger equations described in Section 2. Note that the use of algebraic notation allows one to express not only equations but also integrable boundary conditions in a compact elegant form.

Results of this Section have been obtained by Svinolupov in collaboration with Habibullin [28,29].

Let us remind the definition of the integrable boundary condition. A boundary condition

$$u_x = f(u,t)|_{x=0} \tag{24}$$

for an integrable equation

$$u_t = F(u, u_1, \ldots, u_n) \tag{25}$$

is said to be integrable if it is compatible with higher symmetries

$$u_\tau = G(u, u_1, \ldots, u_m) \tag{26}$$

of (25).

In order to explain the definition given above, we consider boundary conditions of the form

$$u_x = f(u)|_{x=0} \tag{27}$$

for the classical Burgers equation

$$u_t = u_{xx} + 2uu_x, \tag{28}$$

compatible with the fourth order symmetry

$$u_\tau = u_4 + 4uu_3 + 10u_1u_2 + 6u^2u_2 + 12uu_1^2 + 4u^3u_1 \tag{29}$$

of (28).

Differentiating (27) with respect to t and using the Burgers equation one gets the expression for the variable u_3 in terms of u, u_2:

$$u_3 = f'(u)(u_2 + 2uf(u)) - 2uu_2 - 2f(u)^2 = f_1(u, u_2).$$

After one more step of this kind, one can derive a formula for the variable u_5 : $u_5 = f_2(u, u_2, u_4)$. Then, differentiating the constraint (27) with respect to τ in virtue of the higher symmetry (29), we obtain the equality $u_{\tau x} = f_u G$, where G is the rhs of (29). Using expressions given above, we can evaluate from this equality the τ-derivative and the variables u_1, u_3, u_5, and then, by definition, the equality must be satisfied identically. After a calculation, one is led to the following condition which is necessary and sufficient for the boundary condition (27) to be compatible with the symmetry (29): $f_{uu} = -2$, i.e. $f = -u^2 + C_1 u + C_2$. It has been proved in [53] that this boundary condition is compatible with all the even order homogeneous symmetries of the Burgers equation. Furthermore, if (27) is compatible with at least one higher symmetry, then it is of the form $u_x = -u^2 + C_1 u + C_2$.

2 Polynomial systems

One of the most remarkable observations of Svinolupov is the discovery of the fact that polynomial multi-component integrable equations are closely connected to the well-known nonassociative algebraic structures as the left-symmetric algebras, Jordan algebras, Jordan triple systems, etc. This connection allows one to clarify the nature of known vector and matrix generalizations (see, for instance [41,42,43]) of classical scalar integrable equations and to construct some new examples of this kind [24].

In this Section, we will consider some classes of polynomial integrable systems of evolution equations generalizing the following famous scalar integrable equations: the Burgers equation

$$u_t = u_{xx} + 2uu_x,$$

the modified Korteweg-de Vries equation

$$u_t = u_{xxx} + u^2 u_x,$$

the nonlinear Schrödinger equation

$$u_t = u_{xx} + 2u^2 v, \qquad -v_t = v_{xx} + 2v^2 u, \tag{30}$$

and the nonlinear derivative Schrödinger equation

$$u_t = u_{xx} + 2(u^2 v)_x, \qquad -v_t = v_{xx} + 2(v^2 u)_x.$$

Class 2.1. The multi-component Burgers equation introduced by Svinolupov in [11] is of the form

$$u^i_t = u^i_{xx} + 2a^i_{jk} u^j u^k_x + b^i_{jkm} u^j u^k u^m, \qquad i = 1, \ldots, N, \tag{31}$$

where $u^i = u^i(t, x)$, and the parameters a^i_{jk}, b^i_{jkm} are constant. The summation on the repeated indices is assumed.

The following classification statement holds.

Theorem 2.1. In order to the system of equations (31) possesses at least one higher symmetry, it is necessary and sufficient that the set of parameters a^i_{jk}, b^i_{jkm} satisfy the constraints

$$b^i_{jkm} = \frac{1}{3}(a^i_{jr} a^r_{km} + a^i_{kr} a^r_{mj} + a^i_{mr} a^r_{jk} - a^i_{rj} a^r_{km} - a^i_{rk} a^r_{mj} - a^i_{rm} a^r_{jk}), \tag{32}$$

$$a^i_{jr} a^r_{km} - a^i_{kr} a^r_{jm} = a^r_{jk} a^i_{rm} - a^r_{kj} a^i_{rm}. \tag{33}$$

The relation (33) means that a^i_{jk} are the structure constants of a left-symmetric algebra A (see Appendix). Let $e_1, \ldots, e_N$ be a basis of A and $u = u^i e_i$. Then (31) can be written in the following simple form

$$u_t = u_{xx} + 2u \circ u_x + u \circ (u \circ u) - (u \circ u) \circ u, \tag{34}$$

where $\circ$ denotes the multiplication in A. Let us consider two simplest examples of the systems (34).

Example 2.1. The set of all quadratic matrices forms an associative (and, therefore, left-symmetric) algebra. The corresponding equation (34) is the matrix Burgers equation

$$u_t = u_{xx} + 2uu_x. \tag{35}$$

Example 2.2. (V.V. Sokolov) The left-symmetric algebra (89) generates the following vector Burgers equation

$$u_t = u_{xx} + 2 < u, u_x > C + 2 < u, C > u_x + \\ + \|u\|^2 < u, C > C - \|C\|^2 \|u\|^2 u. \tag{36}$$

Every equation of the classes presented below has infinitely many higher symmetries and a zero-curvature representation of the form $U_t - V_x = [U, V]$, where U and V belong to the superstructure Lie algebra of a Jordan algebra or a Jordan triple system (see [38,40]).

Class 2.2. Multi-component generalizations of the nonlinear Schrödinger equation (see [16]) are given by the systems of $2N$ equations of the form

$$\begin{aligned} u^i_t &= u^i_{xx} + 2a^i_{jkm} u^j v^k u^m \\ v^i_t &= -v^i_{xx} - 2a^i_{jkm} v^j u^k v^m, \end{aligned} \tag{37}$$

where $i = 1, \ldots, N$, and a^i_{jkm} are constants. Without loss of generality, we assume that a^i_{jkm}=a^i_{mkj}.

The following statement has been proved in [16].

Theorem 2.2. In order to the system (37) possesses at least one non-degenerate higher symmetry, it is necessary and sufficient that the constants a^i_{jkm} satisfy the following relation:

$$a^i_{jkn} a^n_{msp} - a^i_{msn} a^n_{jkp} - a^i_{nsp} a^n_{jkm} + a^i_{mnp} a^n_{kjs} = 0. \tag{38}$$

The relation (38) just means that a^i_{jkm} are the structure constants of a Jordan triple system. Using this fact, one can write down integrable systems of the form (37) in an invariant compact form. Setting $u = u^i e_i$ and $v = v^i e_i$, we can see that (37) is equivalent to

$$u_t = u_{xx} + 2\{u, v, u\}, \qquad v_t = -v_{xx} - 2\{v, u, v\}. \tag{39}$$

The formulas (94)-(97) given in Appendix yield examples of integrable matrix and vector Schrödinger equations.

Example 2.3. The well-known vector Schrödinger equation [54]

$$u_t = u_{xx} + 2 < u, v > u, \qquad v_t = -v_{xx} - 2 < v, u > v \tag{40}$$

corresponds to the Jordan triple system (97).

Example 2.4. A new integrable vector nonlinear Schrödinger equation

$$\begin{aligned} u_t &= u_{xx} + 4 < u, v > u - 2\|u\|^2 v, \\ v_t &= -v_{xx} - 4 < v, u > v + 2\|v\|^2 u \end{aligned} \tag{41}$$

contained in [24] corresponds to the Jordan triple system (96). Note that (41) looks very similar to equations of the paper [73] devoted to the fibre optics.

Example 2.5. The well-known matrix generalization of the Schrödinger equation

$$u_t = u_{xx} + 2uvu, \qquad v_t = -v_{xx} - 2vuv, \tag{42}$$

where v and v are $m \times m$ matrices, is associated with the Jordan triple system (94).

For the next two types of multi-component equations, we do not formulate any classification results (see [13,16,21]), but write down classes of equations containing all irreducible systems.

Class 2.3. The following generalization of the derivative nonlinear Schrödinger equation

$$u_t = u_{xx} + 2\{v,u,v\}_x, \qquad v_t = -v_{xx} - 2\{u,v,u\}_x \tag{43}$$

is integrable for any Jordan triple system. Matrix and vector examples are constructed using formulae (94)-(97).

Class 2.4. In Introduction we described the multi-component KdV equations. Here we present equations of the MKdV type. Equations of the form

$$u_t = u_{xxx} + \{u,u,u_x\} \tag{44}$$

are integrable for any Jordan triple system. Matrix and vector examples can be obtained in the standard way (see [24]).

3 Geometric type systems

Results of this Section have been obtained by Svinolupov and Sokolov.

3.1 Deformations of the Jordan algebras

It can be shown that for any initial data

$$\alpha^i_{jk}(0) = a^i_{jk}, \tag{45}$$

where a^i_{jk} are the structure constants of a Jordan algebra J_0, a solution $\alpha^i_{jk}(u)$ of (14) exists (for a sufficiently small u) and is unique. Moreover, $\alpha^i_{jk}(u)$ turn out to be the structure constants of a Jordan algebra J_u for any u. Let us denote by $\sigma^i_{jkm}(u)$ the structure constants of a Jordan triple system σ_u generated by the algebra J_u by means of the formula (99). There are two important cases in which the deformation equation can be solved explicitly [19,30].

Construction 3.1. If J_0 possesses the unity element e, then the multiplication $\alpha_u(X,Y)$ in the J_u is given by the formula

$$\alpha_u(X,Y) = -(e-u)^{-1}\circ(X\circ Y)+(X\circ(e-u)^{-1})\circ Y+(Y\circ(e-u)^{-1})\circ X. \quad (46)$$

The definition of the inverse element is contained in Appendix. For every simple Jordan algebra the inverse element can be explicitly found. For example, the simple algebra of the type A_n is defined by the multiplication (91). The inverse of v coincides with the standard matrix inverse v^{-1}. For the algebra of the type D_n (see formula (92)),

$$v^{-1} = \frac{2 < C, v > C - ||C||^2 v}{||C||^4 ||v||^2}.$$

Construction 3.2. Let $\{X,Y,Z\}$ be a Jordan triple system, $\phi(u)$ be a solution of the following overdetermined consistent system

$$\frac{\partial \phi}{\partial u^k} = -\{\phi,\ e_k,\ \phi\}, \quad (47)$$

$k = 1,\ldots,N$. Then the structure constants of $J(u)$ with the multiplication

$$\alpha_u(X,Y) = \{X,\phi,Y\} \quad (48)$$

satisfy the deformation equation (14). The Jordan triple system corresponding to (48) is of the form

$$\sigma_u(X,Y,Z) = \{X,\{\phi,Y,\phi\},Z\}. \quad (49)$$

If $\{X,Y,Z\}$ is given by (94), then one of the solutions of (47) is

$$\phi(u) = u^{-1}. \quad (50)$$

An analog of u^{-1} is well-known in the theory of the Jordan triple systems. Let us define a linear operator P_X by the formula $P_X(Y) = \{X,Y,X\}$. Then, by definition, $u^{-1} = P_u^{-1}(u)$.

Assume that there is a vector X, such that P_X is nondegenerate. Then $\phi(u) = P_{X+u}^{-1}(X+u)$ exists for small u and satisfies (47). Without loss of generality, we assume in this case that

$$\phi(u) = P_u^{-1}(u). \quad (51)$$

In particular, one can choose

$$\phi(u) = \frac{u}{||u||^2} \quad (52)$$

for the Jordan triple system (96).

Let us take (97) for the Jordan triple system σ. It is easy to see that the operator P_X is degenerate for any X, and we must solve (47) straightforwardly. The general solution is

$$\phi(u) = \frac{C}{2 < C, u >}, \tag{53}$$

where C is an arbitrary constant vector.

The formula (53) is a special case of the following formula

$$\phi(u) = C(C^t u)^{-1}, \tag{54}$$

where C is a constant $n \times m$ matrix, corresponding to the Jordan triple system (95)

3.2 Examples of geometric type integrable equations generated by the deformation

We present here some classes of integrable equations closely related to the deformation (14). A class of integrable chains generated by (14) is contained in Section 4. Formulas (50), (52), (53) allow one to build up one matrix and two vector equations for every class, using Construction 3.2. We will write down some of them explicitly. The paper [24] contains examples of integrable equations corresponding to (54).

Class 3.1. Let $J(u)$ be the deformation of a Jordan algebra. Consider the equation

$$u_{xy} = \alpha_u(u_x, u_y), \tag{55}$$

where α_u is the multiplication in $J(u)$. In the matrix case, (55) coincides with the equation of the principal chiral field

$$u_{xy} = \frac{1}{2}(u_x u^{-1} u_y + u_y u^{-1} u_x). \tag{56}$$

For this reason we will call (55) the Jordan chiral field equation.

It is easy to verify that (55) admits the following zero-curvature representation

$$\Psi_x = \frac{2}{(1-\lambda)} L_{u_x} \Psi, \qquad \Psi_y = \frac{2}{(1+\lambda)} L_{u_y} \Psi.$$

Here and below we denote by L_X the left multiplication operator: $L_X(Y) = \alpha_u(X, Y)$. Note that this formula gives us a zero-curvature representation for (56):

$$\Psi_x = \frac{1}{(1-\lambda)} M \Psi, \qquad \Psi_y = \frac{1}{(1+\lambda)} N \Psi,$$

where Ψ is a matrix and

$$M\Psi = -u_x u^{-1}\Psi - \Psi u^{-1}u_x, \qquad N\Psi = -u_y u^{-1}\Psi - \Psi u^{-1}u_y,$$

different from the standard one. It should also be remarked that (55) is linearizable if J_u is the deformation of a left-symmetric algebra (see [24,30]).

All equations of classes presented below have higher symmetries and zero-curvature representations in the superstructure Lie algebra of J_u (see [38]).

Class 3.2. The following equation

$$u_t = u_{xxx} - 3\alpha_u(u_x, u_{xx}) + \frac{3}{2}\sigma_u(u_x, u_x, u_x) \tag{57}$$

of the form (13) is integrable if it corresponds to the deformation of a Jordan algebra. Matrix and vector equations have the following form:

$$u_t = u_{xxx} - \frac{3}{2}u_x u^{-1}u_{xx} - \frac{3}{2}u_{xx}u^{-1}u_x + \frac{3}{2}u_x u^{-1}u_x u^{-1}u_x, \tag{58}$$

where $u(x,t)$ is an $m \times m$ matrix,

$$\begin{aligned} u_t \;=\; & u_{xxx} - 3\frac{<u,u_x>}{\|u\|^2}u_{xx} - 3\frac{<u,u_{xx}>}{\|u\|^2}u_x + 3\frac{<u_x,u_{xx}>}{\|u\|^2}u \\ & -\frac{3}{2}\frac{\|u_x\|^2}{\|u\|^2}u_x + 6\frac{<u,u_x>^2}{\|u\|^4}u_x - 3\frac{<u,u_x>\|u_x\|^2}{\|u\|^4}u, \end{aligned} \tag{59}$$

and

$$u_t = u_{xxx} - \frac{3}{2}\frac{<C,u_x>}{<C,u>}u_{xx} - \frac{3}{2}\frac{<C,u_{xx}>}{<C,u>}u_x + \frac{3}{2}\frac{<C,u_x>^2}{<C,u>^2}u_x.$$

Class 3.3. The following integrable equations

$$v_t = v_{xxx} - \frac{3}{2}\alpha_{v_x}(v_{xx}, v_{xx}) \tag{60}$$

are related to ones of Class 3.2 by the potentiation $u = v_x$. Vector equations are of the form:

$$u_t = u_{xxx} - 3\frac{<u_x,u_{xx}>}{\|u_x\|^2}u_{xx} + \frac{3}{2}\frac{\|u_{xx}\|^2}{\|u_x\|^2}u_x,$$

$$u_t = u_{xxx} - \frac{3}{2}\frac{<C,u_{xx}>}{<C,u_x>}u_{xx}.$$

Class 3.4. The scalar representative of this class is given by the Heisenberg model (see [70])

$$u_t = u_{xx} - \frac{2}{u+v}u_x^2, \qquad v_t = -v_{xx} + \frac{2}{u+v}v_x^2.$$

The following coupled equation

$$u_t = u_{xx} - 2\alpha_{u+v}(u_x, u_x), \qquad v_t = -v_{xx} + 2\alpha_{u+v}(v_x, v_x) \tag{61}$$

is integrable if it is associated with the deformation of a Jordan algebra. The equation (61) has a higher symmetry of the form

$$\begin{aligned} u_t &= u_{xxx} - 6\alpha_{u+v}(u_x, u_{xx}) + 6\sigma_{u+v}(u_x, u_x, u_x), \\ v_t &= v_{xxx} - 6\alpha_{u+v}(v_x, v_{xx}) + 6\sigma_{u+v}(v_x, v_x, v_x). \end{aligned} \tag{62}$$

Note that (62) belongs to Class 3.2. After the reduction $u = v$ and the scaling $2u \to u$, it turns into (57). A matrix equation (61) is of the form

$$u_t = u_{xx} - 2u_x(u+v)^{-1}u_x, \qquad v_t = -v_{xx} + 2v_x(u+v)^{-1}v_x. \tag{63}$$

One of vector equations is

$$u_t = u_{xx} - 4\frac{< u_x, u+v >}{\|u+v\|^2}u_x + 2\frac{\|u_x\|^2}{\|u+v\|^2}(u+v),$$

$$v_t = -v_{xx} + 4\frac{< v_x, u+v >}{\|u+v\|^2}v_x - 2\frac{\|v_x\|^2}{\|u+v\|^2}(u+v).$$

3.3 Classification of integrable equations (13)

Let us consider the systems of the type (13). It is convenient to rewrite (13) in the following way

$$u_t^i = u_{xxx}^i + 3\alpha_{jk}^i u_x^j u_{xx}^k + (\frac{\partial \alpha_{km}^i}{\partial u^j} + 2\alpha_{jr}^i \alpha_{km}^r - \alpha_{rj}^i \alpha_{km}^r + \beta_{jkm}^i)u_x^j u_x^k u_x^m, \tag{64}$$

where $\beta_{jkm}^i = \beta_{kjm}^i = \beta_{mkj}^i$, i.e.

$$\beta(X,Y,Z) = \beta(Y,X,Z) = \beta(X,Z,Y)$$

for any vectors X, Y, Z.

The class (64) is invariant under arbitrary point transformations $u \to \vec{\Phi}(u)$, where $u = (u^1, \ldots, u^N)^t$. It is easy to see that under such a change

of coordinates, α^i_{jk} and β^i_{jkm} are transformed just as components of an affine connection Γ and a tensor, respectively. Let R and T be the curvature and torsion tensors of Γ.

In order to formulate classification results, we introduce the following tensor:

$$\sigma(X,Y,Z) = \beta(X,Y,Z) - \frac{1}{3}\delta(X,Y,Z) + \frac{1}{3}\delta(Z,X,Y),$$

where

$$\delta(X,Y,Z) = T(X,T(Y,Z)) + R(X,Y,Z) - \nabla_X(T(Y,Z)).$$

Using the Bianchi's identity, one can find that

$$\sigma(X,Y,Z) = \sigma(Z,Y,X). \tag{65}$$

Theorem 3.1. Eq. (64) possesses a higher symmetry of the form

$$u_\tau = u_n + \vec{G}(u, u_x, \cdots, u_{n-1}), \qquad n > 3,$$

iff

$$\nabla_X(R(Y,Z,V)) = R(Y,X,T(Z,V)), \tag{66}$$

$$\nabla_X\big(\nabla_Y(T(Z,V)) - T(Y,T(Z,V)) - R(Y,Z,V)\big) = 0, \tag{67}$$

$$\nabla_X(\sigma(Y,Z,V)) = 0, \tag{68}$$

$$\begin{aligned} &T(X,\sigma(Y,Z,V)) + T(Z,\sigma(Y,X,V)) \; + \\ &+T(Y,\sigma(X,V,Z)) + T(V,\sigma(X,Y,Z)) = 0, \end{aligned} \tag{69}$$

and

$$\begin{aligned} &\sigma(X,\sigma(Y,Z,V),W) - \sigma(W,V,\sigma(X,Y,Z)) \; + \\ &+\sigma(Z,Y,\sigma(X,V,W)) - \sigma(X,V,\sigma(Z,Y,W)) = 0. \end{aligned} \tag{70}$$

The identities (65) and (70) mean that $\sigma^i_{jkm}(u)$ are the structure constants of a Jordan triple system for any u.

It follows from (66) that any free-torsion space of this kind is the symmetric one. In the case $T \neq 0$, a generalization of the symmetric spaces gives rise. We don not know if such affine connected spaces have been considered by geometers.

Theorem 3.2. For every Jordan triple system $\{\cdot,\cdot,\cdot\}$ with the structure constants s^i_{jkm}, there exists a unique (up to point transformations) integrable equation (64), such that $T = 0$ and

$$\sigma^i_{jkm}(0) = s^i_{jkm}. \tag{71}$$

In the case $T = 0$, there is a class of integrable equations (64) generated by the deformation (14). If the structure constants $\alpha^i_{jk}(u)$ of a family of Jordan algebras satisfy (14) and $\sigma^i_{jkm}(u)$ are given by (99), then all the conditions (65)-(69) turn out to be satisfied. Corresponding examples are presented above (see Class 3.2). This class contains all integrable equations whose initial Jordan triple system $\{\cdot,\cdot,\cdot\}$ (see Theorem 3.2) can be obtained from a Jordan algebra by the triple product (99).

Equations with initial Jordan triple systems (96) and (97) cannot be obtained in this way. Using some tricks, we have found such equations, solving directly (66) and (68). They are of the form

$$u_t = u_{xxx} + \frac{3}{2}\big(P(u,u_x)(C - \|C\|^2 u)\big)_x + 3\frac{\lambda-1}{\lambda+1}\|C\|^2 P(u,u_x)u_x,$$

where $\lambda = 1$ or $\lambda = 0$, and

$$P(u,u_x) = \|u_x + \frac{< C, u_x > u}{1-(C,u)}\|^2.$$

4 Multi-component generalizations of the differential-difference Schrödinger equation and the Toda model

In this Section we, following the papers [14,23] by Svinolupov and Yamilov, discuss integrable generalizations of the scalar chains (18) and (19). As it has been said above (see Section 2), for any Jordan triple system we can construct the multi-component Schrödinger system (39).

Theorem 4.1. The system of equations (39) admits a Bäcklund transformation of the form

$$\tilde{u}_x = u + \{\tilde{u}, v, \tilde{u}\}, \qquad -v_x = \tilde{v} + \{v, \tilde{u}, v\} \tag{72}$$

iff $\{\cdot,\cdot,\cdot\}$ is a product in the Jordan triple system.

In Theorem 2.2 the condition that a triple algebra associated with (39) must be the Jordan triple system was derived by assuming that (39) possesses

higher order symmetries. Now we have obtained the same condition, using quite different assumption.

As in the scalar case (see Introduction), let us consider a chain of Bäcklund transformations which links together solutions $(u, v) = (u_{n+1}, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_n, v_{n-1})$ of (39). Such a chain is equivalent to the following system of differential-difference equations:

$$u_{nx} = u_{n+1} + \{u_n, v_n, u_n\}, \qquad -v_{nx} = v_{n-1} + \{v_n, u_n, v_n\}. \tag{73}$$

It can be verified that (73) is a differential-difference approximation of the system (39). According to [23], a chain obtained in this way is integrable: conservation laws and higher symmetries of (73) can be constructed using the known ones of (39) (see [16]).

Two vector and one matrix examples of chains of the form (73) can be constructed as in the continuous case. For instance, the matrix example corresponding to (95) is of the form

$$u_{nx} = u_{n+1} + u_n v_n^t u_n, \qquad -v_{nx} = v_{n-1} + v_n u_n^t v_n,$$

where u_n and v_n are $N \times M$ matrices. Vector chains (73) can be written as follows:

$$u_{nx} = u_{n+1} + < u_n, v_n > u_n, \qquad -v_{nx} = v_{n-1} + < u_n, v_n > v_n;$$

and

$$\begin{aligned} & u_{nx} = u_{n+1} + 2 < u_n, v_n > u_n - < u_n, u_n > v_n, \\ - \quad & v_{nx} = v_{n-1} + 2 < u_n, v_n > v_n - < v_n, v_n > u_n. \end{aligned}$$

For all vector and matrix chains there are zero curvature representations [47]; for any integrable chain of the form (73) there is a recursion operator [14]. In the case when the Jordan triple system is generated by a simple Jordan algebra, the corresponding chain is Hamiltonian [14]. The last statement is valid for all chains and partial differential systems we discuss in Sections 2-5.

Theorem 4.2. If a system of the form (39) corresponds to the Jordan triple system $\{\cdot,\cdot,\cdot\}$ generated by (99) from a Jordan algebra with unity element, then this system is invariant under the following transformation:

$$\tilde{u} = u_{xx} - \{u_x, u^{-1}, u_x\} + \{u, v, u\}, \qquad \tilde{v} = u^{-1}, \tag{74}$$

where u^{-1} is the inverse of u.

A chain of transformations (74) which links together solutions $(u, v) = (u_n, v_n)$ and $(\tilde{u}, \tilde{v}) = (u_{n+1}, v_{n+1})$ of (39) can be written in the form

$$u_{nxx} = \{u_{nx}, u_n^{-1}, u_{nx}\} + u_{n+1} - \{u_n, u_{n-1}^{-1}, u_n\}. \tag{75}$$

The chain (75) can be regarded as a Jordan generalization of the Toda model. In particular, in the matrix case we have the well-known matrix Toda chain:

$$(u_{nx}u_n^{-1})_x = u_{n+1}u_n^{-1} - u_n u_{n-1}^{-1}.$$

As Jordan triple system from Theorem 4.2 has to be generated by the Jordan algebra with unity element, we can construct only one vector example

$$u_{nxx} = 2\frac{< u_n, u_{nx} >}{\|u_n\|^2}u_{nx} - \frac{\|u_{nx}\|^2}{\|u_n\|^2}u_n + u_{n+1} -$$

$$-2\frac{< u_n, u_{n-1} >}{\|u_{n-1}\|^2}u_n + \frac{\|u_n\|^2}{\|u_{n-1}\|^2}u_{n-1}.$$

Conservation laws and higher symmetries of the Jordan Toda model (75) can be constructed using ones of (39) [23].

5 Jordan analogs of the Volterra equation, and multi-component local master symmetries

Here we present some results obtained by Svinolupov, Yamilov and Adler in the paper [32]. The Jordan analog of the Volterra equation is given by the following multi-component differential-difference system:

$$u_{nx} = \{u_n, u_{n+1}, u_n\} - \{u_n, u_{n-1}, u_n\}, \tag{76}$$

where $\{\cdot, \cdot, \cdot\}$ is a Jordan triple system. The local master symmetry corresponding to (76) have the form:

$$u_{n\tau} = (n+1)\{u_n, u_{n+1}, u_n\} - (n-1)\{u_n, u_{n-1}, u_n\}. \tag{77}$$

Theorem 5.1 The chain (77) is the master symmetry of (76) if the associated triple system $\{\cdot, \cdot, \cdot\}$ is the Jordan one.

Theoretically we, as usually, can construct one matrix and two vector examples of the Jordan Volterra equations. However, one of the vector examples is degenerate. In fact, in the case of the simplest vector triple system, we are led to the chain

$$u_{nx} =< u_{n+1} - u_{n-1}, u_n > u_n.$$

If $u_n = (u_n^1, \ldots, u_n^N)^t$, the constraint $(\log(u_n^i/u_n^j))_x = 0$ holds for any i and j, and one can easily reduce the multi-component chain under consideration to the system consisting of N scalar equations (22).

It turns out that if the Jordan triple system is generated by a Jordan algebra with unity element, then the chain (76) generates an invertible auto-transformation

$$\tilde{u} = v - (u^{-1})_x, \qquad \tilde{v} = u \tag{78}$$

for the multi-component derivative Schrödinger equation (43) which also has the local master symmetry

$$u_\tau = x(u_{xx} + 2\{u, v, u\}_x) + (a + \frac{3}{2})u_x + 2\{u, v, u\},$$

$$v_\tau = x(-v_{xx} - 2\{v, u, v\}_x) + (a - \frac{3}{2})v_x + 2\{v, u, v\}, \tag{79}$$

where a is an arbitrary constant. As far as we know, equations (77) and (79) are first examples of multi-component local master symmetries. Two following examples explain why these equations are integrable and what is the difference between a usual integrable equation and it's master symmetry.

Example 5.1. In the matrix case the chain (76) admits the usual Lax representation $L_{nx} = [A_n, L_n]$, where

$$L_n = u_n(D + D^{-1}), \qquad 2A_n = u_n u_{n+1}(D^2 + 1) - u_n u_{n-1}(1 + D^{-2}),$$

and D is the shift operator. In the case of (77), there is the representation

$$L_{n\tau} = [B_n, L_n] + \frac{1}{2}L_n^3$$

with the same operator L_n and

$$2B_n = (n + \frac{1}{2})u_n u_{n+1}(D^2 + 1) - (n - \frac{1}{2})u_n u_{n-1}(1 + D^{-2}).$$

This means that in the case of (77) we have the spectral problem

$$L_n\psi_n = \lambda\psi_n, \qquad \psi_{n\tau} = B_n\psi_n$$

with the spectral parameter λ depending on the time τ : $\lambda(\tau) = (\varepsilon - \tau)^{-1/2}$.

The transformation (78) allows one to construct exact solutions for the system (43), starting from a solution (u, v) such that $v = 0$ and u satisfies the

multi-component heat equation $u_t = u_{xx}$. As for system (79), this transformation (78) changes the parameter a: $\tilde{a} = a + 1$. Consequently, to obtain exact solutions of (79), we must use not only (78) but also the Galilei transformation: $\tilde{x} = x + \tau$. As a starting point, we can take a solution (u, v) such that $v = 0$ and u satisfies the following linear equation:

$$u_\tau = xu_{xx} + (a + \frac{3}{2})u_x.$$

6 Integrable boundary conditions for multi-component equations

Here we present a class of integrable boundary conditions for multi-component equations (34), (39) obtained by Habibullin and Svinolupov [28,29].

6.1 Integrable boundary conditions for the multi-component Burgers equations

Theorem 6.1. The following boundary conditions for the equation (34):

$$(\sum_{i=0}^{m}(D_t + L(u_x + u \circ u))^i(K_i(u_x + u \circ u) + M_i u + c_i))|_{x=0} = 0 \qquad (80)$$

are integrable. Here K_i, M_i are arbitrary linear operators satisfying the identities

$$M_i(X \circ Y) - (X \circ M_i Y) = 0, \qquad K_i(X \circ Y) - (X \circ K_i Y) = 0, \qquad (81)$$

$\circ$ is the multiplication in a left-symmetric algebra A, c_i are constant vectors satisfying the condition

$$AS(X, Y, c) = 0. \qquad (82)$$

Both the equalities (81) and (82) must be held for all $X, Y \in A$. The operator D_t is the total t-derivative for (34), and the operator $L(X)$ is defined by $L(X)Y = X \circ Y$.

Since $M_i = 0$, $K_i = Id$, $c = 0$ satisfy (81), (82), the boundary condition $u_x + (u \circ u) = 0$ is integrable for any equation (34).

Example 6.1. The following boundary conditions

$$i) \quad u = 0,$$
$$ii) \quad u_x c_1 + u^2 c_1 + u c_2 + c_3 = 0$$

are integrable for the matrix Burgers equation (35) of Example 2.1. Here c_1, c_2, c_3 are arbitrary constant matrices.

Example 6.2. In the case of the equation of Example 2.2, only scalar (i.e. proportional to the identical) operators satisfy (81). The simplest integrable boundary conditions are

$$\begin{aligned} &i) \quad u = 0, \\ &ii) \quad u_x + < u, C > u + \|u\|^2 C + \lambda u = 0, \end{aligned}$$

where λ is a scalar parameter.

6.2 Integrable boundary conditions for the multi-component nonlinear Schrödinger equations

For equations (39), the following boundary conditions are integrable:

$$\begin{aligned} &i) \quad u = 0, \qquad v = 0; \\ &ii) \quad u_x = cu, \qquad v_x = cv; \\ &iii) \quad u_{xx} - cu_x + 2\{\bar{u}vu_x\} = 0, \qquad v_{xx} - cv_x + 2\{v\bar{u}v_x\} = 0, \end{aligned} \tag{83}$$

where c is an arbitrary constant, $\bar{u}$ is a solution of the equation

$$\{\bar{u}v\bar{u}\} + u - c\bar{u} = 0. \tag{84}$$

These conditions generalize the known boundary conditions [51,52] for the nonlinear Schrödinger equation (30):

$$\begin{aligned} &i) \quad u_x = cu|_{x=0}, \qquad v_x = cv|_{x=0}; \\ &ii) \quad u_{xx} = (c + uv)^{1/2} u_x|_{x=0}, \qquad v_{xx} = (c + uv)^{1/2} v_x|_{x=0}. \end{aligned}$$

Let us discuss the problem how to eliminate the extra variable $\bar{u}$ from the boundary condition. For a large class of Jordan triple systems (for instance, for those which are generated by a Jordan algebra with the unity, (see [56])), we have $\det N(v, v_x) \neq 0$, where $N(X, Y)Z = \{X, Y, Z\}$. This allows one to express the variable $\bar{u}$ from the second equation of (83):

$$\bar{u} = -\frac{1}{2} N(v, v_x)^{-1} (v_{xx} - cv_x). \tag{85}$$

Substituting this expression for $\bar{u}$ to (83) and (84), one obtains a boundary condition in the usual form. But sometimes it's more convenient to express $\bar{u}$ from the equation (84). In the examples given below we just follow such a way.

Example 6.3. The following integrable boundary conditions specified at $x = x_0$ are compatible with the system of Example 2.3:

$$\begin{aligned} &i) \quad u = 0, \qquad v = 0; \\ &ii) \quad u_x = cu, \qquad v_x = cv; \\ &iii) \quad u_{xx} = (c - \frac{< u, v >}{Q})u_x - \frac{< v, u_x >}{Q}u, \\ &\qquad\quad v_{xx} = (c - \frac{< u, v >}{Q})v_x - \frac{< u, v_x >}{Q}v. \end{aligned}$$

Here Q is a solution of the equation $Q^2 - cQ + < u, v >= 0$, and c is a scalar parameter.

Example 6.4. The integrable boundary condition for the vector Schrödinger equation of Example 2.4 is of the form:

$$\begin{aligned} u_{xx} + Pu_x - 2\frac{< v, u_x >}{P}u + 2\frac{< u, u_x >}{P}v = 0, \\ v_{xx} + Pv_x - 2\frac{< u, v_x >}{P}v + 2\frac{< v, v_x >}{P}u = 0, \end{aligned}$$

where P is determined from the equation

$$P^4 + (4 < u, v > -c^2)P^2 + 4 < u, v >^2 -4||u||^2||v||^2 = 0,$$

and c is a scalar parameter.

Example 6.5. Let us write down the integrable boundary condition of the second order for the matrix Schrödinger equation of Example 2.5. It follows from the general formulas (83), (84) that this condition has the form:

$$u_{xx} + Pu_x + u_xQ = 0, \quad v_{xx} + Qv_x + v_xP = 0.$$

Here P and Q satisfy the equations $P^2 = \frac{c^2}{4}Id - uv$ and $Q^2 = \frac{c^2}{4}Id - vu$, c is a scalar parameter, and Id is the unity matrix.

7 Appendix

We present here definitions and the simplest examples we need in the main body of the paper. We refer to [35,36,33,39,38,40,56] for more detail information.

Let $e_1, e_2, \ldots, e_N$ be a basis of a finite dimensional algebra J over $\mathbf{C}$. The multiplication in J is given by the formula

$$(e_j \circ e_k) = a^i_{jk}e_i, \tag{86}$$

where a^i_{jk} are the structure constants of J. If $a^i_{jk} = a^i_{kj}$ then J is commutative.

The formula

$$X \circ Y = \lambda < X, C > Y + \mu < Y, C > X + \nu < X, Y > C, \tag{87}$$

where $< , >$ is the standard scalar product in a vector space J and C is a given vector, gives us for different λ, μ, ν a number of interesting examples of nonassociative algebras. The so called vector nonlinear differential equations are closely related to those.

We shall use the following notation:

$$AS(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z),$$

$$[X, Y, Z] = AS(X, Y, Z) - AS(Y, X, Z).$$

Note that J is associative iff $AS(X, Y, Z) = 0$.

Definition 1. An algebra J is called leftsymmetric if

$$[X, Y, Z] = 0. \tag{88}$$

Any associative algebra is leftsymmetric one. The formula

$$X \circ Y = < X, C > Y + < X, Y > C, \tag{89}$$

gives us an example of leftsymmetric algebra of the type (87).

Definition 2. A commutative algebra J is said to be Jordan if the following identity is fulfilled

$$AS(X \circ X, Y, X) = 0. \tag{90}$$

The set of all matrices is a Jordan algebra with respect to the anticommutator operation

$$X \circ Y = \frac{1}{2}(XY + YX). \tag{91}$$

The formula

$$X \circ Y = < X, C > Y + < Y, C > X - < X, Y > C \tag{92}$$

turns a vector space J to a Jordan algebra. For a Jordan algebra with the unity e the element X^{-1} is defined as a polynomial of X such that $X \circ X^{-1} = e$.

In the article [30] S.I. Svinolupov and V.V.Sokolov have introduced a class of nonassociative algebras defined by the identity

$$[V, X, Y \circ Z] - [V, X, Y] \circ Z - Y \circ [V, X, Z] = 0. \tag{93}$$

The multiplication (87) satisfies (93) if $\nu = 0$. It is interesting to note that all nonassociative algebras naturally arising in connection with integrable systems (Lie algebras, Jordan algebras, left-symmetric algebras, LT-algebras [56]) satisfy the universal identity (93). Furthermore, the class of algebras with identity (93) is invariant with respect to the deformation (14).

While any collection of constants a^i_{jk} can be regarded as a set of structure constants of an algebra J, every collection σ^i_{jkm} defines a triple system $\{X, Y, Z\}$ by the formula

$$\{e_j, e_k, e_m\} = \sigma^i_{jkm} e_i.$$

Definition 3. A triple system $\{X, Y, Z\}$ is said to be Jordan if

$$\{X, Y, Z) = \{Z, Y, X\},$$

and

$$\{X, \{Y, Z, V\}, W\} - \{W, V, \{X, Y, Z\}\} + \{Z, Y, \{X, V, W\}\} - \{X, V, \{Z, Y, W\}\} = 0.$$

The set of $n \times n$-matrices equipped with the operation

$$\{X, Y, Z\} = \frac{1}{2}(XYZ + ZYX), \tag{94}$$

is a Jordan triple system. The vector space of all $n \times m$-matrices is a Jordan triple system with respect to operation

$$\{X, Y, Z\} = \frac{1}{2}(XY^tZ + ZY^tX), \tag{95}$$

where "t" stands for transposition. The following operations

$$\{X, Y, Z) = < X, Y > Z + < Y, Z > X - < X, Z > Y, \tag{96}$$

and

$$\{X, Y, Z\} = < X, Y > Z + < Y, Z > X \tag{97}$$

define two "vector" (cf. (87)) simple Jordan triple systems.

For each Jordan triple system $\{X, Y, Z\}$ and a given vector C the system

$$\sigma(X, Y, Z) = \{X, \{C, Y, C\}, Z\} \tag{98}$$

is also Jordan one.

There exists close relationships between the Jordan algebras and the Jordan triple systems. Namely, any Jordan algebra generates a triple system by the formula

$$\{X, Y, Z\} = (X \circ Y) \circ Z + (Z \circ Y) \circ X - Y \circ (X \circ Z). \tag{99}$$

Conversely, any Jordan triple system $\{X, Y, Z\}$ yields a family of Jordan algebras with the multiplication

$$X \circ Y = \{X, \phi, Y\}, \tag{100}$$

where ϕ is an arbitrary element.

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HOW THE VARIATIONAL METHOD CAN GIVE RISE TO FALSE INSTABILITIES

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When the variational method is applied to determine dynamics of a pulse, is is possible for the method to predict that the pulse is unstable when in fact it is stable. We determine the necessary conditions for this to occur and, consequently, give sufficient conditions for avoiding such false instabilities. We also discuss the general problem of applying the method to a general evolution equation.

1 Introduction

We present here some rigorous results concerning the general scheme of the Rayleigh–Ritz variational method applied to 1-D evolution equations.

Let us start with a brief exposition of the idea of this method ([1]). Let us consider a nonlinear evolution equation which has a Lagrangian density. We will be interested in determining an approximate evolution of a solitary wave of this equation. Then, to model a solitary wave, one takes an ansatz, which is usually some bell-shaped function with parameters which correspond to the pulse's amplitude, width, velocity, and so on. These parameters are called variational parameters and are allowed to vary with time. Then one inserts this ansatz into the Lagrangian density and integrates it over the spatial coordinate. The result is a reduced Lagrangian, which depends only on time. One uses it to derive Euler-Lagrange equations for the variational parameters. Therefore, one reduces the problem of finding the evolution of the solitary wave to the problem of solving a system of ordinary differential equations.

This method is widely used for approximate calculations because of its simplicity. However, is has a very important drawback, namely that its limitations do not follow from the initial assumptions (about the ansatz), as it is the case for perturbation methods, for example. It is usually believed that how well the results of the variational methods approximate reality depends on how close the ansatz is to the actual solution. This fact makes the variational method unsuitable for the study of stability of a solitary wave. Indeed, if one finds an instability, then it is not clear if it is a true one or if it occurred because the ansatz was wrongly chosen, or maybe it is just a general failure of the variational method. In the last two cases, we call the instability a **false**

instability.

The main goal of this work is to present the necessary conditions under which such a FI can occur. Namely, we found two mechanisms which can give rise to a false instability. Consequently, we can give sufficient conditions for the ansatz to never generate false instabilities.

We would like to mention that this work started when we learned of the result of Malomed and Tasgal ([2]), who used the variational method to study small vibrations of the soliton of the Massive Thirring Model (MTM) and found that the soliton was unstable. Since this result was in apparent contradiction with the result of the Inverse Scattering Transform, we decided that it would be interesting to understand why a FI arised in that case.

The rest of this work is organized as follows. First, we discuss the assumptions that we made in our analysis. Next, we present some details of our analysis of the variational method when it is applied to the NLS. This will demonstrate the first mechanism via which a false instability can occur. We will show that a false instability of this first type can be easily avoided in practice. Then we briefly discuss the case of the MTM and present the second mechanism via which a false instability can arise. Finally, we will show how our approach can be used in the case of a non–integrable evolution equation of a rather general form.

2 Assumptions and Usefulness

The assumptions of our analysis are as follows:

(1) We consider a force–free motion of a pulse, thus all the dynamics is due to the deviation of the pulse's initial profile from the exact solitary wave.

(2) Second, we consider the case when the exact form of the solitary wave is known explicitly;

(3) and consider only small deviations of the pulse's profile from the exact solution.

The above assumptions are exactly the ones that one makes when studying the stability of a solitary wave. Moreover, if one is interested in the case when deviations from the exact solutions are not small, then it is necessary to make sure that in the limit of small deviations, false instabilities will not arise.

Now let us discuss the form of the ansatz which we take. Let $u_0(x,t)$ be the exact solitary wave. A rather general form of the ansatz that will allow for small deviations from it is:

$$u_{\text{close}}(x,t) = u_0(x,t;\,\alpha_1(t),\ldots\alpha_n(t)) \approx u_0 + \sum_{j=1}^{n} \alpha_j(t) \frac{\partial u}{\partial \alpha_j} \,|_{\vec{\alpha}=0}\,. \qquad (1)$$

Here, the variational parameters α's may correspond to small deviations of the pulse's width, amplitude, velocity, etc., from their stationary values. When all α's are zero, one gets back the exact solution. We will ignore the first term in (1) and call the second term the ansatz and the partial derivatives variations. Without loss of generality all the variational parameters can be considered to be real. This is the form which is usually taken in the variational studies.

3 NLS

Consider the NLS equation:

$$iu_t + u_{xx} + 2u|u|^2 = 0 \,. \tag{2}$$

Its one–soliton solution is well–known:

$$u_0(x,t|V) = A\,\mathrm{sech}\theta\,\exp\{\frac{iV}{2A}\theta + i(A^2 + \frac{V^2}{4})t + i\varphi_0\}\,, \tag{3}$$

$$\theta = A(x - Vt - x_0)\,.$$

One can set the velocity V to zero because of the Galilean invariance. The Lagrangian density of the NLS is:

$$\mathcal{L} = \frac{i}{2}(u_t^* u - u^* u_t) - |u_x|^2 + |u|^4\,. \tag{4}$$

If one substitutes here $u = u_0 + \delta u(x,t)$ with $|\delta u| \ll |u_0|$, then the main order term in the Lagrangian becomes:

$$\mathcal{L}_2 = \frac{1}{2}\vec{\mathbf{v}}^\dagger \sigma_3 (i\partial_t + A^2 L)\vec{\mathbf{v}}\,, \tag{5}$$

where

$$\vec{\mathbf{v}} = \begin{pmatrix} \delta u\, e^{-iA^2 t} \\ \delta u^*\, e^{iA^2 t} \end{pmatrix}\,, \quad L = \sigma_3(\partial_\theta^2 - 1) + 2\mathrm{sech}^2\theta\,(2\sigma_3 + i\sigma_2)\,. \tag{6}$$

So far we everything we have done was exact. So if we were to take an arbitrary $\delta u(x,t)$ and derive for it the usual Euler–Lagrange equations, we would obtain simply the NLS linearized about a soliton. But, according to (1), we have to take $\delta u(x,t) = \sum_j \alpha_j(t) u_j(x)$ and integrate only over x. So the result is expected to be different.

It is important to note that the operator L is exactly the NLS linearized about the soliton. Therefore we choose to expand our ansatz $\vec{\mathbf{v}}$ in the eigenfunctions of L. These functions satisfy the equations:

$$L\psi_1(k) = \lambda(k)\psi_1(k)\,, \quad \psi_2(k) = \sigma_1\psi_1(k)^*\,, \quad \lambda(k) = k^2 + 1 \qquad (7a)$$

— for the continuous spectrum of L, and for the discrete spectrum we have:

$$L\phi_{1,2} = 0\,, \quad L\phi^{\mathcal{D}}_{1,2} = -2\phi_{1,2}\,. \qquad (7b)$$

It was shown with the help of the IST that the set of these eigenfunctions is complete in the class of sufficiently smooth and rapidly decaying functions. In particular, we know that the discrete spectrum consists only of the neutral modes, which are simply the derivatives of the exact solution with respect to the amplitude, velocity, phase, and the center coordinate.

Then an arbitrary ansatz can be expanded as follows:

$$\vec{\mathbf{v}}_{\rm var} = \sum_{j=1}^{n} \alpha_j(t)\{\int_{-\infty}^{\infty} dk[g_j(k)\psi_1(k,\theta) + g_j^*(k)\psi_2(k,\theta)] + \\ + \sum_{m=1,2} (\beta_{mj}\phi_j + \gamma_{mj}\phi_j^{\mathcal{D}})\}\,. \qquad (8)$$

If we now substitute this expression into the Lagrangian density and integrate over θ in order to obtain the reduced Lagrangian, we see that we will need inner products between various eigenfunctions. These can be obtained from the Wronskian relations for the operator L. Thus, one obtains the expression for the reduced Lagrangian:

$$< \mathcal{L}_2 > = \\ \sum_{j,l=1}^{n} \{i\alpha_l\dot{\alpha_j}(< g_j|g_l > - < g_l|g_j >) - \alpha_l\alpha_j\, A^2(< g_j|\lambda|g_l > + < g_l|\lambda|g_j >) + \\ i\alpha_l\dot{\alpha_j}(\beta_{1l}\gamma_{1j} - \beta_{1j}\gamma_{1l} + \beta_{2l}\gamma_{2j} - \beta_{2j}\gamma_{2l}) + 2\alpha_l\alpha_j\, A^2(\gamma_{1l}\gamma_{1j} + \gamma_{2l}\gamma_{2j})\}\,. \qquad (9)$$

where

$$< f|m|h >= \pi\int_{-\infty}^{\infty} dk f^*(k)m(k)h(k)\,, \quad \dot{\alpha}_j \equiv \frac{d\alpha_j}{dt}\,.$$

From this form, there follow two important conclusions. First, the variations in the ansatz can be chosen so that to decouple the continuous eigenfunctions from the neutral modes in the reduced Lagrangian. If this is done, then the reduced Lagrangian splits into two parts, with one corresponding to the

continuous spectrum and the other, to the neutral modes. The second part yields trivial evolution of its parameters: all the time derivatives are either zero or constant in time. But it is obvious that in the absence of a driving force, which is the case we consider, one can readjust the background soliton's parameters so as to completely eliminate this trivial part, which is equivalent to setting all the β's and γ's to zero. Moreover, we will later present an example of an ansatz which does contain the neutral modes and which strongly couples them to each other as well as with the continuous modes, within the same variation. Then we show that such an ansatz will give a false instability. So we set all the β's and γ's to zero in what follows.

Then, using the first part of the reduced Lagrangian, one can derive the following matrix system of linear equations for the variational parameters:

$$(M+\omega N)\vec{\alpha}=0\,, \quad \vec{\alpha}=e^{-i\omega t}\cdot(\alpha_1,\dots\alpha_n)^T \tag{10}$$

$$N_{jl}=<g_j|g_l>-<g_l|g_j>$$

$$M_{jl}=<g_j|\lambda(k)|g_l>+<g_l|\lambda(k)|g_j>\,.$$

Now, if we can prove that ω is always real, then it will follow that the variational method predicts only stable oscillations of the soliton, as we know it indeed should be for the NLS. The sufficient condition for ω to be real is that either N or M be sign-definite. N is not sign–definite, but M is a Gram matrix, and therefore is known to be positive definite (see, e.g., [3]). Thus the normal frequencies ω, produced by the variational method, will be always real in this case.

We can now formulate the **first necessary condition** for a false instability to occur: The ansatz must couple the continuous and neutral modes in the same variation.

This condition is, of course, not sufficient: if the coupling is too weak, then a false instability may not arise. To demonstrate that it *can* arise if the coupling is strong enough, we produce the following ansatz:

$$u(x,t)=u_0(x,t)\cdot[\,\alpha(t)+\beta(t)(\theta+i)\,]\,.$$

As one can check, the variational method predicts an exponential growth for α and β. This is, of course, a false instability.

Obviously, if the ansatz does not contain neutral modes, a false instability will never occur. Since we assumed in the beginning that we know the exact solution, we can always find the neutral modes by simply differentiating it with respect to its parameters. Then the variations which are orthogonal to the neutral modes can be found by inspection, and thus a false instability arising via this mechanism can be easily avoided.

It is interesting to note that it turns out that a false instability will also never arise if each of the soliton's parameters (e.g., A, V, x_0, and φ_0 for soliton (3)) is allowed by the ansatz to vary independently of all the other variational parameters.

4 MTM case

Now let's consider the Massive Thirring equations:

$$\begin{aligned} i(u_x + u_t) + v + |v|^2 u = 0 \,, \\ i(-v_x + v_t) + u + |u|^2 v = 0 \,, \end{aligned} \tag{12}$$

It's particular one–soliton solution with zero velocity is:

$$\begin{aligned} u_0(x,t) &= \sin Q \,\mathrm{sech}(x \sin Q - \frac{iQ}{2}) \cdot \exp\{-it \cos Q\} \,, \\ v_0(x,t) &= -\sin Q \,\mathrm{sech}(x \sin Q + \frac{iQ}{2}) \cdot \exp\{-it \cos Q\} \,, \end{aligned} \tag{13}$$

where $0 < Q < \pi$. Then one follows exactly the same outline as for the NLS. The only essential difference is in the expansion: if we denote

$$\vec{\mathbf{w}} = (\delta u \, e^{it \cos Q}, \, \delta u^* \, e^{-it \cos Q}, \, \delta v \, e^{it \cos Q}, \, \delta v^* \, e^{-it \cos Q})^T \,,$$

then

$$\vec{\mathbf{w}}\mathrm{var} = \sum_{j=1}^{n} \alpha_j(t) \int_{-\infty}^{\infty} dk [g_j(k)\psi_1(k,\theta) + g_j^*(k)\psi_2(k,\theta) + h_j(k)\psi_3(k,\theta) + h_j^*(k)\psi_4(k,\theta)] \,,$$

where ψ_2 and ψ_4 are related to ψ_1 and ψ_3 by interchanging the first and second and third and forth entries and taking complex conjugation. It is important for the final result that the expansion coefficients g and h are not related to each other. The functions ψ_1 and ψ_3 satisfy the equations:

$$\begin{aligned} L\psi_1(k) = \lambda(k)\psi_1(k) \,, \quad L\psi_3(k) = \mu(k)\psi_3(k) \,, \\ \lambda(k) = \cos Q + \sqrt{k^2+1} > 0 \,, \quad \mu(k) = \cos Q - \sqrt{k^2+1} < 0 \,. \end{aligned} \tag{14}$$

When taken together with the neutral modes, this set of eigenfunctions is complete, which can be proved with the help of the IST. We excluded the neutral modes from the expansion because otherwise a false instability of the first type can occur, while we are interested in avoiding it.

Then the matrix form of the Euler–Lagrange equations for the variational parameters is:

$$((M_1 - M_2) + \omega(N_1 + N_2))\vec{\alpha} = 0 \,, \tag{15}$$

where M_1 and M_2 are positive definite Gram matrices, and M_1 is composed only of the expansion coefficients g and M_2 composed only of the expansion coefficients h. The opposite signs in front of M_1 and M_2 arise because the corresponding eigenfunctions, λ and μ, are of of the opposite sign. Now neither the matrix M nor N is sign–definite, and therefore ω can be complex. If this is the case, then it means that the variational method will produce a false instability.

Thus, the **second mechanism** via which a false instability can arise, requires that there be two different classes of eigenfunctions, like ψ_1 and ψ_3 in the MTM case, to which there would correspond independent expansion coefficients, and their eigenvalues must be of the opposite sign. Then if some variation in the ansatz contains eigenfunctions from both classes, then a false instability can occur.

Consequently, a sufficient condition to guarantee that there will be no false instabilities, is to construct the variations so that each one contains only eigenfunctions from one, but not the other, class. If one knows the explicit form of the eigenfunctions, then it is straightforward to construct such an ansatz. But if one knows the eigenfunctions, then there is small need to use the variational method. We didn't find a way to construct an ansatz which would guarantee no false instability for the MTM, if one doesn't know the eigenfunctions explicitly. This question remains open.

5 General evolution equation

Now let us outline how our method can be applied in the case of an arbitrary evolution equation. By application of our method we mean the analysis which would determine if the variational method can produce a false instability for a given equation, and also what conditions one should impose on the ansatz in order to avoid such a false instability.

Since, by Assumption 2 made in the beginning, one knows the solitary wave explicitly, then one can easily find its neutral modes and then construct the ansatz which doesn't contain them. This will guarantee that a false instability will not arise via the first mechanism. There may also exist additional discrete spectrum modes, and it is these modes which can make the solitary wave truly unstable. For a general evolution equation there is no regular analytic method to determine if such additional discrete spectrum modes are present.

Next one follows exactly the outline which we presented here and determines if the corresponding matrix M is sign-definite or not. Sometimes for this it is enough to show that there are two different classes of eigenfunctions, as it was the case for the MTM, but sometimes this can be a non-trivial problem. If M is not sign-definite, then a false instability of the second type can occur, which means that the results of the variational method can be qualitatively wrong.

Now, if M is sign-definite, then there are still two possibilities. First, if the variational method gives an instability, then this must be a true instability of the solitary wave because we have excluded all the possibilities for a false instability to arise. Second, if the variational method does not give an instability, it is still possible that the instability exists in reality. So there is some uncertainty about the result of the variational method.

To see how large this uncertainty can be, we applied our method to the stability analysis of a solitary wave of the well-known nonlinear fiber coupler problem. The results differ from the exact numerical results obtained in [4] by one or two percent, which is quite good an agreement. However, we would like to emphasize that it is not our main point here whether or not the variational method gives a good approximation for a particular equation. The main point is that now we have been able to determine all the sources which can generate false instabilities for 1-D solitary waves, and if we can eliminate these sources and still have an instability, then it is a true one.

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P-ADIC DESCRIPTION OF CHAOS

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We study stochastic models , where relative frequencies of events oscillate with respect to the ordinary real metric , but stabilize with respect to one of the p-adic metrics. Thus , there is no probability in the usual sense but it is possible to define generalized probabilities ,belonging to the field of p-adic numbers . We estimate the Kolmogorov complexity of such random sequences.

[a]

The p-adic probability theory originated in connection with the problem of the probability interpretation of physical models in which wave functions assume values in the field of p-adic numbers (see Vladimirov, Volovich and Zelenov , 1993, and A.Yu. Khrennikov ,1992-1993 , the main definitions and properties of the p-adic numbers fields Q_p are contained in the appendix). This problem cannot be solved in the framework of Kolmogorov's standard axiomatic probability theory , Kolmogorov, 1933, since real numbers are included into one of Kolmogorov's axioms.

As is known (see the remarks in Kolmogorov ,1933), when he constructed the axiomatic theory in the framework of the measure theory, Kolmogorov used the von Mises frequency theory of probability (the best book on this theory Mises , 1964). Every axiom of Kolmogorov is a theorem in the frequency theory of probability (except for the condition of σ-additivety which Kolmogorov imposed additionally). The p-adic theory of probability was developed in the same way. At the first step , the general frequency theory based on the general (topological) principle of the statistical stabilization of relative frequencies was proposed . By virtue of this principle, the statistical stabilization can be regarded not only in the ordinary real topology on the field of rational numbers Q (and all frequencies are rational) but also in other topologies on Q. These topologies are known as topologies of statistical stabilization and the probabilities defined as the limits of relative frequencies belong to the completion of Q with respect to the stabilization topology.

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1 p-Adic Frequency Theory of Probability

Let us recall main notions of R. von Mises' frequency theory of probability , Mises , 1964 . The basis of this theory is a conception of a collective. Let us consider any experiment S and let us denote by $\Omega = \{\omega_1, ..., \omega_s\}$ the set of all possible results of this experiment (attributes ,labels). We consider only finite sets Ω to simplify our investigations. A set Ω is called a label set or a set of attributes. Let us consider N realizations of S and write a result x_j after each realization.Then we get a finite sequence of observations :

$$x = (x_1, ..., x_N), x_j \in \Omega. \tag{1}$$

A collective is an infinite idealization of (1) :

$$x = (x_1, ..., x_N, ...), x_j \in \Omega, \tag{2}$$

for which two von Mises' principles are valid .

The first one is the principle of the statistical stabilization of relative frequency of each attribute $\omega_i \in \Omega$ in the sequence (2) . Let us compute frequencies $\nu(\omega_i) = n_i/N$,where n_i is the number of realizations of the attribute ω_i after N first observations. The principle of the statistical stabilization of the relative frequencies says :

the frequency $\nu(\omega_i)$ approaches a limit as N approaches infinity for every label $\omega_i \in \Omega$.

This limit $\mathbf{P}(\omega_i) = \lim \nu(\omega_i)$ is said to be a probability of a label ω_i in the frequency theory of probability.

The second von Mises'principle is the principle of randomness of the sequence (2) . But this principle generated a great number of logical problems. Modifications of von Mises' theory without this principle were proposed later . For example , there is the possibility to resolve this problem using the notion of the complexity of the random sequence ,see for example the review Zvonkin and Levin , 1970 . We shall also try to apply to the p-adic theory of probability the Kolmogorov complexity .

All axioms of the Kolmogorov theory of probability are theorems in the frequency theory of probability Mises, 1964 .

Let us develop our theory in the same way. We also investigate infinite sequences (2) of observations. But a new topological principle of the statistical stabilization of relative frequencies is proposed:

the statistical stabilization of the relative frequencies $\nu(\omega_i)$ may be considered with respect to an arbitrary topology τ on Q .

This topology is said to be a topology of the statistical stabilization. Limiting values of $\nu(\omega_i)$ are said to be τ-probabilities. These probabilities belong to the completion of Q with respect to the topology τ.

A sequence (2),for which the principle of the statistical stabilization of the relative frequencies for the topology τ is valid,is said to be a τ-collective.

Let us denote by U_Q the subset of the rational numbers field Q :

$$U_Q = \{q \in Q : 0 \leq q \leq 1\}.$$

And let us denote by U_{Q_τ} the closure of the set U_Q in the completion Q_τ.

The following result is an evident consequence of the topological principle of the statistical stabilization: *Probabilities* $\mathbf{P}(\omega_i)$ *belong to the set* U_{Q_τ} *for an arbitrary* τ*-collective (2).*

As usual let us consider an algebra $\alpha(\Omega)$ of all subsets of Ω and denote by $\mathbf{P}(A), A \in \Omega$, the sum $\sum_{\omega_i \in A} \mathbf{P}(\omega_i)$.It follows from Theorem 1.1 that a probability $\mathbf{P}(A)$ belongs to the set U_{Q_τ} for every $A \in \alpha(\Omega)$.

A p-adic topology τ_p on Q is the most important in applications to physics.

The set U_{Q_p} *is equal to the field* Q_p.

For example,every rational number may be realized as a p-adic probability. There are such "pathological" probabilities (from the point of view of the usual theory of probability) as $\mathbf{P}(A) = 2$, $\mathbf{P}(A) = 100$, $\mathbf{P}(A) = 5/3$,$\mathbf{P}(A) = -1$; it may be possible that $\mathbf{P}(A) = i_p = \sqrt{-1}$,if $p = 1(mod 4)$.

2 A Probability Model of a p-adic Coin

We present a statistical model , where relative frequencies oscillate with respect to the real metric and stabilize with respect to a p-adic one. Thus, we have a possibility to create such random sequences that there is no ordinary frequency probabilities but p-adic probabilities are well defined. Statistical samples with such property we can consider as a new type of randomness. Hence we can consider our investigations as investigations of new types of random sequences. A spectrum of different randomnesses is generated. We wish to present a mathematical model of a device,which can be considered as a p-adic analogue of usual coin.

1.A description of a p-adic coin. Let we have a metal disk with a label "a" on one side and "b" on another. An internal structure of this disk is not so simple as the structure of the usual metal coin. At first, there is an electric apparatus ,which can generate a negative electric charge on the side "a" or on the side "b". A charge can be generated only on one side. So,if it is generated on "a" ,then it is deleted on "b" and vice versa. Secondly, there is a digital computer (or something similar) with a generator of (pseudo)

random numbers, $\xi(\omega) = 0, 1$. For instance, let us consider the case of equal probabilities of realizations of 0 and 1. At the moment we speak about usual generator of random numbers and usual probabilities. Here relative frequencies of realizations of 0 and 1 approach to $1/2$ with respect to the real metric. A digital computer organizes the work of the electric apparatus according to a following algorithm $\mathcal{A}$.

At first, there is the fixed unit of the time ,for instance, $\Delta = 1$ second. If $\xi(\omega) = 0$, then an electric charge (negative) is generated on "a" and it is conserved on "a" during an interval

$$\Delta t' = \Delta t p^2, \tag{3}$$

where Δt is a previous interval during which "a" was electrified. And the first interval $\Delta t_1 = \Delta t_1^a = 1$. Thus, we have intervals $\Delta t = 1, p^2, p^4, ..., p^{2k}, ...$ for the side "a". If $\xi(\omega) = 1$, then an electric charge (also negative) is generated on "b" and an interval during which "b" is electrified is also computed with the aid of (3) , but the first interval $\Delta_1 = \Delta_1^b = p$. Thus, we get intervals $\Delta t = p, p^3, p^5, ..., p^{2k+1}, ...$

This disk with its internal structure is called *a p-adic coin.*

2.A description of coin-tossings. Let us consider a statistical experiment with the p-adic coin. There is an observer $\mathcal{O}$ who does not know anything about an internal structure of this "coin". He see only a metal disk. An observer $\mathcal{O}$ organizes the following experiment with this coin. There is a metal table with the constant positive charge. An observer $\mathcal{N}$ throws the coin on the table. There is one and only one coin-tossing every second. A values of charges are such that the coin always falls in a such way that a top side has not a charge. Besides of all there is yet condition on the algorithms $\mathcal{A}$. This algorithms begins to work after the first coin-tossing.

We are interested in probabilities of realizations of the side "a", $\mathbf{P}(a)$, and the side "b", $\mathbf{P}(b)$.

3.Oscillation of relative frequencies with respect to the real metric. Let $\{\xi_k(\omega)\}$ be a sequence of independent RV ,$\xi_k(\omega) = 0, 1$ with probabilities $1/2$ (Bernoulli scheme) on the standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$ .At the moment we use usual Kolmogorov's definition.

Let us introduce the sums of RV

$$S_n(\omega) = \sum_{k=1}^{n} \xi_k(\omega),$$

$$T_n(\omega) = n - S_n(\omega) = \sum_{k=1}^{n} (1 - \xi_k(\omega)).$$

Set

$$n_m^b(\omega) = \sum_{k=1}^{T_m(\omega)} p^{2(k-1)}, n_m^a(\omega) = \sum_{k=1}^{S_m(\omega)} p^{2k-1}$$

and $N_m(\omega) = n_m^a(\omega)+n_m^b(\omega)$. Introduce relative frequencies $\nu_m^a(\omega) = n_m^a(\omega)/N_m(\omega)$ and $\nu_m^b(\omega) = n_m^b(\omega)/N_m(\omega)$.

Theorem 2. *Relative frequencies $\nu_m^a(\omega)$ and $\nu_m^b(\omega)$ have not a limit in the field of real numbers a.e.(mod$\mathbf{P}$).*

4. Statistical stabilization with respect to the p-adic metric.

Theorem 3. *Relative frequencies $\nu_m^a(\omega)$ and $\nu_m^b(\omega)$ have the limits in Q_p a.e. (mod $\mathbf{P}$).*

Proof. As $S_n(\omega), T_n(\omega) \to \infty$ a.e. (*mod*$\mathbf{P}$), we get that there exist the limits

$$\mathbf{P}_p(a) = \lim_{m\to\infty} \nu_m^a(\omega) =$$

$$[p/(1-p^2)](1-p) = p/(1+p);$$

$$\mathbf{P}_p(b) = \lim_{m\to\infty} \nu_m^b(\omega) = [1/(1-p^2)](1-p) = 1/1+p.$$

These p-adic probabilities do not depend on $\omega \in \Omega$.

For example,if $p = 2$, then 2-adic probabilities of realization of "a" and "b" are equal to $2/3$ and $1/3$ respectively and if $p = 127$,then $\mathbf{P}_a = 127/128$ and $\mathbf{P}_b = 1/128$. There is no problem to generalize this p-adic coin to exchange the algorithms $\mathcal{A}$.

3 On the Kolmogorov Complexity of p-adic Random Sequences

Now we hope to apply to the p-adic case Kolmogorov's idea to define random sequences on the basis of a notion of a complexity of their finite segments.

1. Kolmogorov complexity. Let Ω be the set of all sequences $\omega = (\omega_j)_{j=1}^{\infty}, w_j = 0, 1$. We are interested in functions $f : \Omega \to \Omega$. More precisely we are interested in partially recursive functions.

As usual, finite vectors $x = (x_1, ..., x_n), x_j = 0, 1$, are called words with respect to the alphabet $\{0, 1\}.l(x) = n$ is the length of the word x, .

Definition. (A. N. Kolmogorov) *Let $\mathcal{A}$ be an arbitrary algorithm. A complexity of a word x with respect to $\mathcal{A}$ is*

$$K_{\mathcal{A}}(x) = \min l(\pi),$$

where $\{\pi\}$ are the programs which are able to realize the word x with the aid of $\mathcal{A}$.

This definition depends very much on a structure of $\mathcal{A}$. But A.N.Kolmogorov proved the following theorem, which was a good justification of this definition.

Theorem 4. *There exists such algorithm $\mathcal{A}_*$ (optimal algorithm) that*

$$K_{\mathcal{A}_*}(x) \preceq K_{\mathcal{A}}(x) \tag{4}$$

for every algorithm $\mathcal{A}$.

As usual, (4) means that there exists such constant C that

$$K_{\mathcal{A}_*}(x) \leq K_{\mathcal{A}}(x) + C$$

for all words x. An optimal algorithm $\mathcal{A}_*$ is not unique.

Definition . *The complexity $K(x)$ of the word x is equal to the complexity $K_{\mathcal{A}_*}$ with respect to one fixed (for all considerations) optimal algorithm $\mathcal{A}_*$.*

A.N.Kolmogorov proposed to use the notion of the complexity of a finite word to try to define a random sequences with the aid of complexities of their finite segments. The idea of Kolmogorov was very natural. He proposed to consider a sequence $\omega \in \Omega$ as a random sequence, if finite segments $(\omega)_n = (\omega_1, ..., \omega_n)$ of this sequence had complexities not much less then n. Thus , a sequence ω is a random sequence in the Kolmogorov sense iff it is impossible to find programs π_n ,generating words $(\omega)_n$, with lengths $l(\pi_n) \ll n$. We need a word with a length not less then the length of the segment of ω for coding this segment.

2.An estimate of the complexity of the p-adic coin. We can consider the p-adic coin model in the following way. There is the algorithm $\mathcal{A}$ (now this letter is used for the concrete algorithm considered of the p-adic coin) ,Turing machine, which transforms every sequence $\omega \in \Omega$ into a new sequence $\xi = f(\omega)$. Let $\{s_1(\xi) < s_2(\xi) < ... < s_n(\xi) < ....\}$ are the moments when 0 is changed to 1 or vice versa. Denote by $m_j = |(\xi)_{s_j}|$ a number of 1 in the word $(\xi)_{s_j}$.

Theorem 5. *The following estimate*

$$K((\xi)_{s_j}) \preceq \log_p s_j$$

holds a.e. ω with respect to the standard (real) Bernoulli probability on Ω.

Now we wish to present some philosophical considerations about a nature of stochastics. What is the ordinary stochastics ? This is the stochastics which is considered in the standard real theory of probability on the basis of R.von Mises notion of the collective or on the basis of A.N.Kolmogorov notion of the complexity. It is called randomness. On Kolmogorov's language these are sequences of realizations, where a complexity of a finite segment increases as a length of this segment (really, it is not so simple, but we cannot to study this subject more carefully, see, for example, Zvonkin and Levin , 1970

). We have considered one particular example and have seen that sequences with the property of the p-adic statistical stabilization may be much more simple (in the sense of the Kolmogorov complexity) than the ordinary random sequences. Their complexity increases as $\log_p n$ instead of n in the ordinary theory of probability. Consequently, we could try to classify stochastics more carefully and introduce different types of stochastics :n- stochastics (usual real theory of probability), $\log_p n$- stochastics (p-adic theory of probability) and so on. From the standard point of view we have only random sequences , definite sequences (where the Kolmogorov complexity is bounded) and a large middle class sequences , which are not considered as random or definite. The Kolmogorov complexity of these sequences is unbounded , but it is not increasing as a length . We hope that p-adic theory of probability will generate a classification of this middle class stochastics.

Appendix

The field of real numbers R is constructed as a completion of the field of rational numbers Q with respect to the metric $\rho(x,y) = |x-y|$,where $|\cdot|$ is the usual absolute value ,norm . Fields of p-adic numbers Q_p are constructed in the same way . There is an infinite sequence of p-adic number fields ,there is its own field for every prime number $p = 2,3,5,...$ A p-adic norm $|\cdot|_p$ is defined in the following way . At the first , we define it for natural numbers . Every natural number n can be represented as the product of prime numbers : $n = 2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$. Then $|n|_p = p^{-r_p}$, by the definition $|0|_p = 0$, $|-n|_p = |n|_p$, and $|n/m|_p = |n|_p/|m|_p$. The completion of Q with respect to the metric $\rho_p(x,y) = |x-y|_p$ is a locally compact field Q_p .

And it is an intrinsic fact of the theory of numbers that there is the only possibility to introduce a norm on Q to use a real one $|\cdot|$ or one of p-adic valuations.

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INVERSE PROBLEM, LINEARIZATION AND RELATED TOPICS OF A COUPLED INTEGRABLE, DISPERSIONLESS SYSTEM

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A new coupled integrable, dispersionless system is studied. The system is solved by using the inverse scattering method. Novel solitonic evolution phenomena are recorded. By changing the dependent and independent variables, the system is shown to reduce a set of linearized equations. Furthermore, the connection of our system with an equivalent system is considered by using the fact that the system is related to the symmetry SL(2,R).

1 Introduction

In a recent paper, Oono and one of the authors(K.K)[1] found a new coupled integrable, dispersionless system. Changing variables, we arrive at a simple coupled system:[2]

$$\begin{aligned} &q_{xt} + (rs)_x = 0, \\ &r_{xt} - 2q_x r = 0, \\ &s_{xt} - 2q_x s = 0. \end{aligned} \tag{1}$$

Special cases of $r = s^1$ and $r = s^{*3}$ were solved by the inverse scattering method. Eq.(1) has an important conserved quantity

$$q_x^2 + r_x s_x = q_0^2, \tag{2}$$

where q_0 is constant and determined by the boundary conditions $q_x \to q_0$, $r \to 0$ and $s \to 0$ at $|x| \to \infty$. If $r = s$, (2) is rewritten as

$$q_x^2 + r_x^2 = q_0^2, \tag{3}$$

then (3) is related to the Euclidian group in two dimensions E_2. Eq.(1) was found to be connected to the sine-Gordon equation.[45] If $r = s^*$, (2) is rewritten as

$$q_x^2 + \mathrm{Re}(r_x)^2 + \mathrm{Im}(r_x)^2 = q_0^2, \tag{4}$$

then (4) is related to $O(3) \sim SU(2)$. Eq.(1) was shown to be equivalent to the Pohlmeyer-Lund-Regge equation.[6] By changing variables $r = \rho + \eta$ and $s = \rho - \eta$, (2) yields

$$q_x^2 + \rho_x^2 - \eta_x^2 = q_0^2. \tag{5}$$

(5) is related to $O(2,1) \sim SL(2,R)$. In the last section, we shall discuss the connection of the system (1) with an equivalent system.

At first, we shall solve the system (1) by using the inverse scattering method in the next section. Novel solitonic evolution phenomena will be presented in §3. In §4, we shall give an interesting linearized form of (1) by changing the independent and dependent variables. In §5, we shall consider an equivalent system to our system (1) using the symmetry $SL(2,R)$. The last section will be devoted to the concluding remarks.

2 The Inverse Scattering Method

The inverse scattering scheme of (1) is given by

$$V_x = UV, \quad V_t = WV, \tag{6}$$

where

$$U = -i\lambda \begin{pmatrix} q_x & r_x \\ s_x & -q_x \end{pmatrix}, \quad W = \begin{pmatrix} 0 & -r \\ s & 0 \end{pmatrix} + \frac{i}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

We solve (6) on the same way as Refs.7 and 8 and get a slightly complicated form of the Gel'fand-Levitan equations.

For real λ we define the Jost functions to be

$$\begin{aligned} &\phi \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\lambda q_0 x), \quad \overline{\phi} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(i\lambda q_0 x) \quad \text{for} \quad x \to \infty, \\ &\psi \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\lambda q_0 x), \quad \overline{\psi} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\lambda q_0 x) \quad \text{for} \quad x \to -\infty, \end{aligned} \tag{8}$$

and the scattering coefficients to be

$$\phi = a\,\overline{\psi} + b\,\psi, \quad \overline{\phi} = -\overline{a}\,\psi + \overline{b}\,\overline{\psi}, \tag{9}$$

where $a\,\overline{a} + b\,\overline{b} = 1$. Some conserved quantities are shown as

$$\begin{aligned}
&\sigma_{-1} = q_x^2 + r_x s_x = q_0^2, \\
&\sigma_0 = \frac{r_x}{2q_0}\left(\frac{q_x - q_0}{r_x}\right)_x, \quad \widehat{\sigma}_0 = \frac{s_x}{2q_0}\left(\frac{q_x - q_0}{s_x}\right)_x, \\
&\sigma_1 = \frac{r_x}{2q_0}\left[\frac{\sigma_0^2}{r_x} + \left(\frac{\sigma_0}{r_x}\right)_x\right], \quad \widehat{\sigma}_1 = \frac{s_x}{2q_0}\left[\frac{\widehat{\sigma}_0^2}{s_x} + \left(\frac{\widehat{\sigma}_0}{s_x}\right)_x\right].
\end{aligned} \tag{10}$$

Since there is no dependent relationship between r and s, the relations $\overline{\phi}_1(\lambda) = \phi_2^*(\lambda^*)$, $\overline{\phi}_2(\lambda) = -\phi_1^*(\lambda^*)$, $\overline{\psi}_1(\lambda) = \psi_2^*(\lambda^*)$, $\overline{\psi}_2(\lambda) = -\psi_1^*(\lambda^*)$, $\overline{a}(\lambda) = a^*(\lambda^*)$ and $\overline{b}(\lambda) = b^*(\lambda^*)$ do not hold in the complex λ plane. We must consider two integrals

$$\int_C \frac{d\lambda'}{\lambda' - \lambda}\frac{1}{a(\lambda')}\begin{pmatrix}\phi_1(\lambda')\\ \phi_2(\lambda')\end{pmatrix} e^{i\lambda' q_0 x}, \quad \int_{C'} \frac{d\lambda'}{\lambda' - \lambda}\frac{1}{\overline{a}(\lambda')}\begin{pmatrix}\overline{\phi}_1(\lambda')\\ \overline{\phi}_2(\lambda')\end{pmatrix} e^{-i\lambda' q_0 x}, \tag{11}$$

where the contour C is defined to be the contour in the complex λ' plane, starting from $\lambda' = -\infty + i0^+$, passing over all zeros of $a(\lambda')$ and ending at $\lambda' = +\infty + i0^+$ for λ below C, and the contour C' is defined to be the contour in the complex λ' plane, starting from $\lambda' = -\infty + i0^-$, passing over all zeros of $\overline{a}(\lambda')$ and ending at $\lambda' = +\infty + i0^-$ for λ above C'. To obtain the Gel'fand-Levitan equation, we must introduce two sets of the kernels $K_1(x,z)$, $K_2(x,z)$ and $\widehat{K}_1(x,z)$, $\widehat{K}_2(x,z)$ as

$$\begin{aligned}
\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix}0\\ 1\end{pmatrix} e^{i\lambda q_0 x - \widehat{\mu}_+(x)} + \int_x^\infty \begin{pmatrix}K_1(x,z)\\ i\lambda K_2(x,z)\end{pmatrix} e^{i\lambda q_0 z - \widehat{\mu}_+(x)}\,dz, \\
\begin{pmatrix}\overline{\psi}_1\\ \overline{\psi}_2\end{pmatrix} &= \begin{pmatrix}1\\ 0\end{pmatrix} e^{-i\lambda q_0 x - \mu_+(x)} + \int_x^\infty \begin{pmatrix}i\lambda\widehat{K}_1(x,z)\\ \widehat{K}_2(x,z)\end{pmatrix} e^{-i\lambda q_0 z - \mu_+(x)}\,dz.
\end{aligned} \tag{12}$$

where μ_+ and $\widehat{\mu}_+$ are defined with (10) as

$$\mu_+(x) = \int_x^\infty \sigma_0\,dy, \quad \widehat{\mu}_+(x) = \int_x^\infty \widehat{\sigma}_0\,dy. \tag{13}$$

Solutions are expressed as

$$\begin{aligned}
q_x &= q_0\,\frac{q_0^2 - K_1(x,x)\widehat{K}_2(x,x)}{q_0^2 + K_1(x,x)\widehat{K}_2(x,x)}, \\
r_x &= \frac{2q_0^2 K_1(x,x)}{q_0^2 + K_1(x,x)\widehat{K}_2(x,x)}, \quad s_x = \frac{2q_0^2\widehat{K}_2(x,x)}{q_0^2 + K_1(x,x)\widehat{K}_2(x,x)}.
\end{aligned} \tag{14}$$

The Gel'fand-Levitan equations are given by

$$\begin{aligned}
&G_1(x,y) + q_0^2\,\widehat{F}(x+y) + q_0^2\int_x^\infty \widehat{G}_1(x,z)\widehat{F}(x+z)\,dz = 0,\\
&G_2(x,y) + \int_x^\infty \widehat{G}_2(x,z)\widehat{F}''(x+z)\,dz = 0,\\
&\widehat{G}_1(x,y) + \int_x^\infty G_1(x,z)F''(x+z)\,dz = 0,\\
&\widehat{G}_2(x,y) + q_0^2 F(x+y) + q_0^2\int_x^\infty G_2(x,z)F(x+z)\,dz = 0,
\end{aligned} \tag{15}$$

where $G_1, G_2, \widehat{G}_1$ and $\widehat{G}_2$ are defined as

$$\begin{aligned}
&G_1(x,y) = K_1(x,y)\exp(\mu_+(x) - \widehat{\mu}_+(x)), \quad G_2(x,y) = K_2(x,y),\\
&\widehat{G}_1(x,y) = \widehat{K}_1(x,y), \quad \widehat{G}_2(x,y) = \widehat{K}_2(x,y)\exp(\widehat{\mu}_+(x) - \mu_+(x)).
\end{aligned} \tag{16}$$

Soliton solutions are obtained by assuming that simple zeros in the upper half plane of the scattering data $a(\lambda)$ are pure imaginary $\lambda = i\eta_i$ and that simple zeros in the lower half plane of another scattering data $\overline{a}(\lambda)$ are also pure imaginary $\lambda = -i\xi_i$. Then $F(z)$ and $\widehat{F}(z)$ are given by

$$F(z) = \sum_{i=1}^{N} \frac{C_{0i}}{\eta_i q_0}\exp(-\eta_i q_0 z - \frac{t}{\eta_i}), \quad \widehat{F}(z) = \sum_{i=1}^{M} \frac{\widehat{C}_{0i}}{\xi_i q_0}\exp(-\xi_i q_0 z - \frac{t}{\xi_i}). \tag{17}$$

Giving the scattering data, we can determine $F(z)$ and $\widehat{F}(z)$ and$K_1(x,x)$ and $\widehat{K}_2(x,x)$ through $G_1(x,x)$ and $\widehat{G}_2(x,x)$ from the Gel'fand-Levitan equations. We then obtain the solutions by using the relations (14).

3 Soliton Solutions

We call soliton solutions for q_x, r and s, in turn, Q_x solitons, R solitons and S solitons. R solitons and S solitons are of bright type and Q_x is dark type.

One soliton solution[2] is given under the condition of one simple zero $\lambda = i\eta$ of $a(\lambda)$ in the upper half plane and one simple zero $\lambda = -i\xi$ of $\overline{a}(\lambda)$ in the

lower half plane as

$$
\begin{aligned}
q_x &= q_0\left(1-2\frac{C_0\widehat{C}_0\exp(-2(\eta+\xi)q_0x-(1/\eta+1/\xi)t)}{\eta\xi T^2}\right),\\
r &= -\frac{K_0}{q_0}\frac{\widehat{C}_0\exp(-2\xi q_0x-t/\xi)}{\xi^2T},\\
s &= -\frac{\widehat{K}_0}{q_0}\frac{C_0\exp(-2\eta q_0x-t/\eta)}{\eta^2T},
\end{aligned}
\tag{18}
$$

where $T=1+C_0\widehat{C}_0\exp(-2(\eta+\xi)q_0x-(1/\eta+1/\xi)t)/(\eta+\xi)^2$ and $K_0\widehat{K}_0=q_0^2$.

Profiles of the R and S solitons are not symmetric with respect to their peak position in contrast with that of the Q_x soliton. Velocities of three kinds of soliton are the same $V=-1/2\eta\xi q_0$. At a given time, the maximum or the minimum positions of the Q_x, R and S solitons are different. As these solitons evolve, however, the distance between the peaks of the R and S solitons and the minimum point of the Q_x soliton remarkably remains the same throughout the motion. Their peak high is given by

$$
\begin{aligned}
&Q_x \text{ soliton} \quad -\frac{\eta^2+\xi^2}{2\eta\xi},\\
&\text{R soliton} \quad -\frac{\widehat{K}_0}{2\xi^2}\left[\frac{\xi}{\eta}(\eta+\xi)^2\right]^{\frac{\eta}{\eta+\xi}}\exp\{-2(\xi-\eta)q_0x_0\},\\
&\text{S soliton} \quad -\frac{K_0}{2\eta^2}\left[\frac{\eta}{\xi}(\eta+\xi)^2\right]^{\frac{\xi}{\eta+\xi}}\exp\{2(\xi-\eta)q_0x_0\}.
\end{aligned}
\tag{19}
$$

By depending on relative magnitude of the eigenvalues, amplitude of the R (or S) soliton increases and that of the S (or R) soliton decreases as time evolves.

If we take $\widehat{C}_0=0$, an exponential type of R soliton and constant Q_x exist and S soliton vanishes. Taking $\eta=\xi$, we can reproduce the results of the case of $r=s^1$ where R soliton as well as Q_x soliton are stationary.

Two soliton solution[9] is obtained under the conditions of two simple zeros $\lambda_1=i\eta_1$ and $\lambda_2=\eta_2$ of $a(\lambda)$ in the upper half plane and two simple zeros $\lambda_1=-i\xi_1$ and $\lambda_2=-i\xi_2$ of $\overline{a}(\lambda)$ in the lower half plane. Solving the Gel'fand-Levitan equations, we obtain explicit two soliton solutions. Depending on choices of the eigenvalues, we can typically observe the phenomena such that amplitudes of two solitons increase or decrease, an amplitude of one of the R solitons increases and that of another R soliton decreases, and one of the R solitons is an exponential type and another R soliton is asymmetric bell type. Detailed discussions are given in Ref.9.

4 Linearization of the Coupled System

Let us change the independent variables as

$$\sigma = x + t, \quad \tau = x - t. \tag{20}$$

and introduce new variables

$$V\Psi_\sigma = q_\sigma + rs, \quad \frac{1}{V}\Psi_\tau = q_\tau - rs, \tag{21}$$

we find a linearized form of our system (1):

$$\begin{aligned} &\frac{1}{V^2}\Psi_{\tau\tau} - \Psi_{\sigma\sigma} = 0, \\ &r_{\tau\tau} - r_{\sigma\sigma} - 2\left(V\Psi_\sigma + \frac{1}{V}\Psi_\tau\right) r = 0, \\ &s_{\tau\tau} - s_{\sigma\sigma} - 2\left(V\Psi_\sigma + \frac{1}{V}\Psi_\tau\right) s = 0. \end{aligned} \tag{22}$$

The first equation of (22) is linear and we can obtain solutions and put them into the second and the third equations. We get r and s. And from (21) we solve solution q, in principle. Not all the solutions of (22) give meaningful solutions, but some special solutions give non-trivial solutions.

5 Connection of Our System with an Equivalent System

By changing variables

$$r = \rho + \eta, \quad s = \rho - \eta, \tag{23}$$

(1) is rewritten as

$$q_{xt} + (\rho^2 - \eta^2)_x = 0, \quad \rho_{xt} - 2q_x\rho = 0, \quad \eta_{xt} - 2q_x\eta = 0. \tag{24}$$

Introduce S as

$$S = \begin{pmatrix} -\eta_x & \rho_x + iq_x \\ -\rho_x + iq_x & \eta_x \end{pmatrix}, \tag{25}$$

where $SS = -I$ with $q_0 = 1$ in (5). With an element $g(\alpha, \delta, \beta) \in SL(2, R)$

$$g(\alpha, \delta, \beta) = \begin{pmatrix} \exp(-i(\alpha+\beta)/2)\cosh\delta/2 & -i\exp(-i(\alpha-\beta)/2)\sinh\delta/2 \\ i\exp(i(\alpha-\beta)/2)\sinh\delta/2 & \exp(i(\alpha+\beta)/2)\cosh\delta/2 \end{pmatrix}, \tag{26}$$

we can express S with g as

$$S = gi\sigma_1 g^{-1} \tag{27}$$

with the Pauli's spin matrix σ_1. Then

$$\begin{aligned} q_x &= -\sin\alpha\sin\beta + \cos\alpha\cos\beta\cosh\delta, \\ \rho_x &= \cos\alpha\sin\beta + \sin\alpha\cos\beta\cosh\delta, \\ \eta_x &= -\cos\beta\sinh\delta. \end{aligned} \tag{28}$$

Define

$$R = \frac{1}{2}\int_{-\infty}^{x} [S(y,t), i\sigma_1]dy, \tag{29}$$

then (24) yields the commutation relation

$$[g^{-1}g_t + g^{-1}Rg, \sigma_1] = 0. \tag{30}$$

(30) tells us that $g^{-1}g_t + g^{-1}Rg$ must be expressed as

$$D = d_1\sigma_1 + d_0\sigma_0, \tag{31}$$

with the unit matrix σ_0 and arbitrary functions d_1 and d_0. g follows

$$g_t = -Rg + gD. \tag{32}$$

In special cases we can show that this system includes the sine-Gordon equation and the Pohlmeyer-Lund-Regge equation by assuming some conditions.

6 Concluding Remarks

We have discussed a set of integrable, dispersionless equations (1). We have solved it by using the inverse scattering method and found new solitonic phenomena both in one soliton solution and two soliton solution. Remarkable fact is that the shape of the R and S solitons is asymmetric with respect to their peak position and that the number of R solitons and S solitons is not necessarily the same. Next we have found a set of the linearized equations (22) of (1). We can reproduce one real soliton solution (18) and one complex soliton solution. But we can not succeed to obtain multi-soliton solutions.

We have also discussed relationship between our system and equivalent systems with the element of $SL(2,R)$. The result (32) is very general so we must look for its simple from.

Extending simply our system in such a way as

$$\begin{aligned} &q_{xt} + (\sum_{i=1}^{N} \rho_i^2 - \sum_{j=1}^{M} \eta_j^2)_x = 0, \\ &\rho_{ixt} - 2q_x\rho_i = 0, \quad (i = 1, 2, \ldots, N), \\ &\eta_{jxt} - 2q_x\eta_j = 0, \quad (j = 1, 2, \ldots, M). \end{aligned} \tag{33}$$

which has a conserved quantity

$$q_x^2 + \sum_{i=1}^{N} \rho_{ix}^2 - \sum_{j=1}^{M} \eta_{jx}^2 = q_0^2. \tag{34}$$

(34) is related to $SO(N+1, M)$. For the cases of $N = 1$ and $M = 0$, $N = 2$ and $M = 0$, and $N = 1$ and $M = 1$ the systems are integrable. Are general $SO(N+1, M)$ cases also integrable?

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GENERATING EQUATION FOR CONSTANT MEAN CURVATURE SURFACES AS HAMILTONIAN SYSTEM

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It is shown that the equation whose solutions generate constant mean curvature (CMC) surfaces via the generalized Weierstrass–Enneper formulae have Hamiltonian form. Its simplest finite–dimensional reduction is the integrable Hamiltonian system with two degrees of freedom and corresponds to well–known Delaunay and do Carmo–Dajczer surfaces, i.e., CMC surfaces of revolution and helicoidal CMC surfaces respectively.

Mean curvature plays special role among the characteristics of surfaces and their dynamics in several problems both in physics and mathematics (see, e.g., [1,2]). Surfaces of constant mean curvature have been studied intensively during last years by using of methods of nonlinear analysis ([3,4]) and soliton theory ([5,6]).

In present paper we discuss a new approach for construction of constant mean curvature surfaces. This method is based on the generalized Weierstrass–Enneper inducing ([7,8,9]). It allows to generate constant mean curvature surfaces via integrable dynamical system with two degrees of freedom. The relation between the trajectories of different types and surfaces of different types is established.

This approach also arrives to exact formulae as the other one which is based on soliton theory. Our approach differs from soliton's one because CMC

surfaces can be constructed by direct using of solutions of equations considered below. The method based on sinh–Gordon equation gives us formulae for Gauss mapping of CMC surface and this itself is reconstructed by additional procedure based on the Sym formula ([6]).

The generalization of the Weierstrass–Enneper formulas for inducing minimal surfaces have been proposed in [7] (see also [8]) and rediscovered in different but equivalent form in connection with integrable nonlinear equations in [9]. We will use notation and formulae from [9].

We start with the linear system

$$\psi_{1z} = p\psi_2,$$

$$\psi_{2\bar{z}} = -p\psi_1, \tag{1}$$

where $p(z,\bar{z})$ is a real function, ψ_1 and ψ_2 are, in general, complex functions of the complex variable z, and bar denotes the complex conjugation. By using of the solution of (1), one introduces the variables $(X^1(z,\bar{z}), X^2(z,\bar{z}), X^3(z,\bar{z}))$ as follows

$$\begin{aligned} X^1 + iX^2 &= 2i\int_{z_0}^{z}(\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \\ X^1 - iX^2 &= 2i\int_{z_0}^{z}(\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \\ X^3 &= -2\int_{z_0}^{z}(\psi_2\bar{\psi}_1 dz' + \psi_1\bar{\psi}_2 d\bar{z}'). \end{aligned} \tag{2}$$

In virtue of (1) integrals (2) do not depend on the choice of the curve of integration.

Then one treats $z, \bar{z}$ as local coordinates on a surface and (X^1, X^2, X^3) as coordinates of its immersion in $\mathbf{R}^3$. Formulae (2) induce a surface in $\mathbf{R}^3$ via the solutions of system (1). By using of the well-known formulae, one finds the first fundamental form

$$\tilde{\Omega} = 4(|\psi_1|^2 + |\psi_2|^2)^2 dz d\bar{z} \tag{3}$$

and Gaussian (K) and mean (H) curvatures

$$K = -\frac{(\log(|\psi_1|^2 + |\psi_2|^2))_{z\bar{z}}}{(|\psi_1|^2 + |\psi_2|^2)^2}, \quad H = \frac{p(z,\bar{z})}{|\psi_1|^2 + |\psi_2|^2}. \tag{4}$$

This type of inducing of surfaces is the generalization of the well-known Weierstrass–Enneper inducing of minimal surfaces. Indeed, minimal surfaces $(H \equiv 0)$ correspond to $p \equiv 0$ and in these case formulae (2) in terms of

functions $\psi = \frac{1}{\sqrt{2}}\psi_2$ and $\phi = \frac{1}{\sqrt{2}}\bar{\psi}_1$ are reduced to those of Weierstrass–Enneper.

In this paper we will consider the case of constant mean curvature surfaces. In this case $p = H(|\psi_1|^2 + |\psi_2|^2)$ where $H = const$ and system (1) is reduced to the following

$$\begin{aligned} \psi_{1t} - i\psi_{1x} &= 2H(|\psi_1|^2 + |\psi_2|^2)\psi_2, \\ \psi_{2t} + i\psi_{2x} &= -2H(|\psi_1|^2 + |\psi_2|^2)\psi_1, \end{aligned} \tag{5}$$

where $z = t + ix$.

First we note that system (5) has four obvious real integrals of motion (independent on t):

$$\begin{aligned} C_+ &= \int dx(\psi_1^2 + \psi_2^2 + \bar{\psi}_1^2 + \bar{\psi}_2^2), \\ C_- &= \frac{1}{i}\int dx(\psi_1^2 + \psi_2^2 - \bar{\psi}_1^2 - \bar{\psi}_2^2), \\ P &= \int dx(\psi_{1x}\bar{\psi}_2 - \bar{\psi}_1\psi_{2x}), \\ \mathcal{H} &= \int dx\{\frac{i}{2}(\psi_{1x}\bar{\psi}_2 + \bar{\psi}_1\psi_{2x}) + H(|\psi_1|^2 + |\psi_2|^2)^2\}. \end{aligned} \tag{6}$$

Then this system is Hamiltonian, i.e. it can be represented in the form

$$\psi_{1t} = \{\psi_1, \mathcal{H}\}, \psi_{2t} = \{\psi_2, \mathcal{H}\} \tag{7}$$

where the Hamiltonian $\mathcal{H}$ is given by (6) and the Poisson bracket $\{,\}$ is of the form

$$\{F_1, F_2\} = \int dx\{(\frac{\delta F_1}{\delta\psi_1}\frac{\delta F_2}{\delta\bar{\psi}_2} - \frac{\delta F_1}{\delta\psi_2}\frac{\delta F_2}{\delta\bar{\psi}_1}) - (\frac{\delta F_2}{\delta\psi_1}\frac{\delta F_1}{\delta\bar{\psi}_2} - \frac{\delta F_2}{\delta\psi_2}\frac{\delta F_1}{\delta\bar{\psi}_1})\}. \tag{8}$$

The corresponding symplectic form is

$$\Omega = d\psi_1 \wedge d\bar{\psi}_2 + d\bar{\psi}_1 \wedge d\psi_2$$

and the Lagrangian is given by the following formula

$$\mathcal{L} = \psi_1\bar{\psi}_{2z} - \bar{\psi}_{1\bar{z}}\psi_2 + \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2)^2.$$

Thus formula (2) establishes the correspondence between the trajectories of the infinite–dimensional Hamiltonian system (5) and surfaces of constant mean curvatures.

Let us put

$$H \neq 0$$

to omit the discussion of minimal surfaces.

Let us also restrict ourselves to the particular case of this inducing with $p = p(t)$. It is not difficult to show that under this constraint the only admissible solutions, of system (5), which are representable by finite sums of terms of the type $f(t)\exp i\rho x$ are of the form

$$\psi_1 = r(t)\exp(i\lambda x), \psi_2 = s(t)\exp(i\lambda x), \tag{9}$$

where $\lambda(\neq 0)$ is real parameter and $r(t) = p_1 + ip_2$ and $s(t) = q_1 + iq_2$ are complex-valued functions. System (5) in these variables has the following form

$$\begin{aligned} r_t + \lambda r - 2H(|r|^2 + |s|^2)s = 0, \\ s_t - \lambda s + 2H(|r|^2 + |s|^2)r = 0, \end{aligned} \tag{10}$$

or equivalent system of four equations in terms of real and imaginary parts of r and s. It has the Hamiltonian form

$$\frac{\partial p_i}{\partial t} = \{p_i, \mathcal{H}_0\}_0, \frac{\partial q_j}{\partial t} = \{q_j, \mathcal{H}_0\}_0, \quad i,j = 1,2,$$

with the Hamiltonian function

$$\mathcal{H}_0 = \frac{H}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - \lambda(p_1q_1 + p_2q_2)$$

and with respect to the usual Poisson brackets $\{,\}_0$ generated by the symplectic form

$$\Omega_0 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

It is easy to notice that the Hamiltonian function $\mathcal{H}_0$ can be obtained from $\mathcal{H}$ by using of the finite dimensional reduction (9). Hamiltonian system (10) has another first integral

$$M = p_1q_2 - p_2q_1$$

which is in involution with the Hamiltonian $\mathcal{H}_0$ and moreover these first integrals are functionally independent everywhere except the zero ($p_i = q_j = 0$). Thus we conclude that system (10) is integrable.

This system is not only integrable but also S^1-symmetric. It's Hamiltonian, the additional first integral M and the Poisson structure are preserved by the following S^1-action:

$$\begin{cases} p_1 \to p_1\cos\phi - p_2\sin\phi \\ p_2 \to p_1\sin\phi + p_2\cos\phi \end{cases}, \quad \begin{cases} q_1 \to q_1\cos\phi - q_2\sin\phi \\ q_2 \to q_1\sin\phi + q_2\cos\phi \end{cases}. \tag{11}$$

Let us assume without loss of generality that

$$\lambda = H = \frac{1}{2}.$$

Formulae (2) obtain the following form

$$\begin{aligned}
X^1 &= -2\int\{[(p_1^2+q_1^2-p_2^2-q_2^2)\cos x - 2(p_1p_2+q_1q_2)\sin x]dx \\
&\quad +[2(q_1q_2-p_1p_2)\cos x + (q_1^2+p_2^2-q_2^2-p_1^2)\sin x]dt\}, \\
X^2 &= 2\int\{[2(p_1p_2+q_1q_2)\cos x + (p_1^2+q_1^2-p_2^2-q_2^2)\sin x]dx \\
&\quad +[(p_1^2+q_2^2-p_2^2-q_1^2)\cos x + 2(q_1q_2-p_1p_2)\sin x]dt\}, \qquad (12) \\
X^3 &= -4\int\{(p_1q_1+p_2q_2)dt - (p_1q_2-p_2q_1)dx\}.
\end{aligned}$$

Trajectories of Hamiltonian system (10) which are different modulo symmetry (11) describe different constant mean curvature surfaces by using of formulas (12). It also follows from (12) that these surfaces are invariant under the following helicoidal transform:

$$\begin{cases} X^1 \to X^1\cos\tau - X^2\sin\tau \\ X^2 \to X^1\sin\tau + X^2\cos\tau \\ X^3 \to X^3 + 4M\tau \end{cases}, \qquad (13)$$

and the restriction, of this transform, to the surface coincides with the shift of $Imz = x: \quad x \to x+\tau$.

We see that if $M = 0$ then we obtain a surface of revolution. All these surfaces are equivalent modulo (11) to surfaces with $p_2 \equiv q_2 \equiv 0$. It is not complicated to give a qualitative analysis of the behaviour of the restriction of (10) onto this plane. This vector field has three zeros at points $(0,0)$ and $\pm\frac{1}{2}, \pm\frac{1}{2})$. The second ones correspond to cylinders of revolution. At these points the Hamiltonian $\mathcal{H}_0$ is equal to $-\frac{1}{32}$. These points are bounded by cycles on which Hamiltonian is negative but more than $-\frac{1}{32}$ and which correspond to unduloids (i.e., the Delaunay surfaces which are embedded into $\mathbf{R}^3$ and differ from cylinder and round sphere). Hamiltonian vanishes at the zero point and two separatrices which come from $(0,0)$ and arrive to it. These separatrices correspond to round sphere with a pair of truncated points and bound a domain where Hamiltonian is negative. The domain $\mathcal{H}_0 > 0$ is fibered by cycles of Hamiltonian system (10) and these cycles corresponds to nodoids (i.e., Delaunay surfaces which have selfintersections).

Thus we obtain very natural Hamiltonian interpretation for the well-known family of Delaunay surfaces ([10]).

In the same manner it is shown that the full family of surfaces which corresponds to solutions of (10) with $M \neq 0$ coincides with the family of helicoidal surfaces of constant mean curvature which were constructed in [11].

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SYMMETRY ANALYSIS OF DISCRETE PHYSICAL SYSTEMS

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Two different approaches to symmetries of difference equations and differential-difference equations are reviewed and applied to physical problems. Both make use of a Lie algebraic approach. In one of them the symmetry algebra is realized by differential operators, in the other by difference ones.

1 Introduction

The purpose of this presentation is to review some recent progress made in the study of Lie symmetries of differential difference equations, or differential equations on lattices.

Two different approaches will be discussed. In the first[1,2,3,4] we consider a function $u(x,n)$, $u \in \mathbb{R}^p$, $x \in \mathbb{R}^q$, $n \in \mathbb{Z}^m$, depending on a set of continuous variables x and a set of discrete ones n. The function is subject to a differential-difference equation ($D\Delta E$) of the form

$$E^{\mu}\big(x,n,u(n+k,x),u_{x_i}(n+k,x),u_{x_ix_j}(n+k,x)\big)=0,$$
$$\mu=1,\ldots,N, \quad (1.1)$$

where $k \in \mathbb{Z}^n$ varies between some finite (vector) limits. Continuous and discrete symmetries are considered separately. The continuous ones are found using an infinitesimal method, a modification of the prolongation techniques used for differential equations[5,6,7,8]. Discrete symmetries, specially those acting on the discrete variables n, are then found separately, not using infinitesimal techniques, essentially by inspection. The discrete and continuous symmetries are combined into one symmetry group $g = g_D \rhd g_c$, with the (local) continuous part as an invariant subgroup. Subgroups of g can then be used for

instance to perform symmetry reduction, i.e. to reduce the number of independent variables in the equation. "Nonsplitting subgroups" that are themselves not semidirect products of a subgroup of g_D with a subgroup of g_c are of particular interest in this context.

In the second approach to be discussed, all independent variables are viewed as being continuous, however some, or all, of them vary discretely. Thus, instead of derivatives, the equations will involve variations, or discrete derivatives $\Delta_x u$ with respect to these variables. We have

$$\Delta_x u(x) = \frac{u(x+\sigma) - u(x)}{\sigma}, \tag{1.2}$$

where σ is the finite spacing in the x direction (the spacing can be different for each variable, i.e. σ_i for x_i). We also introduce a shift operator T_x with action

$$T_x u(x) = u(x+\sigma), \tag{1.3}$$

and we then have

$$\Delta_x = \frac{T_x - 1}{\sigma} \tag{1.4}$$

In the case when the increments of all variables x_i are discrete we write the equations to be studied as

$$E^\mu\big(x, \sigma, \quad T^\alpha u(x), \quad T^{\beta_i}\Delta_{x_i} u, \quad T^{\gamma_{ij}}\Delta_{x_i}\Delta_{x_j} u, \ldots\big) = 0 \qquad \mu = 1, \ldots, N. \tag{1.5}$$

with e.g.

$$T^\alpha = T_{x_1}^{\alpha_1} T_{x_2}^{\alpha_2} \ldots T_{x_q}^{\alpha_q} \quad , a_i \leq \alpha_i \leq b_i \tag{1.6}$$

where α_i are integers, varying over some finite ranges and T^{β_i}, $T^{\gamma_{ij}}$, $\cdots$ are defined similarly.

Continuous transformations amongst solutions of eq. (1.5) are now treated infinitesimally, in terms of "discrete vector fields" that can be written in the form

$$X = \xi^i(x,u)\Delta_{x_i} + \phi^a(x,u)\partial_{u_a} \tag{1.7}$$

and below we shall describe prolongation techniques for this case [9].

A similar approach, restricted to linear difference equations, was recently developed in terms of commuting linear difference operators[10,11]. The methods

of Ref. [1,2,3,4,9] and this paper are applicable to both linear and nonlinear equations.

We mention that Dorodnitsyn and collaborators have developed quite different techniques that are used to discretize differential equations while preserving all their symmetries of the continuous ones[12,13].

2 Differential Operator Symmetry Algebras for Difference Equations

2.1 The Formalism

Let us consider eq. (1.1) and find its Lie point symmetries satisfying

$$\widetilde{x} = \Lambda_g(x, u_n) \quad , \quad \widetilde{u}_n = \Omega_g(x, n, u_n), \quad \widetilde{n} = n, \tag{2.1}$$

where g denotes the group parameters, Λ_g and Ω_g are some locally smooth and invertible functions, such that if $u_n(x)$ is a solution and $\tilde{u}_n(\tilde{x})$ exists, it is also a solution. For simplicity, we consider u to be a scalar, n a single discrete variable, while x can be a vector. The variable n at this stage is not transformed ($\widetilde{n} = n$). We shall actually look for infinitesimal transformations, i.e. put

$$g = I + \varepsilon, \quad \widetilde{x} = x + \varepsilon\xi(x, u_n), \quad \widetilde{u}_n = u_n + \varepsilon\phi(x, n, u_n) \tag{2.2}$$

The requirement that solutions be transformed into solutions is best formulated in terms of vector fields

$$\widehat{X} = \xi^i(x, u_n)\partial_{x^i} + \phi(x, n, u_n)\partial_{u_n} \tag{2.3}$$

(same ξ and ϕ as in eq. (2.2)). The prolongation $\mathrm{pr}^{(N)} X$ will have the form

$$\begin{aligned}\mathrm{pr}^{(N)}\widehat{X} = &\sum_{i=1}^{q}\xi_i(x, u_n)\partial_{x_i} + \sum_{k=-a}^{b}\phi(x, k, u_{n+k}\partial_{u_{n+k}} \\ &+ \sum_{i=1}^{q}\sum_{k=-a_i}^{b_i}\phi^{x_i}(x, k, \ldots)\partial_{u_{n+k,x_i}} \\ &+ \sum_{i,j}\sum_{k=-a_{ij}}^{b_{ij}}\phi^{x_i x_j}(x, k, \ldots)\partial_{u_{n+k,x_i x_j}} + \cdots\end{aligned} \tag{2.4}$$

In eq. (2.4) N is the order of the equation, ϕ^{x_i}, $\phi^{x_i x_j}$, ... are expressed in terms of total derivatives of the coefficients ϕ, and ξ just as in the purely continuous

case[5,6,7,8]. The only specific feature, characterizing the fact that eq. (1.1) is a differential-difference equation, is the summation over k in eq. (2.4), i.e. a summation over the different values of the discrete variable, figuring in eq. (1.1).

The invariance requirement for eq. (1.1) is translated into the condition that $\mathrm{pr}^{(N)}\,\widehat{X}$ should annihilate the equation on its solution set

$$\mathrm{pr}^{(N)}\,\widehat{X}E\,|_{E=0}=0 \tag{2.5}$$

The determining equations for the Lie symmetries, i.e. for the coefficients ξ_i and ϕ are then read off from eq. (2.5), as coefficients of linearly independent expressions in the derivatives u_{n,x_i}, u_{n,x_ix_j}, ... and differences, $u_{n+1}-u_n$, u_n-u_{n-1}, ...

2.2 Example of the Two-Dimensional Toda Lattice

We apply the above algorithm to the two-dimensional Toda lattice equation

$$u_{n,xt}=e^{u_{n-1}-u_n}-e^{u_n-u_{n+1}}, \tag{2.6}$$

also called the Mikhailov-Fordy-Gibbons system[14,15]. Details are elsewhere[1], here we just present the result. The general element of the symmetry algebra (2.3) in this case has the form

$$\widehat{X}=T(f)+X(g)+U(k)+W(h) \tag{2.7a}$$

with

$$\begin{aligned} T(f)&=f(t)\partial_t+\dot{f}(t)n\partial_{u_n}\\ X(g)&=g(x)\partial_x+\dot{g}(x)n\partial_{u_n}\\ U(k)&=k(t)\partial_{u_n}\\ W(h)&=h(x)\partial_{u_n} \end{aligned} \tag{2.7b}$$

where f, g, k, and h are arbitrary (locally C^∞) functions of their argument.

The non-zero commutation relations are

$$\begin{aligned} [T(f_1),T(f_2)]&=T(f_1\dot{f}_2-\dot{f}_1f_2)\\ [X(g_1),X(g_2)]&=X(g_1g_2'-g_1'g_2)\\ [T(f),U(k)]&=U(f\dot{k})\\ [X(g),W(h)]&=W(gh') \end{aligned} \tag{2.8}$$

(we have $W(1)=U(1)$).

We see that the symmetry algebra of eq. (2.6) is infinite-dimensional, involving four arbitrary functions $f(t)$, $k(t)$, $g(x)$ and $h(x)$. The algebra has a Kac-Moody-Virasoro structure with $\{T(f)\}$ and $\{X(g)\}$ representing Virasoro algebras and $\{U(k)\}$ and $\{W(h)\}$ $\hat{u}(1)$ Kac-Moody algebras.

Thus the two-dimensional Toda lattice shares a distinctive property of other integrable systems involving three independent variables. Indeed the Kadomtsev-Petviashvili equation, the Davey-Stewartson equation, the 3-wave resonant interaction equation and other integrable equations with three (continuous) variables have centerless Kac-Moody-Virasoro algebras as symmetry algebras[16,17,18,19]. In eq. (2.6) we also have three independent variables, two continuous and one discrete, and the algebra is again a Kac-Moody-Virasoro one.

Eq. (2.6) is also invariant under discrete translations of n : $\tilde{n} = n + N$, $N \in \mathbb{Z}$. Formally, we can treat this discrete symmetry algebraicly, by adding a differential operator $Z = \partial_n$ to the symmetry algebra and requesting that the group parameter corresponding to Z be an integer.

As an application of the symmetries of eq. (2.6), let us consider symmetry reduction by a two-dimensional subalgebra generated by

$$\widehat{X}_1 = \partial_x - \partial_t \quad , \quad \widehat{X}_2 = \partial_n + \frac{a}{2}(\partial_x + \partial_t) + k(x+t)\partial_{u_n} \tag{2.9}$$

The corresponding group transformation is

$$\begin{aligned} &\tilde{x} = x + \lambda + \frac{a}{2}N \quad \tilde{t} = t - \lambda + \frac{a}{2}N \\ &\tilde{u} = u + kN(x+t) + \frac{1}{2}kaN^2 \quad , \quad \tilde{n} = n + N. \end{aligned} \tag{2.10}$$

The invariants are

$$\xi = x + t - an \quad , \quad F = u_n - \frac{k}{2a}(x+t)^2.$$

Putting

$$u_n = \frac{k}{2a}(x+t)^2 + F(\xi) \tag{2.11}$$

and substituting into eq. (2.6) we find that $F(\xi)$ must satisfy a "differential delay" equation

$$F_{\xi\xi} + \frac{k}{a} = e^{F(\eta+a)-F(\eta)} - e^{F(\eta)-F(\eta-a)} \tag{2.12}$$

The well-known Toda lattice soliton[20]

$$u_n(x,t) = \ln \frac{1+\exp(2\xi \sinh\alpha - \alpha)}{1+\exp 2\eta \sinh\alpha} \qquad a = -\frac{\alpha}{\sinh\alpha} \tag{2.13}$$

is a solution of equation(2.12).

Other reductions, using continuous, discrete and mixed subgroups, are discussed in Ref. 1.

2.3 Group Classification of Discrete Dynamical Systems

As an application of the methods of Section 2.1, let us consider the $D\Delta E$

$$E_n \equiv \ddot{u}_n(t) - F_n\big(t, u_{n-1}(t), u_n(t), u_{n+1}(t)\big) = 0, \tag{2.14}$$

The equation is of the form (1.1) with $N = p = q = m = 1$. We wish to classify such equations into conjugacy classes, where each class is characterized by the symmetry group of the equations in the class. The classification will be under "allowed transformations" that take eq. (2.14) into an equation of the same form, but possibly with a different function $\widetilde{F}_n\big(\tilde{t}, \tilde{u}_{n-1}, \tilde{u}_n, \tilde{u}_{n+1}\big)$. We restrict the allowed transformations to the form

$$u_n(t) = \Omega_n(\tilde{u}_n(\tilde{t}), t, g), \quad \tilde{t} = \tilde{t}(t,g), \quad \tilde{n} = n, \tag{2.15}$$

where Ω_n and $\tilde{t}$ are some locally smooth and invertible functions; g are group parameters. Lie point symmetries of eq. (2.4) will be realized by vector fields as in eq. (2.3), or more specifically

$$\widehat{X} = \tau(t,u_n)\partial_t + \phi_n(t,u_n)\partial_{u_n}, \tag{2.16}$$

The invariance condition is as in eq. (2.5).

Equations of the form (2.14) arise in physical applications, for instance in studies of molecular chains[21,22]. The interaction functions F_n are usually not completely specified in the models considered. A symmetry classification of these interactions can be used to single out particularly tractable, or even integrable models.

We make several restrictions in our analysis. The interaction F_n involves only nearest neighbours (u_{n+k} with $k = 0, \pm 1$). The interaction is nonlinear and coupled, i.e.

$$\frac{\partial^2 F_n}{\partial u_i \partial u_k} \neq 0, \quad \left(\frac{\partial F_n}{\partial u_{n-1}}, \frac{\partial F_n}{\partial u_{n+1}}\right) \neq (0,0) \tag{2.17}$$

(the first relation holds for at least one set i, k).

To calculate the Lie point symmetry algebra of eq. (2.14) we calculate the relevant terms of $\mathrm{pr}^{(2)}\,\widehat{X}$:

$$\mathrm{pr}^{(2)}\,\widehat{X} = \tau(t,u_n)\partial_t + \sum_{k=n-1}^{n+1} \phi_k(t,u_k)\partial_{u_k} + \phi_n^{tt}\partial_{u_{n,tt}}, \qquad (2.18)$$
$$\phi_n^{tt} = D_t^2\phi_n - (D_t^2\tau)u_{n,t} - 2(D_t\tau)u_{n,tt},$$

Applying the algorithm (2.5) and substituting for $u_{n,tt}$ from eq. (2.14), we find that the vector field (2.16) has the form

$$\widehat{X} = \tau(t)\partial_t + \left[\left(\frac{1}{2}\dot{\tau}(t) + a_n\right)u_n + \beta_n(t)\right]\partial_{u_n},\ \dot{a}_n = 0. \qquad (2.19)$$

The interaction function F_n, the function of t, $\tau(t)$, the function of n and t, $\beta_n(t)$, and the function of n, a_n, are related by one remaining determining equation, namely

$$\frac{1}{2}\dddot{\tau}u_n + \ddot{\beta}_n + \left(a_n - \frac{3}{2}\dot{\tau}\right)F_n - \tau F_{n,t} - \sum_\alpha \left[\left(\frac{1}{2}\dot{\tau} + a_\alpha\right)u_\alpha + \beta_\alpha\right]F_{n,u_\alpha} = 0. \qquad (2.20)$$

The allowed transformations (2.15) turn out to have the form

$$u_n(t) = \frac{A_n}{\sqrt{\tilde{t}_t}}\tilde{u}_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \tilde{t}(t), \quad A_{n,t} = 0, \quad \tilde{t}_t \neq 0, \quad A_n \neq 0, \quad \tilde{n} = n. \qquad (2.21)$$

where $B_n(t)$, A_n and $\tilde{t}(t)$ can be freely chosen. Allowed transformations will simultaneously transform the equation (2.14) and the elements (2.19) of its symmetry algebra. Since F_n is a priori not specified, we find it more efficient to use the allowed transformations to simplify the vector fields.

Indeed, let us assume that the interaction F_n is such that a nontrivial symmetry algebra L exists, at least a one-dimensional one.

We choose an element $X \in L$; it will have the form (2.19) with coefficients satisfying eq. (2.20). Allowed transformations take X into

$$\widehat{X} = \tau(t)\tilde{t}_t\partial_{\tilde{t}} + \left\{\left[\frac{\tau}{2}\tilde{t}_{tt}(\tilde{t}_t)^{-1} + \frac{1}{2}\dot{\tau} + a_n\right]\tilde{u}_n + (\tilde{t}_t)^{1/2}A_n^{-1}\left[\left(\frac{1}{2}\dot{\tau} + a_n\right)B_n + \beta_n - \tau B_{n,t}\right]\partial_{\tilde{u}_n}. \qquad (2.22)$$

where $\tau(t)$, a_n and $\beta_n(t)$ are to be viewed as given, $\tilde{t}(t)$, A_n and $B_n(t)$ as freely chosable. Using this freedom we can transform the given vector field $\widehat{X}$ into one of three "standard" forms, each of them quite simple. We then view eq. (2.20) as a first order linear partial differential equation for F_n and solve it, using the method of characteristics.

This way we find that three different one-paremeter symmetry algebras exist. Together with the corresponding interactions, they are represented by the following

$$
\begin{aligned}
A_{1,1} &: \widehat{X} = \partial_t + a_n u_n \partial_{u_n} \\
&\quad F_n = e^{a_n t} f_n(\xi_{n-1}, \xi_n, \xi_{n+1}), \qquad (2.23) \\
&\quad \xi_k = u_k e^{-a_k t}, \qquad k = n, n \pm 1 \\
A_{1,2} &: \widehat{X} = a_n u_n \partial_{u_n} \\
&\quad F_n = u_n f_n(t, \xi_{n-1}, \xi_{n+1}), \xi_k = u_k^{a_n} u_n^{-a_k}, k = n \pm 1 \qquad (2.24) \\
A_{1,3} &: \widehat{X} = \beta_n(t) \partial_{u_n} \\
&\quad F_n = \frac{\ddot{\beta}_n}{\beta_n} u_n + f_n(t, \xi_{n-1}, \xi_{n+1}) \qquad (2.25) \\
&\quad \xi_n = \beta_n(t) u_k - \beta_k(t) u_n, \quad k = n \pm 1
\end{aligned}
$$

The algebras $A_{1,1}$, $A_{1,2}$, $A_{1,3}$ correspond to a vector field $\widehat{X}$, originally with $\tau(t) \neq 0$; with $\tau \equiv 0$, $a_n \neq 0$; and with $\tau = 0$, $a_n = 0$, $\beta_n(t) \neq 0$, respectively.

We see that while F_n originally depended on 4 variables, the existence of a one dimensional symmetry algebra restricts the interaction to one involving an arbitrary function of 3 symmetry variables.

Equations (2.23), (2.24) and (2.25) are the key to a complete symmetry classification for eq. (2.14). Indeed, we can first find all interactions with abelian symmetry algebras, starting from each of the three algebras $A_{1,1}$, $A_{1,2}$ and $A_{1,3}$. Thus, element X_1 is already in standard form. Further operators X_j are then constructed, commuting with X_1 (and amongst each other for $\dim L > 2$. The interactions F_n, or rather the functions f_n, are further constrained, using the additional determining equations (2.20). It turns out[2] that abelian symmetry algebras L exist for nonlinear equations and their dimensions satisfy $1 \leq \dim L \leq 4$. Once the abelian symmetry algebras $A_{n,i}$ are known, we can use them to construct nilpotent nonabelian symmetry algebras. The abelian ones will figure as maximal abelian ideals in the nilpotent ones. In turn, abelian and nilpotent symmetry algebras will serve as nilradicals (maximal nilpotent ideals) in solvable Lie algebras. Finally, nonsolvable symmetry algebras of eq. (2.14) will always have a Levi decomposition, with $\mathrm{sl}(2, \mathbb{R})$ as the Levi

factor [23].

For complete analysis, see Ref. 2. Here we shall just present some examples of systems of the type (2.14) having interesting symmetry algebras (and groups).

First of all, the symmetry algebra L at most 7-dimensional. The highest dimensions, $\dim L = 6$ and 7, occur for interactions F_n depending in a prescribed way on the dependent functions u_{n-1}, u_n and u_{n+1} namely via a symmetry variable ξ with

$$\xi = (\gamma_n - \gamma_{n+1})u_{n-1} + (\gamma_{n+1} - \gamma_{n-1})u_n + (\gamma_{n-1} - \gamma_n)u_{n+1}, \gamma_{n+1} \neq \gamma_n, \quad \dot{\gamma}_n = 0, \quad (2.26)$$

The $\dim L = 7$ symmetry algebra occurs for

$$F_n = \frac{1}{\xi^3} \quad (2.27)$$

and we have

$$\begin{aligned} &X_1 = \partial_t, \quad X_2 = t\partial_t + \frac{1}{2}u_n\partial_{u_n}, \quad X_3 = t^2\partial_t + tu_n\partial_{u_n} \\ &X_u = \partial_{u_n}, \quad X_5 = t\partial_{u_n}, \quad X_6 = \gamma_n\partial_{u_n}, \quad X_7 = t\gamma_n\partial_{u_n} \end{aligned} \quad (2.28)$$

The algebra has a nontrivial Levi decomposition, namely $L \sim \{X_1, X_2, X_3\} \triangleright \{X_4, X_5, X_6, X_7\}$ with $\{X_1, X_2, X_3\} \sim sl(2, \mathbb{R})$ and $\{X_4, X_5, X_6, X_7\}$ abelian.

Three different interactions allow symmetry algebras with $\dim L = 6$. The algebras are all solvable, with nonabelian nilradicals. The nilradical is actually always the same one, namely

$$X_1 = \partial_{u_n}, \quad X_2 = t\partial_{u_n}, \quad X_3 = \gamma_n\partial_{u_n}, \quad X_4 = \gamma_n t\partial_{u_n}, \quad X_5 = \partial_t \quad (2.29)$$

Let us just list the invariant interactions and the additional element Y of the symmetry algebra

$$\begin{aligned} S_{6,1} : F_n &= C_n\xi^p \quad p = \frac{a - 3/2}{a + 1/2}, a \neq -\frac{1}{2}, \frac{3}{2} \\ Y &= t\partial_t + \left(\frac{1}{2} + a\right)u_n\partial_{u_n} \end{aligned} \quad (2.30)$$

$$\begin{aligned} S_{6,2} : F_n &= c_n + (a + b\gamma_n)\ln\xi \\ Y &= t\partial_t + \left[2u_n + (a + b\gamma_n)t^2\right]\partial_{u_n} \\ &\quad (a, b) \neq (0, 0) \\ &\gamma_n \neq \gamma_{n+1} \end{aligned} \quad (2.31)$$

$$S_{6,3}: F_n = c_n \exp\left(-\frac{2\xi}{(\gamma_n-\gamma_{n+1})\rho_{n-1}+(\gamma_{n+1}-\gamma_{n-1})\rho_n+(\gamma_{n-1}-\gamma_n)\rho_{n+1}}\right)$$
$$Y = t\partial_t + \rho_n\partial_{un}, \quad \rho_n \neq A + B\gamma_n, \dot{\rho}_n = 0 \tag{2.32}$$

It is easy to check that the three algebras $S_{6,1}$, ..., $S_{6,3}$ are mutually nonisomorphic.

One of the four-dimensional symmetry algebras is of particular interest. The interaction involves an arbitrary function of ξ with ξ as in eq. (2.26).

We have

$$F_n = \exp\left(-2\frac{u_{n+1}-u_n}{\gamma_{n+1}-\gamma_n}\right) f_n(\xi)$$
$$X_1 = \partial_{u_n}, X_2 = \partial_t, X_3 = t\partial_{u_n}, Y = t\partial_t + \gamma_n\partial_{u_n} \tag{2.33}$$

The algebra is solvable, with a nilpotent nilradical $\{X_1, X_2, X_3\}$.

The Toda lattice

$$\ddot{u}_n = e^{u_{n-1}-u_n} - e^{u_n-u_{n+1}} \tag{2.34}$$

correspond to a special case of the interaction (2.33), and is obtained by setting

$$\gamma_n = 2n, \quad f_n(\xi) = -1 + e^{1/2\xi}. \tag{2.35}$$

Thus, the integrable Toda lattice[20] is not singled out by its Lie point symmetry group. It occurs in a family, involving an arbitrary function of ξ and n, and an arbitrary function of n ($f_n(\xi)$ and γ_n, respectively).

It would be of interest to investigate the integrability properties of other equations in this family, as well as those of equations with larger symmetry groups ($5 \leq \dim L \leq 7$).

3 Difference Operator Symmetry Algebras

3.1 Background

In this section we report on a different point of view on symmetries of difference equations. We shall consider equations of the form (1.5) and symmetry algebras realized by "discrete vector fields", as in eq. (1.7). For simplicity (mainly of notation) we consider only scalar equations for a single dependent variable u. The independent variables $x = (x_1, \ldots, x_q)$ are continuous, their increments discrete.

When studying Lie point symmetries of differential equations, we can either use the formalism of ordinary vector fields

$$X = \sum_{i=1}^{q} \xi_i(x,u)\partial_{x_i} + \phi(x,u)\partial_u, \tag{3.1}$$

on the equivalent one of "evolutionary vector fields"

$$X_e = -Q\partial_u, \quad Q = \xi_i(x,u)u_{x_i} - \phi(x,u). \tag{3.2}$$

In this formalism the determining equations are again read of from the equation

$$\mathrm{pr}\, X_e E \,\big|_{E=0} = 0. \tag{3.3}$$

and the prolongation formula is

$$\begin{aligned} \mathrm{pr}^N X_e &= -Q\partial_u - Q^{x_i}\partial_{u_{x_i}} - Q^{x_i x_j}\partial_{u_{x_i x_j}} \\ Q^{x_i} &= D_{x_i}Q, \quad Q^{x_i x_j} = D_{x_i}D_{x_j}Q, \ldots \end{aligned} \tag{3.4}$$

where D_{x_i} are total derivatives.

Commutators of evolutionary fields can be calculated by commuting their first prolongations and then projecting the result onto the Lie algebra.

If we consider a linear differential equation $Lu = 0$, we can apply the formalism of commuting linear operators to construct a class of Lie point symmetries. We have

$$\widehat{L}u = 0, \quad [\widehat{L}, \widehat{X}] = \lambda\widehat{L}, \quad \widehat{X} = \xi_i(x)\partial_{x_i} + f(x), \tag{3.5}$$

where λ can be a function of x, or even a linear operator.

In the evolutionary formalism we then have

$$Q_e = \widehat{X}u = \xi_i(x)u_{x_i} + f(x)u \tag{3.6}$$

Comparing with eq. (3.2) we see that the formalism (3.5) of commuting operators gives only a subclass of all possible Lie point symmetries, namely those with $\xi_i(x,u)$ independent of u (i.e. fiber preserving transformations) and with $\phi(x,u) = -f(x)u$, (i.e. linear and homogeneous in u).

It is the formalism of evolutionary vector fields that is best adapted for the case of difference equations.

3.2 *Prolongation Formalism for Linear Difference Equations*

Let us consider a linear difference equation

$$\begin{gathered} Lu(x) = 0 \quad , \quad x \in \mathbb{R}^q \\ L = A(x, T_x) + B_i(x, T_x)\Delta_{x_i} + C_{ij}(x, T_x)\Delta_{x_i}\Delta_{x_j} + \cdots \end{gathered} \tag{3.7}$$

where T_x is viewed as a vector $T_x = (T_{x_1}, \ldots, T_{x_q})$ with T_{x_i} and Δ_{x_i} defined as the shift and variation operators of eq. (1.3) and (1.2), respectively. Let us now consider a Lie group acting on x and u. We have, for $\varepsilon \ll 1$,

$$\begin{aligned} \widetilde{u}(\widetilde{x}) &= [e^{-\varepsilon X}u](\widetilde{x}) = [1 - \varepsilon X u](x + \varepsilon\xi), \\ &= u(x) + \varepsilon\phi(x, T_x, u, u_x), \end{aligned} \tag{3.8}$$

where

$$\begin{gathered} \phi(x, T_x, u, u_x) = \xi_i^0(x)u_{x_i} - \xi_i(x, T_x)\Delta_{x_i}u - f(x, T_x)u \\ x, \xi_i, \xi_i^0 \in \mathbb{R}^q, \quad \xi_i^0(x) = \xi_i(x, 1) \end{gathered} \tag{3.9}$$

We see that the above infinitesimal transformation on a lattice is not a point one: it involves both derivatives u_{x_i} and variations $\Delta_{x_i}u$. The derivatives come from the expansion of $\widetilde{u}(\widetilde{x})$ into a Taylor series, the variations from the fact that in the linear operator formalism we have

$$\widehat{X} = \sum_i \xi_i(x, T_x)\Delta_{x_i} + f(x, T_x), \tag{3.10}$$

in eq. (3.5) (with L as in eq. (3.7)). In the continuous limit we reobtain $\phi \to -f(x)u$, corresponding to a point transformation.

Returning to evolutionary vector fields, we have

$$X_e = -Q\partial_u = (\phi - \xi_i^0(x)u_{x_i})\partial_u \tag{3.11}$$

we find

$$X_e = -[\xi_i(x, T_x)\Delta_{x_i}u + f(x, T_x)u]\partial_u \tag{3.12}$$

In this formulation, the difference between Q in eq. (3.12) and in the continuous case (3.6) is the presence of the shift operator T_x in ξ_i and f (and $\Delta_{x_i}u$ instead of u_{x_i}.

The algorithm for obtaining the determining equations is given in eq. (3.3), once the prolongation of the discrete evolutionary vector field X_e is constructed.

The prolongation formula does not depend on the equation being linear, so we give it in the general case below.

3.3 *Prolongation Formalism for Arbitrary Difference Equations*

The Ansatz (3.12) is not general enough to treat nonlinear equations, nor even to obtain all symmetries of linear equations. We shall use a more general Ansatz, namely

$$X_e = -Q\partial_u, \quad Q = \sum_i \xi_i(x, T^a u) T^b \Delta_{x_i} u + \phi(x, T^c u), \tag{3.13}$$

with $T^a, \ldots$ as in eq. (1.6).

The transformation generated by X_e will take functions $u(x)$ into $\tilde{u}(\tilde{x})$. The transformations generated by $\mathrm{pr}^N X_e$ must also transform the variations $\Delta_{x_i} u(x)$, $\Delta_{x_i}\Delta_{x_k} u(x)$, $\ldots$ into the variations of $\tilde{u}$ with respect to $\tilde{x}$ (up to order N) and to do this simultaneously at all points of the lattice figuring in the equation.

The appropriate prolongation formula is

$$\mathrm{pr}^N X_e = -\Big\{ \sum_\alpha (T^\alpha Q)\partial_{T^\alpha u} + \sum_{\beta_i} T^{\beta_i} Q^{x_i} \partial_{T^{\beta_i}\Delta_{x_i} u} + \sum_{\gamma_{ij}} T^{\gamma_{ij}} Q^{x_i x_j} \partial_{T^{\gamma_{ij}}\Delta_{x_i}\Delta_{x_j} u} + \ldots \Big\}. \tag{3.14}$$

where the summations are over all sites figuring in the equation (1.5).

In eq. (3.14) we have

$$Q^{x_i} = \Delta^T_{x_i} Q, \quad Q^{x_i x_j} = \Delta^T_{x_i}\Delta^T_{x_j} Q, \tag{3.15}$$

where Δ^T_x is the total variation operator, acting on functions of x, u, $\Delta_x u$, $\ldots$ as

$$\Delta^T_x f\big(x, u(x), \Delta_x u(x), \ldots\big) = \frac{1}{\sigma}\Big[f\big(x+\sigma, u(x+\sigma), (\Delta_x u)(x+\sigma), \ldots\big) - f\big(x, u(x), (\Delta_x u)(x), \ldots\big)\Big] \tag{3.16}$$

3.4 *Example of the Discrete Heat Equation*

Let us consider the equation

$$\big[\Delta_t - \Delta_{xx} + g(x, t, T_x, T_t)\big] u(x,t) = 0, \tag{3.17}$$

where g is some given function of its arguments (g is actually an operator, acting on $u(x,t)$). We apply the linear formalism, putting

$$X_e = -[\tau\Delta_t u + \xi\Delta_x u + fu]\partial_u = -Q\partial_u \tag{3.18}$$

with τ, ξ and f functions of x, t, T_x, T_t. Applying $\mathrm{pr}^2 X_e$ to eq. (3.17) and elimination $\Delta_{xx}u$, $\Delta_{xxx}u$, and $\Delta_{xxt}u$, using eq. (3.17) and its consequences, we obtain an expression depending linearly on $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u. Requesting that the coefficient of each of these expressions should vanish, we obtain four determining equations:

$$\begin{aligned}
&\Delta_x\tau = 0,\\
&-(\Delta_t\tau)T_t + 2(\Delta_x\xi)T_x + [g,\tau] = 0,\\
&-(\Delta_t\xi)T_t + (\Delta_{xx}\xi)T_x^2 + 2(\Delta_x f)T_x + [\xi,g] = 0,\\
&-(\Delta_t f)T_t + (\Delta_{xx}f)T_x^2 + 2(\Delta_x\xi)T_x g + \xi(\Delta_x g)T_x + \tau(\Delta_t g)T_t + [f,g] = 0.
\end{aligned} \tag{3.19}$$

Once g is specified, equations (3.19) can be solved. The task simplifies significantly if the commutators $[g,\tau]$, $[\xi,g]$ and $[f,g]$ vanish.

Let us consider three examples, all of which have interesting continuous limits.

A. The "free" discrete head equation: $g = 0$

The solution is

$$\begin{aligned}
\tau &= t^{[2]}\tau_2 + t\tau_1 + \tau_0,\\
\xi &= \frac{1}{2}x(\tau_1 + 2t\tau_2)T_t T_x^{-1} + t\xi_1 + \xi_0,\\
f &= \frac{1}{4}x^{[2]}\tau_2 T_t^2 T_x^{-2} + \frac{1}{2}t\tau_2 T_t + \frac{1}{2}x\xi_1 T_t T_x^{-1} + \gamma,
\end{aligned} \tag{3.20}$$

Here and below we are making use of factorial powers $x^{(n)}$, satisfying

$$\begin{aligned}
\Delta_x x^{(n)} &= n x^{(n-1)}, \qquad n \in \mathbb{Z}^{\geq},\\
x^{(n)} &= x(x-\sigma_x)\dots\big(x-(n-1)\sigma_x\big),
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\Delta_x x^{(-n)} &= -n x^{(-n-1)}\\
x^{(-n)} &= \frac{1}{x(x+\sigma_x)\cdots(x+(n-1)\sigma_x}
\end{aligned} \tag{3.22}$$

The integration "constants" are functions of T_x and T_t and are thus actually shift operators. The symmetry algebra is hence infinite-dimensional. This is due to the fact that the shift operators are also symmetries: if $u(x,t)$ is a solution, then so is $T_x^n T_t^k u(x,t)$, $(n,k \in \mathbb{Z})$.

By appropriately choosing the coefficients τ_i, ξ_i and γ we can pick out a six-dimensional subalgebra of the symmetry algebra, isomorphic to the symmetry algebra of the continuous heat equation:

$$
\begin{aligned}
&\tau_0 = 1: \quad P_0 = (\Delta_t u)\partial_u \\
&\xi_0 = 1: \quad P_1 = (\Delta_x u)\partial_u \\
&\gamma = 1: \quad W = u\partial_u \\
&\xi_1 = 2T_t^{-1}, \quad \gamma = \frac{\sigma_x}{2} T_x^{-1}: \\
&\qquad B = \left(2tT_t^{-1}\Delta_x u + xT_x^{-1}u + \frac{\sigma_x}{2}T_x^{-1}u\right)\partial_u \\
&\tau_1 = 2T_t^{-1}, \quad \gamma = 1 - \frac{1}{2}T_x^{-1}: \\
&\qquad D = \left[2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + \left(1 - \frac{1}{2}T_x^{-1}\right)u\right]\partial_u, \\
&\tau_2 = T_t^{-2}, \quad \xi_1 = \sigma_x T_x^{-1}T_t^{-1}, \quad \gamma = -\frac{1}{16}\sigma_x^2 T_x^{-2}: \\
&\qquad K = \left[t^2 T_t^{-2}\Delta_t u + txT_t^{-1}T_x^{-1}\Delta_x u + \frac{x^2}{4}T_x^{-2}u \right. \\
&\qquad\qquad \left. + t\left(T_t^{-2} - \frac{1}{2}T_t^{-1}T_x^{-1}\right)u - \frac{\sigma_x^2}{16}T_x^{-2}u\right]\partial_u.
\end{aligned}
\tag{3.23}
$$

B. The discrete harmonic oscillator potential $g = k^2 X^{(2)} T_x^{-2}$, $k \in \mathbb{R}^>$

As in the continuous case, the symmetry algebra is isomorphic to that of the free equation. To solve the determining equations, we must introduce a "discrete shifted exponential" $E_m(t)$ satisfying

$$
\begin{aligned}
\Delta_t E_m(t) &= m E_m(t) T_t^{-1} \\
E_m(t) &= (1 + mT_t^{-1}\sigma_t)^{t/\sigma_t}.
\end{aligned}
\tag{3.24}
$$

The symmetry algebra is generated by the following discrete evolutionary fields:

$$
\begin{aligned}
P_0 &= \tau_0 \Delta_t u \partial_u, \quad L = \gamma u \partial_u, \\
T_1 &= E_{4k}(t)\tau_1[\Delta_t u + 2kxT_x^{-1}\Delta_x u + 2k^2 x^{(2)} T_x^{-2} u + ku]\partial_u, \\
T_2 &= E_{-4k}(t)\tau_2[\Delta_t u - 2kxT_x^{-1}\Delta_x u + 2k^2 x^{(2)} T_x^{-2} u - ku]\partial_u, \qquad (3.25) \\
A_1 &= E_{2k}(t)\alpha_1(\Delta_x u + kxT_x^{-1}u)\partial_u \\
A_2 &= E_{-2k}(t)\alpha_2(\Delta_x u - kxT_x^{-1}u)\partial_u.
\end{aligned}
$$

For other examples, see ref. 9.

4 Conclusions

The overall conclusion that we can draw at this stage is that Lie group theory can be used to analyze and solve difference equations just as well as differential ones.

The formalism presented in Section 2, treats continuous variables infinitesimally, via vector fields of differential operators, and discrete variables globally, by inspection. It has the advantage of being algorithmic, and hence programmable. It is applicable, and has been applied, to both linear and nonlinear equations. Its drawback is that some symmetries are lost. More specifically, if we introduce a continuous limit, in which finite differences like $(u_{n+1} - u_n)$ go into infinitesimals and difference equations into differential ones, the limiting equations can have, and usually do have, larger Lie point symmetry groups.

The formalism of discrete vector fields, presented in Section 3, makes use of symmetries, realized by difference operators. It is more difficult to apply and so far has only been applied to linear equations. It does however recover nonlinear symmetries of linear equations, when they exist. In particular, the linear difference equation

$$(\Delta_x)^2 u(x) = 0 \qquad (4.1)$$

was shown[9] to be invariant under the group $\mathrm{sl}(3,\mathbb{R})$, acting as a group of projective transformations on u and x, just as it does in the continuous case.

We have a formalism of "discrete evolutionary vector fields" that makes it possible to calculate symmetries of difference equations on uniform lattices. Alternatively, as we have shown above for the heat equation, it makes it possible to discretize a differential equation on a uniform lattice, while preserving all its symmetries.

Many problems remain open. Among them we mention applications to discrete nonlinear equations and extensions to other types of lattices, in particular

the exponential lattices occuring in q-difference equations. A powerful technique that should be extended from differential equations to difference ones is the combination of group theory with singularity analysis (Painlevé analysis).

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BINARY NONLINEARIZATION OF LAX PAIRS

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A kind of Bargmann symmetry constraints involved in Lax pairs and adjoint Lax pairs is proposed for soliton hierarchy. The Lax pairs and adjoint Lax pairs are nonlinearized into a hierarchy of commutative finite dimensional integrable Hamiltonian systems and explicit integrals of motion may also be generated. The corresponding binary nonlinearization procedure leads to a sort of involutive solutions to every system in soliton hierarchy which are all of finite gap. An illustrative example is given in the case of AKNS soliton hierarchy.

1 Introduction

Symmetry constraints become prominent because of the important role they play in the soliton theory [1 10 15 16]. A kind of very successful symmetry constraint method for soliton equations is proposed through the nonlinearization technique called mono-nonlinearization [2 9]. However, mono-nonlinearization involves only the Lax pairs of soliton equations. We would like to elucidate that the mono-nonlinearization technique can successfully be extended to the Lax pairs and the adjoint Lax pairs associated with soliton hierarchy. The corresponding symmetry constraint procedure is called a binary nonlinearization technique [12 13 8] because it involves the Lax pairs and the adjoint Lax pairs and puts the linear Lax pairs into the nonlinearized Lax systems. A kind of useful symmetries in our symmetry constraints is exactly the specific symmetries expressed through the variational derivatives of the potentials. The resulting theory provides a method of separation of variables for solving nonlinear soliton equations and exhibits integrability by quadratures for soliton equations. It also narrows the gap between finite dimensional integrable Hamiltonian systems and infinite dimensional integrable soliton equations. An illustrative example is carried out in the case of the three-by-three matrix spectral problem for AKNS soliton hierarchy.

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2 Basic idea of binary nonlinearization

This section reveals how to manipulate a binary nonlinearization procedure for a given soliton hierarchy along with a basic idea for the proof of the main result. Let $\mathcal{B}$ denote the differential algebra of differential vector functions $u = u(x,t)$, and write for $k \geq 0$

$$\mathcal{V}^s_{(k)} = \{(P^{ij}\partial^k)_{s\times s} \mid P^{ij} \in \mathcal{B}\},\ \widetilde{\mathcal{V}}^s_{(k)} = \mathcal{V}^s_{(k)} \otimes C[\lambda, \lambda^{-1}],\ \partial = \frac{d}{dx}.$$

For $U = U(u,\lambda) \in \widetilde{\mathcal{V}}^s_{(0)}$, we choose a solution to the adjoint representation equation $V_x = [U,V]$:

$$V = V(u,\lambda) = \sum_{i\geq 0} V_i \lambda^{-i},\ V_i \in \mathcal{V}^s_{(0)}.$$

Suppose that the isospectral ($\lambda_{t_n} = 0$) compatibility conditions $U_{t_n} - V^{(n)}_x + [U, V^{(n)}] = 0,\ n \geq 0$, of the Lax pairs

$$\begin{cases} \phi_x = U\phi = U(u,\lambda)\phi,\ U \in \widetilde{\mathcal{V}}^s_{(0)} \\ \phi_{t_n} = V^{(n)}\phi = V^{(n)}(u,\lambda)\phi,\ V^{(n)} = (\lambda^n V)_+ + \Delta_n,\ \Delta_n \in \widetilde{\mathcal{V}}^s_{(0)} \end{cases}$$

determine a soliton hierarchy

$$u_{t_n} = K_n = JG_n = J\frac{\delta H_n}{\delta u},\ n \geq 0. \tag{1}$$

If $\phi = (\phi_1, \phi_2, \cdots, \phi_s)^T$ and $\psi = (\psi_1, \psi_2, \cdots, \psi_s)^T$ satisfy the spectral problem and the adjoint spectral problem

$$\phi_x = U(u,\lambda)\phi,\ \psi_x = -U^T(u,\lambda)\psi,$$

and we set the matrix $\bar{V} = \phi\psi^T = (\phi_k\psi_l)_{s\times s}$, then we have the following two basic results used in binary nonlinearization [5 13]:
(i) the variational derivative of the spectral parameter λ with respect to the potential u may be expressed by

$$\frac{\delta\lambda}{\delta u} = \frac{\mathrm{tr}\left(\bar{\mathrm{V}}\frac{\partial \mathrm{U}}{\partial \mathrm{u}}\right)}{-\int_{-\infty}^{\infty}\mathrm{tr}\left(\bar{\mathrm{V}}\frac{\partial \mathrm{U}}{\partial \lambda}\right)\mathrm{dx}}, \tag{2}$$

(ii) the matrix $\bar{V}$ is a solution to the adjoint representation equation $V_x = [U,V]$, i.e. $\bar{V}_x = [U,\bar{V}]$.

Now introduce distinct eigenvalues $\lambda_1, \cdots, \lambda_N$ and let

$$\phi^{(j)} = (\phi_{1j}, \cdots, \phi_{sj})^T, \ \psi^{(j)} = (\psi_{1j}, \cdots, \psi_{sj})^T \ \ (1 \leq j \leq N)$$

denote the eigenvectors and the adjoint eigenvectors corresponding to λ_j $(1 \leq j \leq N)$, respectively. Make the Bargmann **symmetry constraint**

$$K_0 = JG_0 = J\sum_{j=1}^{N} E_j \frac{\delta \lambda_j}{\delta u} \ \text{ or } \ \mathrm{G_0} = \sum_{\mathrm{j=1}}^{\mathrm{N}} \mathrm{E_j} \frac{\delta \lambda_\mathrm{j}}{\delta \mathrm{u}}, \tag{3}$$

where $E_j = -\int_{-\infty}^{\infty} < \bar{V}(\lambda_j), \frac{\partial U}{\partial \lambda_j} > dx$, $\bar{V}(\lambda_j) = \phi^{(j)}\psi^{(j)T}$, $1 \leq j \leq N$. The Bargmann constraint requires the covariant G_0 to be a potential function not including any potential differential and hence from the Bargmann symmetry constraint we may find an explicit nonlinear expression for the potential

$$u = f(\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(N)}; \psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(N)}). \tag{4}$$

Upon instituting (4) into the Lax pairs and the adjoint Lax pairs, we get two nonlinearized Lax systems, i.e. the nonlinearized spatial system

$$\begin{cases} \phi_{jx} = U(f(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}), \lambda_j)\phi_j, \ 1 \leq j \leq N, \\ \psi_{jx} = -U^T(f(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}), \lambda_j)\psi_j, \ 1 \leq j \leq N; \end{cases} \tag{5}$$

and the nonlinearized temporal systems for $n \geq 0$

$$\begin{cases} \phi_{jt_n} = V^{(n)}(f(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}), \lambda_j)\phi_j, \ 1 \leq j \leq N, \\ \psi_{jt_n} = -V^{(n)T}(f(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}), \lambda_j)\psi_j, \ 1 \leq j \leq N. \end{cases} \tag{6}$$

In order to discuss the integrability of (5) and (6), we choose the symplectic structure ω^2 on $\mathbf{R}^{2sN}$

$$\omega^2 = \sum_{i=0}^{s}\sum_{j=0}^{N} d\phi_{ij} \wedge d\psi_{ij} = \sum_{i=0}^{s} dP_i \wedge dQ_i,$$

where $P_i = (\phi_{i1}, \cdots, \phi_{iN})^T$, $Q_i = (\psi_{i1}, \cdots, \psi_{iN})^T$, $1 \leq i \leq s$. We accept the following corresponding Poisson bracket for two functions F, G defined over the phase space $\mathbf{R}^{2sN}$

$$\begin{aligned} \{F, G\} &= \omega^2(IdG, IdF) = \omega^2(X_G, X_F) \\ &= \sum_{i=1}^{s}(< \frac{\partial F}{\partial Q_i}, \frac{\partial G}{\partial P_i} > - < \frac{\partial F}{\partial P_i}, \frac{\partial G}{\partial Q_i} >), \end{aligned} \tag{7}$$

where $IdH = X_H$ represents the Hamiltonian vector field with energy H defined by $i_{IdH}\omega^2 = i_{X_H}\omega^2 = dH$ and $< \cdot, \cdot >$ represents the standard inner product of $\mathbb{R}^N$. Then we accept the following corresponding Hamiltonian system with the Hamiltonian function H

$$\dot{P}_i = \{P_i, H\} = -\frac{\partial H}{\partial Q_i}, \ \dot{Q}_i = \{Q_i, H\} = \frac{\partial H}{\partial P_i}, \ 1 \leq i \leq s. \tag{8}$$

Main Result: *The nonlinearized spatial system (5) is a finite dimensional integrable Hamiltonian system in the Liouville sense, and the nonlinearized temporal systems (6) for $n \geq 0$ may be transformed into a hierarchy of finite dimensional integrable Hamiltonian systems in the Liouville sense, under the control of the nonlinearized spatial system (5). Moreover the potential $u = f$ determined by the Bargmann symmetry constraint solves the n-th soliton equation $u_{t_n} = K_n$ in the hierarchy.*

Idea of Proof: Note that we have

$$(V(f,\lambda))_x = [U(f,\lambda), V(f,\lambda)], \ (\bar{V}(\lambda_j))_x = [U(f,\lambda_j), \bar{V}(\lambda_j)]$$

and when $u_{t_n} = K_n$, we have

$$(V(f,\lambda))_{t_n} = [V^{(n)}(f,\lambda), V(f,\lambda)], \ (\bar{V}(\lambda_j))_{t_n} = [V^{(n)}(f,\lambda_j), \bar{V}(\lambda_j)].$$

Therefore we may show that $F = \frac{1}{2}\mathrm{tr}(\mathrm{V(f,\lambda)})^2$ is a common generating function for integrals of motion of (5) and (6) since $F_x = \frac{1}{2}\mathrm{tr}(\mathrm{V}^2)_\mathrm{x} = \frac{1}{2}\mathrm{tr}[\mathrm{U}, \mathrm{V}^2] = 0$ and $F_{t_n} = \frac{1}{2}\mathrm{tr}(\mathrm{V}^2)_{\mathrm{t_n}} = \frac{1}{2}\mathrm{tr}[\mathrm{V^{(n)}}, \mathrm{V}^2] = 0$. A similar deduction may verify that $\bar{F}_j = \frac{1}{2}\mathrm{tr}(\bar{\mathrm{V}}(\lambda_\mathrm{j}))^2$, $1 \leq j \leq N$, are integrals of motion of (5) and (6), too. Noticing

$$F = \sum_{n \geq 0} F_n \lambda^{-n}, \ \bar{F}_j = \frac{1}{2}(\sum_{i=1}^{s} \phi_{ij}\psi_{ij})^2, \ 1 \leq j \leq N, \tag{9}$$

we get a series of explicit integrals of motion: $\bar{F}_j$, $1 \leq j \leq N$, $\{F_n\}_{n=0}^{\infty}$, which may be proved to be involutive with respect to the Poisson bracket (7). Further it is not difficult to show the Liouville integrability of (5) and (6) when they can be rewritten as Hamiltonian systems with Hamiltonian functions being polynomials in F_m, $m \geq 1$.

In addition, because the compatibility condition of (5) and (6) is still the n-th soliton equation $u_{t_n} = K_n$, $u = f(\phi_j; \psi_j)$ gives an involutive solution to the n-th soliton equation $u_{t_n} = K_n$ once ϕ_j, ψ_j, $1 \leq j \leq N$, solve (5) and (6), simultaneously. This sort of involutive solutions also exhibits a kind of separation of independent variables x, t_n for soliton equations.

3 The case of AKNS Hierarchy

For AKNS hierarchy, we introduce a three-by-three matrix spectral problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} -2\lambda & \sqrt{2}q & 0 \\ \sqrt{2}r & 0 & \sqrt{2}q \\ 0 & \sqrt{2}r & 2\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

In this case, $\phi = (\phi_1, \phi_2, \phi_3)^T$ and $u = (q, r)^T$. A hierarchy of AKNS soliton equations [13]

$$u_{t_n} = K_n = \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \end{pmatrix} = JL^n \begin{pmatrix} r \\ q \end{pmatrix} = J\frac{\delta H_n}{\delta u}, \ n \geq 0 \tag{10}$$

is the compatibility conditions of the Lax pairs

$$\phi_x = U\phi, \ \phi_t = V^{(n)}\phi, \ V^{(n)} = (\lambda^n V)_+ . \tag{11}$$

Here the operator solution V to $V_x = [U, V]$, the Hamiltonian operator J, the recursion operator L, and the Hamiltonian functions H_n for $n \geq 0$ read as

$$V = \begin{pmatrix} 2a & \sqrt{2}b & 0 \\ \sqrt{2}c & 0 & \sqrt{2}b \\ 0 & \sqrt{2}c & -2a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} 2a_i & \sqrt{2}b_i & 0 \\ \sqrt{2}c_i & 0 & \sqrt{2}b_i \\ 0 & \sqrt{2}c_i & -2a_i \end{pmatrix} \lambda^{-i},$$

$$J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \ L = \begin{pmatrix} \frac{1}{2}\partial - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}r \end{pmatrix}, \ H_n = \frac{2a_{n+2}}{n+1}.$$

The operators J and JL constitute a Hamiltonian pair and L^* is hereditary [6].

In this AKNS case, the Bargmann symmetry constraint becomes

$$K_0 = J\frac{\delta H_0}{\delta u} = J\sum_{j=1}^{N} \begin{pmatrix} \sqrt{2}(\phi_{2j}\psi_{1j} + \phi_{3j}\psi_{2j}) \\ \sqrt{2}(\phi_{1j}\psi_{2j} + \phi_{2j}\psi_{3j}) \end{pmatrix}, \tag{12}$$

which engenders an explicit expression for the potential u

$$u = f(\phi_{ij}; \psi_{ij}) = \sqrt{2} \begin{pmatrix} < P_1, Q_2 > + < P_2, Q_3 > \\ < P_2, Q_1 > + < P_3, Q_2 > \end{pmatrix}. \tag{13}$$

Further besides $\bar{F}_j$, $1 \leq j \leq N$, we can directly give the following explicit integrals of motion for the nonlinearized Lax systems

$$
\begin{aligned}
&F := \tfrac{1}{2}\mathrm{trV}^2 = 4(\mathrm{a}^2 + \mathrm{bc}) = \textstyle\sum_{\mathrm{m}\geq 0} \mathrm{F_m}\lambda^{-\mathrm{m}},\\
&F_0 = 4,\ F_1 = -8(< P_1, Q_1 > - < P_3, Q_3 >),\\
&F_m = 4\textstyle\sum_{i=1}^{m-1}[(< A^{i-1}P_1, Q_1 > - < A^{i-1}P_3, Q_3 >)\times\\
&\quad (< A^{m-i-1}P_1, Q_1 > - < A^{m-i-1}P_3, Q_3 >)\\
&\quad +2(< A^{i-1}P_1, Q_2 > + < A^{i-1}P_2, Q_3 >)\times\\
&\quad (< A^{m-i-1}P_2, Q_1 > + < A^{m-i-1}P_3, Q_2 >)]\\
&\quad -8 < A^{m-1}P_1, Q_1 > - < A^{m-1}P_3, Q_3 >,\ m \geq 2,
\end{aligned}
$$

where $A = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. The nonlinearized spatial system (5) is rewritten as an integrable Hamiltonian system

$$
P_{ix} = \{P_i, H\} = -\frac{\partial H}{\partial Q_i},\ Q_{ix} = \{Q_i, H\} = \frac{\partial H}{\partial Q_i},\ i = 1, 2, 3 \qquad (14)
$$

with the Hamiltonian function

$$
\begin{aligned}
&H = 2(< AP_1, Q_1 > - < AP_3, Q_3 >)\\
&-2(< P_1, Q_2 > + < P_2, Q_3 >)(< P_2, Q_1 > + < P_3, Q_2 >),
\end{aligned}
$$

and under the control of the nonlinearized spatial system (5), the nonlinearized temporal systems (6) for $n \geq 0$ can also be rewritten as the integrable Hamiltonian systems

$$
P_{it_n} = \{P_i, H_n\} = -\frac{\partial H_n}{\partial Q_i},\ Q_{it_n} = \{Q_i, H_n\} = \frac{\partial H_n}{\partial P_i},\ \ i = 1, 2, 3 \qquad (15)
$$

with the Hamiltonian functions

$$
H_n = -\frac{1}{4}\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\geq 1}} F_{i_1}\cdots F_{i_{m+1}},
$$

where the constants d_m are defined by

$$
\begin{aligned}
&d_0 = 1,\ d_1 = -\tfrac{1}{8},\ d_2 = \tfrac{3}{128},\\
&d_m = -\tfrac{1}{2}\textstyle\sum_{i=1}^{m-1} d_i d_{m-i} - \tfrac{1}{4}d_{m-1} - \tfrac{1}{8}\sum_{i=1}^{m-2} d_i d_{m-i-1},\ m \geq 3.
\end{aligned}
$$

Moreover following the previous main result, the potential (13) with

$$P_i(x,t_n) = g_H^x g_{H_n}^{t_n} P_i(0,0),\ Q_i(x,t_n) = g_H^x g_{H_n}^{t_n} Q_i(0,0),\ i = 1,2,3,$$

gives rise to a sort of involutive solutions with separated variables x, t_n to the n-th AKNS soliton equation $u_{t_n} = K_n$. Here g_G^y denotes the Hamiltonian phase flow of G with a parameter variable y but $P_i(0,0)$, $Q_i(0,0)$ may be arbitrary initial value vectors. A finite gap property for the resulting involutive solutions may also be shown.

4 Concluding remarks

We remark that the finite dimensional Hamiltonian systems generated by nonlinearization technique depend on the starting Lax pairs. Thus the same equation may be connected with different finite dimensional Hamiltonian systems once it possesses different Lax pairs. AKNS soliton equations are exactly such examples [13].

We also point out that the Neumann symmetry constraint and the higher order symmetry constraints

$$K_{-1} = J\sum_{j=1}^{N} E_j \frac{\delta \lambda_j}{\delta u},\ K_m = JG_m = J\sum_{j=1}^{N} E_j \frac{\delta \lambda_j}{\delta u},\ (m \geq 1), \qquad (16)$$

may be considered. These two sorts of symmetry constraints are somewhat different from the Bargmann symmetry constraints because K_{-1} is a constant vector and the conserved covariants G_m, $m \geq 1$, involve some differentials of the potential. This suggests that a few new tools are needed for discussing them [18]. Similarly, we can consider the corresponding τ-symmetry (i.e. time first order dependent symmetry [3]) constraints or more generally, time polynomial dependent symmetry [7] constraints . Binary nonlinearization may also be well applied to discrete systems and non-Hamiltonian soliton equations such as the Toda lattice and the coupled Burgers equations [11]. Note that in the case of KP hierarchy, the similar Bargmann symmetry constraints have been carefully analyzed as well [15], and the specific symmetries we use in constraints are sometimes called additional symmetries [4] and are often taken as source terms of soliton equations [14]. It should also be noted that the nonlinearized Lax systems are intimately related to stationary equations [17] and the more general nonlinearized Lax systems can be generated from the linear combination of Bargmann symmetry constraints which will be shown in a late publication.

However, in the binary nonlinearization procedure, there exist two intriguing **open problems**. The first one is why the nonlinearized spatial system (5)

and the nonlinearized temporal systems (6) for $n \geq 0$ with the control of the nonlinearized spatial system (5) always possess Hamiltonian structures? The second one is whether or not the nonlinearized temporal systems (6) for $n \geq 0$ are themselves integrable soliton equations without the control of the nonlinearized spatial system (5). These two problems are important and interesting but need some further investigation.

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NON-FUCHSIAN PAINLEVÉ TEST

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We consider closed form meromorphic solutions of nonlinear ordinary differential equations, which make their linearized equation non-Fuchsian at a movable singularity, Fuchsian at infinity, without any other singular point. When the nonlinear ODE possesses movable logarithms, a perturbation à la Poincaré detects them sooner than the perturbative (Fuchsian) Painlevé test; indeed, we can investigate the point at infinity, because the particular solution which we consider is *global*, while the series of the perturbative Painlevé test is *local* by definition.

1 Introduction

Nonlinear differential equations (DE's) with fixed critical points define a natural extension [1] of linear DE's. Let us recall that a singular point is said *critical* if several determinations of the solution are permuted around it, and *movable* (contrary *fixed*) if its location depends on the initial conditions. Solutions of nonlinear DE's may possess both movable and fixed singularities while those of linear DE's only have fixed singular points depending on the analytic structure of their coefficients. In the neighbourhood of a singular regular (or Fuchsian) points local solutions can be represented by *convergent* Laurent series possibly multiplied by a logarithm. On the contrary, singular irregular (or non-Fuchsian) points yield essential singularities in the general solution (GS) locally related to *formal* Laurent series (see, for more details, the classical book of Ince [12]).

The *Painlevé property* (PP) of a DE is defined [1] as the absence of movable critical points in the GS of the DE.

The *Painlevé test* [3] is the set of all methods to build necessary conditions for a DE to possess the PP, without guarantee on their sufficiency. We discuss in section 2 all existing such methods. By removing a restriction shared by the algebraic methods, i.e. the nonconsideration of meromorphic particular solutions which yield non-Fuchsian singular points to the linearised equation,

we present in section 3 a new extension to the perturbative method [7,8] which allows very often to conclude more rapidly to a failure of the Painlevé test. In section 4, two examples are then considered, one equation coming from mathematics and presented by Bureau [19], the second one occurring in a cosmological model of the Universe [14]. In both examples, the GS contains movable logarithms which are exhibited by our method and provide a shorter proof of failure of the Painlevé test than before.

2 Methods for Painlevé test

Let us consider $E(u;x) = 0$ a polynomial DE of order N in the variable u with coefficients analytic in x. We distinguish three methods for testing the Painlevé property.

1. *The method of pole-like expansions* [5,2,6]

This allows to test only the *positive indices* associated with the *local* representation in Laurent series of the solutions of the nonlinear DE. Let $-p$ and $-q$ be the two positive integers equal to the singularity order of u and $E(u)$ in the "movable" variable $\chi = x - x_0$. Then the local representations

$$u = \chi^p \sum_{n=0}^{\infty} u_n \chi^n, \; E = \chi^q \sum_{n=0}^{\infty} E_n \chi^n; \; u_0 \neq 0 \tag{1}$$

imply that the equation $E = 0$ is identically satisfied if

$$E_0 = 0 : \text{ polynomial in } u_0 \tag{2}$$

$$\forall n \geq 1, \quad E_n \equiv P(n)u_n + Q_n(\{u_j; j \leq n-1\}) = 0 : \text{ linear in } u_n \tag{3}$$

the indices being the N zeros of the polynomial $P(n)$. Those relations generate a *finite set* of necessary conditions at the positive integer indices, i.e.

- existence of N distinct integer indices $r_1, r_2, \ldots r_N$
- $Q_{r_i} \equiv 0$ for any positive r_i.

2. *The α-method of Painlevé* [4]

In this method one artificially introduces a movable constant x_0 which is a point of holomorphy of the coefficients of the nonlinear DE and a small non zero parameter α by the perturbation:

$$x = x_0 + \alpha X, \; u = \sum_{n=0}^{\infty} \alpha^{n+p} u_n(X), \; E(u,x) = \sum_{n=0}^{\infty} \alpha^{n+q} E_n(X) \tag{4}$$

where p and q are integer constants which in practice have the same meaning as those introduced in the preceding method. The identification to zero of $E(u,x)$ at each order of the perturbation yields

$$E_0 = 0 \quad : \quad \text{nonlinear scaled autonomous DE for } u_0 \tag{5}$$

$$\forall n \geq 1, E_n = 0 \quad : \quad \text{linear DE for } u_n. \tag{6}$$

Those relations generate an *infinite* set of necessary conditions:

- the GS u_0 of $E_0 = 0$ has to possess the PP
- $\forall n \geq 1$ the GS u_n of $E_n = 0$ does not contain any logarithm.

This method is the most powerful but its difficulty is differential since at each step one must integrate a DE.

3. *The perturbative method* [7,8]

This method that we will detail in the next section has been devised to test both *positive* and *negative* indices. Like the first method its difficulty is only algebraic and therefore it is easy to automatize. It consists in an *infinite* set of necessary conditions for the absence of movable logarithms in the GS of the nonlinear DE.

3 Fuchsian versus non-Fuchsian test

All the methods of the Painlevé test [3] are based on a lemma of Painlevé [4,2,9] which can be considered as a consequence of the classical theorem of perturbations of Poincaré. Let us assume that $E(u;x)$ depends implicitly on a small parameter ε through the general GS u and introduce the notation

$$u = \sum_{n=0}^{\infty} \varepsilon^n u_n; \; E(u;x) \equiv \sum_{n=0}^{\infty} \varepsilon^n E_n = 0 \tag{7}$$

in which the equation $E_0 = 0$ is nonlinear and every equation $E_n = 0, n \geq 1$ is linear in u_n:

$$n = 0 : \; E_0 \equiv E(u_0;x) = 0, \tag{8}$$

$$n = 1 : \; E_1 \equiv E'(u_0;x)u_1 = 0, \tag{9}$$

$$\forall n \geq 2 : \; E_n \equiv E'(u_0;x)u_n + R_n(u_0,\ldots,u_{n-1}) = 0. \tag{10}$$

In case of the *Fuchsian* test [8] (CFP test) :

- u_0 is a particular *local* meromorphic solution at $x = x_0$,

- $E_1 = 0$ is a *Fuchsian* linear equation at $x = x_0$,
- at *every order* of the perturbation, one has to check the absence of logarithm in the GS.

In the *non-Fuchsian* test [17] that we present here as an extension of the CFP test:

- u_0 is a particular *global* meromorphic solution at $x = x_0$ (it depends on $M < N$ arbitrary constants),
- E_1 which must have exactly order N is *non-Fuchsian* at $\chi \equiv x - x_0 = 0$ but *Fuchsian* at $\chi = \infty$ and does not possess any fixed critical points,
- at *every order* of the perturbation, one has to check the absence of logarithm in the GS.

As we will see in the next section, this extension allows us to detect *more rapidly* than in the *Fuchsian* test the presence of movable logarithms.

4 Two examples

1.*Preliminary consideration: Chazy's class III*

The simplified (i.e. scaled) equation of class III of Chazy [11]

$$E(x,u) \equiv u''' - 2uu'' + 3u'^2 = 0. \tag{11}$$

admits the global two-parameter solution

$$u^{(0)} = c\chi^{-2} - 6\chi^{-1},\ \chi = x - x_0,\ (x_0, c) \text{ arbitrary}. \tag{12}$$

For that equation, this solution arises from the *local* search for all the families of movable singularities $u \sim u_0\chi^p,\ E \sim E_0\chi^q,\ \chi = x - x_0 \to 0$ represented by Laurent series with a finite principal part

$$p = -1,\ q = -4,\ u^{(0)} = -6\chi^{-1}, \tag{13}$$
$$p = -2,\ q = -6,\ u^{(0)} = c\chi^{-2} - 6\chi^{-1},\ c \text{ arbitrary}. \tag{14}$$

The linearized equation at $u^{(0)}$ given by (12)

$$E'(x, u^{(0)})u^{(1)} \equiv [\partial_x^3 - 2u^{(0)}\partial_x^2 + 6u_x^{(0)}\partial_x - 2u_{xx}^{(0)}]u^{(1)} = 0, \tag{15}$$

has order 3 and possesses only two singular points $\chi = 0, \infty$. The point $\chi = 0$ is irregular singular of rank two for the non-Fuchsian family (14). However one can obtain a lowering of the order by 2 units with the change of function

$$u^{(1)} = \chi^{-3} v : \ [\partial_x + 3\chi^{-1} - 2c\chi^{-2}]v'' = 0, \tag{16}$$

which yields easily by quadrature the fundamental set of *global* solutions

$$\forall \chi \ \forall c : \ u^{(1)} \ = \ \chi^{-2}, \ \chi^{-3}, \ (e^{-2c/\chi} - 1 + 2c\chi^{-1})\chi^{-2}/(2c^2) \tag{17}$$

and contains no logarithm. This solves the question for the perturbation order $n = 1$ and the point $\chi = \infty$ is in this too simple an example useless to study.

Going on with the formalism of Painlevé's lemma at higher orders constitutes the rigorous mathematical framework of the local representation of the GS obtained by Joshi and Kruskal [13]

$$u = -6\chi^{-1} + c\chi^{-2}(1 + z - z^2/8 + z^3/144 - 7z^4/13824 + O(\varepsilon^5)), \ z = (\varepsilon/c)e^{-2c/\chi}. \tag{18}$$

This representation reduces to the one given by Chazy (Taylor series in $1/\chi$) if one starts from the Fuchsian family (13). The equation (11), which possesses the PP [11] and therefore for which no $u^{(n)}$ is multivalued, only shows the method.

We now illustrate on other examples the interest of non-Fuchsian families to detect the presence of a movable critical singularity, very often as soon as the first perturbation order.

2.*First example: a fourth order equation of Bureau*

The equation (ref. [19] p. 79)

$$E(x, u) \equiv u'''' + 3uu'' - 4u'^2 = 0 \tag{19}$$

which possesses the global two-parameter solution [8]

$$u^{(0)} = c\chi^{-3} - 60\chi^{-2}, \ \chi = x - x_0, \tag{20}$$

has as linearized equation

$$E^{(1)} = E'(x, u^{(0)})u^{(1)} \equiv [\partial_x^4 + 3u^{(0)}\partial_x^2 - 8u_x^{(0)}\partial_x + 3u_{xx}^{(0)}]u^{(1)} = 0, \tag{21}$$

with the two singular points $\chi = 0$ and $\chi = \infty$. Following the remark of Painlevé in [4] (p.209 note 1), it necessarily admits the two solutions $\partial_{x_0} u^{(0)}$ and $\partial_c u^{(0)}$, i.e. $u^{(1)} = \chi^{-4}, \chi^{-3}$, leaving only two other solutions to examine. The lowering by $M = 2$ units of the order of equ.(21) is obtained with

$$u^{(1)} = \chi^{-4} v : \ [\partial_x^2 - 16\chi^{-1}\partial_x + 3c\chi^{-3} - 60\chi^{-2}]v'' = 0, \tag{22}$$

For $c \neq 0$, the point $\chi = 0$ is singular irregular with rank one, and the two non-Fuchsian solutions are *formally* given as

$$\chi \to 0,\ c \neq 0 :\ v'' = e^{\pm\sqrt{-12c/\chi}}\chi^{35/4}(1 + O(\sqrt{\chi})), \tag{23}$$

detecting the presence in (21) of an essential singularity at $\chi = 0$, but the generically null radius of convergence of the formal series forbids to immediately conclude to the multivaluedness of $u^{(1)}$.

The point $\chi = \infty$ is singular regular and the searching in its neighbourhood of two independent solutions yields the *global* expressions

$$c \neq 0 :\ v_1'' = \chi^{-3}{}_0F_1(24; -3c/\chi) = \chi^{17/2}J_{23}(\sqrt{12c/\chi}), \tag{24}$$

$$v_2'' = \chi^{17/2}N_{23}(\sqrt{12c/\chi}), \tag{25}$$

where the hypergeometric function ${}_0F_1(24; -3c/\chi)$ is single valued and possesses an isolated essential singularity at $\chi = 0$, while the function N_{23} of Neumann is multivalued because of a $\mathrm{Log}\,\chi$ term.

One thus concludes here as soon as order one, to be compared with the order seven necessary to ref. [8], after a computation practically intractable without a computer.

3.*Second example: Bianchi IX cosmological model*

The Bianchi IX cosmological model in vacuum [14] is ruled by the sixth order system

$$\sigma^2(\mathrm{Log}\,A)'' = A^2 - (B - C)^2 \text{ and cyclically, } \sigma^2 = \pm 1, \tag{26}$$

and it does not possess the PP [15,16]. Let us prove it with our method.

In the neighborhood of the global solution depending on the four arbitrary parameters $(k_1, k_2, \tau_1, \tau_2)$ [18] (x is here denoted τ)

$$A^{(0)} = \sigma\frac{k_1}{\sinh k_1(\tau - \tau_1)},\ B^{(0)} = C^{(0)} = \sigma\frac{k_2^2 \sinh k_1(\tau - \tau_1)}{k_1 \sinh^2 k_2(\tau - \tau_2)}, \tag{27}$$

the perturbation

$$A = A^{(0)}(1 + \varepsilon A^{(1)} + O(\varepsilon^2)) \text{ and cyclically} \tag{28}$$

generates a linearized system whose order is equal to $N = 6$ and the lowering by $M = 4$ units of its order is obtained by the change of function [16]: $P^{(1)} = B^{(1)} + C^{(1)}, M^{(1)} = B^{(1)} - C^{(1)}$

$$\sigma^2 A^{(1)''} - 2A^{(0)^2}A^{(1)} = 0, \tag{29}$$

$$\sigma^2 P^{(1)''} - 2A^{(0)}B^{(0)}P^{(1)} = 4(A^{(0)}B^{(0)} - A^{(0)^2})A^{(1)}, \tag{30}$$

$$\sigma^2 M^{(1)''} + 2(A^{(0)}B^{(0)} - 2B^{(0)^2})M^{(1)} = 0. \tag{31}$$

Four single valued global solutions

$$(A^{(1)}, P^{(1)}) = \partial_c(\mathrm{Log}\, A^{(0)}, \mathrm{Log}(B^{(0)} + C^{(0)})),\ c = k_1, k_2, \tau_1, \tau_2, \qquad (32)$$

are those of the equations (29)–(30), and there only remains to study the equation (31), the singular points of which (modulo the period of sinh) are $\tau - \tau_2 = 0$ and $\tau = \infty$.

At $\tau - \tau_2 = 0$, the equation (31) generically possesses irregular singular points of rank two since the coefficient $B^{(0)^2}$ has there quadruple poles. Its two non-Fuchsian solutions are *formally* (ref. [12] chap.XVII)

$$\tau - \tau_2 \to 0 \ : \ M^{(1)} = e^{\alpha/(\tau-\tau_2)} \sum_{k=0}^{+\infty} \lambda_k (\tau - \tau_2)^{k+s},\ \lambda_0 \neq 0, \qquad (33)$$

$$\text{with } [16] \ : \ \alpha = \pm 2 k_1^{-1} \sinh k_1(\tau_2 - \tau_1),\ s = 1 \mp 2\cosh k_1(\tau_2 - \tau_1) \qquad (34)$$

The two generically irrational values for the exponents s allow to conclude only if the Taylor series $\lambda_k(\tau - \tau_2)^k$ can be summed, which is the case at least for $k_1 = k_2 = 0$ where the two solutions are globally known [16]:

$$k_1 = k_2 = 0 : \quad \frac{\mathrm{d}^2 M^{(1)}}{\mathrm{d}t^2} + \left(\frac{2}{t^2} - \frac{4(t-1)^2}{t^4}\right) M^{(1)} = 0,\ t = \frac{\tau - \tau_2}{\tau_1 - \tau_2},$$

$$M^{(1)} = e^{-2/t} t^{-1},\ e^{-2/t} t^{-1} \int^{1/t} z^{-4} e^{4z} \mathrm{d}z, \qquad (35)$$

The second solution implies the presence of a logarithmic branch point at $t = 0$. This singularity persists for $(k_1, k_2) \neq (0, 0)$ and this proves the absence of the Painlevé property for the Bianchi IX model in vacuum.

5 Conclusion

The main application of this extension to the Painlevé test is the case of a non-Fuchsian family: if the unperturbed solution is known in closed form and if the linearized equation has no fixed singularity at a finite distance, the study of the point at infinity often allows to conclude to non-integrability (in the sense of absence of the PP) much more rapidly than the above mentioned algebraic methods. The present method constitutes an algorithmic extension to the Painlevé test.

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DARBOUX TRANSFORMATIONS FOR INTEGRABLE LATTICE SYSTEMS

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A framework for a general description of Darboux transformations for Lax representations of discrete integrable systems is presented. The Lax equations are regarded as dynamical systems in the algebra of shift operators which is embedded in an algebra of pseudo-difference symbols. Gauge transformations are given by operators satisfying a dressing equation in this space. Special dressing operators are found which are parameterized by (adjoint) eigenfunctions of the Lax system. They give rise to Darboux like transformations as well as adjoint and binary versions. Reductions to finite operators are discussed.

1 Introduction

From a dynamical point of view Bäcklund transformations [1] may be regarded as invariant relations ("submanifolds") between evolution equations. For an integrable system, given by a Lax representation $L_t = [M, L]$, they usually originate from certain invariances of the Lax operator. These are "similarity transformations" $L \to \tilde{L} = WLW^{-1}$ preserving the Lax equation, which have been termed "gauge transformations", "dressing transformations" or "Darboux transformations" (DTs) in various contexts. Darboux's original observation was [2] that a solution of the Schrödinger equation $L(u)\phi = \lambda\phi$, $L(u) = \partial^2 + u$, can be transformed into a solution $\tilde{\phi} = \phi_0(\phi_0^{-1}\phi)_x$ of another spectral problem $L(\tilde{u})\tilde{\phi} = \lambda\tilde{\phi}$ by means of an eigenfunction ϕ_0 satisfying $L(u)\phi_0 = \lambda_0\phi_0$:

$$\tilde{u} \;=\; u \;+\; 2\,(\ln(\phi_0))_{xx}\;. \tag{1}$$

Its relevance in the context of integrable equations is as follows. If one adjoins the time evolutions $\phi_t = M(u)\phi$, $M(u) = 4\partial^3 + 6u\partial + 3u_x$, for both ϕ and ϕ_0, then one finds $\tilde{\phi}_t = M(\tilde{u})\tilde{\phi}$ after some straightforward computation. By compatibility this implies that both the original potential u and the transformed potential $\tilde{u}$ satisfy the KdV equation $u_t = u_{xxx} + 6uu_x$. The map $u \to \tilde{u}$, triggered by the eigenfunction ϕ_0, may be regarded as the gauge

transformation[a]

$$\begin{aligned} L = L(u) &\rightarrow \tilde{L} = L(\tilde{u}) = WL(u)W^{-1} \\ M = M(u) &\rightarrow \tilde{M} = M(\tilde{u}) = WM(u)W^{-1} + W_tW^{-1} \end{aligned}$$

of the Lax operators with $W = \phi_0 \partial \circ \phi_0^{-1}$ ($\circ$ indicates an operator product). Indeed, one computes

$$WL(u)W^{-1} = L(\tilde{u}) + \left(\frac{L(u)\phi_0}{\phi_0}\right)_x W^{-1}$$

and

$$WM(u)W^{-1} + W_tW^{-1} = M(\tilde{u}) - \left(\frac{\phi_{0t} - M(u)\phi_0}{\phi_0}\right)_x W^{-1} + 6\left(\frac{L(u)\phi_0}{\phi_0}\right)_x .$$

Hence, the spectral problem $L(u)\phi_0 = \lambda_0\phi_0$ is needed to preserve the form of L, and the time evolution $\phi_{0t} = M(u)\phi_0$ grants the invariance of the Lax equation.

The DT (1) is triggered by ϕ_0, a quantity satisfying *linear* equations. Elimination of ϕ_0 yields the (spatial part of the) usual auto-Bäcklund transformation $2(\tilde{u} + u) + (\int^x (\tilde{u} - u))^2 = 4\lambda_0$ of the KdV hierarchy.

Following lessons are to be learned from this well-known example:

i) DTs are "linearized" Bäcklund transformations, arising from a gauge (dressing) transformation of the Lax representation.

ii) The non-invertible dressing operator W needs to be inverted.

Although ii) looks absurd, this can be achieved in a simple technical manner. Embedding the differential operator W (with kernel spanned by ϕ_0) into the algebra of pseudo-differential symbols, $W^{-1} = \phi_0 \partial^{-1} \circ \phi_0^{-1}$ is well-defined. However, W^{-1} may not be applied to L_2-functions, so that the transformation of L is not a similarity transformation in a proper sense and the spectrum of the Lax operator may be changed. Indeed, the DT adds λ_0 to the eigenvalues.

In the following a similar construction is used to define DTs for discrete Lax operators of the form

$$L = u_k T^k + u_{k-1} T^{k-1} + \cdots .$$

The coefficients $u_j = u_j(i)$ are sequences with possibly non-commuting components and $T : u(i) \rightarrow u(i+1)$ is the shift operator. The map

[a] The Lax dynamics is preserved, automatically: $L = [M, L] \Rightarrow \tilde{L} = [\tilde{M}, \tilde{L}]$.

$r : \sum_j u_j T^j \to \sum_{j\geq 0} u_j T^j - \sum_{j<0} u_j T^j$ provides a classical r-matrix [3] on the algebra of such operators, leading to the commuting hierarchy of integrable Lax equations [4]

$$L_{t_n} \;=\; [(L^n)_{\geq 0}, L]\;, \tag{2}$$

where $(.)_{\geq 0}$ denotes the projection onto non-negative powers of T. In the following we will derive DTs for these systems. It is easily verified that operators solving the commuting hierarchy of "dressing equations"

$$W_{t_n} \;=\; -W\,(L^n)_{\geq 0} \;+\; (WL^nW^{-1})_{\geq 0}W \tag{3}$$

generate transformations $L \to \tilde{L} = WLW^{-1}$ keeping the dynamics (2) invariant: $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 0}, \tilde{L}]$. Remarkably, the dressing equations (3) can be linearized, i.e., there are solutions parameterized by objects satisfying linear equations (eigenfunctions and adjoint eigenfunctions). These results are given in section 2.

We introduce the difference operator $\Delta = T - 1$. Formally, its inverse may be written as $\Delta^{-1} = T^{-1} + T^{-2} + \cdots$. However, it is more appropriate to interpret it as a pseudo-difference symbol [4] with the algebraic property

$$\Delta^{-1} \circ u \;=\; u^{(-1)}\Delta^{-1} - (\Delta u^{(-2)})\,\Delta^{-2} + (\Delta^2 u^{(-3)})\,\Delta^{-3} \mp \cdots\;,$$

where $(u^{(k)}(i) = (T^k u)(i) = u(i+k)$.

The evolution equations (2) are compatible with the reduction to Lax operators parameterized by finitely many fields

$$L \;=\; u_k T^k + u_{k-1}T^{k-1} + \cdots + u_m T^m + q\,T^m\Delta^{-1}\circ r^*\;, \tag{4}$$

where the functions q and r inherit the dynamics

$$q_{t_n} \;=\; (L^n)_{\geq 0}\,q\;, \quad r_{t_n} \;=\; -(L^n)^*_{\geq 0}\,r$$

of eigenfunctions and adjoint eigenfunctions, respectively. Here

$$(\,u_nT^n + u_{n-1}T^{n-1} + \cdots)^* \;=\; T^{-n}u_n^* + T^{1-n}u_{n-1}^* + \cdots$$

is the formal adjoint and u_j^* denotes transposition (u_j, q and r may be column vectors or matrices). Obviously the nonlocal term in (4) can be omitted, as the constraint $q = r = 0$ is compatible with the dynamics.

For the DTs $L \to WLW^{-1}$ of reduced Lax operators (4) it is not sufficient that W satisfies (3). This would only grant the invariance of the Lax equations (2), but the form of L may be changed by the transformation. Certain additional conditions will arise to preserve the form (4) (section 3).

We finally present a technical result which will be relevant for the transformations of the next sections. If ϕ and ψ satisfy the dynamics of eigenfunctions and adjoint eigenfunctions associated with an operator $M = m_0 + m_1 T + \cdots$, then there exists a potential integrating the "squared eigenfunction" $\psi^* \phi$:

Lemma: *If $\phi_t = M\phi$ and $\psi_t = -M^*\psi$, then there exists a potential $\Omega = \Omega(\psi^*, \phi)$ defined by the compatible equations*

$$\Omega^{(1)} - \Omega = \psi^* \phi \ , \quad \Omega_t = \mathrm{res}(\, \Delta^{-1} \circ \psi^* \circ M \circ \phi \circ \Delta^{-1}) \ .$$

Here $\mathrm{res}(\sum_j u_j T^j) = u_{-1}$ is the residue of a pseudo-difference operator. The compatibility of the two equations for Ω is easily checked. This potential is uniquely defined up to a constant, which does neither depend on the time nor the lattice site. It should be regarded as a bilinear function of ψ and ϕ.

2 Darboux transformations

In this section we look for solutions of the dressing hierarchy (3). It turns out that there are simple dressing operators parameterized by solutions of the linear equations associated with the Lax hierarchy. In the following the notion of "eigenfunctions" ϕ and "adjoint eigenfunctions" ψ of a Lax operator L refers to the dynamical equations

$$\phi_{t_n} = (L^n)_{\geq 0} \phi \ , \quad \psi_{t_n} = -(L^n)^*_{\geq 0} \psi \ . \tag{5}$$

No spectral equation such as $L\phi = \lambda\phi$ is assumed in this section. The following theorems are stated without proofs. They are the discrete analogues of the general description of (continuous) DTs as given in Ref. 5. The statements may be verified by direct computations, details are to be published elsewhere.

Theorem 1: *Let ϕ_0 and ψ_0 be (adjoint) eigenfunctions of L. Then the operators* [b]

$$\begin{aligned} &a) \quad W = \phi_0^{(1)} \Delta \circ \phi_0^{-1} && \textit{(DT)} \\ &b) \quad W = (\psi_0^{*(-1)})^{-1} \Delta^{-1} \circ \psi_0^* && \textit{(adjoint DT)} \\ &c) \quad W = 1 - \phi_0 \, \Omega(\psi_0^*, \phi_0)^{-1} \Delta^{-1} \circ \psi_0^* && \textit{(binary DT)} \end{aligned}$$

solve the hierarchy of dressing equations (3), so that the dynamics (2) is invariant under the transformation $L \to WLW^{-1}$.

[b] The inverse of the operator c) is $W^{-1} = 1 + \phi_0 \Delta^{-1} \circ (\Omega(\psi_0^*, \phi_0)^{(1)})^{-1} \psi_0^*$.

Darboux's original result for the Schrödinger operator featured a transformation mechanism to map eigenfunctions of L to eigenfunctions of $\tilde{L} = WLW^{-1}$. This, in principle, is achieved by $\phi \to \tilde{\phi} = W\phi$. However, due to the occurrence of the pseudo-difference symbol Δ^{-1} special care has to be taken to give a rigorous meaning to the application of W to a function. The heuristic principle is that $\Delta^{-1}\psi_0^*\phi$ has to be replaced by the squared eigenfunction potential $\Omega(\psi_0^*, \phi)$. The following theorem provides rigorous statements on the transformation of eigenfunctions, adjoint eigenfunctions and their bilinear potential:

Theorem 2: *Let ϕ_0 and ψ_0 be eigenfunctions and adjoint eigenfunctions, respectively, associated with L. Let W be one of the dressing operators of Theorem 1, generated by ϕ_0 and ψ_0. If ϕ is an eigenfunction of L, then*

$$\begin{aligned}
&a) \quad \tilde{\phi} = \phi^{(1)} - \phi_0^{(1)}\phi_0^{-1}\phi && \textit{(DT)}\\
&b) \quad \tilde{\phi} = (\psi_0^{*(-1)})^{-1}\,\Omega(\psi_0^*, \phi) && \textit{(adjoint DT)}\\
&c) \quad \tilde{\phi} = \phi - \phi_0\,\Omega(\psi_0^*, \phi_0)^{-1}\,\Omega(\psi_0^*, \phi) && \textit{(binary DT)}
\end{aligned}$$

is an eigenfunction of $\tilde{L} = WLW^{-1}$. An adjoint eigenfunction ψ of L is transformed into the adjoint eigenfunction $\tilde{\psi}$ of $\tilde{L}$ by

$$\begin{aligned}
&a) \quad \tilde{\psi}^* = (\Omega(\psi^*, \phi_0)\,\phi_0^{-1})^{(1)} && \textit{(DT)}\\
&b) \quad \tilde{\psi}^* = \psi^*\psi_0^{*-1}\psi_0^{*(-1)} - \psi^{*(-1)} && \textit{(adjoint DT)}\\
&c) \quad \tilde{\psi}^* = \psi^* - \Omega(\psi^*, \phi_0)^{(1)}\,(\Omega(\psi_0^*, \phi_0)^{(1)})^{-1}\psi_0^* && \textit{(binary DT).}
\end{aligned}$$

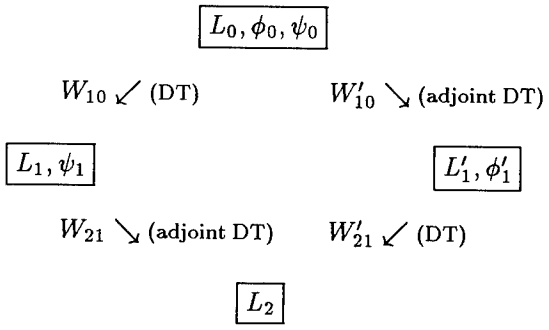
The squared eigenfunction potential of the new pair is given by

$$\begin{aligned}
&a) \quad \Omega(\tilde{\psi}^*, \tilde{\phi}) = \Omega(\psi^*, \phi_0)\phi_0^{-1}\phi - \Omega(\psi^*, \phi) && \textit{(DT)}\\
&b) \quad \Omega(\tilde{\psi}^*, \tilde{\phi}) = \psi^{*(-1)}(\psi_0^{*(-1)})^{-1}\,\Omega(\psi_0^*, \phi) - \Omega(\psi^*, \phi) && \textit{(adjoint DT)}\\
&c) \quad \Omega(\tilde{\psi}^*, \tilde{\phi}) = \Omega(\psi^*, \phi) - \Omega(\psi^*, \phi_0)\,\Omega(\psi_0^*, \phi_0)^{-1}\,\Omega(\psi_0^*, \phi) && \textit{(binary DT).}
\end{aligned}$$

We remark that the adjoint DT may be regarded as an inverse DT: from L_0 with eigenfunction ϕ_0 and trivial adjoint eigenfunction $\psi_0 = 0$ one may construct $L_1 = W_{10}L_0W_{10}^{-1}$ with $W_{10} = \phi_0\Delta \circ \phi_0^{-1}$. According to Theorem 2 there exists the (transposed) adjoint eigenfunction $\psi_1^* = (\phi_0^{(1)})^{-1}$, when putting $\Omega(\psi_0^*, \phi_0) = \Omega(0, \phi_0) = 1$. An adjoint DT $L_2 = W_{21}L_1W_{21}^{-1}$ with

$$W_{21} = (\psi_1^{*(-1)})^{-1}\,\Delta^{-1} \circ \psi_1^* = \phi_0\,\Delta^{-1} \circ (\phi_0^{(1)})^{-1} = W_{10}^{-1}$$

leads back to $L_2 = L_0$. Similarly, the binary DT may be understood as a composition of a DT and an adjoint DT:



Starting with the Lax operator L_0 and a pair of (adjoint) eigenfunctions ϕ_0, ψ_0, one may first use a DT $W_{10} = \phi_0^{(1)} \Delta \circ \phi_0^{-1}$ to obtain L_1 with the (transposed) adjoint eigenfunction $\psi_1^* = (\Omega(\psi_0^*, \phi_0)\phi_0^{-1})^{(1)}$, which is used to trigger an adjoint DT $W_{21} = (\psi_1^{*(-1)})^{-1} \Delta^{-1} \circ \psi_1^*$ to L_2. The diagram commutes: one may just as well start with the adjoint DT triggered by ψ_0 and use the image ϕ'_1 of ϕ_0 for a further DT. This results in the same operator L_2. The composed dressing operator $W_{21}W_{10} = W'_{21}W'_{10}$ mapping L_0 to L_2 defines the binary transformation.

Because of Theorem 2 DT's can be iterated, if a set of eigenfunctions $\phi_1, \ldots, \phi_n$ and/or adjoint eigenfunctions $\psi_1, \ldots, \psi_n$ is known. One may use ϕ_1, ψ_1 for a first transformation, then map the other (adjoint) eigenfunctions to new (adjoint) eigenfunctions and proceed from there. It is easily checked that the result of such an iteration does not depend on the ordering in which the triggering (adjoint) eigenfunctions are used. We present the composed dressing operators for iterated DTs and iterated binary DTs:

Theorem 3: *The iterated DT triggered by eigenfunctions $\phi_1, \ldots, \phi_n$ of L is given by a dressing operator $W = T^n + w_{n-1}T^{n-1} + \cdots + w_0$ satisfying (3). Its components are defined by the linear equations*

$$(w_0, \ldots, w_{n-1}) \begin{pmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{pmatrix} = -(\phi_1^{(n)}, \ldots, \phi_n^{(n)}) \, .$$

With the matrix $\Omega = (\Omega(\psi_i^*, \phi_j))$ the iterated binary DT triggered by $\phi_1, \ldots, \phi_n$

and the adjoint eigenfunctions $\psi_1, \ldots, \psi_n$ is given by the dressing operator

$$W \;=\; 1 \;-\; (\phi_1, \ldots \phi_n)\;\; \Omega^{-1} \begin{pmatrix} \Delta^{-1} \circ \psi_1^* \\ \vdots \\ \Delta^{-1} \circ \psi_n^* \end{pmatrix}$$

which satisfies (3).

These results are immediate consequences of the observation that the dressing operator of an iterated transformation has a kernel spanned by the triggering eigenfunctions. The effect of the iterated DTs on the coefficients of $L = u_k T^k + u_{k-1} T^{k-1} + \cdots$ is easily computed from $LW = W\tilde{L}$.

3 Darboux transformations for $L = u_k T^k + \cdots + u_m T^m + q\, T^m \Delta^{-1} \circ r^*$

So far only the dynamics (5) was imposed on the (adjoint) eigenfunctions generating the dressing transformations of Theorem 1. We now consider reductions of the hierarchy (2) to Lax operators of the form (4), where we have to ensure that the transformation to

$$\tilde{L} \;=\; WLW^{-1} \;=\; \tilde{u}_k T^k + \cdots + \tilde{u}_m T^m + \tilde{q}\, T^m \Delta^{-1} \circ \tilde{r}^* \qquad (6)$$

keeps this form invariant. It turns out that spectral equations $L\phi_0 = \lambda\phi_0$, i.e.,

$$u_k \phi_0^{(k)} + u_{k-1}\phi_0^{(k-1)} + \cdots + u_m \phi_0^{(m)} + q\, \Omega(r^*, \phi_0)^{(m)} \;=\; \lambda\, \phi_0\;, \qquad (7)$$

and $(L^*\psi_0)^* = \mu\psi_0^*$, i.e.,

$$(\psi_0^* u_k)^{(-k)} + (\psi_0^* u_{k-1})^{(1-k)} + \cdots + (\psi_0^* u_m)^{(-m)} - \Omega(\psi_0^*, q)^{(1-m)}\, r^* \;=\; \mu\, \psi_0^*\;, \qquad (8)$$

have to be imposed on the triggering (adjoint) eigenfunctions. The arbitrary constant in the square eigenfunction potential then can be chosen such that

$$\begin{aligned} \lambda\, \Omega(\psi_0^*, \phi_0)^{(-m)} \;-\; \mu\, \Omega(\psi_0^*, \phi_0) \;=\; & \operatorname{res}(T^{-m}\Delta^{-1} \circ \psi_0^* \circ L \circ \phi_0 \circ \Delta^{-1}) \\ & + \Omega(\psi_0^*, q)^{(-m)}\, \Omega(r^*, \phi) \end{aligned} \qquad (9)$$

holds.

Theorem 4: *If ϕ_0 and ψ_0 satisfy the spectral equations (7) and (8), respectively, and $\Omega(\psi_0^*, \phi_0)$ is chosen according to (9), then the DTs of Theorem 1 yield $\tilde{L} = WLW^{-1}$ of the form (6) with*

$$\tilde{q} \;=\; \begin{cases} q^{(1)} - \phi_0^{(1)} \phi_0^{-1} q & \textit{(DT)} \\ (\psi_0^{*(-1)})^{-1}\, \Omega(\psi_0^*, q) & \textit{(adjoint DT)} \\ q - \phi_0 \Omega(\psi_0^*, \phi_0)^{-1}\, \Omega(\psi_0^*, q) & \textit{(binary DT)} \end{cases}$$

$$\text{and} \qquad \tilde{r}^* = \begin{cases} (\Omega(r^*, \phi_0)\, \phi_0^{-1})^{(1)} & (DT) \\ r^* \psi_0^{*-1} \psi_0^{*(-1)} - r^{*(-1)} & (\textit{adjoint DT}) \\ r^* - \Omega(r^*, \phi_0)^{(1)} (\Omega(\psi_0^*, \phi_0)^{(1)})^{-1} \psi_0^* & (\textit{binary DT}). \end{cases}$$

4 Modified Lax hierarchies

Modified Lax hierarchies are obtained by simple gauge transformations $L \to L' = \Phi^{-1} L \circ \Phi$, where Φ is an eigenfunction of L. [4] The new operator satisfies

$$L'_{t_n} = [(L'^n)_{\geq 1}, L'] \,, \tag{10}$$

where $L' = u'_k \Delta^k + u'_{k-1} \Delta^{k-1} + \cdots$ has to be sorted with respect to powers of Δ and $(.)_{\geq 1}$ denotes the projection to positive powers of Δ. Dressing operators now need to satisfy the modified hierarchy

$$W'_{t_n} = -W' (L'^n)_{\geq 1} + (W' L'^n W'^{-1})_{\geq 1} W' \tag{11}$$

in order to preserve (10). Solutions of (11) can be derived systematically from the previous results. We note that the modified eigenfunction dynamics $\phi'_{t_n} = (L'^n)_{\geq 1} \phi'$ admits the trivial function 1. The DTs W of Theorem 1, followed by a gauge transformation with the image $\Phi' = W1$ (given by Theorem 2), can be used to generate dressing operators $W' = \Phi'^{-1} W$ satisfying (11). One finds the modified DTs $L' \to W' L' W'^{-1}$ triggered by (adjoint) eigenfunctions ϕ'_0 and ψ'_0 of L':

$$W' = \begin{cases} (\Delta\, \phi_0'^{-1})^{-1}\, \Delta \circ \phi_0'^{-1} & \text{(modified DT)} \\ \omega'^{-1}\, \Delta^{-1} \circ \psi_0'^* & \text{(modified adjoint DT)} \\ (1 - \phi'_0\, \Omega'^{-1}\, \omega')^{-1}\, (1 - \phi'_0\, \Omega'^{-1} \Delta^{-1} \circ \psi_0'^*) & \text{(modified binary DT)} \end{cases}$$

with $\Omega' = \Omega(\psi_0'^*, \phi'_0)$ and $\omega' = \Omega(\psi_0'^*, 1)$.

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N SOLITON SOLUTION OF HARRY DYM EQUATION BY INVERSE SCATTERING METHOD

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We obtain N soliton solution of the Harry Dym equation with the inverse scattering method due to introduce new independent variable and show concrete one and two soliton solutions.

1 Introduction

The nonlinear evolution equations, so called 'WKI' equations, were found by Wadati, Konno and Ichikawa with a generalization of the inverse scattering method (ISM) in 1979 [1]. In 1980, Wadati, Ichikawa and Shimizu found that one of the WKI equations

$$q_t - 2\left(\frac{1}{(1+q)^{1/2}}\right)_{xxx} = 0 \tag{1}$$

and the Harry Dym (HD) equation

$$r_t + (1-r)^3 r_{xxx} = 0 \tag{2}$$

were related by a transformation

$$(1+q)^{-1/2} = 1 - r, \tag{3}$$

and solved the WKI equation (1) by the generalized ISM, using the relation (3), at last obtained one soliton solution of the HD equation $r(x) = r(x + \varepsilon_+(x))$ [2]. But it was not solved N soliton solution, because except one soliton solution Wadati et al did not obtain $\varepsilon_+(x)$.

On the other hand Dmitrieva solved N soliton solution of the HD equation by the special method so called the 'higher-times approach' [4] that is different from the ISM.

However, we introduce the new independent variable s as

$$s \equiv x + \varepsilon_+(x). \tag{4}$$

Then we can solve N soliton solution of the HD equation by the ISM.

In this paper we find definite $\varepsilon_+(x)$ for N soliton solution due to introduce new independent variable, and solve N soliton solution of the HD equation (2).

2 The inverse scattering problem

First, we solve the WKI equation, using the method that is introduce solving N loop soliton solution by Konno and Jeffrey [3]. We assume the following boundary condition:

$$q(x,t) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5}$$

We consider the following eigenvalue problem for the WKI equation (1)

$$\psi_{xx} + \lambda^2(1+q)\psi = 0, \tag{6}$$

and the time dependence of the eigenfunction

$$\psi_t = 2\lambda^2 \left(\frac{1}{(1+q)^{1/2}} \frac{\partial}{\partial x} - \left(\frac{1}{(1+q)^{1/2}} \right)_x \right) \psi. \tag{7}$$

By assuming $\lambda_t = 0$, a compatibility condition of Eqs. (6) and (7) cause the WKI equation (1).

We now define the Jost functions by

$$\begin{array}{lll} \phi(\lambda,x) \to \exp(-i\lambda x) & \text{as} & x \to -\infty, \\ \psi(\lambda,x) \to \exp(i\lambda x) & \text{as} & x \to +\infty, \end{array} \tag{8}$$

and the scattering coefficients by

$$\phi(\lambda,x) = a(\lambda)\psi(-\lambda,x) + b(\lambda)\psi(\lambda,x). \tag{9}$$

In order to examine the analytic properties of $a(\lambda)$ and the Jost functions for large $|\lambda|$, we introduce

$$\phi = exp(-i\lambda x + \int_{-\infty}^{x} \sigma \, dy), \tag{10}$$

and expand σ in power series of λ :

$$\sigma = \sum_{n=-1}^{\infty} \frac{\sigma_n}{(i\lambda)^n}. \tag{11}$$

Substitution of (10) into (6) and using (11) then yields

$$\sigma_{nx} + \sum_{l=-1}^{n+1} \sigma_l \sigma_{n-l} - 2\sigma_{n+1} - \delta_{n,-2} q = 0. \tag{12}$$

The first two conserved densities are

$$\sigma_{-1} = 1 - (1+q)^{1/2}, \tag{13}$$

$$\sigma_0 = -\frac{1}{4}\frac{\partial}{\partial x}\ln(1+q), \tag{14}$$

where above densities vanish for $|x| \to \infty$. For $\psi(\lambda)$, similar analysis is possible. Thus we have the asymptotic behaviour of ϕ,ψ and a for large $|\lambda|$ such as

$$\begin{aligned} \phi &= \exp[-i\lambda(x-\varepsilon_-)+\mu_-] + O(\frac{1}{\lambda}), \\ \psi &= \exp[i\lambda(x+\varepsilon_+)-\mu_+] + O(\frac{1}{\lambda}), \\ a &= \exp(i\lambda\varepsilon+\mu) + O(\frac{1}{\lambda}), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \varepsilon_- &= \int_{-\infty}^{x} \sigma_{-1}\,dy, \\ \varepsilon_+ &= \int_{x}^{\infty} \sigma_{-1}\,dy, \\ \varepsilon &= \varepsilon_+ + \varepsilon_- = \int_{-\infty}^{\infty} \sigma_{-1}\,dy, \end{aligned} \tag{16}$$

$$\begin{aligned} \mu_- &= \int_{-\infty}^{x} \sigma_0\,dy, \\ \mu_+ &= \int_{x}^{-\infty} \sigma_0\,dy, \\ \mu &= \mu_+ + \mu_- = \int_{-\infty}^{\infty} \sigma_0\,dy. \end{aligned} \tag{17}$$

When q is on compact support, the $\phi\exp[i\lambda(x-\varepsilon_-)]$, $\psi\exp[-i\lambda(x+\varepsilon_+)]$ and $a\exp(-i\lambda\varepsilon)$ are entire functions of λ.

From (9), we consider the integral to be

$$\begin{aligned} \int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{\phi(\lambda')\exp[i\lambda'(x-\varepsilon_-)]}{a(\lambda')\exp(-i\lambda'\varepsilon)} &= \int_C \frac{d\lambda'}{\lambda'-\lambda}\psi(-\lambda')\exp[i\lambda'(x+\varepsilon_+)] \\ &+ \int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')\exp[i\lambda'(x+\varepsilon_+)]. \end{aligned} \tag{18}$$

Here we define an integral path C that is the contour in the complex λ' plane, starting from $\lambda' = -\infty + i0^+$, passing over all zeros of $a(\lambda')$, and ending at $\lambda' = +\infty + i0^+$ for λ below C.

We calculate (18) and obtain

$$\psi(-\lambda)\exp[i\lambda(x+\varepsilon_+)] = \exp(-\mu_+) + \frac{1}{2\pi i}\int_C \frac{d\lambda'}{\lambda' - \lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')\exp[i\lambda'(x+\varepsilon_+)]. \tag{19}$$

Next, in order to derive the Gel'fand-Levitan equation, we introduce a kernel K by

$$\psi = \exp[i\lambda(x+\varepsilon_+)] + i\lambda\int_x^\infty K(x,z)\exp[i\lambda(x+\varepsilon_+(x))]\,dz, \tag{20}$$

where we assume

$$\lim_{z\to\infty} K(x,z) = 0. \tag{21}$$

Substitution of (20) into eigenvalue problem (6), using (3) we get the relation between the kernel K and the solution of WKI or HD equation:

$$1 + q = (1 - K(x,x))^{-4}, \tag{22}$$

$$1 - r = (1 - K(x,x))^2. \tag{23}$$

From (19) and (20), we obtain the following Gel'fand-Levitan equation for $w \geq x$:

$$K(x,w) - F(x+w) - \int_x^\infty K(x,z)F'(w+z)\,dz = 0, \tag{24}$$

where

$$F(z) = \frac{1}{2\pi}\int_C \frac{d\lambda}{i\lambda}\frac{b(\lambda)}{a(\lambda)}\exp[i\lambda(z+2\varepsilon_+(x))], \tag{25}$$

$$F'(z) = \frac{\partial F}{\partial z} = \frac{1}{2\pi}\int_C \frac{b(\lambda)}{a(\lambda)}\exp[i\lambda(z+2\varepsilon_+(x))]\,d\lambda. \tag{26}$$

The time dependences of scattering coefficients are determined from (7) to be

$$\begin{aligned} a(\lambda,t) &= a(\lambda,0), \\ b(\lambda,t) &= b(\lambda,0)\exp(8i\lambda^3 t). \end{aligned} \tag{27}$$

The zeros of $a(\lambda)$ in the upper half λ plane give the bound states. When all the zeros of $a(\lambda)$ in the upper half λ plane are simple, $F(z)$ can be expressed

as

$$F(z,t) = \sum_{k=1}^{N} \frac{c_k(t)}{i\lambda_k} \exp[i\lambda_k(z + 2\varepsilon_+(x))] + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{i\lambda} \rho(\lambda, t) \exp[i\lambda_k(z + 2\varepsilon_+(x))], \tag{28}$$

where

$$\begin{aligned} c_k(t) &= c_k(0) \exp(8i\lambda_k^3 t), \\ \rho(\lambda, t) &= \rho(\lambda, 0) \exp(8i\lambda^3 t). \end{aligned} \tag{29}$$

Given the scattering data $\{\rho(\lambda, 0), \lambda; c_k(0), \lambda_k, k = 1, 2, \ldots, N\}$, we can determine $F(z)$ and then solve the Gel'fand-Levitan equation (24) for $K(x, w; t)$. Then, we obtain the solutions for WKI and HD equation by using the relation (22) or (23).

3 The N soliton solution

Now we shall solve the N soliton solution under the conditions:

$$\begin{aligned} &1) \quad \rho(\lambda, t) = 0, \\ &2) \quad \lambda_k = i\eta_k, \quad \eta_k > 0, \quad k = 1, 2, \ldots, N. \end{aligned} \tag{30}$$

Then $F(z,t)$ became

$$F(z,t) = \sum_{k=1}^{N} \frac{c_k(t)}{\eta_k} \exp[-\eta_k(z + 2\varepsilon_+(x))]. \tag{31}$$

We define the following kernel K:

$$K(x,z) = \sum_{k=1}^{N} A_k(x) \exp[-\eta_k(x + z + 2\varepsilon_+(x))]. \tag{32}$$

Substitution (31) and (32) into the Gel'fand-Levitan equation (24), we obtain

$$A_k = \frac{\|\mathbf{D}_k\|}{\|\mathbf{D}\|}, \tag{33}$$

where $\|\mathbf{D}\|$ and $\|\mathbf{D}_k\|$ means determinants of matrix $\mathbf{D}$ and $\mathbf{D}_k$ respectively. The coefficient matrix $\mathbf{D}$ is defined as

$$\mathbf{D} = \begin{pmatrix} d_{11} & \ldots & d_{1N} \\ \vdots & \ddots & \vdots \\ d_{N1} & \ldots & d_{NN} \end{pmatrix} \tag{34}$$

with

$$d_{kl} = \delta_{kl} + \frac{2\eta_k}{\eta_k + \eta_l} \exp[-2\eta_l(x + \varepsilon_+(x)) + \alpha_k], \tag{35}$$

$$\alpha_k = 8\eta_k^3 t + \ln\left(\frac{c_k(0)}{2\eta_k}\right), \tag{36}$$

while the matrix $\mathbf{D}_k$ is defined as

$$\mathbf{D}_k = \begin{pmatrix} d_{11} & \dots & d_{1k-1} & c_1/\eta_1 & d_{1k+1} & \dots & d_{1N} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ d_{k1} & \dots & d_{kk-1} & c_k/\eta_k & d_{kk+1} & \dots & d_{kN} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{N1} & \dots & d_{Nk-1} & c_N/\eta_N & d_{Nk+1} & \dots & d_{NN} \end{pmatrix}. \tag{37}$$

From (4), we obtain the following relationship between ε_+ and kernel K:

$$\varepsilon_+(s) = \int_{-\infty}^{s} \left(1 - (1 - K(s))^2\right) ds, \tag{38}$$

where

$$K(s) \equiv K(x, x). \tag{39}$$

Thus we have above definite ε_+ for N soliton solution.

4 One and two soliton solutions

On the basis of the results obtained in the previous section, we shall analyze the one and two soliton solutions.

4.1 One soliton solution (N=1)

From (23), (32) and (38), we obtain the one soliton solution of the HD equation

$$(1 - r) = \tanh^2\left(\frac{\gamma_1}{2}\right), \tag{40}$$

and

$$\varepsilon_+ = -\frac{1}{\eta}\left(\tanh\left(\frac{\gamma_1}{2}\right) + 1\right), \tag{41}$$

where

$$\gamma_k = -2\eta_k s + \alpha_k. \tag{42}$$

The shape of one soliton solution of the HD equation is so called 'cusp soliton'.

4.2 Two soliton solution (N=2)

From (23), (32) and (38), we obtain the two soliton solution of the HD equation with the two eigenvalues η_1 and η_2,

$$(1-r) = \frac{T^2}{W^2}, \tag{43}$$

and

$$\varepsilon_+ = -2\frac{V}{W}, \tag{44}$$

where

$$\begin{aligned} W &= 1 + e^{\gamma_1} + e^{\gamma_2} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{\gamma_1+\gamma_2}, \\ T &= 1 - e^{\gamma_1} - e^{\gamma_2} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{\gamma_1+\gamma_2}, \\ V &= \frac{1}{\eta_1}e^{\gamma_1} + \frac{1}{\eta_2}e^{\gamma_2} + \left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{\gamma_1+\gamma_2}. \end{aligned} \tag{45}$$

In order to investigate the asymptotic behaviour of two solitons, we assume

$$\eta_1 > \eta_2. \tag{46}$$

The shifts of the two cusp solitons as a result of a collision are

$$\triangle x_1 = \frac{1}{\eta_1}(2+\beta), \tag{47}$$

and

$$\triangle x_2 = \frac{1}{\eta_2}(2-\beta), \tag{48}$$

where

$$\beta = \ln\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}. \tag{49}$$

5 Concluding remark

We solve the HD equation by the ISM, and obtain N soliton solution due to introduce new independent variable s. We make out that our one and two soliton solutions accord with results by higher-times approach.

We think that similar analysis is possible for the complex HD equation, using the ISM.

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ALGEBRAIC AND GEOMETRICAL PROPERTIES OF INTEGRABLE NONLINEAR FIELD EQUATIONS

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The Estabrook–Wahlquist prolongation method [4] is applied to systems of nonlinear field equations for analyzing the link between the integrability property and loop algebras. It turns out that the integrability property of nonlinear field equations is closely connected with the existence of incomplete prolongation Lie algebras via the introduction of a integrable Cartan–Ehresmann connection on suitable fibered spaces. Quite recently, within the inverse prolongation scheme [3,5,6] a new procedure has been outlined by us [1,2] for generating, starting from a loop algebra realization of the Kac–Moody type, a whole family of nonlinear field equations containing the original systems (and the related linear problems). In the case of the couple of nonlinear Schrödinger equations, we find *and linearize* a system of physical importance describing twisted birefringent optical fibers [7].

1 The incomplete prolongation Lie algebra

Let us consider the continuous isotropic Heisenberg model in $1+1$–dimension

$$(\Sigma\vec{S})_t = \vec{S} \times \vec{S}_{xx} \quad , \quad (\Sigma\vec{S}) \cdot \vec{S} = k^2 \tag{1}$$

where $\vec{S} = \vec{S}(x,t)$, $\Sigma = \mathrm{diag}(1,1,k^2)$, and $k^2 = \pm 1$.

In order to apply the Estabrook–Wahlquist prolongation method, let us introduce the equivalent closed differential ideal defined by the 2-forms

$$\vec{\alpha}_1 = d\vec{S} \wedge dt - \vec{S}_x dx \wedge dt \quad , \quad \vec{\alpha}_2 = d(\Sigma\vec{S}) \wedge dx + \vec{S} \times d\vec{S}_x \wedge dt$$

$$\beta = d(\Sigma\vec{S}) \cdot \vec{S}_x \wedge dt + (\Sigma\vec{S}) \cdot d\vec{S}_x \wedge dt.$$

At this stage, let us consider the prolongation 1-forms

$$\omega^k = -dy^k + F^k(\vec{S}, \vec{S}_x, y^m)dx + G^k(\vec{S}, \vec{S}_x, y^m)dt \tag{2}$$

where $y = \{y^m\}$, $k, m = 1, 2, \ldots, N$ (N arbitrary), and F^k, G^k, are, respectively, the pseudopotential and functions to be determined.

Now, by requiring that $d\omega^k \in \mathcal{I}(\vec{\alpha}_i, \beta, \omega^k)$, we obtain

$$\begin{aligned} F &= \vec{X}(y)\cdot\vec{S} + Y(y) \\ G &= (\Sigma\vec{X}\times\vec{S})\cdot\vec{S}_x - S_1[X_2,X_3] + S_2[X_1,X_3] \\ &\quad -k^2 S_3[X_1,X_2] + Z(y) \end{aligned} \tag{3}$$

and the commutation relations

$$[X_1,X_2]=X_4, [X_1,X_3]=X_5, [X_2,X_3]=X_6, [X_1,X_5]=[X_2,X_6],$$

$$[X_1,X_4]=-k^2[X_3,X_6], [X_2,X_4]=k^2[X_3,X_5], [Y,Z]=-k^2[X_2,X_5],$$

$$[X_1,X_6]=[X_3,X_4]=-[X_2,X_5], \qquad \left[Y,\vec{X}\right]=\left[Z,\vec{X}\right]=\vec{0} \tag{4}$$

Here $\vec{X}=(X_1,X_2,X_3)$, where X_j, Y and Z are arbitrary functions depending on the pseudopotential y only. A homomorphism between this algebra and the $\mathfrak{sl}(2,\mathbb{C})$ algebra $[X_1,X_2]=2i\lambda k^2 X_3, [X_1,X_3]=-2i\lambda X_2, [X_2,X_3]=2i\lambda X_1$, exploiting (2) and (3), yield the spectral problem associated with the model (1). Another possible realization of (4) is given by an infinite-dimensional Lie algebra of the type Kac–Moody:

$$X_1=kT_1^{(1)}, X_2=kT_2^{(1)}, X_3=T_3^{(1)}, X_4=ik^2T_1^{(2)}, X_5=-ikT_2^{(2)}, X_6=ikT_1^{(2)},$$

where the vector fields $T_i^{(m)}(i=1,2,3; m\in\mathbb{Z})$ obey the commutation relations

$$\left[T_i^{(m)}, T_j^{(n)}\right] = i\epsilon_{ijk}T_k^{(m+n)}, \tag{5}$$

ϵ_{ijk} being the Ricci tensor. A representation of (5) in terms of the prolongation variables let us enables to obtain the spectral problem for equation (1).

Likewise let us consider the pair of nonlinear Schrödinger equations

$$i\vec{u}_x + \vec{u}_{tt} + k\sigma\vec{u} + \alpha|\vec{u}|^2\vec{u} = 0 \tag{6}$$

where $\sigma=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\vec{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}$ and $k,\alpha\in\mathbb{R}$. An interesting realization of the Kac–Moody type of the incomplete prolongation algebra can be built up where the vector fields $T_{lm}^{(n)}$ satisfy the commutation relations

$$\left[T_{lm}^{(n)}, T_{l'm'}^{(n')}\right] = i\epsilon_{lml'm'kj}T_{kj}^{(n+n')}, \tag{7}$$

$$\begin{aligned} \epsilon_{lml'm'kj} &= \delta_{lm}\delta_{l'0}\delta_{k0}\delta_{m'j} - \delta_{lm}\delta_{m'0}\delta_{j0}\delta_{l'k} + \delta_{l'm'}\delta_{m0}\delta_{nl}\delta_{j0} \\ &\quad -\delta_{l'm'}\delta_{m0}\delta_{kl}\delta_{j0} + \delta_{m'l}\delta_{kl'}\delta_{jm} - \delta_{ml'}\delta_{kl}\delta_{jm'} \end{aligned}$$

$m, l, m', l', k, j = 0, 1, 2$ and $T_{00}^{(n)} = 0$, $\forall n \in \mathbb{Z}$. It is noteworthy that (7) admits the representation in terms of the prolongation variable:

$$T_{lm}^{(r)} = -i \sum_{k=-\infty}^{+\infty} y_l^{(r+k)} \frac{\partial}{\partial y_m^{(k)}} + i\delta_{lm} \sum_{k=-\infty}^{+\infty} y_0^{(r+k)} \frac{\partial}{\partial y_0^{(k)}}, \tag{8}$$

$(l, m = 1, 2)$, which enables us to determine the spectral problem for (6).

2 The Kac–Moody algebra and a new procedure for the inverse problem.

The prolongation structures of nonlinear field equations can be interpreted as a Cartan–Ehresmann connection. Conversely, starting from a prolongation algebra, one can determine the differential ideal related to a certain nonlinear field equation specifying the form of the connection.

In the following we solve the inverse prolongation problem by adopting a new procedure based on the Kac–Moody realizations (5) and (7). Precisely, let us look for the class of nonlinear field equations whose prolongation structure is assumed to be given by the 1-form of connection

$$\omega = -dy + F dx + G dt \tag{9}$$

where F and G are suitable functions (depending on the generators of the Kac–Moody algebras) to be determined by the integrability condition. Of course, once a representation of the Kac–Moody algebras is furnished, an evolution equation exists which can be obtained simply by equating the 1-forms (9) to zero. Let us expose this new method for the couple of nonlinear Schrödinger equations describing the propagation of waves in optical fibers, where

$$\begin{aligned} F &= i \sum_{k=1}^{2} \left[\alpha_k T_{0k}^{(n)} + \beta_k T_{0k}^{(-n)} + \gamma_k T_{0k}^{(n+l)} + \phi_k T_{k0}^{(-n+l)} \right. \\ &\quad \left. + \psi_k T_{1k}^{(0)} + \chi_k T_{2k}^{(0)} + \rho T_{kk}^{(l)} \right], \\ G &= i \sum_{k=1}^{2} \left[p_k u_k T_{0k}^{(n)} + q_k u_k^* T_{k0}^{(-n)} + \sigma T_{kk}^{(l)} \right] \end{aligned}$$

$\alpha_k, \beta_k, \gamma_k, \phi_k, \psi_k, \chi_k$, are functions of $\vec{u}$ and $\vec{u}^*$ to be determined in such a way that the operators T satisfy the Kac–Moody algebra (7)–(8) and p_k, q_k, ρ, and σ are constants.

The compatibility condition $\vec{y}_{xt} = \vec{y}_{tx}$ yields the constraints

$$\begin{aligned} \beta_{1t} &- (\psi_1 + \chi_2) q_1 u_1^* = \sum_{j=1}^{2} q_j \psi_j u_j^* + q_1 u_{1x}^* \\ \beta_{2t} &- (\psi_1 + \chi_2) q_2 u_2^* = \sum_{j=1}^{2} q_j \chi_j u_j^* + q_2 u_{2x}^* \\ \alpha_{kt} &+ p_k u_k (\psi_1 + \chi_2) + p_1 u_1 \psi_k + p_2 u_2 \chi_k = p_k u_{kx} \end{aligned} \tag{10}$$

and

$$\alpha_k = \frac{\rho}{3\sigma^2} p_k u_{kt}, \quad \beta_k = -\frac{\rho}{3\sigma^2} q_k u^*_{kt}, \quad \gamma_k = \frac{\rho}{\sigma} p_k u_k, \quad \phi_k = \frac{\rho}{\sigma} q_k u^*_k$$
$$\psi_k = -\frac{\rho}{3\sigma^2} q_1 p_k u^*_1 u_k + \eta_k, \quad \chi_k = -\frac{\rho}{3\sigma^2} q_2 p_k u^*_2 u_k + \mu_k. \tag{11}$$

Then, with the help of (11), equations (10) yield

$$i\vec{u}_x + \vec{u}_{tt} + A\vec{u} + \epsilon|\vec{u}|^2\vec{u} = 0 \tag{12}$$

where $A = \begin{pmatrix} \Delta_1 & k \\ k^* & \Delta_2 \end{pmatrix}$, and $\epsilon, \Delta_1, \Delta_2$ are real quantities. We remark that (6) can be found for $\Delta_1 = \Delta_2 = 0$ and $k = k^*$. Furthermore, our inverse prolongation technique is a powerful tool for generating new integrable nonlinear field equations of physical significance. In the present case, by choosing $\Delta_1 = -\Delta_2 \neq 0$ and $k = k^*$, equations (12) become the coupled equations related to twisted birefringent optical fibers, which we are able to obtain as compatibility condition for the linear operators,

$$\mathcal{L}_1 = \begin{pmatrix} -i\left(\frac{\epsilon}{2}|u_1|^2 + \frac{\epsilon}{2}|u_1|^2 + 6\sigma^2\right) & \frac{\epsilon}{2}\left(u^*_{1t} - 3\sigma u^*_1\right) & \frac{\epsilon}{2}\left(u^*_{2t} - 3\sigma u^*_2\right) \\ -u_{1t} - 3\sigma u_1 & i\left(\frac{\epsilon}{2}|u_1|^2 + 3\sigma^2 + \Delta\right) & i\left(\frac{\epsilon}{2} u_1 u^*_2 + k\right) \\ -u_{2t} - 3\sigma u_2 & i\left(\frac{\epsilon}{2} u^*_1 u_2 + k^*\right) & i\left(\frac{\epsilon}{2}|u_2|^2 + 3\sigma^2 - \Delta\right) \end{pmatrix}$$

and

$$\mathcal{L}_2 = \begin{pmatrix} -2\sigma & i\frac{\epsilon}{2}u_1 & i\frac{\epsilon}{2}u^*_2 \\ iu_1 & \sigma & 0 \\ iu_2 & 0 & \sigma \end{pmatrix}.$$

In more than $1+1$ dimensions the inverse method has not been still explored. This is an open problem, together with the attempt of building up a geometrical and algebraic foundation of integrability of nonlinear field equations.

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FROM SOLITARY WAVES TO PERIODIC WAVES VIA NONLINEAR SUPERPOSITION : A BILINEAR APPROACH

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ABSTRACT

In the one hundred years that have passed since D.J. Korteweg and G. de Vries wrote their seminal work on the Korteweg-de-Vries equation, many other equations have been derived that describe various types of nonlinear wave propagation. Some recent studies have focused on those *nonlinear* periodic waves that can be formed by superposing solitary waves in a *linear* fashion. This elegant and revealing structure – which may be interpreted as a very special *nonlinear superposition principle* – is captured in the imbricate series representation of a periodic solution. We illustrate the principle for various nonlinear wave equations – both completely integrable and otherwise – using Hirota's bilinear transformation method.

1. Introduction

It is exactly one hundred years ago that D.J. Korteweg and G. de Vries first reported their equation – the celebrated Korteweg-de-Vries (KdV) equation[1] – that governs the evolution of small-amplitude long waves in shallow water. It is timely, therefore, that the present paper should again focus attention on the two important nonlinear waves enunciated by these authors – the classical sech^2 solitary wave and the periodic cnoidal wave. Some recent studies[2-6] have focused on those *nonlinear* wave equations that admit periodic waves which possess the remarkable property that they can be formed by superposing "solitary waves" in a *linear* fashion. For the KdV equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

a connection between the classical *sech*^{2} solitary wave,

$$u_s(x,t) = \tilde{u}_0 + 3p^2\mathrm{sech}^2\tfrac{1}{2}p(x - ct + x_0), \qquad c = \tilde{u}_0 + p^2, \tag{2a,b}$$

and the periodic *cnoidal* wave has long been known. This connection is made apparent by the limiting process that takes the cnoidal wave into the sech^2 solitary wave as the wavelength becomes infinite. However, further investigation shows that the link between these two solutions is far more intimate, and is one that is realised in a very precise and striking manner. In the first result of its kind, Toda[7] demonstrated that the cnoidal wave of the KdV Eq.1 has the representation

$$u(x,t) = u_0 + 3p^2 \sum_{n=-\infty}^{\infty} \mathrm{sech}^2\tfrac{1}{2}p(\xi - n\sigma), \qquad \xi = x - ct, \tag{3}$$

which expresses the periodic solution as an infinite sum of identical, regularly spaced sech^2 solitary-wave *profiles* (and note our emphasis). (We shall derive this result in Section. 2).

Subsequent investigations have shown that many other nonlinear equations (completely integrable and otherwise) possess this same property, which reveals the beautifully simple structure of certain spatially periodic solutions as arrays of solitary waves[2-6, 8-10]. Boyd[11] has coined the suggestive term "imbricate" – meaning to *decorate with a repeating pattern like overlapping tiles* – with which to describe series such as (3). We have argued elsewhere[4] that the extent of the various known imbricate series permits their interpretation as a very special *nonlinear superposition principle.* The principle may be stated formally as: *"Spatially periodic solutions of nonlinear wave equations have representations as infinite sums of identical, regularly spaced solitary-wave* ***profiles"***.

In this paper we will illustrate the principle for a number of nonlinear wave equations and develop these ideas by means of Hirota's direct, or bilinear transformation method[12,13]. The advantage of the bilinear approach becomes apparent in the ease with which we are able to deduce the imbricate series representations for the various spatially periodic waves.

2. Periodic Solutions and the Bilinear Form

Under the dependent variable transformation

$$u(x,t) = u_0 + 12\partial^2 \ln f(x,t)/\partial x^2, \tag{4}$$

it is readily shown that the KdV Eq.1 has the bilinear form[13]

$$(D_x D_t + u_0 D_x^2 + D_x^4 + B) f \cdot f = 0 \tag{5}$$

where $B = B(t)$ is an arbitrary function of integration and D_x, D_t are the usual Hirota derivatives[13]. In order to draw out the general aspects of the procedure, we write Eq.5 as

$$F f \cdot f = 0, \qquad F(D_x, D_t) = D_x D_t + u_0 D_x^2 + D_x^4 + B. \tag{6a,b}$$

The analysis proceeds without invoking the specific form of F, and so the method applies to any nonlinear wave equation that can be reduced to a single bilinear form, Eq.6a.

We shall seek a solution of Eq.6a in the form of the Riemann theta function[14] θ_4 and set

$$f(x,t) = \theta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \mathrm{e}^{2inz}, \tag{7}$$

where $z = px + \omega t + \alpha$ and $p(\neq 0)$, ω and α are arbitrary (possibly complex) parameters. The nome $q = \exp(i\pi\tau)$ where τ is a complex parameter with $\mathrm{Im}(\tau) > 0$, i.e. $0 < |q| < 1$. Then $\theta_4(z)$ is an entire function[14], which is periodic and even in z. Substituting for f in Eq.6a, we obtain the residual bilinear form[4]

$$Ff\cdot f=\tilde{F}(0)\theta_3(2z,q^2)-q^{-\frac{1}{2}}\tilde{F}(1)\theta_2(2z,q^2)=0, \tag{8}$$

where the residual coefficients $\tilde{F}(r)$, $r=0,1$, are given by

$$\tilde{F}(r)=\sum_{n=-\infty}^{\infty}F[2(2n-r)ip,2(2n-r)i\omega]q^{n^2+(n-r)^2}, \tag{9}$$

and θ_2, θ_3 are defined by Fourier series similar to Eq.7. But as θ_2 and θ_3 are linearly independent[14], $f=\theta_4$ is a solution of the bilinear form Eq.6a if and only if

$$\tilde{F}(0)=0 \quad \text{and} \quad \tilde{F}(1)=0. \tag{10a,b}$$

If we now invoke the functional form of F, Eq.6b, together with Eq.9, the residual Eqs.10 become, respectively,

$$4p(\omega+u_0p)\theta_3''(0,q^2)+16p^4\theta_3^{(4)}(0,q^2)+B\theta_3(0,q^2)=0, \tag{11a}$$

$$4p(\omega+u_0p)\theta_2''(0,q^2)+16p^4\theta_2^{(4)}(0,q^2)+B\theta_2(0,q^2)=0. \tag{11b}$$

Solving these equations, and appealing to various theta identities, we arrive at the particularly neat and compact expression for the dispersion relation,

$$\omega(p)=-u_0p-4p^3\theta_1'''(0,q)/\theta_1'(0,q). \tag{12}$$

If we introduce the wavespeed $c=-\omega/p$ and let $p\to p/2$ (i.e. $z\to z/2$), then Eqs.4 and 7 yield a periodic solution of the KdV Eq.1,

$$u(x,t)=u_0+12\partial^2\ln\theta_4(\tfrac{1}{2}z,q)/\partial x^2, \qquad z=p(x-ct+x_0). \tag{13}$$

We obtain a real periodic solution (i.e. a physical wave) by taking $\tau=is$ $(s>0)$ to be purely imaginary [i.e. $0<q<1$] and all parameters to be real. It is now a relatively straightforward exercise to identify the periodic solution Eq.13 with the familiar KdV cnoidal wave.

The imbricate series representation for the cnoidal wave arises quite naturally from our development in terms of theta functions. The appropriate perturbation parameter for this series is the *complementary nome*, q', that is defined by $q'=\exp(i\pi\tau')$, where $\tau\tau'=-1$. With this parametrisation our solution, Eq.13, becomes

$$u(x,t)=u_0+\frac{6i\tau'}{\pi}p^2+12\frac{\partial^2}{\partial x^2}\ln\theta_2(\tfrac{1}{2}\tau'z,q'). \tag{14}$$

To proceed, we invoke the novel identity

$$\frac{d^2}{dz^2}\ln\theta_2(z,q)=-\sum_{n=-\infty}^{\infty}\operatorname{sech}^2\{i(z-n\pi\tau)\} \tag{15}$$

(with $\tau\to\tau'=i/s$, $q\to q'$) which transforms Eq.14 to

$$u(x,t) = u_0 + 3p^2 \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 \tfrac{1}{2}\left(z - \frac{2n\pi}{s} \right), \qquad z = p(x - ct + x_0). \tag{16}$$

Now, if we compare Eq.16 with the KdV solitary wave, Eq.2, then we see that the cnoidal wave can be expressed as an infinite sum of identical sech^2 solitary-wave *profiles* whose peaks are equally spaced and centred at $z = 0, \pm 2\pi/s, \pm 4\pi/s, \ldots$ Thus, Eq.16 is the desired imbricate series representation of the cnoidal wave, and is our first example of the nonlinear superposition principle that is the main interest of this article.

The temptation in much of the literature has been to regard an imbricate series as a sum of solitary waves or "solitons". However, it is important to recognise that here, and in general, we do not have a (linear) superposition of solitary-wave *solutions*, but only a superposing of solitary-wave *profiles*. This is because the speed of the periodic wave differs from that of the associated solitary wave. Thus, the solitary wave acts as a kind of template which, when replicated in a *linear* fashion at spatially regular intervals, generates a *nonlinear* periodic wave. We have discussed this very special nonlinear superposition principle and its interpretation at greater length elsewhere[4].

Before we consider further examples of imbricate series, let us stress again the general features of the bilinear-theta theory of nonlinear periodic waves, as presented above. Until we write down the residual Eqs.11, we do not require the specific form of the bilinear operator F. We can therefore apply the method to any nonlinear wave equation whose bilinear form is given by Eq.6a. In principle, then , it would appear that the bilinear-theta procedure is quite general and may be used to solve a broad class of nonlinear equations for stationary periodic waves.

3. Periodic Waves and Nonlinear Superposition

We now give some further examples of the nonlinear superposition principle – for equations which reduce to a single bilinear form, Eq.6a, the method of Section 2 applies. We must content ourselves with presenting the main results, leaving the interested reader to pursue the technical details in the references cited.

3.1 The ILW and BO Equations

The intermediate long-wave (ILW) equation[15-17] describes wave propagation in a stratified fluid of finite depth and may be written as

$$u_t + 2uu_x + \tfrac{1}{2}\lambda \int_{-\infty}^{\infty} u(x',t)\{\coth \tfrac{1}{2}\pi\lambda(x'-x) - \operatorname{sgn}(x'-x)\}dx' = 0, \tag{17}$$

where $\lambda > 0$ measures the relative depths of the (two-layer) fluid. The transformation

$$u(x,t) = u_0 + \mathrm{i}\frac{\partial}{\partial x} \ln \frac{f_+(x,t)}{f_-(x,t)}, \qquad f_\pm(x,t) = f(x \mp \mathrm{i}/\lambda, t), \tag{18}$$

yields the bilinear form of the ILW equation[18]

$$\{\mathrm{i}[D_t + (\lambda + 2u_0)D_x]\sinh(\mathrm{i}\lambda^{-1}D_x) + (D_x^2 - B)\cosh(\mathrm{i}\lambda^{-1}D_x)\}f.f = 0. \tag{19}$$

Following the method of Section 2, we obtain a periodic solution[4]

$$u(x,t) = u_0 + p\sum_{n=-\infty}^{\infty} \frac{\sin\gamma}{\cos\gamma + \cosh(z - 2n\pi s)}, \qquad z = px + wt + \alpha, \tag{20}$$

where $\gamma = p/\lambda$ and the dispersion relation is given by

$$\omega(p) = -(\lambda + 2u_0)p + 2\mathrm{i}p^2\theta_1'(2\mathrm{i}\gamma, q)/\theta_1(2\mathrm{i}\gamma, q). \tag{21}$$

We can interpret Eq.20 by noting that the solitary-wave solution of the ILW Eq.17 takes the form[15,19-21]

$$u_s(x,t) = \frac{p\sin\gamma}{\cos\gamma + \cosh(px + \omega_s t + \alpha)}, \qquad \omega_s = -\lambda p + p^2\cot\gamma. \tag{22a,b}$$

It is now apparent that the periodic solution, Eq.20, is an infinite sum of identical, equally spaced solitary-wave *profiles* and is our second example of the nonlinear superposition principle. That the superposition implied by the imbricate series Eq.20 is *nonlinear* becomes evident when we compare the dispersion relations Eqs.21 and 22b.

By taking the "deep-water" limit, $\lambda \to 0$, of the ILW Eq.17 we obtain the Benjamin-Ono (BO) equation[22,23]

$$u_t + 2uu_x + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(x',t)}{x' - x}\,dx' = 0. \tag{23}$$

In this same limit, Eq.20 yields a periodic solution of the BO equation[4],

$$u(x,t) = u_0 + 2p\sum_{n=-\infty}^{\infty} \frac{1}{1 + (z - 2n\pi s)^2}$$

which is an imbricate series of the (Lorentzian) solitary wave[22]

$$u_s(x,t) = \frac{2p}{1+z},$$

and therefore provides yet a further example of the nonlinear superposition principle.

3.2 The Burgers Equation

As an example of an equation which is *not* completely integrable, we consider the well-known Burgers equation[24,25]

$$u_t + uu_x - u_{xx} = 0. \tag{24}$$

Remarkably, Eq.24 can be linearised by the Hopf-Cole transformation[26,27]

$$u(x,t) = -2\partial\ln f(x,t)/\partial x \tag{25}$$

to the classical heat equation $f_t = f_{xx}$. But straightforward differentiation shows that $\theta_4(z|\tau)$, with $z = x/2$ and $\tau = it/\pi$ $(t > 0)$, satisfies the latter equation, whereupon Eq.25 yields

$$u(x,t) = -2\partial \ln \theta_4(x/2|it/\pi)/\partial x, \tag{26}$$

a (real) spatially periodic solution of the Burgers Eq.24.

Using a theta expansion similar to Eq.15, we can rewrite Eq.26 as[5]

$$u(x,t) = \frac{x}{t} - \frac{\pi}{t}\sum_{n=-\infty}^{\infty} \tanh\frac{\pi(x-2n\pi)}{2t}.$$

This is an imbricate series of regularly spaced Taylor shock[28] profiles superimposed on the elementary solution x/t of the Burgers Eq.24, and extends our nonlinear superposition principle to nonintegrable wave equations.

3.3 Periodic Waves in Two Dimensions

We consider the Kadomtsev-Petviashvili (KP) equation[29] (with *positive* dispersion)

$$(u_t + uu_x + u_{xxx})_x - u_{yy} = 0. \tag{27}$$

Taking as our starting point the two-soliton solution[30] of the KP Eq.27 and choosing the (complex) parameters appropriately, we deduce the solution[31]

$$u(x,y,t) = 12\partial^2 \ln[1 - e^{\gamma}\operatorname{sech}\lambda pY \cos\lambda(X + mY)]/\partial x^2, \tag{28a}$$

$$X = x - (m^2 + p^2 - \lambda^2)t, \qquad Y = y + 2mt, \tag{28b}$$

where λ, m, p are real and γ is defined by

$$e^{-2\gamma} = 1 + 3\lambda^2/p^2.$$

We observe that u is spatially periodic in x, and decays exponentially in the y-direction.

If we now introduce the function $\phi(Y)$ defined by

$$\cosh\lambda\phi = e^{-\gamma}\cosh\lambda pY,$$

then the solution, Eq.28a, may be reformulated as[31]

$$u(x,y,t) = 24\sum_{n=-\infty}^{\infty} \frac{[\phi(Y)]^2 - (X + mY - 2n\pi/\lambda)^2}{\{[\phi(Y)]^2 + (X + mY - 2n\pi/\lambda)^2\}^2}. \tag{29}$$

But the rational "lump" solitary wave of the KP Eq.27 has the form[32,33]

$$u_s(x,y,t) = 24\frac{[\tilde{\phi}(Y)]^2 - (X + mY)^2}{\{[\tilde{\phi}(Y)]^2 + (X + mY)^2\}^2},$$

$$X = x - (m^2 + p^2)t, \quad Y = y + 2mt, \quad \phi(Y) = \sqrt{p^2Y^2 + 3/p^2}.$$

Thus, we can interpret the periodic solution Eq.29 as an imbricate series of "lump" solitary-wave profiles, and we have yet one more demonstration of our nonlinear superposition principle.

4. Summary

The remarkable property – whereby a *nonlinear* spatially periodic wave can be represented by a *linear* superposition of identical, regularly spaced solitary-wave profiles – is one that is shared by a large number of nonlinear wave equations[2-10]. Collectively, these results may be interpreted as a very special *nonlinear superposition principle*[4]. The principle – which reveals the elegant and beautifully simple structure of nonlinear periodic waves as arrays of "solitary waves" – is captured, in each case, by expressing the periodic solution in the form of an *imbricate* series. A crucial feature of the nonlinear superposition principle – and the one that distinguishes it from classical linear superposition – is the recognition that an imbricate series is not, in general, a sum of solitary-wave *solutions*, but only of solitary-wave *profiles*.

A second theme of the present work is the development of these ideas by means of Hirota's bilinear transformation method. We saw that, for many nonlinear wave equations, it is sufficient to seek a periodic solution of the bilinear form in terms of a Riemann theta function. This, in turn, leads to elegant and compact expressions for the various dispersion relations which prove useful in both analytical and numerical work. However, the real virtue of the bilinear transform approach lies in the ease with which we are able to derive the imbricate series representations of the various nonlinear periodic waves. Of course, there are many other nonlinear wave equations of physical importance that cannot be reduced to a single bilinear equation of the form Eq.6a. Investigations currently in hand show that it is possible to generalise the bilinear-theta methodology – this holds out the prospect of extending the nonlinear superposition principle to a wider class of nonlinear periodic waves.

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BÄCKLUND AND DARBOUX TRANSFORMATIONS FOR THE ABLOWITZ–LADIK SPECTRAL PROBLEM

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1 Introductory Abstract

The Bäcklund and Darboux transformations have their origin in works by Bäcklund and Darboux in the late nineteenth century and are, therefore, the oldest tool that one can use in exploring nonlinear integrable systems. Much more recently new sophisticated and powerful methods have been developed, in particular the Inverse Spectral (Scattering) Transform and the dressing method. However, in our opinion, the Bäcklund and Darboux transformations remain the simplest way for getting the soliton solutions. Moreover, because, under appropriate circumstances, a reiterated application of the Bäcklund or Darboux transformations generate a sequence of solutions by a purely algebraic superposition principle they can be used for studying the interaction properties of the solitons and, more generally, for adding solitons to a generic background.

We are considering in this paper the Bäcklund and Darboux transformations for the Ablowitz–Ladik spectral problem which can be considered the discretized version of the Zakharov–Shabat spectral problem and which has been associated to the so called discrete nonlinear Schrödinger equation [1,2]. Our work is motivated by the renewed interest in this equation for possible physical application and by the fact the, very recently, we have shown that an integrable equation with nonanalytic dispersion relation can be associated to the Ablowitz–Ladik spectral problem[3].

We show that the so called elementary Bäcklund transformations can be integrated once and, then, explicitly solved in terms of terminating continued fractions.

Darboux transformations for the Ablowitz–Ladik spectral problem have been already considered in the beautiful book on the Darboux transformations

by Matveev and Salle [4]. We show that also the Darboux form of the elementary Bäcklund transformations can be constructed. Unexpectedly we find that it is necessary to consider at the same time a couple of different equivalent Ablowitz–Ladik spectral problems. The fact that one must use different discrete versions of the same continuous spectral equation denotes that discrete integrable evolution equations posses richer symmetry properties than in the continuous case.

2 The Full Bäcklund Transformation

Let us consider the Ablowitz–Ladik spectral problem in the version [3]

$$\psi(n+1,k) = \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix} \psi(n,k) + Q(n+1)\psi(n+1,k). \tag{1}$$

The simplest way to generate a Bäcklund transformation is to use the gauge invariance of the spectral equation and to search for a convenient Bäcklund gauge operator B that transforms the old eigenfunction ψ to a new eigenfunction

$$\psi'(n,k) = B(n,k,Q',Q)\psi(n,k). \tag{2}$$

that satisfies the same equation

$$\psi'(n+1,k) = \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix} \psi'(n,k) + Q'(n+1)\psi'(n+1,k) \tag{3}$$

with a new potential Q'. The B results to be determined by the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} B(n+1) = B(n) \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} - B(n) \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} Q(n+1) + \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} Q'(n+1)B(n+1). \tag{4}$$

The simplest non trivial B has matrix elements which are polynomials of first order in k and $1/k$ that are determined by inserting B into (4) and by looking to the coefficients of the powers of k. One gets $\gamma(n)$

$$B(n) = \begin{pmatrix} \alpha^{(0)}(n) + \alpha k^{-1} & -\delta q'(n+1) - \alpha q(n) k^{-1} \\ -\alpha r'(n+1) - \delta r(n) k & \delta^{(0)}(n) + \delta k \end{pmatrix} \tag{5}$$

where α and δ are arbitrary constants, and

$$\Delta\alpha^{(0)}(n) = \alpha q(n) r(n+1) - \alpha r'(n+2) q'(n+1) \tag{6}$$

$$\Delta\delta^{(0)}(n) = \delta r(n) q(n+1) - \delta q'(n+2) r'(n+1) \tag{7}$$

$$\alpha^{(0)}(n) q(n+1) - \delta^{(0)}(n+1) q'(n+1) + \alpha q(n) - \delta q'(n+2) = 0 \tag{8}$$

$$\alpha^{(0)}(n+1) r'(n+1) - \delta^{(0)}(n) r(n+1) + \alpha r'(n+2) - \delta r(n) = 0 \tag{9}$$

By inserting (6) and (7) into (8) and (9) we obtain $\alpha^{(0)}(n+1)$ and $\delta^{(0)}(n+1)$

$$\alpha^{(0)}(n+1)\,[q(n+1)r(n+1) - q'(n+1)r'(n+1)] = \tag{10}$$
$$-\alpha q(n)r(n+1)\,[1 - q(n+1)r(n+1)] + \alpha q'(n+1)r'(n+2)\,[1 - q(n+1)r(n+1)]$$
$$+\delta q'(n+2)r(n+1)\,[1 - q'(n+1)r'(n+1)] - \delta q'(n+1)r(n)\,[1 - q(n+1)r(n+1)]$$
$$\delta^{(0)}(n+1)\,[q(n+1)r(n+1) - q'(n+1)r'(n+1)] = \tag{11}$$
$$-\alpha q(n)r'(n+1)\,[1 - q(n+1)r(n+1)] + \alpha q(n+1)r'(n+2)\,[1 - q'(n+1)r'(n+1)]$$
$$+\delta q'(n+2)r'(n+1)\,[1 - q(n+1)r(n+1)] - \delta q(n+1)r(n)\,[1 - q(n+1)r(n+1)] \tag{12}$$

By inserting the values of $\alpha^{(0)}(n)$ and $\delta^{(0)}(n)$ of the first two equations into the last two one derives the Bäcklund transformation from Q to Q'.

If one starts from $q(n) \equiv r(n) \equiv 0$ one gets the one soliton solution $Q'(n)$

$$q'(n+2)q'(n) - q'(n+1)q'(n+1) = r'(n+1)q'(n)q'(n+1)q'(n+2) \tag{13}$$
$$r'(n+2)r'(n) - r'(n+1)r'(n+1) = q'(n+1)r'(n)r'(n+1)r'(n+2). \tag{14}$$

In the reduced case $r = \sigma\overline{q}$ ($\sigma = \pm$) if we introduce (dropping the $'$) in (13), (14) the representation

$$q(n) = \frac{1}{\tau(n)}e^{i\phi(n)}, \qquad r(n) = \sigma\frac{1}{\tau(n)}e^{-i\phi(n)} \tag{15}$$

we have

$$2\phi(n+1) - \phi(n) - \phi(n+2) = 0 \tag{16}$$
$$\tau(n+1)\tau(n+1) - \tau(n)\tau(n+2) = \sigma. \tag{17}$$

Then

$$\phi(n) = \phi_0 n + \theta_0 \tag{18}$$
$$\tau(n) = \frac{1}{\sinh b}\cosh(b(n-n_0)), \qquad b > 0, \qquad \text{for } \sigma = -1 \tag{19}$$
$$\tau(n) = \frac{1}{\sinh b}\sinh(b(n-n_0)), \qquad b > 0, \qquad \text{for } \sigma = 1. \tag{20}$$

3 The Elementary Bäcklund Transformation of Second Kind

Let us choose in the full Bäcklund transformation described in section 2 $\alpha = 0$. Then, from (6) we have that $\alpha^{(0)}(n)$ is a constant, say $\alpha^{(0)} = 1$, and we get for the Bäcklund gauge

$$B_{II}(Q', Q, \delta) = \begin{pmatrix} 1 & -\delta q'(n+1) \\ -k\delta r(n) & \delta_{II}^{(0)}(n) + k\delta \end{pmatrix} \tag{21}$$

which we call elementary Bäcklund gauge of second kind. That of first kind is defined in section 5. From (8) and (9) we get

$$\delta_{II}^{(0)}(n)q'(n) = q(n) - \delta q'(n+1) \tag{22}$$

$$\delta_{II}^{(0)}(n)r(n+1) = r'(n+1) - \delta r(n) \tag{23}$$

and, consequently,

$$\delta\left[q'(n+1)r(n+1) - q'(n)r(n)\right] + r'(n+1)q'(n) - r(n+1)q(n) = 0. \tag{24}$$

By inserting the $\delta_{II}^{(0)}(n)$ obtained from (22) and (23) into (7) we have finally the elementary Bäcklund transformation of second kind

$$p'\frac{r'_+ - \delta r}{r_+} = p\frac{r' - \delta r_-}{r} \tag{25}$$

$$p'(\delta q'_+ r - 1) = p(\delta q' r_- - 1) \tag{26}$$

where we used, for convenience, the following notation

$$A = A(n), \qquad A_+ = A(n+1), \qquad A_- = A(n-1), \tag{27}$$

$$p = 1 - qr, \qquad p' = 1 - q'r'. \tag{28}$$

3.1 Solving the Elementary Bäcklund Transformation of Second Kind

The Bäcklund transformation of second kind, by introducing

$$\mathcal{R}' = \frac{r' - \delta r_-}{r}, \qquad \mathcal{Q}' = q'\delta r_- - 1, \tag{29}$$

can be rewritten as

$$p'\mathcal{R}'_+ = p\mathcal{R}', \qquad p'\mathcal{Q}'_+ = p\mathcal{Q}'. \tag{30}$$

From this

$$\frac{\mathcal{R}'_+}{\mathcal{Q}'_+} = \frac{\mathcal{R}'}{\mathcal{Q}'} = c' \tag{31}$$

with c' a constant independent from n. By using this one step explicit integration the Bäcklund transformation can be written as

$$q'_+ = \frac{1}{\delta}\frac{q + c'q'}{1 + c'rq'} \tag{32}$$

$$r' = \delta r_- + c'\delta r r_- q' - c'r. \tag{33}$$

Note that the Bäcklund transformation from (q, r) to q' can be rewritten as

$$q'_+ = \frac{1}{\delta r} - \frac{\frac{p}{c'\delta r^2}}{\frac{1}{c'r} + q'} \tag{34}$$

and, therefore, it can be explicitly solved in terms of a terminating continued fraction

$$q'(n+1) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_m}{b_m + q'(n+1-m)}}}} \tag{35}$$

where

$$b_0 = \frac{1}{\delta r(n)} \tag{36}$$

$$a_\ell = -\frac{p(n+1-\ell)}{c'\delta r^2(n+1-\ell)} \tag{37}$$

$$b_\ell = \frac{1}{c'r(n+1-\ell)} + \frac{1}{\delta r(n-\ell)} \tag{38}$$

The constant of integration of the Bäcklund transformation is fixed by choosing $q'(n+1-m)$ at some $n - m$.

4 The Elementary Bäcklund Transformation of Second Kind à la Darboux

Let us consider the alternative form of the Ablowitz–Ladik spectral equation

$$\begin{aligned} \phi_{1+} &= \phi_1 + q\phi_2 \\ k_0\phi_{2+} &= \phi_2 + r\phi_1 \end{aligned}, \quad \text{for } k_0 = \frac{c'}{\delta} \tag{39}$$

and note that

$$q' = \frac{1}{c'}\frac{\phi_1}{\phi_2} \tag{40}$$

solves (32)

$$q'_+ = \frac{1}{\delta}\frac{q + c'q'}{1 + c'rq'}. \tag{41}$$

Therefore, recalling (31), the elementary Bäcklund transformation of second kind can be written à la Darboux as follows

$$q' = \frac{1}{c'}\frac{\phi_1}{\phi_2} \tag{42}$$

$$r' = \delta r_- + \delta r r_- \frac{\phi_1}{\phi_2} - c' r. \tag{43}$$

This transformation is looking as a Darboux transformation with, however, the relevant difference that the transformed (q', r') are expressed not, as usual in term of (q, r) and the eigenfunction ψ of the originally considered spectral equation (1) but in term of the eigenfunction ϕ of the alternative form (39). The corresponding transformation of the eigenfunctions of the original spectral problem, thanks to (21), results to be

$$\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\delta}{c'}\frac{\phi_{1+}}{\phi_{2+}} \\ -k\delta r & k\delta - c' + \delta r \frac{\phi_{1+}}{\phi_{2+}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{44}$$

5 The Elementary Bäcklund Transformation of First Kind

Let us choose in the full Bäcklund transformation described in section 2 $\delta = 0$. Then, from (7) we have that $\delta^{(0)}(n)$ is a constant, say $\delta^{(0)} = 1$ and we note the elementary Bäcklund of first kind by

$$B_I(\widehat{Q}, Q', \alpha) = \begin{pmatrix} \alpha_I^{(0)}(n) + k^{-1}\alpha & -k^{-1}\alpha q'(n) \\ -\alpha \widehat{r}(n+1) & 1 \end{pmatrix}. \tag{45}$$

From (8) and (9) we get

$$\alpha_I^{(0)}(n)\widehat{r}(n) = r'(n) - \alpha \widehat{r}(n+1) \tag{46}$$

$$\alpha_I^{(0)}(n) q'(n+1) = \widehat{q}(n+1) - \alpha q'(n) \tag{47}$$

and, consequently,

$$\alpha\left[\widehat{r}(n+1)q'(n+1) - \widehat{r}(n)q'(n)\right] + \widehat{q}(n+1)\widehat{r}(n) - q'(n+1)r'(n) = 0. \tag{48}$$

By inserting the $\alpha_I^{(0)}(n)$ obtained from (46) and (47) into (6) we have

$$\left[\widehat{p}(n) - p'(n)\right] = \alpha \widehat{r}(n+1)q'(n)\widehat{p}(n) - \alpha \widehat{r}(n)q'(n-1)p'(n). \tag{49}$$

The solution of these two first order difference equations furnishes the Bäcklund transform $\widehat{Q}$ of Q'.

The Bäcklund transformation can be rewritten as

$$p''\frac{q''_+ - \alpha q}{q_+} = p\frac{q'' - \alpha q_-}{q} \tag{50}$$

$$p''(\alpha r''_+ q - 1) = p(\alpha r'' q_- - 1) \tag{51}$$

Note that the equations for the Bäcklund transformation of first kind can be obtained from those of the Bäcklund of the second kind by the exchanges $q \leftrightarrow r$, $\delta \leftrightarrow \alpha$.

6 The Superposition Formula for Elementary Bäcklund Transformations

We have

$$B_I(\widehat{Q}, Q', \alpha)B_{II}(Q', Q, \delta) = \tag{52}$$
$$\begin{pmatrix} \alpha_I^{(0)}(\widehat{Q}, Q')(n)+\delta\alpha q'(n)r(n)+\alpha k^{-1} & \\ -\alpha\widehat{r}(n+1)-k\delta r(n) & \end{pmatrix}$$
$$\begin{vmatrix} -\delta\left(\alpha_I^{(0)}(\widehat{Q}, Q')(n)q'(n+1)+\alpha q'(n)\right) & -\alpha k^{-1}\left(\delta q'(n+1)+\delta_{II}^{(0)}(Q', Q)(n)q'(n)\right) \\ \delta_{II}^{(0)}(Q', Q)(n)+\delta\alpha q'(n+1)\widehat{r}(n+1)+k\delta & \end{vmatrix}.$$

Comparing with the full Bäcklund gauge transformation we get some identities, some equations previously already obtained giving $\alpha_I^{(0)}$ and $\delta_{II}^{(1)}$, and the relations

$$\alpha^{(0)}(\widehat{Q}, Q)(n) = \alpha_I^{(0)}(\widehat{Q}, Q')(n) + \delta\alpha q'(n)r(n) \tag{53}$$

$$\delta^{(0)}(\widehat{Q}, Q)(n) = \delta_{II}^{(0)}(Q', Q)(n) + \delta\alpha q'(n+1)\widehat{r}(n+1). \tag{54}$$

Analogously, if we impose the commutativity of the elementary Bäcklund transformations, we have

$$B_{II}(\widehat{Q}, Q'', \delta)B_I(Q'', Q, \alpha) = \tag{55}$$
$$\begin{pmatrix} \alpha_I^{(0)}(Q'', Q)(n)+\delta\alpha\widehat{q}(n+1)r''(n+1)+\alpha k^{-1} & \\ -\alpha\left(\delta r''(n)+\delta_{II}^{(0)}(\widehat{Q}, Q'')(n)r''(n+1)\right) & -k\delta\left(\alpha_I^{(0)}(Q'', Q)(n)r''(n)+\alpha r''(n+1)\right) \end{pmatrix}$$
$$\begin{vmatrix} -\delta\widehat{q}(n+1)-\alpha k^{-1}q(n) & \\ \delta_{II}^{(0)}(\widehat{Q}, Q'')(n)+\delta\alpha r''(n)q(n)+k\delta & \end{vmatrix}.$$

Comparing with the full Bäcklund gauge transformation we get some identities, some equations previously already obtained giving $\alpha_I^{(0)}$ and $\delta_{II}^{(1)}$, and the relations

$$\alpha^{(0)}(\widehat{Q},Q)(n) = \alpha_I^{(0)}(Q'',Q)(n) + \delta\alpha\widehat{q}(n+1)r''(n+1) \tag{56}$$

$$\delta^{(0)}(\widehat{Q},Q)(n) = \delta_{II}^{(0)}(\widehat{Q},Q'')(n) + \delta\alpha r''(n)q(n). \tag{57}$$

Comparing the relations obtained in the two cases we derive two equivalent superposition formulae furnishing $\widehat{q}$

$$\widehat{q}_+q_+(1-\delta\alpha r''_+q'_+) = \alpha(q'q_+ - qq'_+) + q'_+(q''_+ - \delta\alpha q'rq_+) \tag{58}$$

$$\widehat{q}_+r''(1-\delta\alpha r''_+q'_+) = \alpha(q'r'' - q'_+r''_+) + q'_+(r - \delta\alpha q'rr''). \tag{59}$$

and two other equivalent superposition formulae furnishing $\widehat{r}$

$$\widehat{r}_+r_+(1-\delta\alpha q'_+r''_+) = \delta(r_+r'' - rr''_+) + r''_+(r'_+ - \delta\alpha r''qr_+) \tag{60}$$

$$\widehat{r}_+q'(1-\delta\alpha q'_+r''_+) = \delta(q'r'' - q'_+r''_+) + r''_+(q - \delta\alpha r''qq'). \tag{61}$$

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LAX PAIRS AND EXACT SOLUTIONS FROM PAINLEVÉ ANALYSIS

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For many completely integrable partial differential equations the singular manifold method of Weiss allows the recovery of the Lax pair and Darboux transformation (DT), and so also the Bäcklund transformation, from a truncated Painlevé expansion. Here we present, using as an example the Broer-Kaup equation, a natural extension of this singular manifold method whereby the DT is now identified with an infinite Painlevé expansion for a certain choice of the arbitrary coefficients. This approach involves only one singular manifold, and leads us to a new and more consistent definition of "singular manifold equation." The summation of an infinite Painlevé expansion is achieved by seeking a truncated expansion in a new Riccati variable Z. This approach also allows us to place within the context of Painlevé analysis a larger class of exact solutions than was possible hitherto.

1 Introduction: the singular manifold method

The Weiss-Tabor-Carnevale (WTC) Painlevé test [1] is now well-known as a test of partial differential equations (PDEs) for complete integrability. This test involves seeking a solution of the PDE, for example an equation in $W(x,t)$,

$$E[W] = 0, \tag{1}$$

as an expansion of the form

$$W = \varphi^p \sum_{j=0}^{\infty} W_j^{\varphi} \varphi^j, \tag{2}$$

where $\varphi(x,t) = 0$ defines an arbitrary non-characteristic [2] movable singular manifold. Here the superscript φ on W_j^{φ} denotes the expansion variable. For each family (choice of leading order behaviour) there is a set of indices $\{j_1, \ldots, j_N\}$ which give the values of j for which the coefficient $W_j^{\varphi}(x,t)$ in (2) remains arbitrary. For any family which represents either the general or a

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particular solution the Painlevé test requires: [1,3] p to be integer; indices to be distinct integers; all compatibility conditions for each index to be satisfied.

However, Painlevé analysis can also provide a constructive proof of integrability: it has been shown [1,4] that from a truncated expansion W_T $(p < 0)$,

$$W_T = \varphi^p \sum_{j=0}^{-p} W_j^{\varphi} \varphi^j, \tag{3}$$

it is possible to obtain the Lax pairs of many integrable PDEs. This singular manifold method of Weiss, [4] later phrased algorithmically by Musette and Conte, [5] can also give the Darboux transformation (DT) — and so together with the Lax pair the Bäcklund transformation (BT) — for the PDE.

The singular manifold method involves seeking a constraint on φ, the so-called "singular manifold equation" (SME), for the constant level truncation (3) to exist. We later give a new definition of "singular manifold equation." For now we simply remark that (3) represents the trivial (because finite) summation of (2) for certain choices of the arbitrary coefficients, and with φ subject to the SME. In particular, all arbitrary coefficients of positive powers of φ in (2) have been set equal to zero.

2 A new expansion variable

2.1 The invariant Painlevé analysis

Conte [6] introduced an expansion variable χ given in terms of the WTC φ as

$$\chi = \left(\frac{\varphi_x}{\varphi} - \frac{\varphi_{xx}}{2\varphi_x} \right)^{-1}. \tag{4}$$

This variable χ satisfies the system of Riccati equations

$$\chi_x = 1 + \frac{1}{2} S \chi^2, \qquad \chi_t = -C + C_x \chi - \frac{1}{2}(C_{xx} + CS)\chi^2, \tag{5}$$

where S, the Schwarzian derivative of φ, and C are given by

$$S = \{\varphi; x\} = \left(\frac{\varphi_{xx}}{\varphi_x} \right)_x - \frac{1}{2} \left(\frac{\varphi_{xx}}{\varphi_x} \right)^2, \qquad C = -\frac{\varphi_t}{\varphi_x}. \tag{6}$$

The cross-derivative condition (identically satisfied in terms of φ) on (5) is

$$S_t + C_{xxx} + 2SC_x + S_x C = 0. \tag{7}$$

The resulting expansion in χ is a resummation [6] of the WTC expansion (2):

$$W = \chi^p \sum_{j=0}^{\infty} W_j^{\chi} \chi^j. \tag{8}$$

2.2 A new expansion variable

The linearization of the Riccati system (5) via $\chi^{-1} = \psi_x/\psi$ is

$$\psi_{xx} = -\frac{1}{2} S\psi, \qquad \psi_t = -C\psi_x + \frac{1}{2} C_x \psi. \tag{9}$$

Thus a natural way to construct a more general Riccati system is to consider the nonlinearization of the most general scalar linear system, given by

$$\eta_{xx} = A\eta_x + B\eta, \qquad \eta_t = -C\eta_x + \left(\int^x D \right) \eta. \tag{10}$$

The corresponding Riccati system, [7] obtained via $Z^{-1} = \eta_x/\eta$, is

$$Z_x = 1 - AZ - BZ^2, \tag{11}$$

$$Z_t = -C + (AC + C_x)Z - (D - BC)Z^2, \tag{12}$$

which has as cross-derivative condition

$$(Z_x)_t - (Z_t)_x \equiv x_1 Z + x_2 Z^2 = 0, \tag{13}$$

where

$$-x_1 \equiv A_t + (AC)_x + C_{xx} - 2D = 0, \tag{14}$$

$$-x_2 \equiv B_t - D_x + 2BC_x + B_x C + AD = 0. \tag{15}$$

It might be objected that it is possible to set $A = 0$ in (11), (12) by using the simple gauge transformation (which we note gives Z in terms of φ and A)

$$Z^{-1} = \chi^{-1} + \frac{1}{2} A, \tag{16}$$

which then gives a Riccati system of the same form as (5) with

$$-\frac{1}{2} S = B + \frac{1}{4} A^2 - \frac{1}{2} A_x, \qquad C = C, \tag{17}$$

and the same linearization to (9). However this does not mean that consideration of the Riccati system (11), (12) is useless. If the truncation is such that the

resulting expansion of $E[W]$ includes positive powers of Z, i.e. it is a higher order truncation, [8] then for $A \neq 0$ there corresponds to such a truncation in Z no *finite* series in χ. Further explanation can be found in [7].

The Riccati system (11), (12) has also the linearization via $Z^{-1} = \psi_2/\psi_1$ onto a matrix spectral problem. This lies behind the "two-singular manifold method" developed in [9,10,11]. Whilst there is clearly a close connection between the singularity structure and the DT (or the Hirota form [12]) of the equations considered, the point of the present article is to show how the DT can be identified with a Painlevé expansion involving only one singular manifold.

2.3 Exact solutions

When seeking a higher order truncation in the case of constant coefficients $A = a$, $B = b$, $C = c$, and $D = 0$, we find that we are able to identify positive and negative powers of Z as

$$Z^{-1} = \frac{k}{2}\left(\tau - \sigma + \frac{a}{k}\right), \qquad Z = \frac{k}{2b}\left(\tau + \sigma - \frac{a}{k}\right), \tag{18}$$

where

$$\sigma = \frac{\sqrt{\mu^2 - 1}}{\cosh(k(x - ct + \gamma)) + \mu}, \qquad \tau = \frac{\sinh(k(x - ct + \gamma))}{\cosh(k(x - ct + \gamma)) + \mu}, \tag{19}$$

$b = (k^2 - a^2)/4$, $\mu = ia/\sqrt{k^2 - a^2}$, and γ is an arbitrary constant. In [13] the recovery of solutions polynomial in σ and τ was considered: we have now therefore extended our previous work [8] (for $a = \mu = 0$) and placed this entire class of solutions within the context of Painlevé analysis (a corresponding WTC expansion can be given). For details see [7]. The advantage of our approach is that we make an expansion in a single variable Z.

3 Example: Broer-Kaup equation

We now consider the recovery of the Lax pair and DT (and so the BT) for the scalar Broer-Kaup (BK) equation from Painlevé analysis. This problem has been considered previously in [11] and [12]. Our interest here is in the representation of the DT as a Painlevé expansion involving only one singular manifold, and in the corresponding SME, which we give for the first time.

The BK equation [14,15] is given by

$$E[W] \equiv (\kappa^2 W_{xxx} - 2W_x^3)_x - 4W_x W_{xt} - 2W_t W_{xx} - W_{tt} = 0, \tag{20}$$

and is the result of the elimination of V from the classical Boussinesq system

$$\left.\begin{aligned} E_1[U,V] &\equiv U_t + \left(V + \tfrac{1}{2}U^2\right)_x = 0 \\ E_2[U,V] &\equiv V_t + \left(\kappa^2 U_{xx} + UV\right)_x = 0 \end{aligned}\right\} \tag{21}$$

via

$$U = 2W_x, \qquad V = -2W_t - 2W_x^2. \tag{22}$$

Equation (20) has two families $W = \kappa \log \chi + \cdots$ (here κ may take either sign) with indices $\{-1, 0, 3, 4\}$. Thus the truncation (for either family) is

$$W_T = \kappa \log Z + w. \tag{23}$$

Substitution into (20) gives

$$E[W_T] \equiv Z^{-4} \sum_{j=0}^{8} E_j^Z Z^j = 0, \tag{24}$$

where the first and last coefficients E_0^Z, E_8^Z are identically zero since we have already set the leading order coefficient in (23).

The solution of the determining equations $E_j^Z = 0$, in conjunction with the cross-derivative conditions (14), (15), is easily obtained as

$$A = \frac{1}{\kappa}(w_x - \lambda), \tag{25}$$

$$B = \frac{1}{2\kappa^2}(\kappa w_{xx} + w_x^2 + w_t), \tag{26}$$

$$C = w_x + \lambda, \tag{27}$$

$$D = \frac{1}{2\kappa}(\kappa w_{xx} + w_x^2 + w_t)_x. \tag{28}$$

Here λ is an arbitrary constant of integration and (15) becomes

$$-x_2 \equiv -\frac{1}{2\kappa^2} E[w] = 0, \tag{29}$$

which tells us that w is a second solution of BK. These results are equivalent to those given in [11]: (11), (12) form the Riccati pseudopotential and (23) the DT (under $Z^{-1} = \psi_2/\psi_1$). However the use of the Riccati system (11), (12) has removed the gauge freedom which complicated the results obtained in [11].

We now rewrite the DT (23) as a Painlevé expansion in χ using (16):

$$\begin{aligned} W_T &= \kappa \log Z + w = \kappa \log \chi - \kappa \log \left(1 + \frac{1}{2} A\chi\right) + w \\ &= \kappa \log \chi + w - \frac{\kappa}{2} A\chi + \frac{\kappa}{8} A^2 \chi^2 + \cdots, \end{aligned} \tag{30}$$

where A is as in (25). For this χ, we obtain S and C from (17) and (25)—(27):

$$S = -\frac{1}{\kappa^2}\left(w_t + \frac{3}{2} w_x^2 - \lambda w_x + \frac{1}{2}\lambda^2\right), \qquad C = w_x + \lambda. \tag{31}$$

These values of S and C are a parameterization of

$$C_t + 3CC_x - 4\lambda C_x + \kappa^2 S_x = 0. \tag{32}$$

Thus we see that the DT corresponds to the summation of an infinite Painlevé expansion (in χ or φ) for certain choices of the arbitrary coefficients. This expansion is subject also to the constraint (32). This prompts us to introduce the following more consistent definition of "singular manifold equation:"

> *The singular manifold equation of a PDE is the constraint on the WTC expansion variable φ such that, for a certain choice of arbitrary coefficients, the Painlevé expansion corresponds to the Darboux transformation of the PDE.*

We therefore claim that (32) is the correct SME for BK. This can be obtained from the Painlevé expansion corresponding to either of the two families, but *not* if we insist on truncation at constant level. Our analysis also proves the convergence of the Painlevé expansion for BK — for this choice of arbitrary coefficients and S, C as in (31) — for χ in the cut plane such that

$$|\chi| < 2\left|\kappa(w_x - \lambda)^{-1}\right|. \tag{33}$$

There is a further argument which may be used to demonstrate that (32) is the correct SME for BK. Substitution of (23) into (22) yields corresponding expansions for the system (21). Rewriting S and C in terms of two functions u and v (related to w in the same way as U, V are related to W in (22)) gives

$$S = -\frac{1}{8\kappa^2}\left[(u - 2\lambda)^2 - 4v\right], \qquad C = \frac{1}{2}(u + 2\lambda). \tag{34}$$

So the expansions for U and V are subject to the SME (32) and the cross-derivative condition (7). For S and C as in (34), the SME (32) gives

$$C_t + 3CC_x - 4\lambda C_x + \kappa^2 S_x = \frac{1}{2} E_1[u, v], \tag{35}$$

and the cross-derivative condition (7) gives

$$S_t + C_{xxx} + 2SC_x + S_xC = \frac{1}{2\kappa^2}E_2[u,v] - \frac{1}{4\kappa^2}(u-2\lambda)E_1[u,v], \qquad (36)$$

and so these two equations are together equivalent to another copy of the classical Boussinesq system, just as for KdV the SME and the cross-derivative condition together provide another copy of KdV. [4,5] Equation (36) tells us that (34) are the values of S and C such that (9) form the scalar Lax pair [15] for the classical Boussinesq system.

If we now consider the SME usually given for BK, i.e.

$$C_t + CC_x + 2\kappa C_{xx} - \kappa^2 S_x = 0, \qquad (37)$$

then substitution of these values of S and C gives

$$C_t + CC_x + 2\kappa C_{xx} - \kappa^2 S_x \equiv \frac{1}{2}E_1[u,v] - (v - \kappa u_x)_x = 0. \qquad (38)$$

So (37) does not with (7) provide us with another copy of the classical Boussinesq system, but instead forces us onto the well-known reduction $v = \kappa u_x$ to Burgers' equation. This incompatibility of (38) and (36), except in this reduction, means that (37) is not the correct SME for BK.

The reason given in [16] for the appearance of (37) as the constraint for a constant level truncation to exist — i.e. that BK admits a factorization as a differential operator acting on Burgers' equation, and (37) is the condition which occurs for such a truncation to exist for this equation — remains valid. However (37) should no longer be considered to be the SME for BK. The correct SME for BK is (32), which cannot be obtained from the standard truncation.

It is now a simple matter to derive the soliton solution [15] W_T of BK by assuming in (25)—(28) that A, B, C, and D take constant values as described in Section 2.3. This solution has previously been obtained [12] using two singular manifolds. The advantage of our approach, which identifies this solution with a WTC expansion and thus place it firmly within the framework of Painlevé analysis, is that we use a single expansion variable Z.

We now make some brief remarks on obtaining the DT. We note that above we did not need to derive the DT for BK in advance (as noted in [11]). The same is true of all examples thus far considered. [7]

The derivation of the DT for modified Korteweg-de Vries (mKdV) and sine-Gordon (s-G) given in [10] relies on the fact that the two functions ψ_1, ψ_2 appearing in the quotient $Z^{-1} = \psi_2/\psi_1$ each satisfy a pair of linear equations such that the pair for ψ_2 is obtained from the pair for ψ_1 via $\{\psi_1, u\} \rightarrow$

$\{\psi_2, -u\}$ (u being the field of mKdV or s-G). The equations satisfied by ψ_1 are (9) with S and C given by (17). Those satisfied by ψ_2 are

$$\psi_{2,xx} = \frac{B_x}{B}\psi_{2,x} + \left(B - \frac{1}{2}A\frac{B_x}{B} + \frac{1}{4}A^2 + \frac{1}{2}A_x\right)\psi_2, \tag{39}$$

$$\psi_{2,t} = \left(\frac{D}{B} - C\right)\psi_{2,x} - \frac{1}{2}\left(A\frac{D}{B} + C_x\right)\psi_2, \tag{40}$$

which have cross-derivative condition satisfied modulo (14) and (15). Thus asking that the mapping between these pairs of equations be $\{\psi_1, u\} \to \{\psi_2, -u\}$ places in advance some restrictions on the coefficients A, B, C, D of our Riccati system (e.g. $B_x = 0$). Lastly, we note that this mapping for mKdV and s-G is a special case of $\{\psi_1, A, C\} \to \{B^{-\frac{1}{2}}\psi_2, (B_x/B) - A, C - (D/B)\}$.

4 Final remarks

We have introduced a new and more consistent definition of SME, as the constraint on φ such that for certain choices of arbitrary data the Painlevé expansion corresponds to the DT. This approach, whereby we use only one singular manifold but no longer expect the DT to correspond to a WTC expansion truncated at constant level, provides a natural extension of the Weiss singular manifold method. We have seen that for BK the SME can be obtained from either one of the two families: we do not need to use these two families in conjunction. This new understanding arose from the introduction of a new Riccati variable which in itself greatly simplifies the process of obtaining the Lax pair and DT from Painlevé analysis. Further examples can be found in [7].

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ON GENERALIZED ERMAKOV SYSTEMS

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Ermakov systems of arbitrary order and dimension are constructed. These inherit an underlying linear structure based on that recently established for the classical Ermakov system.

1 Introduction

In modern applications, Ermakov systems arise most notably in nonlinear optics [1-5] and nonlinear elasticity [6,7]. They consist of a coupled pair of nonlinear ordinary differential equations which originated in work of Ermakov [8] in 1880. Since that time there has been an extensive literature on the subject. The main theoretical interest in the classical Ermakov system resides in the fact that it admits, generically, an integral of motion known as the Lewis-Ray-Reid invariant [9,10]. In a key development, it was shown in Athorne *et al* [11] that, in fact, the classical Ermakov system may be linearized by a procedure in which this Lewis-Ray-Reid invariant plays a pivotal role.

Here, natural extensions of the classical Ermakov system are constructed, both with regard to order and dimension. It is remarked that N-component Ermakov systems have been earlier introduced in Rogers and Schief [12]. These can be iteratively reduced to $N-2$ linear equations augmented by a canonical two-component Ermakov system.

2 The Classical Ermakov System

The classical Ermakov system as extended by Ray and Reid [10] adopts the form

$$\ddot{u}+\omega(t)u=\frac{\bar{f}(v/u)}{u^2v},\quad \ddot{v}+\omega(t)v=\frac{\bar{g}(u/v)}{v^2u} \tag{1}$$

where $\bar{f},\bar{g}$ and ω are arbitrary functions of their indicated arguments. The associated first integral

$$I=\frac{w^2(u,v)}{2}+\int^{u/v}\bar{g}(\lambda)\,d\lambda+\int^{v/u}\bar{f}(\mu)\,d\mu \tag{2}$$

is termed the Lewis-Ray-Reid invariant of the Ermakov system (1). Here, $w(u,v) = u\dot{v} - v\dot{u}$ denotes the Wronskian of u and v with respect to t. The relation (2) implies that w is a function of u/v only, whence (1) assumes the equivalent form

$$\ddot{u} + \omega(t)u = \frac{f(v/u)}{u^3}w^2(u,v), \quad \ddot{v} + \omega(t)v = \frac{g(u/v)}{v^3}w^2(u,v) \tag{3}$$

to be adopted in what follows.

On introduction of the change of dependent and independent variables

$$u = a(z)\phi, \quad v = b(z)\phi, \quad z = \psi/\phi \tag{4}$$

where ϕ and ψ are linearly independent solutions of the linear base equation

$$\ddot{\phi} + \omega(t)\phi = 0, \tag{5}$$

the Ermakov system (3) reduces to the autonomous form

$$a'' = \frac{f(b/a)}{a^3}w^2(a,b), \quad b'' = \frac{g(a/b)}{b^3}w^2(a,b) \tag{6}$$

with the Wronskian $w(a,b) = ab' - ba'$ and the derivative $' = d/dz$. Subsequent application of the involutory transformation [12]

$$\tilde{a} = z/b, \quad \tilde{b} = 1/b, \quad \tilde{z} = a/b \tag{7}$$

may be shown to decouple the canonical system (6) into two equations which are not only linear but are also, remarkably, of the same form, viz

$$\tilde{a}_{\tilde{z}\tilde{z}} - (\ln \mathcal{W})_{\tilde{z}}\tilde{a}_{\tilde{z}} + g(\tilde{z})\tilde{a} = 0, \quad \tilde{b}_{\tilde{z}\tilde{z}} - (\ln \mathcal{W})_{\tilde{z}}\tilde{b}_{\tilde{z}} + g(\tilde{z})\tilde{b} = 0, \tag{8}$$

where

$$\mathcal{W} = \int^{\tilde{z}} [s^{-3}f(s^{-1}) - sg(s)]\, ds. \tag{9}$$

This result was exploited in Rogers and Schief [12] to construct a Darboux transformation linking sequences of Ermakov systems. In particular, the *decoupled* Ermakov-Pinney system

$$\ddot{u} + \omega(t)u + \frac{\nu}{u^3} = 0, \quad \ddot{v} + \omega(t)v = 0 \tag{10}$$

which arises in nonlinear elasticity may be linked via a Darboux transformation to a *coupled* Ermakov system of the type

$$\ddot{u} + \omega(t)u + \left(\frac{\alpha}{u^4} + \frac{\beta}{v^4}\right)u = 0, \quad \ddot{v} + \omega(t)v + \left(\frac{\gamma}{u^4} + \frac{\delta}{v^4}\right)v = 0, \tag{11}$$

which occurs in nonlinear optics [1].

3 Generalized Ermakov Systems

In analogy with (4), we introduce pairs of functions (u, v) and (ϕ, ψ) connected via the relations

$$u = a(z)\phi, \quad v = b(z)\phi, \quad z = \psi/\phi \tag{12}$$

where (a, b) is, again, a solution of the canonical Ermakov system (6) but now ϕ, ψ are, as yet, unspecified functions of independent variables x^i. If

$$L(\partial_i), \quad \partial_i = \partial/\partial x^i \tag{13}$$

denotes a linear differential operator of arbitrary order and dimension then the following result is readily established [13]:

Theorem 1. *If (u, v) is connected to (ϕ, ψ) by the relations (12) where (a, b) is a solution pair of the canonical Ermakov system (6), then the identity*

$$L(D_i)\begin{pmatrix} u \\ v \end{pmatrix} = \Omega L(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{14}$$

holds, where the matrix-valued quantities D_i and Ω are defined by

$$D_i = \partial_i + \frac{1}{w_i}\begin{pmatrix} vX_i & -uX_i \\ vY_i & -uY_i \end{pmatrix}, \quad \Omega = \begin{pmatrix} a - a'z & a' \\ b - b'z & b' \end{pmatrix} \tag{15}$$

with

$$X_i = \frac{f(v/u)}{u^3} w_i^2, \quad Y_i = \frac{g(u/v)}{v^3} w_i^2 \tag{16}$$

and $w_i = uv_{x^i} - vu_{x^i}$ is the Wronskian of u and v with respect to the variable x^i.

The above theorem has the important corollary that, since in the generic case $\det \Omega = w(a, b) \neq 0$, the solution of the nonlinear coupled system

$$L(D_i)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{17}$$

for u and v admits the representation (12) where a and b are governed by the canonical Ermakov system (6) if and only if

$$L(\partial_i)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0. \tag{18}$$

It is the nonlinear coupled system (17) with the operator D_i defined by the relations (15) and (16) that we term a *generalized Ermakov system.*

The preceding establishes the following important results:

- The generalized Ermakov system admits the solution representation (12) as a *nonlinear superposition* of linearly independent solutions ϕ, ψ of the *linear* base system (18). The latter embodies its higher dimensionality and order.
- The nonlinearity of the generalized Ermakov system is encapsulated in the autonomous classical Ermakov system (6). This, in turn, can be reduced to consideration of a single linear Schrödinger equation via (8). This may be shown to admit a Darboux transformation which accordingly may be used to connect generalized Ermakov systems.

The classical Ermakov system is readily retrieved as a specialization of the above formulation. Thus, if we consider the system (17) associated with the operator $L(\partial_t) = \partial_t^n$ then its u-component corresponding to $n = 0, 1, 2, 3, \ldots$ yield, in turn,

$$\begin{gathered} u = 0, \quad u_t = 0, \quad u_{tt} - X = 0, \\ u_{ttt} - [(wX)_t + (vX - uY)X]w^{-1} = 0, \ldots \end{gathered} \tag{19}$$

where the index on X, Y and w has been dropped since there is only the single independent variable t. The linear combination of $(19)_1$ and $(19)_3$ corresponding to the operator $L(\partial_t) = \partial_t^2 + \omega(t)$ produces the classical Ermakov equation

$$u_{tt} + \omega(t)u = X \tag{20}$$

while its companion

$$v_{tt} + \omega(t)v = Y \tag{21}$$

is obtained analogously via the v-component of (17).

The above sequence (19) indicates that a hierarchy of classical Ermakov systems may be generated by the action of D_t on their predecessors and indeed that D_t or, more generally, D_i may be regarded as recurrence operators. Indeed, a recurrence operator formulation of hierarchies of classical Ermakov systems is readily introduced [13].

4 N-dimensional Ermakov systems

Theorem I admits the following corollary corresponding to the specialization

$$L(\partial_i) = \alpha^{nm}\partial_n\partial_m + \beta^n\partial_n + \gamma \tag{22}$$

where α^{nm}, β^n and γ denote arbitrary functions of the N independent variables $x^1, \ldots, x^N$ and Einstein's summation convention has been adopted:

Theorem II. *The coupled N-dimensional Ermakov system*

$$\begin{aligned} L(\partial_i)u + \alpha^{nm}\frac{f(v/u)}{u^3}w_n(u,v)w_m(u,v) = 0 \\ L(\partial_i)v + \alpha^{nm}\frac{g(u/v)}{v^3}w_n(u,v)w_m(u,v) = 0 \end{aligned} \tag{23}$$

admits the solution representation (12) as a nonlinear superposition of linearly independent solutions of (18) with the operator $L(\partial_i)$ given by (22).

The classical Ermakov system (3) is again retrieved corresponding to the choice $L(\partial_t) = \partial_t^2 + \omega(t)$.

It is noted that the N-dimensional Ermakov system as introduced in (23) above has the alternative representation

$$\begin{aligned} L(\partial_i)u + \alpha^{nm}[\partial_n\tilde{f}(v/u)][\partial_m\tilde{f}(v/u)]u = 0 \\ L(\partial_i)v + \alpha^{nm}[\partial_n\tilde{g}(u/v)][\partial_m\tilde{g}(u/v)]v = 0. \end{aligned} \tag{24}$$

As an illustration, if we set $L(\partial_i) = -\mathrm{i}\partial_t + \Delta + \omega(t)$ then a C-integrable coupled system of nonlinear Schrödinger equations is obtained, namely

$$\begin{aligned} \mathrm{i}u_t = \Delta u + \omega(t)u + [\nabla\tilde{f}(v/u)]^2 u \\ \mathrm{i}v_t = \Delta v + \omega(t)v + [\nabla\tilde{g}(u/v)]^2 v \end{aligned} \tag{25}$$

where Δ and ∇ denote the N-dimensional Laplacian and gradient respectively. The system (25) has solution representation (4) as a nonlinear superposition of linearly independent solutions ϕ, ψ of the $N+1$-dimensional linear Schrödinger equation

$$\mathrm{i}\phi_t = \Delta\phi + \omega(t)\phi. \tag{26}$$

5 Ermakov Systems with Constraints

Here, we consider the pair of $N+1$-dimensional coupled wave equations

$$\begin{aligned} \Box u + \omega^2 u + u^{-3}H(v/u) = 0 \\ \Box v + \omega^2 v + v^{-3}J(u/v) = 0 \end{aligned} \tag{27}$$

where

$$\Box = \Delta_N - \partial_t^2 \tag{28}$$

and ω^2 is assumed to be an arbitrary function of the independent variables. Thus, on introduction of the $N+1$-dimensional gradient ∇ and the associated

Wronskian vector $\boldsymbol{w} = u\nabla v - v\nabla u$, the nonlinear system (27) corresponds to the $N+1$-dimensional Ermakov system

$$\begin{aligned} \Box u + \omega^2 u + \frac{f(v/u)}{u^3}\boldsymbol{w}^2 = 0 \\ \Box v + \omega^2 v + \frac{g(u/v)}{v^3}\boldsymbol{w}^2 = 0 \end{aligned} \tag{29}$$

provided we adjoin the constraint

$$\boldsymbol{w}^2 = \boldsymbol{w}\cdot\boldsymbol{w} = \Phi(u/v) \tag{30}$$

whence it is required that

$$(u\nabla v - v\nabla u)^2 = \Phi(u/v). \tag{31}$$

In the preceding, the scalar product of vectors has been taken with respect to the Lorentzian metric $\mathrm{diag}(1,\ldots,1,-1)$ so that $\nabla^2 = \nabla\cdot\nabla = \Box$.

A dimensional reduction

$$u = u(\xi,\eta), \quad v = v(\xi,\eta) \tag{32}$$

of the system (29)-(30) is sought, so that

$$\Box = (\nabla\xi)^2\partial_\xi^2 + 2(\nabla\xi\cdot\nabla\eta)\partial_\xi\partial_\eta + (\nabla\eta)^2\partial_\eta^2 + \Box\xi\partial_\xi + \Box\eta\partial_\eta \tag{33}$$

while the constraint (31) becomes

$$[(uv_\xi - vu_\xi)\nabla\xi + (uv_\eta - vu_\eta)\nabla\eta]^2 = \Phi(u/v). \tag{34}$$

It is required that the constraint (34) be compatible with the coupled system (29) where the d'Alembert operator is now given by (33). To this end, the following conditions on ξ,η are adjoined:

$$\Box\xi = 0, \quad \Box\eta = 0, \quad (\nabla\xi)^2 = 1, \quad (\nabla\eta)^2 = 0, \quad \nabla\xi\cdot\nabla\eta = 0. \tag{35}$$

The system (27) then reduces to the classical one-dimensional Ermakov system

$$\begin{aligned} u_{\xi\xi}|_\eta + \omega^2 u + u^{-3}H(v/u) = 0 \\ v_{\xi\xi}|_\eta + \omega^2 v + v^{-3}J(u/v) = 0 \end{aligned} \tag{36}$$

wherein η enters as a parameter. The condition (34) becomes

$$(uv_\xi - vu_\xi)^2 = \Phi(u/v) \tag{37}$$

and this automatically holds by virtue of the Lewis-Ray-Reid invariant of the Ermakov system (36).

Multi-wave solutions of the system of constraints (35) and thereby of the original non-integrable system (27) may now be constructed in the manner described in Rogers *et al* [14] for the 2+1-dimensional case.

6 A Novel Reduction of a 2+1-Dimensional Ernst-type Equation

Schief [15] recently introduced an integrable 2+1-dimensional Ernst-type equation, namely

$$\left[\partial_t - \frac{\mathcal{E}_t}{\Re(\mathcal{E})} + \mathrm{i}\Re(\rho)\right]\left[\mathcal{E}_{xx} + \mathcal{E}_{yy} - \frac{\mathcal{E}_x^2 + \mathcal{E}_y^2}{\Re(\mathcal{E})}\right] + \mathcal{E}_x\left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_x - \bar{\mathcal{E}}_t\mathcal{E}_x}{2\Re(\mathcal{E})^2} + \mathrm{i}\rho_x\right] + \mathcal{E}_y\left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_y - \bar{\mathcal{E}}_t\mathcal{E}_y}{2\Re(\mathcal{E})^2} + \mathrm{i}\rho_y\right] = 0 \quad (38)$$

$$\mathrm{i}(\rho_{xx} + \rho_{yy}) + \left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_x - \bar{\mathcal{E}}_t\mathcal{E}_x}{2\Re(\mathcal{E})^2}\right]_x + \left[\frac{\mathcal{E}_t\bar{\mathcal{E}}_y - \bar{\mathcal{E}}_t\mathcal{E}_y}{2\Re(\mathcal{E})^2}\right]_y + \left[\frac{\mathcal{E}_x\bar{\mathcal{E}}_x + \mathcal{E}_y\bar{\mathcal{E}}_y}{2\Re(\mathcal{E})^2}\right]_t = 0,$$

where $\mathcal{E}$ and ρ are, in general, complex functions. In the t-independent case with $\Im(\rho) = 0$ the usual Ernst equation of General Relativity is retrieved.

It is readily seen that the ansatz

$$\mathcal{E} = a(\psi), \quad \rho = \mathrm{i}\sigma \tag{39}$$

where ψ and σ are real, reduces (38) to the complex equation

$$a'' = \frac{a'^2}{\Re(a)} \tag{40}$$

augmented by the real coupled system

$$\psi_{xxt} + \psi_{yyt} = \sigma_x\psi_x + \sigma_y\psi_y, \quad \sigma_{xx} + \sigma_{yy} = \frac{1}{2}c(\psi_x^2 + \psi_y^2)_t, \tag{41}$$

where

$$\frac{a'\bar{a}'}{\Re(a)^2} = c \tag{42}$$

is a first integral of (40).

The system (41) represents a disguised elliptic version of a 2+1-dimensional integrable extension of the classical sine-Gordon equation [16]. Thus, the change of variables

$$\varphi_x = \psi_{ty}\cosh\psi - \sigma_y\sinh\psi, \quad \varphi_y = -\psi_{tx}\cosh\psi + \sigma_x\sinh\psi \tag{43}$$

in (41) produces the sinh-Gordon system

$$\begin{aligned}\left(\frac{\psi_{tx}}{\sinh\psi}\right)_x + \left(\frac{\psi_{ty}}{\sinh\psi}\right)_y + \frac{\psi_y\varphi_x - \psi_x\varphi_y}{\sinh^2\psi} = 0\\ \left(\frac{\varphi_x}{\sinh\psi}\right)_x + \left(\frac{\varphi_y}{\sinh\psi}\right)_y - \frac{\psi_y\psi_{tx} - \psi_x\psi_{ty}}{\sinh^2\psi} = 0,\end{aligned} \tag{44}$$

where we have set $c = 1$ without loss of generality. This system has been previously constructed by Schief [15] via an eigenfunction-adjoint eigenfunction constraint applied to the LKR integrable systems [16]. The novel class of solutions (39) of the 2+1-dimensional Ernst-type equation (38) given by the hybrid system (40) and (41) has its genesis in an alignment of (38) with a generalized 2+1-dimensional Ermakov system. The details are given in Schief *et al* [13].

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NOETHER OPERATOR RELATION AND CONSERVATION LAWS FOR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

A class of partial differential equations for which it is possible to define a correspondence between symmetries and conservation laws is considered. The conditions for a differential system to belong to the class are given. The correspondence between symmetries and local conservation laws through a given conservation law is discussed.

1. Introduction

In this paper we will follow the ideology of paper *Ref.*[1] , where by means of a Noether operator relation the correspondence between generalized symmetries and local conservation laws was defined for a certain class of differential equations. Among the equations of this class are many physically interesting equations: Korteweg-de Vries equation, modified KdV, Boussinesq equation, Kadomtsev-Petviashvili equation, nonlinear wave equation, nonlinear diffusion equation, regularized long-wave equation, Monge-Ampere equation, shallow water equation, Navier-Stokes equations, and many others.

In Section 2 basic facts and definitions are given, and the class of differential equations is defined where (generalized) symmetries correspond to local conservation laws.

In Section 3 the conditions for a differential system to be transformed to this class of differential equations with the correspondence between symmetries and conservation laws are examined, and examples are given.

In Section 4 the possibility to establish the correspondence between symmetries and conservation laws through one of the conserved vectors is discussed for more general differential systems.

2. Symmetries and Conservation Laws

We study differential systems of the form

$$\omega^a(x^i, u^b, u^b_i, u^b_{ij}, \ldots) = 0 , \tag{2.1}$$

where x^i and u^a are independent and dependent variables, respectively, and

$$u^a_i \equiv \frac{\partial u^a}{\partial x^i} , \qquad u^a_{ij} \equiv \frac{\partial^2 u^a}{\partial x^i \partial x^j} , \qquad i,j = 1,\ldots,n , \quad a,b = 1,\ldots,m . \tag{2.2}$$

We will consider infinitesimal transformations of the form

$$\begin{aligned} x'^i &= x^i + \epsilon\xi^i(x,u,u_j,\ldots) + O(\epsilon^2) , \\ u'^a &= u^a + \epsilon\eta^a(x,u,u_j,\ldots) + O(\epsilon^2) , \end{aligned} \tag{2.3}$$

where ϵ is an infinitesimal parameter.

The operator X

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^a\frac{\partial}{\partial u^a} + \zeta^a_i\frac{\partial}{\partial u^a_i} + \sum_{i\le j}\sigma^a_{ij}\frac{\partial}{\partial u^a_{ij}} + \ldots \tag{2.4}$$

(where coefficients ζ^a_i, $\sigma^a_{ij}, \ldots$ are known functions of ξ^i, η^a and their derivatives) is a symmetry operator for the differential system (2.1) iff

$$X\omega^a \doteq 0 , \tag{2.5}$$

where the symbol $\doteq$ denotes equality on the solution manifold $\omega^a = 0$, $D_i\omega^a = 0, \ldots$ (see, e.g. *Ref.*[2-6]). Instead of X it is often convenient to consider a canonical operator

$$\begin{aligned} X_\alpha &= \alpha^a\frac{\partial}{\partial u^a} + (D_i\alpha^a)\frac{\partial}{\partial u^a_i} + \sum_{i\le j}(D_iD_j\alpha^a)\frac{\partial}{\partial u^a_{ij}} + \ldots , \\ \alpha^a &= \eta^a - \xi^i u^a_i , \end{aligned} \tag{2.6}$$

where D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x^i} + u^a_i\frac{\partial}{\partial u^a} + u^a_{ij}\frac{\partial}{\partial u^a_j} + \ldots . \tag{2.7}$$

Now the symmetry condition (2.5) reads

$$X_\alpha\omega^a \doteq 0 . \tag{2.5'}$$

By local conservation law we mean the total divergence of some vector vanishing on the solution manifold:

$$D_iM^i \doteq 0. \tag{2.8}$$

The basis for establishing the correspondence between symmetries and local conservation laws for differential systems in *Ref.*[1] is the Noether operator identity

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i} , \tag{2.9}$$

where X_α is the canonical symmetry operator (2.6),

$$E^a = \frac{\partial}{\partial u^a} - D_i \frac{\partial}{\partial u_i^a} + \sum_{i \le j} D_i D_j \frac{\partial}{\partial u_{ij}^a} - \dots . \tag{2.10}$$

is the Euler-Lagrange operator, and

$$\begin{aligned} R_{\alpha i} = {} & \alpha^a \frac{\partial}{\partial u_i^a} + [\sum_{k \ge i} (D_k \alpha^a) - \alpha^a \sum_{k \le i} D_k] \frac{\partial}{\partial u_{ik}^a} + \\ & [\sum_{k \ge j \ge i} (D_j D_k \alpha^a) - \sum_{k \le i \le j} (D_j \alpha^a) D_k + \alpha^a \sum_{j \le k \le i} D_j D_k] \frac{\partial}{\partial u_{ijk}^a} - \dots . \end{aligned} \tag{2.11}$$

Equation (2.9) is the operator relation underlying the general form of the Noether identity (see also *Ref.*[7,4]). Indeed, the application of (2.9) to L (for Lagrangian systems $\omega^a = E^a(L)$) gives

$$X_\alpha L = \alpha^a E^a(L) + D_i(R_{\alpha i} L) , \tag{2.12}$$

where L is an arbitrary Lagrangian function of any order. For first order Lagrangians (2.12) reduces to the familiar form of Noether identity

$$X_\alpha L = \alpha^a E^a(L) + D_i(\alpha^a \frac{\partial L}{\partial u_j^a}) , \tag{2.13}$$

In *Ref.*[1] the class of differential equations (2.1) for which it is possible to relate symmetries and conservation laws was defined by the following conditions on ω^a :

$$E^b(\omega^a) \doteq 0 , \tag{2.14}$$

or

$$E^b(\omega^a) = k^{abc}\omega^c + l_i^{abc} D_i \omega^c + m_{ij}^{abc} D_i D_j \omega^c + \dots \equiv K^{ab} \Delta \omega , \tag{2.15}$$

where $\Delta \equiv (1, D_i, D_i D_j, \dots)$, and $K^{abc} = (k^{abc}, l_i^{abc}, m_{ij}^{abc}, \dots)$. Indeed from (2.9)

$$X_\alpha \omega^a = \alpha^b E^b(\omega^a) + D_i(R_{\alpha i} \omega^a) , \tag{2.16}$$

and because of (2.14) the second term in (2.16) vanishes on the solution manifold. Therefore, for each representative of the class (2.15), we get m conserved currents associated with a symmetry operator X_α ($X_\alpha \omega^a \doteq 0$):

$$D_i(R_{\alpha i} \omega^a) \doteq 0 . \tag{2.17}$$

Among the differential systems satisfying (2.15), there are equations in the form of a total divergence ($K^{ab} = 0$, see (2.15))

$$\omega^a = D_i N_i^a , \qquad a = 1,\ldots,m , \quad i = 1,\ldots,n . \tag{2.18}$$

We list some of them: Korteweg-de Vries (KdV) equation: $u_t + (\frac{\alpha u^2}{2} + u_{xx})_x = 0$,

Boussinesq equation: $u_{tt} - (\frac{1}{3}u_{xxx} + \frac{8}{3}uu_x)_x = 0$,

nonlinear heat equation: $u_t - (D_x K(u))_x = 0$,

Kadomtsev-Petviashvili equation: $(u_t + uu_x + u_{xxx})_x + ku_{yy} = 0$, $k = \pm 1$,

regularized long-wave equation: $(u - u_{xx})_t + (u - \frac{u^2}{2})_x = 0$,

shallow water equation: $(u_{xxt} + \beta u_x u_t - u_x - u_t)_x + \frac{(\alpha - \beta)}{2}(u_x^2)_t = 0$,

Euler equations: $u_t^i + p_i + (u^i u^k)_k = 0$, $u_k^k = 0$, and many others.

Another class of equations satisfying (2.15) (with $K^{ab} \neq 0$) is $E(\omega) = b(u)\omega$, e.g. $\omega = u_{12} - u_1 u_2/2(u+k) - l(u+k)$, $b(u) = 1/(u+k)$, $k = const.$ (see *Ref.*[1]).

In *Ref.*[8] the following commutation relation was derived as an immediate consequence of the Noether operator identity (2.9) and generalized Helmholtz condition: (see also *Ref.*[5]) $(-1)^k D_S(\frac{\partial E^a(f)}{\partial u_S^b}\varphi) = \frac{\partial E^b(f)}{\partial u_S^a}(D_S\varphi)$, (for any functions f, φ)

$$[E^b, X_\alpha]f = (-1)^k D_S(\frac{\partial \alpha^a}{\partial u_S^b}E^a(f)) . \tag{2.19}$$

Let us consider some special cases of (2.19):

1) $\alpha = \alpha(x^i)$,

$$[E^b, X_\alpha] = 0 . \tag{2.20}$$

2) $\alpha = \alpha(x^i, u^a)$, point transformations of dependent variables

$$[E^b, X_\alpha]f = \frac{\partial \alpha^a}{\partial u^b}E^a(f) . \tag{2.21}$$

(see also *Ref.*[9] for the corresponding commutator with noncanonical operator X (2.4)).

3) $\alpha = \alpha(x^i, u^a, u_i^a)$, contact transformations

$$[E^b, X_\alpha]f = E^b(\alpha^a)E^a(f) - \frac{\partial \alpha^a}{\partial u_j^b}(D_j E^a(f)) \tag{2.22}$$

(see *Ref.*[10] for the corresponding identity in mechanics).

Let us note that the application of the expression (2.19) to a Lagrangian function L provides an important relation

$$[E, X_\alpha]L \doteq 0 , \tag{2.23}$$

or $E(X_\alpha L) \doteq 0$, which means that a symmetry transformation of the original Lagrangian leads to the equation with the solution manifold covering (in general) that of the original one.

Applying the relation (2.19) to a differential equation of the class (2.14) we can prove that

$$E^b(X_\alpha \omega^a) \doteq 0 . \tag{2.24}$$

(see *Ref.*[8]). Equation (2.24) means that a symmetry transformation of an original differential system of the class (2.14) belongs to the same class.

3. Transformations of equations

Let us consider the equations that can be transformed into equations of the class (2.14) by a linear transformation.

We will start with the case of a single (scalar) differential equation

$$\omega(x^i, u_i, u_{ij}, \ldots) = 0 . \tag{3.1}$$

Suppose that the condition (2.14) does not hold, $E(\omega) \doteq 0$. We consider the case when

$$E(a\omega) \doteq 0 , \tag{3.2}$$

where in general $a = a(x^i, u, u_i, \ldots)$. We get

$$E(a\omega) = \frac{\partial}{\partial u}(a\omega) - D_j[\frac{\partial}{\partial u_j}(a\omega)] + \sum_{j\leq k} D_j D_k[a\frac{\partial \omega}{\partial u_{jk}}] + \ldots \tag{3.3}$$

and on the solution manifold

$$\begin{aligned} E(a\omega) \doteq\ & aE(\omega) - (D_j a)\frac{\partial \omega}{\partial u_j} + \sum_{j\leq k}(D_j a)(D_k\frac{\partial \omega}{\partial u_{jk}}) \\ & + \sum_{j\leq k}(D_k a)(D_j\frac{\partial \omega}{\partial u_{jk}}) + \sum_{j\leq k}(D_j D_k a)\frac{\partial \omega}{\partial u_{jk}} + \ldots \end{aligned} \tag{3.4}$$

Thus, if it is possible to find a function a that satisfies (for second-order differential equations (3.1))

$$(D_j a)\frac{\partial \omega}{\partial u_j} - (\sum_{j\leq k} + \sum_{j\geq k})[(D_j a)(D_k\frac{\partial \omega}{\partial u_{jk}})] - \sum_{j\leq k}(D_j D_k a)\frac{\partial \omega}{\partial u_{jk}} \doteq aE(\omega) , \tag{3.5}$$

then (3.2) holds, and for the equation $\overline{\omega} = a\omega$ we can find the correspondence (2.17) between the symmetries and conservation laws. For second order equations of the form

$$\omega = c_{ij}u_{ij} + F(x^i, u, u_i) , \qquad c_{ij} = const. \tag{3.6}$$

equation (3.5) takes the much simpler form

$$(D_j a)\frac{\partial \omega}{\partial u_j} - \sum_{j\leq k} c_{jk}(D_j D_k a) \doteq aE(\omega) . \tag{3.7}$$

Similar consideration can be given to systems of differential equations (2.1). In this case we have to find functions $A^{ac} = A^{ac}(x^i, u, u^a_i, \ldots)$ for which

$$E^b(A^{ac}\omega^c) \doteq 0 , \quad \forall a,b . \tag{3.8}$$

(the equivalence problem will not be discussed here). It is possible to solve equations (3.8) in some cases. For example, for the Navier-Stokes equations

$$\begin{aligned} &\omega^j \equiv -\nu u^j_{ii} + u^j_t + u^j_i u^i + p_j = 0 , \\ &\omega^4 \equiv u^i_i = 0 , \\ &u^j = u^j(x^i, t) , \; p = p(x^i, t) , \quad i,j = 1,2,3 , \; \nu = const , \end{aligned} \tag{3.9}$$

functions A^{ab} take the form (see *Ref.*[1])

$$A^{jk} = \delta_{jk}, \quad A^{j4} = u^j, \quad A^{4j} = 0, \quad A^{44} = 1, \quad j = 1,2,3 . \tag{3.10}$$

Indeed,

$$E^b(\omega^j) \doteq u^b_j , \quad E^b(\omega^4) = 0 , \quad E^b(A^{j4}\omega^4) \doteq -D_b A^{j4} , \quad b = 1,\ldots,4 , \tag{3.11}$$

which means that $A^{j4} = u^j$, and $E^b(\omega^j + u^j\omega^4) \doteq 0$, $E^b(\omega^4) = 0$.

4. Correspondence Through a Conservation Law

Let us consider now the situation when the initial equations do not satisfy the condition (2.14) and it does not seem possible to solve equations (3.5) or (3.8). In this case for non-Lagrangian systems (2.1) it is still possible to relate symmetries and conservation laws using a known conserved vector.

Let M_i be a conserved vector for the system (2.1). i.e.

$$D_i M^i = \beta^a\omega^a + b^a_i D_i\omega^a + b^a_{ij} D_i D_j\omega^a + \ldots , \tag{4.1}$$

or

$$\begin{aligned} D_i M^i = {} & (\beta^a - D_i b^a_i + D_i D_j b_{ij} - \ldots)\omega^a \\ & + D_i(b^a_i\omega^a + b^a_{ij} D_j\omega^a - (D_j b^a_{ji})\omega^a + \ldots) . \end{aligned} \tag{4.2}$$

Rewriting equations (4.1) and (4.2) in terms of the operator Δ and the adjoint operator Δ^* (see *Ref.*[1] for details, see also *Ref.*[5])

$$\Delta \equiv (1, D_i, D_i D_j, D_i D_j D_k , \ldots) , \tag{4.3}$$

$$\Delta^* = \sum_r (\Delta^-)_r \tag{4.4}$$

($(\Delta^-)_r$ can be obtained from Δ_r by the change $D_i \to -D_i$ for all i), we obtain

$$D_i M_i = P^a \Delta_r \omega^a = (\Delta^*_r P^a_r)\omega^a \tag{4.5}$$

(M^i , as a conserved vector, is defined up to an equivalence transformation

$M^i \to M^i + r^a_{ij}\omega^a + s^a_{ij}D_j\omega^a + \ldots$ (with arbitrary functions $r^a_{ij}, s^a_{ij}, \ldots$)), where

$$P^a \equiv (\beta^a, b^a_i, b^a_{ij}, \ldots) . \tag{4.6}$$

Thus,

$$D_iM_i = \gamma^a\omega^a , \tag{4.7}$$

where

$$\gamma^a = \Delta^*_r P^a_r = \beta^a - D_ib^a_i + D_iD_jb^a_{ij} - \ldots . \tag{4.8}$$

For any known conserved vector M_i with the characteristic γ^a , we have

$$E^b(\gamma^a\omega^a) = 0 . \tag{4.9}$$

Applying the identity (2.9) to the combination $\gamma^a\omega^a$ we obtain

$$X_\alpha(\gamma^a\omega^a) = \alpha^bE^b(\gamma^a\omega^a) + D_i[R_{\alpha i}(\gamma^a\omega^a)] , \tag{4.10}$$

or with (4.9)

$$(X_\alpha\gamma^a)\omega^a + \gamma^a(X_\alpha\omega^a) = D_i[R_{\alpha i}(\gamma^a\omega^a)] . \tag{4.11}$$

Thus, for any symmetry vector α^a we can get an associated (through the vector γ^b) conservation law

$$D_i[R_{\alpha i}(\gamma^a\omega^a)] \doteq 0 . \tag{4.12}$$

Let us calculate its characteristic. Let

$$X_\alpha\omega^a = v^{ab}\omega^b + v^{ab}_iD_i\omega^b + v^{ab}_{ij}D_iD_j\omega^b + \ldots = V^{ab}_S\Delta_S\omega^b , \tag{4.13}$$

where $V^{ab} = (v^{ab}, v^{ab}_i, v^{ab}_{ij}, \ldots)$. Then

$$\gamma^aX_\alpha\omega^a = \Delta^*_S(\gamma^aV^{ab}_S)\omega^b + D_i(k^{ab}_i\omega^b + l^{ab}_{ij}D_j\omega^b + , \ldots) , \tag{4.14}$$

and

$$D_i[R_{\alpha i}(\gamma^a\omega^a)] = [X_\alpha\gamma^b + \Delta^*_S(\gamma^aV^{ab}_S)]\omega^b . \tag{4.15}$$

Thus, the vector

$$\mu^b = X_\alpha\gamma^b + \Delta^*_S(\gamma^aV^{ab}_S) \tag{4.16}$$

(see (4.8) and (4.13)) is the characteristic of the conservation law (4.12) associated with a symmetry vector α^a through the vector γ^a .

Note that for Lagrangian systems we can consider expression (4.15) as a result of the Noether symmetry transformation X_α of a (Noether) conservation law (4.7). In this case the characteristic μ^b of the resulting conservation law can be easily calculated. Let us apply the identity (3.4) to the Noether condition $X_\alpha L = D_i\tau^i$, to obtain

$$E^a(X_\alpha L) - X_\alpha \omega^a = (-1)^k D_S\left(\frac{\partial \alpha^b}{\partial u^a_S}\omega^b\right) . \qquad (4.17)$$

After simplification we find

$$\Delta^*_S(\gamma^a V^{ab}_s)\,\omega^b = \gamma^a X_\alpha \omega^a = -\,(D_S\gamma^a)\frac{\partial \alpha^b}{\partial u^a_S}\omega^b \qquad (4.18)$$

(within total divergence terms $D_i(p^a_i\omega^a + r^a_{ij}D_j\omega^a + \ldots)$). Thus, for Lagrangian systems we get a familiar result *Ref.*[5]

$$\mu^a = X_\alpha \gamma^a - X_\gamma \alpha^a . \qquad (4.19)$$

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THE METHOD OF THE MULTIPLE SCALES AND THE NONLINEAR SCHRÖDINGER HIERARCHY

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We
review results concerning the propagation of quasi-monocromatic, nondissipative and weakly nonlinear waves using a multiple scales formalism. We show that the perturbative series of the solution is asymptotic iff the modulation of the leading amplitude of the waves is described by the nonlinear Schrödinger hierarchy with respect to the slow space-time variables characteristic of the problem.

1 Introduction

We present a review of results on the propagation of quasi-monocromatic, nondissipative and weakly nonlinear waves, modelled by partial differential equations (PDE's) in 1+1 dimensions. Most of the presentation is based on the results presented in the work [1], co-authored by Degasperis and Manakov.

Consider a nonlinear PDE in 1+1 dimensions

$$L(\partial_t, \partial_x)u = G(u), \qquad u = u(x,t) \in \mathcal{R}, \tag{1.1}$$

where L is a linear dispersive differential operator with constant coefficients and $G(u)$, the nonlinear part of the equation, is an entire function of u and of its x-derivatives (without linear terms). The linear equation: $Lu = 0$ admits the monocromatic wave solution:

$$u = Ae^{i\theta} + c.c., \qquad \theta = kx - \omega(k)t, \quad k \in \mathcal{R} \tag{1.2}$$

for an arbitrary complex constant amplitude A, where the frequency $\omega = \omega(k)$ is given in terms of the wave number k through the dispersion relation

$$L(-i\omega, ik) = 0. \tag{1.3}$$

If one is interested in small solutions: $u = O(\epsilon)$, $\epsilon << 1$ of (1.1), it is known that the effect of the weak nonlinearity $G(u)$ on the approximate solution (1.2)

i) occurs at large space - time scales:

$$\xi = \epsilon x, \qquad t_n = \epsilon^n t, \quad n \in \mathcal{N}, \tag{1.4}$$

ii) determines a redistribution of the energy to the higher harmonics $e^{i\alpha\theta}$, $\alpha \in \mathcal{Z}$ and
iii) determines a dependence of the amplitudes of such harmonics on the slow space and time variables (1.4) (the modulation of the wave packet).

It is well-known, in particular, that, if the following "no-resonance condition"

$$L(-i\alpha\omega, i\alpha k) \neq 0, \quad |\alpha| \neq 1, \tag{1.5}$$

is satisfied, i.e., if no higher harmonics, different from the leading one, solves the linear equation: $Lu = 0$, then the dependence of the leading amplitude ψ of $e^{i\theta}$ on the long space - time variables ξ, t_1, t_2 is given by [2-12]:

$$u(x,t) = \epsilon\psi(\xi - \omega_1 t_1, t_2)e^{i\theta} + c.c. + O(\epsilon^2), \tag{1.6}$$

$$\psi_{t_2} = i\omega_2(\psi_{\xi\xi} - 2c|\psi|^2\psi) =: K_2(\psi), \tag{1.7}$$

where

$$\omega_n = \frac{1}{n!}\frac{d^n\omega(k)}{dk^n}, \qquad n \in \mathcal{N}. \tag{1.8}$$

Namely: **the leading amplitude ψ translates w.r.t. t_1 with the group velocity and evolves in t_2 according to the celebrated cubic nonlinear Schrödinger (NLS) equation.**

Such a striking universality makes the NLS equation **ubiquitous** and **applicable** in several areas of the Natural Sciences, from Fluid Dynamics to Plasma Physics, from Nonlinear Optics to Biology, etc.... [7, 8, 10]. This universality is also responsible for the very special and beautiful mathematical properties possessed by the NLS equation since, as pointed out in [12], it is enough that in the large class of PDE's which reduce to the NLS equation there is an integrable one, to guarantee that the universal model be **integrable.** Indeed, the main ingredients of integrability, like the existence of an associated Lax pair, the existence of ∞-many commuting symmetries and of ∞-many constants of the motion in involution, are preserved by the above reductive - perturbative procedure. It is not surprising, therefore, that the NLS equation (1.7) be integrable if $c \in \mathcal{R}$ [14].

In a recent paper [1] we have shown that:
i) the RHS of equation (1.6) is the first term of the perturbative series

$$u(x,t) = \sum_{\alpha=-\infty}^{\infty}\sum_{n=n_\alpha}^{\infty} \epsilon^n e^{i\alpha\theta} u_n^{(\alpha)}(\xi, t_1, t_2, t_3, ..), \quad n_\alpha = |\alpha|, \ if \ \alpha \neq 0, \quad n_0 = 1. \tag{1.9}$$

ii) The perturbative series (1.9) is asymptotic iff the leading amplitude $\psi = u_1^{(1)}$ satisfies the NLS hierarchy of evolution equations with respect to the slow space-time variables (1.4).

Before giving, in §3, some of the details of the approach presented in [1], I would like to motivate the mathematical formalism and the relevance of the NLS hierarchy in perturbation theory through simple qualitative considerations.

2 Why should the NLS Hierarchy be Relevant in Perturbation Theory?

A) We first show that the relevance of the multiple scales formalism (1.4) in the study of the propagation of quasi-monocromatic weakly nonlinear waves can be easily justified already at a linear level [1].

Suppose, for concreteness, that the propagation of a linear dispersive wave is modelled by the PDE: $Lu = [\partial_t + i\omega(-i\partial_x)]u = 0$, where $\omega(\cdot)$ is an odd function (to assure the reality of the equation). Suppose that we are interested in a quasi-monocromatic wave packet, i.e., suppose that the Fourier amplitude $A(k,t)$ of the solution $u(x,t)$ has a compact support around the wave number k_0: $A(k) = 0$ if k is s.t. $|k - k_0| > \epsilon k_0$, $\epsilon << 1$. Then, expanding the dispersion relation in powers of the small parameter ϵ: $\omega(k_0 + \epsilon k_0) = \sum_{n=0}^{\infty} \epsilon^n \omega_n {k_0}^n$, one obtains: $u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx - \omega(k)t]} dk + c.c. \sim \epsilon\psi(\xi, t_1, t_2, t_3, ..)e^{i[k_0 x - \omega(k_0)t]} + c.c.$,
where:

$$\psi(\xi, t_1, t_2, t_3, ..) = k_0 \int_{-1}^{1} \hat{u}(k_0 + \nu\epsilon k_0) e^{i[k_0\nu(\xi - \omega_1 t_1) - \nu^2 {k_0}^2 \omega_2 t_2 - \nu^3 {k_0}^3 \omega_3 t_3 - ..]} d\nu$$

This shows that **linear quasi-monocromatic waves are described by the multiple scales formalism (1.4)** (which, therefore, is not an option); the amplitude ψ of the monocromatic wave is a slowly varying function of space and time and its modulation is described by an infinite set of linear PDE's of evolutionary type [1]:

$$\frac{\partial \psi}{\partial t_n} = (-i)^{n+1} \omega_n \frac{\partial^n \psi}{\partial \xi^n}, \quad n \in \mathcal{N}. \tag{1.10}$$

We remark that equations (1.10) describe commuting flows, i.e.: $\frac{\partial^2 \psi}{\partial t_m \partial t_n} = \frac{\partial^2 \psi}{\partial t_n \partial t_m}$, as it has to be, since the variables t_n, $n \in \mathcal{N}$ are treated as independent

B) This picture generalizes nicely to the nonlinear case; in particular, the apparition of the NLS hierarchy, proven in [1], can be easily predicted through the following qualitative argument.
1) First, we use the well-established fact that, in the nonlinear case, the dependence of the leading amplitude ψ on the first two times t_1, t_2 is described by equations (1.6) - (1.8), which are, therefore, the proper nonlinear analogues of equations (1.10) for $n = 1, 2$.
2) Second, we must have in mind that the slow times t_n, $n \in \mathcal{N}$ are treated as independent; therefore, if we succeed in describing the dependence of ψ on t_n, $n \in \mathcal{N}$ through evolution equations of the type: $\psi_{t_n} = K_n(\psi)$, $n \in \mathcal{N}$, then such a dependence is necessarily described by commuting flows of the NLS equation, since:

$$\psi_{t_n t_m} = \psi_{t_m t_n} \quad \Leftrightarrow \quad K_n'(\psi) \cdot K_m(\psi) = K_m'(\psi) \cdot K_n(\psi), \quad n, m \in \mathcal{N}, \tag{1.11}$$

where $K'(\psi) \cdot f$ is the so-called "Frechet derivative of K with respect to ψ in the direction f": $K'(\psi) \cdot f := \partial_\epsilon K(\psi + \epsilon f)|_{\epsilon=0}$. In other words, the higher flows $K_n(\psi)$, $n \in \mathcal{N}$, must be (commuting) symmetries of the NLS equation (1.7), i.e. (commuting) solutions σ of the linearized (about a solution ψ) NLS equation:

$$[\partial_{t_2} - K_2'(\psi)\cdot]\sigma = 0, \tag{1.12a}$$

where

$$K_2'(\psi) \cdot \sigma = \partial_\epsilon K_2(\psi + \epsilon\sigma)|_{\epsilon=0} = i\omega_2[\sigma_{\xi\xi} - 2c(\psi^2\bar{\sigma} + 2|\psi|^2\sigma)]. \tag{1.12b}$$

On the other hand, the existence of (higher order) symmetries of a given PDE is a very exceptional fact, and it is typical of integrable PDE's only [15],[16]. It turns out that, if $c \in \mathcal{R}$, the NLS equation (1.7) is integrable and possesses the ∞-many abelian symmetries:

$$\sigma_n(\psi) := \Phi^n \sigma_0(\psi), \quad n \in \mathcal{N}, \tag{1.13}$$

where: $\sigma_0(\psi) := -i\psi$ is the "gauge symmetry" of NLS, and $\Phi f := i[f_\xi - 4c\psi\partial_\xi^{-1} Re(\bar{\psi} f)]$ is the so-called "recursion operator" [17-20]. In particular, the first few commuting symmetries of the NLS equation (1.7), for $c \in \mathcal{R}$, read:

$$\sigma_0 = -i\psi, \quad \sigma_1 = \psi_\xi, \quad \sigma_2 = i(\psi_{\xi\xi} - 2c|\psi|^2\psi), \quad \sigma_3 = -(\psi_{\xi\xi\xi} - 6c|\psi|^2\psi_\xi),$$

$$\sigma_4 = -i[\psi_{\xi\xi\xi\xi} - 2c(3\psi_\xi^2\bar{\psi} + 4|\psi|^2\psi_{\xi\xi} + 2\psi|\psi_\xi|^2 + \psi^2\bar{\psi}_{\xi\xi}) + 6c^2\psi|\psi|^4]. \tag{1.14}$$

We are therefore lead to the conclusion that ψ_{t_n} should be given by a suitable linear combination of the symmetries (1.13) [21].

3) A simple dimensionality argument (ψ and ∂_ξ "rescale" like ϵ and ∂_{t_n} like ϵ^n) imposes that ψ_{t_n} be proportional to the single symmetry $\sigma_n(\psi)$ [21]:

$$\psi_{t_n} = d_n \sigma_n(\psi), \quad n \in \mathcal{N}. \tag{1.15a}$$

4) Comparing the linear limit of equations (1.15a) with the linear equations (1.10) we finally conclude that $d_n = (-)^n \omega_n$; i.e.: **the modulation of the leading amplitude ψ should be described by the NLS hierarchy**:

$$\psi_{t_n} = (-)^n \omega_n \sigma_n(\psi), \quad n \in \mathcal{N}. \tag{1.15b}$$

The above considerations are, of course, qualitative, but can serve as motivation for the results sketched in the §1.3 .

We end these introductory considerations remarking that the first attempts to compute the higher terms of the perturbation expansion at all orders have been presented in [5],[6]. The main difference with respect to our approach is the (artificious) limitation to the dependence on t_1 and t_2 only, and the restriction of the analysis to the pure soliton solution. For an approach similar to ours, applied to the weakly dispersive limit (which gives rise to the KdV hierarchy) and restricted to the one soliton solution, the interested reader is referred to [22].

3 Multiple Scales and the NLS Hierarchy

In this section we apply the method of the multiple scales to a scalar PDE of the type (1.1), concentrating our attention on the following two classes of equations

$$L_+ u = [\partial_t + i\omega(-i\partial_x)]u = G(u), \qquad L_+ L_- u = [\partial_t^2 - \omega^2(-i\partial_x)]u = G(u), \tag{1.16a}$$

where

$$L_\pm := \partial_t \pm i\omega(-i\partial_x), \tag{1.16b}$$

and having always in mind the following two concrete examples:

$$A: \qquad u_t - u_{xxx} = (au^3 + bu^5)_x, \tag{1.17a}$$

$$B: \qquad u_{tt} - u_{xx} + \mu^2 u = au^3 + bu^5, \tag{1.17b}$$

where a, b, μ are arbitrary real constants. Equation (1.17a) is a modified KdV equation with a quintic nonintegrable correction; equation (1.17b) is a nonlinear wave equation.

Inserting the perturbative series (1.9) into equations (1.16) one obtains a double family of equations: $EQ_n^{(\alpha)} = 0$ for the coefficients of the n-th power of ϵ and of the α-th harmonics. For instance, for equation (1.17a) we have:

$$EQ_n^{(\alpha)} = \sum_{m=1}^{n} L^{(\alpha)}_{+(n-m)} u_m^{(\alpha)} - G_n^{(\alpha)} = 0, \tag{1.18a}$$

where

$$G_n^{(\alpha)} = \sum_{\alpha_j=-\infty}^{\infty} \sum_{n_j=1}^{\infty} \{a\Delta(\alpha_1+\alpha_2+\alpha_3-\alpha)[i\alpha k\Delta(n_1+n_2+n_3-n)u_{n_1}^{(\alpha_1)}u_{n_2}^{(\alpha_2)}u_{n_3}^{(\alpha_3)}+$$

$$+\Delta(n_1+n_2+n_3+1-n)(u_{n_1}^{(\alpha_1)}u_{n_2}^{(\alpha_2)}u_{n_3}^{(\alpha_3)})_\xi] + b\Delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5-\alpha)\cdot$$

$$\cdot[i\alpha k\Delta(n_1+n_2+n_3+n_4+n_5-n)u_{n_1}^{(\alpha_1)}u_{n_2}^{(\alpha_2)}u_{n_3}^{(\alpha_3)}u_{n_4}^{(\alpha_4)}u_{n_5}^{(\alpha_5)}+$$

$$+\Delta(n_1+n_2+n_3+n_4+n_5+1-n)(u_{n_1}^{(\alpha_1)}u_{n_2}^{(\alpha_2)}u_{n_3}^{(\alpha_3)}u_{n_4}^{(\alpha_4)}u_{n_5}^{(\alpha_5)})_\xi]\}, \tag{1.18b}$$

$$L^{(\alpha)}_{+\ n} := \partial_{t_n} - (-i)^{n+1}\frac{\omega^{(n)}(\alpha k)}{n!}\partial_\xi^n \tag{1.18c}$$

and $\Delta(n) = 0$, if $n \neq 0$; $\Delta(0) = 1$.

The no-resonance condition (1.5) implies that equations $EQ_n^{(\alpha)} = 0$ for $\alpha \neq \pm 1$ are algebraic and can be solved explicitly for the higher harmonics coefficient in terms of the amplitude $\psi = u_1^{(1)}$ and its higher corrections $u_n^{(1)}$, $n > 1$: $u_n^{(\alpha)} = u_n^{(\alpha)}(\psi, u_2^{(1)}, u_3^{(1)}, ..)$, $|\alpha| \neq 1$. Therefore we shall concentrate in the following on the equations $EQ_n^{(1)} = 0$ for the coefficients $u_n^{(1)}$, $n \in \mathcal{N}$ of the first harmonics $e^{i\theta}$.

At $O(\epsilon)$ we obtain: $L(-i\omega, ik)\psi = 0$, $\psi := u_1^{(1)}$, which gives the dispersion relation (1.3). For our two examples: A: $\omega = k^3$, B: $\omega = (k^2+1)^{\frac{1}{2}}$.

At $O(\epsilon^2)$ we obtain: $L^{(1)}_{+\ 1}\psi = 0 \Rightarrow \psi = \psi(\xi - \omega_1 t_1, t_2, t_3, ..)$.

At $O(\epsilon^3)$:

$$L^{(1)}_{+\ 1} u_2^{(1)} = -[\psi_{t_2} - K_2(\psi)], \tag{1.19}$$

where $K_2(\psi)$ is defined in (1.7); for our two examples: A: $c = -a/2$, B: $c = -(3a/4\omega\omega_2)$.

Since the RHS of equation (1.19) is in the null space of $L_1^{(1)}$, the solution $u_2^{(1)}$ of (1.19) would blow linearly in the variable $\xi + \omega_1 t_1$, unless the RHS of equation (1.19) is zero. Therefore, to avoid this secularity, the t_2 - dependence of ψ must be such that the RHS of (1.19) is zero, i.e. ψ **must** evolves w.r.t. t_2

according to the NLS equation (1.7). Consequently: $L^{(1)}_{+\ 1} u^{(1)}_2 = 0 \Rightarrow u^{(1)}_2 = u^{(1)}_2(\xi - \omega_1 t_1,\ t_2, t_3, ..)$.
At the next orders in ϵ, the above secularity mechanism fixes recursively the t_1 dependence of all the coefficients $u^{(\alpha)}_n$ of the expansion in the same form: $u^{(\alpha)}_n = u^{(\alpha)}_n(\xi - \omega_1 t_1, t_2, t_3, ..)$, i.e.: **the whole solution translates with w.r.t. t_1 with the group velocity $\omega'(k)$.**

One is left with the following equations for the corrections $u^{(1)}_n$, $n \geq 2$ of the NLS field $\psi = u^{(1)}_1$:

$$O(\epsilon^{n+2}): \qquad [\partial_{t_2} - K'_2(\psi)] u^{(1)}_n = F_n(\psi, u^{(1)}_2, .., u^{(1)}_{n-1}), \qquad n \geq 2, \ (1.20a)$$

where $K'_2(\psi)$ is defined in (1.12b) and the forcing F_n, which depends on the fields $\psi,\ u^{(1)}_2, .., u^{(1)}_{n-1}$ determined in the previous iterations, exhibits the following universal character:

$$F_n = -L^{(1)}_{+\ n+1}\psi - \sum_{k=2}^{n-1} L^{(1)}_{+\ n+2-k} u^{(1)}_k + H_n(\psi, u^{(1)}_2, .., u^{(1)}_{n-1}). \qquad (1.20b)$$

The nonlinear part H_n of the forcing is the sum of typical blocks that can be guessed a priori through a dimensionality argument; only the coefficients of such blocks depend on the original equation (1.16). For our two examples: A: $H_2 = 3a(|\psi|^2\psi)_\xi$, B: $H_2 = -\frac{3a\omega_1}{2\omega^2}(|\psi|^2\psi)_\xi$.

Equation (1.20) has to be viewed as a **forced linear evolution equation for the field $u^{(1)}_n$** (we remark that its homogeneous version is equation (1.12) defining the NLS symmetries). The general philosophy remains the same: at each order in ϵ we have to guarantee the asymptotic character of the perturbative series (1.9); therefore:
i) we have to identify in each forcing F_n the (potential) secular terms;
ii) we have to fix the (unknown at $O(\epsilon^{n+2})$) dependence of ψ on t_{n+1} in order to eliminate such a secular terms.

In order to identify the secular terms of the forcing F_n, it is expedient to have under control the (asymptotic) behaviour of the solutions of the NLS equation. We recall that, in the focusing case ($c < 0$), a generic localized initial condition $\psi(\xi, t_2 = 0)$ of the NLS equation (1.7) decomposes asymptotically into a finite number N of solitons plus radiation [14]; solitons dominate on the radiation and are generically well separated [7]:

$$\psi \sim \psi^{(j)}_{sol}(\xi, t_2) := \frac{2p_j}{\sqrt{|c|}} \frac{e^{i\varphi_j(\xi,t_2)}}{\cosh z_j(\xi, t_2)},\ p_j > 0,\ z_j = O(1),\ t_2 >> 1,\ j = 1, .., N, \qquad (21a)$$

$$\varphi_j(\xi,t_2) := -2s_j\xi + 4\omega_2 t_2({p_j}^2 - {s_j}^2) + \alpha_j, \quad z_j(\xi,t_2) := 2p_j(\xi + 4s_j\omega_2 t_2) + \beta_j. \tag{1.21b}$$

In the absence of solitons (like in the defocusing ($c > 0$) case), only radiation is present asymptotically:

$$\psi(\xi,t_2) = \psi_{rad}(\xi,t_2)(1 + O(\frac{\ln t_2}{t_2})), \qquad t_2 >> 1, \tag{1.22a}$$

$$\psi_{rad}(\xi,t_2) := \sqrt{\frac{\rho(y)}{\omega_2 t_2}} e^{i[\frac{y^2}{4\omega_2}t_2 - 2c\rho(y)\ln(\omega_2 t_2) + \delta(y)]}, \qquad y = \frac{\xi}{t_2} = O(1), \tag{1.22b}$$

and the real slowly varying functions $\rho(y)$, $\delta(y)$ are expressed in terms of (the spectral data of) the initial condition $\psi(\xi, t_2 = 0)$ in the following way [23],[8]:

$$\rho(y) = -\frac{1}{4\pi c}\ln[1 - c|R_0(\frac{y}{4})|^2], \tag{1.23a}$$

$$\delta(y) = arg(R_0(\frac{y}{4})) - \frac{\pi}{4} - arg(\Gamma(1 - 2ic\rho(y)) + 2c\rho(y)\ln 2 + 4c\int_y^\infty \rho_{y'}(y')\ln(y' - y)dy', \tag{1.23b}$$

where $\Gamma(\cdot)$ is the Gamma function and $R_0(k)$ is the reflection coefficient of the Zakharov-Shabat spectral problem [14], corresponding to the initial condition $\psi(\xi, t_2 = 0)$.

One can prove by induction the following result [1].

Let $\psi(\xi,t_2)$ be a solution of the NLS equation (1.7) corresponding to pure radiation (the no soliton case). Then:
i) the secular terms of the forcing F_n are the linear ones;
ii) such secular terms are eliminated imposing that the t_n-dependence of the leading amplitude ψ be described by the NLS hierarchy of evolution equations:

$$\psi_{t_n} = (-)^n\omega_n\sigma_n(\psi) =: (-i)^{n+1}\omega_n\frac{\partial^n\psi}{\partial\xi^n} + V_n(\psi), \qquad n \in \mathcal{N}, \tag{1.24}$$

where the symmetries $\sigma_n(\psi)$ are defined in (1.13). Consequently, equations (1.20) take the following secularity-free form:

$$[\partial_{t_2} - K_2'(\psi)]u_n^{(1)} = F_n(\psi, u_2^{(1)}, .., u_{n-1}^{(1)}), \tag{1.25a}$$

$$F_n := -\sum_{k=2}^{n-1} {L_+^{(1)}}_{n+2-k} u_k^{(1)} - V_{n+1}(\psi) + H_n(\psi, u_2^{(1)}, .., u_{n-1}^{(1)}), \qquad n \geq 2. \tag{1.25b}$$

In the pure soliton case, one can follow the analysis in [6], implemented by the formalism of [1], which includes all the slow times (1.4). In particular, the first few iterations for the pure one-soliton solution are presented in [24], [25]. The general case (solitons plus radiation) has not been treated yet.

Referring to [1] for the details of the proof by induction, here we sketch only the first step of the iteration.

The first forcing $F_2 = -\psi_{t_3} + \omega_3\psi_{\xi\xi\xi} + H_2(\psi)$ appears at $O(\epsilon^4)$. We first observe that ψ_{t_3} is a symmetry (still to be fixed at this stage) of the NLS equation (1.7) (i.e., an element of the null space of the linearized operator (1.12a)); therefore it is a secular term, since its presence in the forcing F_2 would cause the solution $u_2^{(1)}$ to exhibit a contribution of the form: $t_2\psi_{t_3}$, which would destroy the asymptotic character of the series (1.9) for $t_2 = O(\epsilon^{-1})$.

Furthermore, in the pure soliton case, we observe that the other linear term of F_2:
$\omega_3\psi^{(j)}_{sol\,\xi\xi\xi} = \omega_3[8is_j(s_j^2 - 3p_j^2) + 8p_j(3s_j^2 - p_j^2)\tanh z_j +$
$48p_j^2(is_j + p_j\tanh z_j)\cosh^{-2} z_j]\psi^{(j)}_{sol}$ contains the two solutions: $\Sigma_1 = i\psi^{(j)}_{sol}$, $\Sigma_2 = \psi^{(j)}_{sol}\tanh z_j$ of the linearized equation (1.12). Therefore also $\omega_3\psi^{(j)}_{sol\,\xi\xi\xi}$ is a secular term, since it forces the solution $u_2^{(1)}$ of equation (1.20) to contain contributions of the type: $z_j\Sigma_1$ and $z_j\Sigma_2$, which would make the terms $\epsilon u_2^{(1)}$ and ψ comparable for $t_2 = O(\epsilon^{-1})$, destroying the asymptotic character of the expansion (1.9). Finally we observe that the nonlinear terms of F_2 do not contain the two solutions Σ_1 and Σ_2 and are nonsecular.

Our multitime formalism gives an effective way to deal with such a secularity; indeed the above identified secular terms are eliminated iff the t_3-dependence of the spectral parameters of the soliton is given by: $p_{j\,t_3} = s_{j\,t_3} = 0$, $\alpha_{j\,t_3} = 8\omega_3 s_j({s_j}^2 - 3{p_j}^2)$, $\beta_{j\,t_3} = 8\omega_3 p_j({p_j}^2 - 3{s_j}^2)$, which corresponds to the t_3-dependence of $\psi = \psi^{(j)}_{sol}$ described by the complex modified KdV (cmKdV) equation:

$$\psi_{t_3} = \omega_3[\psi_{\xi\xi\xi} - 6c|\psi|^2\psi_\xi]. \tag{1.26}$$

Even in the absence of solitons one is lead to the t_3-dependence described by (1.26). Indeed it follows from (1.22) that the linear term: $\omega_3\psi_{\xi\xi\xi} = \omega_3(\frac{iy}{2})^3\psi_{rad}(\xi, t_2)(1 + O(\frac{\ln t_2}{t_2}))$, $y = \xi/t_2 = O(1)$, $t_2 >> 1$ is $O(1/\sqrt{t_2})$, and therefore is secular, since it would force the solution $u_2^{(1)}$ of (1.20) to blow like $O(\sqrt{t_2})$, destroying the asymptotic character of the expansion (1.7). The nonlinear terms of F_2, which are $O(t_2^{-\frac{3}{2}})$ or smaller, are instead nonsecular. The secular terms are eliminated iff the t_3-dependence of $\rho(y)$ and $\delta(y)$ is given by: $\rho_{t_3} = 0$, $\delta_{t_3} = -(\frac{iy}{2})^3\omega_3$, which is equivalent (see (1.23)) to: $R_{0t_3} =$

$-i(2k)^3 R_0$. This t_3- dependence of the reflection coefficient corresponds to equation (1.26). Finally, replacing (1.26) into the forcing F_2, one obtains the secularity free forcing: $F_2 = 6c\omega_3|\psi|^2\psi_\xi + H_2(\psi)$. For our two examples: A: $F_2 = -6c\psi(|\psi|^2)_\xi$, B: $F_2 = \frac{3ak}{2\omega^3}\psi(\psi^*\psi_\xi - \psi^*_\xi\psi)$.

Similar considerations can be made at the higher orders [1]. The concrete consequence of the above result is that the dependence on the higher times: t_n, $n \in \mathcal{N}$ of the soliton parameters $p_j, s_j, \alpha_j, \beta_j$ and of the radiation slowly varying functions ρ, δ is the one described by the NLS hierarchy (1.24).

This theory, although developed in [1] for scalar PDE's of the type (1.16), has a general validity; in particular:
1) it has been successfully applied to the following two physical situations [24],[26]: the one-dimensional propagation of surface waves in an ideal irrotational fluid and the one-dimensional propagation of ion-acoustic waves in a Plasma.
2) It can be extended to the case partial differential equations in 2+1 dimensions [24] and to the case of ordinary differential equations [27].

We finally point out that the above results are obtained starting from large classes of nonlinear PDE's of the type (1.16). Differences between different equations (1.16) do not seem to alter the asymptotic character of the multiscale expansion (1.9) [1]. In order to appreciate such differences and, in particular, in order to distinguish between integrable and nonintegrable equations (1.16), it is necessary to implement properly the method using the above multiple scales expansion to construct **approximate symmetries** of the original equation [28].

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THE GEOMETRY OF THE LKR SYSTEM. APPLICATION OF A LAPLACE-DARBOUX-TYPE TRANSFORMATION TO ERNST-TYPE EQUATIONS

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Classical geometric systems of Darboux, Bianchi and Weingarten are identified as members of the integrable Loewner-Konopelchenko-Rogers (LKR) system. In this connection, constraints which preserve integrability are presented along with compatible solution generation techniques. In particular, a novel Laplace-Darboux-type transformation is discussed, leading to sequences of *nonlinear* Ernst-type equations akin to those obtained in the classical case of *linear* hyperbolic equations.

1 Introduction

The classical Bäcklund transformation for the sine-Gordon equation and its generalizations have proven to be a fundamental ingredient of modern Soliton Theory. Therein, Bäcklund transformations are used as a tool for generating solutions of differential equations. In most cases, however, this application of Bäcklund transformations is somewhat removed from Bäcklund's original geometric approach, i.e. the construction of pseudo-spherical surfaces.

Here, we present an extension of an application of Bäcklund transformations in Continuum Mechanics that predated those in Soliton Theory and proceeded entirely independently for some forty years. It turns out that this approach is closely related to classical differential geometry.

2 The Loewner-Konopelchenko-Rogers (LKR) System. Canonical Parameterization and First Examples

In 1952, Loewner [1] introduced the notion of infinitesimal Bäcklund transformation in connection with the study of the hodograph equations of plane potential gasdynamics. It was only recently that Konopelchenko and Rogers [2] reinterpreted and generalized Loewner's approach, leading to a class of integrable 2+1-dimensional systems. The systems so derived shared the novel feature of a symmetric dependence on the 'spatial' variables x and y. In order to illustrate the relevance of the Loewner-Konopelchenko-Rogers (LKR) system in terms

of classical differential geometry, attention is focussed on the subcase

$$\phi_y = S\phi_x, \quad \phi_{xt} = V\phi_x + W\phi, \quad \phi_{yt} = V\phi_y + SW\phi \tag{1}$$

where S, V and W are matrix-valued functions of x, y and t. The compatibility conditions for the 'eigenfunction' ϕ lead to the nonlinear system

$$S_t = [V, S], \quad V_y - V_x S + [W, S] = 0, \quad W_y - (SW)_x = 0. \tag{2}$$

A characteristic feature of these nonlinear equations is their compact representation in terms of quantities satisfying linear equations. Thus, on setting

$$V = \Phi + \boldsymbol{\omega}^\dagger, \quad W = \Phi_x, \quad SW = \Phi_y \tag{3}$$

we obtain the equivalent system

$$S_t = [\Phi + \boldsymbol{\omega}^\dagger, S], \quad \Phi_y = S\Phi_x, \quad \boldsymbol{\omega}_y = S^\dagger \boldsymbol{\omega}_x. \tag{4}$$

In fact, it has been pointed out by Oevel and Schief [3] that the Loewner system (2) may be regarded as squared eigenfunction symmetry of the multi-component modified Kadomtsev-Petviashvili (mKP) hierarchy. This is indicated by the fact that $(1)_1$ is nothing but the 'scattering problem' of the latter so that Φ and $\boldsymbol{\omega}$ are a corresponding eigenfunction and an (integrated) adjoint eigenfunction respectively.

This observation may now be exploited to reduce the number of dependent variables in the Loewner system (2). For instance, an eigenfunction-adjoint eigenfunction constraint of Kupershmidt-type together with consistent conditions on S read [3]

$$\boldsymbol{\omega} = -\Phi, \quad S^\dagger = S, \quad S^2 = \mathbf{1}, \quad \mathrm{Tr}\, S = 0. \tag{5}$$

In the case of 2×2 matrices, these constraints lead to the important 2+1-dimensional integrable extension of the classical sine-Gordon equation

$$\begin{aligned} &\left(\frac{\theta_{tx}}{\sin\theta}\right)_x - \left(\frac{\theta_{ty}}{\sin\theta}\right)_y + \frac{\theta_x\varphi_y - \theta_y\varphi_x}{\sin^2\theta} = 0 \\ &\left(\frac{\varphi_x}{\sin\theta}\right)_x - \left(\frac{\varphi_y}{\sin\theta}\right)_y + \frac{\theta_x\theta_{ty} - \theta_y\theta_{tx}}{\sin^2\theta} = 0 \end{aligned} \tag{6}$$

as given by Konopelchenko and Rogers [2]. Therein, it has also been pointed out that the 1+1-dimensional systems considered by Bruschi and Ragnisco [4] in connection with the chiral field spectral problem constitute x-independent Loewner systems. Moreover, a deep reduction of these systems of the form

$$S^p = \mathbf{1}, \quad p = \dim(S) \tag{7}$$

leads to the 1+1-dimensional periodic Toda lattice

$$\theta_{ntx} = \mathrm{e}^{\theta_{n+1}-\theta_n} - \mathrm{e}^{\theta_n-\theta_{n-1}}, \qquad \theta_{n+p} = \theta_n \tag{8}$$

so that (6) is gauge equivalent to a 2+1-dimensional version of (8) for $p = 2$.

It is well-known that the integrable Toda lattice system (8) was derived by Darboux [5] in the last century. In this connection, he demonstrated that the Laplace transformation which owes its origin to early work by Laplace [6] on form-invariance and reduction of linear hyperbolic equations has interesting applications in the study of conjugate nets. Particular cases of (8), however, admit further geometric interpretation:

- For $p = 2$, we obtain the classical sine-Gordon equation

$$\theta_{tx} = \sin\theta \tag{9}$$

  descriptive of surfaces of constant negative Gaussian curvature [7].

- A particular reduction of the case $p = 3$ delivers the Tzitzeica-Dodd-Bullough equation

$$\theta_{tx} = \mathrm{e}^{\theta} - \mathrm{e}^{-2\theta}. \tag{10}$$

  This equation was originally derived by Tzitzeica [8] in 1910 in connection with a class of surfaces Σ parameterized in terms of asymptotic coordinates which have the property that their Gaussian curvature is proportional to the fourth power of the distance from the origin to the tangent plane to Σ at a generic point. It has been recovered by Dodd and Bullough [9] in the context of integrability and shown to admit an interpretation in 1+1-dimensional anisentropic gasdynamics by Gaffet [10]. Tzitzeica surfaces associated with soliton and breather solutions of (10) are displayed in Rogers *et al* [11].

3 Reduction to the Classical Darboux System and Sub-Systems

In this section, we show how a systematic reduction of the Loewner system (2) leads to systems of classical differential geometry. The first step in this procedure is to specialize the (adjoint) eigenfunctions Φ and ω to

$$\Phi = \varphi\,(1\ 0), \quad \omega = w\,(1\ 0) \tag{11}$$

where φ and w are column vectors. Then the Loewner representation (1) reduces to a scalar triad of the form [12]

$$\psi_{,i,k} = (\ln H_i)_{,k}\psi_{,i} + (\ln H_k)_{,i}\psi_{,k}, \tag{12}$$

where $_{,i} = \partial/\partial u^i$ with $(u^1, u^2, u^3) = (x+y, x-y, t)$ and ψ denotes the second component of ϕ. Here, $i, k, l \in \{1, 2, 3\}$ take distinct values throughout this note. The compatibility conditions of the above adopt the form

$$H_{l,i,k} = (\ln H_i)_{,k} H_{l,i} + (\ln H_k)_{,i} H_{l,k}. \tag{13}$$

If one now regards three linearly independent solutions $x^\nu(u^i)$ of (12) as a curvilinear coordinate system in three-dimensional Euclidean space then for any given coordinate surface $S_i = \{x^\nu : u^i = \text{const}\}$, the lines of intersection of S_i with any two members of the two other one-parameter families of coordinate surfaces S_k and S_l are conjugate with respect to the second fundamental form of S_i. Hence, under the constraints (11), the Loewner representation (1) encapsulates classical conjugate coordinate systems as discussed by Darboux [13] in 1910.

such as Darboux, Bianchi, Weingarten and Lamé also inquired as to whether the condition of may be consistently supplemented by further geometric constraints on the coordinate surfaces S_i. For instance, the requirements of orthogonality and negative constant Gaussian curvature led to various important systems. Accounts of these are to be found in Eisenhart [7]. A systematic way of obtaining these sub-systems of (13) within the framework of modern Soliton Theory is the following. On introduction of the coefficients

$$X_i = \frac{\psi_{,i}}{H_i}, \quad \beta_{ik} = \frac{H_{k,i}}{H_i} \tag{14}$$

the Darboux system (12)-(13) assumes the equivalent form

$$X_{i,k} = \beta_{ik} X_k, \quad \beta_{ik,l} = \beta_{il}\beta_{lk}. \tag{15}$$

Darboux also introduced a linear system adjoint to $(15)_1$, namely

$$X^*_{i,k} = \beta_{ki} X^*_k \tag{16}$$

which is compatible modulo $(15)_2$. The mixed potential-eigenfunction constraints

$$X^*_i = \mu_i X_{i,i} + \mu_k \beta_{ki} X_k + \mu_l \beta_{li} X_l + \mu H_i \psi \tag{17}$$

then produce the additional equations

$$\mu_i \beta_{ik,i} + \mu_k \beta_{ki,k} + \mu_l \beta_{li}\beta_{lk} + \mu H_i H_k = 0 \tag{18}$$

where μ_i and μ are arbitrary constants. Cross-differentiation shows that (18) is compatible with $(15)_2$. These equations have been studied in detail in Schief [14]. Amongst the systems discussed therein we find

- Riemannian spaces of constant curvature μ given by

$$R_{ik} = \mu g_{ik}, \quad g = H_i^2 (du^i)^2. \tag{19}$$

Since the metric g is diagonal, the coordinate system $x^\nu(u^i)$ is orthogonal. Hence, for vanishing curvature μ the corresponding coordinate surfaces S_i form a system of triply orthogonal surfaces in Euclidean space. The equations (19) were first deduced by the geometer Lamé [7].

- The 2+1-dimensional sine-Gordon equation

$$\theta_{uvt} = \theta_{vt}\theta_u \cot\theta - \theta_{ut}\theta_v \tan\theta \tag{20}$$

as derived by Konopelchenko and Rogers [2] via the identification $\varphi = \theta_t$ in (6). Alternatively, one may give a construction of orthogonal coordinate systems based on this equation. Thus, consider the triad of unit vectors $\boldsymbol{T} = (\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3)$ tangent to the coordinate lines of an orthogonal coordinate system. Then, it is evident that, for instance, the x^3-components of $\boldsymbol{T}$ sweep out a sphere, i.e. $(T_1^3)^2 + (T_2^3)^2 + (T_3^3)^2 = 1$. It was Darboux [a] again, who showed that if one projects this sphere stereographically onto the complex plane labelled by $e^{i\theta}$, θ satisfies the 2+1-dimensional sine-Gordon equation (20). In fact, there is a one-to-one correspondence between the Lamé equations (19) and (20) which in terms of integrability reflects the fact that the latter may be regarded as the eigenfunction equation of the former.

- A generalized Weingarten system [7] descriptive of triply orthogonal systems of surfaces in which one family consists of pseudo-spherical surfaces. Its derivation is based on a superposition of the constraints leading to the first two examples given above.

4 Solution Generation Techniques. An Application to Ernst-Type Sequences

Driven by the squared eigenfunction connection between the multi-component (modified) KP hierarchy and the LKR systems one can show that the latter are amenable to generalized Darboux and Darboux-Levi transformations [16]. Some of these transformations leave invariant the constraints of the type (5) and (11) (see Schief [17] for an explicit example). In this way, one obtains

[a] A discussion and generalization of Darboux's method leading, for instance, to the Konopelchenko-Rogers equations (6) is to be found in Schief [15]

Bäcklund transformations for instance for the 2+1-dimensional sine-Gordon system [18] (6) and the Darboux system (13). Moreover, a Darboux-Levi-type transformation for the latter may be specialized in such a way that the potential-eigenfunction constraints (17) are preserved. This transformation includes classical transformations such as those of Moutard and Ribaucour [14].

Here, by contrast, we focus on a matrix-valued extension of the classical Laplace-Darboux transformation as suggested by Konopelchenko [19]. Thus, it is readily established that the hyperbolic equation

$$\phi_{xt} = V\phi_x + W\phi \tag{21}$$

is form-invariant under the Laplace-Darboux-type transformation

$$\begin{aligned} \phi_- &= \mathcal{L}_-\phi = W^{-1}\phi_x \\ V_- &= W^{-1}VW - W^{-1}W_t \\ W_- &= W + (W^{-1}VW - W^{-1}W_t)_x. \end{aligned} \tag{22}$$

The inverse of $\mathcal{L}_-$ reads

$$\phi_+ = \mathcal{L}_+\phi = (\partial_t - V)\phi, \quad \mathcal{L}_- \circ \mathcal{L}_+ = \mathrm{id}. \tag{23}$$

Now, direct substitution shows that the remaining equations of the LKR representation $(1)_{1,3}$ are equally form-invariant under $\mathcal{L}_-$ with

$$S_- = W^{-1}SW. \tag{24}$$

An alternative proof of this fact is based on the observation that $(22)_1$ may be brought into the form

$$\phi_- = \Phi_x^{-1}\phi_x. \tag{25}$$

Hence, bearing in mind that $(1)_1$ is the scattering problem of the multi-component modified KP hierarchy, it is readily deduced that (25) is a composite gauge and Darboux transformation generated by the eigenfunction Φ [20]. Accordingly, it is seen that the LKR system (2) is form-invariant under the Laplace-Darboux transformation (22) with $(S, V, W) \to (S_-, V_-, W_-)$.

In Schief and Rogers [21], the action of this Darboux-Laplace transformation on the 2×2 Loewner system (2) subject to the compatible constraints

$$S^2 = -\mathbf{1}, \quad \mathrm{Tr}\,(S) = 0 \tag{26}$$

has been investigated. This particular Loewner system represents a 2+1-dimensional extension of the Ernst equation of General Relativity [17]. Thus, in the t-independent case, (2) adopts the form

$$\mathcal{E}_{xx} + \mathcal{E}_{yy} + \frac{\rho_x\mathcal{E}_x + \rho_y\mathcal{E}_y}{\Re(\rho)} = \frac{\mathcal{E}_x^2 + \mathcal{E}_y^2}{\Re(\mathcal{E})}, \quad \rho_{xx} + \rho_{yy} = 0, \tag{27}$$

where the matrices S, V and W have been parameterized according to

$$\begin{gathered} S=\frac{1}{\mathcal{E}+\bar{\mathcal{E}}}\begin{pmatrix} \mathrm{i}(\mathcal{E}-\bar{\mathcal{E}}) & -2\mathcal{E}\bar{\mathcal{E}} \\ 2 & -\mathrm{i}(\mathcal{E}-\bar{\mathcal{E}}) \end{pmatrix} \\ V=\tfrac{1}{2}\varrho S+\tfrac{1}{2}\nu, \quad W=\tfrac{1}{4}(\varrho S+\sigma)_x-\tfrac{1}{4}S(\varrho S+\sigma)_y \\ \rho=\varrho+\mathrm{i}\sigma, \quad \nu_x=\varrho_y, \quad \nu_y=-\varrho_x. \end{gathered} \tag{28}$$

The Ernst equation is retrieved in the case $\Im(\rho)=0$.

As an illustration of the Laplace-Darboux-type transformation $\mathcal{L}_-$ we set $\rho=\varrho-\mathrm{i}n\nu$, where n is an arbitrary constant, which takes (27) to the one-parameter family of Ernst-type equations

$$\mathcal{E}_{z\bar{z}}+\frac{1}{2}(1-n)\frac{\varrho_{\bar{z}}}{\varrho}\mathcal{E}_z+\frac{1}{2}(1+n)\frac{\varrho_z}{\varrho}\mathcal{E}_{\bar{z}}=\frac{\mathcal{E}_z\mathcal{E}_{\bar{z}}}{\Re(\mathcal{E})}, \quad \varrho_{z\bar{z}}=0 \tag{29}$$

with the complex independent variable $z=x+\mathrm{i}y$. In this case, evaluation of the transformation formula (24) for S in terms of $\mathcal{E}$ yields the Laplace transform

$$\mathcal{E}_-=\frac{(1-n)\mathcal{E}\Re(\mathcal{E})\varrho_{\bar{z}}+\varrho\bar{\mathcal{E}}\mathcal{E}_{\bar{z}}}{(1-n)\Re(\mathcal{E})\varrho_{\bar{z}}-\varrho\mathcal{E}_{\bar{z}}}, \quad n_-=n-2, \quad \varrho_-=\varrho. \tag{30}$$

The inverse of this transformation associated with $\mathcal{L}_+$ is readily obtained via the interchange $(\bar{z},n)\leftrightarrow(z,-n)$, i.e.

$$\mathcal{E}_+=\frac{(1+n)\mathcal{E}\Re(\mathcal{E})\varrho_z+\varrho\bar{\mathcal{E}}\mathcal{E}_z}{(1+n)\Re(\mathcal{E})\varrho_z-\varrho\mathcal{E}_z}, \quad n_+=n+2, \quad \varrho_+=\varrho. \tag{31}$$

In particular, it is important to note that the Ernst-type equation (29) may be reduced to the Ernst equation ($n=0$) via a sequence of Laplace transformations $\mathcal{L}_\pm$ iff n is an even integer.

We conclude with the observation that (29) may be linearized via the limit $\mathcal{E}=\hat{\mathcal{E}}+c$, $c\to\infty$, where c is a real constant. We obtain

$$\hat{\mathcal{E}}_{z\bar{z}}+\frac{1}{2}(1-n)\frac{\varrho_{\bar{z}}}{\varrho}\hat{\mathcal{E}}_z+\frac{1}{2}(1+n)\frac{\varrho_z}{\varrho}\hat{\mathcal{E}}_{\bar{z}}=0, \quad \varrho_{z\bar{z}}=0. \tag{32}$$

The same limit applied to the transformation (31) results in the classical Laplace-Darboux transformation

$$\hat{\mathcal{E}}_+=[(1+n)+2\frac{\varrho}{\varrho_z}\partial_z]\hat{\mathcal{E}}, \quad n_+=n+2, \quad \varrho_+=\varrho \tag{33}$$

for (32) up to an irrelevant constant factor. Thus, the transformation (31) for the *nonlinear* Ernst-type equation (29) induced by the Laplace-Darboux transformation $\mathcal{L}_+$ for the *linear* Loewner representation (1) may be regarded as *nonlinear* extension of the classical Laplace-Darboux transformation (33).

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ON THE INTEGRABILITY OF HENON–HEILES TYPE SYSTEMS [a]

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We consider multidimensional extensions of the Hénon–Heiles system obtained as stationary flows from the KdV hierarchy. These systems are shown to have a bi–Hamiltonian structure in an extended phase space and a new integrability structure in their standard phase space. As an example we discuss in detail a Hénon–Heiles type system with four degrees of freedom.

1 Introduction

In the recent years the integrability properties (bi–Hamiltonian structures, Lax pairs, r–matrix) of finite dimensional systems generated as constrained flows of soliton hierarchies has been studied extensively [1,2,3,4,5]. However only very recently a systematic method to find the bi-Hamiltonian formulation of the stationary flows of each order for the KdV hierarchy has been introduced [6]. This method has allowed us to construct integrable multidimensional extensions of the Hénon–Heiles system, different from the ones of [7]; they have a bi–Hamiltonian structure in a suitable extended phase space, as well as the integrable Hénon–Heiles system with two degrees of freedom[2]. However in this framework one is faced with the problem that the bi–Hamiltonian structure cannot be reduced onto the original phase space together with the associated Hamiltonian vector fields. For this reason, as a generalization of a method previously introduced in [8], we have proposed a new integrability criterion holding for a generic finite–dimensional Hamiltonian system [6]. Though weaker than the bi–Hamiltonian scheme, this criterion assures Liouville–integrability of a Hamiltonian system in its standard (i.e. not extended) phase space. It applies both to each stationary flow of the KdV hierarchy and to its first restricted flow (the Garnier system with n degrees of freedom).

The aim of this paper is just to present a Hénon–Heiles type system with four degrees of freedom as an illustration of the previous results.

First of all we give some preliminaries to be used in the following: the aim is mainly to fix notations and terminology. Let M be a n–dimensional manifold. At any point $m \in M$, the tangent and cotangent spaces are denoted by $T_m M$

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and T_m^*M, the pairing between the two spaces by $<,>: T_m^*M \times T_mM \to \mathbf{R}$. For each smooth function $f \in C^\infty(M)$, df denotes the differential of f. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{,\}$: $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$; the associated Poisson tensor P is defined by $\{f,g\}(m) :=< df(m), P_m\, dg(m) >$. So, at each point m, P_m is a linear map $P_m : T_m^*M \to T_mM$, skew–symmetric and with vanishing Schouten bracket [9]. A function $h \in C^\infty(M)$ with a non trivial differential $df \in KerP$ is called a Casimir of P: $P_m\, df(m) = 0$. If $\Phi : M \to M'$ is a differentiable map, Φ_* and Φ^* denote the tangent and the cotangent maps associated to Φ. M is said to be a bi–Hamiltonian manifold [10] if it is endowed with two Poisson tensors P_0 and P_1 such that the associated pencil $P^\lambda := P_1 - \lambda P_0$ be itself a Poisson tensor for any $\lambda \in \mathbf{R}$. A vector field X on M is a bi–Hamiltonian vector field with respect to P_0 and P_1 if $X = P_0\, dh_0 = P_1\, dh_1$ with $h_0, h_1 \in C^\infty(M)$.

2 A new class of integrable Hamiltonian systems

As it is known, the KdV hierarchy is given by the evolution equations $du/dt_j = X_j(u)$, $(j = 0, 1, \ldots)$; $u(x,t)$ is the field function and the bi–Hamiltonian recursion relation is $X_j = P_1\, v_j = P_0\, v_{j+1}$, where the Poisson tensors P_0 and P_1 are $P_0 := \partial_x$, $P_1 := \partial_x^3 + 4u\partial_x + 2u_x$ and v_j are the gradients of Hamiltonian functionals. The first ones are

$$v_0 = 1\ ,\ v_1 = 2u\ ,\ v_2 = 2(u_{xx} + 3u^2)\ ,\ v_3 = 2(u^{(4)} + 5u_x^2 + 10u_{xx}u + 10u^3)\ , \quad (1)$$

$$v_4 = 2\,(35u^4 + 70uu_x^2 + 70u^2u_{xx} + 21u_x^2 + 28u_xu_{xxx} + 14uu^{(4)} + u^{(6)})\ . \quad (2)$$

(Throughout the paper, the subscripts x, xx, xxx and the superscripts $^{(j)}$ denote derivatives with respect to x).

Here we present a new finite–dimensional integrable Hamiltonian system with four degrees of freedom. It can be obtained as the stationary reduction $X_4 = 0$ of the ninth-order KdV flow X_4.

It is known [11] that the stationary flow $X_n(u, u_x, \ldots, u^{(2n+1)}) = 0$ [12] is equivalent to the equation $P^\lambda v(\lambda) = 0$, where P^λ is the KdV Poisson pencil and $v(\lambda) = 1 + \sum_{j=1}^n v_j \lambda^{-j}$ (v_j being the jth gradient of the KdV hierarchy). Developing further this point of view, one can construct a bi–Hamiltonian formulation for each equation $X_n = 0$ [6]. Here we report the new Hamiltonian formulation in the so–called Hénon–Heiles coordinates. For each n, the Hamiltonian function is given by

$$\mathcal{H}_n = \frac{p_n^2}{2} + \frac{q_1 q_n^2}{2} + \frac{a_{2n}}{8q_n^2} + +\Delta_{-n}\left(\frac{p^2(\lambda)}{2} + (q_1 - \frac{\lambda}{4})\frac{q^2(\lambda)}{2} + \frac{a(\lambda)}{8q^2(\lambda)}\right)\ , \quad (3)$$

where $q(\lambda) = 1 + \sum_{j=1}^{n} q_j \lambda^{-j}$, $p(\lambda) = \sum_{j=1}^{n} p_j \lambda^{-j}$, $a(\lambda) = -\lambda + a_n \lambda^{-n} + a_{2n}\lambda^{-2n}$, ($\Delta_k$ means the coefficient of λ^k in a Laurent series, a_n, a_{2n} are free parameters) and $q_j(x)$, $p_j(x)$ are coordinates and momenta respectively. They are related to the function u and to its derivatives through

$$q_k = \Delta_{-k}\left(\sqrt{v(\lambda)}\right), \; p_{n-k} = q_{kx} \; (k = 1, \ldots, n-1); q_n = \sqrt{-v_n}, p_n = q_{nx} \; . \tag{4}$$

The Hamiltonian function H of the new system with four degrees of freedom is obtained taking $n = 4$ in the above general formulation ($H = \mathcal{H}_4$):

$$H = \frac{1}{2}(p_4^2 + 2p_1p_3 + p_2^2) + \frac{3}{4}q_1^5 - \frac{5}{2}q_1^3q_2 + 2q_1q_2^2 + \frac{5}{2}q_1^2q_3 + \frac{q_1q_4^2}{2} - q_2q_3 + \frac{a}{8q_4^2}, \tag{5}$$

where $a := a_8$. The coordinates q_k and the momenta p_k are related to the function u and to their derivatives through (4): $q_1 = u$, $q_2 = u_{xx} + (5/2)u^2$, $q_3 = u^{(4)} + 5u_x^2 + 9u_{xx}u + 5/2u^3$, $q_4 = (-v_4)^{1/2}$; $p_1 = q_{3x}$, $p_2 = q_{2x}$, $p_3 = q_{1x}$, $p_4 = q_{4x}$.

Let us consider the Hamiltonian vector field $Y = E\,dH$, where E is the canonical Poisson matrix in the phase space $S = \mathbf{R^8} - \{q_4 = 0\}$. A set of integrals of motion for Y_1 can be constructed by restricting to the stationary manifold $X_n = 0$ some Gelfand–Dickey (GD) polynomials [12], i.e. the polynomials p_{jk} in u and its derivatives which enjoy the property $\partial_x p_{jk} = v_j X_k$. The GD polynomials p_{14}, p_{24}, p_{34} reduced to the solution manifold of the equation $X_4 = 0$ are given by

$$\begin{aligned}
H_2 &= p_1p_2 + p_2^2q_1 + p_1p_3q_1 + p_4^2q_1 - p_2p_3q_2 - p_3^2q_3 - p_3p_4q_4 + \\
&+ \frac{5}{8}q_1^6 + \frac{5}{4}q_1^4q_2 + q_1^2q_2^2 + q_2^3 + 3q_1q_2q_3 - \frac{1}{2}q_3^2 + \frac{aq_1}{4q_4^2} \; , \\
H_3 &= \frac{1}{2}p_2^2q_1^2 + \frac{1}{2}p_4^2q_1^2 + \frac{1}{2}p_3^2q_2^2 + p_2p_3q_1q_2 + \frac{1}{2}p_3^2q_4^2 - p_3p_4q_1q_4 + \\
&+ 2p_2p_3q_3 + p_4^2q_2 + p_1p_3q_2 + p_1p_2q_1 - p_2p_4q_4 + \frac{1}{2}p_1^2 + \\
&+ \frac{5}{4}q_1^5q_2 - 3q_1^3q_2^2 + \frac{1}{2}q_1^3q_4^2 + \frac{5}{4}q_1^4q_3 + q_1q_2^3 - q_1^2q_2q_3 - \frac{1}{2}q_1q_2q_4^2 + \\
&+ \frac{1}{2}q_3q_4^2 + + q_2^2q_3 + 2q_1q_3^2 + \frac{a}{8q_4^2}(q_1^2 + 2aq_2) \; , \\
H_4 &= -p_2p_4q_1q_4 - p_3p_4q_2q_4 + p_2p_3q_4^2 + p_4^2q_1q_2 + p_4^2q_3 - p_1p_4q_4 +
\end{aligned} \tag{6}$$

$$- \quad \frac{5}{8}q_1^4q_4^2 + \frac{3}{2}q_1^2q_2q_4^2 - \frac{1}{2}q_2^2q_4^2 + -q_1q_3q_4^2 - \frac{1}{8}q_4^4 + \frac{a}{4q_4^2}(q_1q_2 + q_3) \ ,$$

and they are integrals of motion for $Y_1 := Y$, together with $H_1 := H$. Furthermore they can be checked to be independent and in involution, so the vector fields $Y_j := E\,dH_j$, $(j = 1,2,3,4)$ are Liouville–integrable.

An easy way to construct integrals of motion of a given Hamiltonian vector field, automatically in involution, is given by the use of its bi–Hamiltonian structure. In order to construct a second bi–Hamiltonian formulation for the previous vector fields Y one has to embed its phase space S in the manifold $M = \{(q_k, p_k, a)\}$, $(k = 1, \ldots, 4)$, $\Phi : S \hookrightarrow M, (q,p) \mapsto (q,p,a = cost)$, viewing the free parameter a contained in the Hamiltonian functions (6) as an additional coordinate. Hereafter it is convenient to denote by a tilde the tensor fields in M and to make use of block notations. So, for example, we denote with $\tilde{m} = (q,p;a)$ the 9–tuple (q_j, p_j, a), with $\tilde{Y} = [\tilde{Y}^q, \tilde{Y}^p; \tilde{Y}^a]^T$ the generic vector field and with $d\tilde{K} = [\partial\tilde{K}/\partial q, \partial\tilde{K}/\partial p; \partial\tilde{K}/\partial a]^T$ the generic gradient of a function $\tilde{K}$ in M (the superscript T means transposition). In this notation a vector field $\tilde{Y} = \tilde{P}\,d\tilde{K}$ with Hamiltonian function $\tilde{K}$ with respect to a Poisson tensor $\tilde{P}$ will be written

$$\begin{bmatrix} \tilde{Y}^q \\ \tilde{Y}^p \\ \tilde{Y}^a \end{bmatrix} = \begin{bmatrix} P^{qq} & P^{qp} & P^{qa} \\ P^{pq} & P^{pp} & P^{pa} \\ P^{aq} & P^{ap} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial\tilde{K}}{\partial q} \\ \frac{\partial\tilde{K}}{\partial p} \\ \frac{\partial\tilde{K}}{\partial a} \end{bmatrix} . \tag{7}$$

In M we define the vector fields $\tilde{Y}_j$ and the Poisson tensor $\tilde{P}_1$ as natural extensions of the push–forward of the vector fields Y_j and of the canonical Poisson tensor E. So we consider the (trivially) extended Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{Y}_j$, with components $\tilde{Y}_j^q = Y_j^q$, $\tilde{Y}_j^p = Y_j^p$, $\tilde{Y}_j^a = 0$ and the Poisson tensor $\tilde{P}_1$

$$\tilde{P}_1 := \begin{pmatrix} \mathbf{0}_4 & \mathbf{I}_4 & 0 \\ -\mathbf{I}_4 & \mathbf{0}_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{8}$$

$\mathbf{0}_4$ and $\mathbf{I}_4$ being respectively the 4×4 null and identity matrix. The vector fields $\tilde{Y}_j$ have a bi-Hamiltonian formulation [13], inherited from the bi–Hamiltonian structure of the KdV hierarchy, $\tilde{Y}_j = \tilde{P}_1\,d\tilde{H}_j = \tilde{P}_0\,d\tilde{H}_{j+1}$ $(j = 1,2,3,4)$, where $\tilde{H}_5 = -a/8$ and

$$\tilde{P}_0 := \begin{bmatrix} 0 & A & -8Y_4^q \\ -A^T & B & -8Y_4^p \\ 8(Y_4^q)^T & 8(Y_4^p)^T & 0 \end{bmatrix} , \tag{9}$$

$$A = -\frac{1}{q_4}\begin{bmatrix} 0 & 0 & 0 & 1 \\ -q_4 & 0 & 0 & q_1 \\ 0 & -q_4 & 0 & q_2 \\ q_2 & q_1 & 1 & -2q_4^{-1}(q_1q_2+q_3) \end{bmatrix}, \tag{10}$$

$$B = \frac{1}{q_4^2}\begin{bmatrix} 0 & 0 & 0 & -(p_4q_2-p_2q_4) \\ 0 & 0 & 0 & -(p_4q_1-p_3q_4) \\ 0 & 0 & 0 & -p_4 \\ (p_4q_2-p_2q_4) & (p_4q_1-p_3q_4) & p_4 & 0\,, \end{bmatrix}. \tag{11}$$

We remark that the Poisson tensors $\tilde{P}_0$ and $\tilde{P}_1$ are degenerate and have respectively the Hamiltonian functions $\tilde{H}_1$ and $\tilde{H}_5$ as Casimir functions.

In order to have a bi–Hamiltonian formulation also in the original (not extended) phase space for the Henon–Heiles system $dm/dx = Y_1(m)$, one can try to perform a geometrical reduction of this structure following the reduction techniques known from the literature (see for example [14,15]). In particular, two methods can be followed: a *restriction* to the standard phase space or a *projection* onto it.

In the first case, if the restriction submanifold is chosen to be a leaf S_a of the foliation $a = cost$ in M, the Hamiltonians $\tilde{H}_j$, the vector fields $\tilde{Y}_j$ and the Poisson structure $\tilde{P}_1$ can be trivially restricted respectively to H_j, Y_j and E; but it turns out that $\tilde{P}_0$ cannot be restricted. So on S_a one gets four integrable Hamiltonian vector fields but not a bi–Hamiltonian hierarchy.

In the second case, if $\Pi : M \to S, (q,p;a) \mapsto (q,p)$ is the projection map, the two Poisson tensors can be projected onto Poisson structures without Casimirs (symplectic structures), but the Hamiltonians $\tilde{H}_j$ and the vector fields $\tilde{Y}_j$ cannot be projected onto S, because they depend in an essential way on the fiber coordinate. However these two tensors are not invariant along the flow of Y_1, so that they cannot generate a bi–Hamiltonian hierarchy starting with Y_1.

3 A new integrability criterion

In order to bypass the difficulty of constructing a bi–Hamiltonian structure in the standard phase space we introduce a new integrability scheme, weaker than the bi–Hamiltonian one, but living in the standard phase space. We shall define this new scheme for a generic Hamiltonian system with n degrees of freedom; for $n = 2$ it coincides with the one introduced in [8] for the integrable Hénon–Heiles system with two degrees of freedom. As a new example of this integrability structure the Hénon–Heiles type system of the previous Section will be discussed in the following.

Proposition 3.1 *Let M be a $2n$-dimensional Poisson manifold equipped with an invertible Poisson tensor Q_0, and let Z be the Hamiltonian vector field with Hamiltonian h: $Z = Q_0\, dh$. Let there exists a tensor $\mathcal{N} : TM \to TM$ such that the tensor $Q_1 : T^*M \to TM$ defined by $Q_1 := \mathcal{N}Q_0$ be skew-symmetric. Denote by $Z_i := \mathcal{N}^{i-1}Z$ $(i = 1, \ldots)$ the vector fields obtained by the iterated action of the tensor $\mathcal{N}$ on Z and by $\alpha_i := \mathcal{N}^{*^{i-1}}\alpha$ the 1-forms obtained by the iterated action of the adjoint tensor $\mathcal{N}^*$ on $\alpha := dh$*

Let there exist $(n-1)$ independent functions h_i $(i = 2, \ldots, n)$ and $(n(n+1)/2 - 1)$ functions μ_{ij} $(i = 2, \ldots, n; 1 \leq j \leq i)$ with $\mu_{11} = 1$ and $\mu_{ii} \neq 0$ $(i = 2, \ldots, n)$, such that the 1-forms α_i can be written as

$$\alpha_i = \sum_{j=1}^{i} \mu_{ij}\, dh_j \qquad (i = 2, \ldots, n)\ . \tag{12}$$

Under the previous assumptions the following results hold:
i) *the vector fields Z_i satisfy the recursion relations $Z_{i+1} = Q_0\alpha_{i+1} = Q_1\alpha_i$, $(i = 1, \ldots, n-1)$.*
ii) *The functions h_i are in involution with respect to the Poisson bracket defined by Q_0 and they are constants of motion for the fields Z_k .*
iii) *The Hamiltonian system corresponding to the vector field Z_0 is Liouville-integrable. Moreover, if Q_1 is a Poisson tensor, then also Z_1 is an integrable Hamiltonian vector field and the functions h_i are in involution also with respect to the Poisson bracket defined by Q_1.*

Remark We observe that the recursion scheme and the integrability of the vector field Z_1 do not require that the tensor Q_1 be a Poisson tensor; so the manifold M is a Poisson manifold, not a bi-Hamiltonian one.

4 The integrability structure

In this section we construct the integrability structure of the Hénon–Heiles type system of Sect. 2 as an example of the model presented in Prop. 3.1. The first step consists in projecting the two compatible Poisson tensors $\tilde{P}_0$, $\tilde{P}_1$ via the canonical projection $\Pi : M \to S, (q, p; a) \mapsto (q, p)$. The projected tensors are given by

$$P_0 := \Pi_* \tilde{P}_0 \Pi^* = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix}\ , \qquad \Pi_* \tilde{P}_1 \Pi^* = E\ , \tag{13}$$

where A and B are the matrices (10), (11). They give rise to the following Nijenhuis tensor [16]

$$N := EP_0^{-1} = \begin{bmatrix} C & 0 \\ D & C^T \end{bmatrix} , \tag{14}$$

where

$$C = A^{-1} = - \begin{bmatrix} q_1 & -1 & 0 & 0 \\ q_2 & 0 & -1 & 0 \\ 2q_3 & q_2 & q_1 & q_4 \\ q_4 & 0 & 0 & 0 \end{bmatrix} , \tag{15}$$

$$D = -(A^{-1})^T B A^{-1} = \begin{bmatrix} 0 & -p_2 & -p_3 & -p_4 \\ p_2 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 \end{bmatrix} , \tag{16}$$

together with the hierarchy of compatible Poisson tensors $P_k := N^k P_0$, $k \in \mathbf{Z}$. As a matter of fact, one can use N to construct an integrability scheme for Y_1 as in Prop. 3.1. Indeed, let us take

i) $Q_0 = E$, the vector field $Z_1 := Y_1$ with Hamiltonian $h_1 := H$ (5);

ii) the tensor field $\mathcal{N} := N$ (14), and $Q_1 := P_2 = \begin{bmatrix} 0 & C \\ -C^T & D \end{bmatrix}$;

iii) the functions $h_j := H_j$ $(j = 2, 3, 4)$ (6);

iv) the functions μ_{ij} as $\mu_{22} = \mu_{33} = \mu_{44} = 1$, $\mu_{21} = \mu_{32} = \mu_{43} = -2q_1$, $\mu_{31} = \mu_{42} = (3q_1^2 - 2q_2)$, $\mu_{41} = (-4q_1^3 + 6q_1q_2 - 2q_3)$.

Then it is straightforward to check that all the conditions of Prop. 3.1 are satisfied; so, the Hénon Heiles vector field $Y_1 = Z_1$ is Liouville–integrable. Moreover, since in this case the skew—symmetric tensor $Q_1 = P_2$ is Poisson, also the vector field $Z_2 := Q_1 dh_1 = P_2 \, dH_1$ is a new integrable Hamiltonian vector field.

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THE nlBQ-HIERARCHY AS A pq = C REDUCTION OF THE KP-HIERARCHY

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In this paper it is shown that the so called nonlocal Boussinesq equation can be regarded as a $pq = C$ reduction of the KP-hierarchy. The reduction procedure can be carried out explicitly on the N-soliton solutions of the KP-hierarchy once a suitable bilinear form of the nonlocal Boussinesq equation is obtained. It is however necessary to impose an extra constraint on the KP-evolution in order to implement the reduction on the bilinear forms themselves. This special constraint allows for yet another class of solutions.

1 Introduction

An interesting example of a 1+1 dimensional integrable system that is not a straightforward reduction of 2+1 dimensional systems is provided by the so called nonlocal Boussinesq (nlBq)-system[1] :

$$\begin{cases} U_t = V_x \\ V_t = \alpha U_x - U_{3x} - (U^2)_x + \left[\dfrac{V^2 + U_x^2}{U - \alpha/2}\right]_x \end{cases} \tag{1}$$

which can be shown to possess $sech^2$-soliton solutions:

$$U_{sol} = 2\partial_x^2 \ln(1 + \exp\theta) = \frac{1}{2}k^2 sech^2(\frac{\theta}{2}), \tag{2}$$

(with $\theta = -kx + \omega t + \delta; \omega = \pm k\sqrt{\alpha - k^2}$) that are identical to those of the well known "good" Boussinesq (gBq)-equation :

$$\begin{cases} U_t = V_x \\ V_t = \alpha U_x - U_{3x} - 3(U^2)_x. \end{cases} \tag{3}$$

However, the typical 2-soliton solution of the nlBq-system :

$$U_2 = 2\partial_x^2 \ln[1 + \exp(\theta_1) + \exp(\theta_2) + A_{12}\exp(\theta_1 + \theta_2)], \tag{4}$$

with $\theta_i = -k_i x + \omega_i t + \delta_i$, $\omega_i = \pm k_i\sqrt{\alpha - k_i^2}$, $k_i^2 < \alpha$, has a two soliton coupling factor A_{12} :

$$A_{12} = \frac{k_1^2 + k_2^2 - \frac{2}{\alpha}(k_1^2 k_2^2 + \omega_1^{(2)}\omega_2^{(2)})}{(k_1 + k_2)^2} \tag{5}$$

which differs considerably from that found for the gBq-equation, especially with respect to its incapacity to produce resonant behaviour for the corresponding solitons. One is therefore tempted to refer to the nlBq-system as a "resonance-free" alternative to the gBq-equation[1].
The nlBq-system is closely tied to Kaup's higher order water wave equation[2] :

$$w_{2t} - \alpha w_{2x} + w_{4x} + \frac{1}{2}(w_x^2)_t + (w_x w_t + \frac{1}{2}w_x^3)_x = 0. \tag{6}$$

Recently, a direct bilinearization scheme[1] was presented, allowing for a systematic construction of a Hirota bilinear form for this equation (by setting $w = -2i \ln \frac{F}{G}$) :

$$\begin{cases} (iD_t + D_x^2)\ F \cdot G = 0 \\ (iD_x D_t - \alpha D_x + D_x^3)\ F \cdot G = 0. \end{cases} \tag{7}$$

This bilinear system is identical to the bilinear form of the well known "classical" Boussinesq (clBq) equation[3,4] and will therefore be referred to as such. The Hirota D-operators[5] used in these bilinear formulations are defined in the usual manner :

$$D_x^p D_t^q\ F \cdot G = [(\partial_t - \partial_{t'})^q (\partial_x - \partial_{x'})^p F(x,t)G(x',t')]\Big|_{\substack{x' \to x \\ t' \to t}}. \tag{8}$$

Owing to the particular combinatorial properties of the D-operators[6], the bilinear system (7) can be *linearized* by the dependent variable transformation $\Psi = \frac{F}{G}$ and $q = 2 \ln G$, yielding :

$$\begin{cases} i\Psi_t + \Psi_{2x} + q_{2x}\Psi = 0 \\ i\Psi_{xt} + \Psi_{3x} - \alpha\Psi_x + 3q_{2x}\Psi_x + iq_{xt}\Psi = 0. \end{cases} \tag{9}$$

The nlBq-system (1) is then obtained as the compatibility condition of this linear system, expressed in the variables $U = q_{2x}$ and $V = q_{xt}$. Notwithstanding this explicit link between the nlBq-system and the bilinear system (7) (or its linearization (9)), one cannot look upon the system (7) as a suitable bilinearization for the nlBq-system. Nor can it fulfil the role of a (bilinear) Bäcklund transformation for this system, since no spectral parameter can be introduced into it in any nontrivial manner. As it stands - without extra free parameters - the bilinear system simply maps N-soliton solutions to equivalent N-soliton solutions. In section 3 a more suitable bilinear form for the nlBq-system will be presented.
As mentioned earlier the bilinear system (7) can be found in a direct way using the particular combinatorial properties of the Hirota D-operators[1,6]. It was discovered by Hirota that the classical Boussinesq system (7) can be linked to the mKP-hierarchy by a special type of reduction[4,7], most often referred to

as a "$pq = C$" reduction (C a constant). As will be explained in section 2, this type of reduction differs considerably from what is generally understood by "reducing" 2+1 to 1+1 dimensional integrable equations[8]. The main difference lies in the fact that although it provides a way of understanding how the soliton solutions of 1+1 dimensional systems arise from a 2+1 dimensional integrable system, it does not define the actual form of these 1+1 dimensional equations. In this paper it will be shown that once the nlBq-system has been properly bilinearized, its soliton solutions expressed in Wronskian form can be recognized as $pq = C$ reductions of the KP-solitons. Since the reduction can be explicitly performed on the Wronskian level, these results offer an extension of the results previously obtained by Hirota[9]. Special solutions arising from the implementation of the pq reduction on the bilinear forms are also discussed.

2 Reductions of the KP-hierarchy

2.1 ℓ-reductions

Let us consider the first member of the KP-hierarchy in bilinear form[8] :

$$\left[4D_xD_{t_3} - D_x^4 - 3D_{t_2}^2\right]\ \tau\cdot\tau = 0. \tag{10}$$

Demanding that the τ-functions $\tau(x, t_2, t_3, t_4, \cdots)$ appearing in the bilinear form should not depend on a specific time variable t_ℓ ($\frac{\partial\tau}{\partial x_\ell} = 0$), is generally referred to as performing a ℓ-reduction on the equation[8].
This constraint also implies that the τ-functions do not depend on integer multiples of ℓ-labeled time variables, hence it is clear that the reduced τ-functions will satisfy a bilinear equation obtained by setting the corresponding D-operators equal to zero : $D_\ell, D_{2\ell}, D_{3\ell}, \cdots \to 0$. As is well known for example, performing a 2-reduction on the KP-equation (10) gives rise to the KdV-equation in bilinear form[5,8] :

$$\left[4D_xD_{t_3} - D_x^4\right]\ \tau\cdot\tau = 0. \tag{11}$$

The ℓ-reduction can also be implemented on the level of explicit solutions of the KP-equations.
If we take for instance a general N-soliton solution in Wronskian form[10]:

$$\tau_N = W(\varphi_1, \varphi_2, \cdots, \varphi_N) = \begin{vmatrix} \varphi_1 & \varphi_{1,x} & \cdots & \varphi_{1,(N-1)x} \\ \vdots & & \ddots & \vdots \\ \varphi_i & \varphi_{i,x} & \cdots & \varphi_{i,(N-1)x} \\ \vdots & & \ddots & \vdots \\ \varphi_N & \varphi_{N,x} & \cdots & \varphi_{N,(N-1)x} \end{vmatrix}, \tag{12}$$

with:

$$\varphi_j = \exp(p_j x + p_j^2 t_2 + p_j^3 t_3 + \cdots) + \exp(q_j x + q_j^2 t_2 + q_j^3 t_3 + \cdots), \qquad (13)$$

then the N-soliton solutions of the ℓ-reduced equations can be recovered[8,7] by setting $p_i^\ell = q_i^\ell, p_i \neq q_i$ in the exponentials in (13) ; i.e. for the KdV reduction this means setting $p_i = -q_i$.

2.2 pseudo − ℓ-reductions

A second type of reduction one can perform on the KP-equation is a so called pseudo-ℓ-reduction[7], which is designed in such a way that it has a specific effect on the dispersion relations appearing in the exponential entries in the KP-Wronskian solutions (12). Instead of setting $p_i^\ell = q_i^\ell$ in the functions (13), one imposes the constraint $p_i^\ell - q_i^\ell = \beta(p_i - q_i)$, after having performed a suitable rotation in the $(x, t_2, t_3, t_4, \cdots)$ coordinate space.
For example, in the case of a pseudo-2-reduction - restricting ourselves to the variables appearing in (10) - we can perform the following linear transformation in the x, t_2-plane: $x = \frac{x'+\beta t_2'}{2}, t_2 = \frac{x'-\beta t_2'}{2\beta}, t_3 = t_3'$, after which we impose $p_i + q_i = \beta$. This enables us to rewrite the Wronskian (12) by extracting exponential factors from its rows :

$$\begin{aligned}
\varphi_{i,(m)x} &= \exp[\frac{\beta p_i - p_i^2}{2}(t_2' - \frac{x'}{\beta})] \times \\
&\qquad [p_i^m \exp(p_i x' + p_i^3 t_3') + q_i^m \exp(q_i x' + q_i^3 t_3')] \\
&\simeq \exp(p_i x' + p_i^3 t_3') \left[p_i^m + q_i^m \exp\left((q_i - p_i)x' + (q_i^3 - p_i^3)t_3'\right)\right] \qquad (14) \\
&\simeq p_i^m + q_i^m \exp(k_i x' + \frac{k_i^3 + 3\beta^2 k_i}{4} t_3'), \qquad (15)
\end{aligned}$$

(where $k_i = q_i - p_i$). The equivalences in formulas (14) and (15) must be understood as follows : when extracting exponentials in the above way the net result on the entire Wronskian is a multiplicative factor in exponentials with linear arguments in x' and t_2', i.e. a gauge transformation that leaves the bilinear equation invariant. Notice that, due to this invariance, when written in these new variables the Wronskian solution becomes explicitly independent of the t_2' variable. Hence the corresponding D-operator $D_{t_2'}$ can be set equal to zero when one expresses the bilinear equation in the rotated coordinates $(D_x \to D_{x'}, D_{t_2} \to \beta D_{x'})$:

$$\left[4D_{x'}D_{t'_3} - D_{x'}^4 - 3\beta^2 D_{x'}^2\right] \ \tau \cdot \tau = 0. \qquad (16)$$

2.3 $pq = C$ reductions

Similarly to pseudo-ℓ-reductions which are primarily defined on the Wronskian solution level, Hirota[7] introduced a third type of reduction defined with as sole aim the recovery of specific N-soliton solutions from these systems. A typical example of such a reduction is the clBq-system (7). The solutions to this system can be obtained from the N-soliton solutions of the mKP-equation[7] by imposing that $p_i q_i = C$ ($C = -\frac{\alpha}{4}$) on the parameters p_i and q_i appearing in these solutions (up to a scaling $t_m \to (i)^{m-1} t_m$).
Take for example the case of the one soliton solution of the clBq-system :

$$\begin{cases} F = 1 + C \exp\theta \\ G = 1 + \exp\theta, \end{cases} \tag{17}$$

with $\theta = kx \pm k\sqrt{\alpha - k^2}t_2 + (\alpha k - k^3)t_3 + \delta$ and $C = \dfrac{ik^2 - \omega^{(2)}}{-ik^2 + \omega^{(2)}}$. Starting from a function φ_i (13) - playing the role of G in (17) - and extracting an exponential factor as in formula (14), one obtains through the $pq = -\frac{\alpha}{4}$ reduction the following dispersion relation for the t_3 time evolution : $\omega^{(3)} = k^3 - \frac{3\alpha}{4}k$ (instead of the $\omega^{(3)} = \alpha k - k^3$ prescribed for the soliton solutions of clBq). The correct dispersion relation can however be recovered by performing a linear transformation in the (x, t_2, t_3)-plane : $x' = x + \frac{\alpha t_3}{4}, t_2' = it_2, t_3' = -t_3$. Using this transformation, one can reexpress the bilinear form of the mKP-equation into a bilinear system allowing the one soliton solutions (17) :

$$\begin{cases} (D_{t_2} + D_x^2)F \cdot G = 0 \\ (D_x^3 - 4D_{t_3} - 3D_x D_{t_2})F \cdot G = 0 \end{cases} \longrightarrow \begin{cases} (iD_{t_2'} + D_{x'}^2)F \cdot G = 0 \\ (D_{x'}^3 + 4D_{t_3'} - 3iD_{x'}D_{t_2'} - \alpha D_{x'})F \cdot G = 0. \end{cases} \tag{18}$$

The fact that this system differs considerably from the bilinear form of the clBq-equation (7) should not be surprising since up to this point, no actual reduction to a lower dimensional system has been performed : the transformed system (18) is still a 2+1 dimensional soliton system. As can be appreciated in the Hirota papers devoted to this topic[4,7], it is essential to *impose* an extra evolution equation (in this case for the t_2 evolution) whereby reducing the mKP-equation to a 1+1 dimensional soliton system. It must be stressed that this extra constraint cannot be obtained from the mKP-hierarchy through any of the previously cited reduction procedures and one is thus forced to conclude that in contrast to these other types of reductions, the $pq = C$ reduction by itself does not permit to disclose all the links which exist with higher dimensional integrable systems.

3 The nlBq-system as a reduction of the KP-hierarchy

In section 1 it was explained how the nlBq-system is obtained as the compatibility condition for a linear system, itself obtained from the bilinear form of Kaup's water wave equation. The construction of a "canonical" bilinear form[11] for the entire Kaup hierarchy, i.e. of a bilinear system equivalent to the action of the recursion operator (mapping commuting flows onto each), can therefore - in its turn - give rise to a recursion operator formulation for the nlBq-hierarchy : it suffices this canonical bilinear form by the same transformation as was used to obtain formulas (9). The compatibility condition of the resulting linear system will then yield the recursion operator defining the nlBq-hierarchy[12] :

$$\begin{pmatrix} U \\ V \end{pmatrix}_{t_m} = \mathcal{R} \cdot \begin{pmatrix} U \\ V \end{pmatrix}_{t_{m-1}}, \qquad \forall m \geq 2, \quad t_1 = x, \tag{19}$$

where the operator $\mathcal{R}$ is found to be ($\partial_x^{-1} \cdot \partial_x = \partial_x \cdot \partial_x^{-1} = 1$):

$$\mathcal{R} = \begin{pmatrix} \frac{-V}{2U-\alpha} - 2U_x\partial_x^{-1}\frac{V}{(2U-\alpha)^2} & 1 + U_x\partial_x^{-1}\frac{1}{2U-\alpha} \\ \alpha - \partial_x^2 - 2U + \frac{3U_x}{2U-\alpha}\partial_x & \\ + \frac{U_{2x}}{2U-\alpha} - 4\frac{V^2+U_x^2}{(2U-\alpha)^2} - 2V_x\partial_x^{-1}\frac{V}{(2U-\alpha)^2} & \frac{3V}{2U-\alpha} + V_x\partial_x^{-1}\frac{1}{2U-\alpha} \end{pmatrix}. \tag{20}$$

Careful inspection of the first 4 higher order flows in this hierarchy provides the following bilinear formulations of these flows[12],

at weight 4 :

$$\left[4D_xD_{t_3} - \alpha D_x^2 + D_x^4 - 3D_{t_2}^2\right] G \cdot G = 0 \tag{21}$$

at weight 5 :

$$\left[3D_xD_{t_4} - 2D_{t_2}D_{t_3} - \alpha D_xD_{t_2} + D_x^3D_{t_2}\right] G \cdot G = 0 \tag{22}$$

at weight 6 :

$$\begin{aligned} &\left[144D_xD_{t_5} + 20D_x^3D_{t_3} - 80D_{t_3}^2 + 45D_x^2D_{t_2}^2 + D_x^6\right. \\ &\qquad \left. + 4\alpha^2D_x^2 - 5\alpha D_x^4 - 68\alpha D_xD_{t_3}\right] G \cdot G = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &\left[36D_{t_2}D_{t_4} - 32D_{t_3}^2 - 4D_x^3D_{t_3} + D_x^6 + 9D_x^2D_{t_2}^2\right. \\ &\qquad \left. + 4\alpha^2D_x^2 - 5\alpha D_x^4 - 8\alpha D_xD_{t_3}\right] G \cdot G = 0 \end{aligned} \tag{24}$$

and finally :

$$\left[4D_x^3D_{t_3} - \alpha D_x^4 + D_x^6 - 3D_x^2D_{t_2}^2 + 12\alpha(D_xD_{t_3} - D_{t_2}^2)\right] G \cdot G = 0. \tag{25}$$

Notice that the first and last bilinear equations in this list (21 and 25) only depend on the x, t_2 and t_3 variables. Hence, taken together they provide a bilinear formulation of the t_2 and t_3 flows of the nlBq-hierarchy. Indeed[12], substituting $U = 2\partial_x^2 \ln G$ and $V = 2\partial_x \partial_{t_2} \ln G$ into equations (21) and (25) and eliminating the U_{t_3} contributions between them, immediately yields the nlBq-system (1). The t_3-evolution is then obtained from (21).
Using the system (21,25) as a bilinear form of the nlBq-system, it can be shown[13] that the N-soliton solutions (in Wronskian form) of the nlBq-system are in fact $pq = -\frac{\alpha}{4}$ reductions of KP-type N-soliton solutions (12). The constraint $pq = -\frac{\alpha}{4}$ however does not immediately provide the sought after solutions, i.e. with a correct dispersion relation[12] : $\omega^{(m)} = \omega^{(2)}\omega^{(m-1)}/k$, $\omega^{(2)} = \pm k\sqrt{\alpha - k^2}$. As was previously pointed out in the case of the clBq-system, one must correct the obtained dispersionrelations through a specific linear transformation on the coordinates :

$$t_m = (i)^{m-1}(t'_m + \sum_{s=1}^{\infty} \Delta_m^{(m+2s)} t'_{m+2s}) \tag{26}$$

with coefficients Δ defined by the recursion relation ($j \geq 1$):

$$\Delta_{m-2j}^{(m)} \equiv -\left(\frac{\alpha}{4}\right)^j \left[1 - \binom{m-1}{j} - \sum_{s \geq 0, \text{even}}^{2(j-2)} (i)^s \Delta_{m-2-s}^{(m)} \left(-\frac{4}{\alpha}\right)^{1+\frac{s}{2}}\right]. \tag{27}$$

It can be verified that this transformation relates the first 4 bilinear forms of the nlBq-hierarchy (i.e. formulas (21) - (24)) to the 4 lowest weight bilinear equations in the KP-hierarchy[8].
However, as mentioned before (see section 2.3), a pq reduction - merely defined on the level of the solutions - does not fully describe the reduction process from a 2+1 to a 1+1 dimensional evolution equation. The actual reduction to 1+1 dimensions is performed by imposing an extra constraint on the KP equation. The bilinear equation (25) provides such a constraint. Since (25) cannot be obtained from the KP-equations through any of the above manipulations, it is not surprising that it gives rise to solutions of a different nature than the $pq = -\frac{\alpha}{4}$ reduced KP N-soliton solutions we discussed earlier. In the case $\alpha = 0$, the equations in the bilinear system (21,25) we found for the nlBq-system exhibit a special structure :

$$\begin{cases} (4D_x D_{t_3} + D_x^4 - 3D_{t_2}^2) f \cdot f = 0 \\ D_x^2 (4D_x D_{t_3} + D_x^4 - 3D_{t_2}^2) f \cdot f = -4(4D_x D_{t_3} + D_x^4 - 3D_{t_2}^2) f_x \cdot f_x = 0. \end{cases} \tag{28}$$

This peculiar structure automatically ensures the existence of so called symmetric Wronskian solutions[9,14,15] :

$$f = W(\varphi, \varphi_x, \ldots, \varphi_{(N)x}) = \begin{vmatrix} \varphi & \varphi_x & \cdots & \varphi_{(N-1)x} \\ \varphi_x & \varphi_{2x} & \cdots & \varphi_{Nx} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{(N-1)x} & \cdots & \cdots & \varphi_{(2N-2)x} \end{vmatrix} \tag{29}$$

with $\varphi_{t_2} = \pm i\varphi_{2x}$ and $\varphi_{t_3} = -\varphi_{3x}$: the first equation in (28) is automatically satisfied since it is nothing but a scaled version of the KP equation ($t_2 \to it_2, t_3 \to -t_3$) for which the Wronskian solution (29) is just a particular case of more general Wronskian solutions[10] ; the second equation of (28) is also trivially satisfied since the x-derivative of the symmetric Wronskian (29) is again a Wronskian of KP type.
In the same manner as above it can be shown[13] that the clBq-system (7) has N-soliton solutions in Wronskian form that can be obtained through a ($pq = -\frac{\alpha}{4}$)-reduction from Wronskian solutions of the mKP-hierarchy. Similarly it can be easily proven[13] that, in the case $\alpha = 0$, the clBq-system (7)

$$\begin{cases} (iD_{t_2} + D_x^2) f \cdot \tilde{f} = 0 \\ D_x(iD_{t_2} + D_x^3) f \cdot \tilde{f} = 2(iD_{t_2} + D_x^2) f_x \cdot \tilde{f} = 0 \end{cases} \tag{30}$$

also admits symmetric Wronskian solutions[9,14,15] $f = W(\varphi, \varphi_x, \ldots, \varphi_{(N)x})$ and $\tilde{f} = W(\varphi, \varphi_x, \ldots, \varphi_{(N-1)x})$ with $\varphi_{t_2} = \pm i\varphi_{2x}$ and $\varphi_{t_3} = -\varphi_{3x}$.

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Part II
Nonlinear Models. Theory

ORDER AND CHAOS IN SEMICONDUCTOR SUPERLATTICES

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Semiconductor superlattices (SSL's) are artificial, layered materials that have long been known to exhibit striking nonlinear effects in their electron transport properties. We show that, in the presence of an external, time-periodic electric field in the terahertz domain, these nonlinear effects can include both mode-locking and chaotic motions for the electrons. Using a phenomenological "balance equation" approach and accounting properly for the collective excitations of the superlattice electrons, we examine the parameter regions in which chaotic behavior might be observed in experiment and briefly mention similarities to possible the chaotic responses in Josephson junctions.

1 Introduction

More than two decades ago, Esaki and Tsu [1] discovered the first of the striking nonlinear effects in semiconductor superlattices (SSL's) by establishing that the dissipative motion of electrons within a single SSL miniband in the presence of a *static* electric field can produce a *negative differential conductivity* in the stationary current-voltage characteristic. In the past several years, incorporating technological advances in semiconductor nanostructure fabrication and electromagnetic field generation techniques, experiments have observed novel nonlinear phenomena involving electron transport in SSL's in many other situations, including most recently high intensity, high frequency alternating electromagnetic fields (in the terahertz (THz) regime).[2] A number of articles in the semi-popular physics literature provide excellent and rapid access to many of the important issues in this exciting field. [3]

In essence, SSL's consist of alternating layers -- easily as many as 50 of each compound – of two different semiconductors. The individual layers are grown by molecular beam epitaxy (MBE) and range in thickness from 10-100 Å.

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Typical materials are gallium arsenide ($GaAs$) and aluminum arsenide ($AlAs$) because these compounds have nearly equal bond lengths so that the resulting SSL has little strain. Although doping is sometimes used to introduce charge carriers, to avoid the resultant defects in the crystal structure, charge carriers are more commonly introduced by exciting particle-hole pairs with radiation. The basic theory of electrons in semiconductors (see [3] for a brief discussion) implies that in the direction perpendicular to the layers, the electrons "see" a periodic array of potential wells, with a lattice spacing given by the interlayer separation (and thus completely tunable by the MBE growth procedure). Since the motion of an electron in a periodic potential is the fundamental problem of solid state physics, SSL's can usefully be viewed as "artificial solids" with a lattice spacing not determined by intrinsic chemical properties but rather variable over a very wide range. This perspective will be important in all of our later discussion, which briefly summarizes the essential theoretical background and presents our some of new results. [4]

2 Theoretical Formulation

2.1 Minibands, Bloch Oscillations, and Negative Differential Conductivity

For electrons in a periodic lattice, the "dispersion relation" between energy and momentum becomes, in the tight-binding approximation,

$$\varepsilon(k) = \frac{\Delta}{2}(1 - \cos ka) \tag{1}$$

where ε is the energy (which ranges from $-\Delta$ to Δ, so that 2Δ is the band width), k is the Bloch wavenumber, and a is the SSL lattice constant.[4] For an SSL, since a is much larger than the interatomic distance in a normal solid, the coupling between adjacent potential wells is much weaker than in real solids, so that Δ is smaller and the energy bands are narrower: hence they are known as "minibands." Further, the (inverse) *effective mass*, $1/m^\star$ is given by [5]

$$\frac{1}{m^\star(\varepsilon)} = \frac{1}{\hbar^2}\frac{\partial^2\varepsilon(k)}{\partial k^2}, \quad \text{hence} \quad m(\varepsilon) = \frac{m_0}{1 - 2\varepsilon/\Delta}, \quad m_0 \equiv \frac{2\hbar^2}{\Delta a^2}. \tag{2}$$

In the earliest days of solid state physics, Bloch [5] realized that, in the presence of a constant external field, these equations led to an *oscillatory* motion for the electrons. To see how these *Bloch oscillations* arise, consider the *acceleration theorem*, first proved by Bloch [5] using a semiclassical approach: with

$$\hbar\dot{k} = eE, \tag{3}$$

k increases linearly in time for constant E. Since Eq. (1) has period $2\pi/a$ in k, it follows that the energy oscillates with period $T_B = h/eEa$. Physically, what happens to a lattice electron is an *umklapp* process: an electron is accelerated until its wavelength approaches the Brillouin zone boundary, where it is Bragg reflected.

In a normal solid, the energy bands are so wide that electrons are scattered by impurities and phonons before they can reach the top of the band; said another way, the mean time between scatterings, τ, is much shorter than the period of the Bloch oscillations, T_B. In effect, the leading order term of Eq. (1) (*i.e.*, the free electron behavior) dominates. In a superlattice, the minibands are narrow enough that the sinusoidal shape of Eq. (1) and Bloch oscillations are expected to be observable.

Similarly, the narrow bandwidth and consequent strong energy dependence for tunneling between adjacent wells lead to the dramatic nonlinear effect of *negative differential conductivity* first found by Esaki and Tsu [1]. In the present SSL case, an extension of their analysis gives

$$J = \frac{e^2 n E \tau}{1 + (eEa\tau/\hbar)^2} \frac{1}{m^*(0)} \tag{4}$$

so that $\partial J/\partial E < 0$ for $(eaE\tau/\hbar) > 1$. Importantly, both the negative differential conductivity [6] and (possibly, although this remains controversial) the Bloch oscillations [7] have been observed experimentally.

2.2 The Phenomenological Balance Equations for Miniband Transport

Once the dynamics of a single electron has been examined, it is necessary to develop a description of the *collective* electron behavior. We employ a phenomenological "balance equation" approach proposed by Ignatov [8] and by Lei [9], to whose articles interested readers are referred for details.

Our approach generalizes their results by adding the effects of a self-consistent field, E_{sc}, which accounts for collective effects of charge accumulation caused by the drift current. In unscaled form, the balance equations are

$$\dot{V} = -eE_{tot}(t)/m(\varepsilon) - \gamma_v V, \tag{5a}$$

$$\dot{\varepsilon} = -eE_{tot}(t)V - \gamma_\varepsilon(\varepsilon - \varepsilon_0), \tag{5b}$$

$$\dot{E}_{sc} = -4\pi j - \alpha E_{sc}, \tag{5c}$$

where

$$E_{tot}(t) = E_{sc}(t) + E_{ext}(t), \quad \text{and} \quad j = -eNV. \tag{6}$$

where V is the average velocity, ε the average energy of electrons along the SSL axis; ε_0 is the equilibrium energy of carriers; E_{sc} is the self-consistent field; the external applied field is $E_{ext}(t) \equiv E_0 cos(\Omega t)$ unless otherwise specified; N is the number of electrons per unit volume; and γ_v, γ_ε and α are the relaxation constants of the average velocity, energy and the self-consistent field, respectively. [4] To be in the frequency regime comparable to Bloch oscillations in a typical SSL, the external source must produce terahertz radiation; the free-electron laser laboratory at UCSB can, for example, provide a strong field whose frequency is tunable from 120 GHz to 5 THz.[10]

Although the derivation of these equations is well beyond the scope of this article, some motivation for their form is readily given. Eq. (5a) is just the basic equation of motion of electrons belonging to one miniband (with the dispersion law (1)) in the presence of the electric field $E_{tot}(t)$. Eq. (5b) describes the heating (*i.e.*,increase in energy) of the electrons by the field $E_{tot}(t)$ and cooling (*i.e.*, relaxation of energy) to the equilibrium value ε_0. Eq. (5c) is Maxwell's equation for the time evolution of the self-consistent electric field, with an additional term describing its relaxation.

3 Analytic and Numerical Studies

3.1 Limiting Cases

As an important check of the relevance and consistency of the balance equations, we must show that they reproduce the behavior in known limiting cases. This is done in some detail in [4], but interested readers should be able to verify analytically that, if one entirely ignores the self-consistent field (so that one drops (Eq. (5c)), *both* Bloch oscillations (damped, if γ_ε *and* γ_v are not zero, undamped otherwise) and (when damping is included) negative differential conductivity follow from Eqs. (5a) and (5b). With more work (involving numerics), one can show that qualitatively similar results follow even when the self-consistent field effects are included.

3.2 Chaotic Behavior in the Full Equations

To study the balance equations in full generality requires extensive numerical simulations. These simulations, including a description of the various scalings and the methods used, are described in [4]. Here we report only a few of the most significant results.

When the equations are scaled in the proper manner, several insights emerge. First, in addition to the external driving frequency (Ω) and the "Stark frequency" (associated with Bloch oscillations [3] and given by $\omega_s = \frac{eaE_0}{\hbar}$), there

is a third natural frequency, associated with the self-consistent field and given by $\omega_E = \left[\frac{2\pi e^2 N a^2 \Delta}{\hbar^2}\right]^{1/2}$. Since ω_E depends on SSL material parameters, it is natural to hold it fixed and to use the ratios of the other two frequencies to ω_E as parameters to study the system. Second, in different limits, the full equations can be mapped [4] onto (1) the well-known Lorenz equations, famed for their role in the discovery of "butterfly effect" in chaos; and (2) variants of the coupled Maxwell-Bloch equations used to describe bistable optical devices and lasers. Finally, in the case for which $\gamma_\epsilon = \gamma_v = 0$ but $\alpha \neq 0$, our equations reduce *exactly* to those describing the damped, driven Josephson junction, in which chaotic behavior has previously been predicted [11].

To select parameters, we first recall that the scaling suggests that we use ω_E as the unit of (inverse) time, and thus the natural damping parameters that occur in the re-scaled equations are the dimensionless quantities $\gamma_v/\omega_E, \gamma_\epsilon/\omega_E$ and α/ω_E. Since there is considerable uncertainty in the individual values of the phenomenological damping parameters, we have studied [4] a broad range of values of these parameters: $0 \leq \gamma_v/\omega_E = \gamma_\epsilon/\omega_E \leq 0.1$ and $0 \leq \alpha/\omega_E \leq 0.2$. For initial conditions, we take $E(0) = \omega_s, v(0) = 0$, and $w(0) = -1$, corresponding to the initially unexcited SSL just being struck by the incident EM radiation.

The variable most directly related to experimental observables is the average electron velocity, v (the scaled version of V). Accordingly, we will focus on the various different behaviors of v we observe and the regions in which they occur. In essence, we see two types of behavior for v: (1) "regular" motion, in which v oscillates in time periodically, with its Fourier spectrum consisting of (usually two) frequencies commensurate with Ω; and (2) "chaotic" behavior, in which v displays the erratic time-evolution expected for a deterministically chaotic system. The best overview of the qualitative nature – chaotic versus periodic – of the behavior of the system is provided by two-dimensional plot showing, as functions of the two parameters of the external field, ω_s/ω_E and Ω/ω_E, the locations of the regions with *positive* values of the Lyapunov exponent; this is a necessary and sufficient condition for chaos in this system. This sort of "λ topographic plot" – henceforth, "λ-plot" – has been used very effectively in studies of chaos in the damped, driven Josephson junction. [11]

In Fig. 1 we show the "λ-plots" for two sets of damping parameters, the first (Fig. 1a) chosen to coincide with the Josephson junction case (thus providing a non-trivial test of our numerics, see [4,11] for a definition of ρ) and the second (Fig. 1b) illustrating that even for quite strong energy and velocity dissipation, the chaotic regime remains large. Many other cases, including the (dissipationless) Hamiltonian case are presented elsewhere. [4]

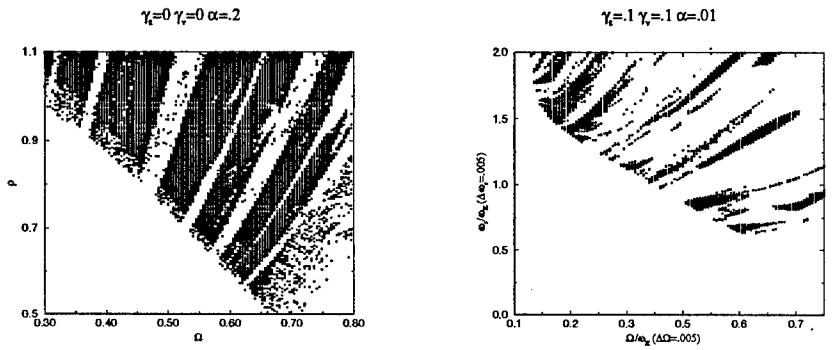

Figure 1: Chaotic (dark symbols) and regular (white areas) regions for parameters corresponding to (a) the damped, driven Josephson Junction; and (b) an SSL with moderate-strength, normal dissipative mechanisms.

4 Conclusions and Open Issues

We have considered the influence of an ac electric field on the motion of ballistic electrons in a miniband of a semiconductor superlattice, establishing within a phenomenological balance equation approach the possibility of chaotic dynamics. Our numerical and analytic results suggest that for a transition to chaos one must satisfy the following conditions: (i) the frequency of the ac field (Ω) should be near the characteristic frequency of the collective electron motion in the SSL (roughly, $0.10 \leq \Omega/\omega_E \leq 1.6$; (ii) also, the Stark frequency $\omega_s = eaE_0/\hbar$, which is determined by the *amplitude* of the external field, can not be too large compared to the collective frequency ($\omega_s \leq 3.0\omega_E$; and (iii) the relaxation rates of the electron's energy and momentum should not be too large ($\gamma_v/\omega_E \simeq \gamma_\epsilon/\omega_E \lesssim 0.1$).

Of the open experimental issues, the two most significant are the extent to which our treatment of dissipation effects – both their magnitude and the assumption that they do not depend on energy – is valid and the precise nature of the experimental indication of chaos. As both these issues involve the details of individual experiments, they are difficult to discuss out of context. Theoretically, the two crucial open issues are the region of validity of the assumption of miniband transport and the relationship between the phenomenological balance equations and more fundamental quantum transport calculations. The former depends on experimental and material parameters and is therefore to some extent controllable, whereas the latter involves more fundamental issues of principle. Given these open issues, it is particularly heartening to note that

a recent calculation, employing widely different assumptions (presumed valid under different experimental/material conditions and involving different simplifications of the underlying quantum transport problem) also predicts that chaotic behavior should be observed in SSL's. [12]

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NONLINEAR WAVES GENERATED BY INSTABILITIES IN PRESENCE OF A CONSERVATION LAW

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The stability of spatially periodic traveling waves, generated by a long-wavelength oscillatory instability, is investigated. One- and two-dimensional models are described and used. In the one-dimensional case the dissipation-modified Korteweg-de Vries equation, derived formerly in a number of physical problems, is used. The stability region and the main instability modes are found. In the two-dimensional case two models are used. No stability region was found in the case of dissipation-modified Kadomtsev-Petviashvili equation. It is shown that stability region is possible for the anisotropic 2-dimensional version of the dissipation-modified Korteweg-de Vries equation.

1 Introduction

The generic one-dimensional amplitude equation describing long dispersive waves near the instability threshold is the *dissipation-modified Korteweg-de Vries equation* (DMKdV) (or *modified Kawahara equation*):

$$U_T + 3(U^2)_X + U_{XXX} + \epsilon \left[U_{XX} + U_{XXXX} + D(U^2)_{XX}\right] = 0. \qquad (1)$$

This equation has been derived in many physical problems, e.g., for oscillatory Marangoni convection [1], instability of a liquid film flowing down a slightly inclined plane [2], Eckhaus instability [3,4], Rayleigh convection [5] and so on.

In two dimensions *for spatially isotropic* systems one obtains the *dissipation-modified Kadomtsev-Petviashvili equation* (DMKP) [6,7]:

$$\frac{\partial}{\partial X}\{U_T + 3(U^2)_X + U_{XXX} + \epsilon[U_{XX} + U_{XXXX} + D(U^2)_{XX}]\} - 3sU_{YY} = 0, \quad (2)$$

$s = \pm 1$. The one-dimensional solutions $U = U(x,t)$ of equation (2) coincide with the ones of equation (1), but their stability properties can be different, because two-dimensional (transverse) disturbances are allowed.

In the *anisotropic case* (e.g., for the Eckhaus instability of one-dimensional waves governed by the Ginzburg-Landau equation [4]), another equation can be derived [8]:

$$U_T + 3(U^2)_X + U_{XXX} + \epsilon[U_{XX} + U_{XXXX} + D(U^2)_{XX} - U_{YY}] = 0, \quad (3)$$

which is a *2-dimensional dissipative modification of the Korteweg-de Vries equation* (2-DMKdV).

In the present paper, we present a brief description of results concerning the stability of waves governed by equations (1-3). The details can be found elsewhere [9,10,8].

2 One-dimensional case

The trivial solution $U = 0$ of DMKdV equation (1) is unstable with respect to disturbances, $u = A\exp[iqx+\sigma T]$, with wavenumbers in the interval $0 < q < 1$. This instability generates a family of nonlinear spatially periodic traveling waves with zero mean value:

$$U(X,T) = u(\xi) = u(\xi + 2\pi/q),\ \xi = X - cT,\ \langle u\rangle = \int_0^{\frac{2\pi}{q}} u(\xi)d\xi = 0. \quad (4)$$

For u we obtain the following nonlinear ordinary differential equation:

$$-cu' + 6uu' + u''' = -\epsilon\left(u'' + u'''' + D(u^2)''\right) \quad (5)$$

where $'$ denotes a differentiation with respect to ξ.

Let us represent the solution in the form of an asymptotic expansion:

$$u = u_0 + \epsilon u_1 + \cdots,\ \ c = c_0 + \epsilon c_1 + \cdots \quad (6)$$

In the leading order, we obtain a family of solutions depending on the parameters c_0, q and ξ_0:

$$u_0 = u_0(\xi - \xi_0; q, c_0) = \frac{c_0}{6} - 2\wp(\xi - \xi_0 + i\omega'; \omega, i\omega') \quad (7)$$

where $\wp(z)$ is Weierstrass elliptic function with periods ω and $i\omega'$ [11]. Let us note that u_0 is an even function of $\xi - \xi_0$.

In the next order in ϵ, we obtain the solvability condition:

$$\langle (u_0')^2(1 + 2Du_0) - (u_0'')^2\rangle = 0 \quad (8)$$

that defines implicitly the dependence of the phase velocity c_0, or one of the wave amplitude, $A = u_{0max} - u_{0min}$, on the wavenumber q [2,9]. Therefore, we obtain a two parameters family $u_0 = u_0(\xi - \xi_0; q)$, $c_0 = c_0(q)$, $A = A(q)$.

If the condition (8) is satisfied, the solution can be written in the form: $u_1 = c_1 \frac{\partial u_0}{\partial q} / \frac{\partial c_0}{\partial q} + \tilde{u}_1 + C u_0'$, where $\tilde{u}_1$ is an odd function of $\xi - \xi_0$. The constant C corresponding to an arbitrary shift of ξ_0 can be chosen as 0. The correction to the phase velocity c_1 is not determined from the equation in the first order in ϵ. In next orders, it was shown [9] that the solvability condition for u_2 is always satisfied if $c_1 = 0$.

Let us consider now the linear stability of stationary solutions (7), (8). We substitute $U = u(\xi) + \tilde{u}$, where $\tilde{u} = \tilde{\Phi}(\xi) \exp[\Omega t]$, into equation (1) and linearize the equation with respect to $\tilde{u}$. The following equation for the disturbance is obtained:

$$\Omega\Phi - c\Phi' + 6(u\Phi)' + \Phi''' + \epsilon(\Phi'' + \Phi'''' + 2D(u\Phi)'') = 0. \tag{9}$$

We assume asymptotic expansions as before:

$$\Phi = \Phi_0 + \epsilon\Phi_1 + \cdots, \quad \Omega = \Omega_0 + \epsilon\Omega_1 + \cdots \tag{10}$$

According to Floquet's theory, the eigenfunctions are in the form: $\Phi_n(\xi + 2\pi/q)/\Phi_n(\xi) = e^{2i\pi\tilde{q}/q}$, $\tilde{q}$ is a quasi-wavenumber.

For $\tilde{q} = O(1)$, the solutions of the problem for Φ_0 were found by Spector [11]:

$$\Phi_0 = \frac{\partial}{\partial\xi}\Psi_0, \quad \Psi_0 = \frac{\sigma^2(\xi + i\omega' + \alpha)}{\sigma^2(\xi + i\omega')\sigma^2(\alpha)} \exp\left[-2(\xi + i\omega')\zeta(\alpha)\right] \tag{11}$$

$$\Omega_0 = -4\wp'(\alpha) \tag{12}$$

$$\tilde{q} = 2i\left[\zeta(\alpha) - \frac{\alpha}{\omega}\zeta(\omega)\right] \tag{13}$$

where σ and ζ are Weierstrass' elliptic functions. Here $Re\alpha$ is equal to $n\omega$ (n is integer), and Ω_0 is an imaginary number.

In the next order, using the solvability condition, Ω_1 is found as a real number:

$$\Omega_1 = -\frac{[(1 + \frac{Dc_0}{3})(\Psi_0', \Psi_0'') + (1 - \frac{D}{3})(\Psi_0', \Psi_0'''') - \frac{D\Omega_0}{3}(\Psi_0', \Psi_0')]}{(\Psi_0', \Psi_0)} \tag{14}$$

In the region $\tilde{q} = O(\epsilon)$, the perturbation theory should be constructed in a different way. It is known that the non-perturbed KdV equation has three acoustic modes [11]. Putting $\tilde{q} = Q\epsilon$, $Q = O(1)$ and performing the asymptotic expansions for the solution of the equation (9), we obtain a cubic dispersion relation determining the growth rate for three resonant modes.

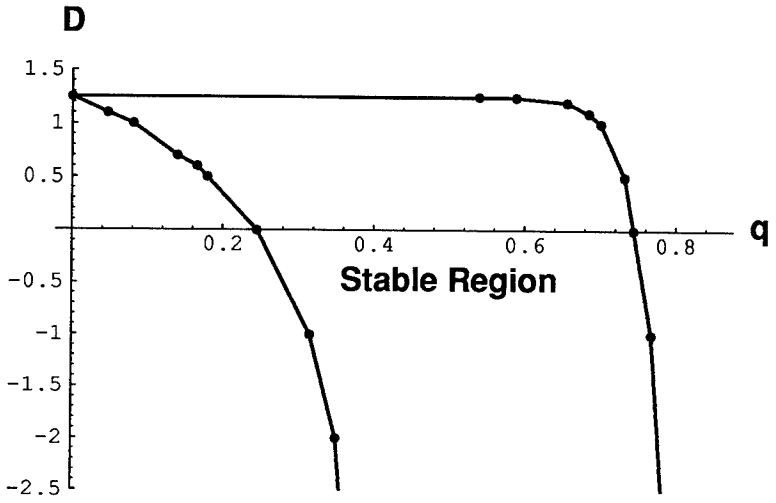

Figure 1: The stable region in the (D, q) plane.

By means of the formula (14), we find the domain in (q, D) plane where $\Omega_1 \leq 0$ for all the possible values of $\tilde{q}$, corresponding to the stability region. These calculations were performed numerically and are presented on the Fig.1.

For $D > D_c$ ($D_c = -6.8 \pm 0.1$) the right boundary has a weakly inhomogeneous translation of the original wave ($\tilde{q} \to 0$). The rest of the right boundary of the stability region ($D < D_c$) corresponds to finite values of $\tilde{q}$. On this boundary, a small-amplitude wave with an incommensurable wavelength starts to grow on the background of the periodic wave. On the left boundary, the unstable disturbance generates a wave with smaller wavelength between the peaks of the original wave. For $D \geq 1.25$ the stability region disappears.

When $D = 0$, the results can be compared with ones of Chang et al.[12] who investigated numerically the stability of periodic waves governed by the Kawahara equation. Our case $\epsilon \ll 1$ corresponds to $\delta \to \infty$ in their notations. The boundaries that we found and the types of the disturbances are close to ones presented in [12].

The case $D \to -\infty$ can be considered analytically. In this case the stable region is $\frac{1}{2} < q < \frac{\sqrt{3}}{2}$. This is seen in Fig.2 for $k_y = 0$.

3 Two-dimensional case

3.1 Isotropic systems

Let us consider now the stability of one-dimensional waves (4), (8) with respect to two-dimensional disturbances in frames of the equation (2).

We obtain the following linear eigenvalue problem

$$\Omega\Phi' - c\Phi'' + 6(u\Phi)'' + \Phi'''' + 3sk^2\Phi + \epsilon[\Phi''' + \Phi''''') + 2D(u\Phi)'''] = 0,$$
$$|\Phi| < \infty \ \ as \ \ |\xi| \to \infty$$

for disturbances $\tilde{u}(\xi, Y, T) = \Phi(\xi)exp(\Omega t + ikY)$ propagating on the background of the periodic waves. The eigenfunctions $\Phi(\xi)$ have the form of Floquet functions.

To demonstrate the instability of the traveling wave solutions, it is sufficient to consider the limit $k \gg 1$.

In this limit, we find the asymptotic expansion for Ω and Φ in powers of k^{-2}:

$$\Omega = k^2\Omega_{-2} + \Omega_0 + \cdots, \ \ \Phi = \Phi_0 + k^2\Phi_2 + \cdots \tag{15}$$

Taking into account that $\langle u \rangle = 0$, we find the growth rate:

$$\Omega = 3isk^2/\tilde{q} + i(\tilde{q}c + \tilde{q}^3) + \epsilon(\tilde{q}^2 - \tilde{q}^4) + O(k^{-2}) + \cdots \tag{16}$$

The instability is found for $0 < |\tilde{q}| < 1$.

The above formula (16) uses no properties of U except $\langle u \rangle = 0$. Hence, U can be actually an arbitrary solution of equation (2) satisfying this condition.

3.2 Anisotropic systems

It is obvious that the stability properties of one-dimensional waves (4),(8) are not changed in frames of the equation (3). In the present subsection, we will consider a more general class of periodic wavy solutions ("oblique waves"):

$$U = u(\xi), \ \ \xi = X + \alpha Y - cT. \tag{17}$$

The trivial solution of the 2-DMKdV equation (3) is unstable with respect to disturbances, $u = Ae^{i(k_x X + k_y Y) + \sigma T}$ inside an egg shape region $k_x^2 - k_x^4 - k_y^2 > 0$, see figure 2. The solutions (17) and the disturbances, $\tilde{u} = \Phi(\xi)e^{ikY + \Omega T}$, are governed by the following equations:

$$-cu' + 3(u^2)' + u''' + \epsilon[(1-\alpha^2)u'' + u'''' + D(u^2)''] = 0 \quad (18)$$

$$\Omega\Phi - c\Phi' + 6(u\Phi)' + \Phi''' + \epsilon[(1-\alpha^2)\Phi'' + \Phi'''' + 2D(u\Phi)'' + k^2\Phi - 2i\alpha k\Phi'] = 0 \quad (19)$$

Here we consider only the analytically tractable limit case $|D| \gg 1$, $D < 0$. Two small parameters appear: $\delta = |D|^{-1/2}$ and $\tilde{\epsilon} = \epsilon|D|^{1/2}$. The stationary traveling wave solution $u = \delta v(\xi)$ is governed by the equation:

$$-cv' + 3\delta(v^2)' + v''' + \tilde{\epsilon}[\delta((1-\alpha^2)v'' + v'''') - (v^2)''] = 0 \quad (20)$$

Assuming that $q = O(1)$ $[v(\xi + 2\pi/q) = v(\xi)]$, we obtain:

$$v = 2A\cos q\xi + \delta\frac{2A^2}{q^2}\cos 2q\xi + \tilde{\epsilon}\frac{4A^2}{3q}\sin 2q\xi + O(\delta^2, \delta\tilde{\epsilon}, \tilde{\epsilon}^2) \quad (21)$$

$$c = -q^2 + O(\delta^2, \delta\tilde{\epsilon}, \tilde{\epsilon}^2) \quad (22)$$

$$A^2 = \frac{1}{6}(q^2(1-\alpha^2) - q^4) \quad (23)$$

For the disturbance we get the following eigenvalue problem:

$$\Omega\Phi - c\Phi' + 6\delta(v\Phi)' + \Phi''' + \tilde{\epsilon}[\delta((1-\alpha^2)\Phi'' + \Phi'''') - 2(v\Phi)'' + \delta(k^2\Phi - 2i\alpha k\Phi')] = 0 \quad (24)$$

Using asymptotic expansions, we get:

$$\Phi = e^{i\tilde{q}\xi} + \delta\frac{2A}{\tilde{q}q}(e^{i(\tilde{q}+q)\xi} - e^{-i(\tilde{q}-q)\xi}) + \tilde{\epsilon}\frac{2A}{3i\tilde{q}q}((\tilde{q}+q)e^{i(\tilde{q}+q)\xi} - (\tilde{q}-q)e^{-i(\tilde{q}-q)\xi}) + \cdots \quad (25)$$

$$\Omega = i(\tilde{q}^3 - \tilde{q}q^2) + \tilde{\epsilon}\delta[-k^2 + \tilde{q}^2(1-\alpha^2) - \tilde{q}^4 - 2\tilde{q}k\alpha - 8A^2] + \cdots \quad (26)$$

From (26), we conclude that a stable region is located inside the region:

$$\frac{1}{4} - \frac{4}{3}q^2(1-\alpha^2) + \frac{4}{3}q^4 < 0 \quad (27)$$

in the $(k_x = q, k_y = \alpha q)$ plane (see figure 2). Let us emphasize, that our analysis is valid only for $q, k, \tilde{q} = O(1)$ and is violated as $q, k, \tilde{q} = O(\epsilon)$. We do not touch here this problem which is physically relevant only in the case of systems of very large size.

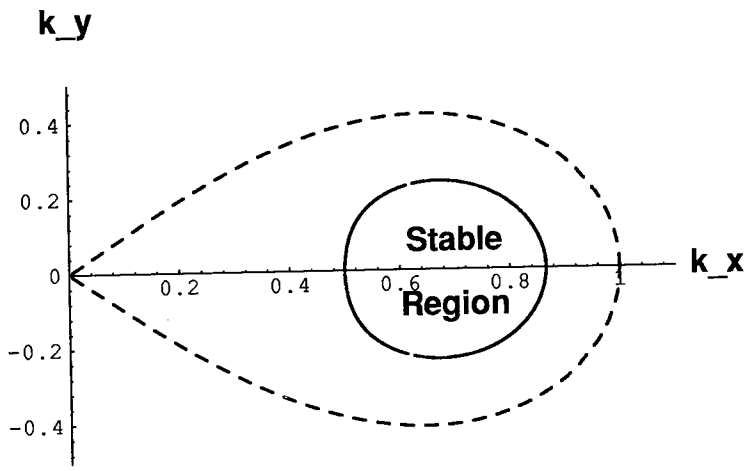

Figure 2: The stable region (in solid line) inside the egg shape region (in dashed line).

4 Conclusions

In the one-dimensional case, as well as for anisotropic two-dimensional systems, we showed that the spatially periodic solutions can be stable in a certain region of wavevectors.

In the region where stable bounded solutions are not found, we can expect that the solutions blow up. This means that the assumptions used for derivation of the weakly nonlinear equations (1),(3) are violated. Depending on the specific features of concrete physical problems, the blow-up instability can lead either to some strongly nonlinear regimes or to the destruction of the system (e.g., to appearance of a "dry spot" in a fluid layer). In the case of modulational instability, the occurrence of a phase slip is the most probable result of such an instability.

In the isotropic two-dimensional case, we found that the instability of smooth solutions with respect to transverse disturbances is an unalienable feature of the DMKP equation.

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CAN PARTICLE-LIKE BREATHING STRUCTURES OCCUR IN SUPERFLUID FILMS?

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We investigate the possibility that the Gross-Pitayevski equation of superfluidity (alias repulsive nonlinear Schrödinger equation on the plane) admits long-lived breather structures similar to the Klein-Gordon pulsons.

1. Recently there has been an upsurge of interest in long-lived breather-like solitons in multidimensions. It has been argued that these structures may play an important rôle in the dynamics of first and second order phase transitions, in particular in cosmological models [1,2,3]. Multidimensional pulsating structures have been known since mid-seventies; they were first observed [4] within the Klein-Gordon equation with the ϕ^4 nonlinearity:

$$\phi_{tt} - \Delta\phi + \lambda\phi(\phi^2 - m^2/\lambda) = 0, \tag{1}$$

with $\phi \underset{r\to\infty}{\longrightarrow} m/\sqrt{\lambda}$. A series of more precise and detailed simulations then revealed the extreme longevity of these particle-like solutions which were subsequently coined "pulsons" [5]. Since then pulsons have been discovered within Klein-Gordon equations with other nonlinearities and a great depth of understanding of their properties has become available [6]. Although there is an extensive literature on the pulsons of the Klein-Gordon equation, no work has been done so far on its nonrelativistic counterpart, the Gross-Pitayevski (alias repulsive nonlinear Schrödinger) equation:

$$i\hbar\Psi_t + \frac{\hbar^2}{2m}\Delta\Psi + \mu\Psi - \lambda|\Psi|^2\Psi = 0. \tag{2}$$

Meanwhile, this equation also has a wide range of important applications, and in the first place the Bose condensate, where pulsons (if exist) could be identified with certain physical excitations [7].

This contribution is a tribute to Professor David Woods, Deputy Vice-Chancellor of the University of Cape Town, to gratefully acknowledge the excellent service he has done to the research community and to express our profound regret at his impending departure from this University.

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It is the aim of this note to explore the existence of breathing solitons in the repulsive NLS equation (2). Here we consider the two-dimensional case ($\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$), where the underlying physical problem is a thin superfluid film. The 3D case will be addressed in future publications.

Before we proceed to the analysis, it is appropriate to mention that the existence of pulsons within eq.(1) does not yet imply that eq.(2) should necessarily exhibit similar structures. One may of course argue that the nonlinear Schrödinger is the nonrelativistic limit of the Klein-Gordon equation. However, this transition uses the assumption that solutions of (1) are small in amplitude, the requirement which is clearly not met in the case of the pulson boundary conditions $\phi(r,t) \underset{r\to\infty}{\longrightarrow} m/\sqrt{\lambda}$.

2. Rescaling the coordinates and field,

$$\Psi = \sqrt{\frac{\mu}{\lambda}}\psi, \quad t = \frac{\mu}{\hbar}\tilde{t}, \quad \mathbf{r} = \frac{\hbar}{\sqrt{2m\mu}}\tilde{\mathbf{r}}, \tag{3}$$

eq. (2) is brought into the dimensionless form

$$i\psi_t + \Delta\psi + (1 - |\psi|^2)\psi = 0, \tag{4}$$

where we have omitted tildes. The condensate boundary condition is now $|\psi|^2 \underset{r\to\infty}{\longrightarrow} 1$. The simplest possible approach is as follows. Take a trial function depending on some set of parameters; assume that the dependence on r is known and parameters are functions of time; substitute into the action functional of eq.(4),

$$S = \int_0^T \int dt d^2r \left\{ \frac{i}{2}(\psi_t\overline{\psi} - \overline{\psi}_t\psi) - |\nabla\psi|^2 - \frac{1}{2}(|\psi|^2 - 1)^2 \right\}; \tag{5}$$

integrate off the spatial dependence and arrive at a finite dimensional dynamical system. This approach is known as the method of collective coordinates, or as the variation-of-action method. (A good exposition is in [8].) Although providing little information on how close the collective coordinate solution is to the real solution, this approach has several obvious advantages. Firstly, it ensures that the resulting finite dimensional evolution is automatically energy-conserving. Secondly, if its results agree with direct numerical simulations, one has an idea of what nonlinear collective modes are responsible for the observed behaviour. We shall confine ourselves to the case of rotationally-symmetric fields, $\psi(\mathbf{r},t) = \psi(r,t)$. To start with, we mention that one cannot simply borrow the trial function of the relativistic problem [3]: $\psi(r,t) = \varphi(r/a)$, with $\varphi(r)$ given function and $a = a(t)$. Such a simple choice would not do as eq. (5)

is linear in ψ_t and $\overline{\psi}_t$, and so the velocity a would enter the resulting effective Lagrangian linearly, as $\dot{a}/a^2$. Since this total derivative can be simply discarded, there will be no dynamics. We have therefore to include a variable phase.

Hence our second guess will be to write $\psi = \rho^{1/2} e^{i\theta}$ with

$$\rho = \rho\left(\frac{r}{a(t)}\right), \qquad \theta = \mu(t) f\left(\frac{r}{a(t)}\right) \tag{6}$$

where $f(r)$ and $\rho(r)$ are some localised functions satisfying $\rho_r(0) = f_r(0) = 0$, and $\rho \to 1$, $f \to 0$ as $r \to \infty$. Substituting into (5) and integrating over r, we arrive at a (linear!) dynamical system with the Lagrangian

$$L = -(\dot{\mu} x I_1 + \mu^2 I_2 + x I_3 + I_0), \tag{7}$$

where we have defined $x = a^2$ and denoted

$$I_0 = \frac{1}{4}\int \frac{\rho_\xi^2}{\rho}\xi d\xi, \quad I_1 = \int \left(f(\xi) + \frac{\xi}{2}\frac{df}{d\xi}\right)\rho(\xi)\xi d\xi,$$
$$I_2 = \int f_\xi^2 \rho(\xi)\xi d\xi, \quad I_3 = \frac{1}{2}\int (\rho(\xi) - 1)^2 \xi d\xi.$$

The Euler-Lagrange equations

$$I_1 \dot{x} = 2 I_2 \mu \tag{8}$$
$$I_1 \dot{\mu} + I_3 = 0 \tag{9}$$

conserve energy,

$$E = I_2 \mu^2 + I_3 x + I_0. \tag{10}$$

Substituting $(I_1/2I_2)\dot{x}$ for μ in eq. (10), we get

$$E = \frac{m}{2}\dot{x}^2 + I_3 x + I_0 \tag{11}$$

where $m = I_1^2/2I_2 > 0$. Consequently we have a motion of a classical particle in a linearly growing potential. This motion is periodic: the particle brakes smoothly at the point $x_{max} = (E - I_0)/I_3$, turns back, bounces off the origin with a finite speed $|\dot{x}| = [4I_2(E - I_0)/I_1^2]^{1/2}$, reaches the turning point x_{max} again and so on. The above collective coordinate description furnishes a characteristic size of the pulson, $a_{max} = \sqrt{x_{max}}$, and its period:

$$a_{max} = \sqrt{\frac{E - I_0}{I_3}}, \quad T = \frac{2I_1}{I_3}\sqrt{\frac{E - I_0}{I_2}}. \tag{12}$$

If a pulsating structure is observed in direct numerical simulations of the partial differential equation, eq. (4), one will be able to compute the energy of this configuration by formula

$$E = \int \left\{ |\nabla\psi|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 \right\} r dr; \tag{13}$$

calculate $I_0, ..., I_3$ with some plausible guess for ρ and f, and eventually evaluate the size and period of the pulson by eqs. (12). A good comparison with the numerically observed size and period would imply that the ansatz (6) captures essentials of the dynamics.

3. How far is the dynamics described by this two-dimensional system from the actual behaviour of the pulson in the partial differential equation (4) (if it exists)? A natural test is to check whether the above collective motions conserve the integrals of the nonlinear Schrödinger equation. Apart from the energy eq.(13), the NLS evolutions conserve the number of particles,

$$N = \int (|\psi|^2 - 1) r dr \tag{14}$$

and the radial component of the linear momentum:

$$P = \frac{i}{2} \int (\overline{\psi}\psi_r - \overline{\psi}_r\psi) r dr. \tag{15}$$

It is straightforward to verify, however, that the ansatz (6) conserves neither N nor P. Indeed, substituting (6) into (14) and (15), one gets

$$N = a^2(t) \int (\rho - 1)\xi d\xi, \quad P = a(t)\mu(t) \int \rho f_\xi \xi d\xi. \tag{16}$$

Can we design a trial function that would conserve integrals? We may try to do so by allowing more degrees of freedom in the finite dimensional dynamics. Take, for instance,

$$\rho(r,t) = 1 - b(t)g\left(\frac{r}{a(t)}\right), \quad \theta(r,t) = \mu(t)f\left(\frac{r}{a(t)}\right), \tag{17}$$

where $g(\xi)$ and $f(\xi)$ are decaying functions of ξ, with $g'(0) = f'(0) = 0$, and $g(\infty) = f(\infty) = 0$. The corresponding Lagrangian is

$$\begin{aligned} L = &-\dot{\mu}a^2(J_1 - bJ_5) + 2a\dot{a}\mu bJ_6 \\ &-a^2b^2J_3 - b^2J_0(b) - \mu^2(J_4 - bJ_2), \end{aligned} \tag{18}$$

where

$$J_1 = \int \left(f(\xi) + \frac{\xi}{2}\frac{df}{d\xi} \right) \xi d\xi, \quad 2J_3 = \int g^2 \xi d\xi,$$
$$2J_6 = -\int f_\xi g \xi^2 d\xi > 0, \quad 4J_0(b) = \int \frac{g_\xi^2}{1 - bg(\xi)} \xi d\xi,$$
$$J_4 = \int f_\xi^2 \xi d\xi, \quad J_2 = \int f_\xi^2 g \xi d\xi, \quad J_5 = \int f g \xi d\xi.$$

Now let us impose the condition that the collective evolutions conserve the number of particles, eq. (14). Substituting (17) into (14) we see that $N = const$ if $a^2 b = const$. Without loss of generality we may identify

$$b = \frac{1}{a^2}$$

and the Lagrangian (18) becomes

$$L = -\dot{\mu} x J_1 + \dot{x} x^{-1} \mu J_6$$
$$-x^{-1} J_3 - x^{-2} J_0(x^{-1}) - \mu^2 (J_4 - x^{-1} J_2), \tag{19}$$

where we have introduced $x = a^2$.

The Lagrangian (19) is linear in velocities and so the conserved energy does not depend on $\dot{x}$ or $\dot{\mu}$:

$$E = \mu^2 (J_4 - x^{-1} J_2) + V(x) \tag{20}$$

where

$$V(x) = J_0(x^{-1})/x^2 + J_3/x. \tag{21}$$

One of the two equations of motion,

$$\dot{x}(J_1 + J_6/x) = 2\mu(J_4 - J_2/x),$$

can be used to eliminate μ from the Hamiltonian (20) which becomes a function of x and $\dot{x}$ only:

$$E = \frac{m(x)}{2}\dot{x}^2 + V(x). \tag{22}$$

Here m is the variable mass of fictitous particle which moves in the potential $V(x)$:

$$m(x) = \frac{1}{2}\left(\frac{xJ_1 + J_6}{xJ_4 - J_2} \right)^2 (J_4 - \frac{J_2}{x}). \tag{23}$$

Notice that the fact that $\rho = 1 - g(r/a)/a^2$ is positive, implies that a cannot take values smaller than a_{min}: $a^2 > a_{min}^2 = g(0)$. Accordingly,

$$xJ_4 - J_2 > \int [g(0) - g(\xi)] f_\xi^2 \xi d\xi > 0$$

and the denominator in (23) can never be zero. The mass is bounded and nowhere vanishing; more precisely,

$$0 < \frac{1}{2}\frac{J_1^2}{J_4} < m(x) < \frac{1}{2g(0)}\frac{[J_6 + J_1 g(0)]^2}{J_4 g(0) - J_2}.$$

The potential $V(x)$ is equal to plus infinity for $x < x_{min} = g(0)$, takes certain finite value $V_0 > 0$ at $x = x_{min}$ and then monotonously decreases to zero as $x \to \infty$. There can be of course no periodic motions in such a potential. If the particle with energy $E < V_0$ approaches the origin from large values of x, it will brake smoothly at the point x where $E = V(x)$, then reverse and escape to infinity. If $E > V_0$, the particle will bounce off the infinitely high wall at $x = x_{min}$ and also escape to infinity. The motion is unbounded.

4. Thus it may seem that the number of particles conservation prohibits the existence of pulson solutions. We will show, however, that this is a consequence of an unrealistic ansatz rather than a fundamental exclusion principle. In the case of asymptotically nonvanishing fields this difficulty can be easily circumvented.

Consider a configuration $\psi_0(r,t)$ which does not conserve the integral (14):

$$\int \{|\psi_0(r,t)|^2 - 1\} r dr = N_0(t) = \tilde{N}_0 + n_0(t),$$

where we have decomposed N_0 into a constant and variable parts. Suppose we add to ψ_0 a "small" function $\delta\psi(r,t)$ such that $\int |\delta\psi|^2 r dr = \epsilon^2 \ll 1$. The energy and action functionals, eqs.(13) and (5) will change by $O(\epsilon^2)$ whereas the integral N will become

$$N = \int \{|\psi_0 + \delta\psi|^2 - 1\} r dr = \tilde{N}_0 + n_0(t) + N_1(t) + \epsilon^2$$

where $N_1 = \int (\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) r dr$.
Can one choose $\delta\psi$ in such a way that the variation of $N_0(t)$ is compensated by $N_1(t)$ and so $N(t) = \tilde{N}_0$, a constant? In the case of fields vanishing at infinity, $\psi_0 \underset{r\to\infty}{\rightrightarrows} 0$, this would contradict the continuity of the functional $N_1[\delta\psi]$. Indeed, by the Schwartz inequality,

$$\left| \int (\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) r dr \right| \leq 2\epsilon \left(\int |\psi_0|^2 r dr \right)^{1/2} \qquad (24)$$

and so if $\delta\psi$ is small, N_1 will also be small and insufficient to compensate changes in $N_0(t)$. However if $|\psi_0| \to 1$ as $r \to \infty$, the integral $\int |\psi_0|^2 r dr$ diverges, the Schwartz inequality is inapplicable and N_1 need not to be small anymore. In fact, it can be rigorously proved that $N_1 = N_1[\delta\psi, \delta\overline{\psi}]$ is a *discontinuous* functional in this case [9] and so it may take arbitrary large values even if $\|\delta\psi\|$ is small. In particular the "small" function $\delta\psi$ can always be chosen in such a way that $N_1[\delta\psi, \delta\overline{\psi}] = -n_0(t)$ and therefore the number of particles in the configuration $\psi_0(r,t) + \delta\psi(r,t)$ is constant, $N = \tilde{N}_0$.

This can be understood in very simple terms. If $\delta\psi$ decays slowly enough as $r \to \infty$, the integral N_1 does not have to be small even if $\int |\delta\psi|^2 d\mathbf{r}$ is small. For example if $\delta\psi \sim \epsilon/r^2$, $\int |\delta\psi|^2 d\mathbf{r}$ is $\sim \epsilon^2$ whereas $\int (\psi_0 \delta\overline{\psi} + \overline{\psi}_0 \delta\psi) d\mathbf{r}$ diverges. Thus we can always "correct" the field configuration [for instance, the one in eq.(6)] by adding a "long tail of small amplitude" such that on one hand, the action of the resulting configuration will be as close to $S[\psi_0]$ as desired, and on the other hand, $\psi_0 + \delta\psi$ will conserve the number of particles. This implies that the fact that our pulson configuration eq.(6) does not conserve the number of particles, does not mean that this cofiguration is far from the actual solution of the Gross-Pitayevski equation (4). In a similar way we can ensure that the momentum eq. (15) is conserved. Adding $\delta\psi$ to ψ_0, the momentum $P_0(t) = (i/2) \int (\overline{\psi_0}\psi_0' - \psi_0\overline{\psi_0'}) d\mathbf{r}$ receives an increment

$$P_1(t) = i \int \left(\frac{\partial \psi_0}{\partial r} \delta\overline{\psi} - c.c. \right) r dr + \frac{i}{2} \int (\psi_0 \delta\overline{\psi} - \overline{\psi_0} \delta\psi) dr.$$

The second integral in the right-hand side is a discontinuous functional which can take arbitrarily large values. Finally, there is yet another way of accomplishing the conservation of N and P. One can simply take the ansatz (6) with $\rho(r)$ satisfying $\int (\rho - 1) r dr = 0$, and $f(r)$ such that $\int \rho f_r r dr = 0$. In view of eq.(16) this ansatz will conserve both momentum and number of particles. It is worthwhile to note that the appropriate $\rho(r)$ will have to be *nodal*, i.e. $(\rho - 1)$ will have to change sign.

5. Thus, the analysis of the conservation laws does not rule out the existence of pulsons. It does indicate, however, that the hypothetic pulson would have to be nodal and/or have a long radiation "tail". This observation is in agreement with the fact that the spectrum of linear oscillations about the homogeneous solution $\psi = 1$ is gapless: $\omega^2 = k^2(2 + k^2)$, and so any localised structure oscillating with the frequency ω will excite radiation waves with the wavenumber k. It is natural to expect, therefore, that the nonrelativistic pulson (if exists) shall have a much shorter lifetime than its relativistic counterpart.

The ultimate answer to whether long-lived pulsating structures occur in the Gross-Pitayevski equation can only be given by the direct numerical simulation.

This work is in progress.

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SOLITON ANALOGS OF ABRIKOSOV-NIELSEN-OLESEN VORTICES

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We investigate Lorentz-invariant models comprising 3-component unit isovector field with easy-axis anisotropy (A3-field) interacting with vector (Maxwell and Chern-Simons) fields. The vector fields acquire masses through interaction with the A3-field governed by $U(1) \times Z(2)$ symmetric Lagrangian. We found 2D stationary solitons with unit topological charge of the A3-field (they break $Z(2)$ symmetry) and compare them with Abrikosov-Nielsen-Olesen vortices.

1. Conception of spontaneous symmetry breaking of the Lagrangian is one of the most fruitful ideas of the contemporary theoretical physics. The mechanism of such breaking within the model describing the Higgs scalar field $\varphi_a(x)$ with selfinteraction of the form $V(\varphi) = A^2(\varphi_b\varphi_b - B^2)^2, a, b = 1, 2, ...N$ was studied by Goldstone [1], and in [2] interaction of the complex Higgs field with the gauge Maxwell field was considered (abelian-Higgs (AH) model). The well-known Ginzburg-Landau model of superconductivity [3] can be considered as a nonrelativistic analog of the AH model. Extended solutions to these models in (2+1) dimensions (namely, Nielsen-Olesen strings [4] and Abrikosov vortices [5], respectively), found in stationary ($\partial/\partial t = 0$) case are identical. These solutions (we shall name them ANO strings for brevity) and their numerous analogs, which describe localized energy distributions, are widely discussed in condensed matter physics, cosmology, particle physics [6,7]. Notice however that ANO strings cannot be referred to as solitons because neither Higgs field nor Maxwell one do not attain unique asymptotic value at $|\mathbf{x}| = \infty$.

2. In the present paper we study $2D$ soliton solutions to Lorentz-invariant models supporting non-Goldstone mechanism of symmetry breaking. Consider a unit isovector field $s_a(x)$ having Lagrangian density

$$\mathcal{L} = \partial_\mu s_a \partial^\mu s_a - V(\mathbf{s}), \quad s_a s_a = 1, \quad V(\mathbf{s}) = 1 - s_3^2,$$

$$\mu = 0, 1, ..., D, \qquad a = 1, 2, 3. \tag{1}$$

(we shall call it the A3-field). It is easily seen that the Lagrangian (1) possesses $U(1) \times Z(2)$ internal symmetry; its vacuum manifold comprises two points on the S^2 sphere: $s_3 = 1$ and $s_3 = -1$ and possesses discrete $Z(2)$ symmetry.

At $D = 1$ the model (1), which we conjecture to be completely integrable generalization of the sine-Gordon equation [8], possesses kink and antikink solutions, which break $Z(2)$ symmetry of the vacuum manifold. Futhermore, it can be shown that $Z(2)$ symmetry is also broken on nonstationary topological solitons of the model (1) at $D = 2$.

Lagrangian (1) can be derived when describing in continuous approximation easy-axis Heisenberg antiferromagnets [9] and ferroelectrics [10] with easy-axis anisotropy; thus, the pattern of the symmetry breaking under discussion is realized in condensed matter physics. Hopefully, the investigation of solitons in $(D+1)$-dimensional field theory models, comprising the A3-field and/or its generalizations will give advantageous results in particle physics as well, in particular, in electroweak interaction theory.

3. Consider $2D$ Lorentz-invariant model which describes "minimal" interaction of the A3-field with vector Maxwell field $A_\mu(x)$ ("MA3 model"):

$$\mathcal{L} = (\bar{\mathcal{D}}_\mu \bar{S}^b)(\mathcal{D}^\mu S^b) - V(\mathbf{S}) - \frac{1}{4}F_{\mu\nu}^2, \tag{2}$$

$$\bar{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - ieA_\mu,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(\mathbf{S}) = \beta[1 - s_3^2],$$

$$\bar{S}^b = (s_1 - is_2, s_3), \quad S^b = (s_1 + is_2, s_3), \quad b = 1, 2,$$

$$\bar{S}^b S^b = s_1^2 + s_2^2 + s_3^2 = 1,$$

where β, e are coupling constants.

Lagrangian (2) is easily transformed to the form

$$\mathcal{L} = \partial_\mu s_a \partial^\mu s_a - V(s_a) - \frac{1}{4}F_{\mu\nu}^2 + 2eA_\mu(s_2\partial^\mu s_1 - s_1\partial^\mu s_2) + e^2 A_\mu A^\mu. \tag{3}$$

Note that due to the interaction with the A3-field described by (2), the vector field becomes massive, and this is the exact result contrary to the case of the Higgs model in which mass of the vector field A_μ is obtained when expanding the Lagrangian density of the model in series in the vicinity of its vacuum manifold.

We begin studying the localized solutions of the (2+1)-dimensional model (3) in the simplest stationary case. Use the hedgehog ansatz for the A3-field

$$s_1 = \frac{x}{R}\sin\theta(R), \quad s_2 = \frac{y}{R}\sin\theta(R), \quad s_3 = \cos\theta(R), \quad R^2 = x^2 + y^2, \tag{4}$$

and look for the vector field solution in the form

$$A_0 = 0, \quad A_1 = A_x = -A_t(R)\frac{y}{R}, \quad A_2 = A_y = A_t(R)\frac{x}{R}. \tag{5}$$

Introducing

$$a(R) = A_t(R)R, \tag{6}$$

and then going over to variables $\alpha(r), r$, given by

$$a = \alpha e^{-1}, \quad R = re^{-1}, \tag{7}$$

we get for stationary Hamiltonian density $\mathcal{H}(r)$:

$$e^{-2}\mathcal{H}(r) = \left(\frac{d\theta}{dr}\right)^2 + \sin^2\theta\left(p + \frac{1}{r^2} - \frac{2\alpha}{r^2}\right) + \frac{1}{2}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)^2 + \frac{\alpha^2}{r^2}, \tag{8}$$

$$p = \frac{\beta}{e^2}. \tag{9}$$

Calculating $\delta\mathcal{H}/\delta\theta$ and $\delta\mathcal{H}/\delta\alpha$ and setting them equal to zero, we get the set of equations for $\theta(r)$ and $\alpha(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \sin\theta\cos\theta\left(\frac{2\alpha - 1}{r^2} - p\right) = 0, \tag{10}$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - 2\alpha + 2\sin^2\theta = 0. \tag{11}$$

We shall look for solutions of Eqs. (10),(11) under the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0, \tag{12}$$

$$\alpha(0) = 0, \quad \alpha(\infty) = 0. \tag{13}$$

Table 1: Soliton energy E vs p

p	E
0.03	21.6437
0.10	23.8876
0.20	24.8351
0.26	25.0589
0.30	25.1328
0.33	25.1479
0.40	25.1600
0.50	25.1833
1.00	25.2449

Notice that Eqs.(4),(12) define the class of mappings $R^2_{comp} \to S^2$, such that $Q_t = 1$, where Q_t is the topological index ("winding number") of localized distributions $s_a(x)$, described by the mappings of this class.

Taking into account boundary conditions at $r = 0$ and expanding Eqs. (10), (11) into series at $r \to 0$, we find

$$\theta(r) = \pi - (C_1 + C_2)r + o(r), \tag{14}$$

$$\alpha(r) = r^2(C_1^2 - \frac{1}{4}C_2^2 r^2) + o(r^4). \tag{15}$$

Choosing appropriate C_1 and C_2 values we find such solutions to Eqs. (10), (11), which satisfy boundary conditions $\theta(\infty) = 0, \alpha(\infty) = 0$ ("shooting method"). Thus we get solution of the boundary value problem (10)-(13).

Numerical studies of these solutions with $Q_t = 1$ have been accomplished for various values of the dimensionless parameter p, given by (9); the most detailed computations have been made on the interval $0 < p < 2$. Soliton solutions are plotted in Fig. 1 for various p. It can be seen that the characteristic width of the soliton and maximum values of $\alpha(r)$ and $A_t(R)$ functions decrease with the growth of p.. Energy density $\mathcal{H}(r)$ for $Q_t = 1$ solitons has a peak at $r = 0$ and monotonously decreases with the increase of r. The dependence of the soliton energy $E = 2\pi \int \mathcal{H}(r) r dr$ on p value is presented in Table 1; note that at $p \approx 0.3$ soliton energy $E = E_0 = 8\pi$ (E_0 is the energy value of Belavin-Polyakov localized solutions in $D = 2$ isotropic Heisenberg magnet [11]).

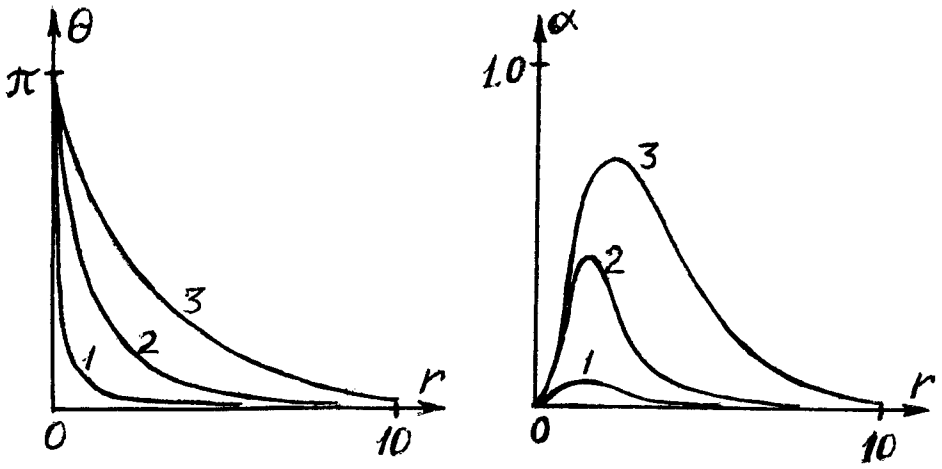

Figure 1: Radial functions $\theta(r)$ and $\alpha(r)$ of solitons with $Q_t = 1$ in the MA3 model; $1 - -p = 0.3$, $2 - -p = 0.1$, $3 - -p = 0.03$.

4. Compare soliton solutions found above in the $D = 2$ MA3 model with ANO strings in the AH model. Both solutions represent field energy lumps, which are exponentially localized in space, both solutions describe distributions of the scalar (with respect to Lorentz transformations) field (A3-field or Higgs one) with nonzero topological indices.

Nevertheless, there exist essential distinctions between string-like MA3 solitons and ANO strings, namely:

1) the A3-field and the vector field constituting MA3 solitons approach unique asymptotic values at $\mathbf{x} \to \infty$, which are independent of the direction of $\mathbf{x}$, namely $s_a(\infty) = (0, 0, 1), \mathbf{A}(\infty) = 0$ (see (12b),(13b)). The latter equality means that

2) the magnetic flux is equal to zero for MA3 solitons, $\int \mathbf{BdS} = 0$,

3) it is easily seen that magnetic field in $D = 2$ MA3 solitons, $B(r) = -(d\alpha/dr)/r$, changes its sign when r increasing.

Statements 1),2),3) are not valid for ANO strings [4-6].

5. Consider another model, which describe interaction of the A3-field with the Chern-Simons (CS) field ("CSA3 model"). The Lagrangian of the CSA3 model is defined by equations, which are obtained from (2),(3) when the Maxwell term $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2$ is replaced in them by the CS term, $\mathcal{L}_{CS} =$

$\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$, $\mu,\nu,\lambda = 0,1,2$. An essential difference between this CSA3 model and the CP^1CS model introduced in [12] and investigated numerically in [13], is the anisotropy of the A3-field, possessing $U(1) \times Z(2)$ symmetry. Adding of an anisotropic potential term to CP^1CS model changes crucially its properties making emergence of exponentially localized solutions possible [14]. The vacuum manifold of the "extended version" of the CP^1CS model considered in [14] comprises the only point, whereas the vacuum manifold of our CSA3 model comprises two points, $s_3 = +1$ and $s_3 = -1$, and possesses Z(2) symmetry.

To find soliton solutions of the CSA3 model describing localized distributions of the A3-field with $Q_t = 1$, we use once again the ansatz given by Eqs. (4),(5). By using variables (4)-(7), we get the following Eqs. for $\theta(r)$ and $\alpha(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \sin\theta\cos\theta\left(\frac{2\alpha-1}{r^2} - p\right) = 0, \tag{16}$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - \alpha + \sin^2\theta = 0. \tag{17}$$

Making scaling transformation $R = r/\sqrt{2}$, we get the set of equations (10),(11) with p replaced by $2p$, thus the solitons in the CSA3 model can be easily obtained from solitons found above within the MA3 model.

Note that the same stationary solitons can be found in the nonrelativistic analogs of the MA3 and CSA3 models, in which the A3-field is replaced by the field of the easy-axis Heisenberg ferromagnet, described by the Landau-Lifshitz equation.

6. In conclusion we considered interaction of the chiral A3-field (3-component unit isovector field having easy-axis anisotropy) with the vector fields (Maxwell and Chern-Simons ones) in (2+1)-dimensional space-time and found soliton solutions with unit topological charge of the A3-field within these models. Both the A3-field and the vector fields approach unique asymptotic values at $|\mathbf{x}| \to \infty$. These localized solutions are characterized by zero magnetic flux.

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STATISTICS OF THE PARTICLES AND QUANTUM GROUPS

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The relevance of the coalgebra, as a useful and required element of the algebraic approach to physics is stressed. The coalgebra of the Heisenberg algebra $h(1)$ is shown to impose a binomial statistics as well as its quantum deformation $h_q(1)$ where the deformation parameter is a measure of temperature. A new crazy statistics is introduced by the superalgebras $osp(1|2)$ and $osp_q(1|2)$.

1 Composed Systems from Hopf Algebras

This contribution has been elaborated in cooperation with M. Rasetti and G. Vitiello [1] and it is in a line of research intended to extend the exploiting of the immense power of algebras in physics. It could look quaint that there is still room for this but two directions are now under study: quantum algebras and full exploiting of related coalgebras. The recent appear in mathematical physics of quantum algebras [2] has given us a new series of tools to work with that can describe from one side an apparent breaking of symmetry controlled by a parameter and from other sides seems related to discrete symmetries (as continuous ones are to Lie algebras). The almost unexplored opportunity given by quantum algebras with $|q| = 1$ (where the properties are completely different from the related Lie algebra) must also to be taken into account. Quantum algebras have also pointed out (to us if, perhaps, not to all physicists) that some of properties of Lie algebras we saw in textbooks are more relevant and deeper related to physics we are used to think. Indeed, if we do not consider the fascinating but fruitless idea of bootstrap, modern physics always works on hierarchic rules: we assume that fundamental objects are simple and that complexity results from putting together simple objects. In other words, elementary systems are simple, while, by definition, non-elementary systems can always be split in subsystems with week or no interaction among themselves. A central point in this approach is the necessity to relate global observables to observables acting on subsystems that, because have the same physical meaning, must close the same algebra: as discussed in the following, this simply means that we need the coalgebra of observables.

We confine ourselves here to implications related to Fock space. From the correct taking into account of coalgebra, it results that the Heisenberg

algebra $h(1)$ and its deformation $h_q(1)$ give the binomial distribution at $T = \infty$ and at temperature finite respectively, while the superalgebras $osp(1|2)$ [4] and $osp_q(1|2)$ [5] are new statistics.

2 The $h(1)$ Algebra and its Quantum Extension

The $h(1)$ algebra is well known. In the textbooks its is usually assumed that its generators are three only: $(a, a^\dagger, E)$, respectively the annihilation and creation operators and the identity while N, the occupation number, is a derived quantity ($N \equiv a^\dagger a$). Because a more accurate mathematical analysis, just related to the coalgebra, shows that this is inconsistent with the physical request of additive occupation numbers, we shall instead consider N not as belonging to the universal enveloping algebra but as an independent generator, the relation with a and $a^\dagger$ being obtained not at algebraic level but on the Fock representation only. We shall start, in this way, from a four generators algebra including N among the fundamentals objects. Of course the assumed commutation relations are the standard ones:

$$[a, a^\dagger] = E, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [E, \bullet] = 0. \tag{1}$$

We are dealing, in such a way, with a contraction of $u(2)$, while the three generators Heisenberg algebra is a contraction of $su(2)$.

The invariant of the algebra is $\mathcal{C} = EN - a^\dagger a$. 1-mode Fock space without Pauli exclusion principle (where $N \simeq a^\dagger a$) is nothing else that the unitary irreducible representation of this algebra with $\mathcal{C} = 0$. Its highest weight $|0>$ is defined by the relation $a|0>= 0$ and, of course,

$$\begin{array}{llll} N|n>= & 0, & E|n>= & |n>, \qquad n \in \mathcal{I}^+ \\ a^\dagger|n>= & \sqrt{n+1}|n+1>, & a|n>= & \sqrt{n}|n-1>, \\ (a^\dagger)^\dagger = & a & (a^\dagger)^n|0>= & \sqrt{n!}|n>. \end{array} \tag{2}$$

The quantum deformation of $h(1)$ is known as $h_q(1)$ [3]. Its generators are, of course, related to the ones of $h(1)$ and they close a slightly different algebra

$$[a, a^\dagger] = [E]_q, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [E, \bullet] = 0 \tag{3}$$

where $[x]_q$'s are the so called q-numbers that have many of the properties of the standard numbers, are known to the mathematician since more then one century and are defined as:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \equiv \frac{sinh(zx)}{sinh(z)} \tag{4}$$

q being a free parameter and $z \equiv ln(q)$. Note that, as $q \to 1$, $h_q(1)$ reduces to $h(1)$. If q is real the Lie algebras and their quantum deformation are at a pure algebraic level, essentially, the same structure. This is particularly obvious in our case where $h_q(1)$ seems (and it is, at the algebra level) trivially equivalent to $h(1)$ substituting simply E with $E' \equiv [E]_q$. The point is that it exist, at a deeper level, a related structure (the Hopf algebra) where things are different, as discussed in section [4].

3 The Superalgebra $osp(1|2)$ and its Quantum Extension

From eqs. (2) it is easy to see that:

$$aa^\dagger|n>=(n+1)|n> \quad a^\dagger a|n>=n|n>, \tag{5}$$

and subtracting the second relation from the first the commutator on the states is obtained, but in a representation both commutator and anticommutator are defined: we can also write $\{a, a^\dagger\}|n>=(2n+1)|n>$. This seems a play but it allows us to see that our Fock space is not only a representation of $h(1)$ but also of another structure known as the superalgebra (the prefix *super-* means that it contains anticommutators) $osp(1|2)$ [4]:

$$\{a, a^\dagger\} = 2H, \qquad [H, a^\dagger] = a^\dagger, \qquad [H, a] = a \tag{6}$$

where $H \equiv N + E/2$. It should be necessary to include among the relations the anticommutator of a (and $a^\dagger$) with itself also but, because we are interested in the universal enveloping algebra, we shall consider the functions in $(a, a^\dagger, H)$ quotiented with (6) which is equivalent to the traditional scheme.

As in the $h(1)$ case, the quantum extension of $osp(1|2)$ is simply obtained deforming the "product" between creation and annihilation operators. The quantum superalgebra $osp_q(1|2)$ is indeed [5]:

$$\{a, a^\dagger\} = [2H]_q, \quad [H, a^\dagger] = a^\dagger, \quad [H, a] = a \tag{7}$$

It is noteworthy that, while both $h(1)$ and $osp(1|2)$ can be used for the Jordan-Schwinger construction of $su(2)$, $osp_q(1|2)$ only (and not $h_q(1)$) can be used to build $su_q(2)$ [6].

4 Coalgebra and its Physical Relevance

When considering Lie algebras, physicists are used to think to them as a set of commutation relations. The physics consists, indeed, into associating physical

observables to well defined elements of the universal enveloping algebra (almost always generators). The algebra of observables is, in such a way, defined by induction from the commutation relations of the Lie algebra. This procedure is, of course, the core of the physical applications of Lie algebras (and of quantum algebras also).

However recent studies on Hopf algebras exhibit that the property of generators of being primitive (i.e. corresponding to additive observables) are not implied by the commutation relations and, if needed, must be separately imposed. To clarify this point let us consider as an example the Poincarè algebra in 1+1 dimension, $e(1,1)$, with three generators (E,P,K) and commutation relations:

$$[K,E]=iP, \quad [K,P]=iE, \quad [E,P]=0. \tag{8}$$

It is almost a reflex, for a physicist, to associate to E the energy and to P the momentum but in all this the fundamental notion of composed system it is not contained. Indeed, to consider a complex system as the addition of simpler systems, we have to relate the observables of the constituents to the global ones. And we have a rule: energy, momentum as well all other observables are simply additive i.e.

$$E_{tot}=E_1+E_2, \quad P_{tot}=P_1+P_2, \quad K_{tot}=K_1+K_2, \tag{9}$$

and so on. This rule is usually considered the only possible one, such that it is seldom discussed in details but the point we are stressing here is that it isn't the only one compatible with eqs.(8). In a more mathematical language, eqs. (8) do not define completely the structure of the Hopf algebra $e(1,1)$. There is no room here to discuss what a Hopf algebra is [7] and we shall restrict to state that it results from four applications: a product operating from $A\otimes A$ into A (nothing else that the commutation relations), a coproduct from A into $A\otimes A$ (it's the one we are interest now, from the algebra into the coalgebra), an antipode from A into A (related, in the Lie case, to the inverse) and a counity from A into K. The coalgebra of the Lie algebra $e(1,1)$ is nothing else that eq. (9), that mathematicians use to write as

$$\Delta(E)=E\otimes 1+1\otimes E, \quad \Delta(P)=P\otimes 1+1\otimes P, \quad \Delta(K)=K\otimes 1+1\otimes K. \tag{10}$$

While the general request that global observables close the same algebra of observables on each subsystem (more formally that coalgebra is isomorphic to the algebra) is one of the axioms of Hopf algebras, the more specific physical request of additivity (eq. (9)) is related to a peculiar form of coalgebra of eq. (10) and, in general, it's not true. Indeed, while coalgebras of quantum algebras looks quite different from eqs.(10), the commutation relation of eq. (8)

are compatible with both a Lie algebra and a quantum one: define for instance, instead of (10):

$$\begin{array}{rcl} \Delta(E) & = & E \otimes q^K + q^{-K} \otimes E, \qquad \Delta(P) = P \otimes q^K + q^{-K} \otimes P, \\ \Delta(K) & = & K \otimes 1 + 1 \otimes K \end{array} \tag{11}$$

the coalgebra still close e(1,1) (as it must and can be easy checked) but the global energy and momentum no longer commute with the corresponding quantities of components: we are no more considering as a whole two non interacting systems but quantum deformation has introduced a some sort of interaction and the physics is completely different.

In conclusion: if we have to operate on a single object the commutation relation are enough, but to work on more spaces together (to consider substructures or a system of two particles or, in second quantization, more then one mode) we must specify the entire Hopf algebra because we can have more then one coalgebra for the same algebra (as in (10) and (11)) or (as in the following) more algebras for the same coalgebra. In particular, while eqs. (2) are the 1-mode representation for both $h(1)$ and $osp(1|2)$, the coalgebra (still always additive) gives for the two structures completely different 2-modes representations (and, so, completely different physical models). This is, of course, not peculiar of second quantization algebras but it is true also for the best known coalgebra: the sum of angular momenta is not simply related to the $su(2)$ commutation relations but to the whole $su(2)$ Hopf algebra that is the only guarantee that we can rotate a system composed by two isolated subsystems rotating independently the two subsystems (and that our usual Clebsch-Gordan coefficients are correct).

5 Coalgebras in Fock space and their physical implications

In Fock space coalgebras are, of course, related to more then one mode. Let us focus on two modes. As well as in eqs. (2) Fock space is builded starting from a vacuum (this time $|0,0>$). The traditional approach implies two steps, first to apply the available creation operators ($a_1^\dagger$ and $a_2^\dagger$), to realize all possible states $|n_1, n_2>$, and then to consider the statistical constrains. This procedure non only it's not necessary but by chance only can be compatible with the algebra structure. Indeed what we obtain before to impose a symmetry is a unitary but reducible representation of the algebra and group theory has its own constraints (the Clebsch-Gordon coefficients) to isolate the irreducible components. Our proposal is simpler: start from $|0,0>$ and apply only $\Delta(a^\dagger)$: all the work will be performed in one step without any problem and especially without any inconsistency.

Let us consider $h(1)$: because it's a Lie algebra its generators are additive (primitive for a mathematician):

$$\begin{array}{rlrl} \Delta(a^\dagger) = & a_1^\dagger + a_2^\dagger, & \Delta(a) = & a_1 + a_2, \\ \Delta(N) = & N_1 + N_2, & \Delta(E) = & E_1 + E_2, \\ \Delta(\mathcal{C}) = & \Delta(E)\Delta(N) - \Delta(a^\dagger)\Delta(a) & . & \end{array} \tag{12}$$

while $\Delta(\mathcal{C})$ is a derived variable (if we had assumed $N \equiv a^\dagger a$, we should also have $\Delta(N) = \Delta(a^\dagger)\Delta(a)$, not additive). From its expression it's easy to see that $\Delta(\mathcal{C})|0,0>=0$ and, as the homomorphism gives us $[\Delta(\mathcal{C}), \Delta(a^\dagger)] = 0$, all the states of the form $(\Delta(a^\dagger))^n|0,0>$ belong to the Fock representation (i.e. the representation with Casimir zero).

In concrete the two modes states are the ones belonging to the representation with highest weight $|0,0>$ of $h(1)$ and are completely determined from eqs. (2) and (12), as well as the one mode ones belong to the representation with highest weight $|0>$ and are determined from eqs. (2):

$$\begin{array}{l} |0,0> \\ \Delta(a^\dagger)|0,0>= \quad (a_1^\dagger + a_2\dagger)|0,0>= \quad |1,0> + |0,1> \\ (\Delta(a^\dagger))^2|0,0>= \quad \sqrt{2}|2,0> + 2|1,1> + \sqrt{2}|0,2> \\ \vdots \\ (\Delta(a^\dagger))^n|0,0>= \quad \sqrt{n!}\sum\sqrt{\binom{n}{m}}|m,n-m>; \end{array} \tag{13}$$

These states are, by inspection, symmetric between the two modes (this means, for a physicist, that the two modes have the same energy or $T=\infty$) but not only: for each n, among the many 2-modes symmetric states only one is chosen by the coproduct. And indeed, if we fix the number n of particles and we define $P_n(m)$ the probability to find m of the n particles in the first mode or level, we have

$$P_n(m) = \frac{1}{2^n}\binom{n}{m} \qquad \left(\sum P_n(m) = 1\right) \tag{14}$$

i.e. the binomial distribution with $p = 1/2$.

To describe finite T we need, indeed, to break this symmetry between the two modes (in mathematical notation the cocommutativity of the algebra), what can be done by $h_q(1)$. Assuming, as usual, that all central operators E, E_1 and E_2 have eigenvalue 1, the $h_q(1)$ coalgebra can be written indeed as

$$\begin{array}{rcll} \Delta(a^\dagger) & = & a_1^\dagger\ q + q^{-1}\ a_2^\dagger, & \Delta(a) = a_1\ q + q^{-1}\ a_2, \\ \Delta(N) & = & N_1 + N_2, & \Delta(E) = E_1 + E_2 \end{array} \tag{15}$$

and probabilities become

$$P_n(m) = \binom{n}{m} p^m (1-p)^{n-m} \qquad \text{with} \quad p \equiv \frac{q^2}{q^2+q^{-2}}. \tag{16}$$

Still the binomial distribution but with p free this time. So p (and consequently q) is a measure of the temperature T of the system: if the two modes have energy $\mathcal{E}_1$ and $\mathcal{E}_2$ with $\mathcal{E}_1 > \mathcal{E}_2$ at T is zero only the level $\{2\}$ will be filled and p and q are zero, at $T = \infty$, $p = 1/2$ and $q = 1$ while $0 < q < 1$ describes finite temperatures ($1 < q$ is related to $\mathcal{E}_1 < \mathcal{E}_2$).

Let us look now what $osp(1|2)$ gives. Because the 1-mode is the same of $h(1)$ we could suppose that no changes can be found at the 2-modes also. But $osp(1|2)$ is a superalgebra and $a^\dagger$ is an odd operator: this implies that $\{a_1^\dagger, a_2^\dagger\} = 0$. So the 2-modes $osp(1|2)$ states are

$$\begin{array}{lcl} |0,0> & & \\ \Delta(a^\dagger)|0,0> & = & (a_1^\dagger + a_2\dagger)|0,0> = \ |1,0> + |0,1> \\ (\Delta(a^\dagger))^2|0,0> & = & (a_1^\dagger + a_2\dagger)^2|0,0> = \\ & = & ((a_1^\dagger)^2 + (a_2\dagger)^2)|0,0> = \sqrt{2}(|2,0> + |0,2>) \end{array} \tag{17}$$

and so on. The peculiar point is that, contrary to $h(1)$, $osp(1|2)$ strongly discriminates among even and odd n and m. A new, unorthodox statistics is, in such a way, suggested; we limit ourselves to give here the limit distribution. Define $\lim_{n\to\infty} P_{2n}(2m) = P^a(m/n)$, $\lim_{n\to\infty} P_{2n+1}(2m) = P^b(m/n)$, $\lim_{n\to\infty} P_{2n+1}(2m+1) = P^c(m/n)$: they look as in fig.1, with divergencies as square root at the extremes of the interval of definition $\{0-1\}$ (still integrable, of course), while $P_{2n}(2m+1)$ is identically zero. It's really a crazy statistics.

Few words about its quantum deformation: for $q \sim 1$, $osp_q(1|2)$ gives, more or less, the same results, while for q far from 1, the q-deformation induces a leakage and parity of n and m becomes less relevant but always asymmetric distributions remain strongly preferred.

6 Conclusions

The fundamental point of our discussion is that, in general, we cannot consider an algebra keeping apart its coalgebra, because the physical scheme results from both, while constraints imposed without taking into account the compatibility are, almost always, inconsistent. Considering Fock space we have shown that Weyl-Heisenberg algebra $h(1)$ contains in itself the binomial distribution (and, of course, is incompatible with other ones). Always in the scheme of

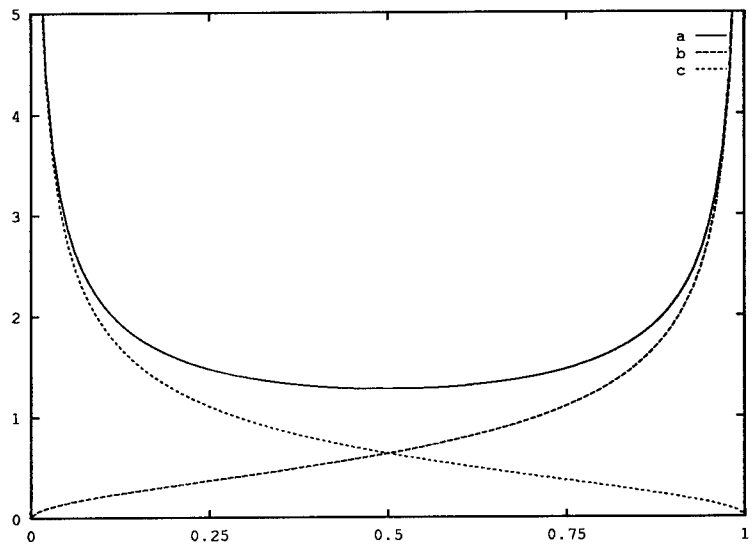

Figure 1: 2-modes limit distributions of $osp(1|2)$ for different parities

binomial distribution, $h_q(1)$ allows to distinguish among the modes and introduces, in this way, temperature as a statistical parameter. $osp(1|2)$ is a new exotic statistics where particles incline to stay together quit more that in any known statistics. Nobody knows if it has something to do with physics but, from a mathematical point of view, it's a statistics like the others.

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BREATHERS IN A NONLOCAL SINE-GORDON MODEL

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It is shown that small amplitude solitons of a nonlocal sine-Gordon model corresponding to different frequencies of the carrier wave can create coupled states. Within the framework of the multiscale expansion such pulses are described by the system of the nonlinear Schrödinger equations which possesses coupled mode solutions in the form of running localized waves (breathers). Such breathers consist of modes with different frequencies and are characterized by two internal frequencies.

Recently great attention has been paid to various nonlinear nonlocal models due to their role in the description of the lattice dynamics [1,2], superconductivity [3], and magnetic systems [4]. Those models take into account long-range interactions which naturally result in new physical phenomena. We consider the nonlocal sine-Gordon (NSG) model

$$u_{xx} - u_{tt} = 2\cos\left[\frac{1}{2}u(x,t)\right]\int dy f(x-y)\sin\left[\frac{1}{2}u(y,t)\right], \qquad (1)$$

introduced in [5], where it has been found that due to the nonlocality kink solutions possessing nonzero topological charge can create zero-topological-charge localized excitations of odd and even parity. These static solutions are shown to be stable at $\sigma > \sigma_{cr}$, where σ is a parameter characterizing the range of interactions [see (4) below] and σ_{cr} is its critical value ($\sigma_{cr} \approx 1.25$). Being static the mentioned solutions differ from another known type of localized excitations which are called breathers. Taking this fact into account and recalling that the small-amplitude breathers of the local Sine-Gordon model are governed by the nonlinear Schrödinger (NLS) equation it is natural to analyze solutions of the NSG model in the limit of small amplitude.

We make use of the method of multiple scales. It implies that the wave field is represented in the form of the series $u = \epsilon u_1 + \epsilon^2 u_2 + ...; \epsilon \ll 1$, where u_n are varying on different space $x_n = \epsilon^n x$ and time $t_n = \epsilon^n t$ scales.

It is assumed that in terms of the scaled variables the function $f(x)$ depends only on the 'rapid' spatial coordinate x_0. Developing $u(x,t)$ in the integrand of (1) in a series with respect to the slow independent variables $\{x_n\}$ $(n \geq 1)$ we find a set of equations of the form

$$Lu_n = F_n; \quad Lu = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial t_0^2} - \int dy f(y) u(x_0 - y, t_0). \tag{2}$$

The dispersion relation associated with the linear operator L is $\omega^2(k) = k^2 + \hat{f}(k)$, where $\hat{f}(k)$ is the Fourier transform of $f(x)$.

First, let us concentrate on the single-mode solution of the first order system $(F_1 = 0)$: $u_1 = A(x_1, ...; t_1, ...)e^{i\theta} + c.c.$; $\theta = kx_0 - \omega(k)t_0$. The condition of the absence of secular terms in the second order leads to the conclusion that A depends on x_1 and t_1 only through the combination $\zeta = x_1 - v_{gr}(k)t_1$ where $v_{gr}(k) = \frac{d\omega}{dk}$ is the the group velocity of the carrier wave. Finally, considering the solutions independent of x_2 in the third order of the small parameter ϵ we obtain the NLS equation

$$2i\frac{\partial A}{\partial t_2} + \omega''\frac{\partial^2 A}{\partial \zeta^2} + \chi|A|^2 A = 0, \tag{3}$$

where $\omega'' = \frac{d^2\omega}{dk^2}$ and $\chi = \frac{1}{2\omega}\hat{f}(k)$. Let us now analyze the features of the problem using as an example the non-local kernel introduced in [5]

$$f(x) = \frac{1}{2\sqrt{\pi}\sigma} e^{-\frac{x^2}{4\sigma^2}}. \tag{4}$$

The first characteristic feature of the problem follows from (3), (4). $\chi \to 0$ with k and has a maximum at $k = 0$, then localized (i.e. soliton) solutions are available only in the limited region of k. This region however becomes infinite when $\sigma \to 0$. The group velocity has the form

$$v_{gr}(k) = k\frac{1 - \sigma^2 e^{-\sigma^2 k^2}}{\sqrt{k^2 + e^{-\sigma^2 k^2}}}. \tag{5}$$

Fig.1 shows the N-shape form of the curve $v_{gr}(k)$ when $\sigma > \sigma_{cr}^{(1)} = 1$. This means the existence of solitons of different types at the different wave numbers. In the vicinity of $k = 0$, where $\omega'' < 0$, there can exist a dark soliton, while at $|k| > k_0$ [k_0 being the wave number corresponding to the minimum of $v_{gr}(k)$] $\omega'' > 0$, what means the existence of bright soliton solutions. Similar phenomenon of co-existence of bright and dark solitons in the nonlinear lattices with long range interactions has been discovered

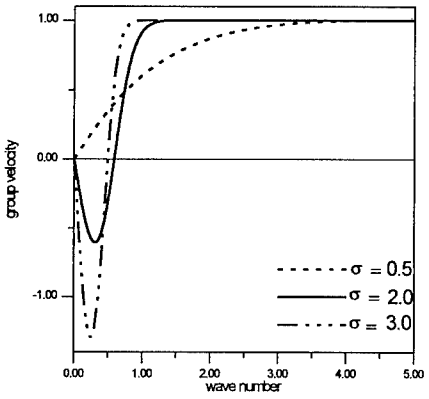

Figure 1: Group velocity at different values of σ.

by Remoissenet and Flytzanis [2]. Further increasing σ leads to appearing linear waves with $v_{gr}(k) > 1$. This happens when $\sigma > \sigma_{cr}^{(2)} = 2e^{\frac{1}{4}}$. The existence of waves with different frequencies and group velocity dispersions but with the same group velocity, v_{gr}, leads to running tied states due to the nonlinearity. In this case the respective solution can be represented by $u_1 = A_1(x_1, ...; t_1, ...)e^{i\theta_1} + A_2(x_1, ...; t_1, ...)e^{i\theta_2} + c.c.$, $(\theta_n = k_n x_0 - \omega(k_n)t_0)$; and excluding secular terms in the equations of the third order we obtain

$$2i\frac{\partial A_j}{\partial t_2} + \omega_j''\frac{\partial^2 A_j}{\partial \zeta^2} + \chi_{j1}|A_1|^2 A_j + \chi_{j2}|A_2|^2 A_j = 0, \quad (j = 1, 2) \qquad (6)$$

where $\chi_{jj} = \frac{1}{2\omega_j}\hat{f}(k_j)$, $\chi_{ij} = \frac{1}{2\omega_i}[\hat{f}(k_1) + \hat{f}(k_2)](i \neq j)$, $\omega_n = \omega(k_n)$ and $\omega_n'' = \omega''(k_n)$. First, we consider k_1, k_2 such that $\omega_2'' = 0$, $\omega_1'' > 0$ (i.e. $k_2 = \pm k_0$). This requirement can be fulfilled when $\sigma_{cr}^{(1)} < \sigma < \sigma_{cr}^{(2)}$. Then a solitary wave solution (we call it breather) of (6) can be written as follows

$$A_1 = \frac{\alpha}{\cosh \beta\zeta} e^{i\frac{\beta^2\omega_1''}{2}t_2}, \quad A_2 = \frac{\alpha}{\cosh \beta\zeta} e^{i\phi(\zeta)t_2}, \qquad (7)$$

where

$$\beta = \sqrt{\frac{1}{\omega_1\omega_1''}[2\hat{f}(k_1) + \hat{f}(k_2)]}\frac{\alpha}{2}, \quad \phi(\zeta) = \frac{1}{4\omega_2}[\hat{f}(k_1) + 2\hat{f}(k_2)]\frac{\alpha^2}{\cosh \beta\zeta}, \qquad (8)$$

and α is the constant amplitude of the breather. An explicit form of u_1, is

$$u_1 = \frac{4\alpha}{\cosh\beta\zeta}\cos\left[\left(\frac{\beta^2\omega_1''}{4}+\frac{\phi(\zeta)}{2}-\frac{\omega_1+\omega_2}{2}\right)t+\frac{k_1+k_2}{2}x\right]\times$$
$$\cos\left[\left(\frac{\beta^2\omega_1''}{4}-\frac{\phi(\zeta)}{2}+\frac{\omega_2-\omega_1}{2}\right)t+\frac{k_1-k_2}{2}x\right]. \qquad (9)$$

Thus the breather, *moving* with group velocity v_{gr}, is characterized by the amplitude and *two* internal frequencies depending on the spatial coordinate.

Another coupled mode solutions appear when $\sigma > \sigma_{cr}^{(2)}$; we can consider the coupling of two modes with $v_{gr} = 1$. If $\omega_1'' > 0$ and $\omega_2'' = \hat{f}(k_2) = O(\epsilon)$, the system (6) is degenerated

$$2i\frac{\partial A_1}{\partial t_2}+\omega_1''\frac{\partial^2 A_1}{\partial\zeta^2}+\frac{\hat{f}(k_1)}{2\omega_1}(|A_1|^2+I_2)A_1=0; \quad I_2=|A_2|^2. \qquad (10)$$

Since now the mode 2 is associate with linear dispersionless propagation $A_2 = \sqrt{I_2}\exp(i\frac{\hat{f}(k_1)}{4\omega_2}|A_1|^2 t_2)$ and the intensity I_2 may depend on ζ in arbitrary way. Thus we have dynamics when the first mode is driven by the second one. In particular, if the mode 2 is the monochromatic wave of a constant amplitude (i.e. I_2 does not depend of t_2) then there exists a breather of the NSG which is exactly the NLS soliton with the frequency defined by $\frac{\hat{f}(k_1)}{4\omega_1}I_2$.

To conclude we have shown that in the limit of small amplitudes there exist coupled localized states originated by long range interactions. In the case of the NSG (1), (4) the properties of the system are characterized by two critical values of σ; $\sigma_{cr}^{(1)}$ is close to the critical value found numerically in [5] for the excitations of large amplitudes. As is evident the coupled mode dynamics is much more reach and is not exhausted by the phenomena described here.

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Forced Lattice Vibrations

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In 1978 Holian and Straub [HS] conducted an extensive series of numerical experiments on a driven, semi-infinite lattice

$$\ddot{x}_n = F(x_{n-1} - x_n) - F(x_n - x_{n+1}), \qquad n = 1, 2, \ldots, \tag{1.1}$$

with initial conditions

$$x_n(0) = nd, \quad \dot{x}_n(0) = 0, \quad n = 1, 2, \ldots, \ d \text{ constant}, \tag{1.2}$$

for a variety of force laws F, and in the case that the velocity of the driving particle x_0 is fixed,

$$x_0(t) = 2at, \qquad t \geq 0, \quad a > 0. \tag{1.3}$$

They discovered, in particular, a striking new phenomenon – the existence of a critical "shock" strength a_{crit}. If $a < a_{\text{crit}} = a_{\text{crit}}(F)$, then in the frame moving with the particle x_0, they observed behavior similar to that shown in Figure 1.4. Thus, the particles come to rest in a regular lattice behind the driver. However, if

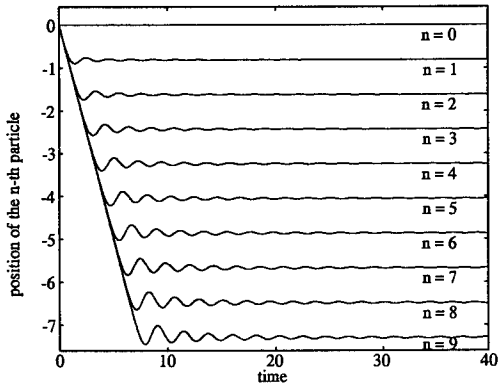

Figure 1.4: *Motion of the first ten particles of a lattice described by the above system (1.1) – (1.3) with* $F(x) = e^x, d = 0, a = .5$, *in the frame of* x_0 *(case* $a < a_{\text{crit}}$*).*

$a > a_{\text{crit}} = a_{\text{crit}}(F)$, then, again in the frame of the driver, they observed behavior as

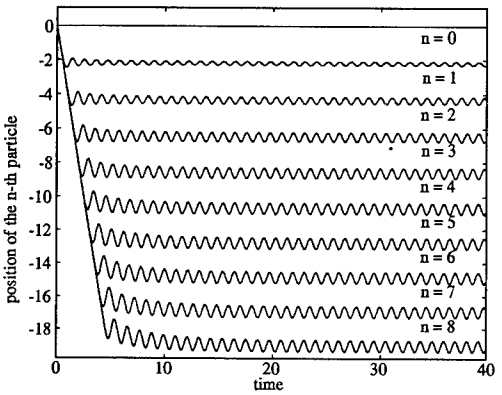

Figure 1.5: *Motion of the first ten particles of a lattice described by the above system (1.1) - (1.3) with* $F(x) = e^x, d = 0, a = 2$, *in the frame of* x_0 *(case* $a > a_{\text{crit}}$*).*

in Figure 1.5. Now the particles do **not** come to rest behind the driver, but execute an on-going binary oscillation (i.e. $x_n(t+T) = x_n(t)$, $x_n(t) = x_{n+2}(t) + \text{const.}$). This marvelous, fundamentally nonlinear phenomenon (if F is linear, the effect is absent) has now been observed for many different singular and nonsingular force laws F, but an explanation from first principles in the general case has not yet been given.

In 1981, Holian, Flaschka and McLaughlin [HFM] considered the shock problem in the special case in which F is an exponential $F(x) = e^x$, the so-called Toda shock problem. They considered this case because the Toda equation

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \tag{1.6}$$

with appropriate boundary conditions, is integrable (a fact discovered by Flaschka [F] and Manakov [Man]; see also [H], [T1], [T2]). Holian, Flaschka and McLaughlin realized that if they went into the frame of the driver, so that (1.2), (1.3) become

$$x_n(0) = nd, \quad \dot{x}_n(0) = -2a, \quad n \geq 1, \tag{1.7}$$

$$x_0(t) \equiv 0, \tag{1.8}$$

and doubled up the system

$$x_n(t) \equiv -x_{-n}(t), \quad n < 0, \tag{1.9}$$

then the full system $\{x_n\}_{n=-\infty}^{\infty}$ solves the **autonomous** Toda equations (1.6) with initial conditions

$$x_n(0) = dn, \quad \dot{x}_n(0) = -2a(\text{ sgn } n), \quad -\infty < n < \infty. \tag{1.10}$$

The solutions of these equations lie in a class to which the method of inverse scattering applies. To see what is involved we use Flaschka's variables,

$$a_n = -\dot{x}_n/2, \quad b_n = \frac{1}{2}\, e^{\frac{1}{2}(x_n - x_{n+1})}, \quad -\infty < n < \infty, \tag{1.11}$$

and arrange these variables into a doubly-infinite tridiagonal matrix

$$\widetilde{L} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & b_{-2} & a_{-1} & b_{-1} & & \bigcirc & \\ & & b_{-1} & a_0 & b_0 & & \\ & & & b_0 & a_1 & b_1 & \\ & \bigcirc & & & b_1 & \ddots & \ddots \\ & & & & & \ddots & \end{pmatrix}, \tag{1.12}$$

which represents the state of the system at any given time, with companion matrix

$$\widetilde{B} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & -b_{-2} & 0 & b_{-1} & & \bigcirc & \\ & & -b_{-1} & 0 & b_0 & & \\ & & & -b_0 & 0 & b_1 & \\ & \bigcirc & & & -b_1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}. \tag{1.13}$$

Then, remarkably, (1.6) is equivalent to the so-called Lax-pair system

$$\frac{d\widetilde{L}}{dt} = [\widetilde{B}, \widetilde{L}] = \widetilde{B}\widetilde{L} - \widetilde{L}\widetilde{B}. \tag{1.14}$$

Thus, the Toda equations are equivalent to an iso-spectral deformation of the matrix operator $\widetilde{L}$. Inverse scattering theory tells us that one can solve (1.14), and hence (1.1) – (1.3), through the scattering map for $\widetilde{L}$. Rescaling time, one sees that it is sufficient to consider the case where the initial spacing $d = 0$. Then at $t = 0$,

$$a_n = a \text{ sgn } n, \quad b_n = \frac{1}{2}, \tag{1.15}$$

Figure 1.16: *The spectrum of* $\tilde{L}(0)$

and one sees that the essential spectrum of $\tilde{L}$ is given by two bands (cf Figure 1.16). The bands overlap if and only if $a < 1$. Holian, Flaschka and McLaughlin observed that supercritical behavior occurred for the Toda shock problem only if the gap was open. Hence they identified $a_{\text{crit}}(F = e^x) = 1$. Using the inverse method they were able to calculate a number of other features of the Toda shock problem, such as the speed and the form of the shock front, and also the form of the binary oscillations. The problem of how to extract detailed information about the long-time behavior of the Toda shock problem from knowledge of the initial data using the rather formidable formulae of inverse scattering theory, however, remained open.

In the early 80's, a very important development took place in the analysis of infinite-dimensional integrable systems in the form of the calculation by Lax and Levermore ([LL]) of the leading order asymptotics for the zero-dispersion limit of the Kortweg de Vries equation, in which the weak limit of the solution as the dispersion coefficient tends to zero is derived and the small scale oscillations that arise are averaged out. This was followed in the late 80's by the calculation of Venakides [V] for the higher order terms in the Lax-Levermore theory which produces the detailed structure of the small scale oscillations. These developments raised the possibility of being able to analyze the inverse scattering formulae for the solution of the Toda shock problem effectively as $t \to \infty$, and in [VDO], Venakides, Deift and Oba proved the following result in the supercritial case $a > 1$:

In addition to the shock speed v_s calculated by Holian, Flaschka and McLaughlin, there is a second speed $0 < v_0 < v_s$. In the frame moving with the driver, as $t \to \infty$,

- for $0 < n/t < v_0$, the lattice converges to a binary oscillation $x_n(t+T) = x_n(t)$, $x_{n+2}(t) = x_n(t)+$ const. (Fig. 1.5). The band structure corresponding to the asymptotic solution is $[-a-1, -a+1] \cup [a-1, a+1]$. The binary oscillation is connected to the driver $x_0(t) \equiv 0$, through a boundary layer, in which the local disturbance due to the driver decays exponentially in n.

- for $v_0 < n/t < v_s$, the asymptotic motion is a modulated, single-phase, quasi-periodic wave with band structure $[-a-1, \gamma(n/t)] \cup [a-1, a+1]$, where $\gamma(n/t)$

varies monotonically from $-a+1$ to $-a-1$ as n/t increases from v_0 to v_s.

- for $n/t > v_s$, the deviation of the particles from their initial motion $-2at$ is exponentially small. The influence from the shock has not yet been felt. As noted in [HFM], for $n/t \sim v_s$, the motion of the lattice is described by a Toda solution with associated spectrum $\{-a-1\} \cup [a-1, a+1]$.

In 1991, again using the techniques in [LL] and [V], Kamvissis ([Kam]) showed that in the subcritical case $a < 1$, in the frame moving with the driver, as $t \to \infty$, the oscillatory motion behind the shock front dies down to a quiescent regular lattice with spacing $x_{n+1} - x_n \to -2\log(1+a)$, (cf Figure 1.4). In [BK], Bloch and Kodama considered the Toda shock problem, both in the subcritical and the supercritical cases, from the point of view of Whitham modulation theory in which the validity of a modulated wave form for the solution is assumed a priori, and the parameters of the modulated wave form are calculated explicitly. More recently in [GN], Greenberg and Nachman have considered the shock problem for a general force law in the weak shock limit; they are able to describe many aspects of the solution, including the modulated wave region where they use a KdV-type continuum limit.

In a slightly different direction, motivated by the so-called von-Neumann problem arising in the computation of shock fronts using discrete approximations, Goodman and Lax [GL] and Hou and Lax [HL] observed and analyzed features strikingly similar to those in [HS]. Finally, Kaup and his collaborators, [Kau], [KN], [WK], use various integrable features of the non-autonomous system (1.1) – (1.3) to gain valuable insight into the Toda shock problem.

Here, we consider the driven lattice (1.1), (1.2) in the case where the uniform motion of the driving particle x_0 is periodically perturbed[1]

$$\text{(1.17)} \qquad \begin{cases} \ddot{x}_n = F(x_{n-1} - x_n) - F(x_n - x_{n+1}), & n \geq 1, \\ x_n(0) = \dot{x}_n(0) = 0, & n \geq 1, \\ x_0(t) = 2at + h(\gamma t). \end{cases}$$

Here $h(\cdot)$ is periodic with period 2π and the frequency $\gamma > 0$ is constant. We restrict our attention to the case where the average velocity of the driver is subcritical, i.e.

[1]The more general initial value $x_n(0) = dn$, $\dot{x}_n(0) = 0$, can clearly be converted to (1.17) by shifting the argument of F, $F(\cdot) \to F(\cdot - d)$: in the case of Toda, as noted above, this shift converts into a rescaling of the time.

$a < a_{\text{crit}}$. We consider a variety of force laws F, but we restrict our attention to forces that are real analytic and monotone increasing in the region of interest.

Typically we observed the following phenomena:

In the frame moving with the average velocity $2a$ of the driver, as $t \to \infty$, the asymptotic motion of the particles behind the shock front, is $\frac{2\pi}{\gamma}$-periodic in time,

$$x_n(t + \frac{2\pi}{\gamma}) = x_n(t), \quad 0 < n \ll t. \tag{1.18}$$

Moreover, there is a sequence of thresholds,

$$\begin{aligned} &\gamma_1 = \gamma_1(a,h,F) > \gamma_2 = \gamma_2(a,h,F) > \cdots > \quad \gamma_k = \gamma_k(a,h,F) > \cdots > 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma_k \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{1.19}$$

- If $\gamma > \gamma_1$, there exist constants c, d such that $x_n - cn - d$ converges exponentially to zero as $n \to \infty$. In other words, the effect of the oscillatory component of the driver does not propagate into the lattice and away from the boundary $n = 0$.

- If $\gamma_1 > \gamma > \gamma_2$, then the asymptotic motion is described by a travelling wave

  $$x_n(t) = c_1 n + X_1(\beta_1 n + \gamma t), \qquad 1 \ll n \ll t, \tag{1.20}$$

  transporting energy away from the driver x_0. Here $c_1 = c_1(a,h,F,\gamma)$ and $X_1(\cdot) = X_1(\cdot\,; a,h,F,\gamma)$ is a 2π-periodic function.

- More generally, if $\gamma_k > \gamma > \gamma_{k+1}$, a multi-phase wave emerges which is well-described by the wave form

  $$x_n = c_k n + X_k(\beta_1 n + \gamma t,\, \beta_2 n + 2\gamma t, \ldots, \beta_k n + k\gamma t), \quad 1 \ll n \ll t, \tag{1.21}$$

  again transporting energy away from the driver. Here $c_k = c_k(a,h,F,\gamma)$ and $X_k(\cdot, \ldots, \cdot; a,h,F,\gamma)$ is 2π-periodic in each of its k variables.

In the case of the Toda lattice, when the driving is constant the doubling-up trick converts the shock problem into an iso-spectral deformation (1.14) for the operator $\widetilde{L}$. When h is non-zero, it is no longer clear how to convert the shock problem (1.17) into an integrable form (although recent results of Fokas and Its [FI] suggest that this may still be possible to do). As a tool for analyzing (1.17) in the Toda case we

consider, rather, the Lax pair of operators

$$(1.22)\qquad L=\begin{pmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & b_2 & \\ & b_2 & \ddots & \ddots \\ 0 & & \ddots & \end{pmatrix}, \qquad B=\begin{pmatrix} 0 & b_1 & & 0 \\ -b_1 & 0 & b_2 & \\ & -b_2 & \ddots & \ddots \\ 0 & & \ddots & \end{pmatrix}$$

for the semi-infinite lattice $a_n = -\dot{x}_n/2,\ b_n = \frac{1}{2}\, e^{\frac{1}{2}(x_n - x_{n+1})},\ n \geq 1$.

A straightforward calculation shows that L solves the equation

$$(1.23)\qquad \dot{L} = [B, L] - 2\, b_0^2(t) P, \quad b_0 = \frac{1}{2}\, e^{\frac{1}{2}(x_0 - x_1)},$$

which we think of as a forced Lax system. Here $P = (P_{ij})_{i,j\geq 1}$, is a matrix operator with $P_{ij} = 0$ unless $i = j = 1$, and $P_{11} = 1$. The equation describes a motion that is almost, but not quite, an iso-spectral deformation of L. As t evolves, the essential spectrum of $L(t)$ remains fixed, $\sigma_{ess}(L(t)) = \sigma_{ess}(L(0)) = [a-1, a+1]$, but eigenvalues may "leak out" from the continuum. This is true, in particular, in the case of constant driving $x_0 = 2at$, as was first observed by Kaup and Neuberger [KN].

In the case of constant driving with $a < 1$, the eigenvalues emerge from the band $[a-1, a+1]$ and eventually fill the larger band $[-a-1, a+1] = [-a-1, -a+1] \cup [a-1, a+1]$. (In the case $a > 1$, the bands $[-a-1, -a+1]$, $[a-1, a+1]$ are disjoint and the spectrum of $L(t)$ fills these two bands separated by a gap). Thus this "ghost" band, which appeared as an artifact of the solution procedure through the introduction of the doubled-up operator $\widetilde{L}$, now emerges in real form, populated by eigenvalues emerging from the original band $[a-1, a+1]$.

In the periodically driven case, $x_0 = 2at + h(\gamma t)$, where $\gamma > \gamma_1$ (and $a < 1$), the evolution of $\sigma(L(t))$ is displayed in Figure 1.24. All eigenvalues emerge from the lowest point of the continuous spectrum. The eigenvalues in the figure, whose trajectories do not start from this point, are the ones that were generated prior to the displayed time-window.

As $t \to \infty$, $\sigma(L(t))$ again converges to a single band and no travelling wave emerges. However, if $\gamma_1 > \gamma > \gamma_2$, we find different behavior (Figure 1.25). We see that $\sigma(L(t))$ converges to two bands separated by a gap, and a single phase wave emerges. For $\gamma_2 > \gamma > \gamma_3$, we see from Figure 1.26 that $\sigma(L(t))$ converges to three bands separated by two gaps, and a two phase wave emerges, etc.

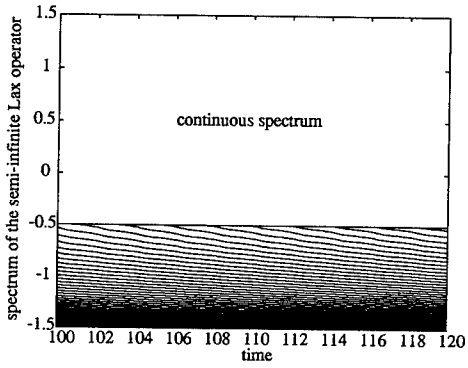

Figure 1.24: *Evolution of* $\sigma(L(t))$*; driver:* $x_0(t) = t + 0.2(\sin\gamma t + 0.5\cos 2\gamma t), \gamma = 3.1 > \gamma_1$

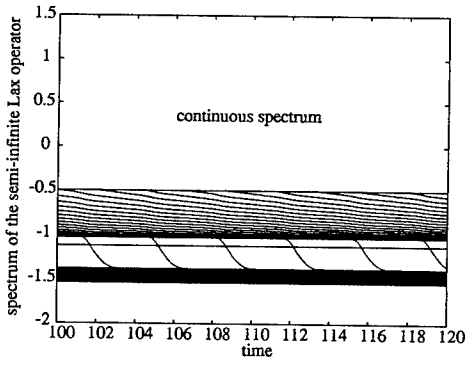

Figure 1.25: *Evolution of* $\sigma(L(t))$*; driver:* $x_0(t) = t + 0.2(\sin\gamma t + 0.5\cos 2\gamma t), \gamma = 1.8, \gamma_1 > \gamma > \gamma_2$

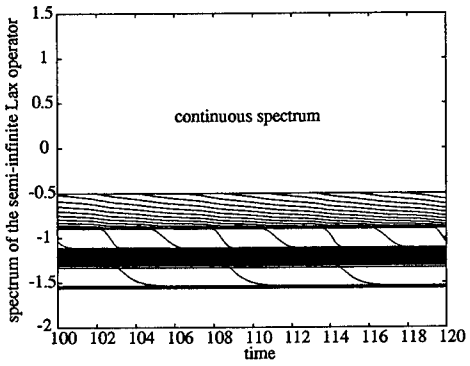

Figure 1.26: *Evolution of* $\sigma(L(t))$*; driver:* $x_0(t) = t + 0.2(\sin \gamma t + 0.5 \cos 2\gamma t), \gamma = 1.1, \gamma_2 > \gamma > \gamma_3$

For $\lambda < \inf \sigma_{ess}(L(0))$, an interesting quantity to compute is

$$J(\lambda) = \lim_{t \to \infty} \frac{\sharp \{\text{eigenvalues of } L(t) \text{ that are } < \lambda\}}{t}. \tag{1.27}$$

Clearly $J(\lambda)$ represents the asymptotic flux of eigenvalues of $L(t)$ across the value λ. It is observed numerically that $J(\lambda)$ indeed exists and for $\gamma_2 > \gamma > \gamma_3$, say, we find that $J(\lambda)$ looks as displayed in Figure 1.28 Thus $J(\lambda)$ is constant in the gaps

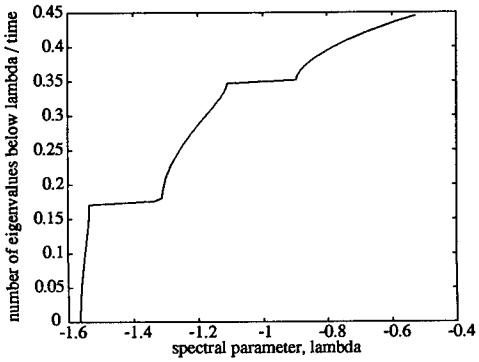

Figure 1.28: *Numerically observed* $J(\lambda)$ *in the case* $\gamma_2 > \gamma > \gamma_3, \gamma = 1.1$

and indeed we observe more generally that

$$J(\lambda) = \frac{j\gamma}{2\pi} \quad \text{for} \quad \lambda \text{ in the } j^{\text{th}} \text{ gap.} \tag{1.29}$$

This is a very intriguing fact, reminiscent of the Johnson-Moser gap labelling theorem [JM] in the spectral theory of one-dimensional Schrödinger operators with almost periodic potentials (see also the analogous gap labelling theorem for Jacobi matrices [B], [S]).

Finally we are at the stage where we can describe our analytical results, whose goal is to explain the above numerical experiments.

I. Strongly nonlinear case.

Here we consider (1.17) in the case of the Toda lattice without any smallness restriction on the size of the oscillatory component h of the driver x_0. The main result is that we compute the normalized density of state $J(\lambda)$ through the solution of a linear integral equation, once the number and endpoints of the bands in $\sigma(L(\infty))$ are known.

At this stage it is not clear how to relate the number and endpoints of the bands to the parameters of the problem a, γ, h. To test our result one reads off the discrete information given by the number and endpoints of the bands from the numerical experiment, and then compares the solution of the integral equation with the normalized density of states $J(\lambda)$ obtained directly from definition (1.27) using the numerically computed eigenvalues of $L(t)$ at large times. The numerical and analytical solutions for $J(\lambda)$ agree to very high order.

The proof of this result proceeds by deriving an equation of motion for the eigenvalues of a truncated version of $L(t)$ of size $N \gg t$ as $t \to \infty$. The continuum limit of the time average of these equations, leads to the linear integral equation mentioned above, which we solve by Riemann-Hilbert techniques.

II. Weakly nonlinear case.

Here we consider general F, but the periodic component h is now required to be suitably small. From the numerical experiments we see that if $h = 0(\varepsilon)$, then as $t \to \infty$, $x_n(t)$ converges to an asymptotic state which is a $\frac{2\pi}{\gamma}$-time periodic solution $x_{\text{asymp},n}(t)$ with $x_{\text{asymp},n}(t) = cn + 0(\varepsilon)$ for some lattice spacing c. The goal here is prove that such time periodic asymptotic states $x_{\text{asymp},n}(t)$ indeed exist for ε small. We proceed by linearizing around the particular solution $x_{\text{asymp},n}(t) = cn$, $n \geq 0$, of the equations $\ddot{x}_n = F(x_{n-1} - x_n) - F(x_n - x_{n+1})$, $n \geq 1$, and use various tools

from implicit function theory.

In our first result, we show that provided a sufficiently large parameter family of travelling wave solutions of the doubly infinite lattice

$$\ddot{x}_n = F(x_{n-1} - x_n) - F(x_n - x_{n+1}), \quad -\infty < n < \infty, \tag{1.30}$$

exist, then the parameters can always be chosen to produce the desired asymptotic states $x_{\text{asymp},n}(t)$ of the driven semi-infinite problem.

Thus the problem of the existence of the observed asymptotic states, reduces to the problem of constructing sufficiently large parameter families of travelling waves of the full lattice equation (1.30). For $\gamma_k > \gamma > \gamma_{k+1}$ $k \geq 1$, we will need $2k$-parameter families of k-phase travelling waves of type (1.21) on the full lattice in order to construct the solution of the driven lattice observed as $t \to \infty$ in the numerical experiments. If $\gamma > \gamma_1$, the requirement of travelling wave solutions of (1.30) trivializes, and our result guarantees the existence of the desired asymptotic states $x_{\text{asymp},n}(t)$ of the driven lattice for sufficiently small h and general real analytic F which are monotone in the region of interest.

The next result concerns general F in the case that $\gamma_1 > \gamma > \gamma_2$. Here we show that a 2-parameter family of one-phase travelling wave solutions of (1.30) always exist for general F. This result, together with the previous one, imply that for $\gamma_1 > \gamma > \gamma_2$ the desired states $x_{\text{asymp},n}(t)$ of the driven lattice exist This 2-parameter family is constructed by deriving an equation for the Fourier coefficients of the travelling wave solution, which can be solved by a Lyapunov-Schmidt decomposition. The infinite dimensional part does not pose any problems and the degenerate finite dimensional equations can be solved by using certain symmetries of the equation.

If we try a similar construction for m_0-phase waves, $m_0 > 1$, we encounter in the infinite dimensional part of the Lyapunov-Schmidt decomposition, a small divisor problem related to the small divisor problem occurring in [CW], where periodic solutions of the nonlinear wave equation are constructed, and which we hope to solve in the near future. In the special case of Toda, however, the family of travelling waves can be constructed explicitly. Indeed in our third and final result, we use the integrability of the doubly infinite Toda lattice and show how the well-known class of g-gap solutions contains a sufficiently large family of travelling waves to construct the desired asymptotics states $x_{\text{asymp},n}(t)$ of the driven lattice for any $\gamma \in \mathbb{R}_+ \backslash \{\gamma_1, \gamma_2, \cdots\}$.

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EXISTENCE AND PROPERTIES OF DISCRETE BREATHERS

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Nonlinear classical Hamiltonian lattices exhibit generic solutions in the form of discrete breathers. These solutions are time-periodic and (at least) exponentially localized in space. The lattices exhibit discrete translational symmetry. Discrete breathers are not confined to certain lattice dimensions. Necessary ingridients for their occurence are the existence of upper bounds on the linear spectrum (of small fluctuations around the groundstate) of the system as well as the nonlinearity. I will present existence proofs, formulate necessary existence conditions, and discuss structural stability of discrete breathers. The following results will be also discussed: the birth of breathers through tangential bifurcation of band edge plane waves; dynamical stability; details of the spatial decay; numerical methods of obtaining breathers; interaction of breathers with phonons and electrons; movability.

1 Introduction

It is well-known that nonlinear media support localized energetical excitations. For integrable nonlinear systems these solutions are the conventional solitons.[a] These solutions have energies larger than the state of lowest energy of the system (thus we coin them excitations). Most of the studies are usually done for systems with a continuous translational symmetry (field equations). If the considered system is allowed to evolve in time according to some dynamical equations of motion, then besides stationary static localized solutions (and the family of related solutions which are generated by the corresponding symmetry of the system) also time-dependent stationary localized solutions may exist. A well-known member of the latter group is the breather solution of the sine-Gordon (sG) equation in 1+1 dimensions. It is this type of solutions we will deal with in the following, although the final results might be well applicable to other (but similar) types of solutions too.

1.1 Some Properties of Breathers in Hamiltonian Field Equations

Let us characterize sG breather solutions with respect to the considerations to follow. The sG equation for a field $\Psi(x,t)$ is a particular example out of the

[a] For nonintegrable systems there seems to be no unique definition of those solutions.

class of nonlinear Klein-Gordon (KG) equations

$$\Psi_{,tt} = C\Psi_{,xx} - F(\Psi) \tag{1}$$

with the choice $F_{sG}(z) = \sin z$. The breather solution is given by

$$\Psi_b(x,t) = 4\tan^{-1}\left[\frac{m}{\omega}\frac{\sin(\omega t)}{\mathrm{ch}(mx)}\right] \ , \ \omega = \sqrt{1-m^2}. \tag{2}$$

It represents a field which is periodically oscillating in time and decays exponentially in space as the distance from the center $x = 0$ is increased.

Most probably these breather solutions are *nongeneric*. This statement is due to the following facts. Birnir showed [1] that sG breathers are *isolated*, i.e. the solutions survive only under a finite number of perturbations $\delta(z)$ of $F_{sG}(z) \to F_{sG}(z) + \delta(z)$. Years of searching for breathers in ϕ^4 systems ($F(z) = -z + z^3$) were terminated by the nonexistence proof of breathers by Segur and Kruskal [2]. Finally we are not aware of existence proofs of similar solutions in 1+2 dimensions. So we deal up to now with structurally unstable solutions, which are not interesting for most of the possible applications. The reason for that lies in the fact that a decomposition of (2) into a Fourier series with respect to time yields higher harmonics with frequencies $k\omega$ (k is the Fourier number). These frequencies resonate with the linear spectrum of (1) $\omega_q = \sqrt{Cq^2 + F'(z=0)}$ (q - wave number), if k is larger than a given number which depends on the choice of ω. Consequently the corresponding separatrix manifolds associated with (1) are of finite dimension. Together with the infinity of the dimension of the corresponding phase space the structural instability follows immediately [3].

1.2 The Lattice Case

Now let us consider a lattice, which is obtained from say (1) by replacing the continuous x-axis with an equidistant set of points labeled with l, and by replacing the second derivative with respect to x in (1) with a second difference. In a more general fashion the system can be described by the following Hamilton function:

$$H = \sum_l \left[\frac{1}{2}\dot{X}_l^2 + \Phi(X_l - X_{l-1}) + V(X_l)\right]. \tag{3}$$

Here $\dot{A}$ denotes a time derivative, and the equations of motion are given by $\ddot{X}_l = -\partial H/\partial X_l$. The main change by going over from (1) to (3) is the change of continuous translational symmetry to discrete translational symmetry. This

reduction of symmetry has several consequencies for the dynamics of the system. The potential functions $V(z)$ and $\Phi(z)$ can be expanded around the energy minimum:

$$V(z) = \sum_{\mu=2,3,..} \frac{v_\mu}{\mu} z^\mu \ , \ \Phi(z) = \sum_{\mu=2,3,..} \frac{\phi_\mu}{\mu} z^\mu. \tag{4}$$

The linear spectrum is now given by $\omega_q^2 = v_2 + 4\phi_2 \sin^2(\pi q/2)$. As opposed to (1) the linear spectrum of (4) is *bounded* from above : $v_2 \leq \omega_q^2 \leq (v_2 + 4\phi_2)$. Clearly this can change the properties of the mentioned separatrix manifolds drastically. Systems of the type (3) are the simplest realizations of models widely used in many areas of physics, as e.g. solid state physics, arrays of Josephson junctions etc, where the discreteness of the system plays an important role.

2 Discrete Breathers

Let us search for spatially localized and time-periodic solutions of (3) which can be coined discrete breathers due to the similarity to the sG breather solutions. The required periodicity in time allows to expand the ansatz in a Fourier series:

$$X_l(t) = \sum_{k=-\infty}^{+\infty} A_{kl} e^{ik\omega t} \ , \ A_{k,l\to\pm\infty} \to 0. \tag{5}$$

2.1 Necessary Existence Condition

Inserting this ansatz into the equations of motion and eliminating time we arrive at a nonlinear coupled set of algebraic equations for the coefficients A_{kl} [4]. This set can be rewritten as a map

$$A_{k,(l+1)} = M(\{A_{k'l}\}, \{A_{k',(l-1)}\}). \tag{6}$$

M has a fixed point $A_{kl} = 0$. Linearizing M around this fixed point we obtain[b]

$$A_{k,(l+1)} = [2\kappa_k + 2]\, A_{kl} - A_{k,(l-1)} \ , \ \kappa_k = \frac{v_2 - k^2\omega^2}{2\phi_2}. \tag{7}$$

The fixed point $A_{kl} = 0$ of (7) is an elliptic one if $k\omega = \omega_q$ and a hyperbolic one if $k\omega \neq \omega_q$. Since only the hyperbolicity of the fixed point suites the required

[b]There are limitations on the linearization procedure which apply when some multiple $k\omega$ comes too close to the linear spectrum [5].

spatial localization property of the ansatz (5) the necessary condition for the existence of a generic discrete breather is

$$k\omega \neq \omega_q. \tag{8}$$

Clearly we can satisfy this condition for a lattice[c] because the linear spectrum is bounded from above - in contrast to the continuous case, where condition (8) can be never fulfilled. The algebraic equations (6) can be solved numerically [5]. From (7) it follows that a discrete breather is characterized by an exponential decay in space with k-dependent exponents [4,5]:

$$A_{kl} \sim [\mathrm{sgn}(\lambda_k)]^l \, \mathrm{e}^{\ln|\lambda_k| l} \, , \; \lambda_k = 1 + \kappa_k \pm \sqrt{(1+\kappa_k)^2 - 1}, \tag{9}$$

where the sign has to be chosen such that $|\lambda_k| < 1$. Numerical solutions nicely reproduce those features [5].

In the limit of large values of k it follows $|\lambda_k| \approx \omega^2 k^2/\phi_2$ and consequently the k-dependence of A_{kl} is given by [5] $A_{kl} \sim k^{-2|l|} s(k)$, where $s(k)$ is a monotonous decreasing function with increasing k. Thus the decay of A_{kl} in the k-space is stronger than $k^{-2|l|}$ (here $|l|$ measures the distance from the center of the solution). This is one of the reasons why the Rotating Wave Approximation used by Takeno [6] and others (it amounts to neglecting all but the first Fourier component) often produces approximate solutions quite close to the exact ones.

2.2 Structural Stability

For the 1d lattice (where (6) applies) it is straightforward to conclude, that in the case of (8) a discrete breather is *structurally stable* [7]. This follows from the circumstance that if (8) applies for an existing discrete breather solution, the stable and unstable invariant manifolds of the fixed point $A_{kl} = 0$ of (6) have common homoclinic points and dimension half of the phase space dimension of map (6) [7]. Then either i) small perturbations of the Hamiltonian (3) preserve the existence of those homoclinic points (and thus the existence of a discrete breather) or ii) there exist 'right' perturbations such that the discrete breather becomes structurally stable in the vicinity of the new perturbed Hamiltonian[d]

If the condition (8) is violated for a given number n of k-values, then the separatrix manifold dimensions are reduced by that number [7]. Consequently

[c]Note that for higher dimensional lattices one can find the same condition through an analysis of the corresponding linearized equations with the only difference that these equations do not constitute a map as in the 1d case.

[d]No analogous result is known for higher dimensions, because the problem can not be reduced to a map; still there is evidence that discrete breathers are structurally stable in higher dimensions too [8].

even if a discrete breather exists, it would be structurally unstable, since always $2n$ perturbations of (3) exist which destroy the solution. In the continuum limit $(C \to \infty)$ $n \to \infty$ for any finite ω, and thus no structurally stable breathers remain in the continuous case - in agreement with the results quoted in the introduction.

Further it follows [4], that no structurally stable localized solutions exist, with quasiperiodic time dependence. This steams from the fact that the condition (8) has then to be replaced by $k_1\omega_1 + k_2\omega_2 + ... + k_m\omega_m \neq \omega_q$, where the ratios of the frequencies ω_i are irrational. It is straightforward to show that there exists an infinite number of combinations $(k_1, k_2, .., k_m)$ such that the new condition is violated for any choice of the frequencies ω_i [4].

2.3 Existence Proofs

Up to now we know about two rigorous existence proofs for discrete breathers. MacKay and Aubry considered weakly coupled anharmonic oscillators, i.e. $V(z)$ anharmonic, $\Phi(z)$ small [9]. Then it is possible to show that time-periodic localized solutions of the trivial case $\Phi(z) = 0$ can be analytically continued into the weakly coupled regime if the condition $\omega/\sqrt{v_2}$ irrational is met. Remarkably this proof goes equally well for any lattice dimension, and is quite robust to variations in the interaction range.

The second proof is due to Flach and it considers a 1d system with homogeneous potentials [7]: $(V(\lambda z) + \Phi(\lambda z)) = \lambda^{2m}(V(z) + \Phi(z))$. Due to this additional symmetry discrete breather solutions have the form $X_l(t) = A_l G(t)$, i.e. space-time separability applies. The resulting two-dimensional map for A_l is analyzed and the existence of homoclinic points is shown to be true [7]. Note that this proof also incorporates the special case $V(z) = 0$, where only the interaction potential remains - in contrast to the weak coupling limit of the existence proof by MacKay and Aubry [9].

2.4 Dynamical Stability

If we perturb the trajectory of an exact discrete breather solution then it is important (with respect to applications) to know how long the new trajectory evolves closely enough to the old localized solution. In the previous chapter we have already shown that quasiperiodic breathers do not exist in general. Consequently a perturbed breather will be an object which radiates energy out of its center (because otherwise it would be a new non-periodic breather solution!). There exists no general approach to the problem of stability - actually in most of the nonintegrable systems we will run into the problem of small denominators and ultimate chaos. In a numerical investigation [10]

perturbed breathers could either radiate a bit of energy and become essentially time-periodic again (and thus exact) or internal resonances could evolve, which eventually lead to chaotic motion *confined* to the breather volume. That chaotic motion leads to an increase in the radiation power of several orders of magnitude. Alltogether a rather complex scenario seems to evolve.

It is useful to linearize the equations of motion for the perturbation $\delta_l = X_l - X_l(t)$ (here $X_l(t)$ denotes the time-periodic breather solution):

$$\ddot{\delta}_l = - \left. \frac{\partial^2 H}{\partial X_l \partial X_{l'}} \right|_{X_l = X_l(t)} \delta_{l'}. \tag{10}$$

These equations are a generalization of Hill's equation [11]. The problem amounts to diagonalizing a matrix. Since the breather is a localized solution, one can immediately state that all extended eigenvectors have frequencies of the linear spectrum. Consequently time-periodic breathers are stable with respect to small amplitude phonon perturbations if the condition $k\omega \neq 2\omega_q$ is fulfilled [10]. Note that this stability condition includes the already obtained existence condition (8). The localized eigenvectors (and their corresponding eigenvalues) have to be found through numerical diagonalizations.

2.5 Breather Birth due to Band Edge Phonon Bifurcations

For small systems (few degrees of freedom) and small amplitudes/energies the phonon orbits are surely stable. So breathers have to occur through certain bifurcations of the linear orbits. The analysis of stability of band edge phonons has been done in a recent work [12]. The result is that for large systems only the first six expansion coeffiecients of the potentials have to be known: $v_2, v_3, v_4, \phi_2, \phi_3, \phi_4$. Then the phonon orbit is characterized by a nonnegative energy. The phonon orbit becomes unstable at a bifurcation energy $E_c \sim N^{-2/d}$ where N is the size of the system and d the dimensionality. The proportionality factor is required to be positive. This condition is in most of the cases equivalent to the condition that *the frequency of the band edge phonon orbit is repelled from the linear spectrum with increase in energy.* The new bifurcating periodic orbits are shown to be *not* invariant under the discrete translational symmetry of the system. These new orbits eventually become discrete breathers [12].

2.6 Movability

It is possible to create moving discrete breathers [13]. We remind the reader that this is nontrivial due to the low symmetry of the lattice as compared

to a field equation. Up to now there exist no proofs of existence of moving breathers. All reports refer to numerical simulations with finite simulation time. We can consider two cases: i) the discrete breather is weakly localized and ii) the discrete breather is strongly localized. In case i) we are close to the continuous description, and moving breathers are not a surprise. However case ii) is nontrivial. It turns out that up to now case ii) has been reported only for one-dimensional lattices with $V(z) = 0$ (so-called Fermi-Pasta-Ulam systems) [13]. All other systems considered did not show up with moving breathers. So the movability in its strict sense appears to be a quite isolated phenomenon.

Due to the circumstance that time-periodic stationary breathers come in one-parameter families, the excitation of a moving entity starting with an exact stationary breather does not necessarily require changes in the energy. Thus the idea of defining a Peierls-Nabarro potential for the moving entity (in analogy to the problem of moving kinks in lattices) is ill-defined. The high dimensionality of the separatrix which separates stationary objects (perturbed breathers) from possible moving entities is the reason for having an infinite number of possibilities to cross the separatrix with increase, decrease or even no change in energy [14].

2.7 Interaction with Phonons and Electrons

Breathers can strongly scatter phonons and electrons. The phonon scattering problem is in essence similar to the problem of dynamical stability of a breather. Numerical studies revealed that the transmission coefficient for phonons (through a breather) decreases exponentially with increasing wave number [15]. The breather interaction with electrons has been considered within adiabatic approximation, where the lattice vibrations are described classically [16]. Then it is straightforward to show that band electrons are scattered by a breather who acts as a time-dependent (periodic) dipole obstacle. Band edge electrons can be trapped by a breather because band edge states become localized due to the localized character of the corresponding potential mediated by the breather.

3 Conclusion

Discrete breathers are time-periodic localized solutions of classical Hamiltonian lattice equations. Discrete breathers are structurally stable and generic - as opposed to their continuum relatives (e.g. the sG breather). The main reason for this is the existence of a finite upper bound of the linear spectrum of the lattice. Consequently it becomes easy to escape from resonances be-

tween multiples of the breather's frequency and the linear spectrum. Discrete breathers can be dynamically stable too. This, together with the fact that discrete breathers come in one-parameter families, makes these objects highly interesting for any physical problem where discreteness is of importance. Discrete breathers appear independently of the lattice dimension - this is a drastic difference as compared to conventional soliton theory. The possibility of local excitation of a lattice whithout subsequent dispersion of the energy pulse makes breathers highly interesting for applications.

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REDUCTIVE PERTURBATION METHOD, MULTIPLE–TIME SOLUTIONS AND THE KdV HIERARCHY

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We apply a multiple–time version of the reductive perturbation method to study long waves as governed by the Boussinesq model equation. By requiring the absence of secular producing terms in each order of the perturbative scheme, we show that the solitary–wave of the Boussinesq equation can be written as a solitary–wave satisfying simultaneously all equations of the KdV hierarchy, each one in a different slow time variable. We also show that the conditions for eliminating the secularities ar e such that they make the perturbation theory compatible with the linear theory coming from the Boussinesq equation.

1 Introduction

As is well known, the intermediate long–wave Boussinesq model equation

$$u_{tt} - u_{xx} + u_{xxxx} - 3(u^2)_{xx} = 0\,, \tag{1}$$

with $u(x,t)$ a one–dimensional field and with the subscripts denoting partial differentiation, is completely integrable[1] and has N–soliton solutions. In particular, its solitary–wave solution is of the form[2]

$$u = -2k^2 \operatorname{sech}^2\left[k\left(x - \sqrt{1-4k^2}\,t\right)\right]\,, \tag{2}$$

where k is the wavenumber.

To study the long waves of eq.(1), we are going to consider a perturbative scheme based on the reductive perturbation method of Taniuti,[3] modified by the introduction of an infinite number of slow time-variables: τ_3, τ_5, τ_7, etc. Then, as a consequence of a natural compatibility condition, we have that any wave field satisfying the KdV equation in the time τ_3 must also satisfy all

equations of the KdV hierarchy,[4] each one in a different slow time variable. The main reason for introducing these time variables, as we are going to see, is that they allow for the construction of a perturbative scheme, valid for weak nonlinear dispersive systems, which is free of solitary–wave related secularities.

In this paper, by making use of the perturbative scheme with multiple slow time–scales, we will show that the solitary–wave of the Boussinesq equation may be written, in the slow variables, as a solitary–wave solution to the whole set of equations of t he KdV hierarchy, each one in a different slow time variable. This result follows both, from the general long–wave perturbation theory, and from the observation that the perturbative series truncates for a solitary–wave solution to the KdV hierarchy equ ations, rendering thus an exact solution for the Boussinesq equation. Furthermore, we will show that the conditions for the elimination of the secular producing terms make the perturbation theory compatible with the linear theory associated to the Boussin esq equation.

2 The Multiple Time Evolution Equations

The long–wave limit is achieved by putting

$$k = \epsilon\kappa\,, \tag{3}$$

where ϵ is a small parameter. Accordingly, we define a slow space

$$\xi = \epsilon(x - t)\,, \tag{4}$$

as well as an infinity of slow time coordinates:

$$\tau_3 = \epsilon^3 t \quad ; \quad \tau_5 = \epsilon^5 t \quad ; \quad \tau_7 = \epsilon^7 t \quad ; \quad \dots \quad . \tag{5}$$

Consequently, we have that

$$\frac{\partial}{\partial x} = \epsilon\frac{\partial}{\partial \xi} \tag{6}$$

and

$$\frac{\partial}{\partial t} = -\epsilon\frac{\partial}{\partial \xi} + \epsilon^3\frac{\partial}{\partial \tau_3} + \epsilon^5\frac{\partial}{\partial \tau_5} + \epsilon^7\frac{\partial}{\partial \tau_7} + \cdots \quad . \tag{7}$$

In addition, we make the expansion

$$u = \epsilon^2\hat{u} = \epsilon^2\left(u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \cdots\right)\,, \tag{8}$$

and we suppose that $u_{2n} = u_{2n}(\xi, \tau_3, \tau_5, \ldots), n = 0, 1, 2, \ldots$, which corresponds to an extension in the sense of Sandri.[5] Substituting eqs.(6), (7) and (8) into

the Boussinesq equation (1), the res ulting expression, up to terms of order ϵ^4, is:

$$\Big[-2\frac{\partial^2}{\partial\xi\partial\tau_3}+\frac{\partial^4}{\partial\xi^4}+\epsilon^2\left(\frac{\partial^2}{\partial{\tau_3}^2}-2\frac{\partial^2}{\partial\xi\partial\tau_5}\right)+\epsilon^4\left(-2\frac{\partial^2}{\partial\xi\partial\tau_7}+2\frac{\partial^2}{\partial\tau_3\partial\tau_5}\right)+\cdots\Big]\hat{u}$$

$$-3\frac{\partial^2}{\partial\xi^2}\Big[(u_0)^2+2\epsilon^2u_0u_2+\epsilon^4(2u_0u_4+(u_2)^2)+\cdots\Big]=0\,. \qquad (9)$$

We proceed now to an order–by–order analysis of this equation. At order ϵ^0, after an integration in ξ, we get

$$\frac{\partial u_0}{\partial\tau_3}=\alpha_3\left[6u_0\frac{\partial u_0}{\partial\xi}-\frac{\partial^3u_0}{\partial\xi^3}\right]=0\quad;\quad\alpha_3=-\frac{1}{2}\,, \qquad (10)$$

which is the KdV equation.

At order ϵ^2, eq.(9) yields

$$\frac{\partial}{\partial\xi}\left[-2\frac{\partial u_2}{\partial\tau_3}-6\frac{\partial}{\partial\xi}(u_0u_2)+\frac{\partial u_2}{\partial\xi^3}\right]=2\frac{\partial^2u_0}{\partial\xi\partial\tau_5}-\frac{\partial^2u_0}{\partial{\tau_3}^2}\,. \qquad (11)$$

Using the KdV equation (10) to express $\partial u_0/\partial\tau_3$, integrating once in ξ and assuming a vanishing integration constant, we obtain

$$\frac{\partial u_2}{\partial\tau_3}+3\frac{\partial}{\partial\xi}(u_0u_2)-\frac{1}{2}\frac{\partial^3u_2}{\partial\xi^3}=-\frac{\partial u_0}{\partial\tau_5}+\frac{1}{8}\frac{\partial^5u_0}{\partial\xi^5}$$

$$-\frac{3}{2}u_0\frac{\partial^3u_0}{\partial\xi^3}-\frac{9}{4}\frac{\partial u_0}{\partial\xi}\frac{\partial^2u_0}{\partial\xi^2}+\frac{9}{2}(u_0)^2\frac{\partial u_0}{\partial\xi}\,. \qquad (12)$$

Equation (12), as it stands, presents two problems. First, the evolution of u_0 in the time τ_5 is not known *a priori*. The second problem is that the term $(\partial^5u_0/\partial\xi^5)$, as a source term, is a secular producing term when u_0 is chosen to be a solitary–wave solution of the KdV equation. In the next sections we will be dealing with these two problems.

3 The Korteweg–de Vries Hierarchy

As we have seen, the field u_0 satisfies the KdV equation (10) in the time τ_3. The evolution of the same field u_0 in any of the higher–order times τ_{2n+1} can be obtained in the following way.[6] First, to have a well ordere d perturbative scheme we impose that each one of the equations for $u_{0\tau_{2n+1}}$ be ϵ–independent when passing from the slow (u_0,ξ,τ_{2n+1}) to the laboratory coordinates (u,x,t).

This step selects all possible terms to appear in $u_{0\tau_{2n+1}}$. For instance, the evolution of u_0 in time τ_5 is restricted to be of the form

$$u_{0\tau_5} = \alpha_5 u_{0(5\xi)} + \beta_5 u_0 u_{0\xi\xi\xi} + (\beta_5 + \gamma_5) u_{0\xi} u_{0\xi\xi} + \delta_5 u_0^2 u_{0\xi} \,, \qquad (13)$$

where α_5, β_5, γ_5 and δ_5 are unknown constants. Then, by imposing the natural (in the multiple time formalism) compatibility condition

$$\left(u_{0\tau_3}\right)_{\tau_{2n+1}} = \left(u_{0\tau_{2n+1}}\right)_{\tau_3} \,, \qquad (14)$$

with $u_{0\tau_3}$ given by eq.(10), it is possible to determine any $u_{0\tau_{2n+1}}$, *i.e.*, to determine all constants appearing in $u_{0\tau_{2n+1}}$. As it can be verified,[6] the resulting equations are those given by the KdV hier archy. In particular, for $u_{0\tau_5}$ and $u_{0\tau_7}$ we obtain respectively

$$u_{0\tau_5} = \alpha_5 \left[u_{0(5\xi)} - 10 u_0 u_{0\xi\xi\xi} - 20 u_{0\xi} u_{0\xi\xi} + 30 (u_0)^2 u_{0\xi}\right] \,, \qquad (15)$$

and

$$\begin{aligned} u_{0\tau_7} = \alpha_7 \big[& - u_{0(7\xi)} + 14 u_0 u_{0(5\xi)} + 42 u_{0\xi} u_{0(4\xi)} + 140 (u_0)^3 u_{0\xi} \\ & + 70 u_{0\xi\xi} u_{0\xi\xi\xi} - 280 u_0 u_{0\xi} u_{0\xi\xi} - 70 (u_{0\xi})^3 - 70 (u_0)^2 u_{0\xi\xi\xi}\big] \,, \end{aligned} \qquad (16)$$

where α_5 and α_7 are free parameters not determined by the algebraic system originated from eq.(14). These free parameters are related to different possible normalizations of the slow time variables.

4 Higher Order Evolution Equations

We return now to eq.(12) for u_2. Substituting $u_{0\tau_5}$ from eq.(15), we obtain

$$\begin{aligned} u_{2\tau_3} + 3(u_0 u_2)_\xi - \frac{1}{2} u_{2\xi\xi\xi} = & \left[\frac{1}{8} - \alpha_5\right] u_{0(5\xi)} + \left[-\frac{3}{2} + 10\alpha_5\right] u_0 u_{0\xi\xi\xi} \\ & + \left[-\frac{9}{4} + 20\alpha_5\right] u_{0\xi} u_{0\xi\xi} + \left[\frac{9}{2} - 30\alpha_5\right] (u_0)^2 u_{0\xi} \,. \end{aligned} \qquad (17)$$

We see thus that the secular–producing term $u_{0(5\xi)}$ can be eliminated if we choose $\alpha_5 = \frac{1}{8}$. In this case, Eq.(17) becomes

$$u_{2\tau_3} + 3(u_0 u_2)_\xi - \frac{1}{2} u_{2\xi\xi\xi} = -\frac{1}{4} \left[-3(u_0)^2 u_{0\xi} + u_0 u_{0\xi\xi\xi} - u_{0\xi} u_{0\xi\xi}\right] \,. \qquad (18)$$

From this point on, we are going to consider some specific solutions to our equations. First of all, we assume the solution of the KdV equation (10) to be the solitary–wave solution

$$u_0 = -2\kappa^2 \mathrm{sech}^2\left[\kappa\xi - 4\alpha_3\kappa^3\tau_3 + \theta\right] , \tag{19}$$

where θ is a phase. However, we have just seen that u_0 must satisfy also the higher order equation (15) of the KdV hierarchy. Actually, as we are going to see, to obtain a perturbative scheme free of secularities at any higher order, we wi ll assume that u_0 be a solitary–wave solution to all equations of the KdV hierarchy, each one in a different slow–time variable. Such a solution is given by

$$u_0 = -2\kappa^2 \mathrm{sech}^2\left[\kappa\xi - 4\alpha_3\kappa^3\tau_3 + 16\alpha_5\kappa^5\tau_5 - 64\alpha_7\kappa^7\tau_7 + \cdots\right] . \tag{20}$$

Using this solitary–wave solution, we see that the right–hand side of eq.(18) vanishes, leading to

$$u_{2\tau_3} + 3(u_0u_2)_\xi - \frac{1}{2}u_{2\xi\xi\xi} = 0 , \tag{21}$$

which is a homogeneous linearized KdV equation. We will assume for it the trivial solution $u_2 = 0$.

At order ϵ^4, and already assuming that $u_2 = 0$, eq.(9) gives

$$u_{4\tau_3\xi} + 3(u_0u_4)_{\xi\xi} - \frac{1}{2}u_{4(4\xi)} = -u_{0\tau_7\xi} + u_{0\tau_3\tau_5} . \tag{22}$$

Using equations (10) and (15) to express $u_{0\tau_3}$ and $u_{0\tau_5}$ respectively, and integrating once in ξ, we obtain

$$u_{4\tau_3} + 3(u_0u_4)_\xi - \frac{1}{2}u_{4\xi\xi\xi} = -u_{0\tau_7} + \frac{1}{16}u_{0(7\xi)} - u_0u_{0(5\xi)} + \frac{45}{8}(u_0)^2u_{0\xi\xi\xi}$$
$$- \frac{35}{8}u_{0\xi\xi}u_{0\xi\xi\xi} - \frac{5}{2}u_{0\xi}u_{0(4\xi)} + \frac{75}{4}u_0u_{0\xi}u_{0\xi\xi} - \frac{45}{4}(u_0)^3u_{0\xi} + \frac{15}{4}(u_{0\xi})^3 . \tag{23}$$

The source term proportional $u_{0(7\xi)}$ is the only resonant, that is, secular producing term to the solution u_4. Then, in the very same way we did before, we first use the KdV hierarchy equation (16) to express $u_{0\tau_7}$. After we do that , we can then choose the free parameter α_7 in such a way to eliminate the resonant term from the right–hand side of eq.(23). This choice corresponds to $\alpha_7 = -\frac{1}{16}$, which brings eq.(23) to the form

$$u_{4\tau_3} + 3(u_0u_4)_\xi - \frac{1}{2}u_{4\xi\xi\xi} = \frac{1}{8}\big[u_{0\xi}u_{0(4\xi)} - u_0u_{0(5\xi)}$$
$$+10u_0u_{0\xi}u_{0\xi\xi} - 5(u_{0\xi})^3 + 10(u_0)^2u_{0\xi\xi\xi} - 20(u_0)^3u_{0\xi}\big] . \tag{24}$$

Substituting again the solitary–wave solution (20) for u_0, we can easily see that the nonhomogeneous term of eq.(24) vanishes, leading to

$$u_{4\tau_3} + 3(u_0 u_4)_\xi - \frac{1}{2} u_{4\xi\xi\xi} = 0 \,. \tag{25}$$

And again, we take the trivial solution $u_4 = 0$.

It is easy to see that this is a general result that will repeat at any higher order: for $n \geq 1$, the evolution of u_{2n} in the time τ_3, after using the KdV hierarchy equation to express $u_{0\tau_{2n+1}}$ and substituting the solitary–wave solution (20) for u_0, is given by a homogeneous linearized KdV equation. Consequently, the solution $u_{2n} = 0$, for $n \geq 1$, can be assumed for any higher order.

5 Back to the Laboratory Coordinates

Let us now take the solitary–wave solution to all equations of the KdV hierarchy,

$$u_0 = -2\kappa^2 \mathrm{sech}^2 \left[\kappa\xi + 2\kappa^3\tau_3 + 2\kappa^5\tau_5 + 4\kappa^7\tau_7 + \cdots\right] , \tag{26}$$

where we have already substituted the (not anymore) free parameters α_{2n+1}, and rewrite it in the laboratory coordinates. First, recall that we have expanded u according to eq.(8). Thereafter, we have found a particular solution in which $u_{2n} = 0$, for $n \geq 1$. Consequently, expansion (8) truncates, leading to an exact solution of the form

$$u = \epsilon^2 u_0 \,, \tag{27}$$

with u_0 given by eq.(26). Moreover, the slow variables $(\kappa, \xi, \tau_{2n+1})$ are related to the laboratory ones, (k, x, t), respectively by eqs.(3), (4) and (5). Then, in the laboratory coordinates, the exact solution (27) is written as

$$u = -2k^2 \mathrm{sech}^2 \, k \left[x - \left(1 - 2k^2 - 2k^4 - 4k^6 - \cdots\right) t\right] . \tag{28}$$

Now, the series appearing inside the parenthesis can be summed:

$$1 - 2k^2 - 2k^4 - 4k^6 - \cdots = \left(1 - 4k^2\right)^{1/2} \,. \tag{29}$$

Consequently, we get

$$u = -2k^2 \mathrm{sech}^2 \left[k \left(x - \sqrt{1 - 4k^2}\, t\right)\right] , \tag{30}$$

which is the well known solitary–wave solution of the Boussinesq equation (1).

6 Relation to the Dispersion Relation Expansion

Let us take now the linear Boussinesq dispersion relation:

$$\omega(k) = k\left(1+k^2\right)^{1/2} . \tag{31}$$

Its long–wave ($k = \epsilon\kappa$) expansion is given by

$$\omega(\kappa) = \epsilon\kappa + \alpha_3\epsilon^3\kappa^3 + \alpha_5\epsilon^5\kappa^5 + \alpha_7\epsilon^7\kappa^7 + \cdots , \tag{32}$$

where the coefficients α_{2n+1}, except for α_3 which arose naturally in the KdV equation (10), coincide exactly with those necessary to eliminate the secular producing terms in each order of the perturbative scheme. With this expansion, the solution of the associated linear Boussinesq equation can be written as

$$u = \exp i\left[\kappa\epsilon(x-t) + \alpha_3\kappa^3\epsilon^3 t + \alpha_5\kappa^5\epsilon^5 t + \alpha_7\kappa^7\epsilon^7 t + \cdots\right] . \tag{33}$$

Therefore, if we define from the very beginning, as given by this solution, the properly normalized slow time coordinates

$$\tau_3 = \alpha_3\epsilon^3 t \quad ; \quad \tau_5 = \alpha_5\epsilon^5 t \quad ; \quad \tau_7 = \alpha_7\epsilon^7 t \quad ; \quad \ldots , \tag{34}$$

the resulting perturbative theory will be automatically free of secularities.[7] Furthermore, the linear limit of the perturbation theory will be compatible with the linear theory coming directly from the Boussinesq equation (1).

7 Final Comments

By applying a multiple time version of the reductive perturbation method to the Boussinesq model equation, and by eliminating the solitary–wave related secular producing terms through the use of the KdV hierarchy equations, we have shown that the solitar y–wave of the Boussinesq equation is given by a solitary–wave satisfying, in the slow variables, all equations of the KdV hierarchy. Accordingly, while the KdV solitary–wave depends only on one slow time variable, namely τ_3, the solitary–wave of the Boussinesq equation can be thought of as depending on the infinite slow time variables.

The same results hold [8] for the shallow water wave (SWW) equation. In other words, the solitary–wave of the SWW equation can also be written as a solitary–wave satisfying simultaneously, in the slow variables, all equations of the KdV hierarchy . It is important to remark that in both cases the resulting secularity–free perturbation theory will be automatically compatible with the corresponding linear theory.

A crucial point of the multiple time perturbative scheme is the return to the laboratory coordinates, which implies in a renormalization of the solitary–wave velocity.[9] In the case of the Boussinesq, as well as of the SWW equation, this renorma lization is such that the KdV hierarchy solitary–wave is led to the corresponding Boussinesq or SWW solitary–waves. However, when the original nonlinear dispersive system does not present an exact solitary–wave solution, the series will not truncate. I n this case, a secularity–free expansion can still be obtained and the process of returning to the laboratory coordinates can be made order–by–order at any higher order, implying in a successive renormalization in the velocity of the solitary–wave, which in this case is represented by the leading order term of the perturbative series.[6]

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SECOND HARMONIC OF NONLINEAR RESPONSE OF MAGNETS

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Over the last years a marked experimental and theoretical progress has been made in study of the second harmonic longitudinal nonlinear response (NLR) (constant and ac magnetic fields are parallel to each other) of magnets. The NLR has been investigated in dilute solid paramagnets and their solutions [1] as well as in concentrated exchange magnets [2,3]. The aim of this paper is to present the basic theoretical approaches and equations which are needed for an analysis of the NLR. The obtained results on definite systems will be described briefly to clarify the possibilities of this method.

1 General symmetry properties of the NLR

The NLR is investigated experimentally by the following way. A sample under study is placed in parallel steady magnetic field H and harmonic ac magnetic field of amplitude h_0 and frequency ω, $h(t) = h_0 \cos \omega t$. Two phase components of the second harmonic of magnetization M_2 are recorded:

$$M_2(t) = M_2' \cos 2\omega t + M_2'' \sin 2\omega t \,. \tag{1}$$

The amplitudes $M_2^{(\prime,\prime\prime)}$ can be investigated as the functions of H, ω, h_0 and temperature T.

The even harmonics are even functions of h_0:

$$M_2 = M_2(h_0^2)\,, \qquad M_2^{(\prime,\prime\prime)}(h_0^2) = m_2^{(\prime,\prime\prime)} h_0^2 + \cdots \tag{2}$$

Here $m_2^{(\prime,\prime\prime)}$ are first terms of $M_2^{(\prime,\prime\prime)}$ expansion in h_0^2.

The M_2 (as a magnetization) is a pseudovector and, being an even function of h_0, it is odd in H in the paramagnetic phase:

$$M_2(H) = -M_2(-H)\,, \qquad M_2(0) = 0\,. \tag{3}$$

Thus, the constant field is needed to observe the NLR in the paramagnetic phase. The M_2 is very sensitive to the appearance of spontaneous magnetization in system because $M_2 \neq 0$ at $H = 0$ in this case.

In the first order in h_0^2, the M_2 can be written

$$M_2(t) = \frac{1}{2}(\chi_2(\omega) e^{2i\omega t} + \chi_2^*(\omega) e^{-2i\omega t}) h_0^2\,. \tag{4}$$

Here $\chi_2(\omega)$ is the second-order dynamic susceptibility. By virtue of the reality of the response:

$$\chi_2(\omega) = \chi_2^*(-\omega)\,, \quad Re\chi_2(\omega) \equiv \chi_2'(\omega) = \chi_2'(-\omega)\,, \quad Im\chi_2(\omega) \equiv \chi_2''(\omega = -\chi_2''(-\omega)\,. \tag{5}$$

Since $\chi_2(\omega)$ describes a retarded response it is analytic in the upper halt ω plane. The two latter properties coincide with those of the linear susceptibility $\chi_1(\omega)$. In contrast to $\chi_1''(\omega)$, $\chi_2''(\omega)$ may change sign at $\omega > 0$ because an average value of energy absorbed by system due to $M_2(t)$ oscillates and vanishes at averaging over a period.

The convenient starting point for an analysis of the NLR is single spin systems which are dilute paramagnets and their solutions.

2 The NLR of single spin systems

The Hamiltonian has the form

$$\mathcal{H} = \mathcal{H}_{S0} + \mathcal{H}_S(t) + \mathcal{H}_{RS} + \mathcal{H}_R\,. \tag{6}$$

Here $\mathcal{H}_{S0}$ describes spin system, $\mathcal{H}_S(t)$ is interaction with the magnetic field ($\vec{H} \parallel \vec{h}_0$), $\mathcal{H}_{RS} \propto R_\alpha T_{\alpha\beta} S_\beta$ is interaction of spin and "lattice" R_α variables which provides spin relaxation, $T_{\alpha\beta}$ is coupling tensor, $\mathcal{H}_R$ is "lattice" Hamiltonian.

There are three mechanisms that leads to the NLR of these systems: 1) influence of ac field on the relaxation processes (relaxation NLR); 2) the presence of the resonance favored transitions in the parallel fields (resonance NLR); 3) nonlinear dependence of magnetization M on H.

The resonance NLR is similar to that which occurs in the traditional EPR studies. This phenomenon is well known and will not be discussed here. Some results on this matter are presented in [1,4].

The NLR of these systems is usually studied at $T \geq 77K$ where nonlinearity of $M(H)$ is a negligible small effect because $g\mu H/T \ll 1$.

The relaxation NLR will be considered in detail. The ac field modulates the energy levels between which the relaxation transitions occur. As a result, the stationary values of the probabilities of these transitions become functions of changing external field. This leads to the NLR. Using perturbative approach to kinetic equation for matrix density one can get the Bloch equation for magnetization of spin system in high temperature approximation [4]:

$$\frac{d}{dt}\Delta M(t) = -\lambda^2 \int_0^\infty d\tau K(\tau)(\Delta M(t) - \chi_0 h(t-\tau))(J_+(t,\tau) + J_-(t,\tau))\,,$$

$$J_{\pm}(t,\tau) = \exp\left(\mp i\omega_0\tau \pm \frac{g\mu h_0}{\omega} Im(e^{-i\omega t}(1-e^{i\omega\tau}))\right), \qquad (7)$$

$$\Delta M(t) = M(t) - < M > .$$

Here $\omega_0 = g\mu H$, $\chi_0 \propto S(S+1)/3T$ is static susceptibility, $\lambda^2 K(\tau)$ is determined by correlators $< R_\alpha(\tau)R_\beta(0) >$ of "lattice" variables, λ is introduced as an effective coupling ($K(0) = 1$), $K(\tau) = K(\tau/\tau_c)$, τ_c is a correlation time of "lattice" variables. The term with h_0 in $J_\pm$ corresponds to modulation of the levels. This equation is valid at high temperatures ($\omega/T \ll 1$, $\Delta E/T \ll 1$, ΔE is the interval between the levels for the favorable transitions) and $(\lambda\tau_c)^2 \ll 1$. At $\omega\tau_c \ll 1$ the eq.(7) is reduced to the Bloch equation with relaxation rate Γ depending on the instantaneous value of $H(t)$:

$$\begin{array}{c} \frac{d}{dt}\Delta M(t) = -\Gamma(\omega_0(t))(\Delta M(t) - \chi_0 h(t)), \\ \Gamma(\omega_0) = \lambda^2 \int_0^\infty K(\tau)\cos(\omega_0\tau)d\tau . \end{array} \qquad (8)$$

Under the condition $(g\mu h_0\tau_c)^2 \ll 1$, we get for $\chi_2(\omega)$:

$$\chi_2(\omega) = -\omega\frac{\partial\Gamma(\omega_0)}{\partial\omega_0}((-2i\omega + \Gamma(\omega_0))(-i\omega + \Gamma(\omega_0)))^{-1} g\mu\chi_0 . \qquad (9)$$

It is the net dynamic response ($\chi_2(0) = 0$) related to dependence of Γ on H. General analysis of eq.(7),(8) can be found in [1,4].

The net relaxation NLR of these systems is usually observed together with the resonance NLR or it is accompanied by effects of spin-spin interactions [1].

To analyse the $\chi_2(\omega)$ beyond the scope of perturbation theory one has to consider its general properties.

3 General properties of $\chi_2(\omega)$

Expression for $\chi_2(\omega)$ is obtained from the response of a spin system to ac field ($\vec{H} \parallel \vec{h}_0 \parallel z$, $M_z \equiv M$) switched off adiabatically a t $t = -\infty$ [4]:

$$\begin{array}{c} \chi_2(\omega) = \int_0^\infty dt_1 \int_0^\infty dt_2 Q(t_1,t_2) e^{i\omega(t_1+t_2)}, \\ Q(t_1,t_2) = Sp([M(t_2-t_1), M(t_2)][M,\rho]), \\ \rho = \exp(-\beta\mathcal{H})(Sp \quad \exp(-\beta\mathcal{H}))^{-1} . \end{array} \qquad (10)$$

Here $\mathcal{H}$ is the Hamiltonian, $\beta = T^{-1}$.

Note, $Q(t_1,t_2)$ is the three spin Green function. Static $\chi_2(0)$ is

$$\chi_2(0) = Sp\left(\int_0^\beta \Delta M(-i\beta_1)d\beta_1 \int_0^{\beta_2} d\beta_2(\Delta M(-i\beta_2))\Delta M\rho\right), \qquad (11)$$

$$\Delta M = M - < M > .$$

One can check that $\chi_2(0) = \frac{1}{2}\partial^2 M/\partial H^2$. Since $\chi_2(\omega)$ is analytic function in the upper-half ω plane,

$$\chi_2(\omega) = \frac{1}{\pi i} \int\!\!\!\!\!\!- _{-\infty}^{\infty} \frac{\chi_2(\omega')d\omega'}{\omega' - \omega} . \tag{12}$$

Using (12) for $\omega\chi_2(\omega)$ and letting $\omega \to 0$, we get the sum rule:

$$\int_{-\infty}^{\infty} \chi_2'(\omega)d\omega = 2\int_0^{\infty} \chi_2'(\omega)d\omega = 0 . \tag{13}$$

It means that $\chi_2(\omega)$ changes sign (at least once) at $\omega > 0$. Integrating in eq.(10) by parts, we find asymptotic behavior of $\chi_2(\omega)$ at $\omega \to \infty$: $\chi_2(\omega) \approx \chi_2''(\omega) \propto \omega^{-3}$.

In the high temperature limit (in the first order in β) we get from (10):

$$\chi_2(\omega) = -i\beta \int_0^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 e^{i\omega(t_1+t_2)} Sp(M[M(-t_1), \dot{M}(-t_2)])(SpI)^{-1} . \tag{14}$$

Integrating over t_2 in (14) gives $\chi_2(0) = 0$. Using this equality, we find from (12):

$$\int_{-\infty}^{\infty} \chi_2''(\omega)\omega^{-1}d\omega = 2\int_0^{\infty} \chi_2''(\omega)\omega^{-1} = 0 . \tag{15}$$

Thus, $\chi_2''(\omega)$ has at least one zero at $\omega > 0$. Eq.(14) gives the following asymptotic behavior: $\chi_2(\omega) \approx \chi_2'(\omega) \propto \omega^{-4}$ at $\omega \to$ "∞" ($\omega \ll \beta^{-1}$). Discussion of relationship of these properties of $\chi_2(\omega)$ and approximate expression (9) (as well as a more general expression obtained from eq.(7)) can be found in [4].

One can transform the eq.(10) to select the two contribution to $\chi_2(\omega)$ with different physics meaning. Using the Kubo formula and integration by parts, we get:

$$\chi_2(\omega) = \Phi(\omega)\chi_2(0) + \chi_{2dyn}(\omega) , \tag{16}$$

$$\Phi(\omega)\chi_2(0) = i\int_0^{\infty} e^{2i\omega t}dt < [M, M(t)]\rho \int_0^{\beta} M(-i\lambda)d\lambda > , \tag{17}$$

$$\chi_{2dyn}(\omega) = i\omega \int_0^{\infty} e^{i\omega t_1}dt_1 \int_{t_1}^{\infty} e^{i\omega t_2}dt_2 \cdot \tag{18}$$

$$\cdot < [M(t_2 - t_1), M(t_2)]\rho \int_0^{\beta} M(-i\lambda)d\lambda > ,$$

$$< \cdots >= Sp(\cdots)(Sp\exp(-\beta\mathcal{H}))^{-1} .$$

Here $\Phi(\omega)$ is the dynamic formfactor ($\Phi(0) = 1$), $\chi_{2dyn}(\omega)$ is the net dynamic part of the response ($\chi_{2dyn}(0) = 0$). The first term in eq.(16) is related to nonlinearity of M(H) whereas the second one is due to magnetic field effect on the relaxation processes. Eq.(9) is the perturbative expression for the latter. The approximate expression for $\chi_2(\omega)$ containing the both contributions is given below (eq.(20)).

4 The NLR exchange magnets in the paramagnetic phase

In these magnets the isotropic exchange interaction is much large than anisotropic (dipolar, for instance) forces that are responsible for relaxation of uniform magnetization and dynamic response to the uniform field. These weak forces can be treated perturbatively in a wide region above temperature of ordering. As a result, the following Bloch equation can be used to analyse the NLR of these magnets [2]:

$$\frac{\partial}{\partial t}\Delta M(t) = -\Gamma(\omega_0(t))(\Delta M(t) - \chi_0 h(t) - \chi_2(0)h^2(t)) . \qquad (19)$$

The equation takes into account two mechanisms of the NLR: dependence of the longitudinal relaxation rate Γ on $H(t)$ and nonlinearity of $M(H)$ (the term containing $\chi_2(0)$). Corresponding $\chi_2(\omega)$ has the form:

$$\chi_2(\omega) = \frac{\Gamma}{-2i\omega + \Gamma}\chi_2(0) - i\omega\frac{g\mu\frac{\partial\Gamma}{\partial\omega_0}}{(-2i\omega + \Gamma)(-i\omega + \Gamma)}\chi_0 . \qquad (20)$$

The conditions, under which this expression is valid, are discussed in [2] where the eq.(19), (20) used to analyse the NLR of a cubic ferromagnet in the critical region above T_c.

Investigation of the NLR of this magnet allowed to obtain dependences of $\chi_2(0)$ and Γ on H and $\tau = (T - T_c)/T_c$ as well as the spin diffusion coefficient. In recent work [3] the first data on the NLR of antiferromagnet $La_2CuO_{4+\delta}$ has been presented. This response was attributed to formation of regions that possess weak ferromagnetic properties.

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INTEGRABLE MODELS AND TOPOLOGICAL FIELD THEORY

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The classical non-abelian pure Chern-Simons action is related to non-relativistic models in (2+1)-dimensions, via a gauge fixing procedure in Hermitian symmetric spaces. In such models the matter fields are coupled to a residual gauge Chern-Simons field, associated with the isotropy group of the considered symmetric spaces. The connection with the completely integrable equations is discussed and the classical and quantum properties of these systems are explored.

1 Introduction

By dealing with theoretical physical models, we are often led to study their completely integrable "reduced" versions. Here we want to describe a different point of view, in which the completely integrable systems arise as particular formulations of the topological gauge field theories (TFT) of the Chern-Simons (CS) type [1]. The idea to adopt this approach comes from the observation that the equations of motion are expressed as the zero-curvature condition for certain linear connections, both in the CS theories and in the completely integrable systems. In the latter case, such connections provide a pair of compatible linear problems, which together with the equations of motion are left invariant by certain gauge transformations. These take the special form of the (auto)Bäcklund Transformations and of the gauge equivalence mappings. In opposition to the former case, in the gauge equivalence we are allowed to change the parametrization of the connections in question. For instance, this happens between the Nonlinear Schrödinger Equation (NLSE) and the continuous Heisenberg model (HM) [2]. Analogously, in 2+1 dimensions, we can map the Ishimori equation (IE) into the Davey-Stewartson equation (DSE) [3]. Actually, such type of transformations exist independently from the integrability properties [4]. In fact, by interpreting the spin variables as elements of a coset space $\mathcal{G}/\mathcal{H}$, the geometrical characterization of this type of transformations

is given in term of zero-curvature conditions for the associated chiral currents. Then, we are led to look for a gauge field theory, in which the equations of motion are expressed as zero-curvature conditions, which admit several specific realizations by a gauge fixing for the current components, taking values in the Lie algebra of $\mathcal{H}$. This can be done by including specific spin dynamics as further restrictions on the currents. Some of them are completely integrable.

2 Chern-Simons theory on symmetric spaces

We suppose that the theory is defined by the CS action given by

$$S[J] = \frac{k}{4\pi} \int_{\mathcal{M}} Tr\,(J \wedge dJ + \frac{2}{3} J \wedge J \wedge J) \quad , \tag{1}$$

where J is a 1-form gauge connection with values in the Lie algebra $\hat{g}$ of a compact non-abelian simple Lie group $\mathcal{G}$ on an oriented closed 3-dimensional manifold $\mathcal{M}$. The corresponding classical equation of motion is the zero-curvature condition

$$F \equiv dJ + J \wedge J = 0 \quad . \tag{2}$$

This model has received a great attention in the last few years, since it provides a general covariant field theory [5]. This is a TFT in the sense that it possesses observables which are metric independent, the vacuum expectation values are invariants under smooth deformations on $\mathcal{M}$ and are related to the Jones polynomials of the knots theory [6]. From another point of view, models of point particles coupled to a CS gauge field in 2+1 dimensions have gained attention, in connection with the study of the fractional quantum Hall effect and of the high temperature superconductivity [7,8]. Within the classical field approach, we are led to equations of the gauged NLSE type. In the static self-dual situation, such systems become the Liouville equation and its integrable multicomponent generalizations [9] and to the static reductions of the IE and the DSE [10].

The action (1) is manifestly invariant under general coordinate transformations. Moreover, it is invariant under infinitesimal gauge transformations $G \simeq I + \lambda$ ($\lambda : \mathcal{M} \rightarrow \hat{g}$), acting on J as $\delta J = [J, \lambda] + d\lambda$.

Now, we choose in $\mathcal{G}$ a closed isotropy subgroup $\mathcal{H}$, such that the Lie algebra $\hat{g} = \hat{l}^{(0)} \oplus \hat{l}^{(1)}$ ($\hat{l}^{(0)}$ is the Lie algebra of $\mathcal{H}$) satisfies the so-called $\mathbf{Z}_2$ - graduation condition and we endow the space $\mathcal{G}$ /$\mathcal{H}$ with a complex structure, i.e. it is a Hermitian symmetric spaces [11]. Hence, the current J has the form

$$J = J^{(0)} + J^{(1)} \quad , \tag{3}$$

where $J^{(i)}$ are 1-forms taking values in $\hat{l}^{(i)}$, and the CS action (1) becomes

$$S[J^{(0)}, J^{(1)}] = \frac{k}{4\pi}\int_{\mathcal{M}} Tr(J^{(0)}\wedge dJ^{(0)}+\frac{2}{3}J^{(0)}\wedge J^{(0)}\wedge J^{(0)}+J^{(1)}\wedge\hat{\mathrm{D}}J^{(1)}) \quad (4)$$

where $\hat{\mathrm{D}} = d + \cdot \wedge J^{(0)} + J^{(0)} \wedge \cdot$ is the covariant exterior derivative. The expression (4) suggests to interpret $J^{(0)}$ as a CS-gauge field and $J^{(1)}$ as a coupled matter field. In this sense we have reformulated a pure $\mathcal{G}$ -invariant non-abelian CS theory (1) as a gauge field theory, with local with group $\mathcal{H}$, interacting with the matter. The whole set and the properties of such theories are directly connected with the solved classification problem of the Hermitian spaces [11].

In order to introduce the evolution equations, we break the space-time covariance assuming locally $\mathcal{M} \simeq \Sigma \times \mathbf{R}$, where Σ is a Riemann surface. We can parametrize Σ by means of $z = x_1 + i x_2$, $\bar{z} = x_1 - i x_2$. Then, its cotangent space will be $\Omega = \Omega^{(1,0)} \oplus \Omega^{(0,1)}$, respectively spanned by dz and $d\bar{z}$. The connection J can be rewritten in the form

$$J = V_0 + V + \bar{V} + M_0 + M + \bar{M} \quad , \quad (5)$$

where V and $M \in \Omega^{(1,0)}$ and $\bar{V}$ and $\bar{M} \in \Omega^{(0,1)}$, with the coefficients belonging to $\hat{l}^{(0)}$ and $\hat{l}^{(1)}$, respectively. Finally, V_0 and M_0 have the same algebraic structure and are 1-forms in $d_0 = dx^0\partial_0$. Rearranging the action (4) we obtain

$$\begin{aligned} S = \tfrac{k}{4\pi}\textstyle\int_{\Sigma\times\mathbf{R}} Tr\,(&V\wedge d_0\bar{V} + \bar{V}\wedge d_0 V + M\wedge d_0\bar{M} + \bar{M}\wedge d_0 M + \\ &+2V_0\wedge\left(\partial\bar{V} + \bar{\partial}V + V\wedge\bar{V} + \bar{V}\wedge V + M\wedge\bar{M} + \bar{M}\wedge M\right) \\ &+2M_0\wedge(\mathcal{D}\bar{M} + \bar{\mathcal{D}}M)) \quad , \end{aligned} \quad (6)$$

where $\mathcal{D} = \partial + V\wedge\cdot + \cdot\wedge V$, $\bar{\mathcal{D}} = \bar{\partial} + \bar{V}\wedge\cdot + \cdot\wedge\bar{V}$, with the holomorphic and anti-holomorphic operators ∂ and $\bar{\partial}$ globally defined on Σ. In the expression (6) V_0 and M_0 are the Lagrange multipliers enforcing, respectively, the Gauss-Chern-Simons (GCS) law and a sort of generalized self-dual condition, which derives from the torsion - free property of the $\mathcal{G}/\mathcal{H}$ manifold. The action (6) and the corresponding equations of motion are invariant under infinitesimal $\mathcal{H}$ -gauge transformations, and the decomposition among V-type fields and M-type matter fields is preserved. This picture leads to break the invariance under generic $\mathcal{G}$ -transformations, in such a way that only the $\mathcal{H}$ -invariance survives, by introducing a supplementary constraint of the form

$$\Gamma\left[M_0, M, \bar{M}, \mathcal{D}M_0, \bar{\mathcal{D}}M_0, \cdots\right] = 0 \quad . \quad (7)$$

If Eq. (7) can be explicitly solved for M_0, we replace it into the equations of motion, obtaining nonlinear evolution equations for the matter fields M

and $\bar{M}$ in interaction with the CS field (generally non-abelian). The same substitution in the action (6) leads to a functional defined on the submanifold of $\mathcal{G}\ /\ \mathcal{H}$ defined by the generalized self-dual condition. Thus, carrying out specific calculations in TFT we can introduce constraints of the form (7), instead of the most commonly used Weyl gauge $V_0 = M_0 = 0$ [5]. In particular, when Eq. (7) is related to certain integrable systems, exact solutions can be given analytically.

In particular we showed [1] how to embed in this scheme the non-abelian CS field models coupled to the matter discussed in [12].

3 Generalized σ-Models in the tangent space

Now, we consider the classical HM [13] in the tangent space formulation and its relation with the theory developed in Sec. 2.

Let us consider the matrix $\mathbf{S}$, which represents a point in the symmetric space $SU(n+m)/S\left(U\left(n\right)\times U\left(m\right)\right)$. The corresponding HM is defined by the equations of motion

$$i\partial_0\,\mathbf{S} = \frac{mn}{m+n}\left[\mathbf{S},\,\nabla^2\mathbf{S}\right] \tag{8}$$

This matrix can be diagonalized by a $U(n+m)$ local transformation g in the form $\mathbf{S} = g\Sigma g^{-1}$, where $\Sigma = diag\left(\frac{1}{n}I_n, -\frac{1}{m}I_m\right)$. The chiral currents $J_\mu = g^{-1}\partial_\mu g = J_\mu^{(0)} + J_\mu^{(1)} \qquad (\mu = 0,1,2)$, where the $\mathbf{Z}_2$ graduation has been used, satisfy the zero-curvature condition by definition. By writing Eq. (8) in terms of J_μ, we can use it as a further constraint for the zero-curvature equation. This provides the classical CS equation of motion (2), supplemented by a constraint of the type (7). In the complex notation Eq. (8) becomes

$$q_0 = 2i\left(D\,\psi_+ + \bar{D}\psi_-\right) \tag{9}$$

where q_0 is defining the "time-like" matter field M_0, $\psi_\pm$ describe the matter fields $M, \bar{M}$ and $D = \partial_z + iv^{(m)}\cdot - \cdot iv^{(n)}, \bar{D} = \partial_{\bar{z}} + iv^{(m)\dagger}\cdot - \cdot iv^{(n)\dagger}$ are the covariant derivatives, which contain the components $v^{(m)}$, $v^{(n)}$ of the gauge fields $V, \bar{V}$. Equation (9) is an explicit example of the constraint (7). The substitution of q_0 in the action (4) gives the formulation of the generalized HM as a specific symmetric reduction from a pure non-abelian CS model, with the supplementary torsion-free condition of the spin phase space

$$\gamma \equiv D\psi_+ - \bar{D}\psi_- = 0 \qquad . \tag{10}$$

Looking at (2+1)-dimensional integrable spin models, we consider the $SU(2)/U(1)$ topological magnet model [14], which comes from the Lax pair

$$L_1 \;=\; \alpha I\partial_2 + \mathbf{S}\partial_1,$$

$$L_2 \;=\; I\partial_0 + 2i\mathbf{S}\partial_1^2 + \left(i\partial_1\mathbf{S} - i\alpha\mathbf{S}\partial_2\mathbf{S} - \alpha\mathbf{S}w_2 + Iw_1\right)\partial_1 \;, \qquad (11)$$

where w_j is a $U(1)$ connection representing a velocity field, with $\alpha^2 = \pm 1$. If the velocity field w_j satisfies also the incompressibility condition $\partial_1 w_1 - \alpha^2\partial_2 w_2 = 0$, and it can be expressed in terms of the stream function ϕ only, by $w_1 = \partial_2\phi$ $w_2 = \alpha^2\partial_1\phi$, we obtain the IE. In the tangent space representation, this system takes a form similar to (9), but non local in the q_i's. This is a new type of integrable gauge fixing condition (7).

4 Canonical structure of the $SU(2)/U(1)$ model

Here we study the canonical structures of the theory described by (6) in the $SU(2)/U(1)$ case.

Following the standard prescriptions in order to obtain the canonical Poisson structure for this case, we have

$$\{v_i(\mathbf{x}), v_j(\mathbf{y})\} = -\frac{\pi}{k}\varepsilon_{ij}\delta(\mathbf{x}-\mathbf{y}) \;, \{\psi_\pm(\mathbf{x}), \bar{\psi}_\pm(\mathbf{y})\} = \pm\frac{\pi i}{k}\delta(\mathbf{x}-\mathbf{y}), \quad (12)$$

where v_i $(i = 1, 2)$ and $\psi_\pm$, $\bar{\psi}_\pm$ are suitable scalar parametrizations of the gauge and matter fields, respectively. In analogy with above, in the action there are two fields, v_0 and q_0, which are Lagrange multipliers enforcing the GCS law Γ_1, and the complex "torsion-free" constraint γ.

Following the Dirac's classification of the constraints [15], we can introduce the set of primary constraints $\Gamma_0 \equiv \pi_0 = 0$, $\Gamma_2 \equiv \pi_q = 0$, where π_0 and π_q are the momenta conjugated to v_0 and q_0, respectively. Hence, we can write the Hamiltonian density in terms of a set of first-class constraints only

$$\mathcal{H} = \frac{k}{\pi}\left(v_0\Gamma_1 + iq_0\gamma^* - iq_0^*\gamma - f_0\Gamma_0 + ig_0\Gamma_2 - ig_0^*\Gamma_2^*\right) \quad , \qquad (13)$$

where $\Gamma_1 = \{\pi_0, H\}$, $\gamma = \{\pi_q^*, H\}$ and $\gamma^* = \{\pi_q, H\}$ are now secondary constraints and f_0, g_0 and g_0^* are arbitrary functions, which characterize the evolution of v_0, q_0 and q_0^*, respectively. We see that all constraints are weakly invariant ($\{\gamma, H\} \approx \{\Gamma_i, H\} \approx 0$ for $i = 0, 1, 2$) and, consequently, also the Hamiltonian (13).

In the present situation, we cannot apply the procedure [15,16], which distinguishes the dynamical variables among gauge independent and pure gauge ones, because of the non-abelian structure of the algebra of constraints. By solving the GCS constraint, we can only introduce $U(1)$-invariant degrees in the Hamiltonian (13).

In doing so, we restrict ourselves to the planar geometry taking $\Sigma \equiv \mathbf{R}^2$ and write the gauge field in the form $v_i(\mathbf{x}) = \partial_i \eta(\mathbf{x}) - \epsilon_{ij}(\partial_j^{-1} B)(\mathbf{x})$, where $\partial_j^{-1} f(\mathbf{x}) = \frac{1}{2\pi}\partial_j^{(x)} \int \ln|\mathbf{x}-\mathbf{y}| f(\mathbf{y}) d^2y$ and $B = \epsilon^{ij}\partial_i v_j$. Now, we look for a canonical transformation, generated by a function W, which satisfies the set of generalized Hamilton-Jacobi equations $\tilde{\pi}_i = \Gamma_i$. Here, the $\tilde{\pi}_i$'s are new momenta and the integrability of this system is assured by the commutativity property $\{\Gamma_i, \Gamma_j\} = 0$ for $i, j = 0, 1, 2$. Thus, we obtain the new $U(1)$-invariant degrees of freedom

$$\Phi_{\pm} = \sqrt{\frac{k}{\pi}}\, \psi_{\pm} \exp(-2i\eta), \tag{14}$$

which fulfill the canonical brackets However, the constraint γ still contains the gauge variables (η, π_1). Thus, the equations of motion for $\Phi_{\pm}$, derived from the transformed Hamiltonian, still involve q_0, which has an arbitrary dynamics.

Nevertheless, restricting the dynamics on the phase space submanifold defined by the equations $\pi_1 = 0$ $\gamma = 0$, we have

$$B = \frac{2\pi}{k}\left(|\Phi_-|^2 - |\Phi_+|^2\right) \tag{15}$$

$$\left(\partial_z - \frac{1}{2}\partial_{\bar{z}}^{-1}(B)\right)\Phi_+ \quad -\left(\partial_{\bar{z}} + \tfrac{1}{2}\partial_z^{-1}(B)\right) \quad \Phi_{\measuredangle} = 0 \quad , \tag{16}$$

where $\partial_{\bar{z}}^{-1}(f(z,\bar{z})) = i\partial_z \int \frac{1}{\pi} ln|z-\xi| f(\xi,\bar{\xi})\, d\xi \wedge d\bar{\xi}$. Since no evolution operator is included into these these equations, we look only for time independent solutions. All dynamics is recovered by general canonical transformations.

Special subcases of Eqs. (15, 16), corresponding to the self-dual CS model [9] and to the self-dual Yang-Mills equations in 4 dimensions [17], are discussed in [1].

Now, if we choose the multiplier q_0 to satisfy Eq. (9) and add the constraint γ to the equations of motion, the arising system coincides with the $SU(2)/U(1)$ HM in the tangent space representation. In the static case, these equations reduce to the self-dual configurations discussed above. In this context, classical self-dual solitons and multiperiodic solutions in the abelian CS system can be interpreted in terms of magnetic bubbles (vortices) of the spin planar model [10].

5 Quantization of the $SU(2)/U(1)$ model

The quantum theory of the $SU(2)/U(1)$ CS model can be carried out by means of the correspondence $\Phi_{\pm} \to \hat{\Phi}_{\pm}$, $\bar{\Phi}_{\pm} \to \hat{\Phi}_{\pm}^{\dagger}$ and replacing, in the boson case, the canonical brackets by the equal-time commutators $[\hat{\Phi}_{\pm}(\mathbf{x}), \hat{\Phi}_{\pm}^{\dagger}(\mathbf{y})] =$

$\mp\delta(\mathbf{x}-\mathbf{y})$. Since the classical energy is vanishing, we assume that the physical quantum states are energy eigenstates with eigenvalue 0. Furthermore, the first class constraints Γ_i become the operators $\hat{\Gamma}_i$, which must annihilate the physical states. In particular, because of $\hat{\Gamma}_1$, the states are independent from v_0 and invariant under time-independent gauge transformations. Finally, we associate to γ an operator $\hat{\gamma}$, which also annihilates the physical states.

For the specific case of the Heisenberg model, in the subspace of the physical states, we can write down a quantum Hamiltonian involving only the operators $\hat{\Phi}_\pm$ and their hermitians:

$$\hat{H} = 4\int\left\{\hat{\Phi}_+^\dagger\left(\partial_z - \frac{1}{2}\partial_{\bar{z}}^{-1}\left(\hat{B}\right)\right)^2\hat{\Phi}_+ - \hat{\Phi}_-^\dagger\left(\partial_{\bar{z}} - \frac{1}{2}\partial_z^{-1}\left(\hat{B}\right)\right)^2\hat{\Phi}_-\right\}dzd\bar{z}, \tag{17}$$

where $\hat{B} = \frac{\pi}{k}\left(\hat{\Phi}_-^\dagger\hat{\Phi}_- - \hat{\Phi}_+^\dagger\hat{\Phi}_+\right)$, and the normal ordering of the operators is used. The quantized free-torsion constraint $\hat{\gamma}$ takes the form

$$\hat{\gamma} \equiv \left(\partial_z - 2\partial_{\bar{z}}^{-1}\left(\hat{B}\right)\right)\hat{\Phi}_+ - \left(\partial_{\bar{z}} - 2\partial_z^{-1}\left(\hat{B}\right)\right)\hat{\Phi}_- \quad . \tag{18}$$

We can prove that $\hat{\Phi}_\pm$ are the $U(1)$-gauge invariant operators, which create a charge-solenoid composite, having magnetic flux equal to $\mp\pi/k$.

Now, defining the quantum vacuum state by the relation $\hat{\Phi}_\pm|\mathbf{0}>=0$, we can introduce two different particles number operators $\hat{N}_\pm = \int\hat{\Phi}_\pm^\dagger\hat{\Phi}_\pm\, d^2$. These operators commute between themselves and with the Hamiltonian operator (17). Thus, we can formally construct the corresponding Fock space. The wave function will be an element of $L_2[\mathcal{R}^{2(N_+ + N_-)}]$ and satisfies a Schrödinger equation for $(N_+ + N_-)$-bodies.

Actually, only states with a finite number of one type of particles can exactly find. For instance, the $|N_+, 0\rangle$ state is described by a bosonic wave function, having the form of the Laughlin multivalued anyonic wave function [18]

$$\Psi\left(\mathbf{x}_1^+, \cdots, \mathbf{x}_{N_+}^+\right) = \tilde{\mathcal{F}}\left(\bar{z}_1^+, \cdots, \bar{z}_{N_+}^+\right)\prod_{i<j}\left(z_i^+ - z_j^+\right)^{-\frac{1}{k}} \quad . \tag{19}$$

This wave function acquires the anyonic phase $\exp\left(\frac{i\pi}{k}\right)$ after the exchange of two particles.

Then, if we consider the model (17) as a quantum version of the magnetic bubble system, we see that it behaves as a quantum anyon system.

6 Conclusions

From this analysis we see that some special planar spin models can be obtained by a generally non-abelian CS theory via a convenient gauge-fixing condition. For example, both the well-known $SU(2)$ Heisenberg model and the IE, which is completely integrable, can be associated with the TFT in special gauges. The symplectic structure of these models has been investigated mainly within the gauge-invariant procedure. Thus, we can provide also the quantum theory of the models.

Our approach could find useful and several interesting applications in the context of the quantum TFT, and in the domain of planar physics, where similar theories have arbitrary real coupling constants, thus they can be considered deformations of our models [9].

Furthermore, since the CS TFT is exactly solvable in the $SU(2)$ case for any three-manifolds , we speculate that our procedure should lead to a field theoretical description of an anyonic system on an arbitrary Riemann surface Σ . It is also physically interesting to consider the extension of our procedure to noncompact groups, like $ISO(2,1)$, the corresponding model of which is equivalent to the TFT studied by Witten in connection with the (2+1)-dimensional quantum gravity [19].

Acknowledgments

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ON DYNAMICS OF THE PULSON COLLISIONS

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The collisions of pulsating gaussons in the scalar field model with logarithmic nonlinearity are investigated on a phase-space both numerically and analitically.

In this note we continue the investigations of the multidimensional localized field configurations in the scalar model with the logarithmic nonlinearity [1-3],

$$\varphi_{tt} - \Delta\varphi + m^2\varphi - \lambda\varphi \log \varphi^2 = 0. \tag{1}$$

Here $\varphi(\mathbf{r}, t)$ is the real scalar field, Δ is the n-dimensional Laplacian, m^2 and λ are positive constants. We assume the field configurations considered have a finite energy $E = (1/2) \int \left\{(\varphi_t)^2 + (\nabla\varphi)^2 + \varphi^2[m^2 + \lambda(1 - \log\varphi^2)]\right\} d^n x$.

According to refs.4-6 the eq.(1) admits the factorized solutions

$$\varphi = a(t)u(\mathbf{r}), \tag{2}$$

where the dynamics is governed by the Newtonian equation

$$a_{tt} = -dV/da, \qquad V(a) = \frac{1}{2}\lambda a^2(1 - \log a^2), \tag{3}$$

and $u(\mathbf{r})$ satisfies the equation $\Delta u - m^2 u + \lambda u \log u^2 = 0$ which has a family of well-localized solutions. The simplest of them is the usual gausson: $u(\mathbf{r}) = \exp[(m^2/\lambda + n - \lambda r^2)/2]$.

The time evolution of the lump (2) can be imaged as the motion of a particle with the Hamiltonian $H = a_t^2/2 + V(a)$. There are two types of these motions. The first one is the oscillations in the potential well with $H < H_{cr} = \lambda/2$ and $a^2 < a_{cr}^2 = 1$. The estimations for their period were obtained in ref.6. The

corresponding lumps (2) are usually called pulsons. Each pulson is characterized by the energy E (rest mass), which takes the form $E = H \int u^2 d^n x$, and by a phase ψ , $\tan\psi = a_t/a$.

The second type of the motion occurs for $H > H_{cr}$ or for $H < H_{cr}$ but $a^2 > a_{cr}^2$. In these cases the amplitude a increases extremely fast as $a(t) \sim \exp(\lambda t^2/2)$. In ref.7 we have named the corresponding lumps explosons. We have also observed the creation of the explosons in head-on collisions of two pulsons.

Now we investigate the collision processes by means of the phase-plane method which permit us to describe clearly the dynamics of the explosion creation, bounded metastable states and elastic scattering versus the phase of colliding pulsons.

Let us first introduce the following two linear functionals

$$Q(t) = \int \varphi(\mathbf{r}, t) d^n x, \qquad P(t) = \int \varphi_t(\mathbf{r}, t) d^n x \tag{4}$$

which are integrals of canonical variables of the system (1) taken over all space. Considering them as dynamical variables define the phase of the field configuration as $\tan\psi = P/Q$. In the case of a single pulson at rest $Q \sim a$, $P \sim a_t$, and hence the introduced dynamical variables become canonical ones. In general case this is not true, and therefore making use of the intergral variables Q and P gives us some "averaged" description of the system.

In the fig.1 the phase trajectories for collisions of the two pulsons are presented on $Q - P$ plane. The numerical procedure is the following. We solve eq.(1) for the two-pulson initial states and calculate the integrals (4) for the current field configurations . Denote as ψ_0 the phase of a free single pulson which it would have in the collision site . In the fig.1a the several phase trajectories are presented which correspond to the various phases ψ_0 . The finite trajectories with one loop describe elastic collisions of the pulsons, the tops of the loops correspond to the moments of the maximum overlapping. The infinite trajectories describe the creation of the explosons. In a narrow intermediate region of phases a multiple loop motion is possible which is associated with appearance of a long-lived metastable state (fig.1b).

To obtain the conditions under which an initial field configuration develops into explosion one needs to consider the quadratic functionals

$$A(t) = \int \varphi^2(\mathbf{r}, t) d^n x, \qquad B(t) = \int \varphi_t^2(\mathbf{r}, t) d^n x. \tag{5}$$

It turns out that in the model (1) these functionals are related to a value of the total energy E by the equation $A_{tt} - 2\lambda A = 4(B - E)$. On the other

hand, the Cauchy-Schwarz inequality gives $A_t^2 \leq 4AB$. Exploiting this fact and introducing new variables

$$e^{q(\tau)} = 2\lambda^{n/2}A(t), \qquad \tau = (2\lambda)^{1/2}t, \qquad \mathcal{E} = 4\lambda^{n/2-1}E \tag{6}$$

we arrive at the Newtonian-like differential inequality

$$q_{\tau\tau} \geq -dU/dq, \qquad U(q) = -q - \mathcal{E}e^{-q} \tag{7}$$

govering the dynamics of A(t) [7].

Note that for the factorized lumps (2) the inequality (7) turn into equality and we return to the dynamics considered above.

Explosons are created when q tends to $+\infty$. Therefore, when $E \leq 0$, due to the positivity of the "acceleration" $q_{\tau\tau}$, the initial field configurations will always develop into the explosons. For configurations with $E > 0$ the analysis of the "particle motion" in the potential U (7) leads to two sets of sufficient conditions for creation of the explosons (see ref.7). Returning to the variable $A(t)$ by (6) and denoting $\mathcal{A} = e^q/\mathcal{E}$ we write them in the following equivalent form.

$$(a) \qquad -S(\mathcal{A}) < \mathcal{A}_\tau \leq S(\mathcal{A}), \qquad \mathcal{A} > 1; \qquad (b) \qquad \mathcal{A}_\tau > S(\mathcal{A}), \tag{8}$$

where $S(\mathcal{A}) = \{2\mathcal{A}[1 - \mathcal{A}(1 - \log\mathcal{A})]\}^{1/2}$

In fig.2 we present the domains (shaded) of the phase plane $(\mathcal{A}, \mathcal{A}_\tau)$ where either the condition (8a) or (8b) are fulfiled. It should emphasized that the conditions obtaned are sufficient. It means that if at some moment a phase trajectory enters the shaded domain then the exploson mode develops necessarily.

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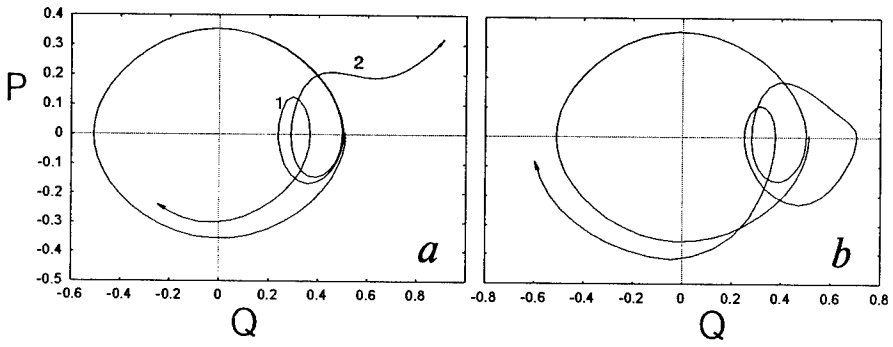

Figure 1: Phase trajectories for head-on collisions of identical pulsons each with the energy $E_0 = 0.2$, velocity $v_0 = 0.5$ and phase ψ_0; (a) - elastic collisions (line $1 - \psi_0 = 3.2$) and appearing of the exploson ($2 - \psi_0 = 3.26$); (b)- inelastic collision and creation of long-lived state in an intermediate region of phases ($\psi_0 = 3.2561$).

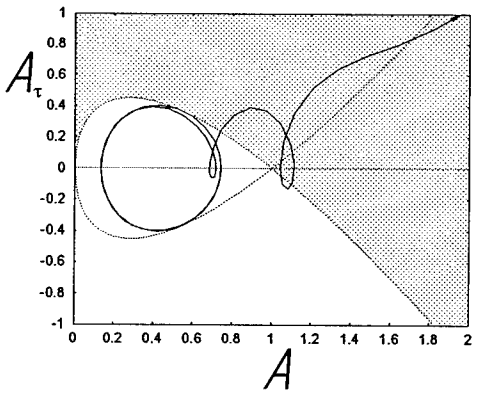

Figure 2: The phase trajectories of the quadratic functional A for creation of exploson (the same collisions as in the fig.1, $\psi_0 = 3.26$).

ON THE EXISTENCE OF SELF–SIMILAR STRUCTURES IN THE RESONANCE DOMAIN

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Abstract

A nonlinear Hamiltonian system with two degrees of freedom is considered which depends on the parameter ϵ and is integrable at $\epsilon = 0$. It is shown that in the case of general position, for any resonance domain there exists a set E, having continual cardinality, of critical values of the parameter ϵ such that at any critical value of ϵ the system under consideration has a self–similar structure of the type "islands around islands". In this situation, the behaviour of solutions in each cell of this structure is determined in the first approximation by a certain standard Hamiltonian. But the fine structure of each cell is described by additional terms of the Hamiltonian which are considered to be small perturbation.

1 Introduction

In the present paper we dwell upon the reasons for appearance of self-similar structures in the Hamiltonian systems that are close to completely integrable ones. In fact, we shall consider only the case of systems with two degrees of freedom, though the construction considered here admits generalization to some other cases with a larger number of degrees of freedom.

In the case of Hamiltonian systems with two degrees of freedom, the appearance of self-similar structures is mostly due to periodic solutions with the so-called threshold nature of creation. This means that each of these periodic solutions is absent not only in an unperturbed system but also in a perturbed one until perturbation remains sufficiently small and appears only when perturbation is large enough. Well-known periodic solutions arising (under the inclusion of perturbation) from invariant tori with rational winding and existing at any sufficiently small perturbation are of a minor role in this case.

From the geometrical point of view, we come across two types of periodic solutions. As it will be seen later, periodic solutions of the first type

are close to homoclinic trajectories and are contained in peculiar domains formed by intersections of separatrix surfaces and called by various authors lobes, tongues, tails, etc. Periodic solutions of the second type can lie near homoclinic trajectories but are outside the above-mentioned domains.

Since any stable periodic solution of the elliptic type is contained inside some invariant torus, on the isoenergetic surface $H = C$ one can find a cross section in which a point corresponds to a periodic solution and a closed curve to an invariant torus. Thus, at some cross section of the isoenergetic surface $H = C$ some domain, invariant with respect to the shift in time, or, as it is sometimes called, an island corresponds to each stable periodic solution of the elliptic type. A self-similar structure of the type "islands around islands" described in this paper contains invariant domains of an infinite number of generations and exists at some, called critical, values of the parameter ε characterizing the value of perturbation. The set E_λ of critical values of the parameter ε belonging to the interval $(0, \lambda)$ has the continual cardinality at any $\lambda > 0$. The structure of this set is unknown at present.

2 An auxiliary system of equations

As a starting point for all further considerations we use the Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

with the Hamiltonian function $H = H(p, q, t, \omega, \varepsilon)$ admitting representation of the form

$$H = H_0(p, q) + \varepsilon H_1(p, q, \omega t, \omega, \varepsilon) \ ,$$

where the variables p and q are scalars, and the function H_1 is periodically dependent with some period T_0 on the product $\theta = \omega t$. We assume that at $\varepsilon = 0$ the behaviour of the trajectories of the system (1) in the domain of changing variables p and q has the form shown in Fig.1.

This means that at the point s_0 the function H_0 has a nondegenerate saddle point, and at the point e_0 the function H_0 has a nondegenerate local extremum. Then, the point e_0 at $\varepsilon = 0$ is surrounded by a family of closed trajectories of the system (1) which are separated from other trajectories of this system by the loop of the separatrix Γ_0 passing through the saddle point s_0. Finally, we do not make any assumptions on a mutual location of the trajectory Γ_+ incoming the point s_0 and the trajectory Γ_- outgoing this point. However, we assume that at sufficiently small $\varepsilon \neq 0$ the behaviour of solutions of the system (1) has the structure specified by the following properties. First of all, we assume that perturbation H_1 is that of "general

position". This means that the loop of the separatrix Γ_0 at $\varepsilon \neq 0$ splits into the stable Γ_0^s and unstable Γ_0^u separatrix surfaces formed by the solutions of the system (1) asymptotically tending at $t \to \infty$ and $t \to -\infty$ to the periodic solution s_ε of hyperbolic type resulting from the hyperbolic point s_0 at sufficiently small $\varepsilon \neq 0$. Then, the stable $\Gamma_+(\varepsilon)$ and unstable $\Gamma_-(\varepsilon)$ separatrix surfaces, arising at $\varepsilon \neq 0$ from the curves Γ_+ and Γ_-, intersect at all sufficiently small $\varepsilon \neq 0$. Moreover, we do not exclude cases when the surfaces $\Gamma_+(\varepsilon)$ and $\Gamma_-(\varepsilon)$ coincide at all or some $\varepsilon \neq 0$.

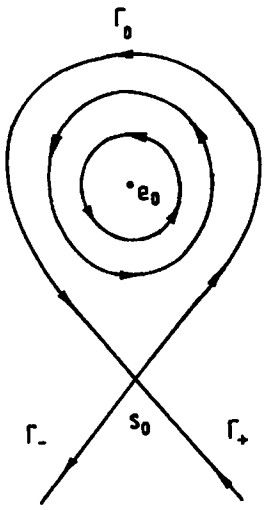

Fig. 1

Now let us take the Poincare mapping T, generated by the system (1), i.e, the shift of points on the plane p, q along the trajectories of the system (1) during the time $\Delta t = \frac{T_0}{\omega}$, and consider the cross section of the phase space (p, q, t) by the plane $t = 0$. Then, the cross section by $t = 0$ of the above-described structure has the form given in Fig.2.

The arrows in this figure show the shift of the points of this cross section when the Poincare mapping T is applied to them. Then, let us take the arc $s_\varepsilon \gamma_{-1} \gamma_0$ of the unstable branch of the separatrix Γ_0^u and the arc $\gamma_0 \gamma_1 s_\varepsilon$ of the stable branch of the separatrix Γ_0^s and consider the open domain Δ_0 bounded by these arcs. With the help of the domain Δ_0 we determine two

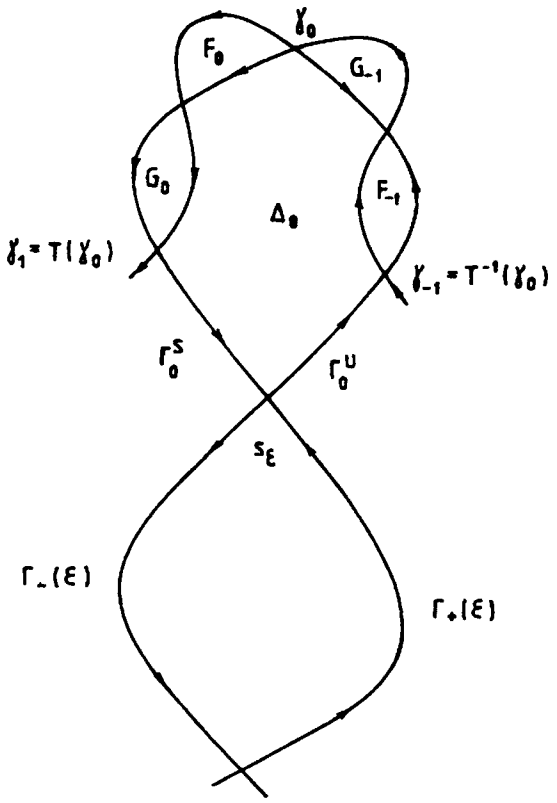

Fig. 2

domains F_0 and G_0 by the equalities

$$F_0 = T(\Delta_0)\backslash\bar{\Delta}_0, \quad G_0 = \Delta_0\backslash T(\bar{\Delta}_0) \ ,$$

where the bar means closure in the topological sense and the skew bar implies that all the points belonging to the second set are rejected from the first one. It is obvious that F_0 is the set of points $\zeta \in T(\Delta_0)$ that left the domain $\bar{\Delta}_0$ and G_0 is the set of points $\eta \in \Delta_0$ that turned out to be uncovered by the domain $T(\bar{\Delta}_0)$. Since the Poincare mapping T preserves the area, the measure of the sets F_0 and G_0 coincides, i.e.,

$$\text{mes } F_0 = \text{ mes } G_0 > 0 \ .$$

Now we assume that

$$F = \bigcup_{n=-\infty}^{\infty} T^n(F_0), \quad G = \bigcup_{n=-\infty}^{\infty} T^n(G_0),$$

$$\Omega = F \cup G, \quad \mathcal{F} = \bar{F}, \quad \mathcal{G} = \bar{G}\,.$$

By virtue of the following easily verified equalities:

$$\text{mes}\left(F_0 \setminus \bigcup_{n=1}^{\infty} T^n(G_0)\right) = 0,$$

$$\text{mes}\left(G_0 \setminus \bigcup_{n=1}^{\infty} T^{-n}(F_0)\right) = 0$$

one can easily be convinced in the validity of $\mathcal{F} = \mathcal{G}$. In what follows we shall call the set $Z = \mathcal{F} = \mathcal{G}$ the separatrix zone. It is obvious that $Z = \bar{\Omega}$.

3 Periodic solutions of the parabolic type

In our conditions the separatrix zone is crossed by an infinite number of periodic solutions of the system (1) with the threshold nature of creation and different periods, integer multiples of the quantity $\frac{T_0}{\omega}$. More exactly, the following statement, specifying some basic properties of these solutions, is valid.

In the case of "general position" there exists an infinite sequence of positive quantities $\varepsilon_m > 0$, $m = 1, 2, \ldots$, $\varepsilon_m \to 0$ if $m \to \infty$ which satisfies the following conditions:

1. At $\varepsilon = \varepsilon_m$ the Hamiltonian system (1) has the periodic solution $p = p_m(t, \omega)$, $q = q_m(t, \omega)$ of the parabolic type with the period $T_m = \frac{T_0}{\omega}\kappa_m$ where κ_m is integer, $\kappa_m \to \infty$ if $m \to \infty$. This periodic solution at $t = 0$ crosses the domain Ω defined above and does not exist at any $\varepsilon \in (0, \varepsilon_m)$. We shall call periodic solutions of this type the periodic solutions of the first generation.

2. In some vicinity of this periodic solution there exists a nonlinear change of variables $(p, q) \to (u, v)$, preserving the area, of the form

$$\begin{aligned} p &= p_m(t,\omega) + S_{11}^{(m)}(t,\omega)u + S_{12}^{(m)}(t,\omega)v + \ldots, \\ q &= q_m(t,\omega) + S_{21}^{(m)}(t,\omega)u + S_{22}^{(m)}(t,\omega)v + \ldots, \end{aligned} \tag{2}$$

such that in the new variables the initial Hamiltonian system (1) has the form

$$\dot{u} = -\frac{\partial H^{(m)}}{\partial v}, \quad \dot{v} = \frac{\partial H^{(m)}}{\partial u} \tag{3}$$

with the Hamiltonian function $H^{(m)}$ of the form

$$H^{(m)} = -\frac{c_m}{2}v^2 + \frac{1}{6}(f_m u^3 + 3g_m u^2 v + 3h_m u v^2 + \\ + k_m v^3) + \hat{H}^{(m)} + (\varepsilon - \varepsilon_m)(r_m u + s_m v + \hat{h}^{(m)}) , \qquad (4)$$

where the function $\hat{H}^{(m)}$ does not contain terms below the fourth order in u and v, and the function $\hat{h}^{(m)}$ does not contain terms below the second order in the same variables, i.e.,

$$\hat{H}^{(m)} \sim O((|u| + |v|)^4), \quad \hat{h}^{(m)} \sim O((|u| + |v|)^2) ,$$

the quantities $c_m, f_m, g_m, h_m, k_m, r_m$ and s_m are independent of time t whereas the change of variables (2) is periodic in time t with the same period $T_m = \frac{T_0}{\omega}\kappa_m$. The dots in equalities (2) denote terms nonlinear in u and v.

This statement is to be explained. First of all, periodic solutions guaranteed by the Birkhoff theorem [1], which at $t = 0$ cross the vicinity of the homoclinic point, are not appropriate for our aim since each of these solutions obviously arises from a closed curve of the unperturbed system (1) and exists at all values of the parameter ε which belong to some vicinity of the point $\varepsilon = 0$. Each of these solutions obviously belongs to the hyperbolic or elliptic type, and a possibility to derive from these solutions periodic solutions of the parabolic type seems to be rather vague at present. At last, none of these periodic solutions at none of the values of time t enters a beam of trajectories outgoing at $t = 0$ from the domain Ω determined above.

The proof of the existence of periodic solutions of the parabolic type, we are interested in, is based on the structure of intersections of the sets F_0 and $F_n = T^n(F_0)$ or G_0 and $G_{-n} = T^{-n}(G_0)$ at sufficiently large values of integer n. A construction arising here is similar in structure to Smale's horseshoe [2] though it is more complex. Since the least integers $m_0 > 0$ and $n_0 > 0$, for which intersections $F_0 \cap F_{m_0}$ and $G_0 \cap G_{-n_0}$ are not empty, depend on ε and satisfy the condition $m_0 \to \infty$, $n_0 \to \infty$ if $\varepsilon \to 0$, any of the periodic solutions thus obtained may exist only at values of the parameter ε which exceed some threshold value. Finally, according to the well-known Poincare theorem, a periodic solution may appear or disappear only in the case when eigenvalues of the corresponding matrix of the monodromy are equal to plus one, i.e., a periodic solution at the moment of appearance or disappearance should belong to a parabolic type.

Let us assume again that the initial system (1) is in "general position" in a sense that the quantities c_m, f_m and r_m entering into the Hamiltonian function $H^{(m)}$ of the form (4) are nonzero, i.e.,

$$c_m \neq 0, \ f_m \neq 0, \ r_m \neq 0, \quad m = 1, 2, \ldots$$

In this case, a scale transformation of the form

$$u = \mu^2 \left(\frac{c_m}{f_m}\right)^{1/3} x, \quad v = \mu^3 y \ ,$$

$$\varepsilon = \varepsilon_m - \frac{\mu^4}{r_m}(c_m^2 f_m)^{1/3} \ , \tag{5}$$

determining transition from the old space variables u, v to the new ones x, y and from the old small parameter ε to the new one μ, transforms the initial Hamiltonian system (3) into the system of the form

$$\dot{x} = -\mu(c_m^2 f_m)^{1/3}\frac{\partial K^{(m)}}{\partial y}, \quad \dot{y} = \mu(c_m^2 f_m)^{1/3}\frac{\partial K^{(m)}}{\partial x}$$

with the function $K^{(m)}$ of the form

$$K^{(m)} = K_0(x, y) + \mu K_1^{(m)}(x, y, \omega t, \omega, \mu)$$

where K_0 is independent of index m and has the standard form

$$K_0 = -\frac{1}{2}y^2 + \frac{1}{6}x^3 - x \ , \tag{6}$$

and $K_1^{(m)}$ is periodically dependent on the quantity $\theta = \omega t$ with the period $\tau_m = T_0 \kappa_m$. Further, we perform the scale transformation of time

$$t \to \mu(c_m^2 f_m)^{1/3} t \ . \tag{7}$$

As a result, our initial Hamiltonian system (1) acquires its final form

$$\dot{x} = -\frac{\partial \hat{K}^{(m)}}{\partial y}, \quad \dot{y} = \frac{\partial \hat{K}^{(m)}}{\partial x} \tag{8}$$

with the Hamiltonian function $\hat{K}^{(m)}$ of the form

$$\hat{K}^{(m)} = K_0(x, y) + \mu K_1^{(m)}(x, y, \omega' t, \omega, \mu) \ , \tag{9}$$

where

$$\omega' = \frac{\omega}{\mu(c_m^2 f_m)^{1/3}} \ .$$

4 Critical values of the parameter ε and the relevant self-similar structure

In the case of "general position", the Hamiltonian system (8) derived has the same properties as the initial system (1). This implies that in the case of "general position" the separatrix loop of the derived system (8) at $\mu = 0$, which is analogous to the loop Γ_0 of the initial system (1) at $\varepsilon = 0$, at all sufficiently small $\mu \neq 0$ splits into the stable and unstable separatrix surfaces with all above-mentioned properties inherent in them. However, analogous Γ_+ and Γ_- asymptotic trajectories existing in the system (8) at $\mu = 0$ at all sufficiently small $\mu \neq 0$ generate analogous $\Gamma_+(\varepsilon)$ and $\Gamma_-(\varepsilon)$ stable and unstable separatrix surfaces and the fact of their intersection (or coincidence) may be strictly proved without any additional assumptions.

It follows from the afore-said that for any $m = 1, 2, \ldots$ there exists an infinite sequence of positive quantities $\mu_{m,n} > 0$, $n = 1, 2, \ldots$, $\mu_{m,n} \to 0$ if $n \to \infty$ satisfying the following conditions:

1. At $\mu = \mu_{m,n}$ the Hamiltonian system (8) has a periodic in time $t' = \mu(c_m^2 f_m)^{1/3} t$ solution $x = x_{m,n}(t', \omega')$, $y = y_{m,n}(t', \omega')$, of the parabolic type with the period $T_{m,n} = \frac{T_0}{\omega'} \kappa_m \kappa_n'$ where κ_n' is integer, and $\kappa_n' \to \infty$ if $n \to \infty$. This periodic solution at $t' = 0$ intersects the domain Ω_m, analogous to Ω, which is contained in the separatrix zone Z_m of the new Hamiltonian system. This periodic solution does not exist at any $\mu \in (0, \mu_{m,n})$. We shall call all these solutions the periodic solutions of the second generation.

2. In some vicinity of this periodic solution there exists an analogous (2), conserving the area, nonlinear change of variables such that in the new variables the Hamiltonian system, derived as a result of the change, has the Hamiltonian function having a structure analogous to (4). Moreover, having changed space variables, time and small parameter $\mu \to \nu$, by analogy with (5) and (7), we derive the Hamiltonian system analogous to (8) with the Hamiltonian function analogous to (9). In the case of "general position" the unperturbed part of the Hamiltonian function derived has the standard form (6).

Thus, we have found a procedure that allows one to derive an infinite sequence of Hamiltonian systems of different generations. According to the KAM theory [3, 4], the initial Hamiltonian system (1) at any $\varepsilon \in (0, \varepsilon^*)$, $\varepsilon^* > 0$ has invariant tori produced from closed trajectories of the system (1) with $\varepsilon = 0$. As a next step, we get the new Hamiltonian system (8) that has invariant tori at all $\mu \in (0, \mu_m^*)$ where $\mu_m^* > 0$. This means that the initial Hamiltonian system (1) has invariant tori near the periodic

solutions $p = p_m(t,\omega)$, $q = q_m(t,\omega)$ at all $\varepsilon \in (\varepsilon_m, \varepsilon_m^*)$ where

$$\varepsilon_m^* = \varepsilon_m - \frac{(\mu_m^*)^4}{r_m}(c_m^2 f_m)^{1/3} > \varepsilon_m \ .$$

Moreover, the new Hamiltonian system (8) has invariant tori near the periodic solution $x = x_{m,n}(t',\omega')$, $y = y_{m,n}(t',\omega')$ of the second generation at all $\nu \in (0, \nu_{m,n}^*)$ where ν is a new small parameter, and $\nu_{m,n}^* > 0$, i.e., at all $\mu \in (\mu_{m,n}, \mu_{m,n}^*)$ where $0 < \mu_{m,n} < \mu_{m,n}^*$. Consequently, our initial Hamiltonian system (1) has invariant tori near an arbitrary periodic solution of the second generation at values of the parameter ε which belong to an interval of the form $(\varepsilon_{m,n}, \varepsilon_{m,n}^*)$, chosen in a proper way, where $0 < \varepsilon_{m,n} < \varepsilon_{m,n}^*$. We can repeat this procedure infinitely many times. As a result, we will get two infinite sequences of positive quantities $\varepsilon_{m_1,\ldots,m_n}$ and $\varepsilon_{m_1,\ldots,m_n}^*$, $n = 1, 2, \ldots$ with the following properties:

1. For any system of positive integers $m_1, \ldots, m_n$ the inequality $0 < \varepsilon_{m_1,\ldots,m_n} < \varepsilon_{m_1,\ldots,m_n}^*$ is valid.

2. For any system of positive integers $m_1, \ldots, m_n$ we have $\varepsilon_{m_1,\ldots,m_n,l} \to \varepsilon_{m_1,\ldots,m_n}$ if $l \to \infty$.

Now we assume that $\varepsilon'_{m_1} = \varepsilon_{m_1}^*$ and consider the interval $\delta_1 = (\varepsilon_{m_1}, \varepsilon'_{m_1})$. The quantity ε_{m_1,m_2} will be called appropriate if $\varepsilon_{m_1,m_2} \in \delta_1$, i.e., $\varepsilon_{m_1} < \varepsilon_{m_1,m_2} < \varepsilon'_{m_1}$. By virtue of the second property of the sequence $\varepsilon_{m_1,\ldots,m_n}$ there is an infinite number of such quantities. Let the quantity ε'_{m_1,m_2} satisfy the condition $\varepsilon_{m_1,m_2} < \varepsilon'_{m_1,m_2} < \min(\varepsilon'_{m_1}, \varepsilon^*_{m_1,m_2})$, and consider the interval δ_2 of the form $\delta_2 = (\varepsilon_{m_1,m_2}, \varepsilon'_{m_1,m_2})$. It is obvious that $\delta_2 \subset \delta_1$. Then, we call the quantity $\varepsilon_{m_1,m_2,m_3}$ appropriate if $\varepsilon_{m_1,m_2,m_3} \in \delta_2$, i.e., $\varepsilon_{m_1,m_2} < \varepsilon_{m_1,m_2,m_3} < \varepsilon'_{m_1,m_2}$ and take the quantity $\varepsilon'_{m_1,m_2,m_3}$ so as to fulfil the inequality $\varepsilon_{m_1,m_2,m_3} < \varepsilon'_{m_1,m_2,m_3} < \min(\varepsilon'_{m_1,m_2}, \varepsilon^*_{m_1,m_2,m_3})$. Finally, we take the interval δ_3 of the form $\delta_3 = (\varepsilon_{m_1,m_2,m_3}, \varepsilon'_{m_1,m_2,m_3})$. It is obvious that $\delta_3 \subset \delta_2$. Repeating this procedure infinitely many times, we get a new infinite sequence of the quantities $\varepsilon_{m_1}, \varepsilon_{m_1,m_2}, \ldots, \varepsilon_{m_1,m_2,\ldots,m_n}, \ldots$, which contains only appropriate terms of the initial sequence, and an infinite sequence of intervals $\delta_1 \supset \delta_2 \supset \ldots \delta_n \supset \ldots$ embedded into each other. Now let us consider a limit of the form

$$\varepsilon_{m_1,\ldots,m_n,\ldots} = \lim_{n\to\infty} \varepsilon_{m_1,\ldots,m_n} \ .$$

This value $\varepsilon = \varepsilon_c$ of the parameter ε we call the critical one. At $\varepsilon = \varepsilon_c$ our initial Hamiltonian system (1) has invariant tori of all generations because $\varepsilon_c \in \left(\bigcap_{n=1}^{\infty} \delta_n\right)$. The cross section by the plane $t = 0$, obtained as a result of the above picture, is shown in Fig. 3.

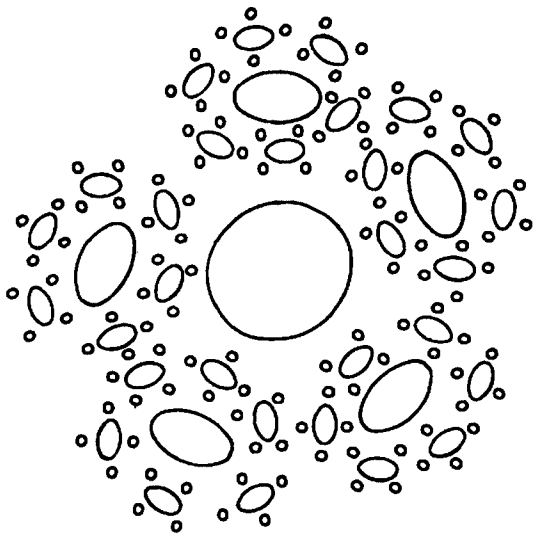

Fig. 3

Now let us take the set E_λ of critical values of the parameter ε which belong to the interval of the form $(0, \lambda)$ where $\lambda > 0$. It immediately follows from the procedure applied to deriving critical values of the parameter ε that the set E_λ has the continual cardinality at any $\lambda > 0$. The following three situations are possible:

1. The set E_λ contains some interval that is contained inside the interval $(0, \lambda)$.
2. The set E_λ is dense in some interval lying inside the interval $(0, \lambda)$.
3. The set E_λ is not dense anywhere in the interval $(0, \lambda)$.

At present, we do not know which of these possibilities really exists.

5 The existence of a self-similar structure in the resonance domain

Let us consider the Hamiltonian system

$$\dot{I} = -\frac{\partial \mathcal{H}}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial \mathcal{H}}{\partial I} \tag{10}$$

with the Hamiltonian function $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \varphi) \ ,$$

where $I = (I_1, I_2)$ and $\varphi = (\varphi_1, \varphi_2)$, and the function $\mathcal{H}_1$ is 2π-periodical with respect to the angle variables φ_1 and φ_2. The point $I' = (I_1', I_2')$ defines a resonance domain of the system (10) if there exist integer numbers m_1 and m_2 such that at the point $I = I'$ the following relations are valid:

$$\text{grad}\, \mathcal{H}_0 \neq 0, \quad m_2 \frac{\partial \mathcal{H}_0}{\partial I_1} - m_1 \frac{\partial \mathcal{H}_0}{\partial I_2} = 0 \, .$$

For definiteness we assume that in some vicinity of the resonance point I' the inequality $\frac{\partial \mathcal{H}_0}{\partial I_2} \neq 0$ holds. Then, the isoenergetic surface $\mathcal{H} = C$, passing through the point I', in some vicinity of this point can be represented as

$$I_2 + H(I_1, \varphi_1, \varphi_2, \varepsilon) = 0 \, ,$$

and the system of equations following from (10) has the form

$$\frac{dI_1}{d\varphi_2} = -\frac{\partial H}{\partial \varphi_1}, \quad \frac{d\varphi_1}{d\varphi_2} = \frac{\partial H}{\partial I_1} \, , \tag{11}$$

where the function H admits the representation

$$H = H_0(I_1) + \varepsilon H_1(I_1, \varphi_1, \varphi_2, \varepsilon) \, .$$

In this case, the equality $\frac{dH_0}{dI_1} = \frac{m_1}{m_2}$ is valid in the resonance point. Thus, in this situation the system (11) describes destruction of an invariant torus with the rational winding. This well–known picture was first described in the paper [5] and has the form depicted in Fig.4.

The results presented above with insignificant changes can be applied to constructing a self-similar structure in the separatrix zone produced after splitting of the separatrix surfaces lying in the resonance domain. It is to be mentioned that if we take two arbitrary resonance domains of the above- described structure, then either the separatrix surfaces of one resonance domain intersect with the separatrix surfaces of the other one or there is an invariant surface separating one resonance domain from the other. Thus, in the case of intersection of the separatrix surfaces of one resonance domain with those of the other resonance domain solutions from one resonance domain may pass into the other one and back. However, this does not necessarily lead to formation of a stochastic layer since self-similar structures arising here prevent from formation of a stochastic layer.

6 Conclusion

In conclusion, I should like to note that the essential part of the results of this paper were obtained as early as 1969 but due to some reasons the

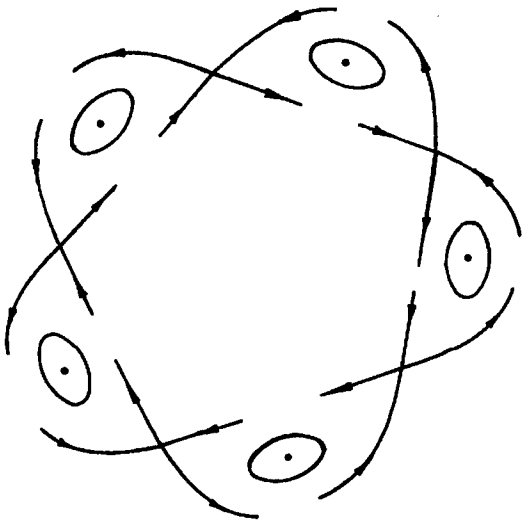

Fig. 4

paper was not accomplished and published. The invitation to visit Courant Institute and to give a talk at a seminar last year stimulated continuation of studies in this field. Taking this opportunity I wish to express my gratitude to Prof. G. Zaslavsky for the invitation and financial support of that visit. I am also grateful to Prof. Y. Ichikawa and J.S.P.S. for financial support.

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Nonlinear Waves in Covariant Shallow Waters

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The vertical integral of the 3-dimensional Euler fluid is discussed as a common basis for covariant shallow water theories including various aspects of vertical variability. In this framework vertical variability appears as internal variability of a strictly 2-dimensional fluid. Under vertical integration the nonlinearity of the Euler fluid generates a hierarchy problem. Several closures are considered in view of the associated wave dynamics.

1 Introduction

At present, numerical ocean models are capable of simulating observable large-scale features of the ocean circulation. Typically these models operate in the framework of the shallow water approximation. With wavelengths of several 10^2 km (for Rossby waves) to several 10^3 km (for Kelvin waves) an ocean depth of rarely more than 4 km renders the system's aspect ratio indeed a small number. There are nevertheless important violations of these scaling characteristics. Most prominently perhaps the deep-water formation in the subpolar Atlantic, driving the "conveyor-belt" circulation, which sets global climate parameters on time-scales of millenia. At high latitudes, surface cooling induces deep convection, which occurs in rotating fluids in narrow "chimneys". Vertical acceleration and vertical shear can generally not be neglected for these processes.

The impact of vertical variability on horizontally propagating waves is extensively studied in nonlinear wave theory. Under appropriate conditions vertical variability introduces dispersive effects, which combined with advection support the formation and propagation of solitary waves and solitons. However, the major body of this theory refers to horizontally 1-dimensional (1D) systems. Generalization of these concepts to the horizontally 2D case frequently leads to severe inconsistencies with respect to the vorticity budget [1].

These problems are readily avoided, if the incorporation of vertical variability into shallow waters observes certain covariance requirements: covariant equations of motion unambiguously determine their vorticity budget. In large-scale oceanography, covariance has besides its general theoretical appeal a very practical advantage. Large-scale motions of the ocean are governed by nonlinear

partial differential equations in curvilinear coordinates in the 2D, non-Euclidean geometry of the planet's (almost) spherical surface. Vorticity- and energy-budgets, wave equations etc. are obtained from these equations of motion with covariance as a helpful guide rather than a mere formal complication [2,3]. Considerations here will nevertheless be restricted to Cartesian coordinates in a nonrotating system.

The primary ingredient of standard shallow water theory is the neglect of internal vertical accelerations relative to the gravitational acceleration. With Greek indices $\mu, \nu, \ldots$ running from 1 to 3 and Latin indices $m, n, \ldots$ running from 1 to 2 over the horizontal coordinates only, the "primitive equations" read

$$\partial^\nu j_\nu = 0$$

$$\partial_t j_n + \partial^\mu j_\mu v_n + \partial_n p = 0$$

$$\partial_3 p = -\rho g$$

where ρ denotes the constant density and $j_\nu = \rho v_\nu$ the mass flux vector. As a mixture of 2D and 3D features, these equations are clearly not covariant. What is more, they no longer satisfy Newton's 1st law: the 3D mass flux vector j_ν in the continuity equation does not equal the 2D momentum density j_n in the prognostic term of the momentum equation. Generally, the "Newtonization" of this system by vertical integration is possible, provided additional information about the vertical shear of v_n is available. From a Newtonian point of view the "primitive equations" thus do not pose a closed dynamical problem. Standard shallow water theory assumes in addition vertical uniformity of v_n and arrives after vertical integration at the spatially 2D, covariant layer equations.

With approximated vertical variability shallow water theories can only be covariant in a spatially 2D, vertically integrated sense. To maintain covariance in the course of the approximation procedure it is hence useful to perform the vertical integration of the 3D Euler equations first and subsequently introduce various assumptions on vertical variability. However, the advective nonlinearity of the Euler fluid generates under vertical integration a hierarchy of products of velocity components. The n-th member of a hierarchy is always controlled by the (n+1)st member. In contrast, the n-th member of an expansion can always be solved if the (n-1)st member is known. The hierarchy problem is a familiar element of turbulence theory and has first been encountered in the present context by Benney [4]. It will here be shown that phenomenologically motivated low-order closures of this hierarchy lead indeed to covariant shallow water theories where vertical dynamics emerge as internal variability of an otherwise strictly 2D fluid.

2 Layer Dynamics

Without approximation, the vertical integral of a 3D, incompressible Euler fluid between a free material surface at $z = h_0(t, x, y)$ and a free (prescribed) material bottom at $z = -h(t, x, y)$ assumes the form

$$\partial_t R + \partial^n J_n = 0 \tag{1.1}$$

$$\partial_t J_n + \partial^m (J_m V_n + R\Pi_{mn} + P\delta_{mn}) = RF_n \tag{1.2}$$

Here, $R = \rho H = \rho(h_0 + h)$ is the column mass and

$$P = \tfrac{1}{2}\gamma R^2$$

with $\gamma = g/\rho$ the layer's effective "equation of state". The mass flux

$$J_n = RV_n = \int_{-h}^{h_0} dz j_n$$

equals momentum density and with the layer velocity V_n the horizontal in situ velocities become $v_n = V_n + w_n$, where the vertical shear is entirely represented by the residual velocities w_n. In contrast to standard shallow waters, the momentum flux of (1) involves a stress tensor

$$R\Pi_{mn} = R(W_{mn} + A_3\delta_{mn}) = \int_{-h}^{h_0} dz\rho(w_m w_n + (h+z)a_3\delta_{mn}) \tag{2}$$

with $a_3 = (\partial_t + v^\nu \partial_\nu)v_3$. This stress tensor is the layer-integrated representation of vertical variability of the 3D Euler fluid. Effects, which cannot be incorporated into the divergence, constitute here an external force

$$F_n = -\partial_n \phi - A_{30}\partial_n h$$

where the potential $\phi = p_0/\rho - gh$ involves surface pressure p_0 and bottom topography h, while

$$RA_{30} = \int_{-h}^{h_0} dz\rho a_3$$

The 2D system (1) is obviously covariant and Newtonian. Thus its conservation properties are formally determined. For the potential vorticity $RZ = \epsilon^{an}\partial_a V_n$, the curl of (1.2) yields

$$RD_t Z = -\epsilon^{an}\partial_a(R^{-1}\partial^m R\Pi_{mn} - A_{30}\partial_n h) \tag{3}$$

with $D_t = \partial_t + V^n \partial_n$. Following a fluid particle the potential vorticity of layer dynamics is generally not conserved due to vertical shear and acceleration. The layer's total energy

$$E = E_k + U + \phi$$

includes in addition to the kinetic energy $E_k = \frac{1}{2} V^n V_n$ and the potential ϕ the positive definite "internal" energy

$$U = E_3 + \tfrac{1}{2} W + P/R$$

where $W = W^n{}_n$ denotes the trace of W_{mn} and

$$RE_3 = \tfrac{1}{2} \int_{-h}^{h_0} dz \rho v_3^2$$

Its budget is obtained as

$$\partial_t RE + \partial^n (BJ_n + \Pi_{mn} J^m + I_n^E) = R\partial_t \phi - RA_{30} \partial_t h \tag{4}$$

with Bernoulli function $B = E + P/R$ and the residual energy flux

$$I_n^E = \int_{-h}^{h_0} dz\, \rho w_n (\epsilon_k + \tfrac{1}{2} v_3^2 + \int_z^{h_0} dr\, a_3)$$

with residual kinetic energy $\epsilon_k = \frac{1}{2} w^n w_n$. Layer energy is conserved, if surface pressure and bottom "topography" are time-independent.

Besides its appealing formal features the major characteristic of (1) is its lack of closure: as a prognostic equation for vertical integrals of 1st order "products" of v_n, the momentum budget (1.2) involves with W_{mn} vertical integrals of 2nd order products. As a prognostic equation for vertical integrals of 2nd order products W, the energy budget (4) involves with I_n^E vertical integrals of 3rd order products $w_n \epsilon_k$. Due to the advective nonlinearity of the 3D Euler equations the (n+1)st member always contaminates the prognostic equation for the n-th member of the layer hierarchy. In the following, simple low-order closures of this hierarchy will be shown to result in 2D, covariant generalizations of well-known 1D advection-dispersion equations.

2.1 Shallow Water

For $a_3 = 0$ and $w_n = 0$ one obtains

$$\Pi_{mn} = 0$$

and (1) reduces to standard shallow water theory with conservation of potential vorticity and nondispersive linear waves

$$\omega^2 = c^2 k^2$$

where $c^2 = \gamma R$. In the 1D case this system has nonlinear, simple wave solutions, while it does not support the propagation of discontinuities. The frequent consideration of turbulent and undular bores in this framework [5] assumes unspecified dissipative effects.

2.2 Boussinesq Equations

If only $w_n = 0$ is assumed, one finds: $W_{mn} = 0$ and $A_{30} = D_t^2 z_1$, where $z_1 = \frac{1}{2}(h_0 - h)$ is the column's center of gravity, while the stress tensor becomes

$$\Pi_{mn} = \tfrac{1}{3} H D_t^2 (H - \tfrac{3}{2} h)\, \delta_{mn}$$

The vorticity budget (3) assumes the form

$$D_t Z = -D_t Y$$

with

$$RY = \epsilon^{mn} (\tfrac{1}{12} \partial_m D_t H \partial_n H + \partial_m D_t z_1 \partial_n z_1)$$

The conservation of $Z_* = Z + Y$ is the covariant expression of the system's "conservation of vertical shear": while vertical acceleration is not entirely neglected, it does not generate vertical shear. The system's total energy with

$$E_3 = \tfrac{1}{2} (D_t z_1)^2 + \tfrac{1}{24} (D_t H)^2$$

is of course positive definite. The dispersion relation for linear waves

$$\omega^2 = c^2 k^2 / (1 + \tfrac{1}{3} H^2 k^2) \approx c^2 k^2 - \tfrac{1}{3} g H^3 k^4 \ldots$$

coincides with the dispersion relation of the familiar 1D Boussinesq system, which leads under appropriate long-term, far-field approximations to the KdV equation. From a different viewpoint the covariant, 2D Boussineq equations were obtained by Green and Naghdi [6].

2.3 Taniuti Equations

The above closures specify vertical profiles of w_n and a_3. Alternatively, the stress tensor (2) may be parameterized without reference to the underlying 3D

quantities, provided such a closure is consistent with the higher order members of the hierarchy, namely the energy budget. Assuming here for simplicity $h = const$, (2) may be written as

$$\Pi_{mn} = \chi \partial_m \partial_n lnR \tag{5}$$

where χ is a free constant. Energetic consistency now requires for the "internal" energy

$$U = U_0 + \tfrac{1}{2}\chi \Delta lnR + P/R$$

where the constant U_0 ensures U to be positive definite, while

$$I_n^E = \tfrac{1}{2}\chi R \Delta V_n$$

for the residual energy flux. These parameterization close the hierarchy such that the energy budget (4) is identically satisfied. Furthermore, since

$$\partial^m R \Pi_{mn} = 2\chi R \partial_n (R^{-1/2} \Delta R^{1/2})$$

it is immediately seen from (3) that this system conserves potential vorticity. Linear waves satisfy the dispersion relation

$$\omega^2 = c^2 k^2 - \chi k^4$$

and in the 1D case appropriate long-term, far-field approximations reduce the system to the KdV equation. Moreover, the nonlinear Schrödinger soliton is an exact solution of the horizontally 1D system [1]. Since the NLS-soliton is a well-known feature of the deep-water approximation, the stress tensor (5) permits the incorporation of aspects of deep-water dynamics into the shallow water equations of large-scale oceanography. In an alternative approach, the spatially 1D version of (5) was first discussed by Taniuti [7].

2.4 Bishallow Water

The shallow 2-layer fluid is the classical model of large-scale oceanography for the study of stratified flows. In the present framework it will be seen that the 2-layer system is in fact a bifluid, similar to Landau's phenomenology of superfluids. For 2 shallow layers with $R_L = \rho_L H_L$, (L=1,2), the total effective density is $R = R_1 + R_2$ and the relative magnitude of the top layer is measured by the concentration-type variable $r = R_1/R$. The "equation of state" of bishallow water is

$$P = P_1 + P_2 + \gamma_2 R_1 R_2 = \tfrac{1}{2}(\gamma r^2 + \gamma_2) R^2$$

with $P_L = \frac{1}{2}\gamma_L R_L^2$ denoting the partial pressures of the individual layers, $\gamma_L = g/\rho_L$ and $\gamma = \gamma_1 - \gamma_2$. In contrast to a mixture of ideal gases, the total pressure of bishallow water is not the sum of its partial pressures, but involves an additional interaction term representing the hydrostatic coupling of both layers. The layer velocities V_n^L form the barycentric mass flux

$$J_n = RV_n = R(rV_n^1 + (1-r)V_n^2)$$

while the interfacial shear $W_n = V_n^1 - V_n^2$ determines the stress tensor (2) as

$$\Pi_{mn} = r(1-r)W_m W_n$$

With this interpretation (1) represents the barycentric components of the shallow 2-layer fluid (besides $a_3 = 0$, here for simplicity p_0=const). Its baroclinic components are governed by the continuity equation of the top layer

$$\partial_t Rr + \partial^n(rJ_n + I_n^M) = 0 \tag{6.1}$$

with baroclinic mass flux $I_n^M = r(1-r)RW_n$, and the shear equation

$$D_t W_n + W^m(\partial_m V_n + (1-2r)\partial_m W_n - W_n \partial_m r) + \partial_n \mu = 0 \tag{6.2}$$

where the quantity $\mu = \mu_1 - \mu_2 = \gamma r R$ plays the role of a "chemical potential". The system (1) and (6) represents a covariant, 2D, 2-component fluid. However, in contrast to binary mixtures, the closure for Π_{mn} and I_n^M is not given diagnostically, but by the prognostic equation (6.2). Hence, the shallow 2-layer system is a bifluid.

The intrinsic phase velocities

$$c_{1/2}^2 = \tfrac{1}{2}c_0^2(1 + \delta r \pm \sqrt{(1-\delta r)^2 + 4\delta r^2})$$

with $c_0^2 = \gamma_2 R$ and $\delta = \gamma/\gamma_2$ refer to barotropic gravity waves (c_1, with dynamics similar to 1st sound) and baroclinic gravity waves (c_2, resembling 2nd sound). Linearization of (1) and (6) around constant (R, r, V_n, W_n) yields the dispersion relation

$$(\Omega^2 - \tfrac{1}{2}A^2)(\Omega^4 - X_2\Omega^2 + X_1\Omega + X_0) = 0$$

where

$$\Omega = \omega - (V^n + \tfrac{1}{2}(1-2r)W^n)k_n$$

is the Doppler-shifted frequency, $A = W^n k_n$ the interfacial shear in propagation direction and

$$X_2 = (c_1^2 + c_2^2)k^2 + \tfrac{1}{2}A^2$$

$$X_1 = c_0^2(1-(2+\delta)r)k^2A$$
$$X_0 = (\tfrac{1}{4}A^2 - c_1^2k^2)(\tfrac{1}{4}A^2 - c_2^2k^2)$$

This dispersion relation provides the basis for the study of critical layers and Kelvin-Helmholtz instability. In the spatially 1D case, bishallow water admits the propagation of baroclinic shocks without further assumptions. The dynamics of this discontinuity are similar to temperature shocks in superfluid Helium [8]. In numerical large-scale oceanography the bishallow water equations find applications in coastal upwelling and mixed-layer physics [9].

3 Summary

Covariant shallow water theories are low-order closures of the layer hierarchy. In compliance with covariance requirements, vertical variability emerges as an internal stress in the otherwise 2D layer equations. Low-order closures of the hierarchy result from approximations of vertical dynamics. Dispersive stresses yield covariant, 2D generalizations of the Boussinesq- and Taniuti-equations. For the purposes of large-scale oceanography the Taniuti equations appear particularly appropriate to incorporate deep-water aspects into shallow water equations. The shallow 2-layer system emerges in the present approach as a bifluid, similar to the phenomenological model of superfluids: barotropic gravity waves take the role of 1st sound, while the dynamics of baroclinic gravity waves resemble those of 2nd sound. The physical wealth of the layer hierarchy warrants its systematic study with the concepts of turbulence theory.

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CHERN-SIMONS GAUGE FIELD THEORY WITH INTEGRABLE DYNAMICS IN 2+1 DIMENSIONS

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The classical spin model in planar condensed media is represented as the $U(1)$ Chern-Simons gauge field theory admitting zero curvature connection. In terms of gauge - invariant fields the evolution is described by the Davey-Stewartson (DS) equations. The Self-dual Chern-Simons solitons are subjected to corresponding integrable dynamics. The physical interpretation of Bäcklund transformations in terms of the Chern-Simons gauge theory is given.

Introduction

Recently, the Chern-Simons (CS) solitons are derived as solutions of the self-dual CS system [2]. These solutions have an important quantum meaning producing excitations known as anyons possessing arbitrary statistics [3] and are relevant to the planar physical phenomena [4,5]. In fact, the effective Ginzburg-Landau theory of the Quantum Hall Effect turns out to be described by the Nonlinear Schrödinger Equation (NLSE) in 2+1 dimensions for a complex matter field which couples with the Abelian Chern-Simons gauge field. When the field's self-interaction coupling strength is related to the Chern-Simons coupling constant the model reduces to the static self-dual CS equations [2]. These equations, being conformal invariant, are completely integrable and admit the Lax pair representation [6].

But analysis of the time dependent solutions with integrable dynamics remains an open problem. Jackiw and Pi [7] mentioned the tantalizing similarity of the problem with the Davey-Stewartson equation, being a well known integrable extension of the NLSE in 2+1 dimensions [14].

In the paper [1] I derived the self-dual CS model from an integrable model in 2+1 dimensions. This model was first obtained by Ishimori and has integrable dynamics of the magnetic vortices [8]. Its formulation in terms of the vorticity for continuous flow tends to clarify the physical meaning of the model and allows an extension for higher dimensions, admitting the bilinear Hirota representation [9,11].

The model is a modification of the classical Heisenberg model for condensed media, having hydrodynamical flow with its vorticity related to a topological

charge density. In this case a hydrodynamical vortex in the magnetic (anyon) fluid is also inducing a magnetic vorticity.

The CS gauge field theory is then obtained by projecting spin variables on the tangent plane to the sphere (spin phase space). This idea has been recently applied to the classical Heisenberg model in [13].

In the case of equality between hydrodynamical vorticity and the "statistical" magnetic field, the theory can be formulated for a zero curvature effective gauge field. This allows me to reformulate the model in terms of gauge invariant matter fields. As shown , their evolution is described by the DS equations and at the same time the fields satisfy the CS system [1]. For the self-dual case, we reproduce CS solitons with an integrable evolution replacing the statistical gauge field with a velocity field.

As a by-product we obtain: a) new reduction conditions for the DS-I model related to the Ishimori-I (IM-I) model, b) the 2+1 dimensional zero curvature formulation for the DS equation.

1 Zero Curvature Condition

The crucial moment of our construction is a zero curvature condition in 2+1 dimensional space

$$F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0, \tag{1.1}$$

for $u(2)(u(1,1))$ Lie algebra valued connection $J_\mu, (\mu = 0,1,2)$. Decomposing the algebra for the diagonal and off-diagonal parts $J_\mu = J_\mu^{(0)} + J_\mu^{(1)}$, and parameterizing

$$J_\mu^{(0)} = \frac{i}{4}(IW_\mu + \sigma_3 V_\mu), J_\mu^{(1)} = \begin{pmatrix} 0 & -\kappa^2 \bar{q}_\mu \\ q_\mu & 0 \end{pmatrix}, \tag{1.2}$$

in terms of two $U(1)$ gauge potentials W_μ and V_μ, and the q_μ matter fields, we have the set of $U(1) \times U(1)$ gauge invariant equations

$$D_0 \psi_\pm = (2\pi^{1/2})^{-1} D_\pm q_0, \tag{1.3a}$$

$$[D_+, D_-] = 8\pi\kappa^2(|\psi_+|^2 - |\psi_-|^2), \tag{1.3b}$$

$$[D_0, D_\pm] = -4\pi^{1/2}\kappa^2(\bar{q}_0\psi_\pm - \bar{\psi}_\pm q_0), \tag{1.3c}$$

$$D_+\psi_- = D_-\psi_+, \tag{1.3d}$$

$$\partial_\mu W_\nu - \partial_\nu W_\mu = 0. \tag{1.3e}$$

Here the covariant derivative is defined as $D_\mu = \partial_\mu - (i/2)V_\mu$, $D_\pm = D_1 \pm iD_2$, $\kappa^2 = \pm 1$. We introduced $\psi_\pm = (2\sqrt{\pi})^{-1}(q_1 \pm iq_2)$, for the spatial part of the matter fields.

These equations can be derived from the Chern-Simons Lagrangian density with the statistical parameters κ^2 and ν [1]. It describes the non-Abelian $U(2)$ CS topological field theory, in terms of parametrization (1.2) [12].

2 Topological magnet in Tangent Space

To resolve the momentum problem in ferromagnets, the simple model of delocalized electrons was considered [10]. In this paper additional hydro-dynamical variables which describe the fermionic background were introduced. This allowed us to formulate a simple model of ferromagnetic continuum with non-trivial background [9]

$$\partial_t \mathbf{S} + v^j \partial_j \mathbf{S} = \mathbf{S} \times \partial E/\partial \mathbf{S}, \tag{2.1a}$$

$$\partial_i v_j - \partial_j v_i = \mathbf{S}(\partial_i \mathbf{S} \times \partial_j \mathbf{S}), (i,j = 1,2), \tag{2.1b}$$

where $E(\mathbf{S}, \partial \mathbf{S}) = (\partial_1 \mathbf{S})^2 + \alpha^2 (\partial_2 \mathbf{S})^2, \mathbf{S}^2 = 1$. The scalar product is defined by a diagonal metric tensor $g_{ij} : A^j B_j = g^{ij} A_i B_j$ with coefficients of exchange interaction constants $g_{ij} = \text{diag}(1, \alpha^2)$, and $\alpha^2 = \pm 1$.

In the case of a restricted flow, when the velocity field $\mathbf{v}$ satisfies the additional, "incompressibility" condition $\partial_1 v_1 + \alpha^2 \partial_2 v_2 = 0$, the model (2.1) can be derived from the linear problem [1]. When this equation is resolved in terms of the stream function ϕ only, $v_1 = \partial_2 \phi, v_2 = \alpha^2 \partial_1 \phi$, the system (2.1) is reduced to the Ishimori model [8].

In order to formulate the model in terms of tangent space variables we diagonalize the spin matrix $S = g\sigma_3 g^{-1}$. and define the left current $J_\mu = g^{-1}\partial_\mu g$, being viewed as a connection (1.2), carrying the zero-curvature condition (1.1) [1]. Equations of motion (2.1) in terms of this current have the form

$$q_0 = i(D^j + iv^j)q_j, \tag{2.2a}$$

$$\partial_i v_j - \partial_j v_i = -4i\kappa^2(\bar{q}_i q_j - \bar{q}_j q_i). \tag{2.2b}$$

The particular choice allows that eq.(1.3e) will have the form of the "incompressibility" condition. If we put $\phi = \frac{1}{4G_0}\gamma$, where G_0 is an arbitrary constant, we find $W_1 = \frac{\alpha^2}{G_0} v_2, W_2 = \frac{1}{G_0} v_1,$. The last relation between $\mathbf{v}$ and $\mathbf{W}$ is just a duality transformation in two dimensions.

3 $\alpha^2 = -1$ Case

We consider the case $\alpha^2 = -1$, with (+,-) signature for the matrix g_{ij} . Eqs.(2.2) written in terms of $\psi_\pm$, allows one to exclude q_0 from the system (1.3).

It turns out that our model (2.1) can be described as a non-relativistic field theory for a pair of $\pm$ charged matter fields, interacting with the Abelian Chern-Simons gauge field V_μ and the velocity field $\mathbf{v}$.

In this theory the excitations carrying charge

$$Q = \int (|\psi_+|^2 - |\psi_-|^2) d^2x,$$

also possess magnetic and hydrodynamical fluxes, determined by statistical parameter κ^2 . The fluxes coincide and can compensate one another. Thus we can introduce new, irrotational, gauge field $\mathcal{A} = \mathbf{V} - \mathbf{v}$ and related covariant derivative $\mathcal{D} = \partial - i/2\mathcal{A}$. Then the system is

$$2iD_0\psi_\pm + (\mathcal{D}_+^2 + \mathcal{D}_-^2)\psi_\pm + [\frac{1}{4}(v_+^2 + v_-^2) \pm \frac{i}{2}(\partial_+ v_+ - \partial_- v_-)]\psi_\pm = 0, \quad (3.1a)$$

$$\partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ = 0, \quad (3.1b)$$

$$\partial_0 \mathcal{A}_\pm - \partial_\pm V_0 = -\dot{v}_\pm - 8\pi\kappa^2\{\psi_\pm[(\bar{\mathcal{D}}_+ - \frac{i}{2}v_-)\bar{\psi}_+ + (\bar{\mathcal{D}}_- - \frac{i}{2}v_+)\bar{\psi}_-] +$$

$$\bar{\psi}_\mp[(\mathcal{D}_+ + \frac{i}{2}v_+)\psi_+ + (\mathcal{D}_- + \frac{i}{2}v_-)\psi_-]\}, \quad (3.1c)$$

$$(\mathcal{D}_+ - \frac{i}{2}v_+)\psi_- = (\mathcal{D}_- - \frac{i}{2}v_-)\psi_+, \quad (3.1d)$$

$$\partial_+ v_- - \partial_- v_+ = 16\pi\kappa^2 i(|\psi_+|^2 - |\psi_-|^2), \quad (3.1e)$$

$$\partial_+ v_- + \partial_- v_+ = 0. \quad (3.1f)$$

The system (3.1) is invariant under the $U(1)$ local gauge transformation [1]. According to (3.1b), the planar gauge field components $\mathcal{A}_\pm$ have a vanishing field strength, and must be of a pure gauge form. We can solve this in terms of real function λ, $\mathcal{A}_j = \partial_j \lambda, (j = 1, 2)$, and introduce the gauge-invariant matter fields

$$\Psi_\pm = \psi_\pm e^{-\frac{i}{2}\lambda(x_1, x_2, t)}. \quad (3.2)$$

Hence the system (3.1) simplifies

$$2i(\partial_0-\frac{i}{2}\mathcal{V}_0)\Psi_\pm+(\partial_+^2+\partial_-^2)\Psi_\pm+[\frac{1}{4}(v_+^2+v_-^2)\pm\frac{i}{2}(\partial_+v_+-\partial_-v_-)]\Psi_\pm=0, \quad (3.3a)$$

$$\dot{v}_\pm=\partial_\pm\mathcal{V}_0-8\pi\kappa^2\{\Psi_\pm(\partial_-\bar{\Psi}_++\partial_+\bar{\Psi}_-)+\bar{\Psi}_\mp(\partial_+\Psi_++\partial_-\Psi_-)\}\pm 4\pi\kappa^2 iv_\mp(|\Psi_+|^2-|\Psi_-|^2), \quad (3.3b)$$

$$(\partial_+-\frac{i}{2}v_+)\Psi_-=(\partial_--\frac{i}{2}v_-)\Psi_+, \quad (3.3c)$$

$$\partial_+v_--\partial_-v_+=16\pi\kappa^2 i(|\Psi_+|^2-|\Psi_-|^2), \quad (3.3d)$$

$$\partial_+v_-+\partial_-v_+=0. \quad (3.3e)$$

Note that the statistical gauge field $\mathbf{V}$ has completely disappeared from our system.

After introducing new fields

$$\mathcal{A}_0^{(\pm)}=\mathcal{V}_0+\frac{1}{4}(v_+^2+v_-^2)\pm\frac{i}{2}(\partial_+v_+-\partial_-v_-), \quad (3.4)$$

we obtain [1] that the model (3.3) contains two coupled Davey-Stewartson equations

$$2i\partial_0\Psi_\pm+(\partial_+^2+\partial_-^2)\Psi_\pm+\mathcal{A}_0^{(\pm)}\Psi_\pm=0, \quad (3.5a)$$

$$\partial_+\partial_-\mathcal{A}_0^{(\pm)}=8\pi\kappa^2(\partial_+^2+\partial_-^2)|\Psi_\pm|^2, \quad (3.5b)$$

for the pairs $(\Psi_+,\mathcal{A}_0^{(+)}),(\Psi_-,\mathcal{A}_0^{(-)})$. These equations are known as the DS-II equations and are an integrable system [14]. The potentials $\mathcal{A}_0^{(\pm)}$ are connected by the relation $\mathcal{A}_0^{(+)}-\mathcal{A}_0^{(-)}=i(\partial_+v_+-\partial_-v_-)$, and the functions $\Psi_\pm$ by $(3.3c-e)$ constraints. These constraints are consistent with the time evolution described by the DS equations (3.5) [1].

On the other hand, to be the solution of model (3.3), a pair of solutions for the DS-II eqs.(3.5) must satisfy further relations. Relation (3.3d) allows one to express $\mathbf{v}$ as a nonlocal form of Ψ_+ and Ψ_- in the "Coulomb" gauge. Then, substituting this $\mathbf{v}$ to eq.(3.3c) we obtain a relation connecting two solutions of the DS-II equations (3.5), which is reminiscent of the Bäcklund transformations for nonlinear evolution equations.

In fact, in 1+1 dimensions (when $\partial_2=0$) the DS system (3.5) reduces to the Nonlinear Schrödinger equation

$$i\partial_0\Psi_\pm+\partial_1^2\Psi_\pm+8\pi\kappa^2|\Psi_\pm|^2\Psi_\pm+(\frac{1}{4}\dot{v}_0(t)x_1+\frac{1}{2}\rho_0(t))\Psi_\pm=0, \quad (3.6)$$

while the model (3.3) produces in addition the first order system

$$\partial_1\Psi_+ - \partial_1\Psi_- = \frac{i}{4}v_0(t)(\Psi_+ - \Psi_-) + \frac{1}{2}v(\Psi_+ + \Psi_-), \qquad (3.7a)$$

$$\partial_0\Psi_+ - \partial_0\Psi_- = -\frac{1}{4}v_0(t)(\partial_1\Psi_+ - \partial_1\Psi_-) + \frac{i}{2}v(\partial_1\Psi_+ + \partial_1\Psi_-) +$$

$$4i\pi\kappa^2(|\Psi_+|^2 + |\Psi_-|^2)(\Psi_+ - \Psi_-) + i(\frac{1}{4}\dot{v}_0(t)x_1 + \frac{1}{2}\rho_0(t))(\Psi_+ - \Psi_-), \quad (3.7b)$$

with the velocity field $v = \pm\sqrt{\alpha_0^2 - 16\pi\kappa^2|\Psi_+ - \Psi_-|^2}$. The system (3.7) is just the Bäcklund transformations for the Nonlinear Schrödinger equation (3.6) and for the time independent v_0, ρ_0 becomes of the well known form [16].

The last results indicate on new, physical, interpretation of the Bäcklund transformations in terms of the Chern-Simons gauge field theory. Moreover, the system (3.3b-e) has the form similar to the Bäcklund transformations for the 2+1 dimensional DS equations [17]. In the next section using these relations we generate a reach class of soliton solutions for the DS-II equations (3.5).

4 Self-Dual Chern-Simons Solitons

If one of the functions $\Psi_\pm$ vanishes the (anti-)holomorphicity condition for the original spin variables (2.1) is satisfied. Let $\Psi_- = 0$. Then as follows $\partial_-\zeta = 0$, where ζ is the stereographic projection of the spin vector $\mathbf{S}$. In this case the system $(3.3c - d)$ reduces to the self-dual Chern-Simons system

$$(\partial_- - \frac{i}{2}v_-)\Psi_+ = 0, \qquad (4.1a)$$

$$\partial_+ v_- - \partial_- v_+ = 16\pi\kappa^2 i|\Psi_+|^2. \qquad (4.1b)$$

We can call it *hydrodynamical*, since the velocity of the planar flow plays the role of the statistical gauge field. This system is connected with the Liouville equation [2]. If we represent the velocity $v_1 = \partial_2\phi + \partial_1\chi, v_2 = -\partial_1\phi + \partial_2\chi$, in terms of the *stream function* ϕ and the *velocity potential* χ, then from (4.1a) the general form of Ψ_+ follows

$$\Psi_+ = \exp(-\frac{1}{2}\phi + \frac{i}{2}\chi)F(\bar{\eta}) = \sqrt{\rho}_+ e^{\frac{i}{2}\omega}, \qquad (4.2)$$

where $F(\bar{\eta})$ is an arbitrary anti-holomorphic function, $\eta = x_1 + ix_2$. Substituting the Ψ_+ in eq.(4.1b), we find for $\rho_+ = |\Psi_+|^2$ the Liouville equation. This equation has regular, nonnegative solutions only for the compact spin model when $\kappa^2 = 1$. According to eq.$(3.3e)$ the function χ should be harmonic $\Delta\chi = 0$, as well as the phase of function Ψ_+.

The auxiliary field $\mathcal{A}_0^{(+)}$ can be find easily from the above relations [1]:

$$\mathcal{A}_0^{(+)} = -(\partial_+^2 + \partial_-^2)(\ln \rho_+ - i\omega).$$

The N - vortex-soliton solution

$$\zeta(\bar{\eta}) = \Sigma_{n=1}^{N} \frac{c_n}{\bar{\eta} - \bar{\eta}_n}, \tag{4.3}$$

is defined by $4n$ real parameters, coded in the complex $c_n(t)$ and $\bar{\eta}_n(t)$, describing the scale, phase and position of solitons on the plane. The corresponding topological charge is $Q = +N$. In the case $\Psi_+ = 0$, we have $\partial_+ \zeta = 0$ and the same Liouville equation for ρ_-. Replacing $\bar{\eta}$ to η in the solution (4.3) we can obtain the N - antivortex soliton having a charge $Q = -N$.

The evolution of these solitons should follow to the DS equations (3.5). In fact, as easy to check, during the time evolution (3.5) the Liouville equation for the modulus and the Laplace equation for the phase of Ψ_+ preserves the form.

It provides the time dependence for parameters $c_n(t)$ and $\bar{\eta}_n(t)$. Combining properly these parameters one can obtain integrals of motion. We note that in the paper [15] the reduction of the DS equation to the Liouville equation was considered. In our model these results have appear naturally and are applicable to the noncompact spin algebra when $\kappa^2 = -1$.

5 $\alpha^2 = 1$ Case

In this section we briefly describe the $\alpha^2 = 1$ reduction of the system (2.1). In this case we introduce new complex fields $\chi_\pm = 1/(2\sqrt{\pi})(q_1 \pm q_2), \bar{\chi}_\pm = 1/(2\sqrt{\pi})(\bar{q}_1 \pm \bar{q}_2)$, and define the covariant derivative and velocity fields as $D_\pm = D_1 \pm D_2, v_\pm = v_1 \pm v_2$. Then excluding q_0 from zero curvature equations (1.1), and introducing the new gauge field as in Sect.3, we find that it must have zero strength, $\mathcal{A}_j = \partial_j \lambda$.

The system can be written in terms of gauge invariant variables [1]

$$X_\pm = \chi_\pm e^{-\frac{i}{2}\lambda}, \tag{5.1}$$

but in contrast to the previous case, with $\alpha^2 = -1$, the potentials $\mathcal{A}_0^{(\pm)}$ are no longer real functions. Then we obtain the DS - I equations for $Q = X_+, R = \bar{X}_-$:

$$2i\partial_0 Q + (\partial_+^2 + \partial_-^2)Q + \mathcal{A}_0^{(+)} Q = 0, \tag{5.2a}$$

$$-2i\partial_0 R + (\partial_+^2 + \partial_-^2)R + \mathcal{A}_0^{(+)} R = 0, \tag{5.2b}$$

$$\partial_+\partial_-\mathcal{A}_0^{(+)} = 8\pi\kappa^2(\partial_+^2 + \partial_-^2)RQ, \tag{5.2c}$$

supplied with constraints

$$(\partial_+ - \frac{i}{2}v_+)\bar{R} = (\partial_- - \frac{i}{2}v_-)Q, \tag{5.3a}$$

$$\partial_+ v_- - \partial_- v_+ = 16i\pi\kappa^2(RQ - \bar{R}\bar{Q}). \tag{5.3b}$$

From the above equations we can conclude that IM-I model is connected with DS-I equation (5.2) but with a new reduction condition (5.3), instead of the usual one $\bar{R} = \pm Q$. The relations (5.3) are the simplest realization of the Bäcklund transformations for the DS-I equations (5.2) [17], with the transformed functions $\tilde{Q} = \bar{R}, \tilde{R} = \bar{Q}$.

If we suppose that $\bar{R} = \pm Q$. in addition to (5.3), then $\mathbf{v} = 0$. Recovering $\mathcal{A}_0^{(+)}$ from eq.(5.2c) we find the one-dimensional Nonlinear Schrödinger equation.

6 Conclusion

In conclusion I like to emphasize some points. Since DS equations arise in hydrodynamical wave phenomena it is important to note the appearance in this context of the self-dual Chern-Simons system (4.2), usually associated with the anyons and exotic statistics.

Moreover, the related zero curvature conditions are the classical equations of motion for the 2+1 dimensional Chern-Simons Topological Quantum Field Theory (TQFT). This allows the mapping of the model to the TQFT with a suitable gauge group. We hope that full 2+1- dimensional solutions for our model will provide a description of the moduli space beyond the perturbative Manton's approach for the slowly moving monopoles [18].

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A PATCHWORK APPROACH TO THE NONLINEARITY INVERSE PROBLEM

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The lecture contains a few remarks on extracting from measurements the informative content of physical systems. Nonlinearities in physical systems usually appear as special features on a linear patchwork but each one can modify very much the information current and thus reduces what can be practically obtained. On the contrary, a patchwork structure in a formally overdetermined linear inverse problem is rather helpful.

1 Introduction

The somewhat ambiguous title of this paper corresponds mainly to a couple of questions. How do nonlinearities appear, as their name suggests it, as accidents in a linear world, and how do their existence modify our way of identifying physical systems.

As a matter of fact, the historical occurrence of nonlinearities in physics looks like a patchwork, whose texture is linear, features are nonlinear aspects. The "linear texture" of physics appears every

response is proportional to excitation
propagation is invariant with respect to the wave amplitude and additive for admissible waves.

Considering that this generic case is the most commonly studied, nonlinearities appeared as figures sewed on this linear tissue, both as regards observed phenomena and mathematical models. Sticking at the 19th century we can trace back obviously nonlinear phenomena to

- *breaking waves*, to be studied later in the framework of *turbulence*
- *bores*, to be studied later as *coherent structures (kinks)*, whereas *solitary waves* inspired *solitons*.
- More refined observations showed nonlinear dynamics in *swell* and in *turbulent flows*, nonlinear *instabilities* in *gravitational* or *thermal evolutions*, and several kinds of nonlinearities in meteorology, oceanography, etc, to begin

the 20th century, where observations continuously increased in number and complexity.

Simultaneously, mathematical physicists reintroduced nonlinear terms in models where these terms came from physics but had been dropped out "for the sake of simplicity". Hence Navier Stokes equation, and others, nonlinear systems with few parameters - and, later, with many parameters, were studied. Curiously, but not unexpectedly, it turned out that solutions of these models showed well defined nonlinear features appearing in most cases *between* wide continuous quasilinear ranges - e.g. *solitons* in "integrable" systems, *bifurcations, attractors, chaotic* behavior, etc - thus, again, a *patchwork* where, contrarily to the "linear texture", the "nonlinear features" strongly depend on amplitude or other parameters ranges, time scales, infinitesimal perturbations of sensible parameters, etc.

Thus we can take as a general observation that nonlinear features appear as isolated figures on our linear world, and in this paper, we concentrate on the second question : how this fact modifies our way of identifying physical systems. We shall go through two cases : that of a system given as a whole (identification problems are then called "inverse problems") and that of a system evolving with time (identification problems are then called "data assimilation"). Our answer will be rather pessimistic, because nonlinearity reduces or partially destroys a large part of the information available within a reasonable precision.

However, the patchwork structure is rather helpful for solving inverse problems, as it can be seen on a linear example given in the last section.

2 Model identification

Unless a physical model is completely guessed from other physical informations, it must be identified, either as a whole ("inverse problem") or progressively ("data assimilation"). In both cases one must first gather the "a priori informations",

- on the linear or quasilinear "texture" of the observed phenomenon
- on its nonlinear features
- on how the information itself is propagated to measure instruments - are there ranges or time scales where signal propagation is linear, or, more generally, where this propagation corresponds to an invariant spectrum, so that convenient methods of signal processing apply ?

2.1 An example of inverse problem

We see all the above questions on a typical physical example, that of a "tsunami event" : it is a case where model equations are reasonably well-known, reasonably stable, but observations are usually sparse, so that a good analysis of information is necessary. Is it possible to do it ?

The event begins at a fixed (of course unexpected) time (say, $t = 0$), with a slide, or an earthquake of small focal depth, perturbing the sea bottom on a range of few kilometers or tenths of kilometers. Inside this "*generation range*", there is no doubt that ground motions are strongly nonlinear, but the excitation of the body of water is quasilinear : we mean that both numerical calculations and models in tanks[1] show that amplitudes in the first times are predicted from a description of the ground motion (amplitude as a function of time) inserted as the input of a linear model (for this amplitude) up to an "admissible error", say δ, at any place except those where usual nonlinear features show (breakers). δ is both smaller than the usual dispersion of results in series of tank experiments and than the usual "noise", i.e waves on sea for average weather. The first disturbance propagates linearly during few minutes, which are enough to extend the phenomenon much outside the generation range, into what we call the generation range.

As a matter of fact, the generation range extends over the deep sea, and strongly depends on the case we consider. But in almost all cases, the characteristic size of the initial disturbance is definitely larger than the average ocean depth (a few kilometers), so that nonlinear features appear in the propagation. There will be quasilinear ranges, where dispersion is strongly dominant : the signal is then on a ring roughly centered in the generation range, extending at constant speed, while the signal amplitude attenuates as the inverse of the ring radius. There will be ranges where the bottom configuration is such that the signal is trapped into a (monodimensional) curve[2]. Nonlinearity and dispersion are therefore of same order and the main signal concentrates into a few solitons. It can be analysed by means of a special nonlinear processing[3]. If we try to appraise the signal informative content, we rapidly realize that the noise is too large in quasilinear ranges. In nonlinear ranges of trapping, the solitons can be analyzed, and their informative content remains after a long propagation. Unfortunately, this content is reduced to a few parameters related to the initial disturbance. Hence the information is either washed out or reduced in the generation range.

Most measurements are not done in the generation range, where a big tsunami shows only a few decimeters amplitude long wave. They are done on beaches of islands or continents, which are related to deep sea by any kind of

sloppy configuration. Because of it, solitons break into pieces, concentration may appear in gulfes, runups achieve a huge increase of amplitude, up to several meters (in extreme cases almost 100 times the deep sea signal !). Thus the information on initial disturbance is still simplified and concealed.

There is another strong difference with the linear case. In the linear case, the response at any measurement would be obtained by a convolution product of the source function (the initial disturbance) and an elementary response (the system admittance, i.e. its response to a Dirac impulse). Identifying the source would be a deconvolution problem, of course ill posed but tractable and invariant under modification of the source : the elementary response would be related once for ever to the basin geometry. In the nonlinear case, the ranges where nonlinear features appear depend not only on the basin geometry but also on the source amplitude : the "patchwork" geometry, in someway, is not invariant.

We conclude that in the typical inverse tsunami problem (get the initial disturbance from realistic measurements) at each step nonlinearity effects come in and they

(a) reduce the information to a few moments of the source function

(b) conceal this useful information behind much overwhelming information on the basin geometry and much noise.

Thanks to them, it is only in the generation range that the source can be precisely estimated !

Remark

Similar conclusions hold for phenomena where signals propagate as quasi-stationary wave packets on a surface, and show envelope solitons in the nonlinear ranges[2].

2.2 The case of data assimilation

We try to identify progressively a complicated system which evolves with time. An example is that of meteorology, where the wind $\mathbf{V}$ is related to the Corolis vector $\mathbf{\Omega}$, the pressure P, the gravity g, the density ρ, by dynamical equations : the system is supposed to have an invariant inertial manifold $\mathbf{M}$ that attracts trajectories. As a matter of fact, in the shallow water equations approximation this attractor expresses a dynamic equilibrium between gravity modes and Rossby waves ; initial conditions should be close to it or gravity motions can mask the short range forecast. Similar systems appear in oceanography. In both cases, and other, the system can be represented by a state vector $\mathbf{X}(t)$

and the model yields the equation

$$\frac{d\mathbf{X}}{dt} = F(\mathbf{X}) \tag{2.1}$$

from the "initial condition" $\mathbf{U} = \mathbf{X}(0)$.

Let $\mathbf{d}$ be a linear "observation" of $\mathbf{X}$, say $d \sim \mathbf{CX}$. The discrepancy at time T between observations and the model solution that corresponds to U is given by a cost functions, e.g.

$$F(\mathbf{U}) = \frac{1}{2}\int_0^T \|\mathbf{CX}(\mathbf{U}) - \mathbf{d}\|^2 dt \tag{2.2}$$

For each T, "assimilating" data is determining U^* which minimizes F on a set of admissible initial conditions U_{ad}.

It is not difficult to understand that the result can be reliable only on a time scale whose value is difficult to appraise, and which corresponds to a quasilinear range in the evolution. After it, parts of the system may evolve to chaos in such a may that prevision becomes impossible. Even in meteorology, where huge networks enable continuous data assimilation, it is too early[4]

(1) to say what time scales will be consistent with good identifications
(2) to appraise what information will remain valid after these times.

3 Approach of a linear problem using its patchwork structure

The patchwork structure is a nuisance to identify systems in the nonlinear case because patchwork "figures", being nonlinear, are very different from the patchwork "texture", so that in some way they chop the whole system into pieces : we can identify the figure only from close measurements, and it is also true for linear parts between them because only a few pieces of information are transmitted through the figures.

A completely different situation exists in linear problems where we try to identify a few isolated figures altogether from close measurements and from remote ones, because here they combine as redundant pieces of information. We shall see now a generic example.

3.1 The problem and its a priori assumptions

We deal with an example of diffraction tomography. Let $\mathcal{D}$ be a wide strip in $\mathbb{R}^3$ limited by two surfaces that can be viewed as two deformed parallel planes

$\mathcal{P}^1$, $\mathcal{P}^2$, not necessarily unlimited, but stopping only at distances much larger than the strip thickness. So as to be simple, we use the wave equation

$$\Delta u + (k^2 - V)\, u = 0 \tag{3.1}$$

the external medium being $\mathbb{R}^3 \backslash \mathcal{D}$.. Sources and measurements lie on $\mathcal{P}^1$ and $\mathcal{P}^2$. Now, the essential a priori assumptions is that V is known "a priori" to get small values in $\mathcal{D}$ that can be managed by a simple Born approximation *everywhere but* in "active subdomains" $\mathcal{A}_i$ of $\mathcal{D}(i = 1, 2, ...N)$, where V may (or may not) be large. The $\mathcal{A}_i's$ are known, and far from each other : clearly, we can assume without loss of generality that each $\mathcal{A}_i$ goes from its boundary C_i^1 on $\mathcal{P}^1$ to its boundary C_i^2 on $\mathcal{P}_i^2$. The remaining part of $\mathcal{D}$ being D_0, with boundaries C_0^1, resp. C_0^2, on $\mathcal{P}^1$, resp. $\mathcal{P}^2$, we set

$$\mathcal{A} = \sum_{i=1}^{N} \mathcal{A}_i \quad ; \quad C_i = C_i^1 + C_i^2 \quad ; \quad C = \sum_{i=0}^{N} C_i \tag{3.2}$$

$$\mathcal{B}_i = \mathcal{A}_i \cap \mathcal{D}_0, i = 1, 2, ...N \tag{3.3}$$

$\mathcal{B}_i$ can be called the "internal boundary of $\mathcal{A}_i$; i.e the boundary where no measurement can be done. In the following we assume for the sake of simplicity that sources impose the value of u on C, such as

$$u(x \in C) = f(x) \tag{3.4}$$

We shall first explain the basic ideas, then the algorithms they suggest for the direct problem, and more tersely for the inverse problem.

3.2 The two key ideas (5)

The first idea arises when the problem is treated on two levels, namely the local one, i.e. separately for each $\mathcal{A}_i$, and the global one, for $\mathcal{D}$, which completes the first one by managing coupling terms between the $\mathcal{A}_i's$. The idea is that this coupling is small enough to be treated iteratively, or, as a reasonable approximation, limited to its first order. Here we justify our idea only by the distance factor, but there are several other heuristic arguments (diffusion due to complicated heterogeneities, absorption, time delays, etc) which all can concur.

The second idea is that if the density of measurements done on C is sufficient, those on each C_i are sufficient to derive a reasonable "first guess" i.e a rough estimate, of V inside $\mathcal{A}_i$ when we manage the local inverse problem for

$\mathcal{A}_i$, forgetting the coupling. Then it should be possible to minimize iteratively a general cost function to solve approximately the global inverse problem. Our justification is that, in $\mathbb{R}^3$, boundary measurements problems are formally overdetermined since we try to derive a function of 3 positions variables from a function of 4 positions variables (kernel of the Dirichlet Neuman or Neuman Dirichlet operator).

3.3 Direct problem management

Let $G(x,y)$ be the Green's function of the global problem which verifies on the boundary C the zero Dirichlet condition (3.4), (or the Sommerfeld one on infinite branches) :

$$\begin{cases} [\Delta_y + k^2]\, G(x,y) = -\delta(x,y) \\ G(x \in C, y) = 0 \end{cases} \tag{3.5}$$

G depends on the geometry of C, and this is a "physical must", since measurements are on C, but it does not depend on V. From (2.1), (3.1), (3.5) we readily derive the fundamental integral equation :

$$u(x) = -\int_C f(y) \frac{\partial}{\partial \nu_y} G(x,y) d\sigma_y - \int_{\mathcal{D}} G(x,y) V(y) u(y) dy \tag{3.6}$$

Assume for the sake of simplicity that V vanishes in $\mathcal{D}_0$, Eq. (3.6) reduces to

$$u(x) = h(x) - \int_{\mathcal{A}} G(x,y) V(y) u(y) dy \tag{3.7}$$

which is an equation for $u(x)$, $x \in \mathcal{A}$, and a simple equality defining the continuation of $u(x)$ from $x \in \mathcal{A}$ to $x \in \mathcal{D}_0$. We used the notation

$$h(x) = -\int_C f(y) \frac{\partial}{\partial \nu_y} G(x,y) d\sigma_y \tag{3.8}$$

If V was small enough inside active domains too, we would be able to get u by means of convergent Born series, and the simple Born approximation u_B of u would be

$$u_B(x) \sim h(x) - \int_{\mathcal{A}} G(x,y) V(y) h(y) dy \tag{3.9}$$

Giving up the assumptions which justify (3.9), let V may be large in active domains. We assume we know how to solve each local problem, i.e each problem where one $\mathcal{A}_i$ is concerned and none other would exist inside $\mathcal{D}$. It means that

we know for $x \in \mathcal{A}_i$, exactly or numerically, the resolvent of the Fredholm equation

$$u(x) = k(x) - \int_{\mathcal{A}_i} G(x,y)V(y)u(y)dy \tag{3.10}$$

i.e the solution of (3.10) is for $x \in \mathcal{A}_i$

$$u(x) = k(x) - \int_{\mathcal{A}_i} R_V^i(x,y)k(y)dy \tag{3.11}$$

Inserting (3.11) into (3.10) would yield the continuation of $u(x)$ for any x such that k and G are defined. Hence, if $\mathcal{A}_i$ was the only active domain in $\mathcal{D}$, so that the fundamental equation (3.7) would be of the form (3.10), (3.11) would give the solution of (3.7), and thus of (3.6). If there is more than one active domain, their coupling comes in. The relation (3.7) remains a way to continue $u(x)$ for $x \notin \mathcal{A}$. For $x \in \mathcal{A}$ and a fixed i, we can write it down as (3.10), with

$$k(x) = h(x) - \sum_{j \neq i} \int_{\mathcal{A}_j} G(x,y)u(y)V(y)dy \tag{3.12}$$

and hence $u(x)$ is given by (3.11), or

$$u(x) = v_i(x) - \sum_{j \neq i} \int_{\mathcal{A}_j} u(y)V(y)H_V^i(x,y)dy \tag{3.13}$$

where

$$v_i(x) = h(x) - \int_{\mathcal{A}_i} R_V^i(x,y)h(y)dy \tag{3.14}$$

$$H_V^i(x,y) = G(x,y) - \int_{\mathcal{A}_i} dz\ R_V^i(x,z)G(z,y) \tag{3.15}$$

We may keep i fixed and iterate on the equation (3.13), with a proper definition of H_V^i to keep it true not only for $x \in \mathcal{A}_i$ but for any $x \in \mathcal{A}$. This is a patchwork algorithm "seen from a particular domain". But we can also construct a patchwork algorithm where no domain is privileged (and there are ways of optimizing it). For this, let us introduce for any function $g(x)$ and any value of i (we do not duplicate the index i if it already appears) the notation

$$g_i(x) = g(x)\theta_i(x) \tag{3.16}$$

where

$$\theta_i(x) = \begin{cases} 1 & x \in \mathcal{A}_i \\ 0 & x \notin \mathcal{A}_i \end{cases} \tag{3.17}$$

The equation (3.13) can be written as the set of coupled equations

$$u_i(x) = v_i(x) - \sum_{j \neq i} \int_{\mathcal{A}} u_j(y) V(y) \mathbf{H}_V^i(x,y) dy, \quad i = 1, 2, ..., N \tag{3.18}$$

and this set of coupled equations can be solved by the iterative algorithm

$$u_i^{n+1}(x) = v_i(x) - \sum_{j \neq i} \int_{\mathcal{A}} u_j^{(n)}(y) V(y) \mathbf{H}_V^i(x,y) dy \tag{3.19}$$

starting at $\left\{u_i^{(0)}\right\} = \{0\}$. In these patchwork algorithms, the ratio of the mean diameter of each set to the distance to its closes neighbour is the factor improving the convergence rate when compared to the simple Born approximation. At large k, this factor is still improved when the wave length is short compared to each diameter. Keeping the first term as an approximation can be then quite good.

3.4 Inverse problem management

We do not go into details. If $F_i(V_i)$ is the local cost functional for $\mathcal{A}_i$, minimizing it for the sources and measurements done on C_i give a first estimate of V_i for each i (local estimate). Then if $F(V)$ is the global cost functional, which includes sources, experimental measurements, and calculated results on $\mathcal{D}_0$ and all the $\mathcal{D}_i's$, the potential equal to zero on $\mathcal{D}_0$ and V_i in each $\mathcal{D}_i$ is a first guess for minimizing $F(V)$. Then either $F(V)$ is linearized in its neighborhood or a Newton Kantorovitch method, using in the direct calculations the patchwork algorithms, can lead us to the "approximate solution" V that makes F minimum.

The patchwork analysis is justified in many practical cases by the history of water waves analyses, which involve similar Green's functions. Although boundary conditions on the frontiers hereby denoted as $\mathcal{A}_i$ are taken almost arbitrarily, the results remain sound. The present analysis, using a 1^{st} order patchwork approximation, should improve them. Computations on the impedance problem give also good results. It is clear that analytic equations are still to be introduced.

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Part III

Nonlinear Models. Applications

PROTON DYNAMICS ALONG THE HYDROGEN BOND IN CHAINS OF PEPTIDE GROUPS: POLARONS OR PROTON TRANSFER ?

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Different theories have been proposed in order to account for the optical anomalies found in the IR and Raman spectra of Acetanilide (ACN) and N-methylacetamide (NMA). This paper discusses some recent IR and NMR experimental data, comparing them with the predictions of the vibrational polaron model and of the proton transfer model. The amide proton dynamics appears very similar in ACN and NMA. The ACN chain has been proved to exhibit only the amidic form of the peptide group, and the examination of the infrared and NMR spectra of NMA does not support the theory of a dynamic exchange between the amide and imidol forms of the peptide group in this compound. So the anomalous amide modes in these peptide chains presumably result from nonlinear coupling between the C = O stretch and the N – H stretch modes.

1 Introduction

The proton dynamics shows anharmonic or anomalous behaviours in some crystalline chains of peptide units linked by $N - H \cdots O = C$ intermolecular hydrogen bonds (such as N-methylacetamide or NMA, Acetanilide or ACN, Uracil, Tyr-[Alanine]$_n$ etc).

Different theories have been proposed, attempting to explain these anomalous properties of the vibrations of the HNCO group.[1] Among these models, two are potentially interesting. The vibrational polaron, or self-trapped state created by coupling an amide mode with phonons, has the properties of a nonlinear excitation (a soliton) and is suggested to be a plausible mechanism for transferring vibrational energy in biomolecules.[2] Alternatively, the proton transfer, invoked in some cases,[3] could be responsible for charge transport along some hydrogen-bonded chains.

Our aim is to confront the predictions of both theories to our recent experimental data,[4,5,6] comparing the proton dynamics in two crystalline chains

of neighbouring structures (ACN and NMA).

2 Acetanilide and N-methylacetamide

Both acetanilide ($C_6H_5 - NHCO - CH_3$, or ACN) and N-methylacetamide ($CH_3 - NHCO - CH_3$ or NMA) crystals have an orthorhombic structure, with parallel chains of intermolecular hydrogen bonds connecting the peptide units. They are very simple model systems for polypeptides. The large interest focused on these substances comes also from the unusual properties of some of the vibrations of the $O = C - N - H$ atom group (also called Amide modes).[7,4] Those unconventional behaviours are sometimes assigned to coherent nonlinear excitations called Davydov solitons or 'polarons',[1,2] suspected to be one of the driving mechanisms for transferring vibrational energy in biomolecules. However these theories are still submitted to discussion, and alternative explanations are invoked, such as Fermi resonance or conformational substates of the amide proton.

It has been recently proposed[3] that the proton involved in intermolecular hydrogen bonding in N-methylacetamide (NMA) and polyglycine I, experiences a local double minimum potential, arising from dynamic exchange between the amide-like ($\cdots O = C - N - H \cdots$) and imidol-like ($\cdots H - O = C - N \cdots$) forms of the peptide units. This suggestion contradicts the customary view of a $N - H \cdots O = C$ bond in which the proton is covalently bonded to the nitrogen. This double-well model rests on the analysis of the inelastic neutron scattering (INS) spectrum and implies that the vibrational frequency of the proton would be almost half the previous estimate based on the optical techniques : the 1595 cm^{-1} vibrational mode (in NMA) would then be attributed to the $N - H$ stretch — or $\nu(NH)$ — rather than the 3250 cm^{-1} infrared mode, now attributed to an overtone.

This lower frequency would be explained by the weakening of the $N - H$ bond resulting from the proton transfer to the neighbouring oxygen. Another consequence would be a significant increase of the $N - H$ bond length.

Our contribution aims at elucidating the origin of the anomalous spectral properties in these structures, and tests the validity of the different theories through structural measurements by neutron diffraction,[5] infrared spectroscopy of selectively deuterated compounds,[4] quasi elastic neutron scattering,[6] and NMR.

3 Results

We have recently demonstrated[5] that *there is no proton transfer in ACN*, though its structure is close to that of NMA. For comparison, the NH $\cdots$ O distance is 2.825 Å in NMA, 2.905 Å and 2.935 Å in ACN at 15K[5] and 295K respectively. Our neutron diffraction measurements have shown that the N – H distance is 1.020 Å in ACN, that this distance does not vary significantly with temperature, and that the proton experiences a simple-well potential in the bond direction.

The strength of the hydrogen bond controls the proton transfer, so it is worth to determine whether a difference of about 0.1 Å in the NH $\cdots$ O distance is sufficient to induce opposite behaviours in NMA and ACN.

We thus have carefully re-examined the infrared spectra of the amide modes in NMA with a special attention to their temperature dependence, especially in the 1500–1700 cm^{-1}, region in which the weakened N–H stretching frequency is expected.[3]

The main band at $\sim$ 1645 cm^{-1} (at 23K) and the neighbouring shoulders have been assigned to the Amide I mode, or dominantly the C=O stretch $\nu(CO)$.[3,8,9]

The attribution of the other structures is under discussion and the detailed interpretation of these spectra will be published elsewhere.[10] Anyway, *the important observation*, in connection with the problem of the frequency of the $\nu(NH)$ mode in NMA, is that *none of the bands* in this range of frequency *could be assigned to a* $\nu(NH)$ *mode* whose frequency is expected to decrease on cooling (the band under discussion at $\sim$ 3250 cm^{-1} undergoes a red-shift of about 25 cm^{-1} on cooling to 23K).

All the bands in the 1500–1700 cm^{-1} region show a blue-shift on cooling, except the mode at $\sim$ 1643 cm^{-1}, firmly assigned to the $\nu(CO)$ or Amide I.[8,9]

A similar behaviour has been observed for ACN, in which it was not possible to detect a red-shifted band in the region of Amide II.[4]

So our analysis of the infrared data does not support the hypothesis of a weakened $\nu(NH)$ mode (i.e., at about 1600 cm^{-1} instead of 3200 cm^{-1}), neither that of a proton transfer.

We conclude that the frequency of the $\nu(NH)$ mode is around 3200 cm^{-1} and not at $\approx$ 1600 cm^{-1}, both in ACN and NMA.

The analysis of the $^{13}C(C = O)$ CP MAS NMR spectra of the peptide groups in both crystals shows a parallel behaviour in ACN and NMA at temperatures below 240K:[10] The chemical shifts are very close, indicating an analogous shielding by the local electronic density. The widths of both signals are also of the same order of magnitude (115 and 119 Hz at 160K, respectively),

and well explained by the splitting due to the direct dipolar coupling to the adjacent quadrupolar ^{14}N nucleus.[11]

So, the magnitude and orientation of the ^{14}N Electric Field Gradient (EFG) tensor are comparable at low temperature in both crystals, indicating similarities in the charge distributions surrounding the nitrogen atom, or in other words similar covalent bonds with the H atom.

On the contrary, at room temperature, above the phase transition undergone by NMA, the structural distortion changes the relative orientations of the EFG tensor and of the (C – N) dipolar vector and consequently modifies the NMR line shape which appears now different from that of ACN.

As a conclusion, the amide proton dynamics appears very similar in ACN and NMA, through their infrared spectra. The ACN chain has been proved to exhibit only the amidic form. The new examination of the infrared spectra of NMA does not support the proposal of a dynamic exchange between the amide and imidol forms of the peptide units in NMA. Examination of the $^{13}C(C = O)$ NMR spectra leads to the same conclusion, at temperatures below the structural phase transition undergone by NMA.

So the anomalous and non-harmonic amide modes in these peptide chains presumably result from nonlinear coupling between modes; in a chain, the C = O stretch and the N – H stretch modes are presumably coupled by the $O \cdots H$ intermolecular hydrogen bond. No evidence for proton transfer has been detected in our measurements.

The next step is now to check the influence of these self-trapped amide modes in biomolecular processes.

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WAVELET ANALYSIS OF THE ELECTROCORTICAL ACTIVITY

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A simple algorithm has been developed in order to extract the coherent structure from noisy chaotic signals, such as the EEG time series. This wavelet-based denoising algorithm has been tested on the well known data set coming from the Lorenz differential system in a chaotic state. The addition (with different signal to noise ratios) and the subsequent removal of gaussian and uniform noise have been investigated, always obtaining a reduction in the Grassberger-Procaccia correlation dimension of the Lorenz strange attractor. The delicate question treated in this paper is if this simple algorithm is able or not to remove the dynamical noise generated perturbing the Lorenz system by a small random amount at each integration time step in the b parameter.

1 Introduction

1.1 The problem

Recently there have been claims that low-dimensional chaos has been found in time sequences obtained measuring the electrocortical activity of the human brain. In our opinion, most of these claims have to be taken with very much caution, essentially for two reasons: first of all many algorithms provided by nonlinear systems analysis have not always been correctly interpreted (due to non stationarity and limited resolution of the data sets for instance) and because any nonlinear determinism present in the behavior of neuronal ensembles is hidden by a background of noise.

Owing to the lack of a reliable mathematical model of the electrocortical activity, most of the above cited claims are based on fractal dimension estimates and Lyapunov exponents calculations of hypothetical *reconstructed* strange attractors. But the state space reconstruction problem (i.e. the recreation of states of the underlying dynamical system when the only information available is contained in a univariate time series), solved by Takens' theorem in absence

of noise, is much more complicated for real-world time sequences (such as those coming from EEG recordings) because for noisy data Takens' theorem doesn't give any clue to what are the best delay coordinates.

Among the factors complicating the reconstruction problem are the observational noise, the dynamical noise and the estimation error. Casdagli et al. [2] have shown that a poor reconstruction provides an amplification of the noise and an increase in the estimation error. Moreover, the brain is a spatially-extended system so it is not clear if measurements at one fixed point give or not enough information for the reconstruction of the full attractor (if any). In fact, if the various neuronal aggregates are coupled with a too small amplitude level, different sequences (i.e. recorded in different points) give significantly different dimensions. Multichannel measurements, necessary to characterize the brain state, will be in any case affected by complicated noise patterns of dynamical nature (probably due to incoherent firings of the neuronal ensembles belonging to different brain regions). So, according to our analysis, it is essential to study the possible reduction of the dynamical noise starting from known nonlinear time series in order to tell something more about the interpretation of the electrocortical activity data in terms of modern nonlinear systems theory.

2 Noise Reduction Methods

2.1 Linear and Non-linear methods

Various noise reduction schemes have been proposed in cases where the time sequence is generated by a low dimensional dynamical system. All these methods, in order to reconstruct the underlying attractor, typically use the time delay embedding procedures. Linear filters, such as low-pass filters, proved to be useful only if the time series is highly oversampled (i.e. if the nonlinear flows are recorded with a small sampling time compared to the internal time scale). Moreover, the dimension of the attractor is left unchanged by moving-average filters and even increased using autoregressive (AR) filters, in case of too weak damping [1].

Kostelich et al. [3], [4], [5] have proposed two-step procedures involving the use of local fits to the dynamics. In the iteration scheme typical of these methods, each step consists of two parts: a dynamics fitting and a trajectory fitting. Schreiber and Grassberger [9] succeeded in simplifying the above cited algorithms obtaining a one-step procedure which makes use of local linear fits. Another approach, called nonlinear noise reduction [7] and aimed to reduce the external noise, is a synthesis of the methods proposed by Schreiber [10] and

Sauer [8].

2.2 *Wavelet-based de-noising methods*

The idea underlying these methods is to represent the signal in an orthogonal basis choosing the basis function maximizing the correlation between it and the information signal contained in the time sequence.

This can be achieved using suitable wavelet bases (wavelet transform) or choosing Wavelet Packet bases which give the smallest entropy expansion of the signal. Then the wavelet coefficients above a suitably chosen threshold are selected and the others eliminated. A synthesis of these ideas is the technique known as Wavelet Shrinkage, developed by Donoho [6] and Johnstone, always for external noise reduction. Our algorithm tries to reconstruct the signal, using the inverse wavelet transform based on the D_{10} Daubechies wavelet, after subsequent removal of the wavelet coefficients, starting from levels corresponding to the fine structure of the signal.

In two previous papers [11] and [12], we have used different wavelet bases. In the first one, spline wavelets were employed to reduce noise present in certain EEG series obtaining no evidence for an effective de-noising of the signals. In the second one instead, the use of Daubechies wavelets was successful in the reduction of external noise added to a chaotic Lorenz time sequence. The results are evident comparing the respective G-P correlation integrals. Present results concern an attempt to use Daubechies wavelets to reduce noise in a chaotic Lorenz attractor, perturbed by a dynamical noise, that is forcing the value of the aspect ratio (parameter b) to change randomly in time as follows: $b(t) = b \pm \alpha \cdot \mathrm{ran}(0,1)$. Small values of the real number α don't significantly change the topological structure of the attractor and consequently the values of the correlation integrals. To get significant variations in the topological structure one must use greater α values (e.g. 3), with strong changes in the correlation integrals too (see Fig. 1).

At this noise level the use of Daubechies wavelets is unsuccessful in signal de-noising (see Fig. 2). While no significant changes in the local slopes at high ϵ are evident some local slope variations are present at low ϵ. In this case our algorithm essentially reduces resolution at low ϵ thus not changing the G-P correlation dimension of the noisy data.

Due to these serious difficulties encountered processing a noisy constructed, high resolution signal in order to reduce this kind of dynamical noise it is therefore quite impossible to try to reduce noise in the low-resolution real-world EEG data without at least a strong hypothesis on the mechanisms underlying the "noise" generation.

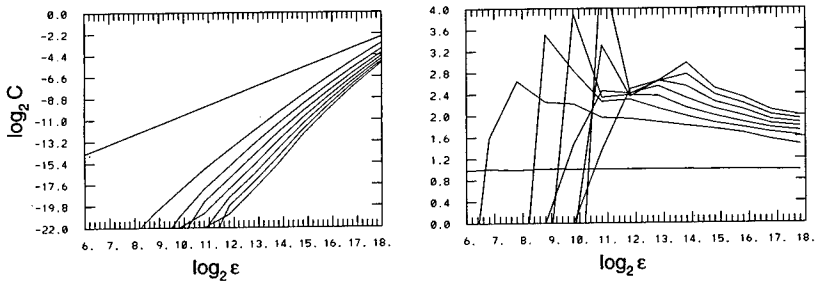

Fig. 1. Correlation integrals (left) and local slope (right) of the noisy Lorenz series. The data consisted of 15000 samples with 20-bit precision. The embedding dimension ranges from 1 to 8 and the maximum norm was used.

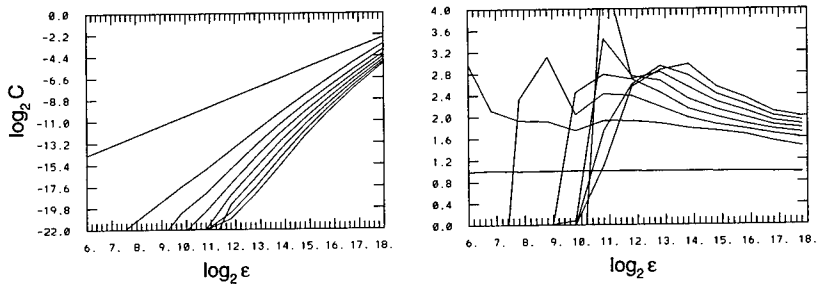

Fig. 2. Correlation integrals (left) and local slope (right) of the "de-noised" Lorenz series. The data consisted of 15000 samples with 20-bit precision. The embedding dimension ranges from 1 to 8 and the maximum norm was used.

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The quantum easy-plane ferro- and antiferromagnet

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The Berezinskii-Kosterlitz-Thouless (BKT) phase transition is peculiar of two-dimensional magnetic systems with easy-plane anisotropy. Their prototype is the classical planar (or XY) model, that neglects the role of the out-of-plane spin component. The latter is accounted for in the (easy-plane) XXZ model, that has been studied on a square two-dimensional lattice. From Monte Carlo simulation for the classical XXZ model the classical transition temperature $T_{\rm BKT}^{\rm (cl)}$ turns out to be considerably lower than that of the planar model. The quantum XXZ model is approached by the *pure-quantum self-consistent harmonic approximation*, which leads to the investigation of an effective classical model. Quantum fluctuations reduce the effective exchange interaction, resulting in a lower BKT transition temperature $T_{\rm BKT}$.

1 The quantum XXZ model

The two-dimensional ferromagnetic (FM) and antiferromagnetic (AFM) XXZ models are described by the general Hamiltonian

$$\hat{\mathcal{H}} = \pm\tfrac{1}{2}\, J \sum_{\mathbf{i},\mathbf{d}} \left(\hat{S}^x_{\mathbf{i}} \hat{S}^x_{\mathbf{i}+\mathbf{d}} + \hat{S}^y_{\mathbf{i}} \hat{S}^y_{\mathbf{i}+\mathbf{d}} + \lambda\, \hat{S}^z_{\mathbf{i}} \hat{S}^z_{\mathbf{i}+\mathbf{d}} \right) , \qquad (1)$$

where the negative (positive) sign refers to the FM (AFM) case. The index $\mathbf{i} \equiv (i_1, i_2)$ runs over the sites of a two-dimensional lattice, and $\mathbf{d} \equiv (d_1, d_2)$ represents the displacements of the z nearest-neighbors of each site. The sum describes an exchange interaction $J > 0$ between nearest-neighbor spins, with

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an easy-plane anisotropy $\lambda \in [0,1)$. The quantum mechanical operators $\hat{\boldsymbol{S}}_{\mathbf{i}}$ satisfy the $\mathcal{SU}(2)$ commutation relations $[\hat{S}^{\alpha}_{\mathbf{i}}, \hat{S}^{\beta}_{\mathbf{j}}] = i\,\delta_{\mathbf{ij}}\,\epsilon^{\alpha\beta\gamma}\,\hat{S}^{\gamma}$ and belong to the spin-S representation, $|\hat{\boldsymbol{S}}_{\mathbf{i}}|^2 = S(S+1)$.

In the AFM case the lattice is such that it can be considered as two interpenetrating identical sublattices (the 'positive' and the 'negative' one), in such a way that the nearest neighbors of any site in a sublattice belong to the other one; in other words, lattices with frustration are not considered here.

If $(-)^{\mathbf{i}} = \pm 1$ denotes the sign of the sublattice containing the site $\mathbf{i}$, the canonical transformation

$$(\hat{S}^{x}_{\mathbf{i}}, \hat{S}^{y}_{\mathbf{i}}, \hat{S}^{z}_{\mathbf{i}}) \quad \longrightarrow \quad ((-)^{\mathbf{i}}\,\hat{S}^{x}_{\mathbf{i}}, (-)^{\mathbf{i}}\,\hat{S}^{y}_{\mathbf{i}}, \hat{S}^{z}_{\mathbf{i}})$$

transforms the AFM XXZ model into the FM XXZ model, but with a negative $\lambda \in (-1,0]$, so that it is apparent that the AFM and the FM cases can be treated simultaneously considering only the FM one with $\lambda \in (-1,1)$, as done in the following. The order parameter $\langle \hat{\boldsymbol{S}}_{\mathbf{i}} \rangle$ describes both the FM magnetization and the AFM staggered magnetization. For $\lambda = 0$ the above Hamiltonian describes the $XX0$ model, often called 'quantum XY model'; of course, the AFM $XX0$ model is equivalent to its FM counterpart.

2 The classical XXZ model

The Hamiltonian (1) has a classical counterpart, that can be obtained by replacing each spin operator $\hat{\boldsymbol{S}}_{\mathbf{i}}$ with a classical vector $\boldsymbol{S}_{\mathbf{i}}$ of a suitable length $\widetilde{S}$ [b]. In terms of unit vectors $\boldsymbol{s}_{\mathbf{i}} = \boldsymbol{S}_{\mathbf{i}}/\widetilde{S}$ the Hamiltonian of the classical XXZ model reads then

$$\mathcal{H} = -\tfrac{1}{2}\,\varepsilon \sum_{\mathbf{i},\mathbf{d}} \left(s^{x}_{\mathbf{i}}\, s^{x}_{\mathbf{i}+\mathbf{d}} + s^{y}_{\mathbf{i}}\, s^{y}_{\mathbf{i}+\mathbf{d}} + \lambda\, s^{z}_{\mathbf{i}}\, s^{z}_{\mathbf{i}+\mathbf{d}} \right) , \tag{2}$$

with the exchange energy $\varepsilon = J\widetilde{S}^2$. It is convenient to make use of the dimensionless temperature $t = T/\varepsilon$.

At variance with the quantum system, in the classical one the FM and the AFM cases are fully equivalent as far as the static properties are concerned, since the classical expression of thermal averages is invariant under independent reflections of the spin components (*i.e.* under the transformation $\lambda \to -\lambda$).

The minimum configuration of the classical Hamiltonian corresponds to the ordered state, where all the spins are aligned in an arbitrary direction in the xy-plane. In the AFM picture this state is the Néel state: the two sublattices are

[b] The choice of $\widetilde{S}$ is not trivial; *e.g.* the values S or $\sqrt{S(S+1)}$ are both reasonable.

Table 1: BKT transition temperatures of the classical XXZ model [11].

λ	0	0.5	0.9	0.95
$t_{\rm BKT}^{\rm (cl)}$	0.695 ± 0.005	0.683 ± 0.005	0.59 ± 0.01	0.55 ± 0.01

ordered ferromagnetically in the xy plane, but in opposite directions. However, this configuration is unstable against thermal fluctuations: as a consequence of the Mermin-Wagner theorem [1] the classical XXZ model in two dimensions cannot have finite magnetization at nonzero temperature.

If one neglects the z-components of the spins, the so-called *planar* (or 'XY') model is obtained. Since the spins are reduced to two-component vectors in the xy plane, $\boldsymbol{s}_i = (\cos\varphi_{\mathbf{i}}, \sin\varphi_{\mathbf{i}})$, its Hamiltonian can be written in terms of the azimuthal angles, $\mathcal{H} = -\frac{1}{2}\varepsilon \sum_{\mathbf{i},\mathbf{d}} \cos(\varphi_{\mathbf{i}} - \varphi_{\mathbf{i}+\mathbf{d}})$. This is the prototype system that undergoes the Berezinskii-Kosterlitz-Thouless (BKT) phase transition [2,3,4] which occurs at the temperature $t_{\rm BKT} = T_{\rm BKT}/\varepsilon \simeq 0.89$ (this value has been calculated by Monte Carlo simulations [5,6]). The order parameter $\langle \boldsymbol{s}_{\mathbf{i}} \rangle$, is vanishing at any temperature, and the mechanism underlying the transition is the unbinding of vortex pairs [3,7]. For $t < t_{\rm BKT}$ the correlation function $\langle \cos(\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}) \rangle \sim |\mathbf{i} - \mathbf{j}|^{-\eta(t)}$ displays a power-law decay, whereas for $t > t_{\rm BKT}$ the decay is exponential; the correlation length ξ and the susceptibility χ have an exponential divergence for $t \to {t_{\rm BKT}}^{+}$,

$$\xi \ , \ \chi \ \sim \ e^{a(t - t_{\rm BKT})^{-1/2}} \ , \tag{3}$$

and stay infinite for all temperatures $t \leq t_{\rm BKT}$, so that an entire line of critical points appears in the phase diagram. The specific heat does not display any divergence, but only a maximum located slightly above the transition temperature (at $t \simeq 1.1\, t_{\rm BKT}$ in the pure planar model [5,6]).

With the inclusion of the third components of the spins, *i.e.* for the generic XXZ model (2), the system symmetry remains unchanged and a BKT transition is still expected at a finite temperature $t_{\rm BKT}(\lambda)$, which vanishes logarithmically [8,9,10] as $t_{\rm BKT} \sim -1/\ln(1 - |\lambda|)$ in the isotropic limit $|\lambda| \to 1$.

We have recently performed Monte Carlo simulations for the classical XXZ model that provide useful data [11] for the XXZ model on the square lattice, at anisotropy values $\lambda = 0$, 0.5, 0.95, and 0.99, with lattice sizes from 32×32 up to 256×256. These observations have lead to identify the transition as BKT, and to locate the corresponding critical temperatures as reported in Table 1. In order to investigate the divergence of ξ and χ in approaching $t_{\rm BKT}$ only outcomes of those simulations not affected by appreciable finite-size effects have been used, *i.e.* the data obtained for lattice sizes $L \geq 6\,\xi$. In order

to ascertain the BKT character of the transition the data were fitted by the chi-squared minimization criterion using Eq. (3), which turns out to reproduce the data better than the power law

$$\xi\ ,\ \chi\ \sim\ (t-t_c)^{-\nu}\ , \tag{4}$$

typical of second-order phase transitions. In Fig. 1 BKT and power-law fits to the data for $\lambda = 0$ are reported.

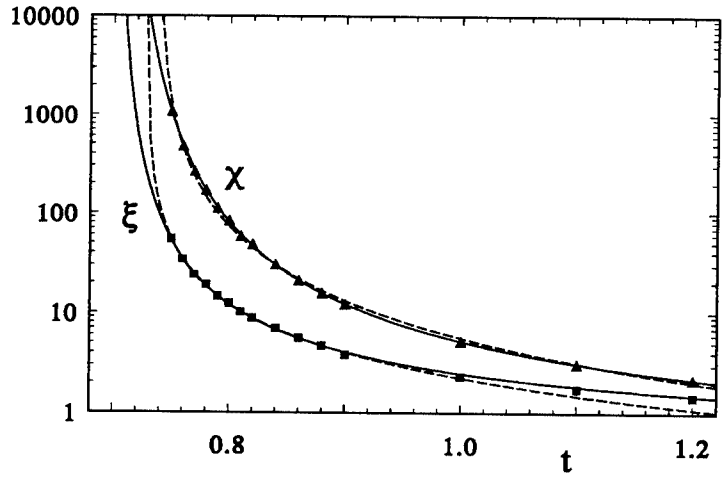

Figure 1: Monte Carlo data for the in-plane susceptibility χ (triangles) and correlation length ξ (squares) at $\lambda = 0$ plotted as function of temperature. The full line and dashed line are the best BKT [Eq. (3)] and power-law [Eq. (4)] fitting functions, respectively.

3 Quantum correction to the BKT transition temperature

The *pure-quantum self-consistent harmonic approximation* (PQSCHA) [12] reduces the thermodynamics of the quantum XXZ model to the study of an *effective* classical problem, that embodies the contribution of the pure-quantum part of the fluctuations (treated in a harmonic approximation) through its temperature-dependent renormalized interaction parameters.

In Refs. [13,14] one finds an outline of the derivation of the effective Hamiltonian $\mathcal{H}_{\rm eff}$ in terms of classical spins, a procedure that involves the *Villain transformation* from quantum spin to bosonic variables [15], and the identification of the classical counterpart of the transformed Hamiltonian by the prescription of *Weyl ordering* [16]. It is just the Weyl ordered form of the Villain-transformed

spin operators that leads [17] to the identification $\widetilde{S} = S + \frac{1}{2}$. Eventually, the following effective Hamiltonian is found:

$$\mathcal{H}_{\rm eff} = \frac{\varepsilon}{2} j_{\rm eff} \sum_{\mathbf{i},\mathbf{d}} \left(s^x_{\mathbf{i}} s^x_{\mathbf{i}+\mathbf{d}} + s^y_{\mathbf{i}} s^y_{\mathbf{i}+\mathbf{d}} + \lambda_{\rm eff} s^z_{\mathbf{i}} s^z_{\mathbf{i}+\mathbf{d}} \right) + N\varepsilon\, G(t) \ . \qquad (5)$$

Within the PQSCHA [12], quantum effects are embodied in

$$j_{\rm eff}(S,\lambda,t) \ = \ (1 - \tfrac{1}{2} D_\perp)^2 \ e^{-\frac{1}{2}\mathcal{D}_\parallel} \ , \qquad (6)$$

$$\lambda_{\rm eff}(S,\lambda,t) \ = \ \lambda \ (1 - \tfrac{1}{2} D_\perp)^{-1} \ e^{\frac{1}{2}\mathcal{D}_\parallel} \ , \qquad (7)$$

while $G(t)$ is an additive renormalization that does not enter the calculation of operator averages. The self-consistent renormalization parameters

$$D_\perp \ = \ \frac{1}{2\widetilde{S}} \frac{1}{N} \sum_{\mathbf{k}} \frac{b_{\mathbf{k}}}{a_{\mathbf{k}}} \left(\coth f_{\mathbf{k}} - f_{\mathbf{k}}^{-1} \right) \ , \qquad (8)$$

$$\mathcal{D}_\parallel \ = \ \frac{1}{2\widetilde{S}} \frac{1}{N} \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}}) \frac{a_{\mathbf{k}}}{b_{\mathbf{k}}} \left(\coth f_{\mathbf{k}} - f_{\mathbf{k}}^{-1} \right) \ , \qquad (9)$$

represent, within the PQSCHA, the *pure-quantum* part of the square fluctuations [12,17] of $\hat{S}^z_{\mathbf{i}}/\widetilde{S}$ and of the relative azimuthal angle between nearest-neighbors, respectively. They decrease with t and S, and vanish both for $t \to \infty$ or $S \to \infty$. The other quantities are $a^2_{\mathbf{k}} = z\, e^{-\mathcal{D}_\parallel/2} \ (1 - \lambda_{\rm eff}\, \gamma_{\mathbf{k}})$, $b^2_{\mathbf{k}} = z(1 - \frac{1}{2} D_\perp)^2 \ e^{-\mathcal{D}_\parallel/2} \ (1 - \gamma_{\mathbf{k}})$, $f_{\mathbf{k}} = a_{\mathbf{k}} b_{\mathbf{k}}/(2\widetilde{S}t)$, $\gamma_{\mathbf{k}} = z^{-1} \sum_{\mathbf{d}} \cos(\mathbf{k}\cdot\mathbf{d})$, and $\mathbf{k}$ is a wavevector varying in the first Brillouin zone.

Therefore, the exchange energy is reduced by the factor $j_{\rm eff}$, and the easy-plane anisotropy is weakened ($|\lambda_{\rm eff}| \geq |\lambda|$), due to the cooperative effect of in-plane and out-of-plane pure-quantum fluctuations. Their typical temperature behavior in the case of the square lattice is reported in Figs. 2, and 3, respectively. For $S \to \infty$, i.e. in the classical limit, $j_{\rm eff} \to 1$ and $\lambda_{\rm eff} \to \lambda$. We notice that the integrals of the pure-quantum fluctuation parameters, Eqs. (8) and (9), get the main contribution from the high-frequency part of the effective magnon spectrum $\omega_{\mathbf{k}} = (J\widetilde{S}/\hbar) a_{\mathbf{k}} b_{\mathbf{k}}$, just because the pure-quantum part of the square fluctuations is obtained by subtracting from the full (harmonically approximated) expression the corresponding classical part (*i.e.* the leading behavior for $f_{\mathbf{k}} \to 0$); on the other hand those effects due to the presence of nonlinear excitations (vortices) would mainly affect the low-frequency part, i.e. they are essentially 'classical' and therefore they cannot sensitively change $D_\perp$ and $\mathcal{D}_\parallel$.

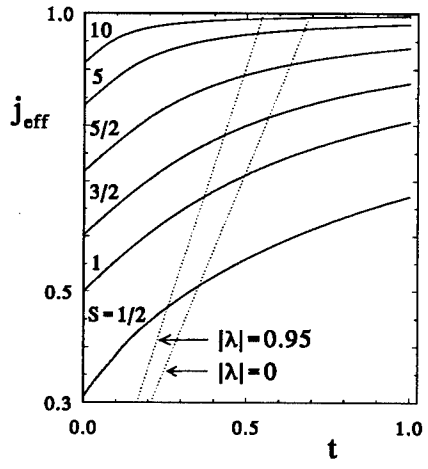

Figure 2: The effective exchange coupling $j_{\rm eff}(S,\lambda=-0.5,t)$ for the (AFM) XXZ model *vs.* temperature and for different spin S. The dotted lines represent $t/t^{\rm (cl)}_{\rm BKT}(\lambda)$ for $|\lambda| = 0$ and 0.95.

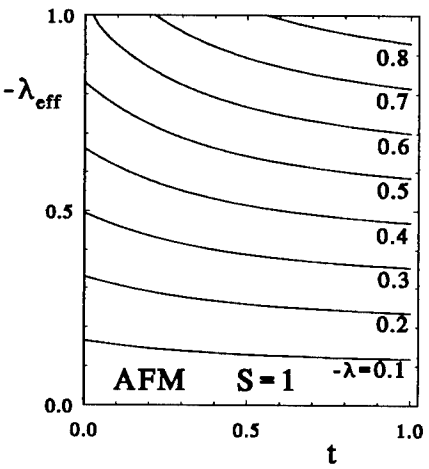

Figure 3: The effective anisotropy parameter $\lambda_{\rm eff}$ for the S=1 AFM XXZ model *vs.* temperature, at different values of λ. For high values of $|\lambda|$ the curves reach the isotropic value $\lambda_{\rm eff} = -1$.

Using the PQSCHA formalism [12] the quantum average of any observable $\hat{\mathcal{O}}$ is reduced to a classical expression as

$$\left\langle \hat{\mathcal{O}} \right\rangle = \left\langle \tilde{\mathcal{O}} \right\rangle_{\rm eff} = \mathcal{Z}^{-1} \left(\prod_{\rm i} \int ds_{\rm i} \right) \tilde{\mathcal{O}}\, e^{-\beta \mathcal{H}_{\rm eff}} \ ,$$

where $\tilde{\mathcal{O}}$ is obtained by Gaussian smearing, on the scale of the pure-quantum fluctuations, of the Weyl symbol $\mathcal{O}$ associated to $\hat{\mathcal{O}}$ [12,17]. In this way one can easily realize [14] that the in-plane correlations $\langle \hat{S}^x_{\rm i} \hat{S}^x_{\rm j} \rangle$ have the same asymptotic behaviour as $\langle s^x_{\rm i} s^x_{\rm j} \rangle_{\rm eff}$, so that the correlation length ξ is just that obtained for the effective classical model, and its divergence signals the occurrence of the BKT transition in the quantum system. The transition temperature $t_{\rm BKT}(S,\lambda)$ can then be estimated from the knowledge of the corresponding classical one $t^{\rm (cl)}_{\rm BKT}(\lambda)$:

$$\frac{t_{\rm BKT}(S,\lambda)}{j_{\rm eff}(S,\lambda,t_{\rm BKT})} = t^{\rm (cl)}_{\rm BKT}\Big(\lambda_{\rm eff}(S,\lambda,t_{\rm BKT})\Big) \ . \qquad (10)$$

It is easy to solve graphically this equation for the $XX0$ model [14], since $\lambda_{\rm eff} = 0$ for $\lambda = 0$. However, it can be solved also for $\lambda \neq 0$, using a rough fit of $t^{\rm (cl)}_{\rm BKT}(\lambda)$

from the available values of Table 1, reported as squares in Fig. 4. Then Eq. (10), rewritten as $j_{\rm eff} = t_{\rm BKT}/t_{\rm BKT}^{\rm (cl)}$, can be solved by recursion. The results are reported for different values of S in Fig. 4, where also the recent quantum Monte Carlo results [18,19] for $S = \frac{1}{2}$ are reported, showing fair agreement with ours for $\lambda = 0$ and 0.5. For $\lambda = 0$ and $S = \frac{1}{2}$ other independent results are available, namely those found by high temperature expansions [20] (0.39) and by real-space renormalization group techniques [21] (0.40).

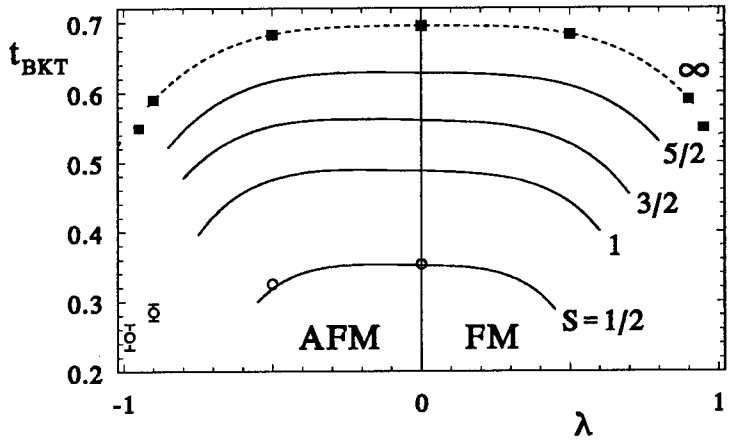

Figure 4: The critical temperature $t_{\rm BKT}$ of the XXZ model *vs.* the anisotropy parameter λ, for different spin S. The circles report quantum Monte Carlo simulation results [19].

When $|\lambda|$ is risen enough, one sees from Fig. 3 that it may happen that $|\lambda_{\rm eff}(S,\lambda,t)| \geq 1$. When $|\lambda_{\rm eff}| = 1$ is reached the effective Hamiltonian becomes isotropic, and the theory therefore predicts the disappearance of the BKT transition when $\lambda/\lambda_{\rm c}(S) \geq 1$ (For the AFM one finds $-\lambda_{\rm c} = 0.58$, 0.75, 0.85, 0.90, 0.92, for $S = \frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, respectively; those for the FM are slightly lower). However, this situation has to be considered with care, because the derivation of the effective Hamiltonian relies on the validity of the Villain transformation [15], which is meaningful only for easy-plane systems. Indeed, the possible break-down of the quantum BKT scenario for sufficiently small anisotropy occurs together with the break-down of the renormalization scheme, since out-of-plane fluctuations become so strong that the assumed easy-plane character becomes meaningless. The break-down does not occur for $\lambda/\lambda_{\rm c} \ll 1$, of course, and therefore it does not affect the results reported in Fig. 4.

The suppression of the BKT transition by 'effective isotropization' is therefore unlikely to be a physical phenomenon. This is suggested also by the cited

quantum Monte Carlo simulations of the S=$\frac{1}{2}$ XXZ antiferromagnet [19]: at $-\lambda = 0.90$ and 0.98 the BKT behavior is still observed, with substantially high transition temperatures, $t_{\rm BKT} = 0.285$ and 0.25, respectively. In order to reach this regime by means of the PQSCHA, one could resort to a different spin-boson transformation, as the Holstein-Primakoff one [22], which is useful also in the isotropic case since it treats the spin fluctuations symmetrically, so that one expects to obtain $|\lambda_{\rm eff}| < 1$ for any $|\lambda| < 1$. On the other hand, the calculation of $\mathcal{H}_{\rm eff}$ becomes much more complicated: work is in progress along this line.

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TWO PHASES BEHAVIOUR OF POROSITY AND SURFACE WIDTH IN GROWTH PHENOMENA

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Using a numerical approach based on parallel processing, we study the Kardar-Parisi-Zhang equation in presence of a spatially gaussian correlated noise with a non-zero mean value. The procedure allows us to study the dependence of the surface width and of the relative porosity on the nonlinear coefficient. A two phases behaviour has been obtained with the same phase diagram for both quantities.

1 Introduction

A large amount of effort has been devoted in recent years to the study of growth patterns and formation of rough surfaces and interfaces[1]. In fact, in addition to its intrinsic mathematical interest, the topic appears to be very important also from an experimental point of view, e.g. concerning thin films deposition, crystal growth, evolution of bacteria colonies, reaction fronts, corrosion and erosion effects ... To study growing surfaces, several models have been developed sharing a common feature: the surface evolves to a steady state exhibiting a non-trivial scaling behaviour. In particular, the surface behaviour may be described by the surface width w

$$w(t) = \sqrt{< \left(u(x,t) - < u(x,t) > \right)^2 >} \qquad (1)$$

where $u(x,t)$ is the height of the growing structure at time t and position x. The surface width evolves as

$$w \approx t^{\beta} \quad \text{for small } t$$
$$w \approx L^{\alpha} \quad \text{for large } t$$

where L is the dimension of the specimen.

2 Theory

The most important continuous model for the study of growth phenomena is the Kardar-Parisi-Zhang equation (KPZ) [2]

$$\frac{\partial u}{\partial t} = \nu \frac{\partial u}{\partial x} + \frac{\lambda}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \eta(x,t) \qquad (2)$$

with a linear diffusion term, a nonlinear term, which keeps into account the effect of lateral growth (introducing the formation of porosity in the structure) and a noise term.

The noise term $\eta(x,t)$ describes the real growth (rain of unspecified particles) and may be chosen arbitrarily. Previous treatments have been restricted to a zero mean value noise (correlated or not), which implies that it is possible to obtain informations only about the interface. Here, by using a non-zero mean value noise, we may also evaluate the relative "porosity" $p(x,t)$

$$p(x,t) = \frac{\sum_x h(x,t) - \sum_{T=0}^{t} \sum_x \eta(x,T)}{\sum_x h(x,t)} \tag{3}$$

which gives informations on the bulk structure.

In addition, since the critical indices $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$ obtained for a white noise are in poor agreement with some experimental results, a noise with spatial gaussian distribution (and therefore spatial gaussian correlations) has been introduced, as more suitable to describe many real structures where growth is not uniform along the sample.

As a result, we find [3] a dependence of the critical index α not only on the noise correlation [4], but also on the ratio between the diffusion and nonlinear coefficients ν and λ. In particular we obtain (for a fixed noise) a variation of α between 0.5 and 1, as found in experimental surface width measurements[5].

A heuristic interpretation of the KPZ equation in its discrete form, based on the Local Interaction Simulation Approach (LISA) [6], allows us to understand better the mechanism of pores formation hidden in the nonlinear term. In particular, pores "annihilation" appears to take place when

$$\frac{\lambda}{\nu} > \frac{1}{|\Delta u|} \tag{4}$$

Therefore, it appears that both relative porosity and surface width have a two phases behaviour with a phase transition occurring when the ratio of the nonlinear and linear coefficients increases.

For both $p(t)$ and $w(t)$ it is possible to write in the steady state region a dependence on λ of the kind

$$w(t=\infty) = \lambda^x \tag{5}$$

$$p(t=\infty) = \lambda^y \tag{6}$$

where x and y are the so called "critical indices".

3 Numerical Results

The KPZ equation has been discretized with a space step ϵ and a time step τ. The evolution of the system has been followed for a very large number of time steps (up to several millions, depending on the specimen dimension L), starting from an empty specimen. The asymptotic values of p and w have been obtained averaging for each simulation over the last 5,000 time steps, supposedly already in the steady state region. To avoid stochastic fluctuations, many simulations (30-50) have been performed.

The KPZ model is easily parallelizable: a direct one-to-one correspondence between lattice sites and processors can be established with a synchronous updating and a uniform law. The use of a parallel architecture (CM-200 with 8192 Processing Elements) decreases drastically the CPU time: since a large number (N) of processors work simultaneously, a reduction of a factor 1/N of the CPU time with respect to serial machines is obtainable.

In Figures 1 and 2 w and p are plotted vs. λ for different values of L and $\nu = 5$. Similar results have been obtained for different values of ν. Since a log-log scale is used, the critical indices x and y are given by the slope of the curves for both phases. They are, for small values of λ, $x \approx 0$ and $y = .86, .82, .83$; for large λ, $x = -.467, -.448$ and $-.451$ for $L = 64, 156, 512$, resp. The intersection of the straight lines fitting the two phases yields the phase transition point, which is the same for the two figures.

4 Conclusions

By applyinh the Local Interaction Simulation Approach (LISA), we have discretized the KPZ equation. For the growth term, we have chosen a non-zero mean value noise, with spatial gaussian correlations. As a result we have predicted the existence of two well defined phases for the dependence on the non-linear parameter of the asymptotic values of both surface width and porosity. For the two phases, the critical indices are universal, i.e. independent from ν and L, as evinced from Figures 1 and 2 (where all the fitting lines belonging to the same phase are parallel). The phase transition point, which is the same for w and p, moves to lower values of λ when L increases and/or ν decreases. This behaviour is consistent with Eq.[4], considering that $|\Delta u|$ is related to w and therefore increase with L.

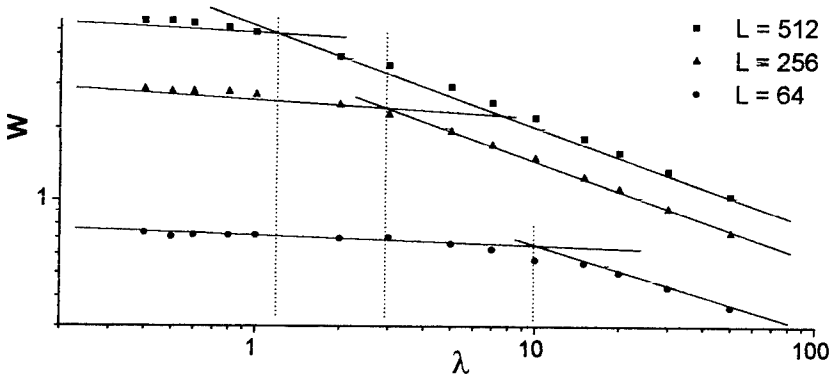

Figure 1: Surface width w vs. λ for different values of L and $\nu = 5$. The intersection of the fitting straight lines identifies the phase transition point.

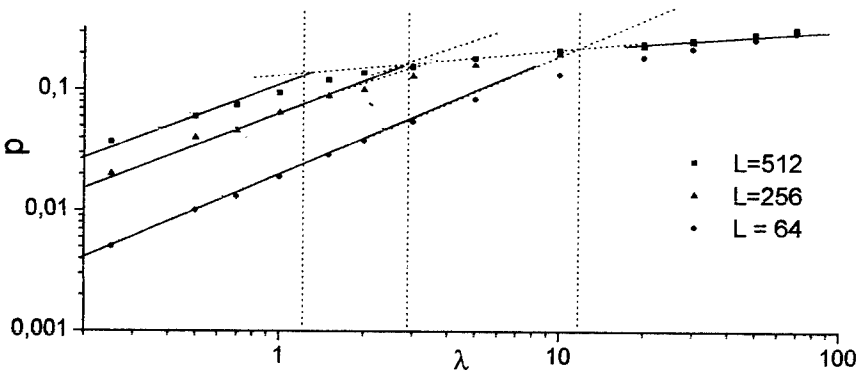

Figure 2: Relative porosity p vs. λ for different values of L and $\nu = 5$.

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INVERSE SCATTERING TRANSFORM ANALYSIS OF STOKES–ANTI–STOKES STIMULATED RAMAN SCATTERING

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A system of Maxwell–Bloch type equations (MBE), describing stimulated Raman scattering (SRS) with both Stokes and anti–Stokes waves taken into account is investigated. We introduce new variables S_3 and $S_\pm$, which are bilinear in the electromagnetic fields and prove, that the corresponding equations possess Lax representation. This fact is used to obtain soliton solutions for the MBE.

1 Introduction

The so–called Maxwell–Bloch equations (MBE) describe a wide variety of interactions of matter with light. These include self–induced transparency, resonance interactions in multilevel media, superfluorescence, amplification of laser pulses etc. In this paper we study a special version of the MBE which describe stimulated Raman scattering when Stokes E_s, anti–Stokes E_a and pump E_p exists and when there is no phase mismatch[1,2,3]:

$$\begin{aligned} &\frac{\partial E_p}{\partial \zeta} = \beta_a Q^* E_a - Q E_s, && \frac{\partial E_s}{\partial \zeta} = Q^* E_p, \\ &\frac{\partial E_a}{\partial \zeta} = -\beta_a Q E_p, && \frac{\partial Q}{\partial \tau} + \tilde{g} Q = E_s^* E_p + \beta_a E_p^* E_a. \end{aligned} \tag{1}$$

Here Q is the normalized effective polarization of the medium, $\zeta = z/L$ and $\tau = t - z/v$ are dimensionless space and retarded time coordinates, v is the wave group velocity. T_2 is the natural damping time of the material excitation, $\tilde{g} = 1/T_2$, β_a is the coupling coefficient which determines the number of anti–Stokes photons relative to number of Stokes photons[1,3]. Below we consider $\beta_a = 1$[1].

For $\beta_a = 0$, i.e. when the anti–Stokes wave E_a is neglected, these equations describe the so called transient stimulated Raman scattering (SRS) equations, which possess Lax representation when $\tilde{g} = 0$[4]. The SRS soliton solutions theoretically discovered by Chu and Scott[4] have been experimentally observed first in[5]. The SRS solitons (regarded as transient solitons[6,7] with a π phase jump at the Stokes frequency) have been extensively studied. In later experiments by Duncan *et al*[8] a careful comparison between theory and experiment showed good agreement. Shortly after this work[8], Hilfer and Menyuk[9] carried out simulations which indicate that in the highly depleted regime the solutions of transient SRS equations always tend toward a self–similar solution. This result[9,10] has been recovered by applying the inverse scattering transform method[11] (ISM) to the transient SRS equations, see[6]. Experiment to observe this solution has been proposed in[9]. Kaup's theory[7] also indicates that the dissipation, which appears for finite T_2 plays a crucial role in soliton formation. The similarity solutions and other group invariant solutions of the SRS equations in the presence of dissipation are studied in[12]. Claude and Leon[13] reformulated the transient SRS equations as an equivalent $\bar{\partial}$–problem and thus were able to treat the inhomogeneous broadening. As a consequence they showed the that *Raman spike* observed in the experiment is not a soliton.

For general system (1) with anti–Stokes wave, phase mismatch, $\beta_a \neq 1$ and dissipation, transient π solitons have been investigated by M. Scalora *et al*[8] using numerical methods. They predict the formation of soliton-like pulses at the anti–Stokes frequency.

2 Lax Representation

Let us introduce the following variables

$$S_3 = \frac{1}{2}(|E_s|^2 - |E_a|^2), \qquad S_+ = \frac{i}{2}(E_s^* E_p + E_p^* E_a), \qquad S_+ = S_-^*. \tag{2}$$

In terms of the new quadratic variables (2) the initial system with $\tilde{g} = 0, \beta_a = 1$ is rewritten as:

$$\frac{\partial S_3}{\partial \zeta} = -iQ^* S_+ + iQS_-, \quad \frac{\partial S_+}{\partial \zeta} = -iQS_3, \quad \frac{\partial Q}{\partial \tau} = -2iS_+. \tag{3}$$

or in matrix form:

$$\frac{\partial q}{\partial \tau} = \frac{1}{\sqrt{2}}\left[\sigma_3, S(\zeta,\tau)\right], \qquad \frac{\partial S}{\partial \zeta} = \frac{1}{\sqrt{2}}\left[q(\zeta,\tau), S(\zeta,\tau)\right], \tag{4}$$

$$q(\zeta,\tau) = \begin{pmatrix} 0 & Q \\ -Q^* & 0 \end{pmatrix}, \qquad S(\zeta,\tau) = \begin{pmatrix} S_3 & -i\sqrt{2}S_+ \\ i\sqrt{2}S_- & -S_3 \end{pmatrix}. \tag{5}$$

The equations (3) and (4) can be written down as the compatibility condition (see[11]) $\partial_\tau U - \partial_\zeta V + [U, V] = 0$, of the following linear systems:

$$L(\lambda)F(\zeta,\tau,\lambda) \equiv \frac{\partial F}{\partial \zeta} - U(\zeta,\tau,\lambda)F(\zeta,\tau,\lambda) = F(\zeta,\tau,\lambda)C(\lambda), \quad (6)$$

$$M(\lambda)F(\zeta,\tau,\lambda) \equiv \frac{\partial F}{\partial \tau} - V(\zeta,\tau,\lambda)F(\zeta,\tau,\lambda) = 0, \quad (7)$$

with

$$U(\zeta,\tau,\lambda) = -\frac{i}{\lambda}\sigma_3 + \frac{1}{\sqrt{2}}q(\zeta,\tau), \qquad V(\zeta,\tau,\lambda) = \frac{\lambda}{2i}S(\zeta,\tau). \quad (8)$$

The matrix $C(\lambda)$ will be fixed up below to our convenience; this is possible, because $C(\lambda)$ in fact does not appear in the compatibility condition. The function $\mathcal{E}^2(\tau) = |E_p(\zeta,\tau)|^2 + |E_s(\zeta,\tau)|^2 + |E_a(\zeta,\tau)|^2$ is the total energy density, which is constant at every point τ as a function of ζ.

3 The one soliton solution

The auxiliary linear problem for the vector NLS equation

$$\frac{\partial}{\partial\zeta}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\\ \tilde\psi_3\end{pmatrix} = \begin{pmatrix}0 & q_1 & q_2\\ -q_1^*, & 0 & 0\\ -q_2^* & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde\psi_1\\ \tilde\psi_2\\ \tilde\psi_3\end{pmatrix} \quad (9)$$

with some additional reduction $q_1 = -Q$, $q_2 = Q^*$, $\psi_1 = E_p$, $\psi_2 = E_s$, $\psi_3 = E_a$ is equivalent to the Stokes–anti-Stokes SRS equations without the last equation for Q. This formal equivalence allows us to recover E_p, E_s, E_a from potential Q. Using the well known one soliton solution of the vector nonlinear Schrödinger equation and the corresponding solution of the linear problem (9) we get

$$Q = \frac{\sqrt{2}\eta e^{i\phi}}{\cosh(z)}, \quad z = \eta\zeta - \frac{1}{\eta}\int_0^\tau \mathcal{E}^2(\tau')d\tau' \quad (10)$$

where $\mathcal{E}(\tau')$ is real, the soliton's eigenvalue is $i\eta$ and ϕ is constant real phase. The direct integration of Eq. (9) gives:

$$E_p = \sqrt{2}\mathcal{E}(\tau)\frac{\tanh(z)}{\cosh(z)}e^{i\phi},\ E_s = \mathcal{E}(\tau)\tanh^2(z),\ E_a = \frac{\mathcal{E}(\tau)}{\cosh^2(z)}e^{2i\phi}. \quad (11)$$

These solutions are similar to the transient SRS solitons obtained in[6] and for $\mathcal{E}(\tau) = 1, \phi = 0$ coincide with the ones in[1].

4 Conclusion

The Lax pair for the Stokes–anti–Stokes SRS equations is presented. New transient solitons (10), (11) are obtained. The Lax operator (7) coincides with the one generating the Heisenberg ferromagnet hierarchy[11]. The expansions over the "squared solutions" in the spirit of[14] can be used to treat the case $\tilde{g} \neq 0$ as a perturbation.

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THE GEOMETRY OF NONLINEAR DYNAMICS FROM TURING PATTERNS TO SUPERCONDUCTORS

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This article reviews work with D.M. Petrich and D.J. Muraki on interface motion in the FitzHugh-Nagumo model of reaction-diffusion dynamics. In the bistable regime where the Turing bifurcation is precluded, finite amplitude perturbations may produce a disordered, highly fingered and branched interface between the coexisting states, but in which the interface connectivity is preserved through self-avoidance. We show how these phenomena derive from an interface dynamics with a competition between motion by mean curvature and motion by a screened Biot-Savart law. The phenomenology of pattern formation is remarkably similar to that seen in recent experiments by Lee, McCormick, Ouyang, and Swinney on the iodide-ferrocyanide-sulfite reaction in a gel reactor, as well as in systems with long-range electromagnetic interactions such as surfactant monolayers and Type-I superconductors. The origins of these similarities are explained.

1 Introduction

In an important recent experiment on pattern formation, Lee, McCormick, Ouyang, and Swinney have found evidence for a new mechanism by which space-filling patterns may appear in reaction-diffusion systems.[1,2] Unlike the scenario proposed originally by Turing,[3] whereby a homogeneous state becomes *linearly* unstable to a periodic modulation, these experiments show that linearly stable states may be *nonlinearly* unstable toward the generation of complex disordered patterns. Moreover, the resulting patterns (see Fig. 1) are seen to have rather sharp fronts between adjacent regions of differing chemical composition, and these fronts are generally observed to be self-avoiding. Hence the name: "Pattern formation by interacting chemical fronts."[1] These experimental results contrast markedly with those which have recently confirmed the Turing mechanism,[4,5] and they have stimulated a number of theoretical explanations. Pearson[6] has presented extensive numerical studies of the Gray-Scott model[7] and demonstrated the existence of disordered lamellar patterns similar to those seen by Lee, *et al.*, and Meron and Hagberg[8] have seen instabilities toward labyrinthine structures in a model similar to that of FitzHugh and Nagumo.[9]

The experiments and simulations show patterns that are essentially *defined* by sharp chemical fronts. A natural question that arises is: What law of motion governs the interfacial evolution from compact configurations to space-filling

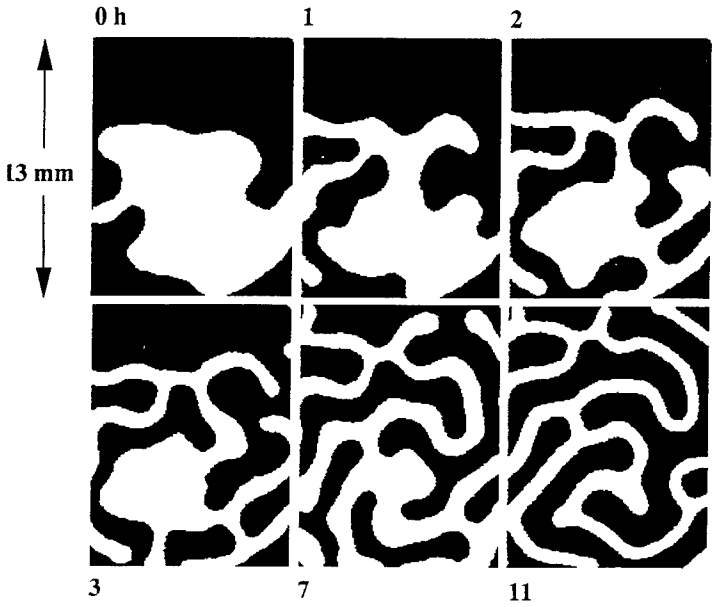

Figure 1: Pattern formation in the iodide-ferrocyanide-sulfite reaction in a gel reactor. Times shown are in hours following a perturbation. An indicator is used to show regions of low (white) and high (black) pH. Adapted from Ref. 2.

labyrinths? Ultimately, one would like to answer this on the basis of the underlying chemical kinetics of the experimental system. Here I summarize recent work[10,11] that may serve as a prototype of such an analysis: the reduction to an interface dynamics of a simplified model of reaction-diffusion phenomena that displays the experimental features. The resulting "contour dynamics" involves an important generalization of the familiar law of motion by mean curvature[12] in reaction-diffusion systems, namely a *nonlocal* coupling in the form of a screened Biot-Savart interaction. In what follows I summarize the model, its reduction to a contour dynamics, mechanisms of instability, and the common features it shares with other systems know to produce labyrinthine interfacial patterns.

2 The reaction-diffusion model

The FitzHugh-Nagumo model describes two chemical species, an *activator* u and an *inhibitor* v. By suitable rescalings the PDEs for the pair are

$$u_t = D\nabla^2 u - F'(u;r) - \rho\,(v-u)$$

$$\epsilon v_t \;=\; \nabla^2 v - (v-u)\ , \tag{1}$$

The diffusion constant for u is assumed much smaller than that of v, so in our rescaled units this means $D \ll 1$. The fronts between the coexisting states of u will then be relatively sharp. Underlying the dynamics is the assumed bistability of u, with the two states ($u = 0$ and $u = 1$) being the minima of a function $F(u;r)$. We may think of these states as corresponding to the experimental ones with different pH values in Fig. 1. The derivative of F is defined to be $F'(u;r) = u(u-r)(u-1)$, so the control parameter r tunes the relative depth of the minima, with $\Delta F \equiv F(1;r) - F(0;r) = (r-1/2)/6$. The coupling constant ρ controls the strength of inhibition of u by v, while the term in the v dynamics due to u represents the stimulation of inhibitor by activator. Finally, the appearance of v on the RHS of the inhibitor dynamics represents a self-limiting effect that prevents runaway of v in regions of high u. These kinds of cross-couplings are typical of what one finds in biochemical systems.

The model (1) has a "mixed" variational structure: a linear combination of a gradient flow and a Hamiltonian system. This feature gives rise to the remarkable diversity of patterns that it can produce. In Eq. 1 we have introduced a characteristic time scale ϵ for the inhibitor. The limit $\epsilon \gg 1$ is that of slow inhibition, where spiral waves are known to form.[13] The fast-inhibitor limit $\epsilon \ll 1$ was first considered by Koga and Kuramoto[14] for one-dimensional problems, and later by others in higher dimensions.[15] Of central interest in these latter works are the properties of "localized states," compact configurations of $u = 1$ in a background of $u = 0$, formed by nonlinear excitations to homogeneous states that are linearly stable to the Turing bifurcation. This is the regime of labyrinth formation.

In the fast-inhibitor limit ($\epsilon \to 0$), the inhibitor equation is a local-in-time relation $\left(\nabla^2 - 1\right) v = -u$. This is just the Poisson equation for a (screened) Coulomb "potential" v due to a distribution of "charge" u. It is solved in terms of the free-space Green's function $\mathcal{G}$ (in two dimensions, $\mathcal{G}(\mathbf{x}) = (1/2\pi)K_0(x)$) in the usual way. Substitution back into the activator equation gives a single *nonlocal* equation of motion for u, that *does* have a the form of a gradient flow, $u_t = -\delta\mathcal{E}/\delta u$, with[15]

$$\mathcal{E} = \int d\mathbf{x} \left\{ \frac{D}{2}|\nabla u|^2 + F(u;r) - \frac{\rho}{2}u^2 \right\} + \frac{\rho}{2}\int d\mathbf{x} \int d\mathbf{x}' u(\mathbf{x})\mathcal{G}(\mathbf{x}-\mathbf{x}')u(\mathbf{x}')\ . \tag{2}$$

Figure 2 shows the results of a direct numerical solution of the fast-inhibition dynamics. The initial condition, a compact domain of $u = 1$ surrounded by $u = 0$, expands and fingers, ultimately filling the (periodic) computational domain without interface self-crossings. The black regions where

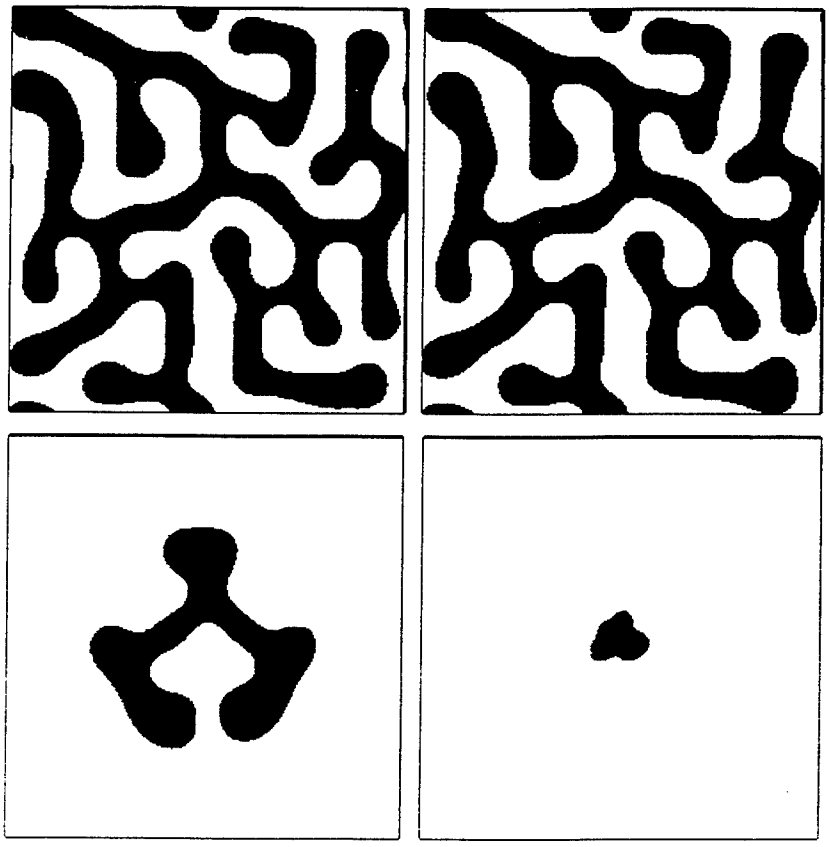

Figure 2: Labyrinth formation in the FitzHugh-Nagumo model, from numerical solution of the fast-inhibitor dynamics. Images are thresholded contour plots of the activator, with $u < 1/2$ shown white and $u > 1/2$ shown black. Evolution proceeds from upper lefto to lower right. Adapted from Ref. 11.

$u \simeq 1$ act as sources for the inhibitor, whose concentration falls off exponentially into the nearby white regions. The overlap of those tails from adjacent black areas leads to a repulsive interaction between the fronts. The pattern apparently settles down to a time-independent state.

3 Interface dynamics

As Fig. 2 shows, the combination of a bistable potential $F(u)$ whose two minima are close in depth, and a small diffusion constant D tends to produce patterns that consist of regions of a nearly constant value of u (0 or 1) joined by narrow interfaces (whose width scales as $\sqrt{D}$). It is then natural to recast the energy functional $\mathcal{E}$ in (2) and dynamics in terms of that boundary alone. We originally advanced a heuristic argument for this,[10] and have now justified it[11] in the limit of slow diffusion, small detuning, and weak inhibition, ($\sqrt{D} \sim \Delta F \sim \rho \ll 1$) using methods of asymptotic analysis employed for the one-component problem.[12] The former derivation is more intuitive, and summarized below.

Consider a single domain with $u \simeq 1$ in a sea of $u \simeq 0$, and recast the three types of contributions in Eq. 2 in terms of the domain boundary $\mathcal{C}$ traced out by the position vector $\mathbf{r}(s)$. The term involving $|\nabla u|^2$ is large only near $\mathcal{C}$, so if L is the length of $\mathcal{C}$ this contribution is γL, with γ the (positive) line tension. The polynomial terms $F(u;r) - \frac{1}{2}\rho u^2$ contribute proportional to the area A of the domain. Finally, from the definition of the Green's function $(\nabla^2 - 1)\,\mathcal{G}(\mathbf{x}) = -\delta(x)$ the nonlocal contribution may be rewritten in terms of an area term and a pairwise interaction between tangent vectors $\hat{\mathbf{t}}(s)$ to the boundary in the form of the self-induction. The result is an energy functional

$$\Delta\mathcal{E}[\mathbf{r}] = \gamma L + \Delta F A - \frac{1}{2}\rho \oint ds \oint ds' \hat{\mathbf{t}}(s) \cdot \hat{\mathbf{t}}(s')\mathcal{G}\left(\mathbf{r} - \mathbf{r}'\right) \ . \tag{3}$$

The boundary dynamics is a simple generalization of the u dynamics, a gradient flow of the form $\mathbf{r}_t = -\Gamma\delta\mathcal{E}/\delta\mathbf{r}$, where Γ is a kinetic coefficient derivable from the original PDEs. This normal velocity of the curve is

$$\hat{\mathbf{n}} \cdot \mathbf{r}_t = -\Gamma\left\{\gamma\kappa(s) + \Delta F - \rho \oint ds' \hat{\mathbf{R}}(s,s') \times \hat{\mathbf{t}}'\mathcal{G}'(R)\right\} \ . \tag{4}$$

where $\hat{\mathbf{R}}(s,s') = (\mathbf{r}(s) - \mathbf{r}(s'))/|\mathbf{r}(s) - \mathbf{r}(s')|$. Apart from the uniform force ΔF, we see in Eq. 4 a competition between two of the fundamental forces in nature that we associate with interfaces: the Young-Laplace force proportional to the curvature κ, arising from tension, and the nonlocal Biot-Savart force,

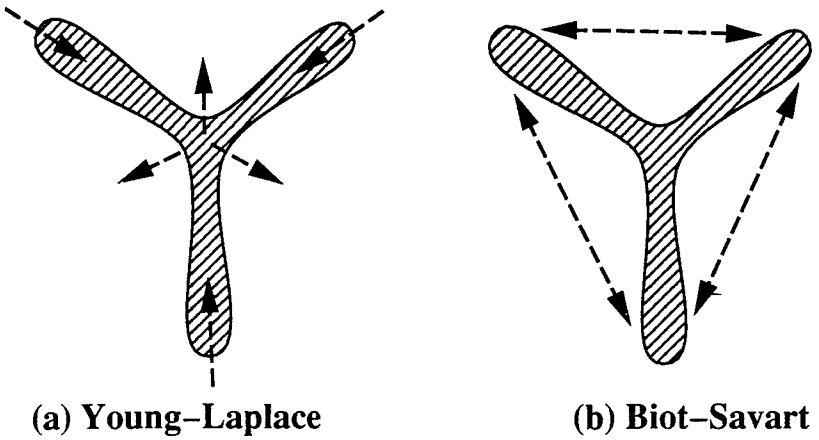

Figure 3: Schematic illustration of the shape-dependent forces acting on chemical fronts in the FitzHugh-Nagumo model. (a) The Young-Laplace force acts inward in regions of positive curvature and outward when $\kappa < 0$. (b) The nonlocal Biot-Savart force between nearby interfaces is repulsive.

representing the interaction mediated by the inhibitor. Figure 3 shows the two opposing tendencies of these forces. The first tends to relax a fingered structure to a circle, while the second leads to repulsion between interface segments on adjacent fingers. While different from the usual unscreened interaction between true current-carrying wires, it nevertheless has the same sign, so that *antiparallel sections of the interface repel.*

4 Instabilities and labyrinths

Possible instabilities of regular geometrical shapes in the FitzHugh-Nagumo model were considered first by Ohta, Mimura, and Kobayashi,[16] who showed that two-dimensional localized states such as discs and stripes could exhibit instabilities of such a form as to suggest the kind of fingering phenomena shown in Fig. 1. In the fast-inhibitor limit, the mechanism of these instabilities is seen in the stability analysis of an infinite, straight chemical front. Of particular interest are perturbations on scales large relative to the $\mathcal{O}(1)$ inhibitor screening length, for which the growth rate as a function of wavevector k is[11]

$$\sigma_{\rm f}(k) \simeq \Gamma \left[-\left(\gamma - \frac{1}{4}\rho\right) k^2 - \frac{3}{16}\rho k^4 + \cdots \right] . \qquad (5)$$

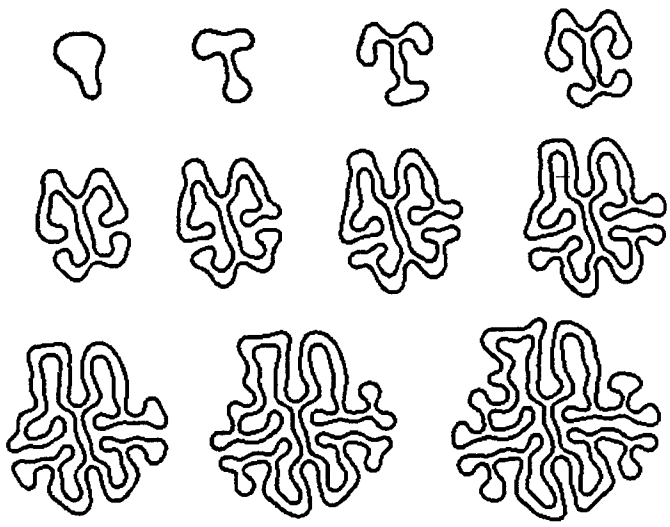

Figure 4: Contour dynamics evolution from a compact initial condition (upper left) to a labyrinthine pattern (lower right). From Ref. 11.

For ρ sufficiently large, the effective line tension $\gamma - \rho/4$ becomes negative, favoring interface proliferation. But the stabilizing term of order k^4 prevents the interface from becoming ramified on arbitrarily fine scales. These competing effects produce fingering at a characteristic length scale.

To study the fully nonlinear regime of the contour dynamics (4) it is necessary to resort to a numerical solution of the equation of motion. Efficient methods have been developed for this.[11] Figure 4 shows how a compact initial condition repeatedly fingers to produce a space-filling, but non-self-crossing pattern. These and other more quantitative studies confirm that the contour dynamics accurately captures the behavior of the FitzHugh-Nagumo model in the limit of fast-inhibition.

5 Interfacial pattern formation and the Biot-Savart Law

Labyrinthine patterns like those in Figs. 1 and 2 are found not only in reaction-diffusion systems but also in a number of quasi-two-dimensional physical and chemical systems with long-range electromagnetic interactions.[17] These include uniaxial ferromagnetic films, Langmuir monolayers of dipolar molecules at the

air-water interface,[18] magnetic fluids in Hele-Shaw flow,[19] and type-I superconductors.[20] In all of these systems, the nonlocal interactions arise from electric or magnetic dipoles (permanent or induced). Through the usual correspondence between magnetic moments and current loops the representation of the field energy as a self-induction integral and the force as a Biot-Savart integral are thus completely natural. From one system to the next the physical meaning of the amplitude of these contribution differs, as do the details of the Green's function, but the form is universal. The way that this energy functional determines the interface dynamics may also vary from system to system, depending on whether the viscous dissipation acts locally on the interface or throughout an incompressible viscous fluid.[21,22] But underlying all those laws of motion is the competition between the Young-Laplace and Biot-Savart terms.

Acknowledgments

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OPTICAL SOLITON TRANSMISSION THROUGH FIBERS, ITS BIRTH AND FUTURE PROSPECTS

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Optical soliton in fibers are now considered as a most promising career of ultra-high speed of optical signals over global distances. Presented here are a brief history of optical solitons and future prospects based on the author's personal view

History of application of optical solitons for high speed signals transmission in fibers may be divided into seven stages, 1) prehistoric developments, 2) discovery of optical solitons, 3) idea and demonstration of all optical soliton transmission systems, 4) derivation of guiding center or average soliton, 5) identification of possible problems, 6) demonstration of soliton control and problem solving, 7) future prospects.

1 Prehistoric Developments

A soliton was first observed by Russell as a water surface wave in 1844 [1]. Korteweg and de Vries in 1895 have derived a model equation (known as the KdV equation) describing a far field property of the surface wave in the lowest order of dispersion and nonlinearity and obtained one humped, localized solution [2]. When the equation was numerically solved in a periodic boundary condition by Zabusky and Kruskal in 1965, a set of solitary waves, whose structure was obtained earlier by Korteweg and de Vries, was found to emerge and stably pass each other. They named these solitary waves as solitons because of their stability [3]. The term "soliton" was justified when the KdV equation was solved analytically by means of inverse scattering method and the solution was described by a set of solitons whose parameters were obtained from discrete eigenvalues of a spectral equation (linear Schrödinger equation in this case) with potential given by the initial wave form [4]. Solitons are regarded as a fundamental unit of mode in nonlinear and dispersive medium and play a role similar to the Fourier mode in a linear medium.

Meanwhile, self-focusing of light intensity in a Kerr medium was demonstrated and, in a one dimensional case, a spatially localized solution analogous to

a soliton was found to emerge by the balance of the cubic nonlinearity and refraction [5]. The model equation, called the nonlinear Schrödinger equation, was later found to be integrable by Zakharov and Shabat [6] also by means of the inverse scattering transform and the solution is given by a set of solitons and dispersive waves.

2 Discovery of Optical Solitons

Existence of optical solitons for complex light wave envelope in lossless fibers with group dispersion and Kerr nonlinearity and their stability were theoretically demonstrated first by Hasegawa and Tappert in 1973; bright solitons appear in an anomalous dispersion regime, while dark solitons appear in a normal dispersion regime [7],[8]. The first experimental demonstration of the existence of a (bright) optical soliton was performed by Mollenauer et al in 1980 [9]. But few people expected importance of solitons in practical use because of erroneous belief that anything "nonlinear" would be too complicated and take enormous power.

3 Idea and Demonstration of All Optical Soliton Transmission System

Since solitons do not distort due to nonlinearity and dispersion which are inherent in fibers, natural next step is to construct and all optical transmission system in which fiber loss is compensated for by amplifications [10]. In the absence of a realistic optical amplifiers, Hasegawa [11] in 1983 proposed to use Raman gain of the fiber itself. The idea was used in the first long distance all optical transmission experiment by Mollenauer and Smith [12] in 1988. This experimental result has ignited serious interests in optical communication society as well as the Bell Labs managements. In particular, the invention of Erbium Doped Fiber Amplifiers (EDFA) has elevated the concept of all optical transmission system to more realistic level as initiated by Nakazawa et al. [13] in the first reshaping experiment of solitons.

4 Derivation of Guiding Center Integrability

In the experiments of long distance soliton transmission with loss compensated periodically by EDFA, it was recognized that the initial soliton amplitude had to be enhanced so that the average soliton intensity over the distance between two amplifiers could provide enough nonlinearity to balance the fiber dispersion [14,15,16].

However, the pulse shape at an arbitrary position along the fiber does not have a soliton structure, that is, the product of amplitude and width does not stay constant. Hasegawa and Kodama, using the Lie-transformation, have succeeded in 1990 [17] in demonstrating that the properly transformed amplitude does satisfy the ideal nonlinear Schrödinger equation to the order $(z_a/z_0)^2$, where z_a and z_0 are the amplifier spacing and dispersion distance respectively and called the transformed amplitude "guiding center soliton". The guiding center soliton warrants the integrability of the periodically amplified soliton transmission system providing the exact causal relation between the input and the output signals.

5 Identification of possible Problems

Concept of all optical transmission by periodic amplifications has been challenged by several authors. Gordon and Haus [18] predicted that the amplifier noise would induce frequency modulation and result in time jitter in soliton position. Dianov et al. [19] predicted that side scattering of phonons would also induce time jitter. Chu and Decem have published series of papers in the effects of interactions between neighboring solitons on the bit limitation in a soliton system [20]. Although the single mode fiber was found needed in the early stage of the soliton concept in fiber, the unavoidable polarization dispersion have produced concern of soliton splitting. However, it was shown later that if the polarization shifts randomly within a distance much shorter than the dispersion distance, this effect may be averaged out due to the nonlinear attractive force and the ideal nonlinear Schrödinger equation can be recovered with somewhat reduced nonlinear coefficient [21]. Collisions between solitons in different channels in soliton WDM systems in the input [22] as well as in amplifiers [23] were predicated to cause time position shifts. It was found later that the effect of initial overlap may be reduced by orthogonalization of the adjacent solitons in neighboring channels, while the former by choosing collision distance to be longer than the amplifier spacing. Higher order terms in linear and nonlinear dispersion were expected to deform solitons and shift soliton velocities [24]. Higher order nonlinear dissipation (self induced Raman effect) was found to decelerate solitons [25,26]. Attempts to solve these problems have since been made and significant progress have been achieved.

6 Problem Solving by Various Means-Proof of Soliton Controllability

Most of the problems stated in 5 can either be reduced or solved by means of soliton control using various means. The idea is based on the robust property

of solitons in that the control of soliton characteristics, amplitude or width, time position, velocity or frequency and phase is sufficient to achieve solutions, that is, although the original wave equation has an infinite dimension, control of finite dimensional (soliton) parameters is sufficient in controlling the soliton transmission systems. This fact presents important merit of a soliton system additional to the causality relation proved in 4. In particular, soliton controls allow separation of soliton signal from linear noise. This fact may present a revolution in communication theory. Control of solitons may be classified to passive and active means. Frequency filters inserted periodically in the transmission systems were found to be effective in reducing soliton time jitter [27,28]. The idea was further refined by Mollenauer et al., who succeeded eliminating linear wave growth produced by the excess gain at the center frequency of the filter by sliding the filter frequency along the transmission [29]. The sliding frequency filters have enabled a record long distance 20 gigabit signal transmission. Soliton parameters were found to be controlled by active means. Nakazawa et al. have demonstrated practically unlimited distance of propagation can be achieved by a combination of temporal gain modulation and filters [30]. Active control was also shown to be possible by means of injection of light waves to the fiber by Grigoryan et al. [31].

7 Future Prospects

Almost a quarter of a century have passed since the discovery of optical solitons in fibers. Experimental data obtained recently have surpassed any other means both in the distance and the rate of error free transmission. Economic impact of a soliton system is also revolutionary because it has practically eliminated costly repeaters and replaced them with low cost amplifiers, achieving a record low cost transmission per bit per distance.

Soliton behaviour is quite different in an adiabatic regime ($z_0\gamma >> 1$) and a non adiabatic regime ($z_0\gamma << 1$) where γ is the fiber loss rate per unit distance. research approach of a soliton system should be differentiated in these two regimes; one, high speed ultra long (intercontinental) distance, and the other for the ultra high speed (> 100Gigabit/sec) long distance communications. The latter is relatively unexplored but should be potentially important as the trunk line for multimedia systems. Creative connection of such a system with wireless (portable) terminals will become an interesting research subjects.

In addition to bright solitons, dark solitons may find a way of being used in ultra high bit rate systems because of being less influenced by perturbations such as amplifier noise [32] fiber loss [33] and mutual interactions [34].

Conclusion

After twenty some years since the discovery of optical solitons in fiber, research on optical solitons has come to a stage when an explosive number of papers are being published in various journals. The research activities are supported by interests in soliton phenomena themselves, interests in technical applications, developments of supporting technologies, as well as by demand from the society for increased speed of communications. Solitons are not yet deployed in a commercial transmission systems as yet but such days will come in any moment. Solitons are expected to play a major role both in intercontinental transmission as well as in domestic multimedia transmission.

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Bose-gas with nontrivial interparticle interaction and exotic magnetic solitons.

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1 Introduction

In weakly nonlinear systems the solitonic excitations (dynamic two-parametric solitons) are studied usually in the framework of nonlinear Schrodinger equation (NSE) [1]:

$$i\partial\psi/\partial t + \partial^2\psi/\partial x^2 - \omega_0\psi + g \mid \psi \mid^2 \psi = 0. \tag{1}$$

Its motionless soliton solution has the simple form

$$\psi = \sqrt{2\epsilon/g}\cosh^{-1}(\epsilon x)\exp(-i\omega t), \epsilon = \sqrt{\omega_0 - \omega}. \tag{2}$$

The 1D Bose-gas with a pair δ-function interaction is a quantum analog of a classical system, which is described by NSE [2]. Schrödinger equation for N-particle wave-function $\Phi(x_1, x_2, .., x_N)$ has the following form ($\hbar = 1, m = 1/2$):

$$-\sum_i d^2\Phi/dx_i^2 + N\omega_0\Phi + \sum_{i<j} U_{ij}\Phi = E\Phi, U_{ij} = -g\delta(x_i - x_j). \tag{3}$$

This quantum system is completely integrable by the Bethe-anzats method [3].Classical soliton solution (2) corresponds to many-bosons bounded states with the energy spectrum $E = \omega_0 N - g^2 N(N^2 - 1)/48$, where $N = 4\epsilon/g$. This spectrum coincides with the result of quasi-classical quantization of (2). The distribution of boson density coincides with an envelope of the function $\mid \psi \mid^2$for the soliton (2) in limit $N \gg 1$.

But the description of solitons in framework of NSE is only for small amplitudes (small N) and the soliton collapse takes place while $N \to \infty$. In reality the limitation of amplitude growth takes place and is connected with a saturation of nonlinearity. The sine- Gordon equation is a typical example

of such system. The generalized NSE with a saturated nonlinearity has a form [1,4]:

$$i\partial\psi/\partial t + \partial^2\psi/\partial x^2 - \omega_0\psi + g \mid \psi \mid^2 \psi - \delta \mid \psi \mid^4 \psi = 0. \tag{4}$$

In this equation nonlinear terms contain only the field variable ψ but not its derivatives. The soliton solution of this equation has a form of a function with a plateous and corresponds to the Bose-drop of condensed matter. The 1D Bose-gas of particles with a two-bodies δ- function attraction and three-particles point repulsion is a quantum analog of classical nonlinear system (4) [2].

But Eq.(4) is not the unique generalization of Eq(1). The well-known Landau-Lifshitz equation for a ferromagnet with the easy-axis on-site anisotropy is another example of such generalization [5]:

$$i\partial\psi/\partial t - l^2\omega_0 \left(m_z\partial^2\psi/\partial x^2 - \psi\partial^2 m_z/\partial x^2\right) + \omega_0\psi m_z = 0, \tag{5}$$

where $\psi = m_x + im_y, m_z = \sqrt{1 - \mid \psi \mid^2}, l = \sqrt{\alpha/\beta}$ is a magnetic length, $\omega_0 = 2\mu_0 M_0\beta/\hbar$ is a frequency of homogeneous ferromagnetic resonance, α is an exchange constant and β is the constant of anisotropy. In small amplitude limit Eq.(5) can be rewritten as

$$\begin{aligned} i\partial\psi/\partial t - l^2\omega_0\partial^2\psi/\partial x^2 + \omega_0\psi - \omega_0 \mid \psi \mid^2 \psi/2 - \omega_0 \mid \psi \mid^4 \psi/8 \\ + l^2\omega_0/2 \left[\mid \psi \mid^2 \partial^2\psi/\partial x^2 - \psi\left(\partial^2 \mid \psi \mid^2 /\partial x^2\right)\right] = 0. \end{aligned} \tag{6}$$

This equation differs from Eq.(4) by the last two terms which are cubic in variable ψ, but contain two spatial derivatives. In small- amplitude soliton the amplitude ψ and the spatial derivative $\partial/\partial x$ are of order of small parameter ϵ and additional terms have the same smallness as saturated nonlinearity $\mid \psi \mid^4 \psi$. (An axistence of these terms is connected with the fact that magnons are not the exact Bose-particles and their interaction is more complicated that point interaction).

Of course, the addition terms $\mid \psi \mid^4 \psi, \mid \psi \mid^2 \partial^2\psi/\partial x^2, \psi\partial^2 \mid \psi \mid^2 /\partial x^2$ are much smaller that the main nonlinear term $\mid \psi \mid^2 \psi$. But situation is quite different in ferromagnets with a strong exchange anisotropy. If the symmetry of on-site and exchange anisotropies is the same (for example, easy-axis anisotropy) the energy density has a form [5]:

$$W = (\alpha/2)\left(\partial \mathbf{M}/\partial x\right)^2 + (\alpha_1/2)\left(\partial M/\partial x\right)^2 - (\beta/2)\, M_z, \tag{7}$$

where $\mathbf{M} = M_0\mathbf{m}$ and $\mathbf{m}$ is the unit vector. The corresponding dynamical equation has a form [5]:

$$i\partial\psi/\partial t - l^2\omega_0\left[m_z\partial^2\psi/\partial x^2 - \left(1+\frac{\alpha_1}{\alpha}\right)\psi\partial^2 m_z/\partial x^2\right] + \omega_0\psi m_z = 0, \quad (8)$$

and in some limiting cases (for some values of $\frac{\alpha_1}{\alpha}$) the above-mentioned terms are of great importance. In the limit of Ising magnet ($\alpha = 0$) and classical XY-magnet ($\alpha_1 = -\alpha$) the differential part of Eq.(8) reduces to the expressions $L^2\omega_0\psi\partial^2 m_z/\partial x^2$ and $-l^2\omega_0 m_z\partial^2\psi/\partial x^2$ correspondingly ($L^2 = \alpha_1/\beta$). In the Ising limit the exchange part of equation does not contain the linear terms and in linear approximation the dispersive terms are absent. This fact leads to essential modification of soliton solution and arising of new type of solitons (*exotic solitons*) in which the deviation of magnetization from the equilibrium state is exactly nonzero only in finite domain of space. Such solitons were investigated previously by P.Rosenau [6,7] for the equations of KdV-type and by Yu.Kosevich [8,9] for nonlinear elastic chains and received the name *"compactons"*. Later Holm and Camassa investigated another type of exotic solitons- *"peakons"*.

The aim of this article is to investigate magnetic compactons and peakons in anisotropic ferromagnets and to study the linkage between exotic solitons and bounded states of bosons in Bose-gas of interacting particles.

2 Generalized model of Bose-gas of interacting particles

We shall investigate 1D Bose-gas with two-particle but more complicated than in (3) potential of interparticle interaction to understand the quantum-mechanical nature of nonlinear dispersion terms in Eqs.(5,6,8). Let us choose the potential function U_{ij} in the following form:

$$U_{ij} = 2V\delta(x_i - x_j) - 2U\left[\delta(x_i - x_j + a) + \delta(x_i - x_j - a)\right]. \quad (9)$$

We consider that the integral interaction is attractive, i.e. $2U - V = W > 0$. Then there are two different cases with $U, V > 0$ (A) and $U, V < 0$ (B). The equation (3) with the potential (9) is completely integrable, but in the limit $a^2UW \ll 1$ we can restrict ourselves to the simple potential

$$U_{ij} = -2W\delta(x_i - x_j) - 2Ua^2\frac{\partial^2}{\partial x^2}\delta(x_i - x_j), \quad (10)$$

which differs from zero only on the system of planes $x_i < x_j$ in N - dimensional coordinate space $(x_1, x_2, ...x_N)$. Using the evident boundary conditions on

these planes we can find the Bethe-type wave function $\Phi = \exp\left(-\sum_{i=1}^{N} k_i x_i\right)$ (for sector $-\infty < x_N \leq x_{N-1} \leq ... \leq x_2 \leq x_1 < \infty$) with the set of parameters $\{k_i\}$,which form the "strings" $k_i - k_{i-1} = W$ and $k_i - k_{i-1} = 1/a^2 U$ in the case A and string $k_i - k_{i-1} = W$ in the case B. The same result is valid for potential (9) as well.

In Hartree approximation ($\Phi = \prod_i \varphi(x_i)$) in the case of potential (10) for one-particle normalized ($\int \mid \varphi \mid dx = 1$) wave-function $\varphi(x_i)$ for stationary states we have the following nonlinear equation:

$$\frac{d^2\varphi}{dx^2} - \epsilon\varphi + 2(N-1)W\varphi^3 + 2(N-1)Ua^2\varphi\frac{d^2}{dx^2}\varphi^2 = 0, \qquad (11)$$

where the Hartree energy $\epsilon = \epsilon(N)$ is equal to

$$\epsilon = \int dx \left[-\left(\frac{d\varphi}{dx}\right)^2 + 2(N-1)W\varphi^4 - 2(N-1)Ua^2\varphi^2\left(\frac{d\varphi}{dx}\right)^2\right]. \qquad (12)$$

We see that Hartree equation (11) includes the nonlinear term of the form $\varphi\left(\frac{d^2}{dx^2}\varphi^2\right)$ as well as Eqs.(6,8) for the dynamics of magnetization.

3 Compactons and peakons in Bose-gas.

For small amplitude solitons in long-wave approximation ($d/dx \ll 1$) the last term in Eq.(11) is small and it is naturally to leave it only for large value of parameter Ua^2/W. As we supposed above that $WUa^2 \ll 1$, we have the system of inequalities: $W^2 \ll WUa^2 \ll 1$.

The soliton solution of Eq.(11) exists only for positive values of Hartree energy. After the first integration this equation reduces to

$$\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2\frac{\epsilon - NW\varphi^2}{1 + 4NUa^2\varphi^2}. \qquad (13)$$

In the case (A) the soliton solution of Eq.(13) in unevident form is

$$\frac{1}{\sqrt{A}}\ln\frac{\sqrt{A} - f}{\sqrt{A} + f} - \arcsin\frac{f^2 - 1}{f^2 + 1} = 2x\sqrt{\epsilon_\star}, \qquad (14)$$

where A is proportional to the Hartree energy $A = \epsilon/\epsilon_\star$, where $\epsilon_\star = W/4Ua^2$ is the characteristic value of this energy, and $f = \left(1/\sqrt{\epsilon_\star}\right)\frac{d}{dx}\ln\varphi$. From Eq.(13) and normalization condition we can find the dependence $N = N(\epsilon)$:

$$\frac{N}{N_\star} = \sqrt{\frac{\epsilon}{\epsilon_\star}} + \left(1 + \frac{\epsilon}{\epsilon_\star}\right) \arcsin \sqrt{\frac{\epsilon}{\epsilon + \epsilon_\star}}, \tag{15}$$

where $N_\star = \sqrt{\frac{1}{4WUa^2}}$ is the characteristic number of bounded in soliton bosons. It is evident from (15) that $(\epsilon/\epsilon_\star) \simeq (N/2N_\star)^2$ for $N \ll N_\star$ and $(\epsilon/\epsilon_\star) \simeq (2N/\pi N_\star)$for $N \gg N_\star$.

The amplitude of soliton is $\varphi_{\max} = \sqrt{\epsilon/NW}$ and asymptotes on infinity have the form $\varphi \sim \exp(-x\sqrt{\epsilon})$. For small N soliton has a standard form of small-amplitude soliton. But in the limit $N \gg N_\star$ the amplitude $\varphi^2_{\max}$ tends to limited value $\sqrt{W/\pi U a^2}$ and the soliton localizes in the region $\Delta = 2x_0 = 2\sqrt{\pi^2 U a^2/W}$. For $N \to \infty$ we have

$$\varphi = \left(\frac{W}{\pi U a^2}\right)^{1/4} \cos\left(\sqrt{\frac{W}{4Ua^2}}x\right), for \mid x \mid \leq x_0,$$
$$\varphi = 0, for \mid x \mid > x_0. \tag{16}$$

This is the typical form of compacton solution. Of course, the density of bosons $\rho = N\varphi^2$ increases to infinity in this compacton when $N \to \infty$, but the collapse differs from those in Bose- gas with usual δ- function interaction and the size of soliton remaines finite.

Note that in compacton limit solutions satisfy the following equation:

$$\varphi^2 \left[\left(\frac{d\varphi}{dx}\right)^2 + \epsilon_\star \varphi^2 - 2\epsilon_\star^{3/2}/\pi\right] = 0. \tag{17}$$

In the case B with $U < 0$ the soliton solution of Eq.(13) can be found exactly as well:

$$\frac{1}{\sqrt{A}} \ln \frac{f - \sqrt{A}}{f + \sqrt{A}} - \ln\left[\left(1 - 4N \mid U \mid a^2\varphi^2\right)(1 - f)^2\right] = 2x\sqrt{\epsilon_\star}, \tag{18}$$

where we must replaces $U \to \mid U \mid$ in $\epsilon_\star$, A, f and $N_\star$.The relation between the Hartree energy and the number of bosons is now somewhat different from (15):

$$\frac{N}{N_\star} = \sqrt{\frac{\epsilon}{\epsilon_\star}} + \left(1 - \frac{\epsilon}{\epsilon_\star}\right) \ln \frac{\sqrt{\epsilon_\star} + \sqrt{\epsilon}}{\sqrt{\epsilon_\star - \epsilon}}, \tag{19}$$

where $0 < \epsilon < \epsilon_\star$ and $0 < N < N_\star$. For small values of N the amplitude of soliton $\varphi_{\max}$ grows ($\varphi_{\max} = WN/4$) and peaks at $N = N_\star$, where $\varphi^2_{\max} =$

$\sqrt{\frac{W}{4|U|a^2}} = \sqrt{\epsilon_\star}$ and amplitude of soliton has the same order as in previous case of compacton with $N \to \infty$.In this limit Eq.(13) reduces to the form:

$$\left(\sqrt{\epsilon_\star} - \varphi^2\right)\left[\left(\frac{d\varphi}{dx}\right)^2 - \epsilon_\star \varphi^2\right] = 0. \tag{20}$$

The soliton solution of this equation

$$\begin{aligned} \varphi &= \epsilon_\star^{1/4} \exp\left(-\sqrt{\epsilon_\star} x\right), x > 0, \\ \varphi &= \epsilon_\star^{1/4} \exp\left(\sqrt{\epsilon_\star} x\right), x < 0 \end{aligned} \tag{21}$$

has a form of peakon. This solution is expressed in terms of exponential functions, as in the case of linear equations, but the amplitude of peakon solution is nonzero and fixed.

It is important that the functions (16) and (21) determine a field distribution in solitons (compactons and peakons) only in some definite intervals. In the sewpoints there are the jumps of derivatives, but we must not introduce δ-functions in these points because of the first factors in Eqs.(17,20). It is connected with the fact that the solutions (16,21) for compactons and peakons are the limiting expressions for analytical functions (14,18).

4 Solitons in ferromagnets with anisotropic exchange interaction.

Let us turn to the investigation of uni-axis ferromagnets with the exchange anisotropy and with the same symmetry of exchange and uni-ion anisotropies, i.e. to the dynamical equation (8). After the first integration in the simplest case of immovable soliton of the type $\psi = \sin\theta(x)\exp(-i\omega t)$ we have

$$\left(l^2 + L^2 \sin^2\theta\right)\left(\frac{d\theta}{dx}\right)^2 = 4\sin^2\left(\frac{\theta}{2}\right)\left[\cos^2\left(\frac{\theta}{2}\right) - \frac{\omega}{\omega_0}\right]. \tag{22}$$

For weak anisotropy $\alpha_1/\alpha \ll 1$ small amplitude solitons have the same structure as in the case of isotropic exchange. But in the case of strong exchange anisotropy the situation is quite different. It is convenient to use the new dynamical variable $\varphi = 1 - \cos\theta$, which describes the density of spin deviations (magnon density). In the terms of this variable Eq.(22) is rewritten as follows:

$$L^2\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2 \frac{(2-\varphi)\left[2\left(1-\omega/\omega_0\right) - \varphi\right]}{(l/L)^2 + 2\varphi - \varphi^2}. \tag{23}$$

For small deviation of frequency from the frequency of homogeneous ferromagnet resonance $(1-\omega/\omega_0 \ll 1)$ and for small difference from Ising limit $(l/L \ll 1)$ this equation is very close to Eq.(13) for many-boson bounded states:

$$L^2\left(\frac{d\varphi}{dx}\right)^2 = \varphi^2\frac{2\left(1-\omega/\omega_0\right)-\varphi}{l^2/2L^2+\varphi}. \tag{24}$$

In the Ising limit $(l \to 0)$ Eq.(22) has a compacton solution [10]:

$$\begin{gathered}\cos\theta = \tfrac{\omega}{\omega_0} - \left(1-\tfrac{\omega}{\omega_0}\right)\cos\left(\tfrac{x}{L}\right), for \mid x \mid \leq \pi L, \\ \cos\theta = 1, for \mid x \mid > \pi L.\end{gathered} \tag{25}$$

It is the magnetic analog of compact bounded state of bosons (16).

The dynamical equation of easy-axis ferromagnet in Ising limit has very simple form in terms of ψ and m_z:

$$i\frac{1}{\omega_0}\partial\psi/\partial t + L^2\psi\partial^2 m_z/\partial x^2 + \psi m_z = 0. \tag{26}$$

Note that this equation is dispersionless in linear approximation, i.e. the dispersion of linear excitations $\left(\partial^2\omega/\partial k^2\right)_{\psi\to 0}$ is zero for the dispersion relation of magnons $\omega = \omega(k)$, where ω and k are the frequency and wave number of magnon. This is the general property of all systems with compacton solutions.

For dynamical compacton Eq.(26) can be rewritten in the form

$$\left(1-m_z^2\right)\left[L^2\left(\frac{dm_z}{dx}\right)^2 + \left(m_z - \frac{\omega}{\omega_0}\right)^2 - \left(1-\frac{\omega}{\omega_0}\right)^2\right] = 0, \tag{27}$$

similar to Eq.(17), and for domain wall of compacton type Eq.(26) may be reduced to the same form for the variable ψ:

$$\psi^2\left[L^2\left(d\psi/dx\right)^2 + \psi^2 - 1\right] = 0. \tag{28}$$

Compact domain wall has an evident form[10]:

$$\begin{gathered}m_z = 1, x > \pi/2, \\ m_z = \sin\left(x/L\right), \mid x \mid < \pi/2, \\ m_z = -1, x < -\pi/2.\end{gathered} \tag{29}$$

In easy-axis Ising-type ferromagnets both compact dynamical solitons and compact domain walls can not move.

In the opposite case of Ising ferromagnets with on-site anisotropy of easy-plane type the dynamical equation (26) changes in the following way:

$$i\frac{1}{\omega_0}\partial\psi/\partial t + L^2\partial^2 m_z/\partial x^2 - \psi m_z = 0. \tag{30}$$

where $L = \sqrt{\frac{\alpha_1}{|\beta|}}$. In this system we have degenerated ground state with $m_z = 0$.

In easy-plane case there are no dynamical exotic solitons but the analog of stationary rotary wave solution [5] has an exotic form in this limit. For stationary rotary wave one can rewrite Eq.(30) in the form quite similar to equation (20) for peakons in Bose-gas:

$$\left(1 - m_z^2\right)\left[L^2\left(dm_z/dx\right)^2 - m_z^2\right] = 0. \tag{31}$$

Magnetic peakon-like soliton has the trivial profile:

$$m_z = \exp\left(- \mid x \mid /L\right), \tag{32}$$

the same as peakon (21) in Bose-gas.

In the limit of classical XY-ferromagnet ($\alpha_1 = -\alpha, L = 0$)we have the following dynamical equations for a magnetization instead of Eqs.(26,30):

$$i\frac{1}{\omega_0}\partial\psi/\partial t - l^2 m_z\partial^2\psi/\partial x^2 + m_z\psi = 0, (easy - axis), \tag{33}$$

$$i\frac{1}{\omega_0}\partial\psi/\partial t - l^2 m_z\partial^2\psi/\partial x^2 - m_z\psi = 0, (easy - plane). \tag{34}$$

In easy-axis XY-ferromagnet there are the peakon-like domain walls and dynamic solitons. In the case of domain wall the profile of the field variable ψ has a peakon form: $\psi = \exp\left(- \mid x \mid /l\right)$, and z-component of magnetization has a usual form of kink. But in the center of this domain wall there is the singularity of the spatial derivative: $dm_z/dx \mid_{x=0} = \infty$. The dynamical soliton of the simple type $\psi = \psi(x)\exp(i\omega t)$ has a form of usual soliton in the frequency interval $1/2 < \omega/\omega_0 < 1$, but the soliton with small frequency $0 < \omega/\omega_0 < 1/2$ has the exotic structure: it represents the bounded state of a pair of peakons of ψ-field.

In easy-plane XY-ferromagnet there are compacton-type stationary rotary waves of magnetization.The equation for ψ is very similar to Eq.(28) :

$$\left(1 - \psi^2\right)\left[l^2\left(d\psi/dx\right)^2 + \psi^2 - 1\right] = 0, \tag{35}$$

and soliton solution has a form (29) but for the variable ψ instead of m_z.

So there is the symmetry in existence of compactons and peakons in Ising -type and XY-type ferromagnets.

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SOLITON SOLUTION OF A NON-LOCAL SINE-GORDON MODEL

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A number of papers has been recently devoted to studying various nonlinear nonlocal problems. A long range interaction is taken into account in the description of the lattice dynamics, magnetic systems, superconductivity and so on. It is clear that the natural account of the long range interaction demands to consider appropriate integral equations. Concerning the sine-Gordon model we will refer to papers [1,2] where two different types of the long range interaction were considered and the different nonlinear integral equations were proposed.

If we are interested in the physical solutions of the problem under consideration which were described with functions slightly changing in the space it is possible to simplify the integral equation. A good approximation can be reached by reduction the integral equation to the corresponding differential equation including the higher spatial derivatives.

The idea of the reduction of the integral equation to a differential one is very simple. Suppose σ is the localization length of the symmetric kernel $K(x)$ of the integral equation and consider solutions of this equation which changes essentially on the length λ. In the case $\sigma \ll \lambda$ we can write in the main approximation

$$\int K(x-y)\,u(y)\,dy = K_0 u(x) + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$

where

$$K_0 = \int K(x)\,dx, \sigma^2 = \int x^2 K(x)\,dx.$$

Such a reduction of the integral equation proposed in Ref.[1] was demonstrated in Ref.[3]. The similar reduction of the integral equation proposed in Ref.[2] leads to the following nonlinear differential equation

$$u_{xx} - u_{tt} - \sin u = \left(\frac{\sigma}{2}\right)^2 \left[u_{xx}(1+\cos u) - \frac{1}{2}u_x^2 \sin u\right]. \qquad (1)$$

Introducing a new variable $z = \frac{2}{\sigma}x$ we obtain

$$u_{tt} + (\cos u - \delta)\, u_{zz} + \left(1 - \frac{1}{2}u_z^2\right) \sin u = 0$$

where

$$\delta = \frac{4 - \sigma^2}{\sigma^2} \cong \frac{4}{\sigma^2} \gg 1.$$

Eq.(1) was independently derived in Ref.[4] for $\delta < 1$ starting from another point of view. This fact allows us to analyse Eq.(1) for any value of the parameter δ not discussing the origin of the equation.

First of all consider the spectrum of small vibrations described by Eq.(1) under the condition $|u| \ll 1$ when

$$u(x,t) = u_0 \exp(ikx - i\omega t) \tag{2}$$

and the dispersion law has the following form

$$\omega^2 = 1 + (\delta - 1)\, k^2. \tag{3}$$

Restrict our consideration to the case $\delta \succ 1$. We see that the minimum phase velocity of the wave (2) is equal to $\sqrt{\delta - 1}$ and the expected velocity of the kink-like excitation must be less than $\sqrt{\delta - 1}$.

The kink solution travelling with a velocity V is described by the equation

$$(\delta - V^2 - \cos u)\, u_{zz} = \sin u \left(1 - \frac{1}{2}u_z^2\right) \tag{4}$$

where $u(z,t) = u(z - Vt)$.

The ordinary 2π-kink corresponds to the first integral of Eq.(4).

$$\frac{1}{2}(\delta - \cos u - V^2)\, u_z^2 = 1 - \cos u \tag{5}$$

and has the following form

$$\pm(\xi - \xi_0) = \sqrt{2} \arccos \frac{\rho}{\left(1 + \frac{\alpha}{2}\right)^{\frac{1}{2}}} + \frac{\alpha^{\frac{1}{2}}}{2} \ln \frac{(1-\rho)\left[\alpha + 2 + 2\rho + \alpha^{\frac{1}{2}}\left(\alpha + 2 - 2\rho^2\right)^{\frac{1}{2}}\right]}{(1+\rho)\left[\alpha + 2 - 2\rho + \alpha^{\frac{1}{2}}\left(\alpha + 2 - 2\rho^2\right)^{\frac{1}{2}}\right]} \tag{6}$$

where

$$u = 2\arccos\rho, \alpha = \delta - 1 - V^2, \xi = z - Vt.$$

However Eq.(4) has also another solution

$$u_{zz} = 0, u_z^2 = 2 \tag{7}$$

which gives the nonlinear dependence of the function $u(z)$ on the coordinate:

$$u(z) = \pm\sqrt{2}(z - z_0) \tag{8}$$

and corresponds to the homogeneous compression or expansion of the chain.

This solution does not allow us to construct a solution-like excitation of the chain under consideration with the exception of the case $V^2 = \delta - 1$. In such a case Eq.(4) transforms into the following one

$$tg\left(\frac{u}{2}\right) u_{\xi\xi} = 1 - \frac{1}{2}u_\xi^2$$

and we can combine the solution (6) and (9) to construct the exotic 2π-kink

$$u(\xi) = \begin{cases} 0, \xi < \xi_0; \\ \sqrt{2}(\xi - \xi_0), \xi_0 < \xi < \xi_0 + \sqrt{2}\pi : \\ 2\pi, \xi > \xi_0 + \sqrt{2}\pi. \end{cases} \tag{9}$$

where ξ_0 is an arbitrary constant.

Besides the kink-like solutions considered above Eq.(1) possesses another soliton solutions. We mean localized vibrations of the breather type. Such solutions can be found using the asymptotic method proposed in Ref.[5].

Consider small amplitude localized vibrations assuming $|u| \ll 1$. Then Eq.(1) can be reduced to the following one

$$u_{tt} + u - (\delta - 1)u_{xx} - \frac{1}{6}u^3 - \frac{1}{4}u\left(u^2\right)_{xx} = 0. \tag{10}$$

It is interesting to note that Eq.(10) has the same form as the nonlinear equation obtained in Ref.[6] and describing localized short wave length vibrations of the anharmonic crystal chain.

If ω is the fundamental frequency of the localized excitation we suppose $1 - \omega^2 = \epsilon^2 \ll 1$. Then we can write down the asymptotic expansion

$$u(x,t) = A(x)\sin\omega t + B(x)\sin 3\omega t + ... \tag{11}$$

where

$$A(x) \sim \epsilon, B(x) \sim \epsilon^3$$

The amplitude of the first harmonic of the expansion is described by the following nonlinear equation:

$$\left(1-\omega^2\right) A - (\delta - 1) A_{xx} - \frac{1}{8} A^3 - \frac{3}{16} A \left(A^2\right)_{xx} = 0. \tag{12}$$

Eq.(12) has the following solution

$$x - x_0 = \sqrt{\frac{6\eta_0}{\eta_m}} arc \tanh\left(\sqrt{\frac{\eta_m}{\eta_0}}\Phi\right) + \sqrt{6} arc \cot \Phi \tag{13}$$

where

$$\eta_0 = \frac{8(\delta - 1)}{3}, \eta_m = 16\epsilon^2, \Phi = \left(\frac{\eta_0 + u^2}{\eta_m - u^2}\right)^{\frac{1}{2}}.$$

Consider two limiting cases: (1) $\delta \gg 1$ and (2) $\delta = 1$.
In the first case ($\delta \gg 1$)

$$A(x) = \frac{4\epsilon}{\cosh\left[\frac{\epsilon(x-x_0)}{\sqrt{\delta}}\right]}. \tag{14}$$

In the second case ($\delta = 1$)

$$A(x) = \begin{cases} 4\epsilon \cos\left[\frac{x-x_0}{\sqrt{6}}\right], |x - x_0| < \pi\sqrt{\frac{3}{2}}; \\ 0, |x - x_0| \geq \pi\sqrt{\frac{3}{2}}. \end{cases} \tag{15}$$

Eq.(15) gives an example of the so-called compactons.

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FAMILIES OF DISCRETE SOLITARY WAVES IN NONLINEAR SCHRÖDINGER EQUATIONS

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The physical descriptions of many nonlinear systems result in discrete models. Typical examples are: optical pulse propagation in arrays of coupled optical waveguides, proton dynamics in hydrogen-bonded chains, transport of exitation energy in biophysical sytems, Scheibe aggregates, the Hubbard model, electrical lattices, DNA dynamics, molecular crystals, and so on. In this contribution we investigate some new techniques for the existence and stability of discrete solitary waves, based on a one-dimensional discrete nonlinear Schrödinger equation. We start with a discussion of stationary states by examining generating functions. The latter technique makes use of solutions of a continuous difference equation and allows for solutions with different symmetry properties. Actually, whole families of stationary solutions are found. In the second part we investigate the stability properties of the newly found solutions.

1 Introduction

Nonlinear localized modes in discrete systems have been a subject of intense but mainly numerical investigations during the past years[1-10]. Different types of localized states were found, and very elegant and efficient schemes have been developed for calculating solitary wave solutions. The broad and discrete solutions may be approximated by the corresponding continuum solutions, but there exist other types of discrete modes that definitely will not obey the continuum limit. Some of the latter show stable behavior in numerical experiments. However, from the principle point of view, numerical simulations cannot prove stability in the strict sense. Thus, analytical or at least semi-analytical criteria are strongly needed, and it is the primary motivation of this paper to develop systematic analytical methods for examining the existence and stability of solitary wave solutions in discrete nonlinear systems.

We choose a DNS equation with arbitrary power nonlinearity because the continuous one-dimensional nonlinear Schrödinger equation also posseses solitary wave solutions when its power nonlinearity is changed from cubic to other algebraic forms. Such different types of nonlinearities might appear for at least two reasons: (i) The physical model may require a strong anharmonic coupling which does not result in a cubic nonlinearity. The latter is a characteristic of the integrable Schrödinger equation. (ii) From the mathematical point of view,

it may be advisable to raise the exponent of the (cubic) power nonlinearity in order to mimic the multi-dimensional behavior of the Schrödinger model.

Solitary wave solutions of a one-dimensional continuous nonlinear Schrödinger equation with arbitrary power nonlinearity can be stable (corresponding, e.g., to the stable solitons) or unstable. The latter means that dispersion balances nonlinear steepening only in the stationary case. Small perturbations around the solitary wave may break this balance leading to instability and perhaps collapse. Because of that rich dynamical behavior, our DNS model will be the exact discrete analogue of the general continuous nonlinear Schrödinger equation.

Let us, for a moment, consider the continuum nonlinear Schrödinger equation

$$i\partial_t \psi + \partial_x^2 \psi + (\sigma+1)|\psi|^{2\sigma}\psi = 0 \; . \tag{1}$$

This equation reduces for $\sigma = 1$ to the famous cubic nonlinear Schrödinger equation. At this stage we would like to mention already that in the following we shall always consider the focussing case [with a plus sign in front of the nonlinearity of Eq. (1)], but all the considerations will also work in the de-focussing case. The (continuous) Schrödinger equation (1) posseses, for arbitrary power nonlinearities, stationary solitary wave solutions. If we introduce

$$\psi(x,t) \equiv G(x)\exp(i\eta^2 t) \; , \tag{2}$$

with the stationary envelope $G(x)$ and a nonlinear frequency shift η^2, stationary localized solutions of Eq. (1) can be presented explicitly, i.e.

$$G(x) = \eta^{1/\sigma}\mathrm{sech}\left[\sigma\eta(x-x_0)\right] \; , \tag{3}$$

where x_0 is a free parameter. When we now turn to a discrete version of Eq. (1) [which will be the main object of demonstration in this paper],

$$i\partial_t \psi_j + \psi_{j+1} - 2\psi_j + \psi_{j-1} + (\sigma+1)|\psi_j|^{2\sigma}\psi_j = 0 \tag{4}$$

for $j = 0, \pm 1, \pm 2, \ldots$ and with boundary conditions $|\psi_j| \to 0$ for $|j| \to \infty$, the situation has completely changed. We do not know analytically any non-trivial (localized) solitary wave solution. Of course numerically solutions have been found (also for finite systems with Dirichlet or periodic boundary conditions). We shall discuss some strategy to construct stationary solutions of Eq. (4) in the form

$$\psi_j = G_j \exp(i\lambda t) \tag{5}$$

in the next section.

2 Generating functions for families of stationary solutions

We will now outline a general procedure to construct stationary solutions. Consider the difference equation ($-\infty < x < +\infty$):

$$-F(x+1) + 2F(x) - F(x-1) + \lambda F(x) = (\sigma+1)\,[F(x)]^{2\sigma+1}\,, \qquad (6)$$

where here and in the following we assume vanishing boundary conditions at infinity. Note that the structure of Eq. (6) is determined by the structure of the equation for the stationary states G_j. Two facts are important: First, the existence of solutions to Eq. (6) can be proven by making use of the formula

$$F(x) = e^{-\delta x} - \frac{\sigma+1}{\sinh(\delta)} \sum_{j=1}^{\infty} [F(x+j)]^{2\sigma+1} \sinh(\delta j)\,, \qquad (7)$$

where δ is determined via the ansatz $F(x) \sim e^{-\delta x}$ in the linear regime $x \to \infty$, i.e. with $\eta = \sqrt{\lambda}$:

$$\sinh\frac{\delta}{2} = \frac{\eta}{2}\,. \qquad (8)$$

One can show that the Fredholm type equation (7) can be solved by iteration,

$$F_{n+1}(x) = e^{-\delta x} - \frac{\sigma+1}{\sinh(\delta)} \sum_{j=1}^{\infty} [F_n(x+j)]^{2\sigma+1} \sinh(\delta j)\,, \qquad (9)$$

with $F_0 \equiv e^{-\delta x}$. The iterates show an alternating and converging behavior for large x, i.e. $0 < F_1 < F_0$, $F_1 < F_2 < F_0$, $F_1 < F_3 < F_2$, and so on. The solution of (7) obeys Eq. (6).

Secondly, after knowing that a solution to Eq. (6) exists, the latter can be easily obtained numerically. Then, from $F(x)$ solutions to G_j can be obtained in a straightforward manner.

3 Explicit forms of families of solutions

We have solved Eq. (6) with vanishing boundary conditions for $x \to +\infty$. Typical results are shown in Fig. 1. Starting from the asymptotic solution (in the linear regime) the numerical evaluation is quite simple, and one is not faced with any numerical problems. Of course, the numerics will fail for $x \to -\infty$ but as we shall show below, that behavior is not needed for the determination of

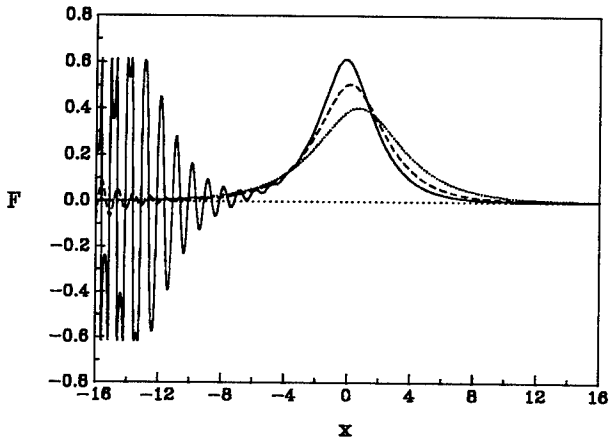

Fig.1: Generating functions for $\sigma = 1$, η equal to 0.4 (dotted line), 0.5 (broken line), and 0.6 (solid line), respectively.

most members of the families of solutions. The general form of the generating function is quite surprising at the first glance. It has an oscillatory behavior, which allows to construct several types of solutions in the form

$$G_j = G(j + \xi) \tag{10}$$

with properly chosen ξ values. The oscillatory behavior is, however, directly related to the absence of a continuous translation symmetry. In the language of dynamical systems, the oscillations belong to a homoclinic structure in the neighbourhood of a separatrix, and the solitary waves can be viewed as points on this structure. First, let us look for symmetric solutions being centered on-site. We define the auxiliary function

$$F_{so}(x) := G(x + 1) - G(x - 1) \, . \tag{11}$$

Its zero points ξ_i^{so} are easy to determine,

$$F_{so}(\xi_k^{so}) = 0 \, . \tag{12}$$

Specifying the index k (out of the family of zeros for $k = 0, 1, 2, ...$) we define

$$G_j^{(k)} = G(\xi_k^{so} + j) \text{ for } j \geq 0 \, , \quad G_j^{(k)} = G_{-j}^{(k)} \text{ for } j < 0 \, . \tag{13}$$

This set is a solution and has the desired symmetry properties by construction.

Now it is clear that due to the existence of a whole set of zero points a whole family of centered symmetric discrete solutions exists. Next, symmetric solutions with inter-site centers are calculated. We define

$$F_{si}(x) := G(x) - G(x-1) \tag{14}$$

and solve for

$$F_{si}(\xi_k^{si}) = 0\ , \quad \text{for } k = 0, 1, 2, ... \tag{15}$$

The solutions G_j are obtained from

$$G_j^{(k)} = G(\xi_k^{si} + j) \text{ for } j \geq 0\ , \quad G_j^{(k)} = G_{-j-1}^{(k)} \text{ for } j \leq -1\ . \tag{16}$$

The third type of solutions, being anti-symmetric and centered on-site, follows by

$$F_{ao}(x) := G(x)\ , \quad F_{ao}(\xi_k^{ao}) = 0\ ,\ k = 0, 1, 2, ... \tag{17}$$

$$G_j^{(k)} = G(\xi_k^{ao} + j) \text{ for } j \geq 0\ , \quad G_j^{(k)} = -G_{-j}^{(k)} \text{ for } j < 0\ . \tag{18}$$

Finally, we determine the family of anti-symmetric and inter-site centered solutions. They follow from

$$F_{ai}(x) := G(x) + G(x-1)\ , \quad F_{ai}(\xi_k^{ai}) = 0\ ,\ k = 0, 1, 2, ... \tag{19}$$

It is straightforward to construct

$$G_j^{(k)} = G(\xi_k^{ai} + j) \text{ for } j \geq 0\ , \quad G_j^{(k)} = -G_{-j-1}^{(k)} \text{ for } j \leq 0\ . \tag{20}$$

This completes the discussion on stationary localized solutions of the discrete nonlinear Schrödinger equation. The stability of these solutions is considered next.

4 Stability considerations

We now go back to the time-dependent Eq. (4) in order to discuss the dynamical behavior of the just found stationary solutions in the presence of perturbations. Introducing

$$\psi_j = (G_j + a_j + ib_j)e^{i\lambda t}\ , \tag{21}$$

and using the operators H_+ and H_-,

$$(H_+\phi)_i := -\phi_{i+1} + 2\phi_i - \phi_{i-1} + \lambda\phi_i - (\sigma+1)|G_i|^{2\sigma}\phi_i\ , \tag{22}$$

$$(H_-\phi)_i := -\phi_{i+1} + 2\phi_i - \phi_{i-1} + \lambda\phi_i - (2\sigma+1)(\sigma+1)|G_i|^{2\sigma}\phi_i$$

$$= (H_+\phi)_i - 2\sigma(\sigma+1)|G_i|^{2\sigma}\phi_i \,, \tag{23}$$

we find in the linear limit

$$\partial_t^2 a_j = -(H_+ H_- a)_j \,. \tag{24}$$

We know that for symmetric ground state solutions G_j

- H_+ is positive semi-definite with $(H_+ G)_j = 0$;
- H_- has only one negative eigenvalue, and $\sum_j G_j (H_- G)_j < 0$.

Under these conditions it is well-known[11-14] that instability occurs provided

$$\Gamma^2 := \sup_{C(\varphi)} \frac{-\sum_j \varphi_j (H_-\varphi)_j}{\sum_j \varphi_j (H_+^{-1}\varphi)_j} > 0 \,, \tag{25}$$

where the supremum is determined for all possible φ_j under the condition

$$C(\varphi): \quad \sum_j \varphi_j G_j = 0 \,. \tag{26}$$

Of course, the occurence of instability depends on the further properties of H_-. If, and only if, under the condition (26) the expression $\sum_j \varphi_j (H_-\varphi)_j$ can become negative, instability will occur. Easier to calculate for the latter behavior is the condition

$$\sum_j G_j (H_-^{-1} G)_j > 0 \,. \tag{27}$$

Since

$$\left(H_- \frac{\partial}{\partial\lambda} G\right)_j = -G_j \tag{28}$$

we can reformulate (27) as

$$\frac{\partial}{\partial\lambda} \sum_j G_j^2 < 0 \quad \Rightarrow \quad \text{instability for even perturbations} \tag{29}$$

Let us comment once more on an additional restriction. The definiteness properties being used here assume symmetric (even parity) ground states with centers either on-site or inter-site. Thus the criterion (29) gives an answer to the question of the (initial time-) dynamics of an even ground state with respect to even perturbations, i.e. perturbations of the same parity.

Now we briefly present the results of the evaluation of (29) which are depicted in Fig. 2.

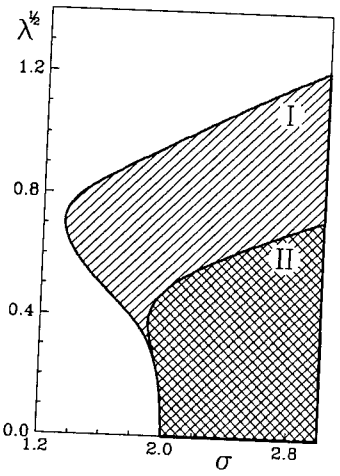

Fig.2

We have stable and unstable regimes which are separated in Fig. 2 by the border lines named I (for type-I solutions, i.e. with even parity and maximum on-site) and II (for type-II solutions, i.e. with even parity and maximum inter-site), respectively. The localized ground states of types I or II are unstable in the right neighbourhoods of the curves marked I or II, respectively, i.e. in the hatched areas. One can see that the discreteness changes the critical value (σ_{cr}) of σ that separates stable and unstable solitons. In the continuum limit $\sigma_{cr} = 2$. Here we find $\sigma_{cr} \approx 1.4$.

Perturbations can have other symmetry properties, i.e. the stationary solutions can be even more unstable, and then we need additional informations which usually are only available through numerical calculations. In the case of ground states of types I or II the stability investigations with respect to arbitrary perturbations can be based on a discussion of the spectral properties of H_-. But for the more general cases, i.e. all the solutions constructed by the generating functions, we have to rely on (simple) numerical procedures to determine the spectral behaviors of H_+ and H_-. We do not discuss more special cases separately but summarize the results.

For ground states of type I

no additional negative eigenvalue of H_- enters the stability considerations, compared to the situation discussed above. Thus, curve I of Fig. 2 is the exact (and completely general) stability boundary.

The situation is different for ground states of type II. These solutions are always unstable with respect to odd perturbations.

Other members of the families starting from types I or II, respectively, (i.e. those being constructed from F_{so} and F_{si}, respectively) are in general not stable, although for some of the solutions with even parity the possible upper limits for the growth rates are so small that physically those solutions can be considered as quasi-stable. On the contrary, all the even symmetric solutions, centered inter-site, are unstable. Note again that some of the so-called unstable solutions have extremely small growth rates so that from the application point

of view they may be called quasi-stable.

For the odd symmetric solutions our analyis has shown that all these solutions are unstable. The calculated growth rates are significant so that there are no quasi-stable solutions.

The methods presented here can be generalized to other physical systems. For Klein–Gordon and Sine–Gordon systems e.g. we have found completely stable states, which do not exist in the continuum limit.

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The Spectral Transform on the Finite Interval and Interpretation of SRS Experiments

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The spectral transform is used for solving the equations of transient stimulated Raman scattering (SRS) on finite interval $x \in [0, L]$, for arbitrary boundary values. The results are used to interpret the long-pulse experiments of Drühl, Wenzel and Carlsten [1].

This contribution is a short summary of results obtained recently on the nonlinear theory of stimulated Raman scattering of high energy long laser pulses in two-level media [2] by using the spectral transform method extended to evolution with non-analytic dispersion relations and arbitrary boundary values [3] and applying it to the *finite interval* problem.

This provides a unified global interpretation of the experiments of Drühl, Wenzel and Carlsten [1] as it is proved that the model, resulting from the slowly varying envelope approximation, and including phase mismatch and detuning from Raman resonance, is solvable. The essential variable in the theory is the phase of the Stokes pulse relatively to the pump. For long duration input pump pulses and when pump and Stokes pulses experience a rapid π-phase shift, an anomalous spike of pump radiation is created in the pump depletion zone. Since its first observation in 1983 [1], the Raman spike has stimulated a large amount of studies both on the experimental and theoretical sides.

The main reason for such an interest in the Raman spike is that it is the macroscopic manifestation of large fluctuations of the phase of the initial Stokes wave. Raman spike generation has been predicted [4], has been given a coherent-mode description [5], and experiments have been preformed where SRS grows spontaneously on initial fluctuations [6]. The Raman spike hence appears as a means to study the quantum properties of the Stokes wave initiation, which gives informations on the *phase of the electromagnetic vacuum.* This comes in addition to the process of Stokes growth which amplifies the quantum fluctuations of the medium. A general discussion of the *quantum coherence properties of SRS* is given in [7].

After many attempts to model the Raman spike as a soliton [8,9], it has been proved that actually the Raman spike occurs in the spectral transform theory as a manifestation of the continuous spectrum only (hence it is not a

soliton) when for a short period of time the *reflection coefficient* becomes close to zero[2]. It has been shown in particular that the decay of Raman spikes is related to the *velocity* of the π-phase shift of the initial Stokes seed.

When a laser pump pulse (frequency ω_L, envelope A_L) interacts with the optical phonons (eigenfrequency ω_V, envelope Q) to give rise to a down-shifted laser pulse (Stokes emission, frequency ω_S, envelope A_S) according to the selection rules

$$\omega_S = \omega_L - \omega_V + \Delta\omega, \quad k_S = k_L - k_V + \Delta k, \tag{1}$$

the slowly varying envelope approximation of the equations governing the interaction of light with a material medium can be written (for technical details see [2])

$$\partial_x A_L = q A_S \, e^{-i\Delta\omega t} \, e^{2ikx}, \quad \partial_x A_S = -q^* A_L \, e^{i\Delta\omega t} \, e^{-2ikx}, \tag{2}$$

$$q_t + \gamma q = -e^{i\Delta\omega t} \int dk g(k) A_L A_S^* \, e^{-2ikx}, \tag{3}$$

with

$$q(x,t) = i\frac{N\alpha_0'}{2\eta}\frac{\sqrt{\omega_S\omega_L}}{c^2} \, Q(x,t), \quad 2k = \Delta k - \frac{\eta}{c}\,\Delta\omega. \tag{4}$$

Hereafter $\Delta\omega$ is the frequency mismatch which originates in the detuning of the Stokes seed from the exact Raman resonance, Δk is the phase mismatch which comes from the inhomogeneous broadening of the spectral width of the response of the medium, (k is the *essential mismatch*). Next, α_0' is the differential polarizability at equilibrium, c/η is the light velocity in the medium and N is the density of oscillators. The damping coefficient γ is phenomenologically introduced to represent homogeneous broadening effects while the distribution $g(k)$ (even function) represents the inhomogeneous broadening and it is sharply distributed around $k = 0$ (the resonance). Finally t denotes the retarded time and x is the usual physical space variable.

For a medium initially in the ground state

$$q(x,0) = 0, \tag{5}$$

and for an arbitrary set $I_L(k,t)$, $I_S(k,t)$ of input pump and Stokes envelopes profiles

$$x = 0 \; : \; A_L = I_L, \quad A_S = I_S, \tag{6}$$

the *explicit output values* of both light waves profiles $A_L(k,L,t)$ and $A_S(k,L,t)$ read

$$x = L \; : \; A_L = \frac{1}{\tau}\left(I_L - \rho I_S e^{-i\Delta\omega t}\right), \quad A_S = \frac{1}{\tau^*}\left(I_S + \rho^* I_L e^{i\Delta\omega t}\right). \tag{7}$$

This formula actually gives the solution to the physical problem (compute the output values from the input values) as soon as the function $\rho(k,t)$ is calculated. The parameter τ is expressed from ρ through

$$\tau = \sqrt{1+|\rho|^2}\, e^{i\theta}, \tag{8}$$

$$\theta = -\frac{1}{2\pi} P\!\int \frac{d\lambda}{\lambda - k}\, \ln(1+|\rho|^2) \tag{9}$$

The evolution of the main parameter $\rho(k,t)$ reads in the undamped case ($\gamma = 0$)

$$\rho_t = \rho\left(\frac{\pi}{2} g(|I_L|^2 - |I_S|^2) - \frac{i}{2} P\!\int \frac{d\lambda}{\lambda - k}\, g(|I_L|^2 - |I_S|^2)\right) + \pi g I_L I_S^* e^{i\Delta\omega t}. \tag{10}$$

A medium initially at rest corresponds to $q(x,0) = 0$, and hence to $\rho(k,0) = 0$. Consequently the above equation can be solved explicitly an furnishes the explicit values of the output (7). This is an exact solution of the system (2)(3) for $\gamma = 0$.

The above time evolution of ρ can also be derived from the linear limit obtained for q and A_S small, for which $\rho(k,t)$ becomes the Fourier transform of $q(x,t)$. Then the natural modification of the evolution of $\rho(k,t)$ to account for the damping coefficient γ, consists in replacing hereafter ρ_t with $\rho_t + \gamma\rho$ such as to respect the linear limit. The resulting general solution reads

$$\rho(k,t) = \rho(k,0) e^{\phi(k,t)} - \int_0^t dt' \Lambda(k,t') e^{\phi(k,t) - \phi(k,t')} \tag{11}$$

$$\phi(k,t) = -\gamma t + \int_0^t dt" \, [\frac{\pi}{2} g(|I_L|^2 - |I_S|^2) - \frac{i}{2} P\!\int \frac{d\lambda}{\lambda - k}\, g(|I_L|^2 - |I_S|^2)] \tag{12}$$

$$\Lambda(k,t) = \pi g I_L I_S^*\, e^{i\Delta\omega t}. \tag{13}$$

For the vanishing initial datum the above solution depends only of the inputs I_L and I_S, but more important, depends on the length L *only through* γ (which is proportional to L/T_2, where T_2 is the mean collisional dephasing time, see e.g. [4]).

To conclude we apply the above formula in the sharp line limit obtained for $g(k) \to g_0\delta(k)$, that is for a distribution $g(k)$ very sharp around $k = 0$. Then we evaluate all quantities in $k = 0$ and, because $g(k)$ is even, the integral in (12) vanishes in $k = 0$ which simplifies the formula. The resulting output

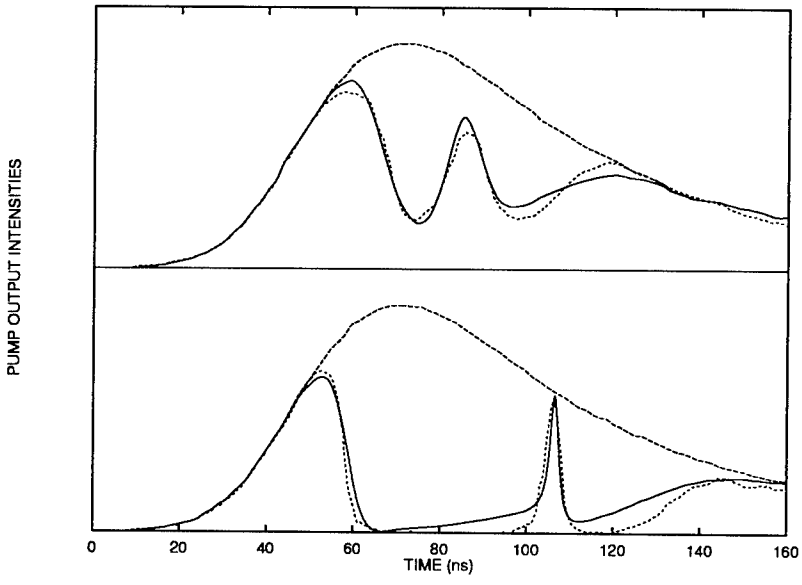
modulus can be written in that case (we denote $\rho(0,t)$ by $\rho(t)$ and $A_L(0,x,t)$ by $A_L(x,t)$):

$$|A_L(L,t)|^2 = \frac{1}{1+|\rho(t)|^2}\ |I_L(t) - I_S(t)\rho(t)e^{-i\Delta\omega t}|^2, \tag{14}$$

with

$$\rho(t)e^{-i\Delta\omega t} = \pi g_0 \int_0^t dt'\ I_L(t')I_S^*(t')\ e^{\Delta(t,t')}, \tag{15}$$

$$\Delta(t,t') = -(i\Delta\omega + \Gamma)(t-t') + \frac{\pi}{2}g_0 \int_{t'}^t d\tau(|I_L(\tau)|^2 - |I_S(\tau)|^2). \tag{16}$$



The figure shows the application of our model, namely formulae (14) with (15),

to the experimental input of [1]. *In both figures, the dot lines are the drawing of the digitalized experimental data. The solid lines represent the output pump intensity obtained through (14) with the parameter values indicated in the text.*

We reproduce in the figure the pump output when the input pump pulse envelope is precisely that of the experiments of Drühl, Wenzel and Carlsten [1]. Thanks to the courtesy of J.L. Carlsten we have used in the above formula the digitalized data of the 1983 experiment for $I_L(t)$ together with the following parameters for the upper figure

$$g_0 = 8.1, \qquad \gamma = 12.6, \qquad I_S(t) = 0.345 \exp[-\frac{(t-90)^2}{600}] \exp[i\zeta(t)] \tag{17}$$

$$\zeta(t) = \frac{\pi}{2}[1 + \tanh[0.11875(t - 76.64)]]. \tag{18}$$

and for the lower figure:

$$g_0 = 1.245, \qquad \gamma = 1.4, \qquad I_S(t) = 0.294 \exp[-\frac{(t-100)^2}{1200}] \exp[i\zeta(t)] \tag{19}$$

$$\zeta(t) = \frac{\pi}{2}[1 + \tanh[0.1875(t - 64)]]. \tag{20}$$

In both cases the detuning $\Delta\omega$ is set to zero. The experimental output is drawn (dashed line) for comparison.

Comments

The fact that IST can be applied to SRS on the finite interval has been first proposed by Kaup in 1983 [9]. However the evolution of the spectral data given there does not correspond to the boundary problem of physical interest (actually the related boundary values was not specified). In particular, as this evolution was *homogeneous*, it does not allow for the growth of the Stokes seed on a medium initially at rest.

IST has been later used to solve an initial-boundary value problem on the half-line for the nonlinear Schrödinger equation (NLS) by Fokas [10], but in this case the required data in $x = 0$ renders *nonlinear* the evolution of the spectral transform.

The property of SRS of being solvable on the finite interval results simply from the nature of the equations (2)(3) for which the initial-boundary value problem (5)(6) is well posed and does not require new constraints when it is given on the finite interval (this is not so for NLS for which the vanishing boundary values at infinity become some prescribed boundary value in $x = 0$). Consequently the method applies for every other case of solvable evolutions with nonanalytic dispersion relations when precisely passing to the finite interval does not imply adding information or constraint.

Other examples of such solvable nonlinear coupled evolutions on the finite interval will be published soon, in particular the *discrete* version of SRS [11].

Acknowledgements

It is a pleasure to thank A.V. Mikhailov for enlightening discussions, and J. Carlsten for having sent the digitalized data of his experiments.

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ANOMALOUS LEVY PROCESSES AND SIGN-SINGULARITY IN ELECTRO-CONVECTIVE TURBULENCE

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We report the scaling laws obtained by investigating the motion of dielectric testing particles in turbulent Nematic Liquid Cristal films.

1 Introduction

The motion of testing particles in turbulent flows is an interesting field of research because it gives informations on many physical phenomena. In particular the physical behavior of some systems cannot be described by the laws of the classical diffusive motion, rather the phenomenon can be described by a Lévy process [1]. If the PDF of a series of independent random variables τ behaves like a power–law $P(\tau) \sim \tau^{-(1+\gamma)}$ we can have different phenomena. In fact when $0 < \gamma < 1$ a walker can perform very long jumps before to be deviated (sub–diffusive or ballistic motion), while if $1 < \gamma < 2$ the tracers can have long jumps and Lévy phenomena occur. This occurs in situations where the broadness of the distribution $P(\tau)$ is dictated by the physical nature of the problem, and the process is intermittent since rare events have a probability of occurrence grater than a Gaussian distribution. In the present paper we experimentally study the motion of tracers in MBBA Nematic Liquid Crystals (NLC) under turbulent electro–convection. In this situation the wandering of buoyant particles is altered by some sticking regions randomly distributed in times and in space. The sticking is due to convective vortices of different amplitude which originate a broad PDF both in the sticking and flight times. This can be studied as a Lévy phenomenon. Moreover we will study the behavior of the stickings and flights alternate by using the concept of singularities of signed measures.

2 The Lévy process

The experimental setup has been reported in previous papers [2] [3]. The sample cell consists in a sandwich–type cell: a flat capillary of MBBA is formed from two square glass plates (with a thickness $d = 36$ μm) with Indium-thin Oxide transparent electrodes. The sample is illuminated by a white light beam polarized along the alignment direction, and is observed by a microscope connected

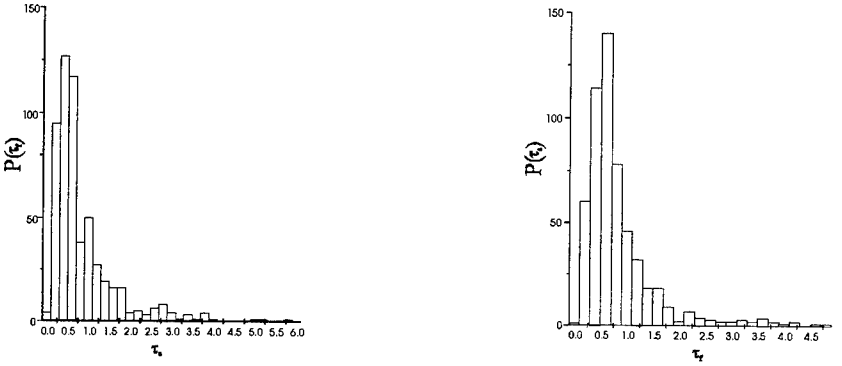

Figure 1: The probability distribution histogram for the times $\tau_s(n)$ and $\tau_f(n)$.

to the images acquisition system. Dielectric buoyant particles (with a diameter of about 1 μm) are suspended in the NLC. An oscillating electric field $\vec{E}_0$ at a fixed frequency $\omega = 70$ Hz. is applied across the sample. Under the action of the applied electric field the electro–convective phenomenon makes the particles to move [3], the behavior depends on the value of the applied voltage V_0. The motion of a large number of these particles are tracked up to $T = 30$ min., making sure that the focal plane of the microscope coincides with the upper surface of the sample [2]. In the weak turbulent regime [2] some random vortices originate in the NLC. These vortices are randomly distributed both in time and space, so that their structure tends to stick the particle into the bulk of the NLC for a certain time. The sticking of the particles alternates with periods where they float on the surface. We collected the series $\tau_f(n)$ $(n = 1, 2, .., N)$, made by the sequence of the subsequent N flight times during which a given particle buoyant on the surface, and the series $\tau_s(n)$, made by the subsequent time intervals between two flights. Both the series appear to be randomly distributed [3], and this is a signature of the random occurrence of the vortices in the turbulent regime. The PDF's of both the sticking and flight times are determined from normalized histograms of both τ_s and τ_f for all the particles, whose tails are used to determine the relations $P(\tau_s) \sim \tau_s^{-\nu}$ and $P(\tau_f) \sim \tau_f^{-\mu}$. In figure 1 we show the histogram of $P(\tau_s)$ along with the power–law fit. The tail of the PDF indicates that the process is not gaussian, that is long–times are favored with respect to a process which follows a gaussian PDF. The best–fit scaling exponent we found is given by $\nu \simeq 2.2 \pm 0.1$, that is the sticking process appears to be a Lévy process [1]. The flight times show the same behavior. In figure 1 we show also the histogram of $P(\tau_f)$

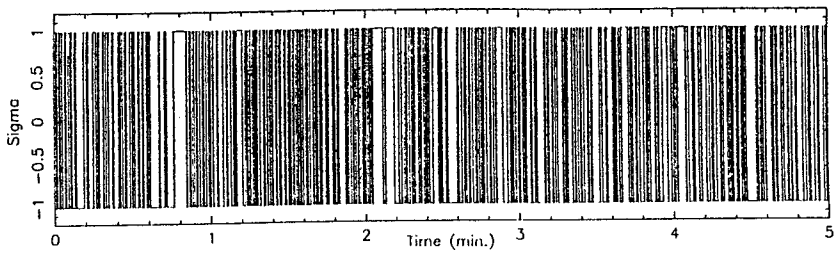

Figure 2: We show the field σ_n vs. n at $V_0 = 30$ Volts.

along with the fit with a power–law obtained with $\mu \simeq 2.7 \pm 0.1$.

3 Sign-singular measure

How the stickings and flights alternate on all the dynamical scales of time can be described by using the concept of singularities of signed measures [4]. Let us consider the behavior of a particle and let us built up the random series made by the "spin–like" field σ_n, where $1 \leq n \leq T/\Delta t$, and $\sigma_n = -1$ when at the time $n\Delta t$ the particle sticks into the bulk of the NLC, while $\sigma_n = 1$ if at the time $n\Delta t$ the particle flights on the surface. The field σ_n is plotted in figure 2. The oscillating phenomenon can be described with a signed measure $\mu(T)$[5]. These measures can be sign-singular, and can be described by a cancellation exponent [4]. Let us consider the sequence of disjoint sub–intervals $S_i \subset T$ with equal lengths r, and let us define the signed measure $\mu_r(S_i) = \sum_{n \in S_i} \sigma_n / |\sum_n \sigma_n|$ where the sum in the numerator is made on the points belonging to S_i. By introducing the partition function $\chi(r) = \sum_i |\mu_r(S_i)|$ (the sum is extended to all the intervals of a given scale r) the cancellation exponent κ can be defined through $\chi(r) \sim r^{-\kappa}$. The increase of $\chi(r)$ when r decreases occurs only because cancellation of positive and negative contributions are reduced for smaller r. In this case $\kappa > 0$ and the measure is said to be sign–singular. Ott et al. [4] found some example of physical situations where the cancellation exponents is positive, and a lot of papers appeared on the subject in the last years (for a review see Carbone [6]). The presence of sign–singularity is shown in figures 3, in which we plot the curves $\log \chi(r)$ vs. $\log(1/r)$ for three values of the external voltage V_0, calculated for a given particle. The cancellation exponents are given by the slopes of the curves in the linear range. For a fixed voltage we found no differences in the slope between different particles. The different behavior of the signed measure, which depends on the applied voltage, is interesting from a physical point of view. As it can be seen for low values of V_0 (figure 4a), when the turbulence is not fully developed, the

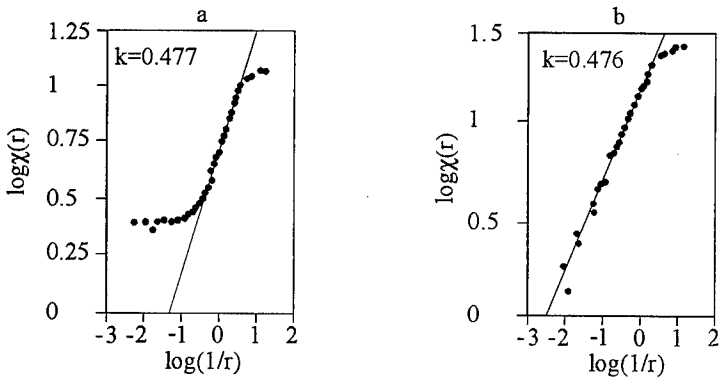

Figure 3: The values of $\log \chi(r)$ vs. $\log(1/r)$ for a given particle, for a) $V_0 = 14$ Volts, where the field is not fully turbulent, and b) $V_0 = 30$ Volts.

range where the cancellation exponent can be calculated is very limited. On the contrary the singularity is well evidenced by the increase of $\chi(r)$ over more than two decades as the turbulence is developed. We found the value $\kappa \simeq 0.48$, which is in agreement with the cancellation exponent found in random fields describing the Brownian motion. The difference which is visible between figure 4a and figures 4b at large time scales, is due to the fact that the flow is not yet fully turbulent at lower values of the applied voltage. In fact some large–scale structures are visible on the microscope [3]. These structures yield an almost complete cancellation of positive and negative contribution of the measure at the largest time scales. This is to say that the sticking and flight motion of the particle over large times, is driven by the distorted Williams domain, and is almost periodic.

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DESTRUCTION OF A FUNDAMENTAL SOLITON IN A PERIODICALLY MODULATED WAVEGUIDE

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We report results of systematic numerical simulations of the nonlinear Schrödinger equation whose dispersion coefficient is a periodic function of the propagation distance. We demonstrate that the solitons whose dispersion length is comparable to the modulation period suffer a sudden splitting into a pair of secondary solitons after an initial period of steady evolution. The splitting takes place provided that the modulation amplitude exceeds a certain critical value. Drawing a separatrix between the nonsplitting and splitting solitons in a parametric plane, we find a complicated structure. A set of stability islands is found inside the splitting regime. Furthermore, we demonstrate that, at large values of the modulation amplitude, there exists a stability "isthmus" between two splitting areas, in which the soliton retrieves its stability. Unexpectedly, the soliton's stability is again restored at large values of the modulation amplitude.

1 INTRODUCTION

The current activity in the field of nonlinear guided light propagation [1] has invoked a renewed interest in soliton dynamics in nonlinear waveguides. In particular, propagation of optical solitons in inhomogeneous fibers was theoretically considered some time ago with an intention to use a natural compression of the soliton in a waveguide with decreasing dispersion for obtaining ultrashort pulses [2]. This problem has gained practical importance with the arrival of technology allowing the fabrication of variable-cross-section silica fibers with a well-controlled local dispersion [3].

Thus, analysis of soliton propagation in nonlinear inhomogeneous waveguides is a relevant topic, with possibilities for real applications. Among problems of this type, consideration of the evolution of a soliton in a waveguide with a periodically modulated dispersion coefficient is especially interesting, as it should allow the testing of hidden dynamical properties of the soliton. On the other hand, fabrication of a long optical fiber with a periodically modulated

dispersion is quite possible, and, in combination with the availability of powerful laser sources of subpicosecond solitons, theoretical analysis of this problem can suggest ideas for new experiments. The objective of the present paper is to report results of systematic numerical simulations of the propagation of the fundamental soliton in a model of a nonlinear optical fiber with a periodically modulated dispersion coefficient. It will be demonstrated that, while a broad soliton with small energy evolves in this model rather trivially, very slowly decaying into radiation, evolution of a shorter soliton with a larger energy, whose characteristic dispersion length is comparable to or smaller than the modulation period, may be quite nontrivial, giving rise to an abrupt destruction of the soliton after a steady initial period. This seems to be qualitatively the same effect which has recently been predicted semi-analytically in [4]. The nontrivial behavior sets in above a certain critical value (threshold) of the modulation amplitude, in accord with [4], where the abrupt destruction of the soliton was predicted in the form of its rapid "spreading out" into radiation. In this work, we demonstrate that the actual destruction mode is different; the soliton splits into two secondary solitons, which is accompanied by emission of radiation. This particular mode could not be predicted in [4], as the simplest variational approximation employed in that work did not have enough degrees of freedom.

After a proper rescaling, the model considered in this work keeps only two nontrivial parameters, viz., the modulation amplitude and the initial energy of the soliton. Our simulations demonstrate that the boundary between splitting and nonsplitting solitons in the corresponding parametric plane has a complicated shape. Moreover, inside the splitting region we have discovered a set of small stability islands. We make a conjecture that the exact shape of the border is "fuzzy", possibly having a fractal dimension, and that distribution of the stability islands is also fractal. However, despite the complexity of the border between the two parametric regions, it demonstrates a simple gross feature: above a certain minimal energy, the mean position of the border almost does not depend upon the energy. This fact was predicted in [4], and, moreover, the nearly constant threshold value of the modulation amplitude corresponding to the mean position of the border is quite close to that predicted in [4].

Our simulations reveal another novel and unexpected feature of the soliton's evolution: with further increase of the modulation amplitude, we find a region in which the soliton effectively retrieves its stability, and then we encounter another threshold beyond which the soliton gets destroyed once again. Thus, there is a stability isthmus between two ranges of the splitting (see Fig. 2 below). Nonetheless, as we keep increasing the modulation amplitude, we reach a new border, beyond which the soliton is again stable. It is noteworthy that,

close to the first splitting threshold, then inside the above-mentioned stability isthmus, and, finally, beyond the uttermost stability border, the soliton is essentially different from the unperturbed one; it seems more like a breather, demonstrating persistent long-period shape oscillations. The period of the oscillations is typically almost ten times as large as the underlying modulation period.

2 THE MODEL

Recently, evolution of a soliton in a model with a harmonically modulated dispersion was considered in a semi-analytical approximation in [4]. The model was taken in the form of the perturbed nonlinear Schrödinger (NLS) equation,

$$iu_z + \frac{1}{2}\left(1 + \epsilon \sin(kz)\right) u_{\tau\tau} + |u|^2 u \;=\; 0\,. \tag{1}$$

In terms of the optical fibers, z is the propagation distance, τ is the so-called reduced time, $u(z,\tau)$ is the complex envelope of the electromagnetic field in the fiber, and the amplitude ϵ measures the depth of modulation of the local dispersion coefficient, the modulation period being $2\pi/k$. The obvious scale invariance of the NLS equation (1) allows us to set $k \equiv 1$.

In [4], emphasis was made on the search for a mechanism of destruction of the soliton based on resonant excitation of the soliton's internal oscillations. A fully analytical approximate description of the pulse's oscillations, completely neglecting the slow radiative attenuation, was put forward on the basis of a variational approximation in [5]. This approximation allows one to predict the frequency of the shape oscillations as a function of initial amplitude and width of the pulse. It is assumed that initially the pulse has the form

$$u_0(\tau) \;=\; A\,\mathrm{sech}(\tau/a) \tag{2}$$

with a "wrong" relation between the amplitude A and width a, different from that for the exact fundamental soliton, $Aa = 1$. The variational approximation is based on postulating that, at any value of z, the pulse has the shape

$$u(z,t) \;=\; A\,\mathrm{sech}(\tau/a)\exp\left(i\phi + ib\tau^2\right)\,, \tag{3}$$

in which the free parameters A,a,ϕ, and b are allowed to be arbitrary functions of z. Further analysis assumes substitution of the *ansatz* (3) into the Lagrangian of the NLS equation and integrating out the explicit τ-dependence, thus arriving at an effective Lagrangian, which is a function of the free parameters and their first derivatives in z. Finally, one can deduce from this Lagrangian a

system of variational ordinary differential equations which govern evolution of all these parameters in z. Further, all the parameters can be excluded from the equations in favour a, which measures the instantaneous width of the pulse according to (3). The remaining evolutional equation for $a(z)$ takes the form of the Newton's equation of motion for a unit-mass particle in the effective potential

$$U(a) = \frac{2}{\pi^2}\left(a^{-2} - 2Ea^{-1}\right), \tag{4}$$

where $E \equiv A^2 a$ is an integral of motion of the variational equations. Actually, E coincides with the exact integral of motion of the underlying NLS equation, which is usually called the energy of the pulse:

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} |u(\tau)|^2 d\tau . \tag{5}$$

If the initial pulse (2) gets into the potential well described by (4), the pulse is predicted to vibrate indefinitely long. Of course, in reality its vibrations will be subject to radiative damping, which is ignored in this approximation. If the initial pulse is outside the potential well, it is expected to spread out unlimitedly, which actually implies its decay into radiation.

Returning to [4], the same variational approximation was used there to analyze the soliton's dynamics in the periodically modulated fiber described by (1). This form of the perturbed NLS equation still admits a Lagrangian representation, and the energy (5) remains an exact integral of motion. The analysis, based on the same *ansatz* (3), yields the final evolution equation in the form

$$\frac{d}{dz}\left[D^{-1}(z)\frac{da}{dz}\right] = -\frac{\partial U}{\partial a}, \tag{6}$$

where $D(z) \equiv 1+\epsilon \sin z$ is the local dispersion coefficient in (1) and the effective potential is [cf. (4)]

$$U(a,z) = \frac{2}{\pi^2}\left[D(z)a^{-2} - 2Ea^{-1}\right] . \tag{7}$$

The basic idea of the analysis developed in [4] was that a possible resonance between the natural oscillations in the above-mentioned potential well and the periodic modulation of the local dispersion might give rise to a strong dynamical response of the soliton to the modulation. The resonantly driven oscillations were analyzed in detail near the well's bottom. However, the main emphasis was made on numerical simulations of the ordinary differential equation (6) without assuming anything small, with the objective of detecting

a transition from the oscillations of the particle trapped in the well to another type of motion, viz., progressive motion of the free particle to infinity. In the framework of the approximation adopted, this transition clearly implies a decay of the soliton into radiation.

Numerical simulations were done in [4] for the solitons with initial energies ~ 1. It was found that for each value of E, there was a critical value $\epsilon_{\rm cr}$, at which the transition took place. The decay observed when ϵ slightly exceeded $\epsilon_{\rm cr}$ can be described as follows: the particle would perform a number of seemingly chaotic oscillations, with rather large amplitudes inside the well, and then it would be suddenly kicked out from the well, finding itself moving to infinity. In all the range of the energies considered, $\epsilon_{\rm cr}$ was found to be nearly constant, taking values between 0.20 and 0.25.

As concerns the physical realizability of this phenomenon, it is relevant to emphasize that the resonant conditions imply that the modulation period must be of the same order of magnitude as the so-called soliton period, which, in the notation adopted here, is

$$T = 4\pi E^{-2} \tag{8}$$

for the ideal soliton. For subpicosecond optical solitons, the soliton period can be made as small as several dozens meters, which should open the way to real experiments.

3 NUMERICAL RESULTS

In all the simulations, we solved (1) numerically with the initial condition

$$u_0(\tau) = E\,\mathrm{sech}(E\tau)\,, \tag{9}$$

which gives rise to an exact soliton with the energy E in the absence of the modulation. Equation (1) was solved with periodic boundary conditions in τ; an artificial dissipation was installed at the edges of the integration domain, in order to absorb the emitted radiation.

First of all, we will mention relatively trivial results obtained at small values of the modulation amplitude ϵ. Due to the radiative losses, the soliton gradually becomes broader and lower, but no dramatic events happens. A remarkable fact is that the stability of the broad solitons persists up to quite large values of the modulation amplitude ϵ. Notice that when ϵ becomes larger than one, the local dispersion coefficient in (1) can change sign. Evidently, in this case the approximation based on (6) is irrelevant. Nevertheless, our simulations demonstrate that the evolution of broad solitons remains steady even in this case.

Now, we proceed to a more interesting case of solitons with a larger initial energy, so that the soliton period (8) becomes comparable to or smaller than the modulation period 2π. For a better understanding of the theoretical results, it is convenient to gradually increase ϵ at a fixed value of E. The structure of the soliton becomes qualitatively different from the simple structure observed at small ϵ: the soliton gradually turns into a breather, demonstrating long-period persistent oscillations of its width and height.

However, the most important fact is that, once ϵ has attained a certain critical value $\epsilon_{\rm cr}$, the soliton suddenly splits after a steady initial period of evolution. A typical example of the splitting at ϵ slightly larger than the corresponding critical value is shown in Fig. 1. For all the soliton's peak powers $E^2 > 2.5$, the value $\epsilon_{\rm cr}$ was found to vary between 0.15 and 0.20, which should be compared to the above-mentioned critical value produced by the variational approximation, which was varying between 0.20 and 0.25.

Now, we aim to follow changes which happen with further increase of ϵ. The results are summarized in the phase diagram (Fig. 2). The first remarkable fact is the existence of a whole set of small stability islands inside the splitting region. We conjecture that the genuine border between the splitting and nonsplitting regions may have a fractal structure.

It is also noteworthy that the left splitting region in Fig. 2 (the right one will be discussed below) is almost exactly bounded from below by the value $E^2 = 2$. On the other hand, (8) yields, at this value of E^2, the soliton period equal 2π, i.e., the one coinciding with the underlying modulation period. This gives an additional argument in favor of interpreting the splitting as a resonant effect, in accordance with [4].

When ϵ is essentially larger than the above-mentioned $\epsilon_{\rm cr}$, we encounter a new parametric border separating different types of evolution of the soliton. Beyond this border, the solitons retrieve their stability, but at still larger ϵ there is another threshold, beyond which the solitons split again. Thus, we have found a "stability isthmus" of a nearly constant width, sandwiched between two splitting areas (Fig. 2). It is noteworthy that almost all of the "isthmus" lies at $\epsilon > 1$, so that the corresponding local dispersion coefficient in (1) changes its sign periodically.

Another unexpected fact is that, keeping increasing ϵ, we once again encounter a stability domain, which is an extension of the stability area contiguous to the axis $E = 0$. Subsequent increase of ϵ up to the value 2.0 did not reveal any new splitting threshold bounding this domain. Thus, instead of pulling the instability border closer to the axis $E = 0$, which would be a naive expectation, increase of ϵ pushes it off.

4 CONCLUSION

In this work, we have reported results of systematic numerical simulations of evolution of a soliton governed by the NLS equation (1) with a periodically ~dulated dispersion coefficient. We have found that broad low-energy solitons, lispersion length is much larger than the modulation period, are quite stabi~ n this model, even in the case when the local dispersion coefficient periodically changes its sign. Contrary to this, solitons with the dispersion length comparable to the modulation period demonstrate a sudden splitting into a pair of secondary solitons and radiation, provided that the modulation amplitude exceeds a certain threshold. This fundamental result qualitatively agrees with the prediction made on the basis of the variational approximation in [4], although this approximation could not predict that formation of a pair of the secondary solitons is a generic outcome of the splitting. The critical value of the modulation amplitude which renders the splitting possible proves to be close to the value predicted by the variational approximation, although somewhat *smaller*.

Another remarkable fact revealed by the simulations is that, with a further increase of the modulation amplitude, there is an "isthmus" in which the soliton retrieves its stability. At still larger modulation amplitudes, the splitting sets in again. However, subsequent increase of the modulation amplitude leads to an uttermost stability area, which seems to be unlimited. In most cases, the quasi-stable solitons are actually breathers performing long-period shape oscillations.

The analysis presented in this work suggests an idea for experiments with subpicosecond optical solitons, Another implication of this work is that it lends additional credit to the variational approximation, demonstrating that, although it fails to grasp many important details, it is able to qualitatively correctly predict gross features of dynamical processes in complex systems. A full version of this work will be published in Physica Scripta.

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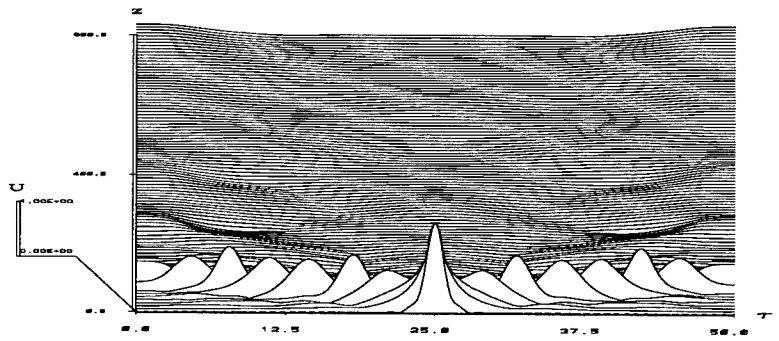

Figure 1: Typical examples of the soliton splitting at ϵ slightly exceeding ϵ_{cr}: $\epsilon = 0.3$; $E^2 = 2.9$.

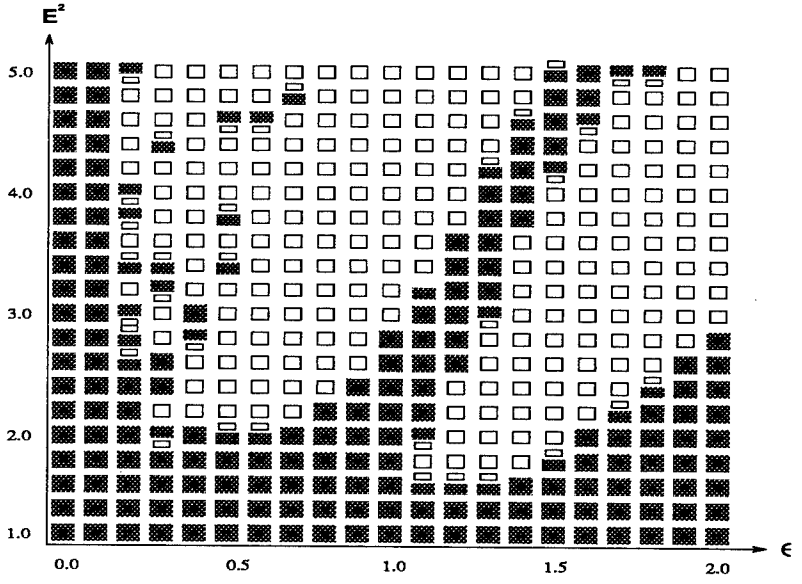

Figure 2: The phase diagram on the parametric plane (ϵ, E^2). The filled and unfilled rectangles correspond, respectively, to the stable and splitting solitons.

SUPERCONDUCTING AND SPIN-GLASS INTERACTION IN COUPLED LAYERED STRUCTURES

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For studying the mutual effects of superconductivity and magnetism in coupled layered structures, multilayers of Nb (superconducting) and CuMn (spin glass) with different Mn concentrations were grown on Si (100) substrates. The superconducting transition temperature T_c showed a pronounced nonmonotonic dependence on the spin glass layer thickness.

1 Why studying superconducting/spin glass multilayers?

A multilayer is a system composed of two or more materials alternated to form a periodic structure. Choosing a superconducting and a magnetic material is possible to study the interplay of superconductivity and magnetism in a controlled manner, varying the thicknesses of the two materials. The interest in this kind of research nowadays lies expecially in the role that the coexistence of superconductivity and magnetism seems to play in many properties of the high temperature superconductors (HTS). Among the magnetic materials we have focused on the spin glasses, because of their lower exchange energy I when compared to the ferromagnets, which causes a reduced pair breaking effect when coupled with a superconductor. Moreover, some compounds, like $(Nd_{1-X}Th_X)Ru_2$ and some HTS, are characterized by a superconducting/spin glass coexistence region in their phase diagrams [1].

2 Fabrication and characterization of the samples

The samples have been deposited using a magnetically enhanced dc triode sputtering system with an oscillating substrate holder alternately passing over the Nb and the CuMn targets. In all multilayers we kept the Nb layer thickness fixed ($\sim$ 250 Å) while varying the CuMn layer thickness d_{CuMn}.

To make a careful study of the T_c vs. d_{CuMn} dependence, we developed a suitable technique for the simultaneous realization of a complete series of

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multilayers, to minimize the variation in material parameters [2]. We used $Si(100)$ substrates and, to obtain different d_{CuMn} values along a diameter of the wafer, we let it pass in controlled steps over the CuMn target, while with a continuous velocity over the Nb target. At the end of the deposition the substrate is cut into strips perpendicularly to the chosen diameter. In each deposition we let a zone of the substrate to pass only over the Nb target, in order to obtain always a Nb sample, to know the properties of the superconducting layers. Here we show the results for two series with two different Mn concentrations:

$R16295$ – with 1.3% of Mn and d_{CuMn} varying between 8 Å and 132 Å,

$R22295$ – with 0.7% of Mn and d_{CuMn} values in the range $21 \div 184$ Å.

We prepared also a Nb/Cu series to compare the magnetic and nonmagnetic cases. The multilayers were electrically characterized using a standard four probe technique.

3 Comparison with existing theories

It is well known that the T_c of S/N (superconducting/normal metal) multilayers decreases monotonously with increasing N layer thickness [3], also in the case of N metal layers with randomly distributed magnetic impurities [4]. A completely different dependence is foreseen by Radovic et al. [5] for superconducting/ferromagnetic multilayers. Because of the periodicity of the multilayer, the Green function $F(x,\omega)$, describing the condensate of pairs, is subjected to the Bloch condition $F(x+L,\omega) = e^{-i\varphi}F(x,\omega)$, with $L = d_S+d_N$ and $d_{S,N}$ superconducting and ferromagnetic layer thicknesses. Obviously the same relation holds for the order parameter $\Delta(x)$. After calculations, they get the characteristic ground-state configurations with non trivial phase difference $0 \leq \varphi \leq \pi$ between neighboring superconducting layers, that give rise to an unusual oscillatory dependence of T_c on the ferromagnetic layer thickness. In suitable limits, the equation for the critical temperature T_c of the multilayer is [6]:

$$\frac{(d_N/2)(d_S/2)}{\gamma\xi_N\xi_S}\frac{T_{cS}-T_c}{T_{cS}} = 3G(\varphi, k_N d_N) \qquad (1)$$

where $\xi_{N,S} = \sqrt{D_{N,S}/2\pi T_{cS}}$ are the correlation lengths in N and S, with $D_{N,S}$ the diffusion coefficients and T_{cS} the critical temperature of the bulk superconductor, $\gamma = \sigma_N\xi_S/\sigma_S\xi_N$, with $\sigma_{N,S}$ the normal state conductivities of the magnetic (N) and superconducting (S) layers, $k_N = (1+i)\sqrt{I/D_N}$ and G is a function defined in Ref. [6]. The values for T_c are obtained minimizing G with respect to φ. As d_N is varied, transitions between the 0 phase and π phase occur causing a T_c vs. d_N oscillating behaviour.

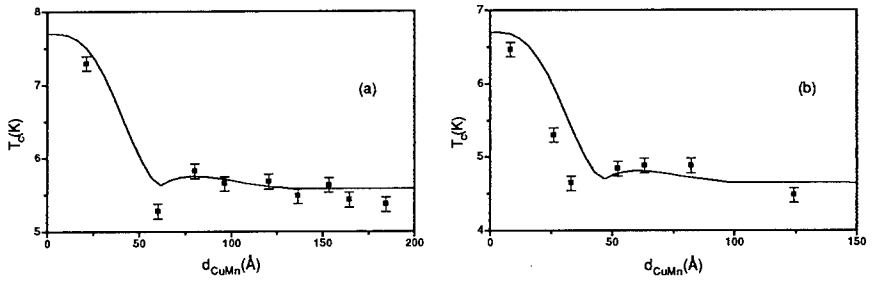

Figure 1: T_c vs. d_{CuMn} dependence for the $R22295$ series (a) and $R16295$ series (b). The solid lines are the best fit curves obtained with equation 1.

For our multilayers we generally measured $\sigma_N/\sigma_S \sim 1$, so that $\xi_S \sim \gamma\xi_N$. Then we have three fit parameters: γ, ξ_N and $a = \sqrt{2\pi T_{cS}/I}$. The solid lines in fig. 1 are the best fit curves obtained using the equation 1. We have obtained $a = 0.7$ ($I = 7.4meV$) and $\xi_N = 27$ Å, for the series $R16295$, and $a = 0.85$ ($I = 5.8meV$) and $\xi_N = 30$ Å, for the series $R22295$. These exchange energy values scale in the right way with the Mn concentrations and are about three orders of magnitude lower than the values measured in the case of superconducting/ferromagnetic multilayers.

Moreover, a great deal of interest has been generated in the field of dimensional behaviour of the superconducting multilayers because of their layered structure similar to the HTS one. We started this kind of study by considering the rounding observed at the onset of the transition in the $R(T)$ curves, due to the phenomenon of paraconductivity. When T approaches T_c from above, some electrons begin to couple to form Cooper pairs, because of thermodynamical fluctuations near a critical point, causing a raising in the conductivity just before the transition. This exceeding conductivity $\Delta\sigma$ behave differently on $\varepsilon = (T - T_c)/T_c$ according to the dimensionality of the system [7]:

$$3D \rightarrow \Delta\sigma \propto \varepsilon^{-1/2} \; ; \; 2D \rightarrow \Delta\sigma \propto \varepsilon^{-1} \qquad (2)$$

where 3D and 2D mean, respectively, a system with coupled and decoupled superconducting layers.

For some of our samples we estimated $\Delta\sigma(T) = 1/\rho(T) - 1/\rho_N$ where $\rho(T)$ and ρ_N are respectively the resistivities obtained by the experimental values of $R(T)$ and R_N, with R_N the normal resistence just before the onset of the transition. Considering the $ln(\Delta\sigma/\sigma_0)$ vs. $ln(\varepsilon)$ plot, with σ_0 the room temperature conductivity, we obtained a linear behaviour to compare with that expected by eqs. 2. In fig. 2 we show the obtained slope values together with

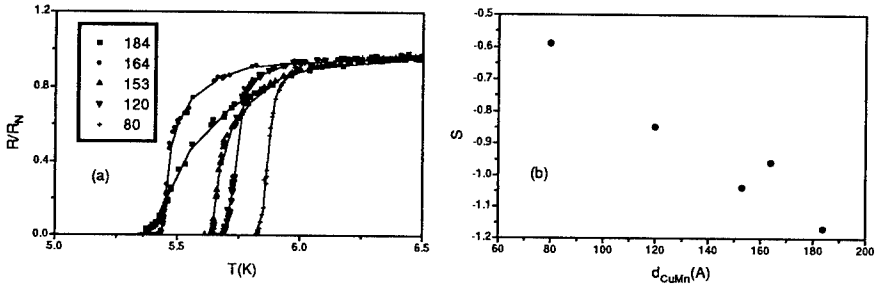

Figure 2: (a) Normalized superconducting transition curves for some samples of the $R22295$ series. The numbers in the legend are the CuMn layer thicknesses in Å. (b) Dependence on d_{CuMn} of the $ln(\Delta\sigma/\sigma_0)$ vs. $ln(\varepsilon)$ plot slope S for some samples of the $R22295$ series.

the superconducting transition curves for some samples of the $R22295$ series: a crossover $3D \rightarrow 2D$ with increasing d_{CuMn} values is clearly visible.

4 Conclusions

We have developed a technique for depositing series of Nb/CuMn samples under identical conditions with identical Nb but different CuMn layer thicknesses. The superconducting transition temperature T_c shows a pronounced nonmonotonic dependence on the spin glass CuMn layer thickness which is well described by Radovic et al. theory. This result provides the first evidence for the validity of this theory in layered superconducting systems with weak magnetic coupling, like the spin glasses.

By considering the paraconductivity effects on the electrical resistivity above the superconducting transition due to thermodynamic fluctuations, we have also studied the influence of the reduced dimensionality of our multilayers on superconductivity, to check similar effects observed on HTS.

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NONLINEAR SIMULATION OF THE ELECTROCORTICAL ACTIVITY

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The low dimensionality of the possible dynamics underlying the brain activity is justified in terms of synchronization, this achievement giving support to rhythm simulation by few "effective" synchronized neurons.

A large amount of literature is presently aimed to the setting up of clinical indicators deduced from time-EEG series by advanced spectral techniques or methods proper of non-linear dynamics analysis [1]. It is an hazardous challenge to draw relevant information on a system with about ten billion of degrees of freedom from an extremely limited sampling. On the other hand, the relatively low measured dimensionality of brain activity encourages similar approaches. We have consequently tried to explain this paradox in terms of synchronization [2-5]. The simplest model of this kind could be based on a network of few "effective" neurons retaining the main features of the underlying structure and producing rhythms similar to the EEG ones.

In recent literature a kind of neural network model producing synchronization, based on a system of oscillators with impulsive coupling, has been successful for its simplicity and calculus efficiency. As an electrical current is injected into a neuron cell the membrane potential $V_i(t)$ grows from the rest value $u_0 < u_{tr} \cong -50$ mV to the value u_{tr} corresponding to the threshold for the generation of the action potential: a potential spike of about 2 ms. After the spike the potential resets to a lower value $u_{res} \cong -70$ mV to restart the growth towards the threshold value. So, as a consequence of a constant current injection, a train of action potentials with regular time spacing (near 200 ms) is generated. The time scales involved justify the impulse approximation of the action potentials. Electric networks behaving in this fashion, with an impulsive modelization of the spikes, are called, according to Hopfield [6], integrate and fire neuron networks. Their dynamics is essentially described by a differential set as

$$\frac{dV_i}{dt} = \frac{1}{\tau_1}(-V_i + u_0 + RI_i) + (V_i - u_{res})\, f_i(t), \qquad (1a)$$

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = -\frac{I_i}{\tau_2} + \sum_{j \neq i} S_{ij} f_j(t) + \eta_i \qquad (1b)$$

including the matching conditions at the firing times t_{ik} roots of the equation $V_i(t) = u_{tr}$. In equations 1a,b $f_j(t) = \sum_k \delta(t - t_{jk})$ where $\delta(t - t_{ik})$ is the Dirac distribution simulating an instantaneous pulse fired by the i-th neuron at time t_{ik} in correspondence to an action potential activation. Quantities S_{ij} weighing the information coming from the j-th to the i-th cell can excite or inhibit the firing according to their sign. At last, η_i expresses the sensory driving. It can be noticed how the coupling of the equation 1b to the 1a takes place by means of impulsive terms dependent on the unknown firing times t_{ik}. This kind of coupling, involving matching conditions, contains unknown values of time and prevents a simple use of numerical approach. This problem can be overcome if the differential equations 1a,b, after the change of variable in the Dirac functions $\delta(t - t_{ik}) = \left|\dot{V}_i(t_{ik})\right| \delta(V_i - u_{tr}) = \mathrm{d}\theta(V_i - u_{tr}) / \mathrm{d}t$, where θ is the Heaviside step function, are converted into a differential system (autonomous in absence of sensor stimuli). The approximation of time-derivatives with first order finite differences converts, on its turn, the differential system into an iterate map also able to express the cell potential jump from u_{tr} to u_{res} . The numerical experiments, illustrated in the figures and reported in F. Ferro-Milone et al. [7] highlight the integrate and fire synchronization mechanism and give support to the explanation here attempted of the low dimensionality of brain activity in terms of this phenomenon. In Fig.2 clusters of neurons with the same global firing rate are compared, exhibiting indistinguishable rhythms after few iterations.

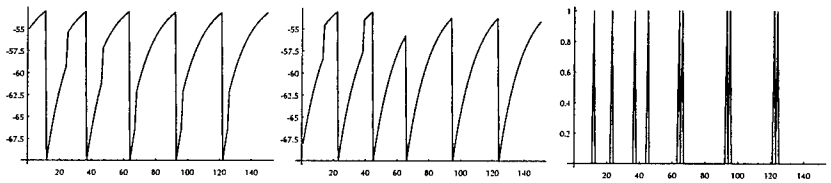

Fig. 1. Synchronization and firing of two neurons.

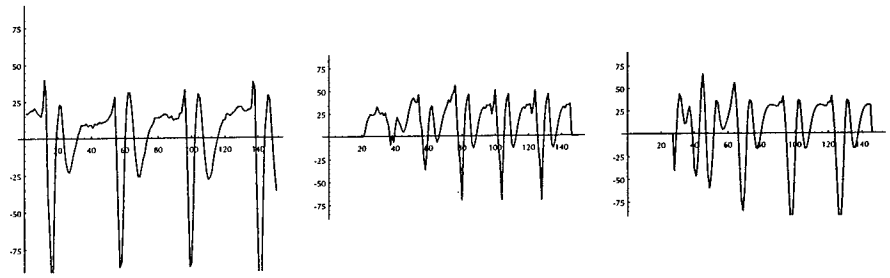

Fig. 2. Detail of the rhythm of ref.[8] (left), simulation by convolution of fires produced by ten neurons (center) and by the neurons of Fig. 1 (right).

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PERIODIC INVERSE SCATTERING TRANSFORM ANALYSIS OF ADRIATIC SEA SURFACE WAVES

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ABSTRACT

Herein I identify a new technique for nonlinearly analyzing shallow water oceanic wave trains: The method is based upon the inverse scattering transform (IST) for the Korteweg-deVries (KdV) equation with periodic boundary conditions. I specifically address the *θ-function representation* of the equation and exploit the following Theorem: All solutions to the periodic KdV equation can be constructed as a sum of cnoidal waves (the travelling wave solution of the equation) plus their mutual, pair-wise nonlinear interactions. One therefore has a kind of *nonlinear Fourier analysis* for long wave motions in which the usual linear sum of sine waves is extended and generalized to the *linear superposition of cnoidal waves plus nonlinear interactions*. These results provide a new approach for the nonlinear Fourier analysis of unidirectional, shallow-water laboratory and oceanic wave trains. I apply the method to analyze a measured time series obtained in the Adriatic Sea in a depth of 16.5 m about 20 km from Venice, Italy. The research discussed herein is covered in much greater detail in a paper in preparation, for which this note is simultaneously an abridgement and an advertisement.

1. Introduction

During the last quarter of a century a number of important discoveries have been made with regard to the physical and mathematical structure of particular nonlinear wave equations. These include the identification of large classes of nonlinear, partial differential equations (PDEs) which are said to be *integrable* by a relatively new method of mathematical physics known as the *inverse scattering transform* [1-4]. Of particular interest in the present paper is the prototypical PDE, known as the Kortweg-deVries (KdV) equation [5], which describes *nonlinear wave motion in shallow water*. The importance of this equation rests with the fact that it derives from the Euler equations in one spatial dimension using a singular perturbation expansion about *zero wave number*, $k \sim 0$. The expansion is carried out to third order in the dispersion relation, e.g. $\omega = c_o k - \beta k^3$; the physical situation corresponds to small but finite-amplitude, long waves in shallow water. Here $c_o = \sqrt{gh}$, $\beta = c_o h^2 / 6$, where g is the acceleration of gravity and h is the water depth. The KdV equation is valid as long as the motion is (sufficiently) unidirectional and the second term in the dispersion relation is small with respect the first, i.e. for $h^2 k^2 / 6 << 1$ [5, 6-8]. The dimensional form of the KdV equation is given by:

$$\eta_t + c_o \eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0 \qquad (1)$$

$\eta(x,t)$ is the space/time evolution of the free surface elevation and $\alpha = 3c_o / 2h$. Note that (1) can be linearized by removal of the nonlinear term $\alpha\eta\eta_x$; the resultant PDE has the linear dispersion relation discussed above. For extended reviews on equation (1) see Whitham [9] and Miles [10].

While the KdV equation was first found to be integrable for *infinite-line boundary conditions* [11], herein I am primarily interested in the associated inverse scattering transform for *periodic boundary conditions* [12-17]. Periodic IST has yielded rather general solutions of many large classes of nonlinear wave equations. From the point of view of mathematical physics and applied mathematics this work has revealed remarkable relationships among the Riemann theory of Abelian functions, the spectral theory of Schroedinger operators and algebraic geometry. Because of the considerable technical nature of periodic IST, the theory has not been very amenable to physical applications over the last 20 years. Herein I take a significant step to correct this unfortunate situation.

The focus of the present work rests with the application of the method to the study of nonlinear, shallow water wave data obtained in the Adriatic Sea. This is accomplished by virtue of the fact that periodic IST can be cast in terms of a kind of *nonlinear Fourier analysis*. The present paper is an attempt to provide a simple, physical perspective for some of the previous results leading to the time series analysis of shallow water waves using the hyperelliptic function representation [18-25] and to more recent results using the θ–function representation of the KdV equation [26, 27].

There are several reasons why periodic IST is potentially an important tool in the analysis of unidirectional, shallow water wave data:

(1) Linear Fourier analysis, in the guise of the discrete Fourier transform (numerically implemented as the fast Fourier transform (FFT)), has played an important and ubiquitous role in the processing of a large variety of signals [28]. Furthermore, the small-amplitude, *linear limit of periodic IST* is given by the discrete Fourier transform, a linear, periodic algorithm [20-22].

(2) Many types of nonlinear wave motion can be assumed to be *stationary* and *ergodic*. An important *physical subset* of these nonlinear stochastic systems is integrable by periodic IST [26, 27].

(3) Many kinds of natural, nonlinear, wave phenomena have a dominant period, such as the 12.4 hour tidal period often found for internal wave motions in the ocean [29].

(4) Particular nonlinear wave phenomena, such as oceanic surface waves, are typically assumed to be approximate realizations of infinitely long stochastic processes [28]. This problem is often addressed with the periodic FFT algorithm. In this case the temporal period, T, of the wave train is *formally* taken in the limit $T \to \infty$; however, for practical considerations T is most often assumed to be the period of the measured wave train and averaging procedures (over sub intervals either in time or in frequency) are employed to improve spectral estimates of the $T \to \infty$ limit. On the basis of previous results and on work presented herein a simple guiding criterion for the development of numerical data analysis algorithms for IST has been exploited: *a numerical, nonlinear (IST) Fourier algorithm (numerical inverse scattering transform, NIST) should be discrete and*

periodic and reduce to the discrete Fourier transform (DFT) in the small amplitude limit [20-23].

2. Cnoidal Waves and Nonlinear Fourier Analysis

Kortweg and deVries found a simple periodic solution of (1) which is known as the *cnoidal wave*:

$$\eta(x,t) = 2\eta_o cn^2\{(K(m)/\pi)[k(x-Ct)];m\} \tag{2}$$

The modulus m of the Jacobian elliptic function cn and the nonlinear phase speed C are well-known relations which are a function of the amplitude η_o; these formulas are given elsewhere (see for example [18]). When the modulus $m \to 0$ the cnoidal wave reduces to a sine wave; intermediate values of the modulus correspond to the Stokes wave; when $m \to 1$ the cnoidal wave approaches a solitary wave or soliton.

One may view the cnoidal wave as a kind of *basis function* for the KdV equation and a long sought goal has been to develop a simple approach to nonlinear Fourier analysis in terms of these fundamental wavelets [22]: presumably the technique would be a generalization of linear Fourier analysis, which has ordinary sine wave basis functions. Does there exist a *nonlinear Fourier analysis for cnoidal waves*? Formally speaking the discovery of periodic IST resolved this problem about 20 years ago [12-16], but practical implementation of the approach has only recently been realized [27]. To see how this formulation comes about I address the general solution to the KdV equation (1) in terms of the so-called *θ-function representation* [12-16]

$$\eta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(\eta_1,\eta_2,\ldots\eta_N) \tag{3a}$$

where the θ-function is given by:

$$\Theta_N(\eta_1,\eta_2,\ldots\eta_N) = \sum_{M_1,\ldots,M_N=-\infty}^{\infty} \exp\left[i\sum_{n=1}^{N} M_n\eta_n + \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} M_m B_{mn} M_n\right] \tag{3b}$$

Here $\lambda = \alpha/6\beta$. The integer N is the number of degrees of freedom (i.e. the number of cnoidal waves) in a particular solution to the KdV equation. The θ–function phases resemble those of linear Fourier analysis:

$$\eta_n = k_n x - \omega_n t + \phi_n$$

The *period matrix* $\mathbf{B} = \{B_{mn}\}$, the *wave numbers* k_n, the *frequencies* ω_n and the *phases* ϕ_n are constants which depend upon algebraic-geometric loop integrals whose determination is discussed elsewhere [27]. The period matrix **B** has elements which are *negative definite*. The *cnoidal wave amplitudes* are determined by the diagonal elements and the *nonlinear interactions* are determined by the off-diagonal terms.

3. Nonlinear Fourier Analysis Theorem

The θ-function solution (3) to the KdV equation (1) is readily written in the following form [27]:

$$\eta(x,t) = \frac{2}{\lambda}\frac{\partial^2}{\partial x^2}\ln\Theta_N(x,t) = \underbrace{\eta_{cn}(x,t)}_{\substack{\text{Linear superposition}\\ \text{of cnoidal waves}}} + \underbrace{\eta_{\text{int}}(x,t)}_{\substack{\text{Nonlinear interactions}\\ \text{among the cnoidal waves}}} \tag{4}$$

Therefore: *Unidirectional, shallow water wave trains can be represented by a linear superposition of cnoidal waves plus a term which includes the mutual nonlinear interactions amongst the cnoidal waves.* The explicit formulation of the nonlinear interaction terms is given elsewhere [27]. An important limit occurs when the wave amplitudes are small with respect to the depth: the cnoidal waves tend to sine waves and the interactions tend to zero. In this way one recovers ordinary, linear Fourier analysis. The latter result provides one of the reasons why periodic IST can be viewed as a *nonlinear generalization of linear Fourier series.*

4. Analyzing Shallow Water Waves from the Adriatic Sea with the Nonlinear Fourier Technique

I now consider a surface wave train measured in the Adriatic Sea in 16.5 m depth about 20 km from Venice, Italy [24, 25]. A time series of N = 1000 points has been selected for the present study and is shown in Fig. 1(a). The data were digitized at temporal intervals of $\Delta t = 0.25$ sec for a total period of $T = 250$ sec. The linear Fourier transform of the series is given in Fig. 1(b). Note that the dominate spectral energy lies in an interval from about 8 to 10 sec; the significant wave height is $H_s = 2.5$ m and the zero crossing period is $T_z = 10.2$ sec. A number of preliminary tests and pre-processing steps are necessary for establishing that the measured wave train is governed by the physics of the KdV equation; these are discussed in detail elsewhere [30].

A full investigation of the nonlinear physics in the measured wave train, using the *theta function representation,* can now be made. This is accomplished by first computing the *interaction* or *period matrix* **B**, which has been taken to be 50 by 50 in the present case. The matrix elements B_{ij} are graphed as a function of i and j in Fig. 2. First note that all the elements in the **B** matrix are negative definite. The matrix elements determine the *nome* of each cnoidal wave and hence, through a simple computation, provide the amplitude and modulus of the cnoidal wave (Fig. 3). The spectrum itself is somewhat similar to the linear Fourier spectrum (Fig. 1(b)). However, the cnoidal wave amplitudes are *larger* than the linear Fourier modes.

In Fig. 4 are shown the results of the θ–function analysis of the (filtered) Adriatic Sea wave train from Fig. 1. I have used the theorem discussed in Section 3, i.e. a shallow water wave train can be decomposed into its constituent cnoidal waves plus their mutual nonlinear interactions. The upper three-fourths of Fig. 4 shows the 50 cnoidal waves extracted from the filtered wave train. Below these are the linear sum of the cnoidal waves,

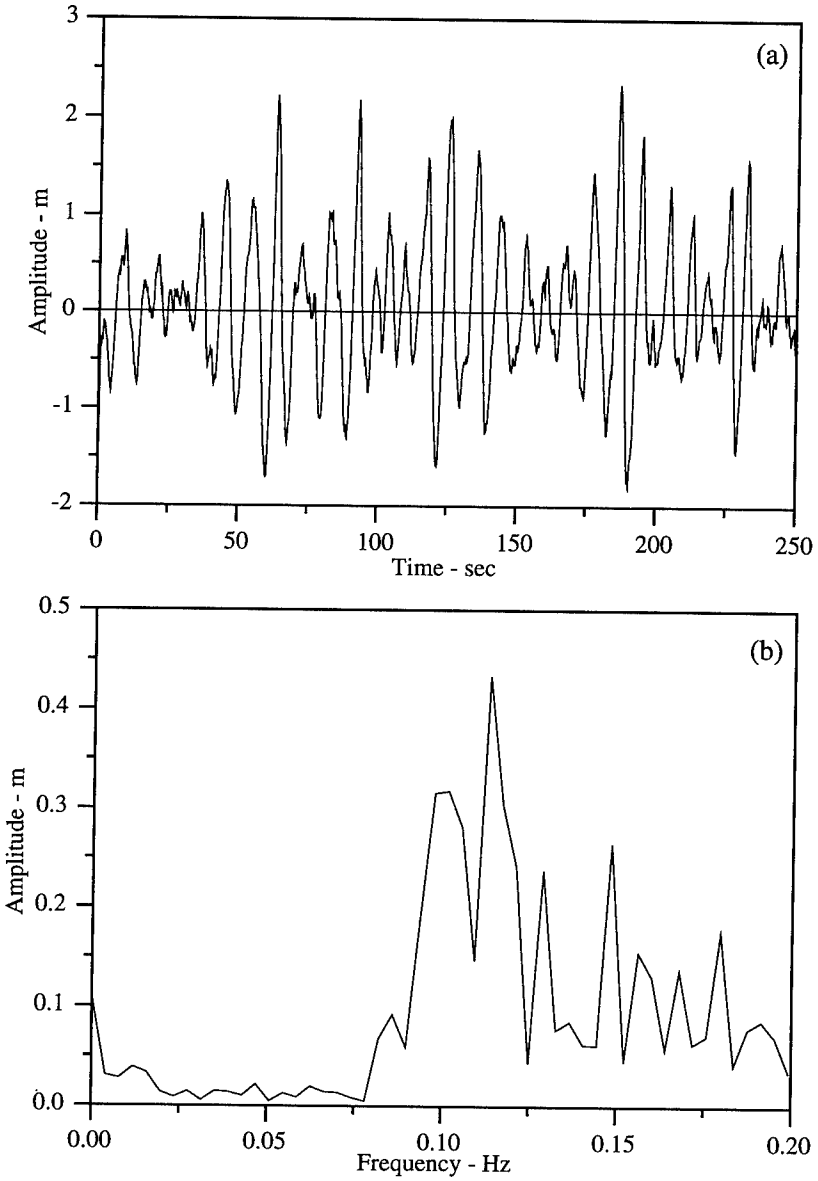

Figure 1. Time series measured in the Adriatic Sea (a). The linear Fourier transform of the measured time series (b).

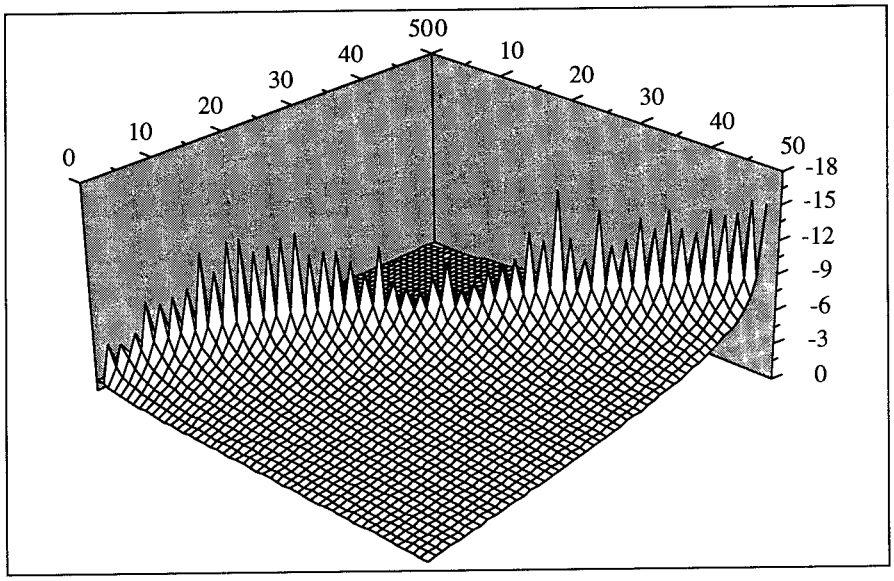

Figure 2. The amplitudes of the elements of the period matrix, which in the present application is 50 by 50. Note that the diagonal elements, directly related to the cnoidal wave amplitudes, form a jagged "backbone" on the surface. The higher the diagonal elements, the smaller are the associated cnoidal waves.

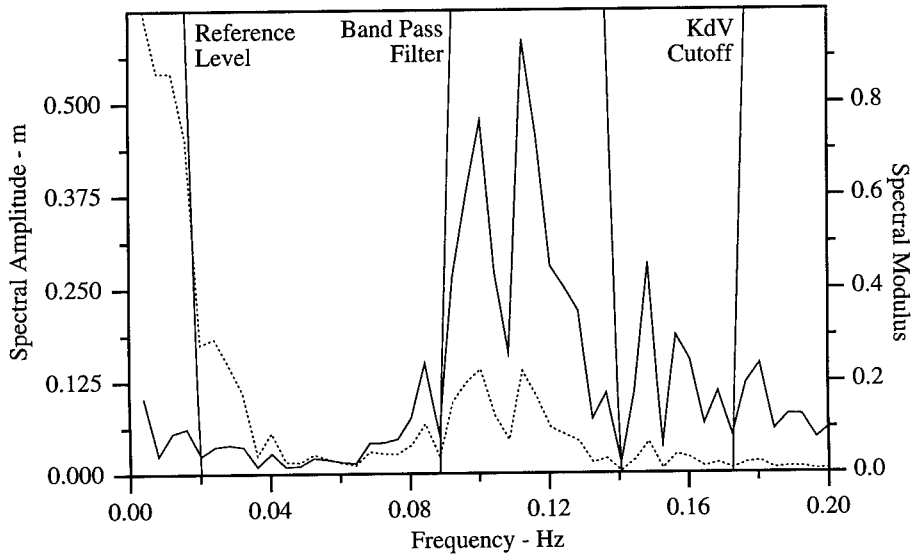

Figure 3. The cnoidal wave spectrum as a function of frequency (solid line) and the associated spectral modulus (dotted line).

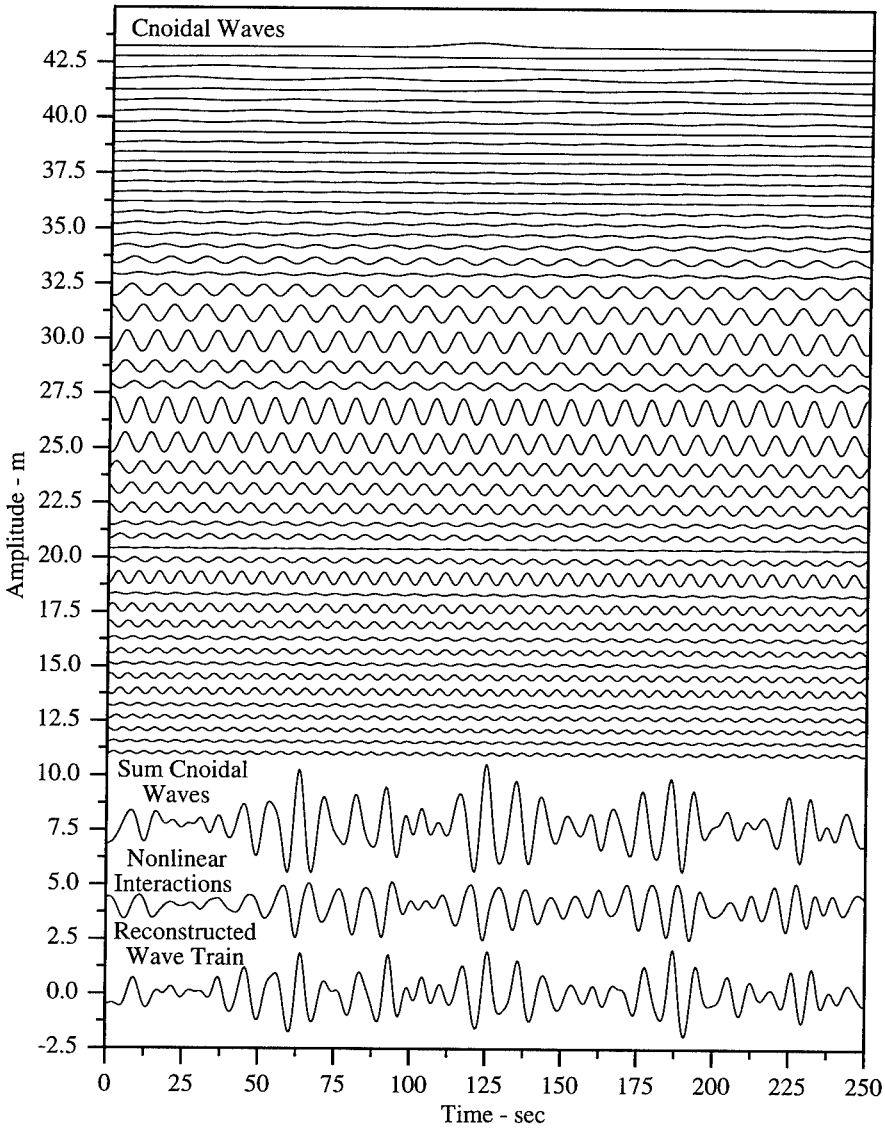

Figure 4. The cnoidal waves in the inverse scattering transform spectrum of the filtered Adriatic Sea wave train of Fig. 4. The 50 cnoidal waves are shown in the upper three-fourths of the figure. The lower three curves are the sum of the cnoidal waves, the nonlinear interactions and the reconstructed, filtered Adriatic Sea wave train.

their mutual nonlinear interactions and, finally, the reconstructed Adriatic Sea wave train of Fig. 4 (in the pre-processing step [30] this signal has been low-pass filtered on the interval (0, 0.2 Hz)). Note that the wave train corresponding to the linear superposition of the cnoidal waves is somewhat larger than the low-passed filtered wave train itself: *the effect of the nonlinear interactions reduces the amplitudes of the summed cnoidal waves in the reconstruction process*. While this may at first seem surprising, it is enough to recall the simple two-soliton interaction in which the soliton amplitudes *decrease* during a collision (see for example the discussion in Osborne [27]). One may attribute the nonlinear interactions to an effective *global spatial/temporal phase shifting* of the summed cnoidal waves relative to one another.

5. Discussion

For the Adriatic Sea time series analyzed herein the nonlinearity is not very large, because most of the energy in the wave train can be characterized as not being very nonlinear: a majority of the cnoidal wave components can at most be viewed as moderate amplitude Stokes waves which interact only weakly with one another.

While the nonlinear interactions in the present analysis are relatively small, one would expect larger, longer waves in shallower water to be considerably more nonlinear. This is because of the dependence of the Ursell number on the parameters H_s (significant wave height), L (wave length) and h (depth), e.g. $U = 3H_sL^2 / 32\pi^2h^3$. Therefore, future experiments with higher Ursell number will likely reveal even more interesting physical results with regard to nonlinear dynamics. In particular the nonlinear evolution of shoaling waves as they propagate into shallow water will be an important area of investigation.

6. Acknowledgements

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DYNAMICS OF NONLINEAR BLOOD PRESSURE WAVES IN ARTERIES

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We present a nonlinear quasi-one-dimensional model which describes blood pressure propagation in large arteries. In the limit of an ideal fluid and for slowly varying arterial parameters a Boussinesq-type equation is obtained. Numerical simulation reveal the influence of changing diameter and Young modulus on the pressure pulse shape. Physiological features such as "peaking" and "steepening" show that the pulse can be seen as a wave whose shape evolves between a solitary and a shock wave.

1 Introduction

We know from measurements that the blood pressure pulse propagates along the major arteries with characteristic shape changes[1]. Indeed, the blood motion is accompanied by an increase in amplitude with the rapid developement of a steep front for the pressure pulse (noted as "peaking" and "steepening" in the literature). These two phenomena are combined with an increase of the wave celerity and are in accordance with the development of a dicrotic wave.

Recently, using various asymptotic methods Hashizume [2,3] and Yomosa [4] studied the motion of weakly pressure waves in a thin nonlinear elastic tube filled with an incompressible fluid. They showed that the dynamics can be governed by the Korteweg-de Vries (KdV) equation.

With the motivation of a better understanding of the typical features of blood flow and to elucidate the role of the different types of structures that influence blood motion, we study the propagation of pressure and wall displacement pulses in large elastic arteries. We take into account several effects neglected in previous studies such as variations of the radius and Young modulus of the arterial wall.

2 The model, governing equations

By virtue of the anatomical geometry, the pulse canbe treated as a one-dimensional wave. Following previous authors we assume that the blood can be regarded as an incompressible and nonviscous fluid. Further, blood vessels are assumed to be uniform cone-shaped cylindrical tubes having nonlinear elastic walls. With the above assumptions, the laws of hydrodynamics governing the blood transport are the conservation of mass (1), the momentum equation (2) and the equation (3) describing the the radial motion of the artery wall under the forces exerted by the fluid:

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial z}(Su) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial P}{\partial z} = 0, \tag{2}$$

and

$$\rho_w h \frac{\partial^2 R}{\partial t^2} = P - P_e - \frac{h}{R}\sigma_T, \tag{3}$$

where P_e is the external pressure, h the thickness of the tube of radius $R(z,t)$ and where $\rho \approx 1.06$ gr/cm^3 and $\rho_w \approx \rho$ are, respectively, the fluid and material wall densities.

Implicit in this representation is the approximation of uniform blood velocity $u(z,t)$ and pressure $P(z,t)$ for all points on each section $S(z,t)$ of the artery. The transverse stress $\sigma_T = E\gamma(1+\alpha\gamma)$ ($\alpha \approx 2.5$) is a nonlinear function of the small radial elongation of the arterial wall $\gamma(z,t) = \Delta R/R$.

Defining the differential pressure $p = P - P_0$ and assuming the icompressibility of the wall (hR =constant= $h_0 R_0$) and that the variation of the at rest radius R_0 and Young's modulus E are slow along the z-axis, that is, $R_0 \equiv R_0(Z)$, $E \equiv E(Z)$ where $Z = \varepsilon z$ is a slow variable. With these considerations, we have two characteristic space variables and, performing a two-space expension, we have the transformation $\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial z} + \varepsilon \frac{\partial}{\partial Z}$. Conserving only terms up to order $O(\gamma^2)$ and $O(\varepsilon^0)$ in our development, we can reduce the problem to a nonlinear partial differential equation describing the motion of the arterial wall [5]:

$$\gamma_{tt} - v_0^2(\varepsilon z)\gamma_{zz} = \frac{1}{2}(\gamma^2)_{tt} + \beta v_0^2(\varepsilon z)(\gamma^2)_{zz} + CR_0^2(\varepsilon z)\gamma_{zztt} \ , \tag{4}$$

with $\eta = h_0/R_0 \approx 0.12$ has a constant value in any artery and wher the phase-velocity of linear waves is given by the Moens-Korteweg formula:

$$v_0(\varepsilon z) = \sqrt{\frac{\eta}{2\rho} E(\varepsilon z)} \ , \tag{5}$$

and where $C = (\eta\rho/2\rho_w)$ is a parameter measuring the dispersion and $\beta = \alpha - 2$.
We Note that, contrary to Yomosa and Hashizume who have directly derived a KdV equation we obtain a Boussinesq-type one which contains two nonlinear terms and a dispersion term. The nonlinear term $\frac{1}{2}(\gamma^2)_{tt}$ is related to the intrinsic nonlinearities of the fluid dynamic equations and the second one $\beta\gamma_{zz}^2$ is proportional to the nonlinear coefficient of elasticity. The dispersion term $CR_0^2(\varepsilon z)\gamma_{zztt}$ is related to the inertial effects of the wall and is proportional to the arterial cross-sectional area.

3 Analytical calculations

In the case of constant radius and Young's modulus, Eq. (4) can be solved analyticaly. We look for a localised solution with permanent profile and celerity v. The result is the one-solitary wave solution

$$\gamma(z,t) = Am \ sech^2\left(\frac{1}{L}(z - vt)\right), \tag{6}$$

with an amplitude $Am_{Bsq} = 3(v^2 - v_0^2)/(v^2 + 2\beta v_0^2)$ and a width $L_{Bsq} = 2R_0\sqrt{C}(1 - v_0^2/v^2)^{-1/2}$. Qualitatively, expression (5) shows that a decrease of vessel diameter results in a reduction of pulse width for propagation in a cone-shaped vessel. Likewise, an increase in Young's modulus will be in accordance with an evolution of the wave velocity. The expression for the pressure pulse is easily derived by substituting Eq. (6) into (3) : $p = \rho v^2(2 - \gamma)\gamma$.
The *in vivo* profile of the arterial pulse may be regarded as a "two-pulse wave" solution. Such a solution can be approched if we consider the KdV approximation of the Bsq equation (4). Specifically, using the reductive perturbation method[6], Eq. (4) is reduced to

$$\frac{\partial\gamma}{\partial\tau} + (b + \beta)\gamma\frac{\partial\gamma}{\partial\xi} + \frac{s}{2}\frac{\partial^3\gamma}{\partial\xi^3} = 0. \tag{7}$$

Here we have introduced a small parameter and performed the following change of variables: $\xi = \varepsilon^{1/2}(z - v_0 t)$ and $\tau = \varepsilon^{3/2}t$. In terms of the original laboratory coordinates (z,t) the solution is identical to Eq. (5) with an amplitude $Am_{KdV} = 3(v - v_0)/(b + \beta)v_0$ and a width $L_{KdV} = R_0\sqrt{(C/2)(v_0/(v - v_0))}$. In the weak-amplitude limit, we use the two-soliton solution of Eq. (7) as an initial condition for our numerical simulations. In the laboratory coordinates (z,t), this two-soliton solution, obtained by the Hirota method [7] is:

$$\gamma(z,t) = 8\frac{\mu_1^2 f_1 + \mu_2^2 f_2 + (\mu_1 - \mu_2)^2\left[\frac{(\mu_1^2 f_1 f_2^2 + \mu_2^2 f_2 f_1^2)}{(\mu_1+\mu_2)^2} + 2f_1 f_2\right]}{\left[1 + f_1 + f_2 + \frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2} f_1 f_2\right]^2} \ ,$$

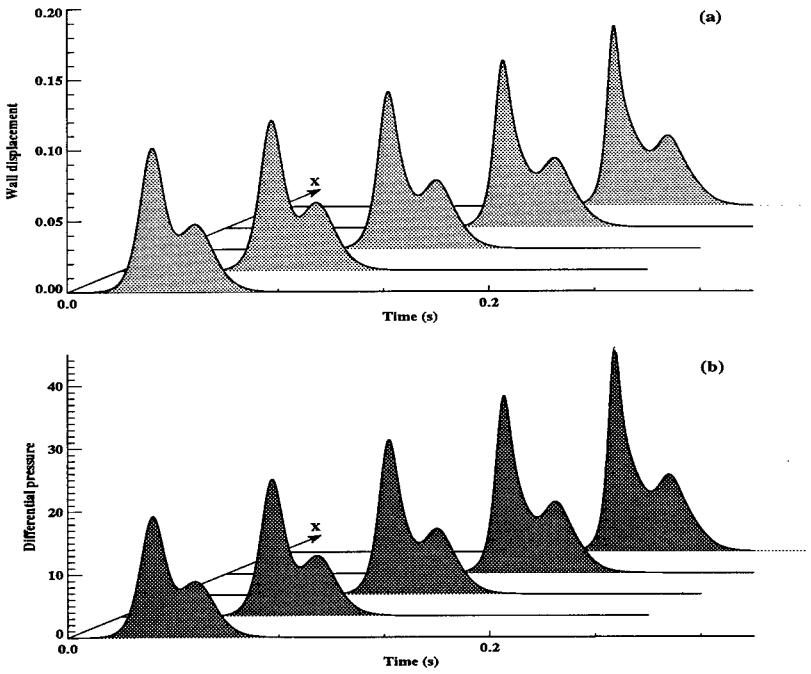

Figure 1: Caculated wall displacement and differential pressure profiles at different locations along an artery for a cone-shaped artery with an increasing Young's modulus. $z = 0$ is the position of the aortic valve.

where

$$f_i = \exp\left[\sqrt{\frac{2}{a}\left(\frac{v_i}{v_o} - 1\right)}\,(z - v_i(t - t_i))\right] \quad \text{and} \quad \mu_i = \frac{1}{2}\sqrt{\frac{6}{b+\beta}\left(\frac{v_i}{v_o} - 1\right)}\,.$$

In the above expressions, $v_i, i = 1, 2$ are, respectively, the initial velocities of the two pulses constituing the wave.

4 Numerical simulations

Knowing an approximate two pulse solution of (4) we now investigate numericaly the influence of the variation of the radius and Young modulus. In order to compare theory and experiments, at the boundary corresponding to the root of the aorta, we impose the ejection pattern of the left ventricle in the form of the radial displacement $\gamma(z = 0, t)$, whereby the shape of the curve is assumed to be equal to expression (7) with an initial amplitude of 0.1 and the following parameters : $v_1 = 3.70$ m/s, $t_1 = 40$ms for the first pulse and $v_2 = 3.55$m/s, $t_2 = 57$ms for the second one. At $z = 0$, we impose $R_0(z = 0) = 1.5$ cm and $E(z = 0) = 30$ N/cm^2. These values correspond to a sound velocity of $v_0 \approx 3.5$ m/s.

Fig. 1 shows the conbined influence of increasing Young modulus and decreasing radius of the artery on the evolution of the pressure (Fig. 1b) and wall displacement pulses (Fig. 1a) corresponding to 20% parameters

variation. We note the pressure pulse maintains its form as a solitary wave right out the periphery while its amplitude increase with its velocity. This is in agreement with in vivo and supports the hardeness role of the vessel wall on the shape changes of the pressure pulse. The generating process of a steep front is also clearly shown in this case. This can be explain by the fact that the dispersion term decreases with the radius. As a result, the nonlinearity becomes dominant and the wave front steepens.

5 Discussion

The qualitative agreement between the experimental observations and the numerical calculations given by the model is satisfactory. Our results show that the "steepening" and "peaking of the wave front occur in accordance with increasing in wave velocity. The pulses with large amplitude move rapidly and are narrower in width as observed *in vivo*. If the dicrotic wave is taken into account, a two-pulse wave model, perturbed by the taper effect and the Young's modulus variation is valid. In this way, the blood pressure pulse can be seen as a wave whose shape evolves between a solitary wave and a shock-like wave. However, we have neglected several effects in order to have a complete quantitative agreement and this model can be extended by including the radial inertia of the liquid and that of the tissues surrounding the artery. The first effect increases the dispersion coefficient that becomes equal to $C' = C + \frac{1}{8}$. The second effect is discussed with a radial approximation of the tissue displacements and the calculation of the effective stress on the wall adds a extra term to the dispersion coefficient $C'' = C' + \frac{1}{2}\ln(R_{ext}/R_0)$ (R_{ext}: approximate radius of the moving tissue volume). Both effects greatly increase dispersion and allows us to compare our results with available data. With this new dispersion term, the displaced blood volume $V = 4\pi R_0^2 AmL$ is $V = 51$ cm^3 and is consistent with the 60 cm^3 injected in the aorta by the heart during one cardiac cycle [8].

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DEFECT-LIKE SOLUTIONS OF 2D SINE-GORDON EQUATION AS A MODEL OF INHOMOGENEOUS STATES IN LARGE AREA JOSEPHSON JUNCTIONS

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Stability of the singular defect-like solutions of 2D damped and driven sine-Gordon (SG) equation is investigated by computer methods. It is found that the driven instability of the defects gives rise to the new dynamical structures such as spirals and target patterns. The results are applied to large area Josephson junctions where "defects" are induced by Abrikosov vortex penetrations or strong spatial localizations of bias currents.

1 Introduction.

The SG equation is a wide spread model in condensed matter physics. The familiar applications are Josephson junctions [1] , magnetics [2] and liquid crystals [3,4] . The one-dimensional SG equation is a well investigated model but its two-dimensional (2D) generalization has some features which allow one to say about new 2D physics of the SG model.

The simplest case under discussion is elliptic equation:

$$\Delta\varphi = \sin(\varphi), \qquad \Delta = \partial_x^2 + \partial_y^2, \tag{1}$$

which is a stationary version of the general SG model. According to well-known Derrick theorem [5] eq.(1) has no stable smooth solutions that are bounded in space. Therefore in general time-dependent case the bounded states can be only dynamic ones like pulsons. Nevertheless if we abandon a smoothness and will treat solutions as "distributions" then we acquire a new class of nontrivial solutions [6] of eq.(1) which, in particular, can be bounded in space. The solutions have point-like singularities with the cores of two basic types:

$$\varphi_V = k\ \tan^{-1}(x/y), \qquad \varphi_S = \alpha\ \log(1/r), \qquad r^2 = x^2 + y^2, \tag{2}$$

i.e. "vortex"- and "source"-like types. Here k-topological charge $(\pm1, \pm2, \pm3, \pm4, ...)$ of vortex and α– intensity of source.

These singular solutions have important applications in condensed matter as a model of defects [2,6,12]. The remarkable example is a large area

Josephson junction with sizes much greater than the Josephson penetration depth where the vortex-like core φ_V with $k = \pm 1$ associates with penetration of a unite Abrikosov vortex from superconductive strip into the junction [7] and the source-like core φ_S corresponds [8] to point injection of a bias current $j = 2\pi\alpha\delta(r)$. Both cases are experimentally observed situations [7,9]. One notes that the second effect of the strong spatial localization of bias currents is a typical phenomenon for simple crossed [8] and overlapped [10] junctions with wide current-feeding superconductive strips. Because feeding currents are located along the edges of the strips then the bias currents concentrate near the corners of a rectangular junction and may be approximated as point-like "sources" for sufficiently wide strips [11].

The goal of this paper is to investigate the driven instability of the defect-like patterns in the scope of damped and driven generalization of eq.(1)

$$-\varphi_t + \Delta\varphi = \sin(\varphi) + f, \tag{3}$$

which gives rise to the new dynamical structures like spiral and target patterns.

2 Defect-like solutions of elliptic SG equation.

The following theorem (Serrin [13]) gives exhaustive description of the singular cores for solutions of elliptic SG eq.(1): *equation* $\Delta\varphi = g(\varphi)$ *with sublinear* $g : \mid g(x) \mid \leq \mid x \mid$ *has nonremovable singularities only with asymptotic* $\varphi \sim \varphi_V$ *or* $\varphi \sim \varphi_S$ *near the singular point.* One notes that functions φ_V and φ_S (2) are well-known fundamental solutions of the linear Laplacian equation $\Delta\varphi = 0$. Therefore the cores are basically linear objects and nonlinearity has an effect only on regions far from the singular cores of patterns. Of course, the more complex singular cores with arbitrary linear combinations of fundamental solutions φ_V and φ_S are admitted. The computer simulations have shown [2,6] that the generic solution $\varphi(x, y)$ of eq.(1) are determined totally by the structure of singularities and their locations on the (x, y)-plane.

The simplest pattern is a fluxon line [a] starting from the singular point with vortex-like core (see Fig.1,$k = 1$, where the function $J(x, y) = \sin(\varphi)$ proportional to Josephson current is pictured). The fluxon transforms asymptotically to a quasi-one-dimensional soliton

$$\varphi(x, y) \approx 4 \tan^{-1}(e^{-x}), \qquad y \to \infty. \tag{4}$$

The bounded dipole-like defect is also possible which consist of a segment of fluxon line connecting two vortex-like cores with opposite topological charges

[a] We use the term "fluxon" bearing in mind the Josephson junction application. Another terms — domain wall or kink

$k = +1$ and $k = -1$ (see Fig.1,dipole). The patterns with $k = 2$ and $k = 4$ are presented in Fig.1. They consist of two and four fluxons starting from the singular cores with corresponding topological charges. One notes that the solution with $k = 4$ has the simple analytic form [14]

$$\varphi(x,y) = 4 \ \tan^{-1}\left(\sqrt{\frac{1-\gamma}{\gamma}}\frac{sh(\sqrt{\gamma}\ x)}{sh(\sqrt{1-\gamma}\ y)}\right), \qquad 0 < \gamma < 1. \qquad (5)$$

Now a lot of complex patterns generating by finite or infinite (periodic) distributions of singular cores with $k = \pm 4$ are obtained in analytic form by Hirota method [15] and θ-function formalizm [16]. But I point out that all these solutions consist of vortices with topological charges multiple by 4. Moreover it was found [17] that solutions with $k = 1, 2$ or dipoles belong to continous spectrum of some elliptic generalization of Inverse Scattering Transform method and then have no appropriate analytic form. So the most solutions of eq.(1) are remained up to now as the subject of computer simulations.

Another type of solutions are source defects. The simple one is shown in Fig.1. It may be interpreted as snugly packing ring fluxons with common center in singular core of the source type φ_S. As it was found by computer simulation the radius of the pattern grows linearly with α as $R \approx (3/4)\alpha - 2$ for $\alpha > 5$, and $R \to 0$ for $\alpha \to 0$. For $r > R$ the solution disappears exponentially $\varphi \sim exp(-r)$.

3 Instability of isolated defects and pattern formation.

The defect-like solutions are stable in a framework of eq.(3) if the driving force $f = 0$, but the nonzero force can produce their instability and appearance of new dynamical patterns. The problem of stability is very important for Josephson junction physics. The stationary solutions correspond to superconductive (Meissner) state of a junction and instability can force the fluxon motion which leads to transition of a junction to the dynamic resistive state [12] . In following investigations we will fix coordinates of the defects cores. This assumption is natural for Josephson junctions where source-like cores can be fixed by peculiarities of junction border and vortex cores are usually fixed by pinning of Abrikosov vortices. We will use also the conditions $(\partial\varphi/\partial\mathbf{n})_\Gamma = 0$ on the boundary Γ of the rectangular region of simulation. Moreover the sizes of the region will be large enough that the influence of boundary conditions on dynamics can be negligible.

The source type defect is stable under the influence of driving force $f < f_c(\alpha)$, where the critical value decreases with growing α as $\sim 1/\alpha$. If $f > f_c(\alpha)$

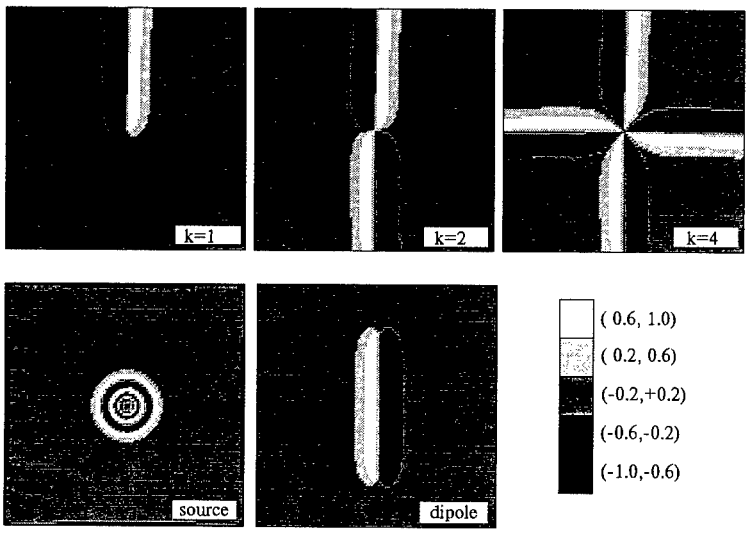

Figure 1: Defect-like solutions of the elliptic SG equation (1). The destribution of $J(x,y) = \sin(\varphi)$ is shown by the grey-level scale.

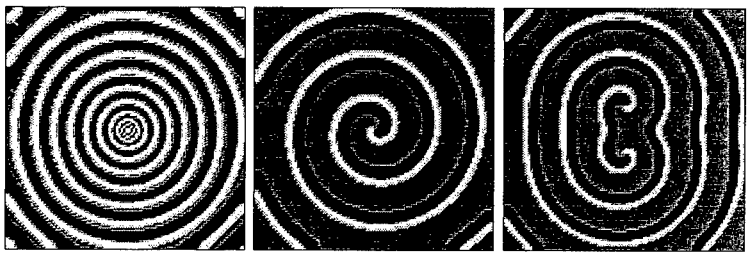

Figure 2: Target (a,c) and spiral (b) patterns as a result of driven instability of source (a), vortex (b) and dipole (c) defects; $f = 0.9$.

the defect loses stability, and ring fluxons begin to move outside with constant velocity forming the steady structure like target pattern (Fig.2a).

The stationary driving force has the most destructive effect on the isolated vortex solution. In this case $f_c = 0$ and even small force rolls up the fluxon to a spiral pattern with center in the defect core. In final stage the spiral rotates with a constant frequency around the core and has a constant radial step (Fig.2b).

The physical mechanism of pattern formation as a result of driven instability of source or vortex solutions is simple enough. Really, it is well known that driving force f in SG Eq. (3) causes moving of a quasi-one-dimensional fluxon in direction defined by its topological charge . The ultimate velocity of the moving fluxon is established as a balance between driving and dissipation. The same effect of driving force to fluxons remains valid in the 2D case, but now the core of the vortex is fixed and the fluxon line will be reeling around the core. Because the normal velocity of the moving fluxon is bounded, the steady spiral pattern shown in Fig. 2b is formed. This supposition for physics of spiral formation is confirmed by simulation.

The target patterns are formed for $f\alpha > 0$ when driving force acts on ring fluxons outside the center and seeks for its divergence. If the force exceeds some critical value the ring fluxons leave the core and a divergent pattern forms (Fig. 2a). The target pattern can be induced also by a dipole defect if driving force $f > f_c(L)$, where the critical value f_c is decreased with grows of L — distance between the vortex cores making up the dipole (Fig.2c).

4 Boundary induced patterns.

Let us consider an influence of boundary conditions on the pattern formation. We set $f = 0$ and impose non-zero boundary conditions: $(\partial\varphi/\partial\mathbf{n})_\Gamma = j(s)$, $s \in \Gamma$. This case is the converse of previous one in Sec.3 and more typical for Josephson junction applications where bias currents are usually boundary injected rather than distributed over the junction area (i.e. for $f \neq 0$).

The first case is extremely nonuniform boundary conditions which can be approximated as point-like sources located in corners of a rectangular junction with ratio of intensities: $\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4$. For crossed junctions the typical ratio is [8] $3/4 : 1/4 : -1/4 : 1/4$ and for overlapped junctions it is $1/4 : 1/4 : 1/4 : 1/4$. The pattern formation in a crossed junction have been investigated in Ref.11. Now we will investigate an overlapped junction.

For small currents (parameter α) the stationary source-like defects are formed near each corner of the junction. The defect radius increases with α and for $\alpha > \alpha_{cr}$ the defects in adjoining corners begin interact one with

another. After that the stationary state becomes impossible and collapsing ring fluxons are formed periodically in the center of the junction which then transfer into dynamic resistive state. Some stages of the dynamics covering one period are shown in Fig.3. The critical value of α was found to be subjected to the simple rule $\alpha_{cr} \approx 0.6L$ $(L > 5)$ where L is the size of the square junction.

Let us now consider the uniform distribution of the boundary injected currents $(\partial\varphi/\partial\mathbf{n})_\Gamma = j = const$. It can be possible for junctions with sizes smaller than width of current feeding strips and located in central part of the intersection region of the strips. There is a critical value j_{cr} of current which defines the state of a junction. If $j < j_{cr}$ the Josephson currents are concentrated in narrow region along the junction boundary and stationary Meissner state is achieved. But if $j > j_{cr}$ the boundary generated fluxons penetrate into the junction forming a resistive dynamic state. These patterns will be deformed considerably by penetration of Abrikosov vortices that is by appearance of vortex-like defect. The process of forming some new dynamical patterns are pictured in Fig.4. A current free $(j = 0)$ junction with penetration of one Abrikisov vortex is assumed at initial moment of time. Then the boundary currents j are switched on and the fluxon line begins interact with the boundary forming dynamical pattern. If $j < j_{cr}$ the fluxon slides slowly along the boundary in anti-clockwise direction (Fig.4a) and stationary Meissner state is destroyed. If $j > j_{cr}$ the spiral pattern is formed (Fig.4b) which rotates in direction opposite to the spiral shown in Fig.2b.

5 Conclusion.

The application of dynamical patterns described above to the Josephson junction physics is justified by clear mechanism of defect creation in the junction area [12]. Now I point out another interesting applications of defect- like solutions of SG eq.(3) in magnetics and liquid crystals.

First example is an easy-plane magnetic with anisotropy in basic plane. Here the magnetization vector lies in the plane (x, y) and variable $\Phi = \frac{1}{2}\varphi(x, y)$ represents the angle of its inclination relative to anisotropy axis. The vortex-like solution with $k = 2$ (see Fig.1) describes now a domain wall with horizontal Bloch line as a singular core [2]. The pattern formation in magnetics was observed experimentally in Ref.18 but it was not an easy-plane magnetics. Nevertheless one can hope that the physical nature of processes of pattern formation has many common features.

Another example is nematics located in a rotating magnetic field [3]. This case is very similar to magnetics where magnetization vector is exchanged be director one. The SG model was proposed in Ref.3 and computer simulations

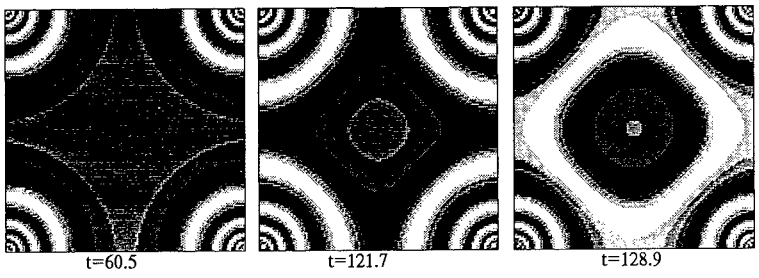

Figure 3: The dynamics of the overlap junction in the unstable region ($\alpha > \alpha_{cr}$). The boundary injected currents are approximated as source-like defects ($\alpha = 10$).

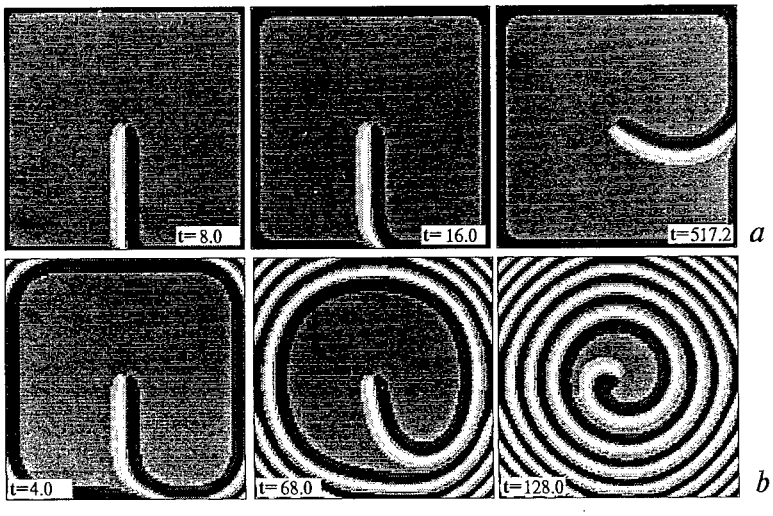

Figure 4: Dynamics of a vortex-like defect in a square junction with uniform destribution of boundary currents; (a)— $j = 1.1 < j_{cr}$ ($j_{cr} \approx 1.2$), (b)— $j = 1.9 > j_{cr}$.

confirmed [4] that the model really can describe the pattern formation closed to the experimental dynamics.

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HARMONIC AND ANHARMONIC DYNAMICS IN PROTEINS AND MOLECULAR CRYSTALS.

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Simulations using fast computers enable a detailed description of the dynamics of molecules in condensed phases to be obtained. The calculated dynamical trajectories can be used to derive experimental quantities such as scattering intensities. These quantities can then be compared with experiment and interpreted in detail using the simulations. We examine a variety of motions in systems primarily of biological interest, including intramolecular vibrations, crystal phonons, diffusive motions and protein internal dynamics.

Introduction

The computer simulation of atomic dynamics in biological molecules is now a flourishing field of research and has many applications of interest in basic and applied biological and materials science. Nonlinear effects are plentiful[1] and the derivation of simplified models for the dynamics that can describe the detailed simulation results will always be useful. Molecular dynamics simulation and normal mode analysis, performed with an empirical potential energy function, can be used to examine harmonic and anharmonic dynamics of proteins, nucleic acids and other molecules of biological interest. In each case simulation results can be compared with experiment, by calculating directly measurable quantities such as scattering intensities. Our group has been involved in characterizing atomic dynamics in proteins and molecular crystals. The crystalline state provides molecular systems that are well defined structurally and can be used for the detailed examination of a wide range of intermolecular interactions. In what follows a summary of some recent results in this field is given.

Acetanilide.

Crystalline acetanilide has been a candidate system for the presence of solitons for some years. However, until recently, a complete description of the dynamics

of this system, using computer simulation, has been lacking. In a recent article[2] a model for the dynamics of acetanilide was derived in the full configurational space of the crystal *i.e.*, including intramolecular and intermolecular degrees of freedom. The model enabled crystallographic and spectroscopic results to be reproduced and attributed. However, a splitting of the amide I absorption at low temperatures, which might be the signature of a soliton, was not seen.

L-alanine.

L-alanine provides a system in which coherent inelastic neutron scattering experiments can be combined with simulation analyses to examine low-frequency collective vibrations. It has also been suggested that this system might harbour nonlocalized dynamical phenomena. In recent work[3] experiments were performed to determine the phonon dispersion curves and scattering intensities for the acoustic and low-frequency optic modes of this system. The potential function was parameterized using *ab initio* quantum chemical calculations of hydrogen-bonding interactions and torsional barriers. A harmonic approximation to the parameterized energy function was used to derive the theoretical phonon dispersion relationships and neutron scattering intensities. A comparison between the theoretical and experimental curves is presented in Fig. 1.

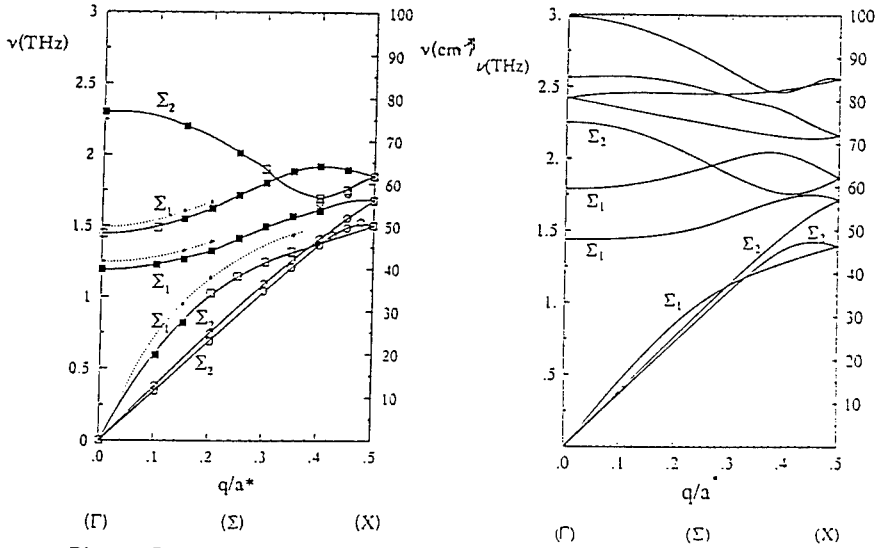


Figure 1: Phonon Dispersion Curves in L-alanine in the a* direction. a) measured b) calculated

Diffusional Dynamics in Urea-Alkane Inclusion Compounds.

In urea-alkane inclusion compounds urea forms a host structure made of parallel channels that incorporate alkane guest molecules. These compounds are incommensurate. That is to say that, if c_h is the repeating distance of the host structure in the direction of the channels and c_g the mean distance between two consecutive molecules in a channel, one cannot find two small integers n and m, such that $nc_h = mc_g$.

The dynamical behaviour of the alkane chains has been studied by incoherent neutron scattering and molecular simulation[4]. In Fig. 2 experimental neutron scattering curves are compared with curves calculated from simulations with different values of c_g : a simulation with one alkane per channel (i.e. $c_g = +\infty$) , and two simulations in which c_g=29.38 Å and c_g=26.44 Å with 3 and 5 alkanes per channel, respectively. The last value c_g=26.44 Åis the closest to the experimental estimate. One can see that it is necessary to include the experimental compression of the alkanes (c_g) to obtain a reasonable agreement with experiment. In the simulations where c_g is greater than the experimental value, the alkane chains perform too-large amplitude motions. Full details of this work will be published elsewhere.

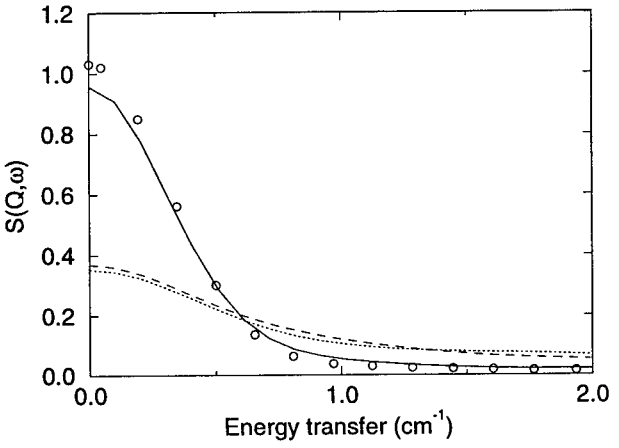


Figure 2: Neutron scattering dynamic structure factor, $S(\vec{q}, \omega)$, for urea-alkane inclusion conpound : Experiment (o) Simulations c_g=26.44 Å(—) c_g=29.38 Å(···) c_g=∞ (- -).

The Alanine Dipeptide.

At very low temperatures atomic motions are effectively harmonic, and normal mode analyses can be used to compute neutron scattering intensities and to compare them with experiment. Here we show an example of such a calculation. The system is a crystal of "alanine dipeptide ", a molecule of 22 atoms used as a model for the basic structural unit of peptides and proteins. The computed $S(\vec{q},\omega)$ is shown in Fig. 3 and can be compared with corresponding experimental data.

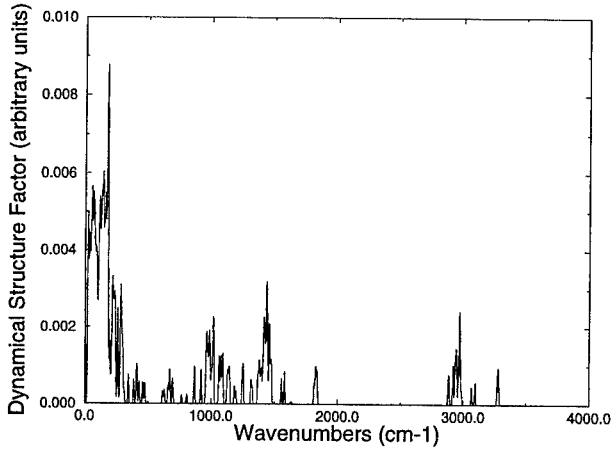


Figure 3: Vibrational neutron scattering dynamic structure factor, for crystalline alanine dipeptide.

By adjusting terms in the potential energy function so as to fit experiment, insight can be gained into the basic interactions present in the peptide system[5]. This work is in progress.

Dynamics of Proteins: X-ray diffuse scattering from lysozyme.

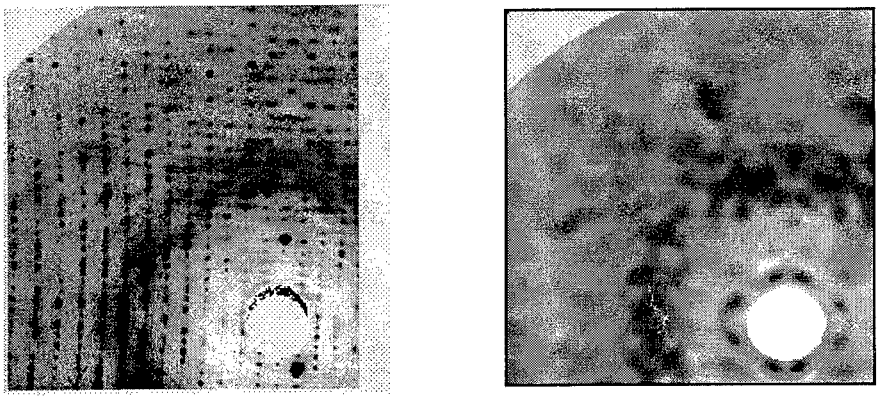
X-ray crystallography has provided much information on the atomic organization of proteins and crystal structures, but in most X-ray analyses, only the Bragg reflexion intensities have been used. To make full use of protein crystal scattering data, the relatively weak, diffuse scattering intensity observed should also be exploited. Diffuse scattering can arise from displacements of

atoms from their average positions in the crystal and therefore contains dynamical information that is of particular interest as the motions in proteins play an essential role in their function.

The diffuse scattering intensity $I^{\mathcal{D}}$ can be written in terms of the structure factors as :

$$I^{\mathcal{D}}_{hkl}(\vec{q}) = |F_{hkl}(\vec{q})|^2 - | < F_{hkl}(\vec{q}) > |^2 \quad (1)$$

Recently, studies on intramolecular dynamics in several proteins and in particular in lysozyme crystals combining molecular dynamics simulations with X-ray scattering experiments have shown that diffuse X-ray scattering can be related to intramolecular correlated movements and atomic displacements[6]. Figure 4a shows a 10 minute exposure X-ray scattering pattern from orthorhom-



a *b*

Figure 4: X-ray diffuse scattering from lysozyme crystals a)Experimental b)Calculated.

bic hen egg-white lysozyme (from ref 6) Figure 4b shows an X-ray diffuse scattering pattern obtained from a 400 *ps* molecular dynamics simulation. Comparison between these patterns can be used to examine correlated, functional motions in proteins.

Protein Dynamics: Low-temperature Vibrational Inhomogeneity?

At physiological temperatures there is a marked anharmonic contribution to picosecond-timescale internal dynamics of globular proteins[7]. Both diffusive and vibrational motions exist, the former making up the major part of the average atomic mean-square displacements. The diffusive motions result

in the protein exploring regions of the potential surface associated with different potential energy minima. This suggests that cooling an experimental sample of protein molecules might lead to a structurally heterogeneous protein population in which individual molecules are trapped in different minima. A corollary of the existence of structural inhomogeneity is the presence of concomitant 'dynamical inhomogeneity' in a protein. At low temperatures ($<\sim$ 200K) the picosecond-timescale motions are essentially vibrational. The forms and frequencies of the vibrational modes of a protein molecule in a given potential energy minimum depend on the specific interatomic interactions present.

A theoretical investigation into low-temperature vibrational heterogeneity in a small protein, the bovine pancreatic trypsin inhibitor (BPTI), has been performed[8]. In Fig.5 is shown the vibrational density of states , $g(\omega)$, in several

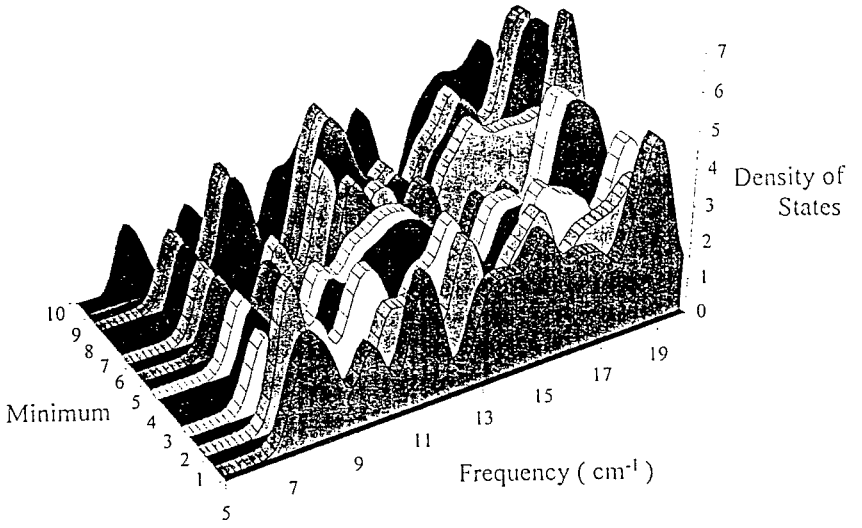


Figure 5: Vibrational density of states, $g(\omega)$, in different potential energy minima of a small protein, the Bovine Pancreatic Trypsin Inhibitor.

protein minima. The peaks correspond for the most part to modes which, in the depicted frequency range, are mostly collective vibrations distributed over the protein. Clear differences in the frequencies of these lowest-frequency modes are seen in the different potential energy minima, testifying to the presence of low-temperature dynamical inhomogeneity in the macromolecule.

Conclusions.

The results presented above illustrate some of the many facets of the dynamics of condensed phase materials that can be probed using molecular dynamics simulation. This field of activity can provide a stepping stone from experiment to a simple physical picture of atomic motions, and from the world of interatomic chemical physics to the examination of functional mechanisms in biological macromolecules. The elucidation and description of nonlinear effects in biomolecular dynamics remains a significant goal for the future.

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COUPLED SOLITONS IN CONTINUOUS AND DISCRETE JOSEPHSON TRANSMISSION LINES

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We review recent progress in experiments and modelling of soliton (fluxon) dynamics in coupled Josephson transmission lines. Our main focus is to draw parallels and to emphasize the differences between the continuous and discrete cases. Single-fluxon dynamics in discrete Josephson transmission lines differs essentially from that in continuous lines due to the modified dispersion relation for small-amplitude linear waves which interact with moving fluxons. Coupled fluxons in continuous lines (long Josephson junctions) recently have been investigated experimentally and theoretically for so-called stacked Josephson junctions, which are also addressed in this review. One may expect similar features for coupled fluxons in planar-coupled discrete Josephson lines, which are now only beginning to be studied. Applications of these concepts to practical Josephson devices such as high-frequency oscillators are discussed.

1 Introduction

The dynamics of solitons in Josephson transmission lines (JTL) appears often as a topic in various contexts within superconductivity and nonlinear physics. Potential applications and the appearance of high-T_c superconductivity are clearly some of the reasons for this recently increased interest. A soliton in a JTL is often called a "fluxon" since it accounts for a magnetic flux quantum moving between two superconducting electrodes. Although much can be intuited about the behavior of fluxons in spatially discrete JTLs from the behavior of the corresponding continuum lines, discreteness introduces a number of aspects into the dynamics that have no counterparts in the continuum systems, a prime example being localized dynamic states. In this brief review we try to follow parallels and to emphasize the differences between continuous and discrete cases, and also to address the recently developed field of coupled solitons in these lines. Several comprehensive overviews of the field of solitons in JTLs[1-3] exist to date, and some recent developments have been

addressed in Ref. 4. We note that progress in thin film and photolithographic technology now permits the construction of large one- and two-dimensional planar arrays of Josephson tunnel junctions having quite precisely designed characteristics; moreover, the ever-increasing availability of computing power also permits large-scale simulations of such arrays, even using small desk-top machines.

2 Continuous and discrete lines

2.1 Continuous sine-Gordon model

For a one-dimensional, overlap-geometry[2] continuum JTL, schematically shown in Fig. 1(a), the perturbed sine-Gordon equation that describes the dynamics of the system, in normalized form, is

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \beta\phi_{xxt} - \gamma\,. \tag{1}$$

In Eq. (1), ϕ is the quantum phase difference between the two superconducting electrodes of the junction, ϕ_t is the voltage, α is a dissipative term due to quasiparticle tunneling (normally assumed ohmic), β is a dissipative term due to surface resistance of the superconductors, γ is a normalized bias current, and x and t are normalized space and time, respectively. The time t is measured in units of ω_0^{-1}, where ω_0 is the plasma frequency, and length x is measured in units of the Josephson penetration depth λ_J. To account for the behavior of a real system, Eq. (1) must be supplemented by appropriate boundary conditions that take into account, for example, the magnetic field applied in the plane of the junction.[1-3]

An important solution to Eq. (1) is the fluxon or soliton, which is a localized 2π kink in the phase difference ϕ, moving with a velocity u. This velocity is determined by a balance between the losses, governed by α and β, and input due to the bias γ. The velocity u is measured in units of $\bar{c}$, where $\bar{c}$ is the velocity of light in the junction, also called the Swihart velocity.

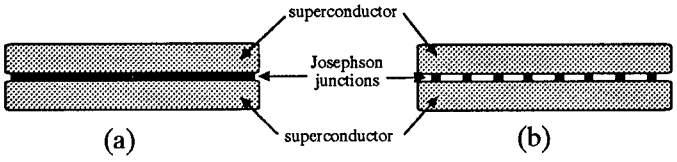

Figure 1: Schematic view of (a) a continuous and (b) a discrete Josephson transmission line.

With a perturbational approach u is approximately given by the expression[1-3]: $u = 1/\sqrt{1+(4\alpha/\pi\gamma)^2}$, and thus may assume values $0 < u < 1$. Note that for low values of the bias current $u \sim \gamma$, while for large values of γ/α a saturation occurs such that the velocity approaches $\bar{c}$, *i.e.*, a fluxon behaves like a relativistic particle. In experiments, the signatures of soliton motion are so-called steps in the current-voltage ($I-V$) characteristics: the profile of the $I-V$ curve is given by the dependence of the average soliton velocity $u \propto V$ on the driving force $\gamma \propto I$.

2.2 Discrete sine-Gordon model

A standard approach for the numerical integration of Eq. (1) involves spatial discretization, *i.e.*, replacing the spatial derivatives with finite-difference approximations. Having a lattice spacing of a, this yields directly

$$\frac{dV_n}{dt} = \frac{1}{a^2}(\phi_{n-1}-2\phi_n+\phi_{n+1})+\frac{\beta}{a^2}(V_{n-1}-2V_n+V_{n+1})-\sin\phi_n-\alpha V_n+\gamma\,, \quad (2)$$

for the $N-2$ interior points of the junction, and

$$\frac{d\phi_n}{dt} = V_n\,,\ \ 1 \le n \le N\,. \quad (3)$$

In this way we obtain $2N$ first-order ordinary differential equations in $2N$ time-dependent variables (N phases and N voltages), which are just the Kirchhoff circuit-law equations for an array of N discrete Josephson junction elements interconnected via a parallel resistance/inductance combination.

A parallel biased array of small Josephson junctions, described by Eqs. (2,3) and schematically shown in Fig. 1(b), represents an experimental realization of the spatially discrete sine-Gordon lattice. Although the discrete sine-Gordon equation is much simpler for numerical studies, its non-integrability is probably a major reason for which very few experiments have been performed in this field. Peyrard and Kruskal[5] were the first to point out that, even with large discreteness, the dynamics of a localized nonlinear kink in the sine-Gordon lattice may exhibit some features of solitonic nature very close to the properties of the continuum sine-Gordon solitons. In general, the kink motion through a discrete lattice leads to radiation of small-amplitude linear waves.[5] The dispersion relation for such linear waves $\phi_n = \phi^{(0)}\exp[i(\omega t - kan)]$ is

$$\omega^2 = 1+\frac{4}{a^2}\sin^2\left(\frac{ka}{2}\right)\,, \quad (4)$$

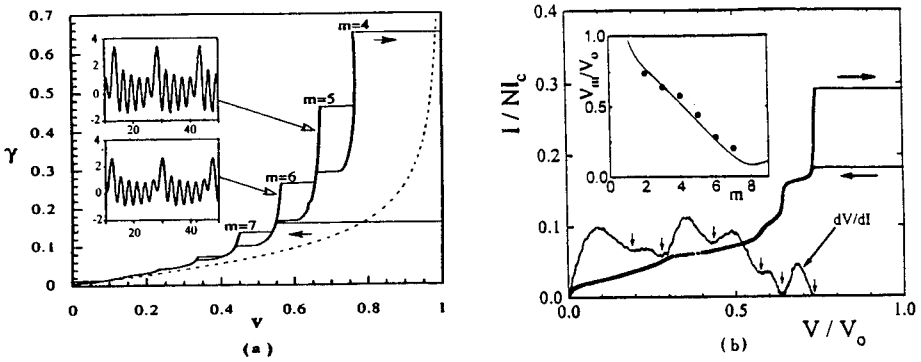

Figure 2: (a) Numerically simulated $I-V$ characteristics of a discrete annular JTL with one trapped fluxon.[6] The dashed line shows the $I-V$ curve for a continuum junction with the same normalized length $L = 10$. The insets show the instantaneous voltage as a function of time for two different points of the $I-V$ curve measured in the middle junction of the array for the modes $m = 5$ and $m = 6$. (b) Experimentally measured $I-V$ characteristics of the annular array of 8 junctions with one trapped fluxon measured by van der Zant et al.[7] *(reprinted with permission).* The solid line shows dV/dI; six dips correspond to six resonant steps in the $I-V$ curve. Inset: voltage position of these steps *vs.* mode number m (points); the solid curve is a fit to Eq. (5).

where ω is the frequency of the linear waves (lattice phonons) and k is their wave number.

A resonance between the emitted waves and the moving periodic chain of fluxons can occur when the phase velocity of the excited linear waves coincides with the fluxon velocity. Imposing further that the period of the linear waves and the period of the fluxon chain must commensurate, a resonance condition can be obtained for fluxon steps:[6]

$$v_m = \frac{L_f}{2\pi m}\sqrt{1+\frac{4}{a^2}\sin^2(\frac{\pi m a}{L_f})}\,. \tag{5}$$

where m is an integer. Eq. (5) gives the values of the velocity v_{fl} for which, with a given spatial period L_f, the fluxon chain (or just one fluxon passing through the same point of the periodic array with the length $L = L_f$) generates resonant super-radiant emission. Fig. 2(a) shows the calculated $I-V$ characteristics[6] of an annular array having parameters $N = 10$, $a = 1.0$, and $\alpha = 0.1$. The dashed line shows the $I-V$ curve for the continuum case calculated for a system

with the same normalized length. The difference between the discrete and the continuum cases is clearly seen. The two curves (solid line and dashed line) lie close to each other for low values of the bias current, but the $I-V$ curve for the discrete array contains of a series of equally spaced current singularities that are not present in the continuum characteristic. The numerically calculated voltage positions of these steps are in good agreement[6] with the kinematic formula (5).

Theoretical predictions made in Ref. 6 have recently been confirmed experimentally by Van der Zant et al.[7] in measurements on 8–junction annular Nb-Al/AlO_x-Nb arrays. Fig. 2(b) shows their experimental data with one fluxon trapped in the array. The positions of the resonances (well visible on the derivative dV/dI plot) are in agreement with the theoretical prediction (5).

In Fig. 2(a), the shift of the discrete array curve at high bias towards lower velocities is very clear. One may expect this type of behavior from the simple intuitive argument that the Lorentz contracted size of a fluxon cannot become smaller than the discreteness parameter a. Thus, even for large driving force the fluxon in the discrete case cannot move faster than a certain maximum velocity $\tilde{v}(a) < 1$.

3 Coupled lines

Soliton dynamics in coupled JTLs has recently become a subject of intensive theoretical and experimental investigation. Due to the interaction between the lines, solitons moving in different lines may form new coherent structures that propagate along the system. Using modern technology, the most efficient coupling of JTLs can be achieved by placing several lines directly on top of one another. These structures are called stacked Josephson junctions.

3.1 Stacked junctions

Two inductively coupled long Josephson junctions are described as derived by Sakai et al.[8] by a system of two coupled sine-Gordon equations:

$$\begin{aligned} \tfrac{1}{1-S^2}\,\phi^A_{xx} - \phi^A_{tt} &= \sin\phi^A + \alpha\phi^A_{tt} + \gamma^A + \tfrac{S}{1-S^2}\,\phi^B_{xx}\,; \\ \tfrac{1}{1-S^2}\,\phi^B_{xx} - \phi^B_{tt} &= \sin\phi^B + \alpha\phi^B_{tt} + \gamma^B + \tfrac{S}{1-S^2}\,\phi^A_{xx}\,. \end{aligned} \tag{6}$$

Here $\phi^A(x,t)$ and $\phi^B(x,t)$ are the superconducting phase differences across the junctions A and B, respectively, and γ^A and γ^B are the bias currents. The coupling strength S ($S<0$) can be calculated from experimental parameters[8], which are essentially the middle electrode thickness d and the London penetration depth λ_L. For simplicity, here it is assumed that junctions A and B

Figure 3: Sketches of (a) a two-junction stack of continuum junctions and (b) an inductively coupled pair of discrete JTLs.

are identical. A typical experimental value for S in a two junction stack lies in the interval from -0.2 to -0.7 (see for example Ref. 9). A schematic drawing of two stacked long junctions is shown in Fig. 3(a).

Eqs. (6) lead to two different modes – with in-phase and with anti-phase oscillations in two junctions. The splitting of the velocity $\bar{c}$ of small amplitude electromagnetic waves (Swihart velocity) in a double Josephson junction stack was predicted in the linear approximation many years ago by Ngai[10], but only recently observed experimentally.[9] It can be shown[11] from Eqs. (6) that for the two-junction system there exist two wave propagation modes with velocities

$$\bar{c}_{\pm} = \frac{\bar{c}}{\sqrt{1 \pm S}} \,. \tag{7}$$

By applying a magnetic field H parallel to the Josephson barrier one finds the so-called Fiske steps in the $(I - V)$ characteristics of the stacks. Fiske steps correspond to resonances between the Josephson frequency and the frequency of one of the cavity modes of the junctions. By measuring the voltage spacings between neighboring Fiske steps $\Delta V_{\pm} = \bar{c}_{\pm}\Phi_0/(2L)$ the characteristic velocities $\bar{c}_-$ and $\bar{c}_+$ have been measured experimentally for double-junction stacks with different thickness d of the common superconducting layer. With decreasing d, the coupling was found to increase, thereby increasing the difference between $\bar{c}_-$ and $\bar{c}_+$. A detailed analysis of experimental data is found to be in good quantitative agreement with theory.[11]

Stacked junctions could serve as model systems for layered high-T_c superconductors.[12] For such systems consisting of many stacked junctions a new type of resonant modes has recently been predicted by Kleiner.[13] In addition to the standing waves (Fiske modes) in the layer plane ($x-$direction), in his model Kleiner also obtained a standing wave pattern in the direction normal to the layers ($z-$direction). Kleiner modes should lead to mutual phase shifts between Josephson oscillations in different coherently operated layers and, thus, should be observable in radiation detection experiments.

3.2 Inductively coupled discrete JTL

In two-dimensional (2D) Josephson junction arrays, inductive interactions play an important rôle both for the statics[14] and for the dynamics[15]. Because the complete dynamics of these systems is rather complicated, it is natural to split the problem into parts and to study first the interaction between fluxons moving in two neighboring elementary rows. First experimental results using this approach have recently been reported.[16,17] Here one deals with planar inductively coupled discrete JTLs, in the form of parallel one-dimensional arrays of Josephson junctions.

By analogy with continuous stacked junctions, coupled discrete arrays are expected to show the splitting of the characteristic velocities of electromagnetic wave propagation. Indeed, the experiments with two coupled arrays[16,17] showed two Fiske-step families characterized by different voltage spacings. For coupled discrete lines, the velocity splitting is modified due to the dispersion relation which differs from the continuum case[18], as mentioned above. Detailed analyses of these effects and further experiments are presently in progress. A next step would be to extend this work toward true two-dimensional "hybrid"[19] (or "shorted") arrays consisting of planar inductively coupled discrete JTLs.

4 Applications

Millimeter-wave oscillators using JTLs continue to attract active research interest. A significant stimulus for the study of Josephson millimeter-wave amplifiers and oscillators is undoubtedly the fact that another Josephson element, the SIS (superconductor-insulator-superconductor) mixer[20], is already firmly established as the best choice as a low-noise front-end detector in the range from $\sim$ 100 GHz to $\sim$ 1 THz, since its intrinsic noise temperature seems to limited only by fundamental quantum-uncertainty effects. Consequently, the idea of a fully integrated superconducting receiver assumes considerable importance, especially for space-borne communications and radio-astronomical systems in which high sensitivity and low weight and volume are crucial. Recently, JTLs operating in the flux-flow mode have shown very promising performance as local oscillators in integrated sub-millimeter wave band superconducting receivers[21].

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THERMODYNAMIC FUNCTIONS OF NONLINEAR 1-D SYSTEMS

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The aim of this paper is to present a technique to obtain the equilibrium thermodynamic properties of nonlinear kink-bearing scalar fields[a] Using known asymptotic methods from the theory of differential equations depending on a large parameter, one calculates temperature and lattice corrections to the free energy. The method allows us to make a distinct separation between the contribution of anharmonic optical phonon field and that of the dilute gas of kinks, and was used to calculate the multi-kink contribution to the free energy. The static correlation functions can also be calculated.

It is clear that this is a classical theory, in which the soliton has the properties of a classical particle, with a considerable stability against perturbations. This is a pure analytical method. The free energy will be calculated in the displacive (continuum) limit, and at low temperatures.

From the phenomenological point of view, the elementary excitations are the renormalized kinks and the low amplitude extended states (phonons). Obviously, the ideal gas phenomenology represents only an approximation of the real situation, but at low temperatures it is in perfect agreement with the exact results of the transfer integral operators.

In summary, the method consists in the following steps:
-The partition function, calculated using the transfer integral operator (TIO) method is determined in the thermodynamic limit by the lowest eigenvalue of TIO.
-In the continuum limit these eigenvalues can be found solving a Schrödinger-type equation, and at low temperatures they can be written as asymptotic series expansions in the small parameter $t = (\beta E_k^0)^{-1}$, where $E_k^{(0)}$ is the rest energy of the kink, $\beta = \frac{1}{k_B t} \gg 1$.
-To solve the Schrödinger-type equation with the potential $V(\phi)$ and with a large effective mass, one follows two steps:

-based on Langer's transformation [2], one looks for an uniform valid expansion of the solution near a degenerate minimum of $V(\phi)$;

[a] For a recent review of the subject we can refer to [1]

-and in the second step, the existence of the other minima are taken into account using symmetry properties of the wavefunction in a symmetry point of the potential. This second stage describes the tunneling between the degenerate minima of $V(\phi)$ and each eigenvalue of an isolated potential well is correspondingly splitted[b]

1 Method

The class of 1-D systems mostly discussed is described by the Hamiltonian

$$H = \sum_i lA\left(\frac{1}{2}\phi_i^2 + \frac{c_0^2}{2l^2}(\phi_{i+1}-\phi_i)^2 + \omega_0^2 V(\phi_i)\right) \qquad (1)$$

where the nonlinearity enters only through the potential $V(\phi)$, assumed to have at least two degenerate minima with nonvanishing curvature. The TIO gives us the integral equation:

$$exp(-\gamma V(\phi))exp\left(\frac{\gamma}{2m*}D^2\right)\Phi_n(\phi) = exp(-\gamma\tilde{\epsilon}_n)\Phi(\phi), \qquad (2)$$

where $D = \frac{d}{d\phi}$, and

$$\begin{aligned} m^* &= \beta^2 A^2 c_0^2 \omega_0^2 \\ \gamma &= \beta l A \omega_0^2 \\ \tilde{\epsilon}_n &= \frac{1}{\sqrt{m^*}}\left(\frac{1}{2}+n\right) + \frac{1}{\gamma} ln\sqrt{\frac{2\pi l^2}{\gamma d^2}}. \end{aligned}$$

Equation (2) is of the form $e^M e^N$ with $M = -\gamma V(\phi)$ and $N = \frac{\gamma}{2m^*}\frac{d^2}{d\phi^2}$ two non-commuting operators. In order to write it in the form of a single exponential, e^C, we have to use a Baker-Hausdorf formula which turns to be a series expansion in two small parameters $\frac{l^2}{d^2}$ and $\frac{\gamma}{m}$.

In the lowest order in $\frac{l^2}{d^2}$, a Schrödinger-like equation is obtained [3]

$$-\frac{1}{2m^*}\frac{d^2\Psi}{d\phi^2} + V(\phi)\Psi = \tilde{\epsilon}\,\Psi, \qquad (3)$$

[b]The method can be included in the class of the improved WKB techniques and for more details and comparison with other works we refer to our previous papers [3], [5,6,7]

Taking into account the first order terms in $\frac{l^2}{d^2}$ an effective potential V_{eff} is obtained [3, 4]

$$V_{eff} = V(\phi) - \frac{l^2}{24d^2}\left(\frac{dV}{d\phi}\right)^2 + O\left(\frac{l^2}{d^2}\right). \tag{4}$$

In order to find the lowest eigenvalue of TIO, one applies a Langer-Cherry transformation [2]. This is a transformation of both dependent and independent variables. The new dependent variable has a harmonic oscillator behaviour near the minimum of the potential and gives an uniform valid expansion of the solution in an isolated potential well. For the ϕ^4 case the lowest eigenvalue is:

$$\tilde{\epsilon^{(0)}} \sim \frac{1}{2\sqrt{m^*}}\left[1 - \frac{1}{2\sqrt{m^*}} + O\left(\frac{1}{m^*}\right)\right], \tag{5}$$

and for the sine-Gordon case

$$\tilde{\epsilon}_0 \sim \frac{1}{2\sqrt{m^*}}\left[1 - \frac{1}{16\sqrt{m^*}} + O\left(\frac{1}{m^*}\right)\right]. \tag{6}$$

In the second step, the other degenerate minima are taken into account. From boundary conditions one obtains the tunneling term. For the ϕ^4 case we get

$$-1 = 2\sqrt{\pi e}\frac{e^{-i\pi\nu}}{\Gamma(-\nu)} exp\left[2\lambda\int_0^{\mu}\sqrt{V-\tilde{\epsilon}}d\phi + \nu(1+ln2) - \frac{1}{2}(1+2\nu)ln(1+2\nu)\right], \tag{7}$$

where $\nu = a\lambda - \frac{1}{2}$ and $a \approx \frac{\tilde{\epsilon}}{\sqrt{2}}\left[1 + \frac{3}{4}\tilde{\epsilon} + O(\tilde{\epsilon}^2)\right]$.

$$E_0 \to \tilde{\epsilon} = E_0 \pm t_0 \qquad t_0 = \frac{1}{\sqrt{m^*}}\nu. \tag{8}$$

For the periodic sine-Gordon problem each eigenvalue of the isolated well splits into a narrow band. The boundary conditions gives us:

$$\begin{aligned} -1 &= 2\sqrt{\pi e}\frac{e^{-i\pi\nu}}{\Gamma(-\frac{1}{2})}\frac{\Gamma(\frac{1}{2+\nu})}{\Gamma(\frac{1}{2})} exp\left[2\lambda\int_{\phi}^{\pi}\sqrt{V-\tilde{\epsilon}}d\phi + 2\nu(1+ln2)\right. \\ &- \left.\frac{1}{2}(1+4\nu)ln(1+4\nu)\right], \end{aligned} \tag{9}$$

where $\nu = \frac{1}{2}a\lambda - \frac{1}{4}$ and $a \approx \frac{\tilde{\epsilon}}{\sqrt{2}}\left[1 + \frac{\tilde{\epsilon}}{16} + O(\tilde{\epsilon}^2)\right]$.

$$E_0 \to \tilde{\epsilon} = E_0 \pm t_0 \qquad t_0 = \frac{2}{\sqrt{m^*}}\nu. \tag{10}$$

The method was used for discussing
a) Lattice corrections; [5]
b) Temperature corrections; [6]
c) Multi-kink contributions to the free energy; [7], [8]
d) Scalar-complex fields; [9]
e) Long-range interaction potential. [10]

2 Multi-Kink Contributions

In order to apply and test the method for a real physical system, one writes the results for the multi-kink contribution to the free energy.

Previous calculations of various authors have led to an intuitive image of the free energy density of nonlinear systems described by the Hamiltonian (1). Two distinct group of terms can be distinguished in the expression of the free energy.

$$f = f_0 + f_k + f_{kk} + f_{kkk} + \ldots \tag{11}$$

(phonons) (kinks)

Here, f_0 can be written as a series expansion in power of $\frac{1}{\lambda}$, ($\lambda^2 = 2m^*$) and is directly related to the contribution of harmonic and anharmonic phonons. Our results for f_0 are in complete agreement with those found by other authors.

The second group of terms represents the kinks contribution to the free energy density. If the nonlinear system has a single type of kinks, with the rest energy $E_k^{(0)}$, then the kink contribution can be written as a series expansion in $e^{-\frac{1}{t}}$, where $t = (\beta E_k^{(0)})^{-1}$.

The kink contributions are related to the tunneling term. In order to find them, one writes a formal expansion for ν:

$$\nu = \nu_k + \nu_{kk} + \nu_{kkk} + \ldots \tag{12}$$

$$\nu_k \sim e^{-\frac{1}{t}} \qquad \nu_{kk} \sim e^{-\frac{2}{t}} \qquad \nu_{kkk} \sim e^{-\frac{3}{t}} \tag{13}$$

$$For\ \phi^4:$$

$$\begin{aligned}
\nu_k &= -\sqrt{\frac{6}{\pi t}}\, e^{-\frac{1}{t}} \\
\nu_{kk} &= \left(ln\frac{12}{t} + c_2 + i\pi\right)\nu_k^2 \\
\nu_{kkk} &= \left[\frac{3}{2}\left(ln\frac{12}{t} + c_2 + i\pi\right)^2 + \left(\frac{c_2^2}{2} - c_3\right)\right]\nu_k^3
\end{aligned}$$

$$for\ \ \mathbf{sine-Gordon}:$$

$$\begin{aligned}
\nu_k &= -\sqrt{\frac{2}{\pi t}}\, e^{-\frac{1}{t}} \\
\nu_{kk} &= 2\left(ln\frac{4}{t} + c_2 + i\pi\right)\nu_k^2 \\
\nu_{kkk} &= 2\left[3\left(ln\frac{4}{t} + c_2 + i\pi\right)^2 - \frac{c_2^2}{2} + c_3\right]\nu_k^3
\end{aligned}$$

where $\frac{1}{\Gamma(-\nu)} = \sum_{n=1}^{\infty} c_n(-\nu)^n$, c_2 =Euler's constant.

Once the various kink contributions were made obvious in the expression of ν, it is an easy task now to make a similar separation in the free energy.

The same picture is revealed in the calculation of the instanton contributions to the eigenvalues of a nonlinear quantum system (Zinn-Justin, [11]).

3 Three soliton contribution to the specific heat in sine-Gordon systems

The easy-plane ferromagnetic chain in an in-plane magnetic field was intensely studied in the last decades, because of the expected contribution of nonlinear excitations to its statistical properties. There are at least one real compound, $CsNiF_3$, which shows one-dimensional behaviour above the Neél temperature, and was classically described by the sine-Gordon model, by Mikeska [12], Leung [13], Maki and Takayama [14]. However, the classical SG model has failed for giving a quantitative explanation of the experiments. The relevance of quantum effects and out-of-plane fluctuations was demonstrated by Tognetti, Giachetti, Vaia, Cuccoli [15,16].

In order to improve the predictions of the semiclassical sine-Gordon theory, the soliton-soliton interaction was taken into account by Sasaki and Tsuzuki [17], [18]. They observed that the two soliton interaction, together with the non-linear temperature corrections, plays a significant role to reproduce the correct position of the peak.

Here one investigates the role of three soliton interactions, together with a first order temperature correction.

The hamiltonian can be written as

$$H = -J\sum_n \vec{S}_n \cdot \vec{S}_{n+1} + D\sum_n (S_n^z)^2 - g\mu_B H \sum_n S_n^x \tag{14}$$

θ describes the deviation from the easy-plane. The ideal easy-plane behaviour demands $D \to \infty$.

$$\vec{S}_n = S(\cos\theta_n \cos\phi_n, \cos\theta_n \sin\phi_n, \sin\theta_n).$$

In the continuum limit, one can map the ferromagnetic chain in a sine-Gordon system like in the Hamiltonian 15:

$$H = A\int \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}c_0^2\left(\frac{d\phi}{dz}\right)^2 + \omega_0^2(1-\cos\phi)\right] dz, \tag{15}$$

where $c_0 = S\sqrt{2DJ}$, $\quad m = \sqrt{\frac{g\mu_B H}{JS}}$, $\quad \omega_0^2 = \frac{m^2}{c_0^2}$

and $\quad t = (\beta E_S^0)_{-1}$, $\quad E_S^{(0)} = 8Ac_0\omega_0$, $\quad d = \frac{c_0}{\omega_0}$.

The partition function is calculated using the transfer integral method. One can see the contribution of one-soliton sector

$$f_S = -\frac{2}{\beta d}\sqrt{\frac{2}{\pi t}}e^{-\frac{1}{t}}(1+\frac{7}{8}t), \tag{16}$$

of two-soliton

$$f_{SS} = \frac{1}{\beta d}\left[2\left(ln\frac{4}{t}+\gamma\right) - i\pi\right]\left(\sqrt{\frac{2}{\pi t}}e^{-\frac{1}{t}}\right)^2, \tag{17}$$

and of three-soliton sector

$$f_{SSS} = -\frac{1}{\beta d}\left[\frac{3}{2}\left((ln\frac{4}{t}+\gamma) - i\pi\right)^2 - \frac{1}{2}\gamma^2 + c_3\right]\left(\sqrt{\frac{2}{\pi t}}e^{-\frac{1}{t}}\right)^3. \tag{18}$$

The specific heat per spin is written as a sum of spin-wave contribution, plus one-soliton, two- and three-soliton contributions.

$$c' = -T\frac{\partial^2 f}{\partial t^2}$$

$$c' = 1 + c_{sw} + c_S + c_{SS} + c_{SSS}.$$

The spin-wave contribution to the specific heat per spin is taken from the paper of Sasaki [16] :

$$\frac{d}{l}c_{sw} = \frac{1}{2}t + \frac{3}{4}t^2 + \frac{9}{4}t^3 + \frac{245}{2^5}t^4 + \frac{4455}{2^7}t^5 + ...$$

As usual,

$$f = -\frac{1}{2\beta l}ln\frac{2\pi Al}{\beta\bar{h}^2} + A\omega_0^2\epsilon_0.$$

$$\frac{d}{l}c_S = \frac{4}{\sqrt{2\pi}}t^{-\frac{5}{2}}e^{-\frac{1}{t}}(1 - a't), \tag{19}$$

where $a' = -\frac{15}{8}t - \frac{203}{2^7}t^2 - \frac{2985}{2^{10}}t^3 - ...$

$$\frac{d}{l}c_{SS} = -\frac{32}{\pi}t^{-3}e^{-\frac{2}{t}}\left[\left(ln\frac{4}{t}\right) - a''t\right], \tag{20}$$

where $a'' = -\frac{9}{4}(1 + \gamma + ln\frac{4}{t}) + t\left[\frac{11}{8} - \frac{13}{32}(\gamma + \frac{1}{t}ln4)\right] + ...$

The one and two soliton contributions we obtained, are the same as those of Sasaki. For the three soliton contribution we get

$$\frac{d}{l}c_{SSS} = \frac{72\sqrt{2}}{\pi^{\frac{3}{2}}}t^{-\frac{7}{2}}e^{-\frac{3}{t}}\left[3\left(ln^2\frac{4}{t} + 2\gamma ln\frac{4}{t} + b\right)\left(1 - (a+2)t - 4t(ln\frac{4}{t} + \gamma)\right)\right], \tag{20}$$

$b = \frac{1}{3}\left(\frac{5}{2}\gamma^2 - \frac{3}{4}\pi^2 + c_3\right)$; $a = \frac{(21+ln4)}{16}$.

The results obtained at this level for the excess of specific heat are plotted in the figure, together with the exact numerical transfer matrix calculations.

Formula (20) gives us the three soliton correction. The contribution maintains the same height and position of the peak, resulting in a good agreement with the exact results up to $t \simeq 0.3$.

However, the two and three soliton contributions are no longer sufficient to describe the specific heat at increasing temperatures. At low temperatures the soliton gas picture gives a good answer to the thermodynamic properties, but becomes less and less valid at intermediate temperatures.

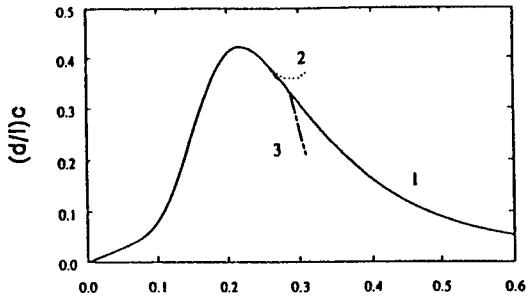

Fig.1. The specific heat capacity c of the sG model as a function of temperature t, where the linear spin-wave contribution was subtracted. Curve (1) exact (numerical) results [9]; curve(2) the results with the two soliton contribution [17]; curve (3) analytic results including the three soliton interactions.

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