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A COMPARISON OF THE TWO-SIDED LAPLACE TRANSFORM AND THE MELLIN TRANSFORM

by

GLORIA L. PORTER
B.S. Alabama State University, 1992

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
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in the College of Arts and Sciences
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Piotr Mikusinski, Chair

Roy Choudhury

Larry C. Andrews

FINAL EXAMINATION COMMITTEE

John R. Cannon
Department Chair

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ABSTRACT

The reals with addition and the positive reals with multiplication are isomorphic as groups. From that point of view, the two-sided Laplace transform and the Mellin transform are different representations of the same transform. This allows us to easily derive the properties of the Mellin transform from the properties of the two-sided Laplace transform. The method extends to functions of several variables as well.

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INTRODUCTION

In this paper, two integral transforms are under consideration: the two-sided Laplace transform and the Mellin transform. Our purpose is to show that these two transforms are actually different versions of the same transform. Consequently, it suffices to study properties of one of them, and then simply translate the results to the language of the other. An additional advantage of this approach is that it extends nicely to the case of the transforms of functions of n variables.

The equivalence of the two-sided Laplace transform and the Mellin transform has its roots in the isomorphism between the additive group of all real numbers $\mathbb{R}$ and the multiplicative group of all positive reals $\mathbb{R}_+$. We begin our investigation by defining and establishing a group isomorphism between these groups. The defined isomorphism transfers the Lebesgue measure on $\mathbb{R}$ to a new measure on $\mathbb{R}_+$. Next, we define convolution operations for functions on $\mathbb{R}$ and for functions on $\mathbb{R}_+$ which are related to the group operations in $\mathbb{R}$ and $\mathbb{R}_+$. Finally, we seek an operation that takes functions in one variable on $\mathbb{R}$ and redefines them in another variable on $\mathbb{R}_+$. This operation helps determine the simple substitution that is needed to show the equivalence between the properties of the two-sided Laplace transform and those properties of the Mellin transform.

In Chapter 2, we discuss the Laplace transform and its properties. Chapter 3 is devoted to the Mellin transform. We do not use the integral definition to prove the properties of the Mellin transform; instead, we use substitution and properties of the

Laplace transform. Finally, in Chapters 4 and 5, we extend this approach to functions of several variables.

CHAPTER 1

PRELIMINARY CONCEPTS

The equivalence of the two-sided Laplace transform and the Mellin transform has its roots in the isomorphism between the additive group of all real numbers $\mathbb{R}$ and the multiplicative group of all positive reals $\mathbb{R}_+$. In this chapter, we take a look at the general definition of a group and the notion of a group isomorphism. Then, we establish an isomorphism between the two groups mentioned above and define a measure on $\mathbb{R}_+$ which is induced by the isomorphism. Finally, we define two convolution operations: for functions defined on $\mathbb{R}$ and for functions defined on $\mathbb{R}_+$.

1.1. Groups $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$

Definition 1.1. A *group* $(G, *)$ consists of a set G and an operation $*$ such that

(a) for every ordered pair (x, y) of G there is a unique element $x * y$ also in G,

(b) $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$,

(c) There exists an identity element $e \in G$ such that $x * e = e * x = x$ for all $x \in G$,

(d) For every $x \in G$ there exist $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$,

$(x^{-1}$ is called the inverse of $x)$ (Kim, 67-68).

In this paper we consider two groups: the reals with the operation of addition, $(\mathbb{R}, +)$, and the positive reals with the operation of multiplication $(\mathbb{R}_+ ,\cdot)$. It is easily seen that in both cases all conditions in the definition of a group are satisfied. Clearly, $\mathbb{R}_+$

is a subset of $\mathbb{R}$; however, $(\mathbb{R}_+, \cdot)$ is not a subgroup of $(\mathbb{R}, +)$ since the operations are different.

Definition 1.2. Let $(U, *_u)$ and $(V, *_v)$ be groups. A bijective function $\phi : U \rightarrow V$ with the property that for any two elements x and y in U,

$$\phi(x *_u y) = \phi(x) *_v \phi(y)$$

is called a *group isomorphism* from U and V. If a group isomorphism exists, we say that the groups are *isomorphic*.

Properties of an isomorphism:

(a) $\phi(e_U) = e_V$, where e_U is the identity of U and e_V is the identity of V,

(b) $\phi(x^{-1}) = (\phi(x))^{-1}$ for all x in U,

(c) U and V have the same cardinality, and

(d) x and y commute in U if and only if $\phi(x)$ and $\phi(y)$ commute in V.

The proofs of (a), (b), and (c) are easily constructed using the previous definitions. To prove (d), note that $xy = yx$ implies

$$\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x).$$

Therefore, if x and y commute then $\phi(x)$ and $\phi(y)$ commute. If $\phi(x)\phi(y) = \phi(y)\phi(x)$, then $\phi(xy) = \phi(yx)$. Since ϕ is 1-1, $xy = yx$ is evident (Shapiro, 48-49).

If two groups are isomorphic, we can say that they are replicas of each other. Although they may be defined by different elements and operations, they should still have the same structure with the same properties. To prove that $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$ are isomorphic consider the function $\phi(x) = e^x$. We know that ϕ is bijective and $e^{a+b} = e^a e^b$, or equivalently $\phi(a+b) = \phi(a)\phi(b)$. The function ϕ is indeed an isomorphism from $(\mathbb{R}, +)$ and $(\mathbb{R}_+, \cdot)$. Note that the inverse of ϕ, $\phi^{-1}(x) = \ln x$, is an isomorphism from $(\mathbb{R}_+, \cdot)$ to $(\mathbb{R}, +)$.

1.2. Measures on $\mathbb{R}$ and $\mathbb{R}_+$

We proceed with defining measures on $\mathbb{R}$ and $\mathbb{R}_+$. A measure assigns a number to each set in a certain class. The objective of this section is to define a measure μ on $\mathbb{R}_+$ such that $\mu(\phi(S)) = \lambda(S)$, where λ is the Lebesgue measure on $\mathbb{R}$.

The Lebesgue measure is determined by the measure of intervals. The Lebesgue measure of any interval (a, b) is equal to the length of that interval, $b - a$. This measure can be calculated for any open set, since an open set is the union of open intervals. Standard techniques allow us to extend λ to all measurable Borel sets, or all measurable sets (Ash, 3-26).

Now we define a measure μ on $\mathbb{R}_+$. Since we want μ to satisfy the condition $\mu(\phi(S)) = \lambda(S)$ for a set $S \subset \mathbb{R}_+$, we first apply ϕ^{-1} to that set to get $\phi^{-1}(S) \subset \mathbb{R}$, then find the λ-measure of $\phi^{-1}(S)$. This is the μ-measure of the set $S \subset \mathbb{R}_+$.

Definition 1.4. The μ-measure defined on $\mathbb{R}_+$ is

$$\mu(S) = \lambda(\phi^{-1}(S)).$$

If $S = (a, b) \subset \mathbb{R}_+$, then $\phi^{-1}(a, b) = (\ln a, \ln b)$ and $\lambda(\ln a, \ln b) = \ln b - \ln a = \ln \frac{b}{a}$. Therefore, $\mu(a, b) = \ln \frac{b}{a}$. Similarly, we now choose the set S to be an interval $(a, b) \subset \mathbb{R}$ which implies $\phi(S) = (e^a, e^b)$. Then $\lambda(S) = \lambda(a, b) = b - a$ and $\mu(\phi(S)) = \ln e^b - \ln e^a = b - a$. Indeed, the two measures are equivalent, that is, $\lambda(S) = \mu(\phi(S))$.

Intervals in $\mathbb{R}$ do not change measure under translation. If we take any open interval $(a,\ b)$ and add a constant c to every element in the interval, then the new interval $(a+c, b+c)$ will have the same measure as the old interval. By the definition of Lebesgue measure, we know that $\lambda(a,b) = b-a$ and

$$\lambda(a+c, b+c) = b+c-(a+c) = b-a.$$

Since the Lebesgue measure of an arbitrary measurable set S is determined by the measure of intervals, we have $\lambda(S+c) = \lambda(S)$, where $S+c = \{s+c, s \in S\}$. Since addition in $\mathbb{R}$ corresponds to multiplication in $\mathbb{R}_{+}$, we expect the measure μ to be "multiplication invariant".

Example 1.1. Take $S = (0,1)$ on $\mathbb{R}$ with $\lambda(S) = 1$. Then $\phi(0,1) = (1,e)$ for which $\mu(\phi(S)) = \ln\frac{e}{1} = \ln e - \ln 1 = 1$. Now add a constant $c = 2$ to S to get $(2,3)$ for which $\lambda(S+2) = 3-2 = 1$. Using a specific numerical example we see that λ is indeed translation invariant on $\mathbb{R}$. However, μ is not translation invariant since $\mu(\phi(S)+2) = \mu(3, e+2) = \ln\frac{(e+2)}{3}$ clearly does not equal $\mu(\phi(S)) = 1$.

Example 1.2. Let us start with a set on $\mathbb{R}_{+}$, say $(2,3)$. Then $\phi^{-1}(S) = (\ln 2, \ln 3)$. Clearly, $\mu(2,3) = \ln\frac{3}{2}$ and $\lambda\ (\phi^{-1}(S)) = \ln 3 - \ln 2 = \ln\frac{3}{2}$. Now let's multiply S by $c = 2$ to get $(4,6)$ for which $\mu(2S) = \ln\frac{6}{4} = \ln\frac{3}{2}$, which is the same as the μ-measure of the original set. Therefore, we can say that μ is multiplication invariant on $\mathbb{R}_{+}$, but λ is not multiplication invariant on $\mathbb{R}$.

1.3. Convolution of Functions on $\mathbb{R}$ and $\mathbb{R}_+$

As elements of $\mathbb{R}$ and $\mathbb{R}_+$ can be identified via the isomorphism ϕ, the function on $\mathbb{R}$ can be identified with the functions on $\mathbb{R}_+$. Let T be an operation such that $(Tf)(t) = f(\ln t)$. In other words, T takes functions in one variable in $\mathbb{R}$ and redefines them in another variable in $\mathbb{R}_+$. If $f, g \in C(\mathbb{R})$, then $Tf, Tg \in C(\mathbb{R}_+)$.

In this section we define two convolution operations: one for functions on $\mathbb{R}$ and the other for functions on $\mathbb{R}_+$.

Definition 1.5. The *convolution* of two functions f and g on $\mathbb{R}$ is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds,$$

if the integral exists almost everywhere.

The objective here is to find a convolution for functions on $\mathbb{R}_+$ such that

$$T(f*g) = (Tf) \odot (Tg).$$

In other words, we want the following diagram to commute:

$$\begin{array}{ccc} f, g & \xrightarrow{T} & Tf, Tg \\ * \downarrow & & \downarrow \odot \\ f * g & \xrightarrow{T} & T(f * g) \end{array}$$

Figure 1.1

This figure shows that the end result will be the same if we apply T to the functions before performing the $\odot$-convolution or perform the $*$-convolution first and then apply T.

However, before beginning this task, we first show by example that $*$ will not have the desired property, that is,

$$T(f*g) \neq Tf * Tg.$$

Example 1.3. Let

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases} \qquad \text{and} \qquad g(x) = x.$$

Taking the convolution of f, g we have

$$\begin{aligned} (f*g)(x) &= \int_{-\infty}^{\infty} f(x-s)g(s)ds \\ &= \int_0^1 (x-s)ds \\ &= \left[xs - \frac{1}{2}s^2\right]_0^1 \\ &= x - \frac{1}{2}. \end{aligned}$$

Now using $(Tf)(t) = f(\ln t)$, we have $T(f*g)(t) = \ln t - \frac{1}{2}$.

Applying T to f, g first yields,

$$(Tf)(t) = \begin{cases} 1 & 1 < t < e \\ 0 & \text{elsewhere} \end{cases} \qquad \text{and} \qquad (Tg)(t) = \ln t.$$

Now taking the convolution of Tf, Tg, we have

$$\begin{aligned} Tf(t) * Tg(t) &= \int_0^\infty (Tg)(t-s)(Tf)(s)ds \\ &= \int_1^e \ln(t-s)ds \\ &= \int_{t-e}^{t-1} \ln u du = [u(\ln u - 1)]_{t-e}^{t-1} \\ &= (t-1)\ln t - (t-e)\ln(t-e). \end{aligned}$$

Clearly, from the example, $*$ does not work. So we are still seeking an operation $\odot$ for functions on $\mathbb{R}_+$ that will give us the same convolution as $*$ on $\mathbb{R}$.

We want to derive the convolution $\odot$ using T^{-1} on $C(\mathbb{R}_+)$. The operation T^{-1} takes functions on $\mathbb{R}_+$ to functions on $\mathbb{R}$:

$$(T^{-1}f)(x) = f(e^x).$$

For f, g defined on $\mathbb{R}$, if we want

$$T(f*g) = (Tf) \odot (Tg),$$

we must have

$$T(T^{-1}f * T^{-1}g) = f \odot g$$

for f, g defined on $\mathbb{R}_+$. This can be used to find a formula for $f \odot g$. Indeed, we have

$$\begin{aligned} (f \odot g)(t) &= T(f(e^x) * g(e^x)) \\ &= T\left(\int_{-\infty}^\infty f(e^{x-s})g(e^s)ds\right) \\ &= \int_{-\infty}^\infty f\left(e^{\ln t - s}\right)g(e^s)ds \\ &= \int_0^\infty f\left(\frac{t}{r}\right)g(r)\frac{dr}{r} \qquad \text{(allowing } e^s = r\text{)}. \end{aligned}$$

Definition 1.6. The *convolution* of two functions f and g on $\mathbb{R}_+$ is defined as

$$(f \odot g)(t) = \int_0^\infty f(\tfrac{t}{r}) g(r) \tfrac{dr}{r},$$

if the integral exists almost every where.

The convolution $\odot$ is the standard convolution in the space of functions on the group $(\mathbb{R}_+, \cdot)$ with respect to measure μ.

Theorem 1.1. *Given two functions f, g for which the convolution exists,*

$$T(f*g) = Tf \odot Tg.$$

Proof of Theorem 1.1 follows from the above calculations of $f \odot g$.

CHAPTER 2

TWO-SIDED LAPLACE TRANSFORM

2.1. Basic Definitions

In this chapter, we discuss the two-sided Laplace transform, its interval of convergence, and some basic operational properties.

Definition 2.1. (*Two-sided Laplace transformable functions*). A function f is called two-sided Laplace transformable, if $\alpha_f < \beta_f$ where

$$\alpha_f = \inf\left\{ \omega \in \mathbb{R} : \int_{-\infty}^{0} |f(x)| e^{\omega x} dx < \infty \right\},$$

and

$$\beta_f = \sup\left\{ \omega \in \mathbb{R} : \int_{0}^{\infty} |f(x)| e^{\omega x} dx < \infty \right\}.$$

The interval (α_f, β_f) will be called the *interval of convergence,* denoted by $\Omega_{\mathbb{R}}(f(x))$, and the subset of the complex plane $\{x + iy : x \in (\alpha_f, \beta_f)\}$ will be called the *strip of convergence,* denoted by $\Omega_{\mathbb{C}}(f(x))$.

The following theorem is a direct consequence of the above definition.

Theorem 2.1. *Let $f(x)$ be a two-sided Laplace transformable function. Then the integral*

$$\int_{-\infty}^{\infty} e^{sx} f(x) dx$$

converges for every $s \in \Omega_{\mathbb{C}}(f(x))$.

Proof: Note that

$$\int_{-\infty}^{\infty} e^{sx} f(x) dx = \int_{-\infty}^{0} e^{sx} f(x) dx + \int_{0}^{\infty} e^{sx} f(x) dx.$$

Thus, the integral $\int_{-\infty}^{\infty} e^{sx} f(x) dx$ converges if and only if $\int_{-\infty}^{0} e^{sx} f(x) dx$ and $\int_{0}^{\infty} e^{sx} f(x) dx$ both converge. We have

$$\begin{aligned}
\int_{-\infty}^{0} |e^{sx} f(x)| dx &= \int_{-\infty}^{0} |e^{(\mathrm{Re}\, s + i\mathrm{Im}\, s)x} f(x)| dx \\
&= \int_{-\infty}^{0} |e^{(\mathrm{Re}\, s)x}||e^{(\mathrm{Im}\, s)x}||f(x)| dx \\
&= \int_{-\infty}^{0} e^{(\mathrm{Re}\, s)x} |f(x)| dx < \infty,
\end{aligned}$$

since $s \in \Omega_{\mathbb{C}}(f(x))$ and thus $\mathrm{Re}\, s > \alpha_f$. Similarly,

$$\begin{aligned}
\int_{0}^{\infty} e^{sx} f(x) dx &= \int_{0}^{\infty} |e^{(\mathrm{Re}\, s + i\mathrm{Im}\, s)x} f(x)| dx \\
&= \int_{0}^{\infty} |e^{(\mathrm{Re}\, s)x}||e^{(\mathrm{Im}\, s)x}||f(x)| dx \\
&= \int_{0}^{\infty} e^{(\mathrm{Re}\, s)x} |f(x)| dx < \infty,
\end{aligned}$$

since $s \in \Omega_{\mathbb{C}}(f(x))$ and thus $\operatorname{Re} s < \beta_f$. □

Definition 2.2. *(Two-sided Laplace transform).* Let f be a function of the real variable x, then

$$\mathcal{L}\{f(x); s\} = \int_{-\infty}^{\infty} e^{sx} f(x) dx$$

is called the two-sided Laplace transform of f.

Thus, the two-sided Laplace transform of f is a function defined in the strip of convergence of f.

This definition is not the conventional definition of the Laplace transform. The standard definition is $\mathcal{L}\{f(x); s\} = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$. We could say that we are using the mirror image of the standard transform. Our definition facilitates the construction of the proofs. Since the two-sided Laplace transform is the only form of a Laplace transform considered in this paper, we will simply call it the Laplace transform. Likewise, two-sided Laplace transformable functions will be called Laplace transformable.

2.2. Basic Operational Properties of the Laplace Transform

Theorem 2.2. *(Shifting Property). If $f(x)$ is Laplace transformable and a is a real constant, then $e^{ax} f(x)$ is Laplace transformable and and we have*

L1 $$\mathcal{L}\{e^{ax} f(x); s\} = \mathcal{L}\{f(x); s+a\}.$$

Moreover, if $\Omega_{\mathbb{R}}(f(x)) = (\alpha_f, \beta_f)$, then $\Omega_{\mathbb{R}}(e^{ax} f(x)) = (\alpha_f - a, \beta_f - a)$.

Proof: From the definition,

$$\begin{aligned}\mathcal{L}\{e^{ax}f(x);s\} &= \int_{-\infty}^{\infty} e^{sx}e^{ax}f(x)dx \\ &= \int_{-\infty}^{\infty} e^{(s+a)x}f(x)dx \\ &= \mathcal{L}\left\{f(x);s+a\right\}.\end{aligned}$$

This proves **L1**. Moreover, since $\mathcal{L}\left\{f(x);s+a\right\}$ is defined whenever $s+a \in (\alpha_f,\beta_f)$, $\mathcal{L}\{e^{ax}f(x);s\}$ must converge for $s \in (\alpha_f - a, \beta_f - a)$. $\square$

Theorem 2.3. *(Translation Property). If $f(x)$ is Laplace transformable and $a > 0$, then $f(x+a)$ is Laplace transformable and we have*

L2 $$\mathcal{L}\left\{f(x+a);s\right\} = e^{-as}\mathcal{L}\left\{f(x);s\right\},$$

Moreover, $\mathcal{L}\{f(x);s\}$ and $\mathcal{L}\{f(x+a);s\}$ have the same interval of convergence.

Proof: We have

$$\begin{aligned}\mathcal{L}\{f(x+a);s\} &= \int_{-\infty}^{\infty} e^{sx}f(x+a)dx \\ &= \int_{-\infty}^{\infty} e^{s(u-a)}f(u)du \\ &= e^{-as}\int_{-\infty}^{\infty} e^{su}f(u)du \\ &= e^{-as}\mathcal{L}\left\{f(x);s\right\}.\end{aligned}$$

This proves **L2**. Moreover, since $e^{-as}\mathcal{L}\left\{f(x);s\right\}$ is defined whenever $s \in (\alpha_f,\beta_f)$, $\mathcal{L}\{f(x+a);s\}$ must also converge for $s \in (\alpha_f,\beta_f)$. $\square$

Theorem 2.4. *(Scaling Property).* *If $f(x)$ is Laplace transformable and $a \neq 0$, then $f(ax)$ is Laplace transformable and we have*

L3 $$\mathcal{L}\{f(ax); s\} = \frac{1}{a}\mathcal{L}\left\{f(x); \frac{s}{a}\right\}.$$

Moreover, if $\Omega_{\mathbb{R}}(f(x)) = (\alpha_f, \beta_f)$, then $\Omega_{\mathbb{R}}(f(ax)) = (a\alpha_f, a\beta_f)$.

Proof: A simple substitution yields

$$\begin{aligned} \mathcal{L}\{f(ax); s\} &= \int_{-\infty}^{\infty} e^{sx} f(ax)dx \\ &= \frac{1}{a}\int_{-\infty}^{\infty} e^{s\frac{u}{a}} f(u)du \\ &= \frac{1}{a}\int_{-\infty}^{\infty} e^{u\frac{s}{a}} f(u)du \\ &= \frac{1}{a}\mathcal{L}\left\{f(x); \frac{s}{a}\right\}. \end{aligned}$$

Thus, **L3** holds. Moreover, since $\frac{1}{a}\mathcal{L}\left\{f(x); \frac{s}{a}\right\}$ is defined whenever $\frac{s}{a} \in (\alpha_f, \beta_f)$, $\mathcal{L}\{f(ax); s\}$ must converge for $s \in (a\alpha_f, a\beta_f)$. □

Theorem 2.5. *(Derivatives of the Laplace Transform).* *If $f(x)$ is Laplace transformable, then $xf(x)$ is Laplace transformable and we have*

L4 $$\mathcal{L}\{xf(x); s\} = \frac{d}{ds}\mathcal{L}\{f(x); s\}.$$

Moreover, $\Omega_{\mathbb{R}}(xf(x)) = \Omega_{\mathbb{R}}(f(x))$.

Proof: By differentiating within the integral sign, we obtain

$$\begin{aligned}
\frac{d}{ds}\mathcal{L}\{f(x);s\} &= \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\, e^{sx} f(x)dx \\
&= \int_{-\infty}^{\infty} x e^{sx} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{sx} x f(x)dx \\
&= \mathcal{L}\left\{x f(x);s\right\}.
\end{aligned}$$

This proves **L4**. Moreover, since $\frac{d}{ds}\mathcal{L}\{f(x);s\}$ is defined whenever $s \in (\alpha_f, \beta_f)$, $\mathcal{L}\{xf(x);s\}$ must also converge for $s \in (\alpha_f, \beta_f)$. □

Corollary 2.5. *If $f(x)$ is transformable and $p(x)$ is a polynomial, then $p(x)f(x)$ is Laplace transformable and we have*

$$\mathcal{L}\{p(x)f(x);s\} = p(\tfrac{d}{ds})\mathcal{L}\{f(x);s\}.$$

Theorem 2.6. *(Laplace Transform of Derivatives). If $f(x)$ is Laplace transformable and $\lim_{x\to\infty} e^{sx}f(x) = \lim_{x\to-\infty} e^{sx}f(x) = 0$ for all $s \in \Omega_{\mathbb{R}}(f(x))$, then $f'(x)$ is Laplace transformable and we have*

L5 $$\mathcal{L}\{f'(x);s\} = -\,s\mathcal{L}\{f(x);s\}.$$

Moreover, $\Omega_{\mathbb{R}}(f'(x)) = \Omega_{\mathbb{R}}(f(x))$.

Proof: Using integration by parts, we get

$$\begin{aligned}
\mathcal{L}\{f'(x);s\} &= \int_{-\infty}^{\infty} e^{sx} f'(x)dx \\
&= [e^{sx}f(x)]_{-\infty}^{\infty} - s\int_{-\infty}^{\infty} e^{sx} f(x)dx \\
&= \lim_{x\to\infty} e^{sx}f(x) - \lim_{x\to-\infty} e^{sx}f(x) - s\mathcal{L}\{f(x);s\} \\
&= -s\mathcal{L}\{f(x);s\}.
\end{aligned}$$

This proves **L5**. Moreover, since $-s\mathcal{L}\{f(x);s\}$ is defined whenever $s \in (\alpha_f, \beta_f)$, $\mathcal{L}\{f'(x);s\}$ must also converge for $s \in (\alpha_f, \beta_f)$. □

Theorem 2.7. *(Convolution Theorem). If $f(x)$ and $g(x)$ are both Laplace transformable and $\Omega_{\mathbb{R}}(f(x)) \cap \Omega_{\mathbb{R}}(g(x)) \neq \emptyset$, then $f(x)*g(x)$ is Laplace transformable and we have*

L6 $$\mathcal{L}\{f(x)*g(x);s\} = \mathcal{L}\{f(x);s\}\mathcal{L}\{g(x);s\}.$$

Moreover, if $\Omega_{\mathbb{R}}(f(x)) = (\alpha_f, \beta_f)$ and $\Omega_{\mathbb{R}}(g(x)) = (\alpha_g, \beta_g)$, then $\Omega_{\mathbb{R}}(f(x)*g(x)) \subset (\alpha_f, \beta_f) \cap (\alpha_g, \beta_g)$.

Proof: Letting $u = x - y$ and changing the order of integration, we obtain

$$\begin{aligned}
\mathcal{L}\{f(x)*g(x);s\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx} f(x-y)g(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)e^{s(u+y)}f(u)dudy \\
&= \int_{-\infty}^{\infty} e^{sy}g(y)dy \int_{-\infty}^{\infty} e^{su}f(u)du \\
&= \mathcal{L}\{f(x);s\}\mathcal{L}\{g(x);s\}.
\end{aligned}$$

This proves **L6**. Moreover, since $\mathcal{L}\{f(x); s\}$ and $\mathcal{L}\{g(x); s\}$ are defined whenever $s \in (\alpha_f, \beta_f) \cap (\alpha_g, \beta_g)$, $\mathcal{L}\{f(x)*g(x); s\}$ must also converge for $s \in (\alpha_f, \beta_f) \cap (\alpha_g, \beta_g)$. □

Theorem 2.8. *(Inversion Formula for the two-sided Laplace transform). If $f(x)$ is integrable over every finite interval and is two-sided Laplace transformable, then*

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)e^{-sx_o}ds = \frac{f(x_o+) + f(x_o-)}{2},$$

where $F(s) = \mathcal{L}\{f(x); s\}$, $f(x)$ is of bounded variation in some neighborhood of x_o, and $c \in (\alpha_f, \beta_f)$. In particular, if f is continuous at x_o, then

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)e^{-sx_o}ds = f(x_o).$$

The proof can be found in *Integral Transforms* by Lokenath Debnath.

CHAPTER 3
MELLIN TRANSFORM

In this chapter, the Mellin transform of $f(t)$ and its properties are derived using simple substitutions and basic properties of the Laplace transform.

3.1. Basic Definitions and Comparison with the Laplace Transform

Definition 3.1. *(Mellin transformable functions).* The function $f(t)$ is Mellin transformable if $A_f < B_f$ where

$$A_f = \inf\left\{\omega \in \mathbb{R} : \int_0^1 |f(t)| t^{\omega-1} dt < \infty\right\},$$

and

$$B_f = \sup\left\{\omega \in \mathbb{R} : \int_1^\infty |f(t)| t^{\omega-1} dt < \infty\right\}.$$

Therefore, the interval of convergence of the Mellin transform of f is (A_f, B_f), denoted by $\Delta_{\mathbb{R}}(f(t))$.

Theorem 3.1. *A function $f : (0, \infty) \to \mathbb{R}$ is Mellin transformable if and only if $f(e^x)$ is two-sided Laplace transformable. Moreover, $\Delta_{\mathbb{R}}(f(t)) = \Omega_{\mathbb{R}}(f(e^x))$.*

Proof: Note that

$$\begin{aligned}
\int_{-\infty}^{0} |f(e^x)||e^{sx}|dx &= \int_{-\infty}^{0} |f(e^x)||e^{(\mathrm{Re}\, s)x}||e^{i(\mathrm{Im}\, s)x}|dx \\
&= \int_{-\infty}^{0} |f(e^x)|e^{(\mathrm{Re}\, s)x}dx \\
&= \int_{0}^{1} |f(t)|t^{\mathrm{Re}\, s}\frac{dt}{t} \\
&= \int_{0}^{1} |f(t)|t^{\mathrm{Re}\, s-1}dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{0}^{\infty} |f(e^x)||e^{sx}|dx &= \int_{0}^{\infty} |f(e^x)|e^{(\mathrm{Re}\, s)x}dx \\
&= \int_{0}^{1} |f(t)|t^{\mathrm{Re}\, s}\frac{dt}{t} \\
&= \int_{0}^{1} |f(t)|t^{\mathrm{Re}\, s-1}dt.
\end{aligned}$$

Therefore,

$$\inf\left\{\omega \in \mathbb{R} : \int_0^1 |f(t)|t^{\omega-1}dt < \infty\right\} = \inf\left\{\omega \in \mathbb{R} : \int_{-\infty}^0 |f(e^x)|e^{\omega x}dx < \infty\right\}$$

and

$$\sup\left\{\omega \in \mathbb{R} : \int_1^\infty |f(t)|t^{\omega-1}dt < \infty\right\} = \sup\left\{\omega \in \mathbb{R} : \int_0^\infty |f(e^x)|e^{\omega x}dx < \infty\right\}.$$

This proves that $\Delta_{\mathbb{R}}(f(t)) = \Omega_{\mathbb{R}}(f(e^x))$ and $f(t)$ is Mellin transformable if and only if $f(e^x)$ is Laplace transformable. $\square$

Theorem 3.2. *If $f(t)$ is a Mellin transformable function, then $\int_0^\infty t^{s-1} f(t)dt$ converges for every $s \in \Delta_{\mathbb{C}}(f(t))$.*

Proof: Since $\int_0^\infty t^{s-1} f(t)dt = \int_0^\infty e^{x(s-1)} f(e^x)e^x dx = \int_{-\infty}^\infty f(e^x)e^{sx} dx$, this theorem follows from Theorem 3.1. $\square$

Definition 3.2. *(The Mellin Transform).* Let f be a function of a real variable t, then

$$\mathcal{M}\{f(t); s\} = \int_0^\infty t^{s-1} f(t)dt$$

is called the Mellin transform of $f(t)$.

Suppose we take $f(x)$, substitute $x = \ln t$, then apply the Mellin transform:

$$\begin{aligned}
\mathcal{M}\{f(\ln t); s\} &= \int_0^\infty t^{s-1} f(\ln t)dt \\
&= \int_{-\infty}^\infty e^{xs-1} f(\ln e^x)e^x dx \\
&= \int_{-\infty}^\infty e^{sx} f(x)dx.
\end{aligned}$$

In other words, the Laplace transform of $f(x)$ is the same as the Mellin transform of $f(\ln t)$ and we have the relation,

$$\mathcal{L}\{f(x); s\} = \mathcal{M}\{f(\ln t); s\}.$$

Conversely, the Mellin transform of $f(t)$ is the same as the Laplace transform of $f(e^x)$, and we have the relation

$$\mathcal{M}\ \{f(t); s\} = \mathcal{L}\{f(e^x); s\}.$$

These two equalities will be useful in proving the basic operational properties of the Mellin transform. We formulate them more precisely in the following two theorems.

Theorem 3.3. *If $f(x)$ is Laplace transformable, then $f(\ln t)$ is Mellin transformable and*

$$\mathcal{L}\{f(x);s\} = \mathcal{M}\{f(\ln t);s\}.$$

Theorem 3.4. *If $f(t)$ is Mellin transformable, then $f(e^x)$ is Laplace transformable and*

$$\mathcal{M}\{f(t);s\} = \mathcal{L}\{f(e^x);s\}.$$

Taking into account our previous discussion, we can say that the two-sided Laplace transform and the Mellin transform are two versions of the same transform. The first one is defined for functions on the group $(\mathbb{R}, +)$, and the second one is defined on the isomorphic group $(\mathbb{R}_+, \cdot)$. The group isomorphism identifies the two transforms.

3.2. Basic Operational Properties of the Mellin Transform

The construction of the following proofs of the basic properties of the Mellin transform will be quite different from the standard method of integral substitution. The procedure consists of allowing $t = e^x$ and rewriting the Mellin transform in terms of the Laplace transform, $\mathcal{M}\{f(t);s\} = \mathcal{L}\{f(e^x);s\}$. Then, by carefully applying appropriate properties of the Laplace transform, we prove the properties of the Mellin transform.

Theorem 3.5. *(Scaling Property).* *If $f(t)$ is a Mellin transformable function and $a > 0$, then the function $f(at)$ is Mellin transformable and we have*

M1 $$\mathcal{M}\{f(at);s\} = a^{-s}\mathcal{M}\{f(t);s\}.$$

Moreover, $\Delta_{\mathbb{R}}(f(at)) = \Delta_{\mathbb{R}}(f(t))$.

Proof: By replacing t with e^x, we have

$$\begin{aligned}
\mathcal{M}\{f(at);s\} &= \mathcal{L}\{f(ae^x);s\} \\
&= \mathcal{L}\{f\left(e^{\ln a}e^x\right);s\} \\
&= \mathcal{L}\{f\left(e^{x+\ln a}\right);s\} \\
&= e^{-s\ln a}\mathcal{L}\{f(e^x);s\} \qquad \text{(by } \mathbf{L2}\text{)} \\
&= a^{-s}\mathcal{M}\{f(t);s\}.
\end{aligned}$$

This proves **M1**. Moreover, since $a^s\mathcal{M}\{f(t);s\}$ is defined whenever $s \in \Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\{f(at);s\}$ must also converge for $s \in \Delta_{\mathbb{R}}(f(t))$. □

Theorem 3.6. *(Translation Property).* *If $f(t)$ is a Mellin transformable function, then the function $t^a f(t)$ is Mellin transformable and we have*

M2 $$\mathcal{M}\{t^a f(t);s\} = \mathcal{M}\{f(t);s+a\}.$$

Moreover, $\Delta_{\mathbb{R}}(t^a f(t)) = \Delta_{\mathbb{R}}(f(t)) - a$.

Proof: Using **L1**, it is clear that

$$\begin{aligned}
\mathcal{M}\{t^a f(t);s\} &= \mathcal{L}\{(e^x)^a f(e^x);s\} \\
&= \mathcal{L}\{e^{ax} f(e^x);s\} \\
&= \mathcal{L}\{f(e^x);s+a\} \\
&= \mathcal{M}\{f(t);s+a\}.
\end{aligned}$$

This proves **M2**. Moreover, since $\mathcal{M}\,\{f(t); s+a\}$ is defined whenever $s+a \in \Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\,\{t^a f(t); s\}$ must converge for $\Delta_{\mathbb{R}}(f(t)) - a$. □

<u>Theorem 3.7.</u> *If $f(t)$ is a Mellin transformable function, then the function $f(t^a)$ is Mellin transformable and we have*

M3
$$\mathcal{M}\,\{f(t^a); s\} = \frac{1}{a}\mathcal{M}\left\{f(t); \frac{s}{a}\right\}.$$

Moreover, $\Delta_{\mathbb{R}}(f(t^a)) = a\Delta_{\mathbb{R}}(f(t))$.

Proof: For this particular proof, **L3** is used to show

$$\begin{aligned}\mathcal{M}\{f(t^a); s\} &= \mathcal{L}\,\{f((e^x)^a); s\}\\ &= \mathcal{L}\,\{f(e^{ax}); s\}\\ &= \frac{1}{a}\mathcal{L}\left\{f(e^x); \frac{s}{a}\right\}\\ &= \frac{1}{a}\mathcal{M}\left\{f(t); \frac{s}{a}\right\}.\end{aligned}$$

This proves **M3**. Moreover, since $\frac{1}{a}\mathcal{M}\,\{f(t); \frac{s}{a}\}$ is defined whenever $\frac{s}{a} \in \Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\{f(t^a); s\}$ converges for s $\in a\Delta_{\mathbb{R}}(f(t))$. □

<u>Theorem 3.8.</u> *If $f(t)$ is a Mellin transformable function, then the function $\frac{1}{t}f\left(\frac{1}{t}\right)$ is Mellin transformable and we have*

M4
$$\mathcal{M}\left\{\frac{1}{t}f\left(\frac{1}{t}\right); s\right\} = \mathcal{M}\left\{f\left(\frac{1}{t}\right); s-1\right\}.$$

Moreover, $\Delta_{\mathbb{R}}(\frac{1}{t}f(\frac{1}{t})) = 1 - \Delta_{\mathbb{R}}(f(t))$.

Proof: Using **L1** , we have

$$\begin{aligned}\mathcal{M}\left\{\frac{1}{t}f\left(\frac{1}{t}\right);s\right\} &= \mathcal{L}\{e^{-x}f(e^{-x});s\} \\ &= \mathcal{L}\{f(e^{-x});s-1\} \\ &= \mathcal{M}\{f(t^{-1});s-1\}.\end{aligned}$$

This proves **M4**. Moreover, since $\mathcal{M}\ \{f(t^{-1});s-1\}$ is defined whenever $s-1\ \in\ -\Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\{\frac{1}{t}f(\frac{1}{t});s\}$ converges for $s\in 1-\Delta_{\mathbb{R}}(f(t))$. $\square$

Theorem 3.9. *If $f(t)$ is a Mellin transformable function, then the function* $(\log t)f(t)$ *is Mellin transformable and we have*

M5 $$\mathcal{M}\{(\log t)f(t);s\} = \frac{d}{ds}\mathcal{M}\{f(t);s\}.$$

Moreover, $\Delta_{\mathbb{R}}((\log t)f(t)) = \Delta_{\mathbb{R}}(f(t))$.

Proof: Here, **L4** is required to show

$$\begin{aligned}\mathcal{M}\{(\log t)f(t);s\} &= \mathcal{L}\{(\log e^{x})f(e^{x});s\} \\ &= \mathcal{L}\{xf(e^{x});s\} \\ &= \frac{d}{ds}\mathcal{L}\{f(e^{x});s\} \\ &= \frac{d}{ds}\mathcal{M}\{f(t);s\}.\end{aligned}$$

This proves **M5**. Moreover, since $\frac{d}{ds}\mathcal{M}\{f(t);s\}$ is defined whenever $s\in\Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\{(\log t)f(t);s\}$ must converge for $s\in\Delta_{\mathbb{R}}(f(t))$. $\square$

Theorem 3.10. *(Mellin Transforms of Derivatives). If $f(t)$ is a Mellin transformable function, then the function $f'(t)$ is Mellin transformable and we have*

M6 $$\mathcal{M}\{f'(t);s\} = -(s-1)\mathcal{M}\{f(t);s-1\}.$$

Moreover, $\Delta_{\mathbb{R}}(f'(t)) = \Delta_{\mathbb{R}}(f(t)) + 1.$

Proof: Note that if we differentiate $f(e^x)$, we simply get $e^x f'(e^x)$. Therefore, to prove the above relation we let $h(x) = f(e^x)$ implying $h'(x) = e^x f'(e^x)$, and we get

$$\begin{aligned}
\mathcal{M}\{f'(t);s\} &= \mathcal{L}\{f'(e^x);s\} \\
&= \mathcal{L}\{e^{-x}h'(x);s\} \\
&= \mathcal{L}\{h'(x);s-1\} && \text{(by } \mathbf{L1}\text{)} \\
&= -(s-1)\mathcal{L}\{h(x);s-1\} && \text{(by } \mathbf{L5}\text{)} \\
&= -(s-1)\mathcal{L}\{f(e^x);s-1\} \\
&= -(s-1)\mathcal{M}\{f(t);s-1\}.
\end{aligned}$$

This proves **M6**. Moreover, since $-(s-1)\mathcal{M}\{f(t);s-1\}$ is defined whenever $s-1 \in \Delta_{\mathbb{R}}(f(t))$, $\mathcal{M}\{f'(t);s\}$ must converge for $s \in \Delta_{\mathbb{R}}(f(t)) + 1$. □

Corollary 3.10. *If $f(t)$ is Mellin transformable, then $\frac{d^n}{dt^n}f(t)$ is Mellin transformable and we have*

$$\mathcal{M}\left\{\frac{d^n}{dt^n}f(t);s\right\} = (-1)^n(s-1)(s-2)\cdots(s-n)\mathcal{M}\{f(t);s-n\}.$$

Theorem 3.11. *(Convolution Property). If $f(t)$ and $g(t)$ are Mellin transformable functions and $\Delta_{\mathbb{R}}(f(t)) \cap \Delta_{\mathbb{R}}(g(t)) \neq \emptyset$, then the function $f(t) \odot g(t)$ is Mellin transformable and we have*

M7 $$\mathcal{M}\{f(t) \odot g(t); s\} = \mathcal{M}\{f(t); s\}\, \mathcal{M}\{g(t); s\}.$$

Moreover, $\Delta_{\mathbb{R}}(f(t) \odot g(t)) \subset \Delta_{\mathbb{R}}(f(t)) \cap \Delta_{\mathbb{R}}(g(t))$.

Proof: Using Theorem 1.1, we can write

$$\begin{aligned} \mathcal{M}\{f(t) \odot g(t); s\} &= \mathcal{L}\{f(e^x) * g(e^x); s\} \\ &= \mathcal{L}\{f(e^x); s\}\mathcal{L}\{g(e^x); s\} \qquad \text{(by } \mathbf{L6}\text{)} \\ &= \mathcal{M}\{f(t); s\}\, \mathcal{M}\{g(t); s\} \end{aligned}$$

This proves **M7**. Moreover, since $\mathcal{M}\{f(t); s\}$ and $\mathcal{M}\{g(t); s\}$ are defined whenever $s \in \Delta_{\mathbb{R}}(f(t)) \cap \Delta_{\mathbb{R}}(g(t))$, $\mathcal{M}\{f(t) \odot g(t); s\}$ converges for $s \in \Delta_{\mathbb{R}}(f(t)) \cap \Delta_{\mathbb{R}}(g(t))$.

□

Theorem 3.12 *(Inversion Formula for the Mellin transform). If $f(t)$ is integrable over every finite interval and is Mellin transformable, then*

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^s F(s) ds,$$

where $F(s) = \mathcal{M}\{f(t); s\}$, $f(t)$ is a real valued function on the positive half line, s is a complex number, and $c \in \Delta_{\mathbb{R}}(f(t))$.

Proof: Using the Laplace Inversion Formula, we have

$$f(e^x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(e^x);s\}e^{sx}\,ds$$

and hence
$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}\{f(e^x);s\}t^s\,ds$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}\{f(t);s\}t^s\,ds.$$

□

CHAPTER 4

TWO-SIDED LAPLACE TRANSFORM OF FUNCTIONS OF n VARIABLES

In this chapter, we want to extend the concept of the Laplace transform of functions of one variable as discussed in Chapter 2 to that of n variables.

4.1. Groups ($\mathbb{R}^n$,+) and ($\mathbb{R}^n_+$, $\diamond$)

We will use the following notation: $\boldsymbol{s} = (s_1, \ldots, s_n)$, $\boldsymbol{t} = (t_1, \ldots, t_n)$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{r} = (r_1, \ldots, r_n)$, $\boldsymbol{u} = (u_1, \ldots, u_n)$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$, $\boldsymbol{a} \diamond \boldsymbol{t} = (a_1 t_1, \ldots, a_n t_n)$, $\boldsymbol{s} \cdot \boldsymbol{x} = s_1 x_1 + \ldots + s_n x_n$, and for a set $\boldsymbol{S} \subset \mathbb{R}^n$ and $\boldsymbol{a} \in \mathbb{R}^n$, define $\boldsymbol{S} - \boldsymbol{a} = \{(s_1 - a_1, \ldots, s_n - a_n) : (s_1, \ldots, s_n) \in \boldsymbol{S}\}$. The operation $+$ is addition on $\mathbb{R}^n$, $\diamond$ is the operator on $\mathbb{R}^n_+$, and $\cdot$ is the dot product of two vectors.

In this chapter we consider two groups: ($\mathbb{R}^n$, $+$) under the operation of addition, and ($\mathbb{R}^n_+$, $\diamond$) under the $\diamond$-operation. In both cases all conditions in the definition of a group are satisfied. Clearly, $\mathbb{R}^n_+$ is a subset of $\mathbb{R}^n$; however, ($\mathbb{R}^n_+$, $\diamond$) is not a subgroup of ($\mathbb{R}^n$, $+$) since the operations are different.

Furthermore, the two groups are isomorphic, and they share the same structure with the same properties. In fact, $\Phi(\boldsymbol{x}) = (e^{x_1}, \ldots, e^{x_n})$ is a group isomorphism from ($\mathbb{R}^n$, $+$) to ($\mathbb{R}^n_+$, $\diamond$). Note that the inverse of Φ, $\Phi^{-1}(\boldsymbol{t}) = (\ln t_1, \ldots, \ln t_n)$, is an isomorphism from ($\mathbb{R}^n_+$, $\diamond$) to ($\mathbb{R}^n$, $+$).

4.2. Convolution of Functions on $\mathbb{R}^n$ and $\mathbb{R}_+^n$

Definition 4.1. The *convolution* of two functions f and g on $\mathbb{R}^n$ is defined as

$$(f * g)(\boldsymbol{t}) = \int_{\mathbb{R}^n} f(\boldsymbol{t} - \boldsymbol{s})g(\boldsymbol{s})d\boldsymbol{s},$$

if the integral exists almost every where.

Extending the concept from Chapter 1, our objective here is to find a convolution for functions of several variables on $\mathbb{R}_+^n$ such that

$$\boldsymbol{T}(f*g)(\boldsymbol{t}) = (\boldsymbol{T}f)(\boldsymbol{t}) \odot (\boldsymbol{T}g)(\boldsymbol{t}),$$

where $\boldsymbol{T}$ is the operation defined by $(\boldsymbol{T}f)(\boldsymbol{t}) = f(\ln t_1, \ldots, \ln t_n)$. Therefore, $\boldsymbol{T}$ takes functions of n variables in $\mathbb{R}^n$ and redefines them as functions of n variables in $\mathbb{R}_+^n$.

We want to derive $\odot$ using $\boldsymbol{T}^{-1}$on $C(\mathbb{R}_+^n)$. The operation $\boldsymbol{T}^{-1}$ takes functions on $\mathbb{R}_+^n$ to functions on $\mathbb{R}^n$: $(\boldsymbol{T}^{-1}f)(\boldsymbol{x}) = f(e^{x_1}, \ldots, e^{x_n})$. For f, g defined on $\mathbb{R}^n$, if we want

$$\boldsymbol{T}(f*g)(\boldsymbol{t}) = (\boldsymbol{T}f)(\boldsymbol{t}) \odot (\boldsymbol{T}g)(\boldsymbol{t}),$$

we must have

$$\boldsymbol{T}(\boldsymbol{T}^{-1}f * \boldsymbol{T}^{-1}g) = f \odot g$$

for f, g defined on $\mathbb{R}_+^n$.

Therefore, we have

$$
\begin{aligned}
(f \odot g)(\boldsymbol{t}) &= \boldsymbol{T}(f(e^{x_1}, \dots, e^{x_n}) * g(e^{x_1}, \dots, e^{x_n})) \\
&= \boldsymbol{T}\left(\int_{\mathbb{R}^n} f(e^{x_1 - s_1}, \dots, e^{x_n - s_n}) g(e^{s_1}, \dots, e^{s_n}) d\boldsymbol{s}\right) \\
&= \int_{\mathbb{R}^n} f\left(e^{\ln t_1 - s_1}, \dots, e^{\ln t_n - s_n}\right) g(e^{s_1}, \dots, e^{s_n}) d\boldsymbol{s} \\
&= \int_0^\infty \cdots \int_0^\infty f\left(\frac{t_1}{r_1}, \dots, \frac{t_n}{r_n}\right) g(\boldsymbol{r}) \frac{dr_1}{r_1} \cdots \frac{dr_n}{r_n} \qquad ((e^{s_1}, \dots, e^{s_n}) = \boldsymbol{r}).
\end{aligned}
$$

Definition 4.2. The *convolution* of two functions f and g on $\mathbb{R}_+^n$ is defined as

$$
(f \odot g)(\boldsymbol{t}) = \int_0^\infty \cdots \int_0^\infty f\left(\frac{t_1}{r_1}, \dots, \frac{t_n}{r_n}\right) g(r_1, \dots, r_n) \frac{dr_1}{r_1} \cdots \frac{dr_n}{r_n},
$$

if the integral exists almost everywhere.

Theorem 4.1. *Given two functions f, g for which the above convolution exists,*

$$
\boldsymbol{T}(f * g) = (\boldsymbol{T}f) \odot (\boldsymbol{T}g),
$$

The proof of this theorem follows from the above calculations of $f \odot g$.

4.3. Basic Definitions

Definition 4.2. *(Two-sided Laplace transformable functions).* A function $f(\boldsymbol{x})$ is called two-sided Laplace transformable, if the set

$$
\Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) = \left\{\boldsymbol{w} \in \mathbb{R}^n : \int_{\mathbb{R}^n} |f(\boldsymbol{x})| e^{\boldsymbol{w} \cdot \boldsymbol{x}} d\boldsymbol{x} < \infty\right\},
$$

has a non-empty interior. $\Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$ is called the region of convergence of $f(\boldsymbol{x})$.

The following theorem is a consequence of the above definition.

Theorem 4.2. *Let $f(\boldsymbol{x})$ be a two-sided Laplace transformable function. Then the integral*

$$\int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x}$$

converges for every $\boldsymbol{s}$ *such that* $(\operatorname{Re} s_1, \ldots, \operatorname{Re} s_n) \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$.

Proof: If $(s_1, \ldots, s_n) \in \mathbb{C}^n$ and $(\operatorname{Re} s_1, \ldots, \operatorname{Re} s_n) \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, then

$$\int_{\mathbb{R}^n} |e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x})| d\boldsymbol{x} = \int_{\mathbb{R}^n} e^{(\operatorname{Re} s_1)x_1} \cdots e^{(\operatorname{Re} s_n)x_n} |f(\boldsymbol{x})| d\boldsymbol{x} < \infty,$$

since $f(\boldsymbol{x})$ is two-sided Laplace transformable. $\square$

Definition 4.3. *(Two-sided Laplace transform).* Let $f(\boldsymbol{x})$ be a function of $(x_1, \ldots, x_n)$, then

$$\mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\} = \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x}$$

is called the two-sided Laplace transform of $f(\boldsymbol{x})$.

Thus, the two-sided Laplace transform of $f(\boldsymbol{x})$ is a function defined in the region of convergence of f.

4.4. Basic Operational Properties of the Laplace Transform

Theorem 4.3. *(Shifting Property). If $f(\boldsymbol{x})$ is Laplace transformable and $\boldsymbol{a} \in \mathbb{R}^n$, then $e^{\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x})$ is Laplace transformable and we have*

L1 $$\mathcal{L}\{e^{\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x}); \boldsymbol{s}\} = \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s} + \boldsymbol{a}\}.$$

Moreover, $\Omega_{\mathbb{R}^n}(e^{\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x})) = \Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) - \boldsymbol{a}$.

Proof: From the definition,

$$\begin{aligned} \mathcal{L}\{e^{\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x}); \boldsymbol{s}\} &= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} e^{\boldsymbol{a}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} e^{(\boldsymbol{s}+\boldsymbol{a})\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x} \\ &= \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s} + \boldsymbol{a}\}. \end{aligned}$$

Thus, **L1** holds. Since $\mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s} + \boldsymbol{a}\}$ is defined whenever $\boldsymbol{s} + \boldsymbol{a} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, $\mathcal{L}\{e^{\boldsymbol{a}\boldsymbol{x}} f(\boldsymbol{x}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) - \boldsymbol{a}$. □

Theorem 4.4. *(Translation Property). If $f(\boldsymbol{x})$ is Laplace transformable and $\boldsymbol{a} \in \mathbb{R}^n$, then $f(\boldsymbol{x} + \boldsymbol{a})$ is Laplace transformable and we have*

L2 $$\mathcal{L}\{f(\boldsymbol{x} + \boldsymbol{a}); \boldsymbol{s}\} = e^{-\boldsymbol{a}\cdot\boldsymbol{s}} \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}.$$

Moreover, $\Omega_{\mathbb{R}^n}(f(\boldsymbol{x} + \boldsymbol{a})) = \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$.

Proof: Allowing $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{a}$, we obtain

$$\begin{aligned}
\mathcal{L}\{f(\boldsymbol{x}+\boldsymbol{a});\boldsymbol{s}\} &= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}+\boldsymbol{a})d\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot(\boldsymbol{u}-\boldsymbol{a})} f(\boldsymbol{u})d\boldsymbol{u} \\
&= e^{-\boldsymbol{a}\cdot\boldsymbol{s}} \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{u}} f(\boldsymbol{u})d\boldsymbol{u} \\
&= e^{-\boldsymbol{a}\cdot\boldsymbol{s}} \mathcal{L}\left\{f(\boldsymbol{x});\boldsymbol{s}\right\}.
\end{aligned}$$

L2 holds. Since $e^{-\boldsymbol{a}\cdot\boldsymbol{s}}\mathcal{L}\{f(\boldsymbol{x});\boldsymbol{s}\}$ is defined whenever $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, $\mathcal{L}\{f(\boldsymbol{x}+\boldsymbol{a});\boldsymbol{s}\}$ must also be defined for $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$. □

Theorem 4.5. *(Scaling Property). If $f(\boldsymbol{x})$ is Laplace transformable, then $f(x_1,\ldots,ax_k,\ldots,x_n)$ is Laplace transformable and we have*

L3 $$\mathcal{L}\{f(x_1,\ldots,ax_k,\ldots,x_n);\boldsymbol{s}\} = \frac{1}{a}\mathcal{L}\Big\{f(\boldsymbol{x});\Big(s_1,\ldots,\frac{s_k}{a},\ldots,s_n\Big)\Big\}.$$

Moreover, $\Omega_{\mathbb{R}^n}(f(x_1,\ldots,ax_k,\ldots,x_n)) = \boldsymbol{a}_k \diamond \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, where $\boldsymbol{a}_k = (1,\ldots,1,a,1,\ldots,1)$ with a being in the kth place.

Proof: We are applying the scalar a to only one component in the vector,

$$\begin{aligned}
\mathcal{L}\{f(x_1,\ldots,ax_k,\ldots,x_n);\boldsymbol{s}\} &= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(x_1,\ldots,ax_k,\ldots,x_n)d\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} e^{s_1x_1}\cdots e^{s_kx_k}\cdots e^{s_nx_n} f(x_1,\ldots,ax_k,\ldots,x_n)d\boldsymbol{u} \\
&= \frac{1}{a}\int_{\mathbb{R}^n} e^{s_1x_1}\cdots e^{\frac{s_k}{a}x_k}\cdots e^{s_nx_n} f(x_1,\ldots,x_n)d\boldsymbol{u} \\
&= \frac{1}{a}\mathcal{L}\Big\{f(\boldsymbol{x});\Big(s_1,\ldots,\frac{s_k}{a},\ldots,s_n\Big)\Big\}.
\end{aligned}$$

L3 holds. Since $\frac{1}{a}\mathcal{L}\{f(\boldsymbol{x}); (s_1, \ldots, \frac{s_k}{a}, \ldots, s_n)\}$ is defined whenever $(s_1, \ldots, \frac{s_k}{a}, \ldots, s_n) \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, $\mathcal{L}\{f(x_1, \ldots, ax_k, \ldots, x_n); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \boldsymbol{a}_k \diamond \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$. □

Corollary 4.5. *If $f(\boldsymbol{x})$ is Laplace transformable and $\boldsymbol{a} \neq 0$, then $f(\boldsymbol{a} \diamond \boldsymbol{x})$ is Laplace transformable and we have*

$$\mathcal{L}\{f(\boldsymbol{a} \diamond \boldsymbol{x}); \boldsymbol{s}\} = \frac{1}{a_1} \cdots \frac{1}{a_n} \mathcal{L}\left\{ f(\boldsymbol{x}); \left(\frac{s_1}{a_1}, \ldots, \frac{s_n}{a_n}\right)\right\}.$$

Moreover, $\Omega_{\mathbb{R}^n}(f(\boldsymbol{a} \diamond \boldsymbol{x})) = \boldsymbol{a} \diamond \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$.

Theorem 4.6. *(Derivatives of the Laplace Transform). If $f(\boldsymbol{x})$ is Laplace transformable, then $x_k f(\boldsymbol{x})$ is Laplace transformable and we have*

L4 $$\frac{\partial}{\partial s_k} \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\} = \mathcal{L}\{x_k f(\boldsymbol{x}); \boldsymbol{s}\},$$

where $k \in \{1, 2, \ldots, n\}$. Moreover, $\Omega_{\mathbb{R}^n}(x_k f(\boldsymbol{x})) = \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$.

Proof: By differentiating within the integral sign, we obtain

$$\begin{aligned} \frac{\partial}{\partial s_k} \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\} &= \int_{\mathbb{R}^n} \frac{\partial}{\partial s_k} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} x_k e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} x_k f(\boldsymbol{x}) d\boldsymbol{x} \\ &= \mathcal{L}\{x_k f(\boldsymbol{x}); \boldsymbol{s}\}. \end{aligned}$$

Clearly, **L4** holds. Since $\mathcal{L}\{x_k f(\boldsymbol{x}); \boldsymbol{s}\}$ is defined whenever $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, $\frac{\partial}{\partial s_k}\mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}$ must also be defined for $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$. □

Theorem 4.7. *(Laplace Transform of Derivatives).* *If $f(\boldsymbol{x})$ is Laplace transformable and $\lim_{x_k \to \infty} e^{s_k x_k} f(\boldsymbol{x}) = \lim_{x_k \to -\infty} e^{s_k x_k} f(\boldsymbol{x}) = 0$ for all s_k such that there is $(s_1, \ldots, s_k, \ldots, s_n) \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$ and all $\boldsymbol{x} \in \mathbb{R}^n$, then $\frac{\partial}{\partial x_k} f(\boldsymbol{x})$ is Laplace transformable and we have*

L5
$$\mathcal{L}\left\{ \frac{\partial}{\partial x_k} f(\boldsymbol{x}); \boldsymbol{s} \right\} = -\, s_k \mathcal{L}\left\{ f(\boldsymbol{x}); \boldsymbol{s} \right\},$$

where $k \in \{1, 2, \ldots, n\}$. Moreover, $\Omega_{\mathbb{R}^n}(\frac{\partial}{\partial x_k} f(\boldsymbol{x})) = \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$.

Proof: Using integration by parts in the integral with respect to x_k,

$$\begin{aligned} \mathcal{L}\left\{ \frac{\partial}{\partial x_k} f(\boldsymbol{x}); \boldsymbol{s} \right\} &= \int_{\mathbb{R}^n} e^{\boldsymbol{s} \cdot \boldsymbol{x}} \frac{\partial}{\partial x_k} f(\boldsymbol{x}) d\boldsymbol{x} \\ &= -\, s_k \mathcal{L}\left\{ f(\boldsymbol{x}); \boldsymbol{s} \right\}. \end{aligned}$$

This proves **L5**. Since $-\, s_k \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}$ is defined whenever $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$, $\mathcal{L}\{\frac{\partial}{\partial x_k} f(\boldsymbol{x}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$. □

Theorem 4.8. *(Convolution Theorem).* *If $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ is Laplace transformable and $\Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) \cap (\Omega_{\mathbb{R}^n}(g(\boldsymbol{x})) \neq \emptyset$, then $f(\boldsymbol{x}) * g(\boldsymbol{x})$ is Laplace transformable and we have*

L6
$$\mathcal{L}\{f(\boldsymbol{x}) * g(\boldsymbol{x}); \boldsymbol{s}\} = \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}\mathcal{L}\{g(\boldsymbol{x}); \boldsymbol{s}\}.$$

Moreover, $\Omega_{\mathbb{R}^n}(f(\boldsymbol{x}) * g(\boldsymbol{x})) \subset \Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) \cap (\Omega_{\mathbb{R}^n}(g(\boldsymbol{x}))$.

Proof: Allowing $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y}$ and changing the order of integration, we get

$$\begin{aligned}
\mathcal{L}\{f(\boldsymbol{x}) * g(\boldsymbol{x}); \boldsymbol{s}\} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot(\boldsymbol{u}+\boldsymbol{y})} f(\boldsymbol{u})g(\boldsymbol{y})d\boldsymbol{u}d\boldsymbol{y} \\
&= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{y}} g(\boldsymbol{y})d\boldsymbol{y}\int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{u}} f(\boldsymbol{u})d\boldsymbol{u} \\
&= \mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}\,\mathcal{L}\{g(\boldsymbol{x}); \boldsymbol{s}\}.
\end{aligned}$$

Thus, **L6** holds. Since $\mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\}$ and $\mathcal{L}\{g(\boldsymbol{x}); \boldsymbol{s}\}$ are defined whenever $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x}))$ and $\boldsymbol{s} \in \Omega_{\mathbb{R}^n}(g(\boldsymbol{x}))$ respectively, $\mathcal{L}\{f(\boldsymbol{x}) * g(\boldsymbol{x}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s}$ $\in \Omega_{\mathbb{R}^n}(f(\boldsymbol{x})) \cap \Omega_{\mathbb{R}^n}(g(\boldsymbol{x}))$. □

Remark. The two-sided Laplace transform of functions with n variables can be viewed as an iterated application of n two-sided Laplace transforms with respect to variables $(x_1, \dots, x_n)$. Therefore, the inversion formula can be obtained by iterated application of the two-sided Laplace transform of functions of a single variable (see Theorem 2.8).

CHAPTER 5
MELLIN TRANSFORM OF FUNCTIONS OF n VARIABLES

In this chapter, the Mellin transform of $f(\boldsymbol{t})$ and its properties are derived using simple substitutions and basic properties of the Laplace transform of functions of n variables.

5.1. Basic Definitions and Comparison with the Laplace Transform

Definition 5.1. *(Mellin transformable functions).* The function $f(\boldsymbol{t})$ is Mellin transformable if the set

$$\Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) = \left\{ \boldsymbol{w} \in \mathbb{R}^n_+ : \int_{\mathbb{R}^n_+} |f(\boldsymbol{t})| t_1^{w_1 - 1} \cdots t_n^{w_n - 1} d\boldsymbol{x} < \infty \right\}$$

has a non-empty interior. $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$ is called the region of convergence of $f(\boldsymbol{t})$.

Theorem 5.1. *A function $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ is Mellin transformable if and only if $f(e^{x_1}, \ldots, e^{x_n})$ is Laplace transformable. Moreover, $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) = \Omega_{\mathbb{R}^n}(f(e^{x_1}, \ldots, e^{x_n}))$.*

Proof: Note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(e^{x_1},\ldots,e^{x_n})||e^{\boldsymbol{s}\cdot\boldsymbol{x}}|d\boldsymbol{x} \\
&\quad= \int_{\mathbb{R}^n} |f(e^{x_1},\ldots,e^{x_n})||e^{(\operatorname{Re} s_1)x_1}\cdots e^{(\operatorname{Re} s_n)x_n}||e^{(i\operatorname{Im} s_1)x_1}\cdots e^{(i\operatorname{Im} s_n)x_n}|d\boldsymbol{x} \\
&\quad= \int_{\mathbb{R}^n} |f(e^{x_1},\ldots,e^{x_n})|e^{(\operatorname{Re} s_1)x_1}\cdots e^{(\operatorname{Re} s_n)x_n}d\boldsymbol{x} \\
&\quad= \int_{\mathbb{R}^n_+} |f(\boldsymbol{t})|t_1^{\operatorname{Re} s_1}\cdots t_n^{\operatorname{Re} s_n}\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n} \\
&\quad= \int_{\mathbb{R}^n_+} |f(\boldsymbol{t})|t_1^{\operatorname{Re} s_1-1}\cdots t_n^{\operatorname{Re} s_n-1}d\boldsymbol{t}.
\end{aligned}$$

This proves that $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) = \Omega_{\mathbb{R}^n}(f(e^{x_1},\ldots,e^{x_n}))$ and $f(\boldsymbol{t})$ is Mellin transformable if and only if $f(e^{x_1},\ldots,e^{x_n})$ is Laplace transformable. □

<u>Theorem 5.2.</u> *If $f(\boldsymbol{t})$ is a Mellin transformable function, then the integral*

$$\int_{\mathbb{R}^n_+} t_1^{s_1-1}\cdots t_n^{s_n-1}f(\boldsymbol{t})d\boldsymbol{t}$$

converges for every $\boldsymbol{s} \in \mathbb{C}^n$ such that $(\operatorname{Re} s_1,\ldots,\operatorname{Re} s_n) \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$.

Proof: Since

$$\int_{\mathbb{R}^n_+} |t_1^{s_1-1}\cdots t_n^{s_n-1}f(\boldsymbol{t})|d\boldsymbol{t} = \int_{\mathbb{R}^n} |f(e^{x_1},\ldots,e^{x_n})|e^{s_1x_1}\cdots e^{s_nx_n}d\boldsymbol{x},$$

this proof follows from Theorem 5.1. □

Definition 5.2. *(The Mellin Transform).* Let f be a function of n variables $(t_1,\dots,t_n)$ then

$$\mathcal{M}\{f(\boldsymbol{t});\boldsymbol{s}\} = \int_{\mathbb{R}^n_+} t_1^{s_1-1}\cdots t_n^{s_n-1} f(\boldsymbol{t})d\boldsymbol{t}$$

is called the Mellin transform of $f(\boldsymbol{t})$.

Thus the Mellin transform of $f(t)$ is a function defined in the region of convergence of f.

Suppose we take $f(\boldsymbol{x})$, substitute $\boldsymbol{x} = (\ln t_1,\dots,\ln t_n)$, then apply the Mellin transform

$$\begin{aligned} \mathcal{M}\{f(\ln t_1,\dots,\ln t_n);\boldsymbol{s}\} &= \int_{\mathbb{R}^n_+} t_1^{s_1-1}\cdots t_n^{s_n-1} f(\ln t_1,\dots,\ln t_n)d\boldsymbol{t} \\ &= \int_{\mathbb{R}^n} e^{s_1x_1}\cdots e^{s_nx_n} f(\ln e^{x_1},\dots,\ln e^{x_n})d\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} e^{\boldsymbol{s}\cdot\boldsymbol{x}} f(\boldsymbol{x})d\boldsymbol{x}. \end{aligned}$$

In other words, the Laplace transform of $f(\boldsymbol{x})$ is the same as the Mellin transform of $f(\ln t_1,\dots,\ln t_n)$and we have the relation,

$$\mathcal{L}\{f(\boldsymbol{x});\boldsymbol{s}\} = \mathcal{M}\ \{f(\ln t_1,\dots,\ln t_n);\boldsymbol{s}\}.$$

These two equalities will be useful in proving the basic properties of the Mellin transform. We formulate them more precisely in the following two theorems.

Theorem 5.3. *If* $f(\boldsymbol{x})$ *is Laplace transformable, then* $f(\ln t_1, \ldots, \ln t_n)$ *is Mellin transformable and*

$$\mathcal{L}\{f(\boldsymbol{x}); \boldsymbol{s}\} = \mathcal{M}\{f(\ln t_1, \ldots, \ln t_n); \boldsymbol{s}\}.$$

Theorem 5.4. *If* $f(\boldsymbol{t})$ *is Mellin transformable, then* $f(e^{x_1}, \ldots, e^{x_n})$ *is Laplace transformable and*

$$\mathcal{M}\ \{f(\boldsymbol{t}); \boldsymbol{s}\} = \mathcal{L}\ \{f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\}.$$

Taking into account our previous discussion, we can say that the two-sided Laplace transform of n variables and the Mellin transform of n variables are two versions of the same transform. The first one is defined for functions on the group $(\mathbb{R}^n,\ +)$, and the second one is defined on the isomorphic group $(\mathbb{R}^n_+,\ \diamond\ )$. The group isomorphism,

$$\Phi(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n}),$$

identifies the two transforms.

5.2. Basic Operational Properties of the Mellin Transform

As in the single variable case, we will prove the following properties by using a similar procedure: allowing $\boldsymbol{t} = (e^{x_1}, \ldots, e^{x_n})$ and rewriting the Mellin transform in terms of the Laplace transform, $\mathcal{M}\ \{f(\boldsymbol{t}); \boldsymbol{s}\} = \mathcal{L}\ \{f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\}$, we then construct the proofs using only Laplace properties. Finally, once the proof is completed in the Laplace space, we transform the end result to the Mellin space using the same substitution.

Theorem 5.5. *(Scaling Property). If $f(\boldsymbol{t})$ is a Mellin transformable function and $a \in \mathbb{R}_+^n$, then the function $f(\boldsymbol{a} \diamond \boldsymbol{t})$ is Mellin transformable and we have*

$$\textbf{M1} \qquad \{f(\boldsymbol{a} \diamond \boldsymbol{t}); \boldsymbol{s}\} = (a_1^{-s_1} \cdots a_n^{-s_n})\mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\},$$

Moreover, $\Delta_{\mathbb{R}^n}(f(\boldsymbol{a} \diamond \boldsymbol{t})) = \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$.

Proof: By replacing $\boldsymbol{t} = (e^{x_1}, \ldots, e^{x_n})$, we have

$$\begin{aligned}
\mathcal{M}\{f(\boldsymbol{a} \diamond \boldsymbol{t}); \boldsymbol{s}\} &= \mathcal{L}\{f(\boldsymbol{a} \diamond (e^{x_1}, \ldots, e^{x_n})); \boldsymbol{s}\} \\
&= \mathcal{L}\{f\big((e^{\ln a_1}, \ldots, e^{\ln a_n}) \diamond (e^{x_1}, \ldots, e^{x_n})\big); \boldsymbol{s}\} \\
&= \mathcal{L}\{f\big(e^{x_1 + \ln a_1}, \ldots, e^{x_n + \ln a_n}\big); \boldsymbol{s}\} \\
&= e^{-s_1 \ln a_1} \cdots e^{-s_n \ln a_n} \mathcal{L}\{f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \qquad \text{(by } \textbf{L2}\text{)} \\
&= a_1^{-s_1} \cdots a_n^{-s_n} \mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\}.
\end{aligned}$$

This proves **M1**. Since $a_1^{-s_1} \cdots a_n^{-s_n} \mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\}$ is defined for $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{f(\boldsymbol{a} \diamond \boldsymbol{t}); \boldsymbol{s}\}$ must also be defined for $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$. □

Theorem 5.6. *If $f(\boldsymbol{t})$ is a Mellin transformable function and $\boldsymbol{a} \in \mathbb{R}_+^n$, then the function $t_1^{a_1} \cdots t_n^{a_n} f(\boldsymbol{t})$ is Mellin transformable and we have*

$$\textbf{M2} \qquad \mathcal{M}\{t_1^{a_1} \cdots t_n^{a_n} f(\boldsymbol{t}); \boldsymbol{s}\} = \mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s} + \boldsymbol{a}\}.$$

Moreover, $\Delta_{\mathbb{R}^n}(t_1^{a_1} \cdots t_n^{a_n} f(\boldsymbol{t})) = \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) - \boldsymbol{a}$.

Proof: Using **L1** for n variables, it is clear that

$$\begin{aligned}
\mathcal{M}\{t_1^{a_1} \cdots t_n^{a_n} f(\boldsymbol{t}); \boldsymbol{s}\} &= \mathcal{L}\{e^{a_1 x_1} \cdots e^{a_n x_n} f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \\
&= \mathcal{L}\{e^{\boldsymbol{a} \cdot \boldsymbol{x}} f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \\
&= \mathcal{L}\{f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s} + \boldsymbol{a}\} \qquad \text{(by } \textbf{L1}\text{)} \\
&= \mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s} + \boldsymbol{a}\}.
\end{aligned}$$

This proves **M2**. Since $\mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}+\boldsymbol{a})\}$ is defined for $\boldsymbol{s}+\boldsymbol{a} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{t_1^{a_1}\cdots t_n^{a_n} f(\boldsymbol{t}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) - \boldsymbol{a}$. □

Theorem 5.7. *If $f(\boldsymbol{t})$ is a Mellin transformable function, then the function $f(t_1^{a_1},\ldots,t_n^{a_n})$ is Mellin transformable and we have*

$$\textbf{M3} \qquad \mathcal{M}\{f(t_1^{a_1},\ldots,t_n^{a_n}); \boldsymbol{s}\} = \frac{1}{a_1}\cdots\frac{1}{a_n}\mathcal{M}\Big\{f(\boldsymbol{t}); \Big(\frac{s_1}{a_1},\ldots,\frac{s_n}{a_n}\Big)\Big\}.$$

Moreover, $\Delta_{\mathbb{R}^n}(f(t_1^{a_1},\ldots,t_n^{a_n})) = \boldsymbol{a} \diamond \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$.

Proof: For this particular proof, **L3** for n variables, is used to show

$$\begin{aligned}
\mathcal{M}\{f(t_1^{a_1},\ldots,t_n^{a_n}); \boldsymbol{s}\} &= \mathcal{L}\{f(e^{a_1x_1},\ldots,e^{a_nx_n}); \boldsymbol{s}\} \\
&= \frac{1}{a_1}\cdots\frac{1}{a_n}\mathcal{L}\Big\{f(e^{x_1},\ldots,e^{x_n}); \Big(\frac{s_1}{a_1},\ldots,\frac{s_n}{a_n}\Big)\Big\} \\
&= \frac{1}{a_1}\cdots\frac{1}{a_n}\mathcal{M}\Big\{f(\boldsymbol{t}); \Big(\frac{s_1}{a_1},\ldots,\frac{s_n}{a_n}\Big)\Big\}.
\end{aligned}$$

This proves **M3**. Since $(\frac{1}{a_1}\cdots\frac{1}{a_n})\mathcal{M}\{f(\boldsymbol{t}); (\frac{s_1}{a_1},\ldots,\frac{s_n}{a_n})\}$ is defined for $(\frac{s_1}{a_1},\ldots,\frac{s_n}{a_n}) \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{f(t_1^{a_1},\ldots,t_n^{a_n}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \boldsymbol{a} \diamond \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$.

□

Theorem 5.8. *If $f(\boldsymbol{t})$ is a Mellin transformable function, then the function $\frac{1}{t_1}\cdots\frac{1}{t_n}f\big(\frac{1}{t_1},\ldots,\frac{1}{t_n}\big)$ is Mellin transformable and we have*

$$\textbf{M4} \qquad \mathcal{M}\left\{\frac{1}{t_1}\cdots\frac{1}{t_n}f\Big(\frac{1}{t_1},\ldots,\frac{1}{t_n}\Big); \boldsymbol{s}\right\} = \mathcal{M}\left\{f\Big(\frac{1}{t_1},\ldots,\frac{1}{t_n}\Big); \boldsymbol{s}-(1,\ldots,1)\right\}.$$

Moreover, $\Delta_{\mathbb{R}^n}(\frac{1}{t_1}\cdots\frac{1}{t_n}f\big(\frac{1}{t_1},\ldots,\frac{1}{t_n}\big); \boldsymbol{s}) = \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) + (1,\ldots,1)$.

Proof: Using **L1** for n variables, we have

$$\begin{aligned}
\mathcal{M}\left\{\frac{1}{t_1}\cdots\frac{1}{t_n}f\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right);\boldsymbol{s}\right\} &= \mathcal{L}\{e^{-x_1}\cdots e^{-x_n}f(e^{-x_1},\ldots,e^{-x_n});\boldsymbol{s}\} \\
&= \mathcal{L}\{f(e^{-x_1},\ldots,e^{-x_n});\boldsymbol{s}-(1,\ldots,1)\} \\
&= \mathcal{M}\left\{f\left(\frac{1}{t_1},\ldots,\frac{1}{t_n}\right);\boldsymbol{s}-(1,\ldots,1)\right\}.
\end{aligned}$$

This proves **M4**. Since $\mathcal{M}\{f(\frac{1}{t_1},\ldots,\frac{1}{t_n});\boldsymbol{s}-(1,\ldots,1)\}$ is defined for $\boldsymbol{s}-(1,\ldots,1)\in\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{\frac{1}{t_1}\cdots\frac{1}{t_n}f(\frac{1}{t_1},\ldots,\frac{1}{t_n});\boldsymbol{s}\}$ must be defined for $\boldsymbol{s}\in\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))+(1,\ldots,1)$. $\square$

Theorem 5.9. *If $f(\boldsymbol{t})$ is a Mellin transformable function, then the function $(\log t_k)f(\boldsymbol{t})$ is Mellin transformable and we have*

M5 $$\mathcal{M}\{(\log t_k)f(\boldsymbol{t});\boldsymbol{s}\} = \frac{\partial}{\partial s_k}\mathcal{M}\{f(\boldsymbol{t});\boldsymbol{s}\},$$

where $k\in\{1,2,\ldots n\}$. Moreover, $\Delta_{\mathbb{R}^n}((\log t_k)f(\boldsymbol{t}))=\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$.

Proof: Here, **L4** for n variables is required to show that

$$\begin{aligned}
\mathcal{M}\{(\log t_k)f(\boldsymbol{t});\boldsymbol{s}\} &= \mathcal{L}\{\log e^{x_k}f(e^{x_1},\ldots,e^{x_n});\boldsymbol{s}\} \\
&= \mathcal{L}\{x_kf(e^{x_1},\ldots,e^{x_n});\boldsymbol{s}\} \\
&= \frac{\partial}{\partial s_k}\mathcal{L}\{f(e^{x_1},\ldots,e^{x_n});\boldsymbol{s}\} \\
&= \frac{\partial}{\partial s_k}\mathcal{M}\{f(\boldsymbol{t});\boldsymbol{s}\}.
\end{aligned}$$

Clearly, **M5** holds. Since $\frac{\partial}{\partial s_k}\mathcal{M}\{f(\boldsymbol{t});\boldsymbol{s}\}$ is defined for $\boldsymbol{s}\in\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{(\log t_k)f(\boldsymbol{t});\boldsymbol{s}\}$ must also be defined for $\boldsymbol{s}\in\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$. $\square$

Theorem 5.10. *(Mellin Transforms of Derivatives). If $f(\boldsymbol{t})$ is a Mellin transformable function, then the function $\frac{\partial}{\partial t_k} f(\boldsymbol{t})$ is Mellin transformable and we have*

$$\textbf{M6} \qquad \mathcal{M}\left\{\frac{\partial}{\partial t_k} f(\boldsymbol{t}); \boldsymbol{s}\right\} = -(s_k - 1)\mathcal{M}\{f(\boldsymbol{t}); (s_1, \ldots, s_k - 1, \ldots s_n)\},$$

where $k \in \{1, 2, \ldots n\}$. Moreover, $\Delta_{\mathbb{R}^n}(\frac{\partial}{\partial t_k} f(\boldsymbol{t})) = \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) + (0, \ldots, 1, \ldots, 0)$, with 1 being in the k-th place.

Proof: Note that if we differentiate $f(e^{x_1}, \ldots, e^{x_n})$, we simply get $e^{x_k} f_k(e^{x_1}, \ldots, e^{x_n})$ where f_k is the k-th partial derivative. Therefore, to prove the above relation, we let $h(\boldsymbol{x}) = f(e^{x_1}, \ldots, e^{x_n})$ implying $\frac{\partial}{\partial x_k} h(\boldsymbol{x}) = e^{x_k} f_k(e^{x_1}, \ldots, e^{x_n})$, and we get

$$\begin{aligned}
\mathcal{M}\{f_k(\boldsymbol{t}); \boldsymbol{s}\} &= \mathcal{L}\{f_k(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \\
&= \mathcal{L}\left\{e^{-x_k} \frac{\partial}{\partial x_k} h(\boldsymbol{x}); \boldsymbol{s}\right\} \\
&= \mathcal{L}\left\{\frac{\partial}{\partial x_k} h(\boldsymbol{x}); (s_1, \ldots, s_k - 1, \ldots s_n)\right\} && \text{(by } \mathbf{L1}\text{)} \\
&= -(s_k - 1)\mathcal{L}\{h(\boldsymbol{x}); (s_1, \ldots, s_k - 1, \ldots s_n)\} && \text{(by } \mathbf{L5}\text{)} \\
&= -(s_k - 1)\mathcal{L}\{f(e^{x_1}, \ldots, e^{x_n}); (s_1, \ldots, s_k - 1, \ldots s_n)\} \\
&= -(s_k - 1)\mathcal{M}\{f(\boldsymbol{t}); (s_1, \ldots, s_k - 1, \ldots s_n)\}.
\end{aligned}$$

This proves **M6**. Since $-(s_k - 1)\mathcal{M}\{f(\boldsymbol{t}); (s_1, \ldots, s_k - 1, \ldots s_n)\}$ is defined for $(s_1, \ldots, s_k - 1, \ldots s_n) \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$, $\mathcal{M}\{\frac{\partial}{\partial t_k} f(\boldsymbol{t}); \boldsymbol{s}\}$ must be defined for $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) + (0, \ldots, 1, \ldots, 0)$. □

Theorem 5.11. *(Convolution Property). If $f(\boldsymbol{t})$ and $g(\boldsymbol{t})$ are Mellin transformable function and $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) \cap \Delta_{\mathbb{R}^n}(g(\boldsymbol{t})) \neq \emptyset$, then the function $f(\boldsymbol{t}) \odot g(\boldsymbol{t})$ is Mellin transformable and we have*

M7 $$\mathcal{M}\{f(\boldsymbol{t}) \odot g(\boldsymbol{t}); \boldsymbol{s}\} = \mathcal{M}\{f(\boldsymbol{t})\}\mathcal{M}\{g(\boldsymbol{t})\}.$$

Moreover, $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t}) \odot g(\boldsymbol{t})) \subset \Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) \cap \Delta_{\mathbb{R}^n}(g(\boldsymbol{t}))$.

Proof: Using Theorem 4.1, we can rewrite

$$\begin{aligned}
\mathcal{M}\{f(\boldsymbol{t}) \odot g(\boldsymbol{t}); \boldsymbol{s}\} &= \mathcal{L}\{f(e^{x_1}, \ldots, e^{x_n}) * g(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \\
&= \mathcal{L}\{f(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\}\mathcal{L}\{g(e^{x_1}, \ldots, e^{x_n}); \boldsymbol{s}\} \qquad \text{(by } \mathbf{L6}\text{)} \\
&= \mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\}\, \mathcal{M}\{g(\boldsymbol{t}); \boldsymbol{s}\}.
\end{aligned}$$

Clearly, **M7** holds. Since $\mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\}$ and $\mathcal{M}\{f(\boldsymbol{t}); \boldsymbol{s}\}$are defined for $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(f(\boldsymbol{t}))$ and $\boldsymbol{s} \in \Delta_{\mathbb{R}^n}(g(\boldsymbol{t}))$, $\mathcal{M}\{f(\boldsymbol{t}) \odot g(\boldsymbol{t}); \boldsymbol{s}\}$ must be contained in $\Delta_{\mathbb{R}^n}(f(\boldsymbol{t})) \cap \Delta_{\mathbb{R}^n}(g(\boldsymbol{t})$. □

Remark. The Mellin transform of functions with n variables can be viewed as an iterated application of n Mellin transforms with respect to variables $(t_1, \ldots, t_n)$. Therefore, the inversion formula can be obtained by iterated application of the inverse Mellin transform of functions of a single variable.

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