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STATISTICAL ERROR IN ABSORPTION EXPERIMENTS

by

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ABSTRACT

In certain exponential absorption experiments, notably measurements of cross sections by transmission, it is important to achieve minimum statistical error in a limited time, or to minimize the counting time required to measure the absorption coefficient with a preassigned accuracy. The conditions required to attain these ends, i.e., the geometry for optimum transmission, and the best apportionment of counting times among the incident and transmitted beams and background, have been investigated for a wide range of relative backgrounds (10^{-3} to 10^{-2}), and for two geometries: I. Beam area fixed, absorber thickness alone is varied. II. Beam area and absorber thickness are both disposable parameters, while the total amount of absorber intercepting the beam remains fixed. In both cases the incident flux density and the background rate are assumed constant. The optimum transmissions are shown to be, in general, considerably smaller than those commonly used in absorption experiments. Thus, in Case I, a useful rule is to employ a transmission of about 0.1 for low backgrounds, 0.2 for moderate backgrounds, and 0.3 for high backgrounds. The following have also been determined: (a) minimum statistical error for a given total counting time, (b) statistical error and the best distribution of counting times for nonoptimum geometry, and (c) sensitivity of the accuracy or total counting time to deviations from optimum transmission.

INTRODUCTION

There are many physical measurements of absorption or transmission of radiation in which the transmission is an exponential function of the absorber thickness. Measurements of this kind occur, for example, in the fields of optics, x-rays, and nuclear physics. In the latter field the purpose is usually to determine a total cross section.

In most measurements of this type, the thickness of absorber can be chosen at will within reasonable limits, i.e., the transmission is a disposable parameter. Suppose for the sake of definiteness that the detector is a counter. If a thick absorber is interposed in the beam, the transmitted intensity is low, and a relatively long time is required to collect an adequate count. In fact, for sufficiently low transmissions, the transmitted beam may become comparable in magnitude to the background. This obviously results in inefficient counting. On the other hand, if a very thin absorber is used, the intensity of the transmitted beam approaches that of the incident beam, and again the geometry is statistically unfavorable. Therefore some intermediate value of the transmission should be employed, and of course this is commonly done.

It may sometimes be desirable to know what value of the transmission is best if the greatest accuracy is to be obtained in a given time or if a preassigned accuracy is to be attained in the least time. Moreover, the dependence of this optimum transmission on background is essential. In addition, it is important to determine the sensitivity of the precision of measurement or the total counting time to deviations from optimum geometry. Further questions are concerned with the best apportionment of the total counting time among incident beam, transmitted beam, and background, and its dependence on the background and the thickness of absorber. These questions become important when the total available counting time is strictly limited by radioactive decay of the absorbing material, or by other considerations. They are also relevant in other situations in which high counting efficiency is desirable. The answers to these questions depend upon the way in which the background varies and upon certain geometric properties of the beam, the absorber, and the detector.

The present discussion is based on the following assumptions. (a) The flux density in the incident beam is constant in time and independent of geometry. (b) The total background rate is likewise constant. (c) The detector completely intercepts the beam. (d) The efficiency of the detector is the same for the incident and transmitted beams. (e) The transmission is an exponential function of the absorber thickness; i.e., the absorber thickness is small compared to the scattering mean free path. To the extent that the optimum transmission is significantly less than unity, which will almost always be true, this also implies that the absorption cross section must be much larger than the scattering cross section.

Two cases will be considered:

I. The total counting rate of the incident beam is fixed (i.e., the cross-sectional area of the beam is fixed); the thickness of absorber can be varied. This is the usual situation.

II. The total counting rate of the incident beam is not fixed but disposable by changing the area of the beam, subject to the condition that the total amount of absorber in the path of the beam remains fixed. Adjustment of the beam area may be advantageous when the amount of absorber available for the measurement is small, and it is desired to exploit the absorber fully by placing all of it in the path of the beam.

CALCULATION OF OPTIMUM CONDITIONS

By virtue of the assumption of constant background a transmission measurement consists in obtaining three counting rates. This simplicity may not be present in the case of individual experimental arrangements. For example, the background may depend on absorber thickness, in which case at least one additional counting rate must be determined. However, such cases can easily be treated by the procedure described below.

Designating the counting times for the incident and transmitted beams, including background, by t_0 and t_1 respectively, and the counting time for the background by t_2, we have for the counting rates

$$r_i = c_i/t_i \qquad (1)$$

Where c_0, c_1, and c_2 are the total counts for these three measurements, respectively. The flux densities in the incident and transmitted beams, exclusive of background, are denoted by s_0 and s_1 respectively, so that

$$s_i = (r_i - r_2)/a \qquad (i = 1.2) \qquad (2)$$

where a is the cross-sectional area of the beam. The transmission T can then be written as

$$T = s_1/s_0 = (r_1 - r_2)/(r_0 - r_2) = e^{-x} \qquad (3)$$

where

$$x = N\sigma/a \qquad (4)$$

N is the total number of atoms in the path of the beam and σ is the total cross section per atom of absorber.

We use the symbol Δ prefixed to any quantity to indicate the standard error in that quantity. The relative standard error $\Delta\sigma/\sigma$ in the cross section σ is obtained by applying the law of propagation of error to

$$\sigma = -a/N \ln T \tag{5}$$

so that

$$\left(\frac{\Delta\sigma}{\sigma}\right)^2 = \left(\frac{\partial\sigma}{\partial a}\frac{\Delta a}{\sigma}\right)^2 + \left(\frac{\partial\sigma}{\partial N}\frac{\Delta N}{\sigma}\right)^2 + \left(\frac{\partial\sigma}{\partial T}\frac{\Delta T}{\sigma}\right)^2$$

We shall confine our attention to the contribution of the third term, i.e., assume that Δa and ΔN are negligible. Subject to this condition, we have

$$p = \left|\frac{\Delta\sigma}{\sigma}\right| = \left|\frac{\partial\sigma}{\partial T}\frac{\Delta T}{\sigma}\right| \tag{6}$$

From (4), (5), and (6),

$$p^2 = (\Delta T/T\ln T)^2 = (\Delta T/T)^2/x^2 \tag{7}$$

To evaluate p^2, we apply the law of propagation of error to $(\Delta T/T)^2$ using (3):

$$(\Delta T/T)^2 = \frac{(\Delta r_1)^2 + (\Delta r_2)^2}{(r_1 - r_2)^2} + \frac{(\Delta r_0)^2 + (\Delta r_2)^2}{(r_0 - r_2)^2} \tag{8}$$

Since the errors in measuring the times t_i are almost invariably negligible, we have, from (1),

$$\Delta r_i r_i = \Delta c_i / c_i \tag{9}$$

Here and in the following i takes on values 0, 1, and 2. With $(\Delta c_i)^2 = c_i$, it follows that

$$(\Delta r_i)^2 = r_i/t_i \tag{10}$$

Substituting from equations (2), (3), and (10) into (8),

$$\left(\frac{\Delta T}{T}\right)^2 = \frac{1}{s_0^2 a^2}\left[e^{2x}\left(\frac{r_1}{t_1} + \frac{r_2}{t_2}\right) + \frac{r_0}{t_o} + \frac{r_2}{t_2}\right] \tag{11}$$

It is convenient to introduce the total counting time τ

$$\tau = t_0 + t_1 + t_2 \tag{12}$$

and to replace the partial counting times t_i by the relative counting times α_i:

$$\alpha_i = t_i/\tau \tag{13}$$

with

$$\sum_i \alpha_i = 1 \tag{14}$$

Using equations (11) and (13), and multiplying both sides by τ, (7) can be written:

$$p^2\tau = (1/xs_0a)^2\left[(r_2 + s_2a)/\alpha_0 + (r_2e^{2x} + s_0ae^{x})/\alpha_1 + (r_2e^{2x} + r_2)/\alpha_2\right] \qquad (15)$$

As might have been expected, the total counting time is inversely proportional to the square of the relative error in the cross section.

We shall now investigate the optimum geometry and apportionment of counting times, treating in turn the two cases previously described.

I. Cross-sectional Area of Beam Fixed

Let the relative background m be defined by

$$m = r_2/(r_0 - r_2) = r_2/s_0a \qquad (16)$$

Substitution of this in equation (15) yields

$$p^2\tau = (1/x^2s_0a)\left[(m + 1)/\alpha_0 + (me^{2x} + e^{x})/\alpha_1 + m(e^{2x} + 1)/\alpha_2\right] \qquad (17)$$

From this it is evident that whether it be required (a) to minimize τ for a preassigned p or (b) to minimize p for a fixed τ, the optimum conditions are the same. The adjustable parameters in either case are x, which refers to the geometrical arrangement, and the α_i, which refer to the apportionment of counting times.

Requirement (a) may occur when there is no essential restriction on the total counting time, and an upper limit is preassigned to the relative error p. To be sure, p can then be made arbitrarily small, regardless of geometry or apportionment of counting times, merely by counting long enough. However, it is sometimes worth while to select experimental conditions which will keep p below the prescribed limit in the least time. For example, in a long series of transmission measurements the time saved can be significant.

Requirement (b) applies when the total available counting time is limited, e.g., by radioactive decay of the absorber material, or in cosmic-ray experiments at high altitudes. Here p cannot be made arbitrarily small, but the experimental conditions can be chosen so as to minimize it.

For the calculation it is convenient to introduce the notation:

$$\begin{aligned} f_0^2 &= (m + 1)/x^2 \\ f_1^2 &= (me^{2x} + e^{x})/x^2 \\ f_2^2 &= m(e^{2x} + 1)/x^2 \end{aligned} \qquad (18)$$

so that

$$p^2\tau = (1/s_0a)\sum_i (f_i^2/\alpha_i) \qquad (19)$$

Designating differentiation with respect to x by primes, the solutions are given by

$$\sum_i f_if_i'/\alpha_i = 0 \qquad (20)$$

Minimizing with respect to the α_i and taking equation (14) into account, we obtain

$$\alpha_i = f_i / \sum_i f_i \tag{21}$$

Substitution of (21) into (20) gives

$$\sum_i f_i' = 0 \tag{22}$$

or, more explicitly,

$$me^{2x}\left[(1/f_1) + (1/f_2)\right] + e^x/2f_1 - x \sum_i f_i = 0 \tag{22a}$$

It is seen that the optimum values of the parameters x and α_i depend on the relative background m alone.

a) Optimum Transmission

By numerically solving equation 22a for x as a function of m, the optimum transmission $T_{opt} = e^{-x_{opt}}$ is evaluated as a function of the relative background m. In Figure 1, T_{opt} is plotted over a wide range of m, from 10^{-3} to 10^2. It will be noticed that even for very large backgrounds, T_{opt} does not exceed 0.31, and, in general, the optimum transmissions are considerably smaller than those frequently used in absorption experiments.

The results in Figure 1 may be crudely summed up by the following useful rule. From the point of view of counting efficiency, a transmission of about 0.1 should be employed for low backgrounds, 0.2 for moderate backgrounds, and 0.3 for high backgrounds. The meaning of "low," "moderate," and "high" will be evident from Figure 1. This rule is applicable when the cross-sectional area of the beam is fixed and the absorber thickness can be adjusted; moreover, the conditions (a) to (e) specified in the introduction are assumed to hold at least approximately. If the background changes significantly during an experiment, and it is inconvenient to change the absorber thickness, then a fairly good procedure is to use a transmission appropriate to an average background. This is a consequence of the insensitivity of T_{opt} to m. As will be evident from the subsequent discussion,(Section (d) below, also Figure 5) these rules have wider applicability than might be supposed at first glance.

When the background is absent (m = 0), equation (22a) reduces to

$$(x-2)^2 = 4e^{-x} \tag{23}$$

and the solution is

$$e^{-x} = T_{opt} = 0.076$$

This special case has been previously considered by Rainwater and Havens.*

b) Optimum Counting Times for Optimum Transmission

The best apportionment of counting times for any value of m is obtained by substituting into (21) the root of (22a) for that m, i.e., by putting the optimum transmission into (21). Although the optimum fractional times α_i are functions of m alone, they can be plotted against T_{opt} instead of m, since to each m there corresponds a unique T_{opt}. In Figure 2 the α_i are so plotted. The curve labeled α_0 gives the fractional time which should be devoted to counting the incident beam, α_1 the transmitted beam, and α_2 the background. It will be recalled from Figure 1 that increasing values of T_{opt} correspond to increasing m. Figure 2 shows that at low T_{opt} (and therefore low backgrounds) the background rate r_2 is relatively unimportant and need not be measured with great precision; most of the time τ should be devoted to measuring r_1 which is small compared with r_0. As the background increases, r_2 contributes more significantly to the quantities $r_0 - r_2$ and $r_1 - r_2$ (the true counting rates for incident and transmitted beams), and it becomes necessary to measure r_2 more and more accu-

* Phys. Rev. 70:146 (1946).

rately. The required time is obtained mainly at the expense of r_1, which of course increases with T_{opt} so that it can be adequately measured in less time.

c) Deviations from Optimum Geometry

Optimum counting times when arbitrary transmission is used: In some experiments it may be too difficult or even unwise* to employ the optimum transmissions derived here. For any transmission T which may be used, however, there is still an optimum apportionment of counting times. These α_i are given by (21), as before, but they are now functions of two independent parameters, T and m, instead of m alone (or T alone). Accordingly, for all nonoptimum (as well as optimum) transmissions a family of curves can be plotted for each α_i, as in Figures 3 and 4. Figure 3 gives the fractional time α_0 which should be devoted to counting the incident beam, as a function of T for various relative backgrounds corresponding to the values of m with which the curves are labeled. Figure 4 shows the fractional time for the transmitted beam. The fractional background time α_2 is, of course, simply $1-\alpha_0-\alpha_1$.

d) Deviations from Optimum Geometry

Effect on p or τ: It may now be asked, how sensitive is the relative error p or the total counting time τ to deviations from the optimum thickness of absorber; i.e., what price is paid in increased counting time or decreased accuracy for a given departure from T_{opt}? The answer depends in part upon whether the best apportionment of times or some arbitrary set α_i is used. Since situations in which the optimum α_i should not be used (when counting efficiency matters) are difficult to conceive, we shall assume that this partial optimum condition is realized.† The sensitivity can then be measured by the deviation of the ratio $(p^2\tau)/(p^2\tau)_{opt}$ from its minimum value of unity. Large values of this ratio are of course undesirable, as they imply large uncertainty or an unduly long time required to attain a preassigned accuracy.

Combining equations (19) and (21),

$$p^2\tau = (1/s_0a)(\Sigma f_i)^2 \tag{24}$$

$p^2\tau$ depends on both m and T (i.e., on m and x). The minimum of $p^2\tau$ is $(p^2\tau)_{opt}$, and this is evaluated from the solution of equation (22a) for each value of m.

The ratio $p^2\tau/(p^2\tau)_{opt}$ is plotted in Figure 5 for various values of m (numbers affixed to each curve) as a function of T. It is evident from the flatness of the minima in Figure 5 that moderate deviations from T_{opt} are only slightly disadvantageous; i.e., it matters little if the geometry used is not very close to the optimum. On the other hand, the steep portions of the curves, and especially those for $T < T_{opt}$ show that for large departures from T_{opt}, a significant and rapidly rising increase in error or counting is time incurred. Thus, from the curve for the smallest relative background, $m = 10^{-3}$, it is clear that a transmission as small as 0.01 is decidedly disadvantageous, as would be expected. The same curve, however, also shows that using a transmission of 0.61 instead of the optimum (≈ 0.1), necessitates a 6-fold increase in the counting time to achieve a given accuracy; or, in a given counting time, increases the relative error p by a factor $\sqrt{6}$.

In general, for low backgrounds, transmissions exceeding 0.6 should be avoided if possible, and even for very high backgrounds, it is unwise to exceed 0.75. Inspection of Figure 5 confirms the working rule previously deduced from Figure 1, i.e., that transmissions of about 0.1, 0.2, and 0.3, respectively,

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* For example, in spectrometry, where higher-order effects introduce errors which are sometimes difficult to measure, the magnitude of these effects may be reduced by using higher transmissions. Another example is given in Phy. Rev. 70:147 (1946).

† It should be noted that the best apportionment of α_i for nonoptimum T is in general different from that shown in Figure 2, as discussed in the previous section. The apportionment for any T is given in Figures 3 and 4.

are suitable for use with low, m derate, and high backgrounds.

Figure 5 can be used to ascertain the permissible range of transmissions if $p^2 \tau$ is not to exceed $(p^2\tau)_{opt}$ by more than some preassigned factor. For example, suppose m = 0.1, and it is desired not to exceed the minimum counting time (for a given accuracy) by more than a factor 2. From the curve labeled 0.1 in Figure 5 it is seen that the transmission should lie in the range $0.06 < T < 0.5$.

The permissible range of transmissions as a function of relative background is shown more directly in Figure 6, for several maximum values, 1.2, 2, and 4, of $p^2\tau/(p^2\tau)_{opt}$. The numbers on the curves are values of this ratio. Thus, if $p^2 \tau/(p^2\tau)_{opt}$ is not to exceed 2, the highest permissible transmission is shown by the upper curve "2", and the lowest by lower curve "2". For example, it the relative background is 0.01, and it is desired not to spend more than twice the minimum time in obtaining a given accuracy, the allowable range of transmissions runs from 0.03 to 0.42. If the total counting time is fixed, the relative error in σ will then exceed the minimum error by at most $\sqrt{2}$.

e) Evaluation of $(p^2\tau)_{opt}$

For a given relative background or a given T_{opt}, it may be desired to estimate quickly the value of $(p^2\tau)_{opt}$. The latter depends on the absolute background as well as on m (or k).

Case 1: Using (16), equation (24) can be rewritten

$$p^2\tau r_2 = m(\Sigma f_i)^2 \tag{24a}$$

When for a given m the corresponding x_{opt} is substituted in the right-hand side of equation 24a, this becomes

$$J = (p^2\tau)_{opt} r_2 = m\ (\Sigma f_i)^2_{opt} \tag{24b}$$

The right-hand side can be expressed as a function of m alone or of T_{opt} alone. In Figure 7, J is plotted against T_{opt}.

II. Beam of Variable Cross-sectional Area

When only a small sample of absorber is available for a cross-section measurement by transmission, it may be advantageous to place all of it in the path of the beam. We suppose that the cross-sectional area a of the beam and the area of the absorber sample are adjustable so that the two can be kept equal. It is further assumed that the total counting rate of the incident beam, unlike that in Case I, is not fixed, but varies in proportion to a, subject to the condition that the total amount of absorber in the path of the beam remains fixed. This condition implies, of course, that the thickness of absorber varies as 1/a. The problem is whether to make the absorber thick and the beam narrow, or the absorber thin and the beam wide. Here again, to a given relative background there corresponds an optimum transmission; however, the latter now determines not only the optimum absorber thickness but also the optimum beam area. It is fairly evident that the value of the best transmission here will differ from that for Case I. It is true that in both cases the thickness of absorber changes, but in Case II the beam area also changes, while the total number of atoms in the path of the beam remains fixed. In Case I the reverse is true; the beam area is fixed, and the number of atoms in the path of the beam varies with the thickness.

The derivation of T_{opt} is parallel to that in Case I. $p^2 \tau$ is again minimized with respect to x and two of the α_i. It is convenient here to introduce a new parameter in place of m (See (16)). The latter is suitable when a is independent of T, but this is not true in Case II, where a choice of a at once determines T. We define instead a new "relative background,"

$$k = r_2/s_0 N\sigma \tag{25}$$

in which the product $N\sigma$, which is constant here, takes the place of a, which is constant in Case I.*

*Comparison of definitions (25) and (16) shows that m = kx.

Replacing x^2a^2 by $N^2\sigma^2$ (See (4)), and introducing k, equation (17) can be rewritten

$$p^2\tau = (1/s_0N\sigma)\left[(k + 1/x)/\alpha_0 + (ke^{2x} + e^x/x)/\alpha_1 + k(e^{2x} + 1)/\alpha_2\right] \tag{26}$$

which corresponds to equation (17) in Case I. Similarly, corresponding to the f_i^2 in (18), we define

$$\begin{aligned} \phi_0^2 &= k + 1/x \\ \phi_1^2 &= ke^{2x} + e^x/x \\ \phi_2^2 &= k(e^{2x} + 1) \end{aligned} \tag{27}$$

so that equation (26) can be written

$$p^2\tau = (1/s_0N\sigma)\,\Sigma\,\phi_i^2/\alpha_i \tag{28}$$

Minimizing with respect to x, α_0, and α_1, subject to (14), gives

$$\Sigma\,(\phi_i\phi_i'/\alpha_i) = 0 \tag{29}$$

$$\alpha_i = \phi_i/\Sigma\,\phi_i \tag{30}$$

and substitution of equation (30) into (29) gives the result

$$\Sigma\,\phi_i' = 0 \tag{31}$$

which can be written explicitly as

$$2kx^2e^{2x}(1/\phi_1 + 1/\phi_2) + e^x/\phi_1(x-1) = 1/\phi_0 = 0 \tag{31a}$$

a) Optimum Transmission

This equation in x and k has been solved numerically, and the results are given in Figure 8. Here $e^{-x} = T_{opt}$ is plotted against k. The optimum transmissions are higher than those shown in Figure 1 for all values of the relative background; in particular, they approach 1.0 rather than 0.31 for large k. The difference can be explained as follows: As we have seen, a transmission is statistically unfavorable when either of the true counting rates r_1-r_2 or r_0-r_1 is too small. In both cases which we have discussed a larger background can be partly offset by increasing r_1; hence T_{opt} increases with relative background. In Case I, however, as r_1 increases it approaches r_0 which is fixed (since a is fixed) so that beyond a certain point (which turns out to be T = 0.31), it is no longer advantageous to increase r_1. In Case II, for larger relative backgrounds, r_1 is increased as before by thinning the absorber. However, a is simultaneously enlarged so that the increase in the transmitted counting rate is automatically accompanied by an increase in the incident rate. Thus, although $(r_0-r_1)/r_0$ still decreases with increasing T, it does so more slowly than in Case I, and the decrease is offset by the increased counting rates of both incident and transmitted beams. Hence, in Case II, it is advantageous for very large backgrounds to increase T up to the maximum value allowed by expanding the beam area.

When the background is zero (k = 0), equation (31a) reduces to

$$(x-1)^2 = e^{-x} \tag{32}$$

and the solution is

$$e^{-x} = T_{opt} = 0.228$$

b) Optimum Counting Times When T_{opt} Is Used

Equation (30) gives the best apportionment of counting times for a given value of k, provided that the root of (31a) corresponding to that k is substituted in (30). Since there is a one-to-one correspondence between k and T_{opt}, the optimum fractional times α_0, α_1, and α_2 can be plotted against the optimum transmission (instead of against k). This has been done in Figure 9.

c) Deviations from Optimum Geometry

Optimum α_i for Arbitrary T: When a nonoptimum transmission is used, the best α_i are, as might be expected, identical with those deduced for Case I. These α_i could be obtained by substituting into equation (30) the relevant value of x (i.e., $-\ln T$) and k. But it is no longer necessary or even useful to employ k, since T, and therefore a, is now fixed (k was introduced in place of m in deriving T_{opt} because the latter contains a, which in Case II depends on T). It is better instead to revert to m as a measure of the relative background.* Moreover, equation (22) is identical with equation (30) for a given value of T. This can be seen by replacing† k with m/x in (27), and substituting the resulting ϕ_i into (30). Thus, for nonoptimum transmissions, the best α_i are the same regardless of whether the experimental conditions are those of Case I or Case II. Two of the α_i are plotted, as we have seen, in Figures 3 and 4, respectively, and the third is obtained by subtracting the sum of the first two from 1 (See (14)).

d) Deviations from Optimum Geometry

Effect on p or τ: As in Case I, the sensitivity of p or τ to departures from optimum geometry can be deduced from the variation of the ratio $p^2\tau/(p^2\tau)_{opt}$ with transmission. In the present case $p^2\tau$ is evaluated from equations (28) and (30) with the result:

$$p^2\tau = (1/s_0 N\sigma)(\Sigma\phi_i)^2 \tag{33}$$

$(p^2\tau)_{opt}$ for a given k is evaluated by substituting in (33) the solution of (31a) for that value of k. Equation (33) can be used to compute the permissible range of transmissions if $p^2\tau$ is not to exceed its minimum value $(p^2\tau)_{opt}$ by more than some preassigned factor. In Figure 10 (the analogue of Figure 6) this permissible range of T is given as a function of k for the values 1.2, 2, and 4 of $p^2\tau/(p^2\tau)_{opt}$. As in Figure 6, there are two curves for each of these values, the upper curve showing the upper limit on T, and the lower one showing the lower limit.

e) Evaluation of $(p\tau)_{opt}$

In the same way as in section I(e), but now using (25), equation (33) can be rewritten

$$(p^2\tau)_{opt}\, r_2 = k(\Sigma\phi_i)^2 \tag{25a}$$

and substituting x_{opt}, obtained from equation (31), this becomes

* To determine m, only r_0 and r_2 need be measured; to determine k, on the other hand, r_1 must be measured as well, as is evident from the definitions of these quantities. In fact, by equations (16), (25), and (3),

$$k = \frac{m}{x} = -\frac{r_2}{(r_0 - r_2)\ln \dfrac{r_1 - r_2}{r_0 - r_2}}$$

† Comparison of definitions (25) and (16) shows that M = KX.

$$G = (p^2\tau)_{opt}\, r_2 = k\,(\Sigma\phi_i)^2_{opt} \tag{25b}$$

In Figure 11, G is plotted against T_{opt}.

DISCUSSION

The utility of this investigation depends largely on the insensitivity of $p^2\tau$ to moderate deviations from T_{opt}, as shown by the flatness of the minima in Figure 5. This is evident from the following considerations: T_{opt} depends on m in Case I, and on k in Case II. A knowledge of m, however, depends on the measurement of r_0 and r_2 (See (16)); k depends on these two parameters and on r_1 as well. (See preceding footnote*) Thus we are confronted with a paradoxical requirement of measuring two (or, in Case II, all three) of the counting rates from which T is deduced before we can calculate the statistically favorable conditions for making these very measurements. Moreover, in order to choose the optimum absorber thickness, once T_{opt} has been determined, the cross section (or, more generally, the absorption coefficient) must be known. Since the latter depends on T, all three of the counting rates must be known in advance even in Case I. In practice, however, this is not a real difficulty. It has been shown that $p^2\tau$ changes so slowly in the neighborhood of T_{opt} that moderate deviations from T_{opt} are unimportant. Thus a rough preliminary measurement of the counting rates suffices to give an estimated absorber thickness which can be used for practical purposes. The time consumed by these preliminary measurements should in general be quite short compared with that for the final measurements, since the time required for a count is inversely proportional to the square of the permissible error, and the latter may be fairly large without leading to a significantly poor estimate of T_{opt}.

The results presented here apply when the primary considerations are the statistical ones described above. It is recognized that circumstances may arise in which other considerations are important. It may then be desirable or even necessary to use nonoptimum transmissions. The foregoing remarks about deviations from optimum geometry indicate to what extent this may be done without incurring unduly large statistical errors. In addition, regardless of what transmission is used, the optimum apportionment of counting times given in Figures 3 and 4 will still be applicable.

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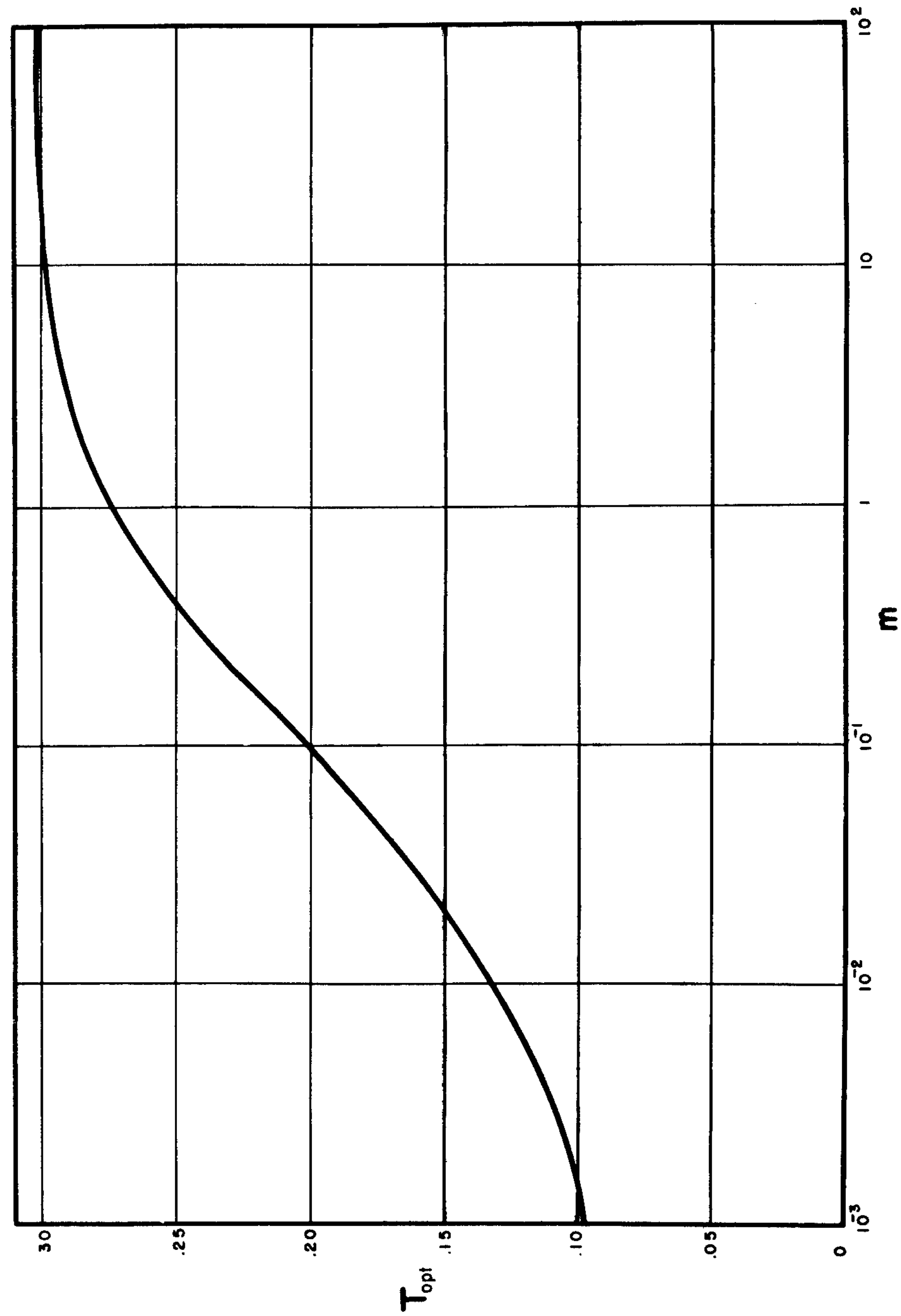
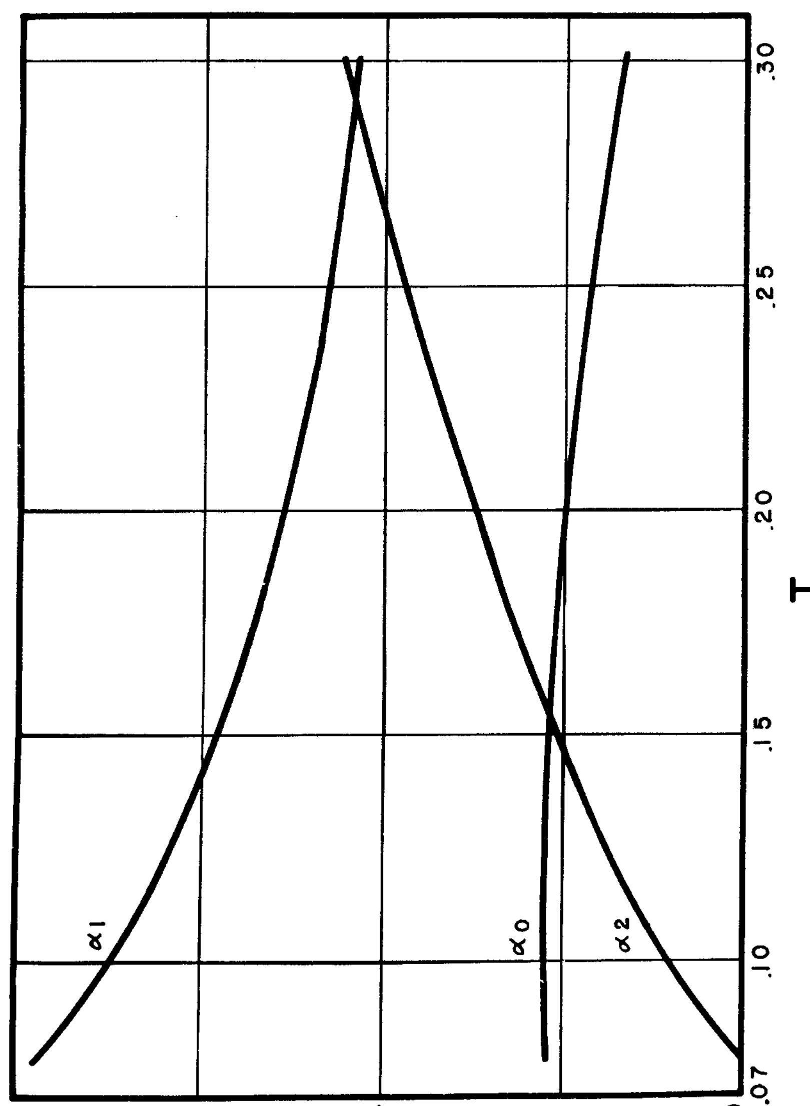


Figure 1.



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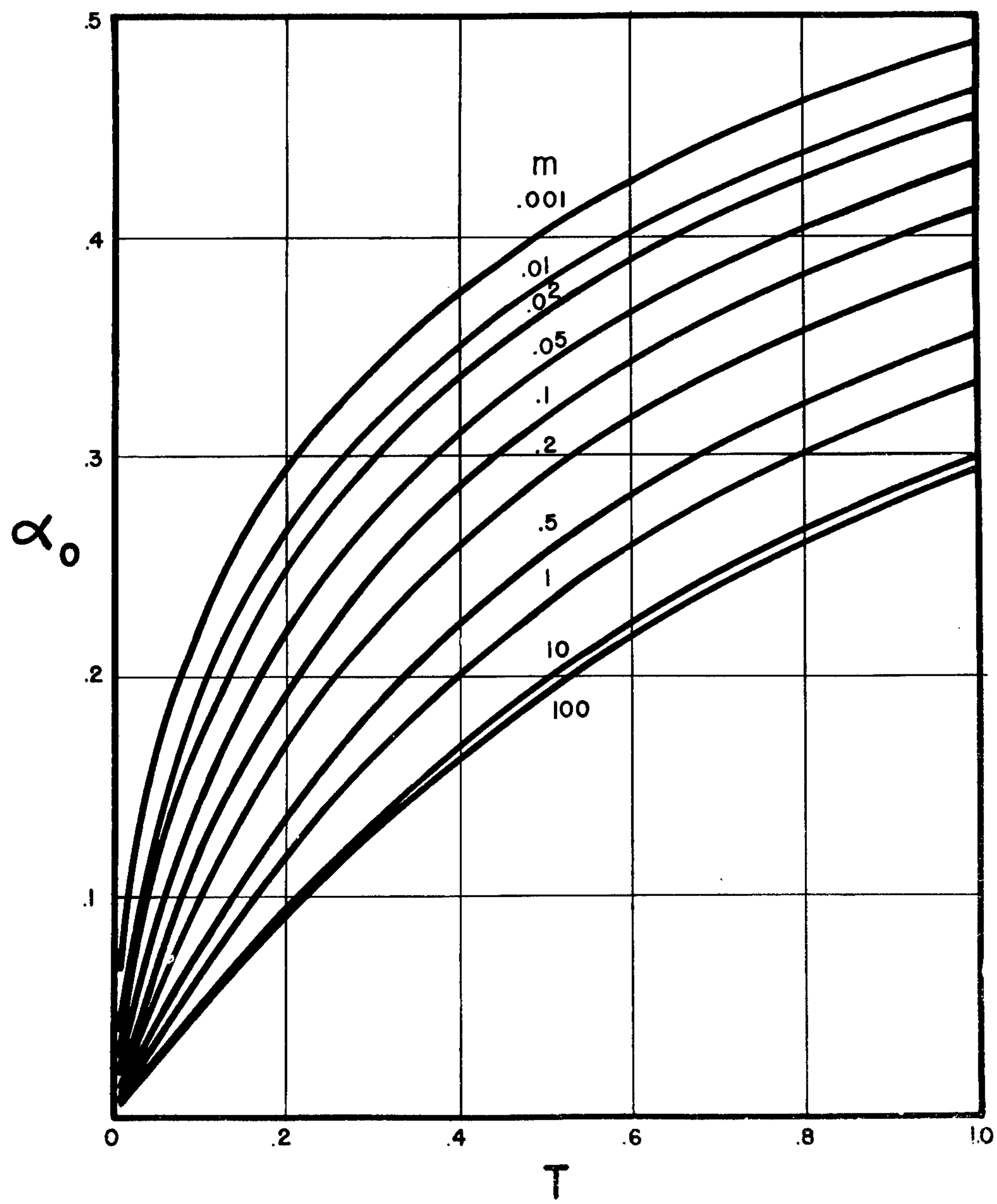

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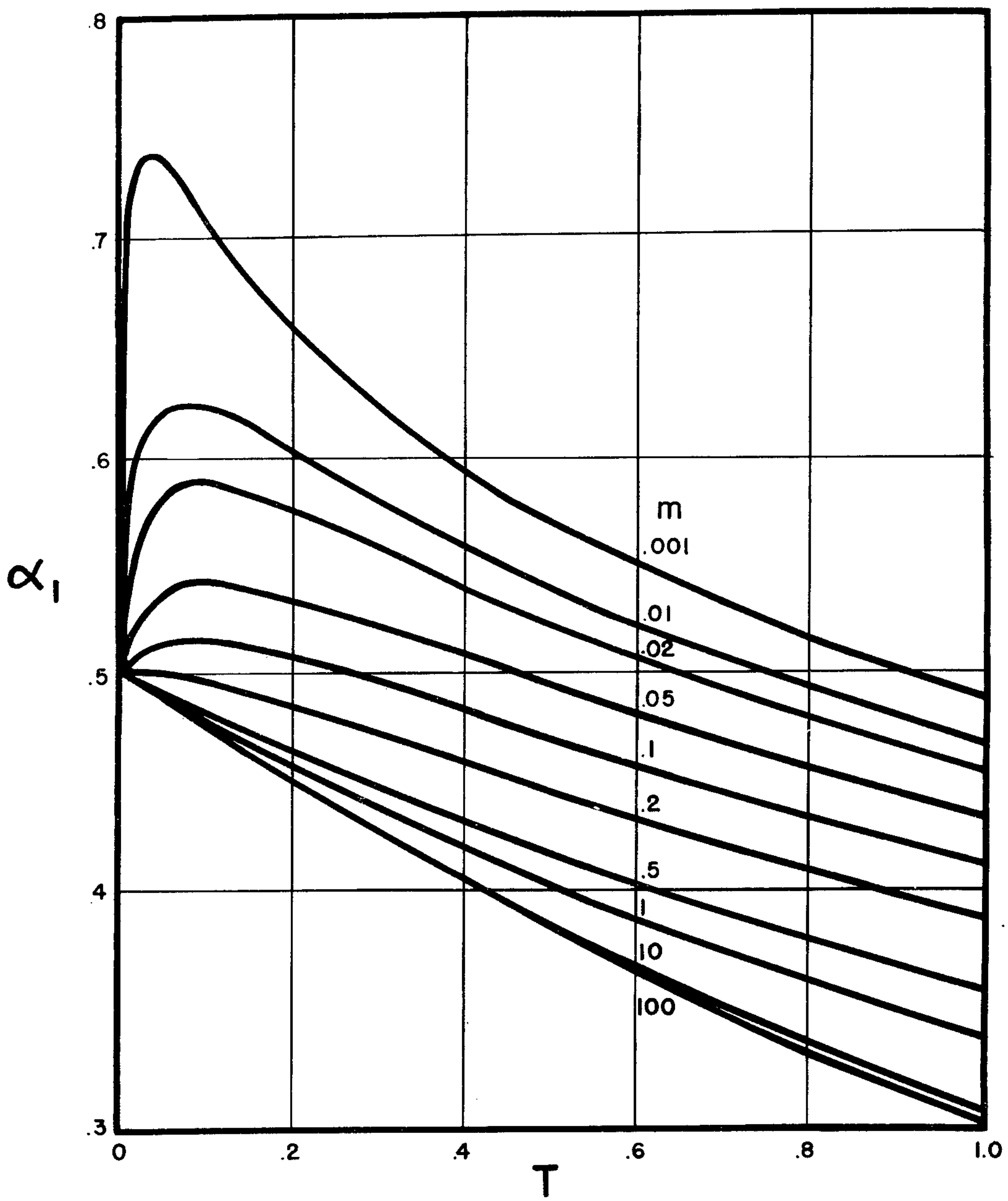

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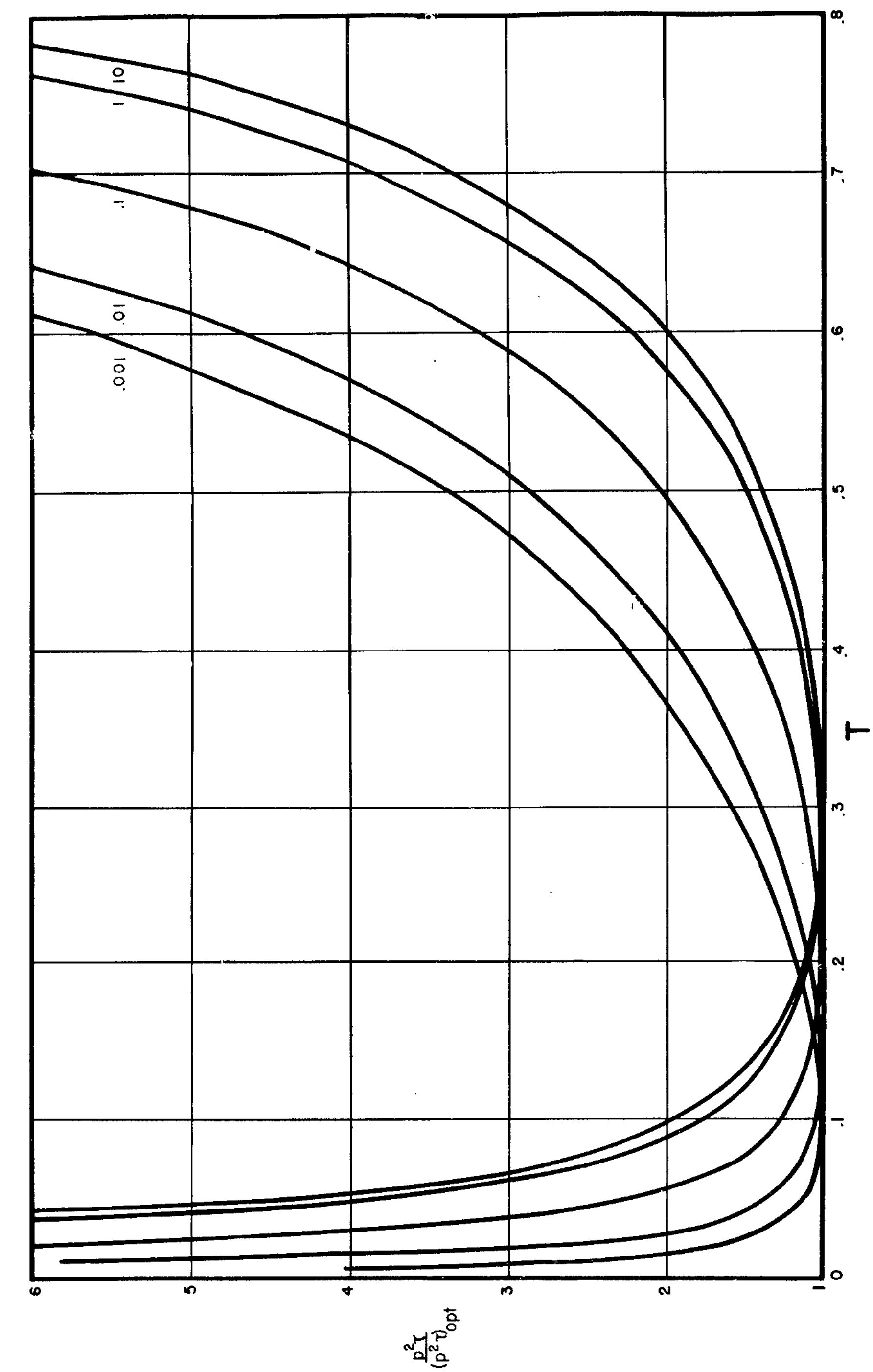

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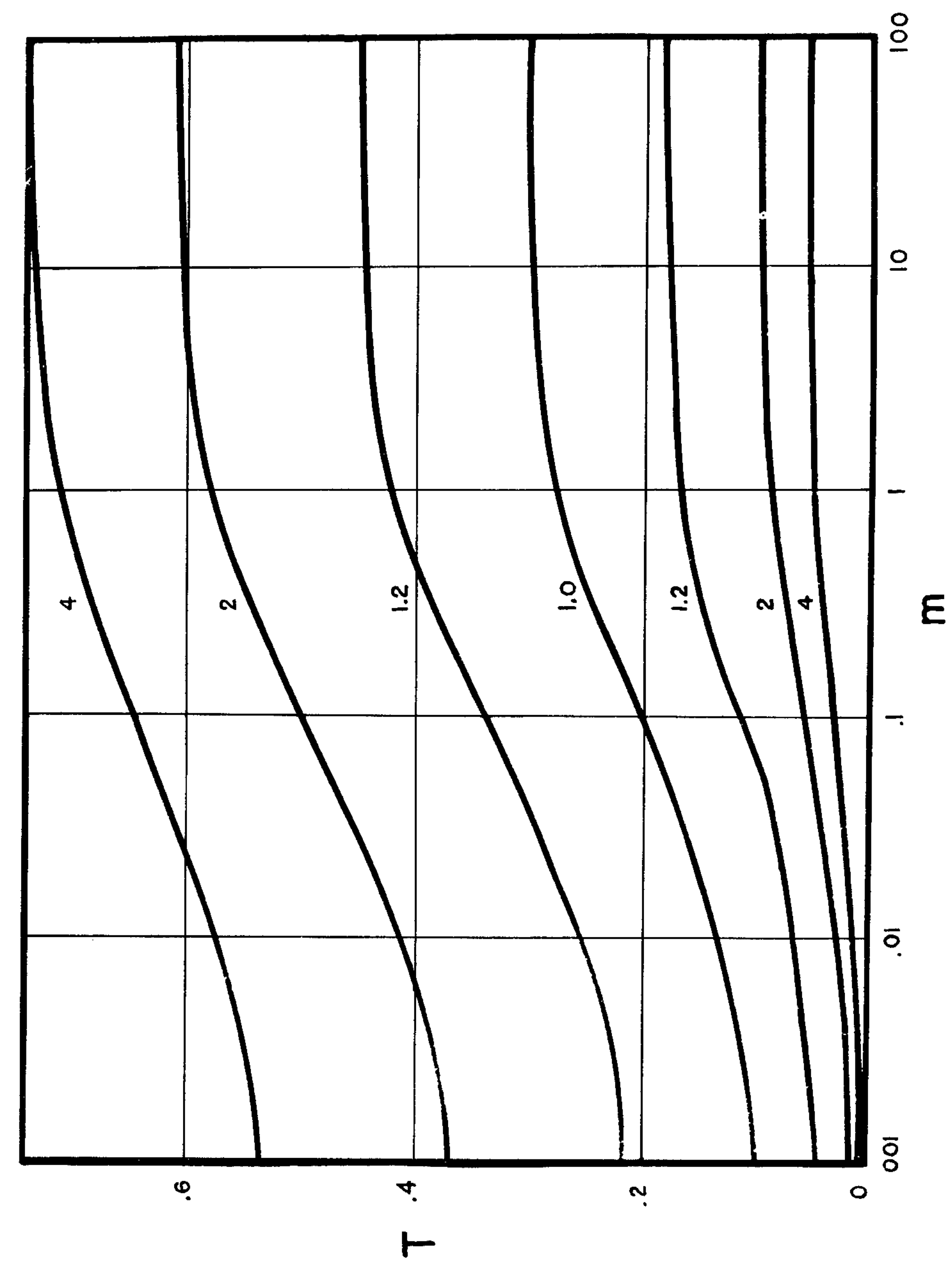

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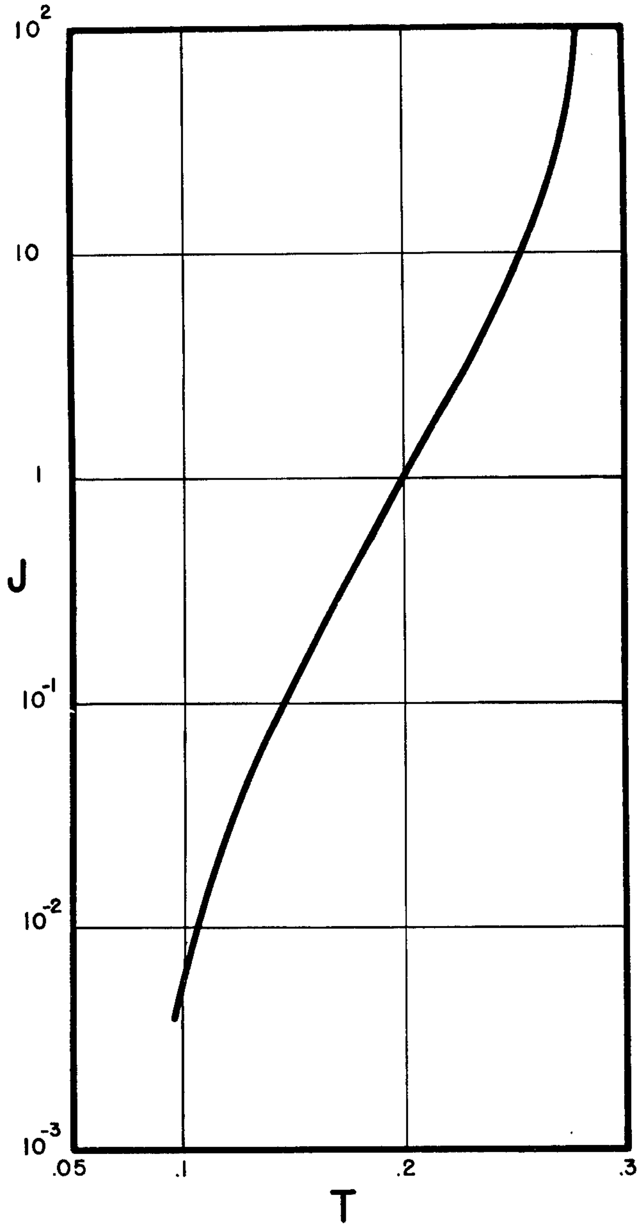

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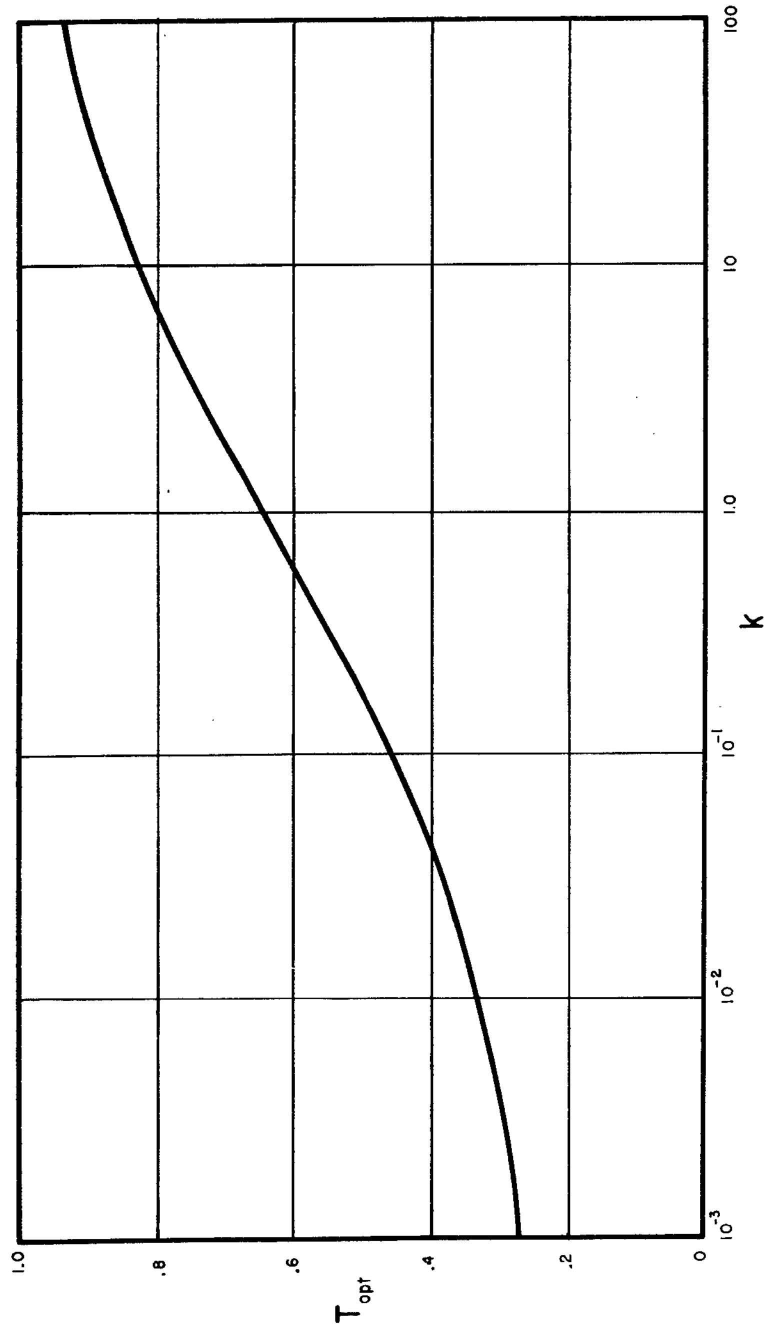

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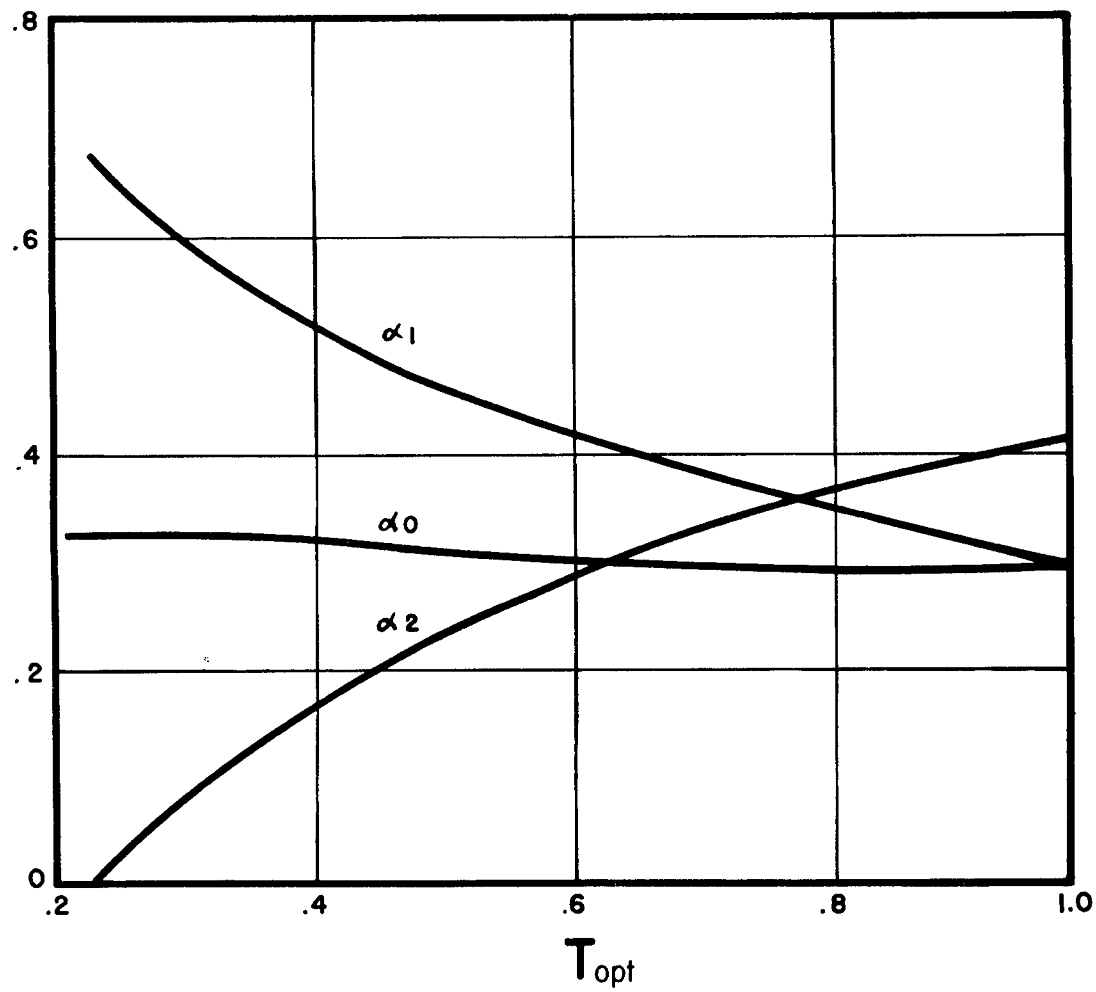

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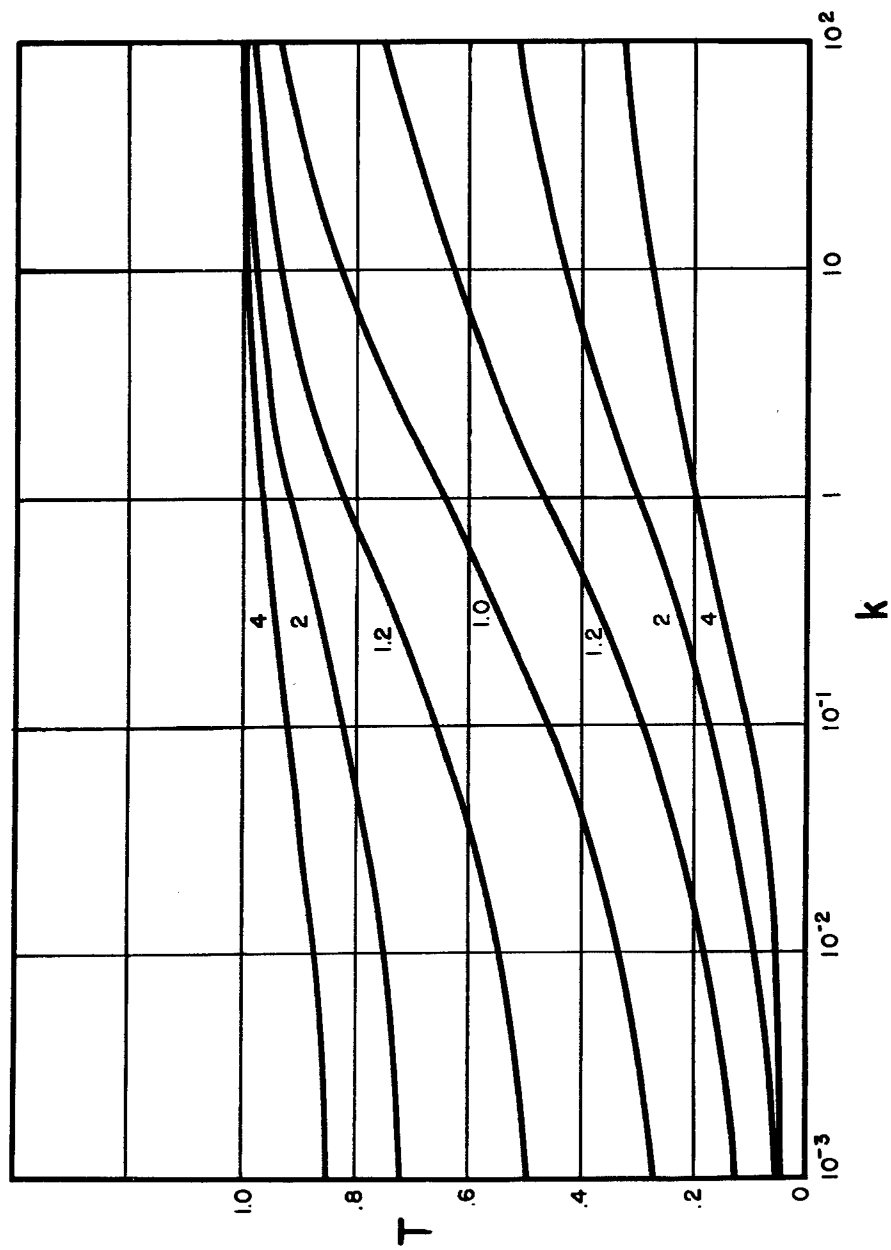

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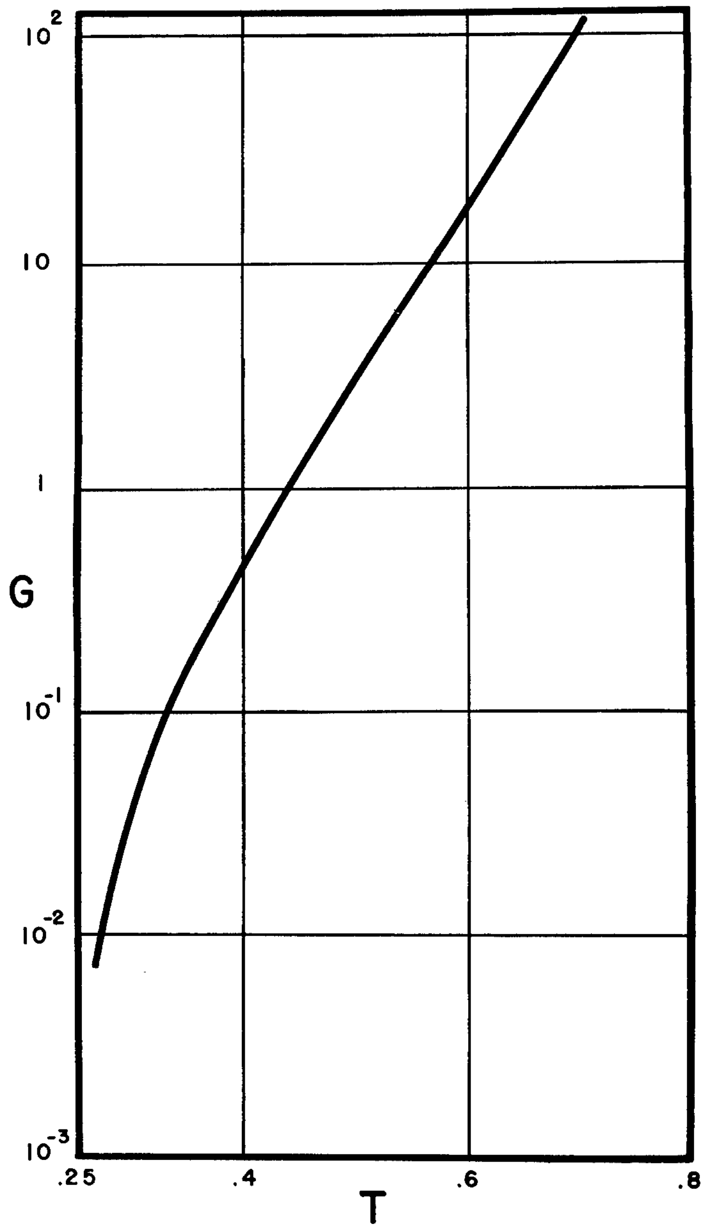

Figure 11.

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