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MEAN MASS-FIELD IN A NON-HOMOGENEOUS OCEAN

by

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RELATIONSHIPS BETWEEN THE WIND-FIELD, THE  
TRANSPORT FIELD AND THE MEAN MASS-FIELD  
IN A NON-HOMOGENEOUS OCEAN

V.B.SHTOKMAN

(Presented by P.P.SHIRSHOV,  
Member of the Academy, December 9, 1947)

The classical theory of the horizontal circulation in a homogeneous ocean, as constructed by V.EKMAN {1}, suffers from the defect that it considers only the frictional forces acting in horizontal planes. In consequence, once the water on the sea-bottom has acquired a motion, vertical velocity-gradients must be developed just as great as those in the surface-layer of the sea. This result does not agree with observed data, which show that the vertical velocity-gradients in a non-homogeneous ocean are rapidly damped out with increasing depth, and at the bottom are to all practical intents and purposes insignificant. This rapid attenuation of the vertical velocity-gradients is a result of intensive horizontal exchange and transfer of motions, setting up, in vertical planes parallel to the horizontal components of flow, large frictional forces which almost entirely counter-balance the tangential stresses due to the wind on the surface of the ocean. Thus the distinctive feature of an established motion in a non-homogeneous ocean is the fact, confirmed by observation, that the frictional forces at the bottom are extremely small in comparison with the stresses set up by the turbulent "side-friction". On the basis of these findings, the equations for the components of the steady-state transport \* in a non-homogeneous ocean may be written in the following form:

$$\begin{aligned} A_1 \left( \frac{\partial^2 S_x}{\partial x^2} + \frac{\partial^2 S_x}{\partial y^2} \right) + T_x + c\bar{\rho}S_y &= g \frac{\partial Q}{\partial x}, \\ A_1 \left( \frac{\partial^2 S_y}{\partial x^2} + \frac{\partial^2 S_y}{\partial y^2} \right) + T_y - c\bar{\rho}S_x &= g \frac{\partial Q}{\partial y}, \end{aligned} \quad (1)$$

---

\* "Transport" in this translation corresponds to Ekman's Strommenge (quantity of flow), as defined in equations 1a. (Translator)

Here  $A_1$  represents the coefficient of horizontal turbulent friction (which is of the order of  $10^8$  CGS),  $c = 2\omega \sin \phi$  is the Coriolis parameter,  $g$  is the acceleration of gravity,  $T_x$  and  $T_y$  are the horizontal  $x$ - and  $y$ -components of the tangential wind-pressure at the surface of the ocean,  $\bar{\rho}$  is the mean density of the water in a column extending from the surface ( $z = 0$ ) to the bottom ( $z = h$ ), and  $S_x$  and  $S_y$  are the components of the transport:

$$S_x = \int_0^h u dz, \quad S_y = \int_0^h v dz, \quad (1a)$$

where  $u$  and  $v$  are the horizontal components of the velocity of flow. Again, we have

$$Q = \int_0^h dz \int_0^z \rho dz.$$

The equation of continuity for the transport may be written in the form:

$$\frac{\partial}{\partial x} (\bar{\rho} S_x) + \frac{\partial}{\partial y} (\bar{\rho} S_y) = 0.$$

Since the variations of  $\bar{\rho}$  along  $x$  and  $y$  are very small in comparison with the corresponding variations of  $S_x$  and  $S_y$ , we may neglect the quantity  $\bar{\rho}$  in the last equation, and write

$$\text{div } S = \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = 0. \quad (2)$$

For the same reason we may, in differentiating equations (1) with respect to  $x$  and  $y$ , put  $\bar{\rho}$  approximately equal to unity. Eliminating the quantities  $\partial Q / \partial x$  and  $\partial Q / \partial y$  from equations (1), we have by (2):---

$$\nabla^2 \text{rot } S = -\text{rot } T / A_1, \quad (3)$$

where

$$\nabla^2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2, \quad \text{rot } T \equiv \partial T_y / \partial x - \partial T_x / \partial y, \quad \text{rot } S \equiv \partial S_y / \partial x - \partial S_x / \partial y.$$

If we introduce the transport function  $\psi$ , related to  $x$ - and  $y$ -components of the transport by the expressions:---

$$S_x = -\partial\psi/\partial y, \quad S_y = \partial\psi/\partial x, \quad (4)$$

then (3) is transformed into the equation

$$\nabla^4\psi = -\text{rot } T/A_l, \quad (5)$$

analogous to the equilibrium-equations of a plate under the action of applied loads (to which the quantity  $\text{rot } T$  is analogous.) In equation (5),  $\nabla^4$  represents the biharmonic operator. The boundary conditions for the transport on the contour  $\Gamma$  of the ocean shore are analogous to those for a plate held by pinching it edge-on:---

$$(\psi)_\Gamma = 0, \quad \left(\frac{\partial\psi}{\partial n}\right)_\Gamma = 0, \quad (6)$$

where  $n$  is the normal to the contour. Conditions (6) are derived by considering that the tangential and normal components of the transport at the shore-line must reduce to zero. Let us now find the equation relating the mass-field to the wind-field. Finding  $S_x$  from the first equation of (1) and substituting it in the second equation, we solve the latter for  $S_y$ :

$$cS_y = g \frac{\partial Q}{\partial x} - T_x + \frac{A_l g}{c} \frac{\partial}{\partial y} \nabla^2 Q - \frac{A_l}{c} \nabla^2 T_y - \frac{A_l^2}{c} \nabla^4 S_y. \quad (7)$$

Similarly:

$$cS_x = -g \frac{\partial Q}{\partial y} + T_y + \frac{A_l g}{c} \frac{\partial}{\partial x} \nabla^2 Q - \frac{A_l}{c} \nabla^2 T_x - \frac{A_l^2}{c} \nabla^4 S_x. \quad (8)$$

Differentiating equation (7) with respect to  $y$ , equation (8) with respect to  $x$ , and combining the results, we get:

$$c \text{div } S = \text{rot } T + \frac{A_l g}{c} \nabla^4 Q - \frac{A_l}{c} \nabla^2 \text{div } T - \frac{A_l^2}{c} \nabla^4 \text{div } S$$

or, by condition (2):---

$$\nabla^4 Q = \frac{1}{g} \left( -\frac{c \operatorname{rot} T}{A_t} + \nabla^2 \operatorname{div} T \right). \quad (9)$$

Thus the distribution of masses, represented by  $Q$ , depends not only on the Coriolis parameter and  $\operatorname{rot} T$ , but also on the quantity  $\nabla^2 \operatorname{div} T = \operatorname{div} \operatorname{grad} \operatorname{div} T$ . The latter is usually very small in comparison with the first term of (9). However, it cannot be neglected in regions of marked convergence or divergence of air-currents (for instance the center of a cyclone or anti-cyclone).

If to describe the mass-field we  $\overline{[}$ choose to $\overline{]}$  use the quantity  $P$ , related to the dynamic height by the expression

$$P = \int_0^h D dz = \bar{D} h,$$

where the  $\overline{[}$ geopotential $\overline{]}$  height  $\bar{D}$  is measured in dynamic centimeters, then the last equation will be written in the form:

$$\nabla^4 \bar{D} = \frac{1}{10^3 h} \left( -\frac{c \operatorname{rot} T}{A_t} + \nabla^2 \operatorname{div} T \right). \quad (10)$$

The boundary conditions to which  $Q$  or  $\bar{D}$  are subject will be found from equations (7) and (8). If at a given point we take  $x$  as the direction tangential to the shore-line and  $y$  as the normal thereto, and if we bear in mind that at the shore-line  $S_x = S_y = 0$ , then we obtain the following conditions for  $Q$ :

$$g \frac{\partial Q}{\partial x} - T_x + \frac{A_t g}{c} \nabla^2 \frac{\partial Q}{\partial y} - \frac{A_t}{c} \nabla^2 T_y - \frac{A_t^2}{c} \nabla^4 S_y = 0, \quad (11)$$

$$-g \frac{\partial Q}{\partial y} + T_y + \frac{A_t g}{c} \nabla^2 \frac{\partial Q}{\partial x} - \frac{A_t}{c} \nabla^2 T_x - \frac{A_t^2}{c} \nabla^4 S_x = 0. \quad (12)$$

Recalling (4), we may write:

$$\nabla^4 S_y = \frac{\partial}{\partial x} \nabla^4 \psi, \quad \nabla^4 S_x = -\frac{\partial}{\partial y} \nabla^4 \psi.$$

Replacing  $\nabla^2 \psi$  by its expression in terms of  $\text{rot } T$  as given by (5), we obtain

$$\frac{A_l^2}{c} \nabla^4 S_y = -\frac{A_l}{c} \frac{\partial}{\partial x} \text{rot } T; \quad \frac{A_l^2}{c} \nabla^4 S_x = \frac{A_l}{c} \frac{\partial}{\partial y} \text{rot } T. \quad (13)$$

Substituting (13) in (11) and (12), we obtain equivalent boundary conditions in the following form:---

$$\frac{\partial Q}{\partial x} = \frac{T_x}{g} - \frac{A_l}{c} \nabla^2 \frac{\partial Q}{\partial y} + \frac{A_l}{cg} \nabla^2 T_y - \frac{A_l}{c} \frac{\partial}{\partial x} \text{rot } T, \quad (14)$$

$$\frac{\partial Q}{\partial y} = \frac{T_y}{g} + \frac{A_l}{c} \nabla^2 \frac{\partial Q}{\partial x} - \frac{A_l}{cg} \nabla^2 T_x - \frac{A_l}{cg} \frac{\partial}{\partial y} \text{rot } T. \quad (15)$$

Finally, let us substitute expression (15) for  $\frac{\partial Q}{\partial y}$  under the  $\nabla^2$  sign in (14):

$$\begin{aligned} \frac{\partial Q}{\partial x} = \frac{T_x}{g} - \frac{A_l}{c} \nabla^2 \left( \frac{T_y}{g} + \frac{A_l}{c} \nabla^2 \frac{\partial Q}{\partial x} - \frac{A_l}{cg} \nabla^2 T_x - \frac{A_l}{cg} \frac{\partial}{\partial y} \text{rot } T \right) + \\ + \frac{A_l}{cg} \nabla^2 T_y - \frac{A_l}{cg} \frac{\partial}{\partial x} \text{rot } T. \end{aligned}$$

Expanding the operators and reducing similar terms, we see that all terms except the first in the right-hand side of this equation cancel each other. Thus we obtain a very simple boundary condition on the shore-line contour:

$$\partial Q / \partial x = T_x / g \quad (16)$$

and in the same way, a second condition:---

$$\partial Q / \partial y = T_y / g. \quad (17)$$

If we use  $\bar{D}$ , the boundary conditions will obviously be written in the form:

$$\frac{\partial \bar{D}}{\partial x} = \frac{T_x}{10^3 h}; \quad \frac{\partial \bar{D}}{\partial y} = \frac{T_y}{10^3 h}. \quad (18)$$

Let us now establish the relation between the field of masses and the transport. Differentiating the first equation of (1) with respect to  $x$ , the second with respect to  $y$ , and combining the results ( $\bar{\rho} = 1$ ), we obtain

$$g\nabla^2 Q = c \operatorname{rot} S + \operatorname{div} T + A_1 \nabla^2 \operatorname{div} S$$

or, by condition (2):---

$$\nabla^2 Q = \frac{1}{g} (c \operatorname{rot} S + \operatorname{div} T). \quad (19)$$

From (19) it follows that the adaptation of the mass-field to the transport field takes place independently of the size of the coefficient of horizontal turbulent exchange  $A_1$ . At the equator, where  $c = 0$ , the adaptation of masses is regulated exclusively by the quantity  $\operatorname{div} T$ . We note that formulae (7) and (8) may be used for calculating the transport components from observed mass-field and wind-field data, if we substitute expression (13) in (7) and (8). In this way we obtain the formulae:---

$$S_y = \frac{g}{c} \frac{\partial Q}{\partial x} - \frac{T_x}{c} + \frac{A_1 g}{c^2} \frac{\partial}{\partial y} \nabla^2 Q - \frac{A_1}{c^2} \nabla^2 T_y + \frac{A_1}{c^2} \frac{\partial}{\partial x} \operatorname{rot} T. \quad (20)$$

$$S_x = - \frac{g}{c} \frac{\partial Q}{\partial y} + \frac{T_y}{c} + \frac{A_1 g}{c^2} \frac{\partial}{\partial x} \nabla^2 Q - \frac{A_1}{c^2} \nabla^2 T_x - \frac{A_1}{c^2} \frac{\partial}{\partial y} \operatorname{rot} T, \quad (21)$$

replacing the well-known formulae of Ekman {2} and Jakhelln {3}. An estimate of the order of magnitude of the terms in (20) and (21) will show that the first three terms on the right-hand side are often identical in amount.

Thus if we take "side-friction" and wind into account, it may introduce substantial corrections in Ekman's formulae. When there is no wind, or when its effect may be neglected, expressions (20) and (21) are a good deal simplified.

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