VOLUME II
FLYING QUALITIES PHASE

CHAPTER 3
DIFFERENTIAL EQUATIONS

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3.1 INTRODUCTION

This chapter reviews the mathematical tools and techniques required to solve differential equations. Study of these operations is a prerequisite for courses in aircraft flying qualities and linear control systems taught at the USAF Test Pilot School. Only analysis and solution techniques which have direct application for work at the School will be covered.

Many systems of interest can be represented (mathematically modeled) by linear differential equations. For example, the pitching motion of an aircraft in flight displays motion similar to a mass-spring-damper system as shown in Figure 3.1.

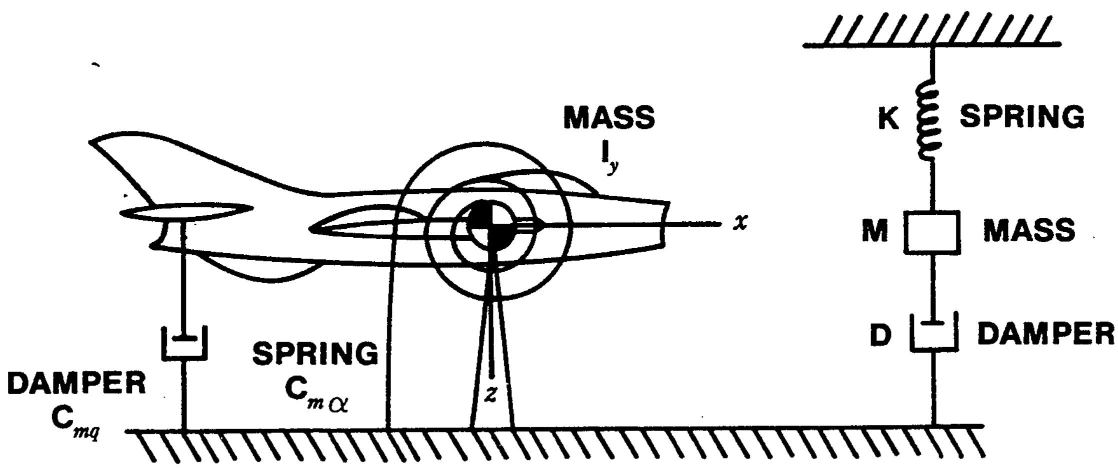

FIGURE 3.1. AIRCRAFT PITCHING MOTION

The static stability of the aircraft is similar to the spring, the moment of inertia about the y-axis is similar to the mass, and the airflow (aerodynamic forces) serves to damp the aircraft motion. Chapter 4 shows that stability derivatives can be used to represent the static stability and damping terms. These derivatives are C_{m_α} and C_{m_q}. In this chapter, M, K, and D will be used to represent mass, spring, and damper terms respectively.

The following terms will be used extensively:

Differential Equation: An equation relating two or more variables in terms of derivatives.

Independent Variables: Variables that are not dependent on other variables.

Dependent Variables: Variables that are dependent on other variables. In a differential equation, the dependent variables are the variables on the left-hand side of the equation that have their derivatives taken with respect to another variable. The other variable, usually time in our study, is the independent variable.

Solution. Any function without derivatives that satisfies a differential equation.

Ordinary Differential Equation. A differential equation with only one independent variable.

Partial Differential Equation. A differential equation with more than one independent variable.

Order. An n^{th} derivative is a derivative of order n. A differential equation has the order of its highest derivative.

Degree. The exponent of a differential term. The degree of differential equation is the exponent of its highest order derivative.

Linear Differential Equation. A differential equation in which the dependent variable and all its derivatives are only first degree, and the coefficients are either constants or functions of the independent variable.

Linear System. Any physical system that can be described which satisfies a differential equation of order n which contains n arbitrary constants.

General Solution. Any function without derivatives which satisfies a differential equation of order n which contains n arbitrary constants.

3.2 BASIC DIFFERENTIAL EQUATION SOLUTION

Unfortunately, there is no general method to solve all types of differential equations. The solving of a differential equation involves finding a mathematical expression without derivatives which satisfies the differential equation. It is usually much easier to determine whether or not a candidate solution to a differential equation is a solution than to determine a likely candidate. For example, given the linear first order differential equation

$$\frac{dy}{dx} - x = 4 \tag{3.1}$$

and a possible candidate solution

$$y = \frac{1}{2} x^2 + 4x + C \tag{3.2}$$

it is easy to differentiate Equation 3.2 and substitute into Equation 3.1 to see if Equation 3.2 is a solution of Equation 3.1. The derivative of Equation 3.2 is

$$\frac{dy}{dx} = x + 4 \tag{3.3}$$

Substituting Equation 3.3 into Equation 3.1,

$$(x + 4) - x = 4 \tag{3.4}$$

$$4 = 4$$

Therefore, Equation 3.2 is a solution of Equation 3.1.

It is interesting that, in general, solutions to linear differential equations are not linear functions. Note that Equation 3.2 is not an equation of the form

$$y = mx + b \tag{3.5}$$

which represents a straight line. As shown in Equation 3.2, y is a function of x and x^2.

There are several methods in use to solve differential equations. The methods to be discussed in this chapter are:

1. Direct Integration
2. Separation of Variables
3. Exact Differential Integration
4. Integrating Factor
5. Special Procedures, to include Operator Techniques and Laplace Transforms.

3.2.1 Direct Integration

Since a differential equation contains derivatives, it is sometimes possible to obtain a solution by anti-differentiation or integration. This process removes the derivatives and provides arbitrary constants in the solution. For example, given

$$\frac{dy}{dx} - x = 4 \tag{3.1}$$

rewriting

$$dy - xdx = 4dx \tag{3.6}$$

integrating

$$\int dy - \int xdx = \int 4dx + C$$

$$y - \frac{x^2}{2} = 4x + C \tag{3.7}$$

or, solving for y

$$y = \frac{x^2}{2} + 4x + C \tag{3.8}$$

where C is an arbitrary constant of integration.

Unfortunately, application of the direct integration process fails to work in many cases.

3.2.2 Separation of Variables

If direct integration fails for a first order differential equation, then the next step is to try to separate the variables. Direct integration may then be possible. When a differential equation can be put in the form

$$f_1(x)dx + f_2(y)dy = 0 \tag{3.9}$$

where one term contains function of x and dx only, and the other functions of y and dy only, the variables are said to be separated. A solution of Equation 3.9 can then be obtained by direct integration

$$\int f_1(x)dx + \int f_2(y)dy = C \tag{3.10}$$

where C is an arbitrary constant. Note, that for a differential equation of the first order there is one arbitrary constant. In general, the number of arbitrary constants is equal to the order of the differential equation.

EXAMPLE

$$\frac{dy}{dx} = \frac{x^2 + 3x + 4}{y + 6}$$

$$(y + 6)dy = (x^2 + 3x + 4)dx$$

$$\int (y + 6)dy = \int (x^2 + 3x + 4)dx + C$$

$$\frac{y^2}{2} + 6y = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + C$$

Not all first order equations can be separated in this fashion.

3.2.3 Exact Differential Integration

If direct integration, or direct integration after separation is not possible, then it still may be possible to obtain a solution if the differential equation is an exact differential. Associated with each suitably differentiable function of two variables f(x,y), there is an expression called its differential, namely

$$df \equiv \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (3.11)$$

that can be written as

$$df = M(x,y)dx + N(x,y)dy = 0 \qquad (3.12)$$

and is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (3.13)$$

If the differential equation is exact, then for all values of C

$$\int_a^x M(x,y)dx + \int_b^y N(x,y)dy = C \qquad (3.14)$$

is a solution of the equation, where a and b are dummy variables of integration.

EXAMPLE

Show that the equation

$$(2x + 3y - 2)dx + (3x - 4y + 1)dy = 0 \qquad (3.15)$$

is exact and find a general solution.

Applying the test in Equation 3.13

$$\frac{\partial M}{\partial y} = \frac{\partial(2x + 3y - 2)}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial(3x - 4y + 1)}{\partial x} = 3$$

Since the two partial derivatives are equal, the equation is exact. Its solution can be found by means of Equation 3.14.

$$\int_a^x (2x + 3y - 2)dx + \int_b^y (3x - 4y + 1)dy = C$$

The integration is performed assuming y is a constant while integrating the first term.

$$(x^2 + 3xy - 2x) \Big|_a^x + (3xy - 2y^2 + y) \Big|_b^y = C$$

$$(x^2 + 3xy - 2x) - (a^2 + 3ay - 2a) + (3xy - 2y^2 + y) - (3xb - 2b^2 + b) = C$$

$$x^2 + 6xy - 2x - 2y^2 + y + 3ay + 3xb = C + a^2 - 2a - 2b^2 + b = C_1 \qquad (3.16)$$

The same result can be obtained with less algebra and probably less chance of error by comparing Equation 3.15 with the differential form in Equation 3.11.

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \qquad (3.11)$$

$$(2x + 3y - 2)dx + (3x - 4y + 1)dy = 0 \qquad (3.15)$$

Comparing these two equations,

$$\frac{\partial f}{\partial x} = 2x + 3y - 2 = 0 \qquad (3.17)$$

and

$$\frac{\partial f}{\partial y} = 3x - 4y + 1 = 0 \qquad (3.18)$$

Since Equation 3.15 is an exact differential, then Equations 3.17 and 3.18 can be obtained by taking partial derivatives of the same function f. To find the unknown function f, first integrate Equations 3.17 and 3.18 assuming that y is constant when integrating with respect to x and that x is constant when integrating with respect to y.

$$f = x^2 + 3xy - 2x + f(y) + C = 0 \tag{3.19}$$

$$f = 3xy - 2y^2 + y + f(x) + C = 0 \tag{3.20}$$

Note that if Equation 3.17 had been obtained from Equation 3.19, any term that was a function of y only, f(y), and any constant term, C, would have disappeared. Similarly, obtaining Equation 3.18 from Equation 3.20, the f(x) and C terms would have vanished. By a direct comparison of Equation 3.19 and 3.20 the total function f can be determined.

$$f = x^2 + 3xy - 2x - 2y^2 + y + C = 0 \tag{3.21}$$

Note that the unknown f(y) term in Equation 3.19 is $(-2y^2 + y)$ and the unknown f(x) term in Equation 3.20 is $x^2 - 2x$. Redefining the constant of integration, Equation 3.21 can be written as

$$x^2 + 3xy - 2x - 2y^2 + y = C_1 \tag{3.16}$$

and was obtained earlier by integrating using dummy variables of integration.

3.2.4 Integrating Factor

When none of the above procedures or techniques work, it may still be possible to integrate a differential equation using an integrating factor. When some unintegrable differential equation is multiplied by some algebraic factor which permits it to be integrated term by term, then the algebraic factor is called an integrating factor. Determining integrating factors for arbitrary differential equations is beyond the scope of this course; however, two integrating factors will be introduced in later sections of this chapter when developing operator techniques and Laplace transforms. These two factors will be e^{mx} and e^{-st}.

3.3 FIRST ORDER EQUATIONS

The solution to a first order linear differential equation can be obtained by direct integration. Consider the form

$$\frac{dy}{dx} + R(x)y = 0 \tag{3.22}$$

where R(x) is a function of x only or a constant. To solve, separate variables

$$\frac{dy}{y} + R(x)dx = 0 \tag{3.23}$$

Integrating

$$\int \frac{dy}{y} = -\int R(x)dx + C' \tag{3.24}$$

where

$$C' = \ln C$$

Thus

$$\ln y = -\int R(x)dx + \ln C \tag{3.25}$$

or

$$y = Ce^{-\int R(x)dx} \tag{3.26}$$

If R(x) is a constant, R, then

$$y = Ce^{-Rx} \tag{3.27}$$

From this result, it can be concluded that a first order linear differential equation in the form of Equation 3.22 can be solved by simply expressing the solution in the form of Equation 3.27.

EXAMPLE

$$\frac{dy}{dx} + 2y = 0 \tag{3.28}$$

then the solution can be written directly as

$$y = Ce^{-2x} \tag{3.29}$$

EXAMPLE

$$\frac{dy}{dx} + x^3 y = 0 \tag{3.30}$$

is in the form

$$\frac{dy}{dx} + R(x)y = 0 \tag{3.22}$$

which has the solution

$$y = Ce^{-\int R(x)dx} \tag{3.26}$$

Therefore, the solution to Equation 3.30 can be obtained directly

$$y = Ce^{-\int x^3 dx}$$

$$y = Ce^{-\frac{1}{4}x^4}$$

3.4 LINEAR DIFFERENTIAL EQUATIONS AND OPERATOR TECHNIQUES

A form of differential equation that is of particular interest

$$A_n \frac{d^n y}{dx^n} + A_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + A_1 \frac{dy}{dx} + A_0 y = f(x) \tag{3.31}$$

If the coefficient expression A_n, A_{n-1}, . . . , A_0 are all functions of x only, then Equation 3.31 is called a linear differential equation. If the coefficient expressions A_n, . . . , A_0 are all constants, then Equation 3.31 is called a linear differential equation with constant coefficients.

EXAMPLE

$$x^2 \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + xy = \sin x$$

is a linear differential equation.

EXAMPLE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^x$$

is a linear differential equation with constant coefficients. Linear differential equations with constant coefficients occur frequently in the analysis of physical systems. Mathematicians and engineers have developed simple and effective techniques to solve this type of equation by using either "classical" or operational methods. When attempting to solve a linear differential equation of the form

$$A_n \frac{d^ny}{dx^n} + A_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + A_1 \frac{dy}{dx} + A_0 y = f(x), \qquad (3.32)$$

it is helpful to first examine the equation

$$A_n \frac{d^ny}{dx^n} + A_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ . \ . \ . + A_1 \frac{dy}{dx} + A_0 y = 0 \qquad (3.33)$$

Equation 3.33 is the same as Equation 3.32 with the right-hand side set equal to zero. Equation 3.32 is known as the general equation and Equation 3.33 as the complementary or homogeneous equation. Solutions of Equation 3.33 possess a useful property known as superposition, which may be briefly stated as follows: Suppose $y_1(x)$ and $y_2(x)$ are distinct solutions of Equation 3.33. Then any linear combination of $y_1(x)$ and $y_2(x)$ is also a solution of Equation 3.33. A linear combination would be $C_1y_1(x) + C_2y_2(x)$.

EXAMPLE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

It can be verified that $y_1(x) = e^{3x}$ is a solution, and that $y_2(x) = e^{2x}$ is another solution which is distinct from $y_1(x)$. Using superposition, then, $y(x) = c_1e^{3x} + c_2e^{2x}$ is also a solution.

Equation 3.32 may be interpreted as representing a physical system where the left side of the equation describes the natural or designed state of the system, and where the right side of the equation represents the input or forcing function.

The following line of reasoning is used to find a solution to Equation 3.32:

1. A general solution of Equation 3.32 must contain n arbitrary constants and must satisfy the equation.

2. The following statements are justified by experience:

   a. It is reasonably straightforward to find a solution to the complementary Equation 3.33, containing n arbitrary constants. Such a solution will be called the <u>transient solution</u>. Physically, it represents the response present in the system regardless of input.

   b. There are varied techniques for finding the solution of a differential equation due to a forcing function. Such solutions do not, in general, contain arbitrary constants. This solution will be called the <u>particular</u> or <u>steady state solution</u>.

3. If the transient solution which describes the response already existing in the system is added to the response due to the forcing function, it would appear that a solution so written would blend the two responses and describe the total response of the system represented by Equation 3.32. In fact, the definition of a general solution is satisfied under such an arrangement. This is simply an extension of the principle of superposition. The transient solution contains the correct number of arbitrary constants, and the particular solution guarantees that the combined solutions satisfy the general Equation 3.32. A general solution of Equation 3.32 is then given by

$$y = y_t + y_p \tag{3.34}$$

where y_t is the transient solution and y_p is the particular solution.

3.4.1 Transient Solution

Equation 3.28 is a complementary or homogeneous first order linear differential equation with constant coefficients. A quick and simple method of solving this equation was found. The solution was always of exponential form; hopefully, solutions of higher order equations of the same family take the same form.

$$\frac{dy}{dx} + 2y = 0 \tag{3.28}$$

Next, a second order differential equation with constant coefficients will be examined to determine if the candidate solution

$$y = e^{mx} \tag{3.35}$$

is a solution of the equation

$$ay'' + by' + cy = 0 \tag{3.36}$$

when the prime notation indicates derivatives with respect to x. That is,

$$y' = \frac{dy}{dx}, \; y'' = \frac{d^2y}{dx^2}$$

Substituting

$$y = e^{mx}$$

$$am^2e^{mx} + bme^{mx} + ce^{mx} = C \tag{3.37}$$

or

$$(am^2 + bm + c)e^{mx} = 0. \tag{3.38}$$

Since

$$e^{mx} \neq 0$$

$$am^2 + bm + c = 0 \tag{3.39}$$

and, using the quadratic formula

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.40}$$

Substituting these values into the assumed candidate solution, it is a solution when m_1 and m_2 are defined by Equation 3.40.

$$y_t = C_1 e^{m_1 x} + C_2 e^{m_2 x} \tag{3.41}$$

Equation 3.41 represents a transient solution since there is no forcing function in Equation 3.36. When working numerical problems, it is not necessary to take the derivatives of e^{mx}. This will be true for any order differential equation with constant coefficients. From the foregoing, it is seen that the method for first order complementary equations has been extended to higher order complementary or homogeneous equations. Again an integration problem has been traded for an algebra problem (solving Equation 3.39 for m's).

There are four possibilities for m_1 and m_2, and each is discussed below.

3.4.1.1 Case 1: Roots Real and Unequal. If m_1 and m_2 are real and unequal, the desired form of solution is just as given by Equation 3.41.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = 0,$$

rewriting in operator form where

$$my = \frac{dy}{dx}$$

and

$$m^2y = \frac{d^2y}{dx^2}$$

$$(m^2 + 4m - 12)y = 0.$$

Solving for the values of m,

$$m^2 + 4m - 12 = 0$$

gives

$$m = \frac{-4 \pm \sqrt{16 + 48}}{2} = \frac{-4 \pm 8}{2} = -6,2$$

and the required transient solution is

$$y_t = c_1 e^{-6x} + c_2 e^{2x}.$$

3.4.1.2. Case 2: Roots Real and Equal. If m_1 and m_2 are real and equal, an alternate form of solution is required.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0, \tag{3.42}$$

rewriting in operator form

$$(m^2 - 4m + 4)y = 0.$$

Solving for the values of m,

$$m = \frac{4 \pm \sqrt{16 - 16}}{2} = \frac{4}{2} = 2$$

or m = 2. But this gives only one value of m, and two values of m are required to result in a solution of the form of Equation 3.41 which has two arbitrary constants. The operator expression

$$m^2 - 4m + 4 = 0$$

can also be written

$$(m - 2)^2 = 0$$

or

$$(m - 2)(m - 2) = 0$$

now a repeated polynomial factor resulting in two (repeated) roots

$$m = 2,2.$$

Writing the solution in the form of Equation 3.41 when the roots are repeated does not give a solution because the two arbitrary constants can be combined into a single arbitrary constant as shown below.

$$y_t = c_1 e^{2x} + c_2 e^{2x} = (c_1 + c_2)e^{2x} = c_3 e^{2x}$$

To solve this problem one of the arbitrary constants is multiplied by x. The solution now contains two arbitrary constants which cannot be combined, and it is easily verified that

$$y_t = c_1 e^{2x} + c_2 x e^{2x}$$

is a transient solution of Equation 3.42.

3.4.1.3 Case 3: Roots Purely Imaginary.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0,$$

rewriting in operator form

$$(m^2 + 1)y = 0.$$

Solving,

$$m = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm\sqrt{-1}$$

In most engineering work $\sqrt{-1}$ is given the symbol j. (In mathematical texts it is denoted by i.). Now,

$$m = \pm j$$

and the solution is written

$$y_t = c_1 e^{jx} + c_2 e^{-jx} \qquad (3.43)$$

This is a perfectly good solution from a mathematical standpoint, but Euler's identity can be used to put the solution in a more useable form.

$$e^{jx} = \cos x + j\sin x \qquad (3.44)$$

This equation can be restated in many ways geometrically and analytically, and can be verified by adding the series expansion of cos x to the series expansion of j sin x. Now Equation 3.43 may be expressed

$$y_t = c_1(\cos x + j\sin x) + c_2[\cos(-x) + j\sin(-x)]$$

$$y_t = (c_1 + c_2)\cos x + j(c_1 - c_2)\sin x \qquad (3.45)$$

or without loss of generality

$$y_t = c_3 \cos x + c_4 \sin x \qquad (3.46)$$

An equivalent expression to Equation 3.46 is

$$y_t = \sqrt{c_3^{\,2} + c_4^{\,2}} \left(\frac{c_3}{\sqrt{c_3^{\,2} + c_4^{\,2}}} \cos x + \frac{c_4}{\sqrt{c_3^{\,2} + c_4^{\,2}}} \sin x \right) \qquad (3.47)$$

If the arbitrary constants c_3 and c_4 are related as shown in Figure 3.2,

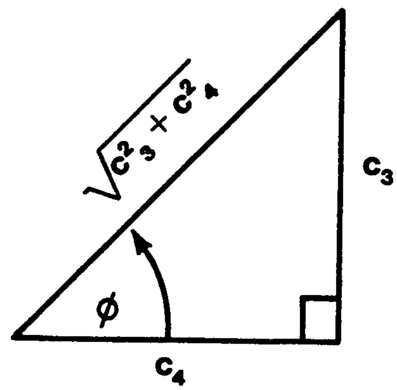

FIGURE 3.2. DEFINITION OF c_3 AND c_4

then

$$\frac{c_3}{\sqrt{c_3{}^2 + c_4{}^2}} = \sin \phi$$

$$\frac{c_4}{\sqrt{c_3{}^2 + c_4{}^2}} = \cos \phi$$

and

$$\sqrt{c_3{}^2 + c_4{}^2} = A$$

where A and ϕ are also arbitrary constants, Equation 3.47 becomes

$$y_t = A\ (\sin \phi \cos x + \cos \phi \sin x)$$

or using a common trigonometric identity

$$y_t = A \sin(x + \phi) \tag{3.48}$$

Note also that Equation 3.48 could be written in the equivalent form

$$y_t = A \cos(x - \theta) \tag{3.49}$$

where

$$\theta = 90° - \phi$$

To summarize, if the roots of the operator polynomial are purely imaginary, they will be numerically equal but opposite in sign, and the solution will have the form of Equation 3.46, 3.48, or 3.49.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

rewriting in operator form

$$(m^2 + 4)y = 0$$

which gives the roots

$$m = \pm 2j$$

Alternate solutions can immediately be written as

$$y_t = c_3 \cos 2x + c_4 \sin 2x$$

or

$$y_t = A \sin(2x + \phi)$$

where c_3, c_4, A, and ϕ are arbitrary constants.

3.4.1.4 Case 4: Roots Complex Conjugates.

EXAMPLE

Given the homogeneous equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

rewriting in operator form

$$(m^2 + 2m + 2)y = 0$$

Solving gives a complex pair of roots

$$m = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \sqrt{-1}$$

or

$$m = -1 + j, -1 - j$$

The solution can be written

$$y_t = c_1 e^{(-1 + j)x} + c_2 e^{(-1 - j)x}$$

Factoring out the exponential term gives

$$y_t = e^{-x}(c_1 e^{jx} + c_2 e^{-jx})$$

or, using the results from Equations 3.46 and 3.48, alternate solutions can immediately be written as

$$y_t = e^{-x}(c_3 \cos x + c_4 \sin x) \tag{3.50}$$

or

$$y_t = e^{-x} A \sin(x + \phi) \tag{3.51}$$

3.4.2 Particular Solution

The particular solution to a linear differential equation can be obtained by the method of undetermined coefficients. This method consists of assuming a solution of the same general form as the input (forcing function), but with undetermined constant coefficients. Substitution of this assumed solution into the differential equation enables the coefficients to be evaluated. The method of undetermined coefficients applies when the forcing function or input is a polynomial, or of the form

$$\sin ax, \cos ax, e^{ax}$$

or combinations of sums and products of these terms. The general solution to the differential equation with constant coefficients is then given by Equation 3.34,

$$y = y_t + y_p \tag{3.34}$$

which is the summation of the solution to the complementary equation (transient solution), plus the particular solution.

Consider the equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \qquad (3.52)$$

The particular solution which results from a given input, f(x), can be solved for using the method of undetermined coefficients. The method is best illustrated by considering examples.

3.4.2.1 Constant Forcing Functions.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 3y = 6 \qquad (3.53)$$

The input is a constant (trivial polynomial), so a solution of form $y_p = K$ is assumed.

Then

$$\frac{dy_p}{dx} = \frac{dK}{dx} = 0$$

and

$$\frac{d^2y_p}{dx^2} = \frac{d^2K}{dx^2} = 0$$

Substituting into Equation 3.53,

$$0 + 4(0) + 3K = 6$$

$$y_p = K = 2$$

Therefore, $y_p = 2$ is a particular solution. The homogeneous equation can be solved using operator form

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0 \tag{3.54}$$

$$(m^2 + 4m + 3)y = 0$$

or

$$m = -1, -3$$

and the transient solution can be written as

$$y_t = c_1 e^{-x} + c_2 e^{-3x} \tag{3.55}$$

The general solution of Equation 3.53 is

$$y = \underbrace{c_1 e^{-x} + c_2 e^{-3x}}_{\text{transient solution}} + \underbrace{2}_{\text{particular (or steady state) solution}} \tag{3.56}$$

3.4.2.2 Polynomial Forcing Function.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x^2 + 2x \tag{3.57}$$

The form of f(x) for Equation 3.57 is a polynomial of second degree, so a particular solution for y_p of second degree is assumed:

$$y_p = Ax^2 + Bx + C$$

Then

$$\frac{dy_p}{dx} = 2Ax + B$$

and

$$\frac{d^2y_p}{dx^2} = 2A$$

Substituting into Equation 3.57,

$$(2A) + 4(2Ax + B) + 3(Ax^2 + Bx + C) = x^2 + 2x$$

or

$$(3A)x^2 + (8A + 3B)x + (2A + 4B + 3C) = x^2 + 2x$$

Equating like powers of x,

$$x^2: \quad 3A = 1$$

$$A = 1/3$$

$$x: \quad 8A + 3B = 2$$

$$3B = 2 - 8/3$$

$$B = -2/9$$

$$x^0: \quad 2A + 4B + 3C = 0$$

$$3C = 8/9 - 2/3$$

$$C = 2/27$$

Therefore,

$$y_p = 1/3\ x^2 - 2/9\ x + 2/27$$

The total general solution of Equation 3.57 is given by

$$y = c_1e^{-x} + c_2e^{-3x} + 1/3\ x^2 - 2/9\ x + 2/27 \tag{3.58}$$

since the transient solution is Equation 3.55. As a general rule, if the forcing function is a polynomial of degree n, assume a polynomial solution of degree n.

3.4.2.3 EXPONENTIAL FORCING FUNCTION.

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x} \tag{3.59}$$

The forcing function is e^{2x} so assume a solution of the form

$$y_p = Ae^{2x}$$

$$\frac{d}{dx}(Ae^{2x}) = 2Ae^{2x}$$

$$\frac{d^2}{dx^2}(Ae^{2x}) = 4Ae^{2x}$$

Substituting into Equation 3.59,

$$4Ae^{2x} + 4(2Ae^{2x}) + 3(Ae^{2x}) = e^{2x}$$

$$e^{2x}(4A + 8A + 3A) = e^{2x}$$

The coefficients on both sides of the equation must be the same. Therefore, $4A + 8A + 3A = 1$, or $15A = 1$, and $A = 1/15$. The particular solution of Equation 3.59 then is $y_p = 1/15\ e^{2x}$. The transient solution is still Equation 3.55. A final example will illustrate a pitfall sometimes encountered using this method.

3.4.2.4 Exponential Forcing Function (special case).

EXAMPLE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-x} \tag{3.60}$$

The forcing function is e^{-x}, so assume a solution of the form $y_p = Ae^{-x}$. Then

$$\frac{d}{dx}(Ae^{-x}) = -Ae^{-x}$$

and

$$\frac{d^2}{dx^2}(Ae^{-x}) = Ae^{-x}$$

Substituting into Equation 3.60,

$$Ae^{-x} + 4(-Ae^{-x}) + 3(Ae^{-x}) = e^{-x}$$

$$(A - 4A + 3A)e^{-x} = e^{-x}$$

$$(0)e^{-x} = e^{-x}$$

Obviously, this is an incorrect statement. To locate the difficulty, the procedure to solve differential equations will be reviewed.

To solve an equation of the form

$$(m + a)(m + b)y = e^{-x}$$

solve the homogeneous equation to get

$$(m + a)(m + b)y = 0$$

$$m = -a, -b$$

$$y_t = c_1e^{-ax} + c_2e^{-bx}$$

If $y_p = Ae^{-ax}$ is assumed for a particular solution, then

$$y = y_t + y_p = c_1 e^{-ax} + c_2 e^{-bx} + Ae^{-ax} = (c_1 + A)e^{-ax} + c_2 e^{-bx}$$

$$= c_3 e^{-ax} + c_2 e^{-bx}$$

$$= y_t$$

However, y_t is the solution only when the right side of the equation is zero, and will not solve the equation when there is a forcing function of the form given. Assuming a particular solution of the form

$$y_p = Axe^{-ax}$$

will lead to a solution, then

$$y = y_p + y_t = c_1 e^{-ax} + c_2 e^{-bx} + Axe^{-ax} = (c_1 + Ax)e^{-ax} + c_2 e^{-bx} \neq y_t$$

Similarly, the equation

$$(m + aj)(m - aj)y = \sin ax$$

has the transient solution

$$y_t = c_1 \sin ax + c_2 \cos ax$$

If $y_p = A \sin ax + B \cos ax$ is assumed for a particular solution, then

$$y = y_t + y_p = (c_1 + A)\sin ax + (c_2 + b)\cos ax$$

$$y = c_3 \sin ax + c_4 \cos ax = y_t$$

which, as in the previous example, does not provide a solution when there is a forcing function of the form given. But, assuming a solution of the form

$$y_p = Ax \sin ax + Bx \cos ax$$

does lead to a solution

$$y = (c_1 + Ax)\sin ax + (c_2 + Bx)\cos ax \neq y_t$$

Continuing with the solution of Equation 3.60, a valid solution can be found by assuming $y_p = Axe^{-x}$, then

$$\frac{d}{dx}(Axe^{-x}) = A(-xe^{-x} + e^{-x})$$

and

$$\frac{d^2}{dx^2}(Axe^{-x}) = A(xe^{-x} - 2e^{-x})$$

Substituting into Equation 3.60,

$$A(xe^{-x} - 2e^{-x}) + 4A(-xe^{-x} + e^{-x}) + 3(Axe^{-x}) = e^{-x}$$

$$(A - 4A + 3A)xe^{-x} + (-2A + 4A)e^{-x} = e^{-x}$$

$$(0)xe^{-x} + 2Ae^{-x} = e^{-x}$$

and

$$A = 1/2$$

Thus,

$$y_p = (1/2)xe^{-x}$$

is a particular solution of Equation 3.60, and the general solution is given by

$$y = c_1 e^{-x} + c_2 e^{-3x} + 1/2\ xe^{-x}$$

The key to successful application of the method of undetermined coefficients is to assume the proper form for a trial or candidate particular solution. Table 3.1 summarizes the results of this discussion. When f(x) in Table 3.1 consists of a sum of several terms, the appropriate choice for y_p is the sum of y_p expressions corresponding to these terms individually. Whenever a term in any of the y_p's listed in Table 3.1 duplicates a term already in the complementary function, <u>all</u> terms in that y_p must be multiplied by the lowest positive integral power of x sufficient to eliminate the duplication.

TABLE 3.1
CANDIDATE PARTICULAR SOLUTIONS

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

Forcing Function $f(x)$	Assumed Solution y_p
Constant: K_1	A
Polynomial: $K_1 x^n$	$A_0 x^n + A_1 x^{n-1} + \ldots + A_{n-1} x + A_n$
Sine: $K_1 \sin K_2 x$	$A \cos K_2 x + B \sin K_2 x$
Cosine: $K_1 \cos K_2 x$	
Exponential: $K_1 e^{K_2 x}$	$A e^{K_2 x}$

3.4.3 Solving For Constants of Integration

As discussed previously, the number of arbitrary constants in the solution of a linear differential equation is equal to the order of the equation. The constants of integration can be determined by initial or boundary conditions. That is, to solve for the constants the physical state (position, velocity, etc.) of the system must be known at some time. The number of initial or boundary conditions given must equal the number of constants to be solved for. Many times these conditions are given at time equal to zero, in which case they are called initial conditions. A system which has zero initial condition, i.e., initial position, velocity, and acceleration all equal to zero, is frequently called a quiescent system.

The arbitrary constants of the solution must be evaluated from the total general solution, that is, the transient plus the steady state solution. The method of evaluating the constants of integration will be illustrated with an example.

EXAMPLE

$$\ddot{x} + 4\dot{x} + 13x = 3 \tag{3.61}$$

where the dot notation indicates derivatives with respect to time, that is, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$. The initial conditions given are $x(0) = 5$, and $\dot{x}(0) = 8$. The transient solution is given by

$$m^2 + 4m + 13 = 0$$

$$m = -2 \pm \sqrt{4 - 13} = -2 \pm j3$$

$$x_t = e^{-2t}(A \cos 3t + B \sin 3t)$$

Assume the particular solution of the form

$$x_p = D$$

$$\dot{x} = \frac{dx_p}{dt} = 0$$

$$\ddot{x}_p = 0$$

Substituting into Equation 3.61, $D = 3/13$ for the total general solution

$$x(t) = e^{-2t}(A \cos 3t + B \sin 3t) + 3/13$$

To solve for A and B, the initial conditions specified above are used.

$$x(0) = 5 = A + 3/13$$

or

$$A = 62/13$$

Differentiating the total general solution,

$$\dot{x}(t) = e^{-2t}[(3B \cos 3t - 3A \sin 3t) - 2e^{-2t}(A \cos 3t + B \sin 3t)]$$

Substituting the second initial condition

$$\dot{x}(0) = 8 = 3B - 2A$$

$$B = \frac{76}{13}$$

Therefore, the complete solution to Equation 3.61 with the given initial conditions is

$$x(t) = e^{-2t}[(62/13) \cos 3t + (76/13) \sin 3t] + 3/13$$

First and second order differential equations have been discussed in some detail. It is of great importance to note that many higher order systems quite naturally decompose into first and second order systems. For example, the study of a third order equation (or system) may be conducted by examining a first and a second order system, a fourth order system analyzed by examining two second order systems, etc. All these cases are handled by solving the characteristic equation to get a transient solution and then obtaining the particular solution by any convenient method.

A few remarks are appropriate regarding the second order linear differential equation with constant coefficients. Although the equation is interesting in its own right, it is of particular value because it is a mathematical model for several problems of physical interest.

$$a \frac{dy^2}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad \text{mathematical model} \tag{3.62}$$

$$M \frac{d^2x}{dt^2} + D \frac{dx}{dt} + Kx = f(t) \quad \text{describes a mass spring damper system} \tag{3.63}$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad \text{describes a series LRC electrical circuit} \tag{3.64}$$

Equations 3.62, 3.63, and 3.64 are all the same mathematically, but are expressed in different notation. Different notations or symbols are employed to emphasize the physical parameters involved, or to force the solution to appear in a form that is easy to interpret. In fact, the similarity of these last two equations may suggest how one might design an electrical circuit to simulate the operation of a mechanical system.

3.5 APPLICATIONS AND STANDARD FORMS

Up to this point, differential equations in general and linear differential equations with constant coefficients have been considered. Methods for solving first and second order equations of the following type have been developed:

$$a \frac{dx}{dt} + bx = f(t) \tag{3.65}$$

$$a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t) \tag{3.66}$$

These two equations are mathematical models or forms. These same forms may be used to describe diverse physical systems. This section will concentrate on the transient response of the systems under investigation.

3.5.1 First Order Equation

EXAMPLE

$$4\dot{x} + x = 3 \tag{3.67}$$

Physically, x can represent distance or displacement, where t is used to represent time. The transient solution can be found from the homogeneous equation.

$$4\dot{x} + x = 0$$

$$(4m + 1)x = 0$$

$$4m + 1 = 0$$

$$m = -1/4$$

Thus

$$x_t = ce^{-t/4}$$

The particular solution is found by assuming

$$x_p = A$$

$$\frac{dx_p}{dt} = 0$$

Substitute

$$A = 3$$

or

$$x_p = 3$$

The total general solution is then

$$x = ce^{-t/4} + 3 \tag{3.68}$$

The first term on the right of Equation 3.68 represents the transient response of the physical system described by Equation 3.67, and the second term represents the steady state response if the transient decays. A term useful in describing the physical effect of a negative exponential term is time constant which is denoted by τ. The time constant is defined by

$$\tau = -\frac{1}{m}$$

Thus, Equation 3.68 could be rewritten as

$$x = ce^{-t/\tau} + 3 \tag{3.69}$$

where $\tau = 4$.

Note the following points:

1. The time constant is discussed only if m is negative. If m is positive, the exponent of e is positive, and the transient solution will not decay.

2. If m is negative, τ is positive.

3. τ is the negative reciprocal of m, so that small numerical values of m give large numerical values of t (and vice versa).

4. The value of τ is the time, in seconds, required for the displacement to decay to 1/e of its original displacement from equilibrium or steady value. To get a better understanding of this statement, examine Equation 3.69

$$x = ce^{-t/\tau} + 3 \tag{3.69}$$

and let $t = \tau$. Then

$$x = ce^{-1} + 3 = c\frac{1}{e} + 3 \tag{3.70}$$

Thus, when $t = \tau$, the exponential portion of the solution has decayed to 1/e of its original displacement as shown in Figure 3.3.

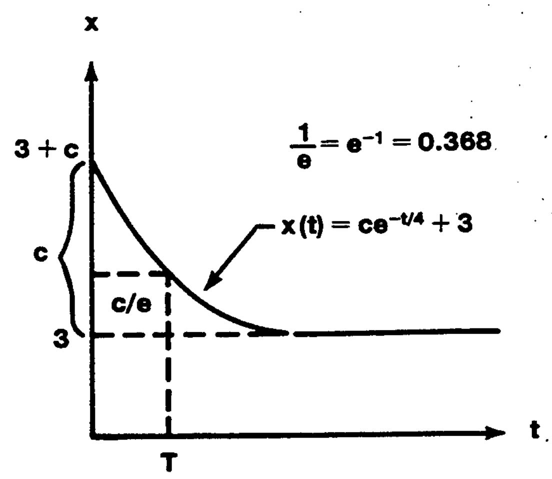

FIGURE 3.3. EXAMPLE OF FIRST ORDER EXPONENTIAL DECAY WITH AN ARBITRARY CONSTANT

Other measures of time are sometimes used to describe the decay of the exponential of a solution. If T_1 is used to denote the time it takes for the transient to decay to one-half its original amplitude, then

$$T_1 = 0.693\,\tau \tag{3.71}$$

This relationship can be easily shown by investigating

$$x = c_1 e^{-at} + c_2 \tag{3.72}$$

By definition, $\tau = 1/a$. T_1 is the value of t at which $x_t = 1/2\, x_t(0)$. Solving

$$x_t = c_1 e^{-at}$$

$$1/2\, x_t(0) = 1/2\, c_1 = c_1 e^{-aT_1}$$

$$e^{-aT_1} = 1/2$$

$$-\ln 1/2 = aT_1$$

$$T_1 = \frac{-\ln 1/2}{a} = \frac{0.693}{a} = 0.693\tau$$

The solution of Equation 3.67 can be completed by specifying a boundary condition and evaluating the arbitrary constant. Let $x = 0$ at $t = 0$.

$$x = ce^{-t/4} + 3$$

$$x(0) = 0 = c + 3$$

$$c = -3$$

The complete solution for this boundary condition is

$$x = -3e^{-t/4} + 3$$

as shown in Figure 3.4.

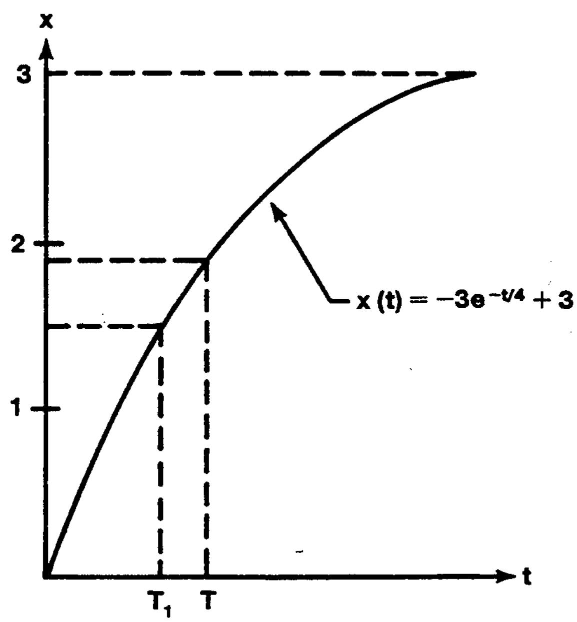

FIGURE 3.4. EXAMPLE OF FIRST ORDER EXPONENTIAL DECAY

3.5.2 Second Order Equations

Consider an equation of the form of Equation 3.66

$$a \frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t) \tag{3.66}$$

As discussed earlier, the characteristic equation can be written in operator notation as

$$am^2 + bm + c = 0 \tag{3.39}$$

where roots can be represented by

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.40}$$

These quadratic roots determine the form of the transient solution. The physical implications of solutions for various values of m will now be discussed.

3.5.2.1 Case 1: Roots Real and Unequal. When the roots are real and unequal, the transient solution has the form

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3.73}$$

When m_1 and m_2 are both negative, the system decays and there will be a time constant associated with each exponential as shown in Figure 3.5.

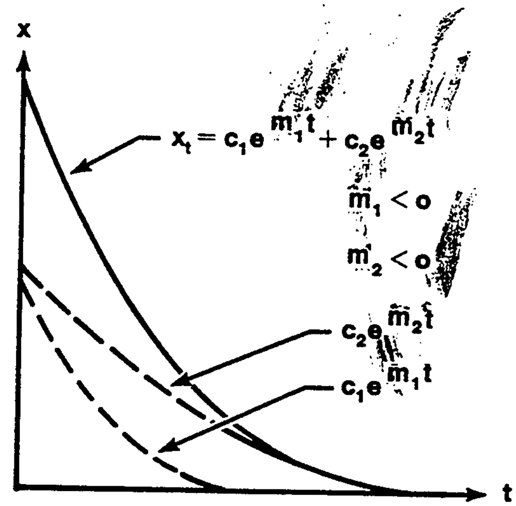

FIGURE 3.5. SECOND ORDER TRANSIENT RESPONSE WITH REAL, UNEQUAL, NEGATIVE ROOTS

When m_1 or m_2 (or both) is positive, the system will generally diverge as shown in Figures 3.6 and 3.7.

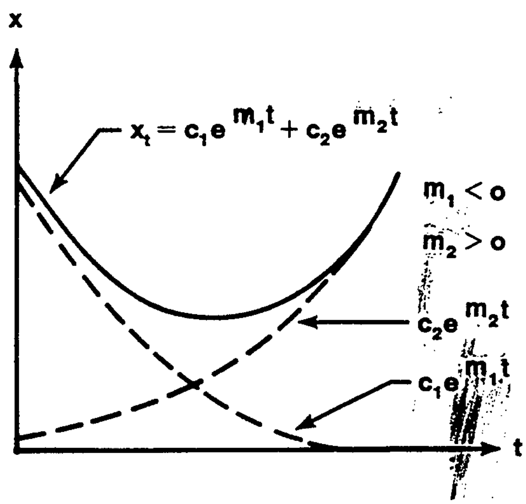

FIGURE 3.6. SECOND ORDER TRANSIENT RESPONSE WITH ONE POSITIVE AND ONE NEGATIVE REAL, UNEQUAL ROOTS

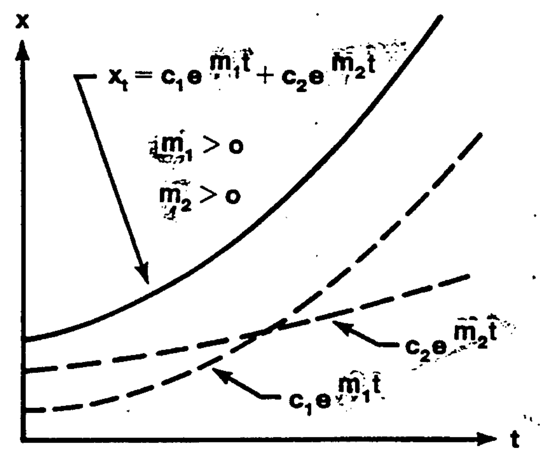

FIGURE 3.7. SECOND ORDER TRANSIENT RESPONSE WITH REAL, UNEQUAL, POSITIVE ROOTS

3.5.2.2 <u>Case 2: Roots Real and Equal</u>. When $m_1 = m_2$, the transient solution has the form

$$x_t = c_1 e^{mt} + c_2 te^{mt} \tag{3.74}$$

When m is negative, the system will usually decay as shown in Figure 3.8. If m is very small, the system may initially exhibit divergence.

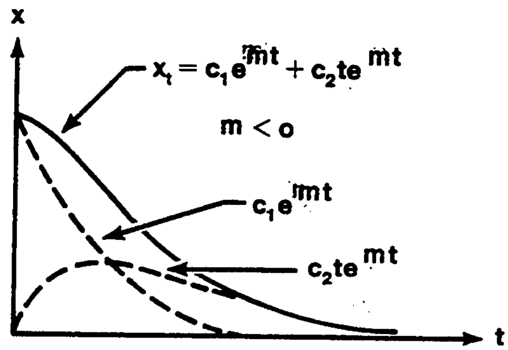

FIGURE 3.8. SECOND ORDER TRANSIENT RESPONSE WITH REAL, EQUAL, NEGATIVE ROOTS

When m is positive, the system will diverge much the same way as shown in Figure 3.7.

3.5.2.3 <u>Case 3: Roots Purely Imaginary</u>. When $m = \pm jk$, the transient solution has the form

$$x_t = c_1 \sin kt + c_2 \cos kt \tag{3.75}$$

or

$$x_t = A \sin(kt + \phi) \tag{3.76}$$

or

$$x_t = A \cos(kt + \theta) \tag{3.77}$$

The system executes oscillations of constant amplitude with a frequency k as shown in Figure 3.9.

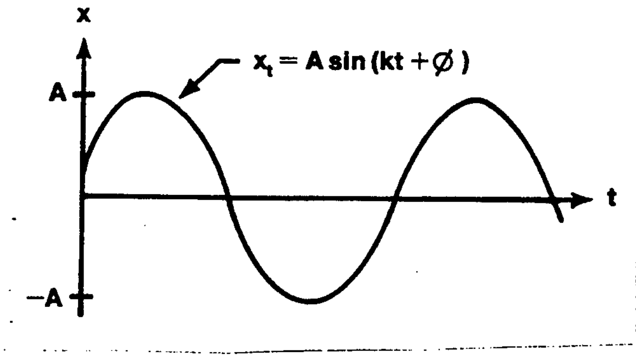

FIGURE 3.9. SECOND ORDER TRANSIENT RESPONSE WITH IMAGINARY ROOTS

3.5.2.4 <u>Case 4: Roots Complex Conjugates</u>. When the roots are given by $m = k_1 \pm jk_2$, the form of the transient solution is

$$x_t = e^{k_1 t}(c_1 \cos k_2 t + c_2 \sin k_2 t) \tag{3.78}$$

or

$$x_t = Ae^{k_1 t} \sin(k_2 t + \phi) \tag{3.79}$$

or

$$x_t = Ae^{k_1 t} \cos(k_2 t + \theta) \tag{3.80}$$

The system executes periodic oscillations contained in an envelope given by $x = \pm e^{k_1 t}$.

When k_1 is negative, the system decays or converges as shown in Figure 3.10. When k_1 is positive, the system diverges as shown in Figure 3.11.

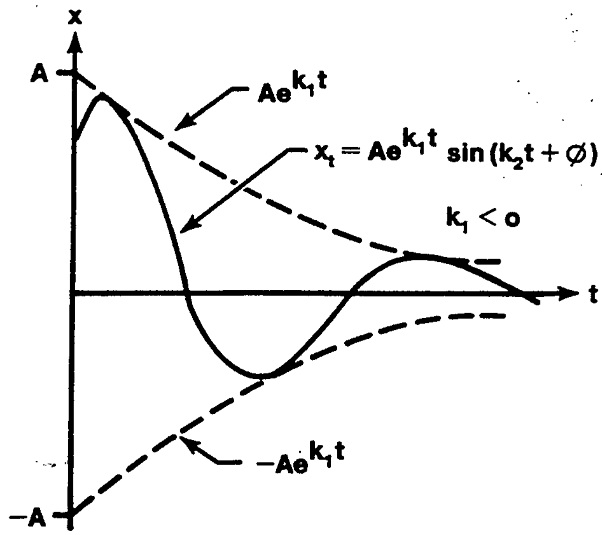

FIGURE 3.10. SECOND ORDER CONVERGENT TRANSIENT RESPONSE WITH COMPLEX CONJUGATE ROOTS

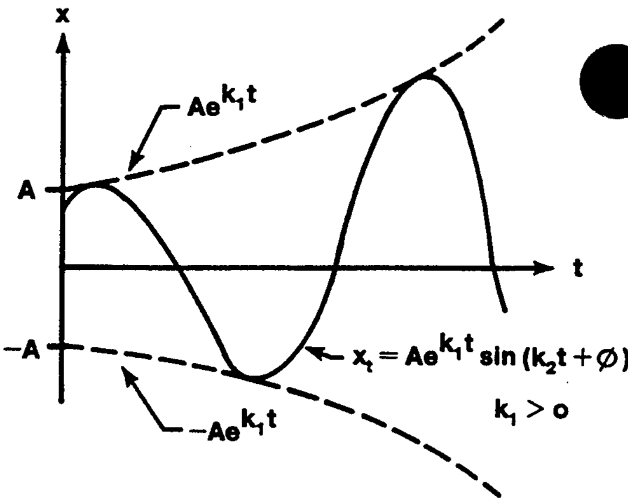

FIGURE 3.11. SECOND ORDER DIVERGENT TRANSIENT RESPONSE WITH COMPLEX CONJUGATE ROOTS

The discussion of transient solutions above reveals only part of the picture presented by Equation 3.66. The input or forcing function is still left to consider, that is, f(t). In practice, a linear system that possesses a divergence (without input) may be changed to a damped system by carefully selecting or controlling the input. Conversely, a nondivergent linear system with weak damping may be made divergent by certain types of inputs. Chapter 13, Linear Control Theory, will examine these problems in detail.

3.5.3 Second Order Linear Systems

Consider the physical model shown in Figure 3.12. The system consists of an object suspended by a spring, with a spring constant of K. The mass represented by M may move vertically and is subject to gravity, input, and damping, with the total viscous damping constant equal to D.

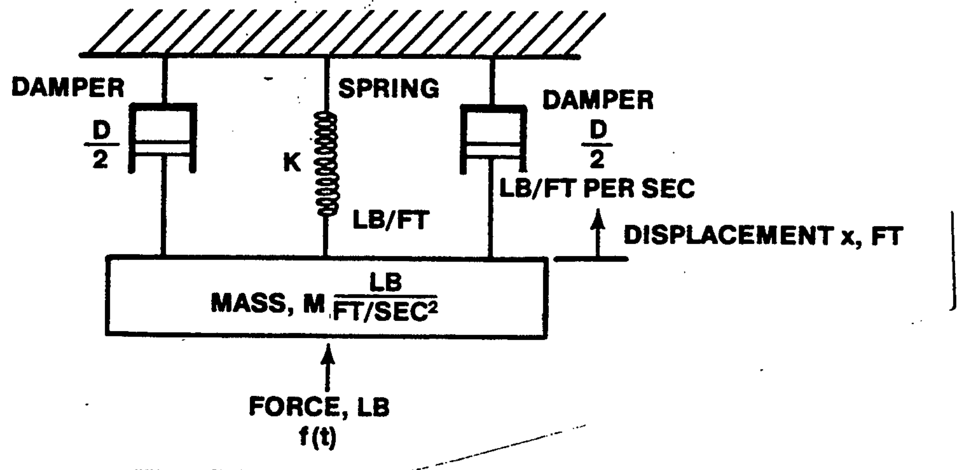

FIGURE 3.12. SECOND ORDER MASS, SPRING, DAMPER SYSTEM

The equation for this system is given by

$$M\ddot{x} + D\dot{x} + Kx = f(t) \tag{3.81}$$

The characteristic equation in operator notation is given by

$$Mm^2 + Dm + K = 0 \tag{3.82}$$

The roots of this equation can be written

$$m_{1,2} = \frac{-D}{2M} \pm \sqrt{\left[\frac{D}{2M}\right]^2 - \frac{K}{M}} \tag{3.83}$$

$$m_{1,2} = \frac{-D}{2M} \pm \sqrt{\frac{K}{M}}\,\sqrt{\frac{D^2}{4KM} - 1} \tag{3.84}$$

For simplicity, and for reasons that will be obvious later three constants are defined

$$\zeta \equiv \frac{D}{2\sqrt{MK}} \tag{3.85}$$

the term ζ is called the damping ratio, and is a value which indicates the damping strength in the system.

$$\omega_n \equiv \sqrt{\frac{K}{M}} \tag{3.86}$$

ω_n is the undamped natural frequency of the system. This is the frequency at which the system would oscillate if there were no damping present.

$$\omega_d \equiv \omega_n \sqrt{1 - \zeta^2} \tag{3.87}$$

ω_d is the damped frequency of the system. It is the frequency at which the system oscillates when a damping ratio of ζ is present.

Substituting the definitions of ζ and ω_n into Equation 3.84 gives

$$m_{1,2} = -\zeta\,\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \tag{3.88}$$

With these roots, the transient solution becomes

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3.89}$$

which can be written as

$$x_t = e^{-\zeta\omega_n t} \left(c_3 \cos \omega_n \sqrt{1 - \zeta^2}\, t + c_4 \sin \omega_n \sqrt{1 - \zeta^2}\, t\right) \tag{3.90}$$

or

$$x_t = A\, e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \phi\right) \tag{3.91}$$

The solution will lie within an exponentially decreasing envelope which has a time constant of $1/(\zeta\omega_n)$. This damped oscillation is shown in Figure 3.13. From Equation 3.91 and Figure 3.14, note that the numerical value of damping ratio has a powerful effect on system response.

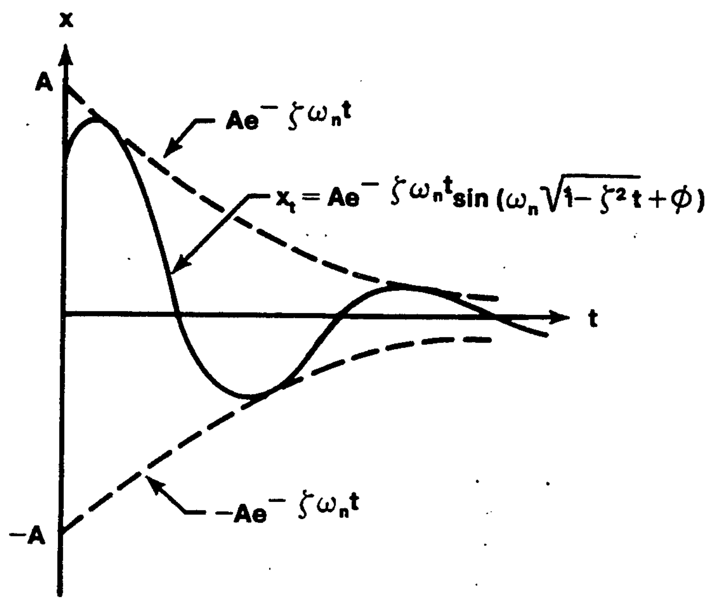

FIGURE 3.13. SECOND ORDER DAMPED OSCILLATIONS

If Equation 3.81 is divided by M

$$\ddot{x} + \frac{D}{M}\dot{x} + \frac{K}{M}x = \frac{f(t)}{M} \tag{3.92}$$

or, rewriting using ω_n and ζ defined by Equations 3.85 and 3.86

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \tag{3.93}$$

Equation 3.93 is a form of Equation 3.81 that is useful in analyzing the behavior of any second order linear system. In general, the magnitude and sign of damping ratio determine the response properties of the system. There are five distinct cases which are given names descriptive of the response associated with each case. These are:

1. $\zeta = 0$, undamped
2. $0 < \zeta > 1$, underdamped
3. $\zeta = 1$, critically damped
4. $\zeta > 1$, overdamped
5. $\zeta < 0$, unstable.

Each case will be examined in turn, making use of Equation 3.88, repeated below

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \tag{3.88}$$

3.5.3.1 Case 1: $\zeta = 0$, Undamped. For this condition, the roots of the characteristic equation are

$$m_{1,2} = \pm j\omega_n$$

giving a transient solution of the form

$$x_t = c_1 \cos \omega_n t + c_2 \sin \omega_n t \tag{3.94}$$

or

$$x_t = A \sin (\omega_n t + \phi) \tag{3.95}$$

showing the system to have the transient response of an undamped sinusoidal oscillation with frequency ω_n. Hence, the designation of ω_n as the "undamped natural frequency." Figure 3.9 shows an undamped system.

3.5.3.2 Case 2: $0 < \zeta < 1.0$, Underdamped. For this case, m is given by Equation 3.88.

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2} \tag{3.88}$$

The transient solution has the form

$$x_t = A e^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t + \phi) \qquad (3.96)$$

This solution shows that the system oscillates at the damped frequency, ω_d, and is bounded by an exponentially decreasing envelope with time constant $1/(\zeta\omega_n)$. Figure 3.14 shows the effect of increasing the damping ratio from 0.1 to 1.0.

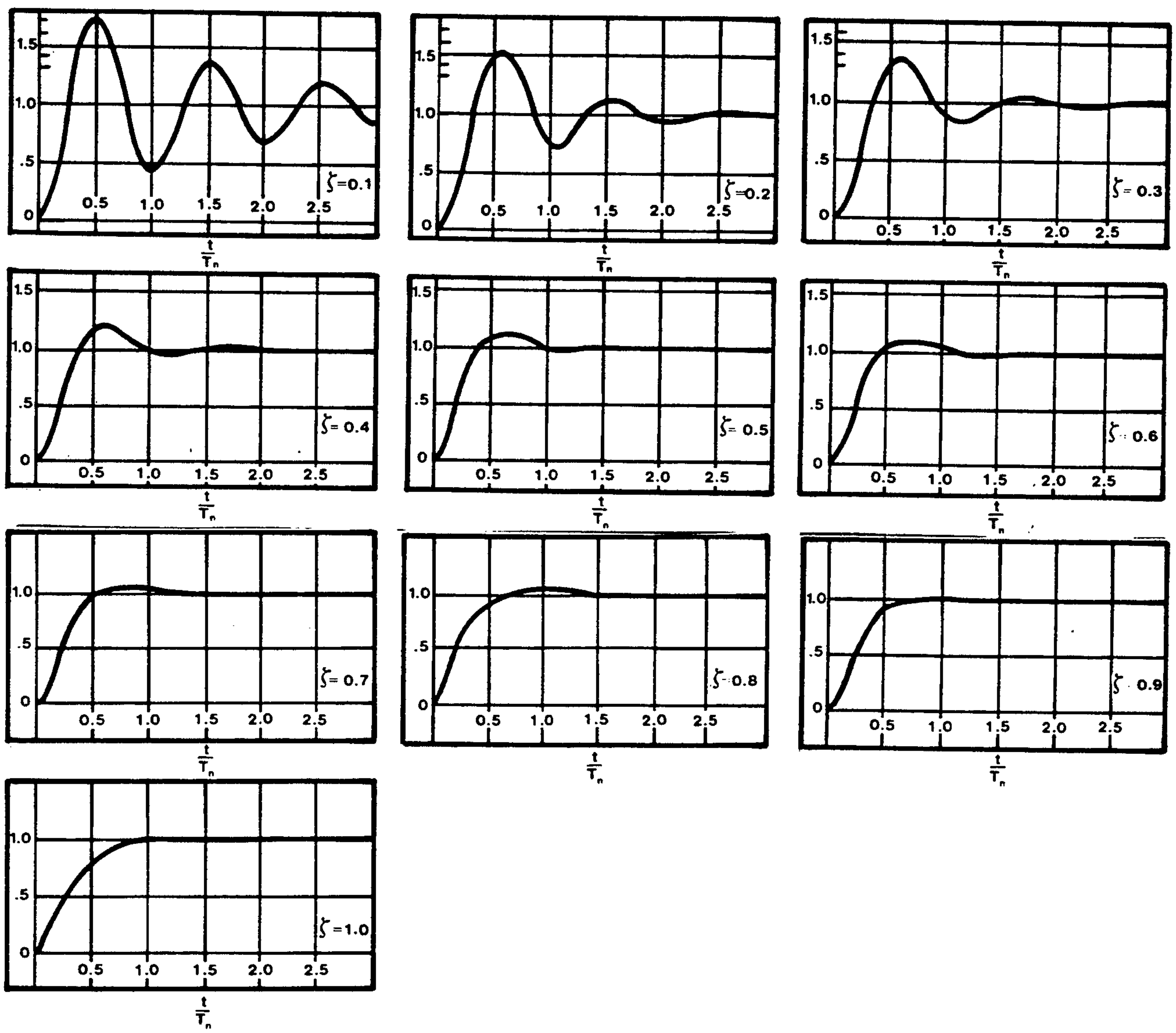

FIGURE 3.14. SECOND ORDER SYSTEM RESPONSE FOR DAMPING RATIOS BETWEEN ZERO AND ONE

3.5.3.3 Case 3: $\zeta = 1.0$, Critically Damped. For this condition, the roots of the characteristic equation are

$$m_{1,2} = -\omega_n \tag{3.97}$$

which gives a transient solution of the form

$$x_t = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \tag{3.98}$$

This is called the critically damped case and generally will not overshoot. It should be noted, however, that large initial values of x can cause one overshoot. Figure 3.14 shows a response when $\zeta = 1.0$.

3.5.3.4 Case 4: $\zeta > 1.0$, Overdamped. In this case, the characteristic roots are

$$m_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{3.99}$$

which shows that both roots are real and negative. The system will have a transient which has an exponential decay without sinusoidal motion. The transient response is given by

$$x_t = c_1 e^{-\omega_n(\zeta - \sqrt{\zeta^2 - 1})t} + c_2 e^{-\omega_n(\zeta + \sqrt{\zeta^2 - 1})t} \tag{3.100}$$

This response can also be written as

$$x_t = c_1 e^{-t/\tau_1} + c_2 e^{-t/\tau_2} \tag{3.101}$$

where τ_1 and τ_2 are time constants for each exponential term.

This solution is the sum of two decreasing exponentials, one with time constant τ_1 and the other with time constant τ_2. The smaller the value of τ, the quicker the transient decays. Usually the larger the value of ζ, the larger τ_1 is compared to τ_2. Figure 3.5 shows an overdamped system.

3.5.3.5 Case 5: $-1.0 < \zeta < 0$, Unstable. For the first Case 5 example, the roots of the characteristic equation are

$$m_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.102}$$

These roots are the same as for the underdamped case, except that the exponential term in the transient solution shows an exponential increase with time.

$$x_t = e^{-\zeta\omega_n t}(c_1 \cos \omega_n\sqrt{1-\zeta^2}\, t + c_2 \sin \omega_n\sqrt{1-\zeta^2}\, t) \tag{3.103}$$

Whenever a term appearing in the transient solution grows with time (and especially an exponential growth), the system is generally unstable. This means that whenever the system is disturbed from equilibrium the disturbance will increase with time. Figure 3.11 shows an unstable system

Case 5: $\zeta = -1.0$, Unstable. For this second Case 5 example, the roots of the characteristic equation are

$$m_{1,2} = +\omega_n \tag{3.104}$$

and

$$x_t = e^{\omega_n t}(c_1 + c_2 t) \tag{3.105}$$

This case diverges much the same way as shown in Figure 3.7.

$$m_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{3.99}$$

The response can be written as the sum of two exponential terms

$$x_t = c_1 e^{m_1 t} + t c_2 e^{m_2 t}$$

where the values of m can be determined from Equation 3.99.

<u>Case 5: $\zeta < -1.0$, Unstable</u>. This third Case 5 example is similar to Case 4, except that the system diverges as shown in Figure 3.7.

$$m_{1,2} = -\zeta\,\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \tag{3.99}$$

The response can be written as the sum of two exponential terms

$$x_t = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

where the values of m can be determined from Equation 3.99.

Five examples will illustrate some of these system response cases.

EXAMPLE

Given the homogeneous equation,

$$\ddot{x} + 4x = 0$$

from Equation 3.93,

$$\zeta = 0$$

and

$$\omega_n = 2.0$$

The system is undamped with a solution

$$x_t = A \sin(2t + \phi)$$

where A and ϕ are constants of integration which could be determined by substituting boundary conditions into the total general solution.

EXAMPLE

Given the homogeneous equation

$$\ddot{x} + \dot{x} + x = 0$$

from Equation 3.93,

$$\omega_n = 1.0$$

and

$$\zeta = 0.5$$

Also from Equation 3.87, the definition of damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.87$$

The system is underdamped with a solution

$$x_t = Ae^{-0.5t} \sin(0.87\ t + \phi)$$

EXAMPLE

Given the homogeneous equation

$$\frac{\ddot{x}}{4} + \dot{x} + x = 0$$

Multiply by four to get the equation in the form of Equation 3.93.

Then

$$\ddot{x} + 4\dot{x} + 4x = 0$$

and

$$\omega_n = 2.0$$

$$\zeta = 1.0$$

The system is critically damped and has a solution given by

$$x_t = c_1 e^{-2t} + c_2 t e^{-2t}$$

EXAMPLE

Given the homogeneous equation

$$\ddot{x} + 8\dot{x} + 4x = 0$$

from Equation 3.93,

$$\omega_n = 2.0$$

and

$$\zeta = 2.0$$

The system is overdamped and has a solution

$$x_t = c_1 e^{-7.46t} + c_2 e^{-0.54t}$$

EXAMPLE

Given the homogeneous equation

$$\ddot{x} - 2\dot{x} + 4x = 0$$

from Equation 3.93,

$$\omega_n = 2.0$$

and

$$\zeta = -0.5$$

From Equation 3.87, the definition of damped frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1.7$$

The solution is unstable (negative damping) and has the form

$$x_t = Ae^{t} \sin (1.7\ t + \phi)$$

In summary, the best damping ratio for a system is determined by the intended use of the system. If a fast response is desired and the size and number of overshoots is inconsequential, then a small value of damping ratio would be desired. If it is essential that the system not overshoot and response time is not too critical, a critically damped (or even an overdamped) system could be used. The value of damping ratio of 0.7 is often referred to as an optimum damping ratio since it gives a small overshoot and a relatively quick response. The optimum damping ratio will change as the requirements of the physical system change.

3.6 ANALOGOUS SECOND ORDER LINEAR SYSTEMS

3.6.1 Mechanical System

The second order equation which has been examined in detail represents the mass-spring-damper system of Figure 3.12 and has a differential equation which was given by

$$M\ddot{x} + D\dot{x} + Kx = f(t) \qquad (3.81)$$

Using the definitions

$$\zeta \equiv \frac{D}{2\sqrt{MK}} \quad (3.85) \quad \text{and} \quad \omega_n \equiv \sqrt{\frac{K}{M}} \qquad (3.86)$$

Equation 3.81 was rewritten as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^{\,2}\, x = \frac{f(t)}{M} \qquad (3.93)$$

3.6.2 Electrical System

The second order equation can also be applied to the series LRC circuit shown in Figure 3.15.

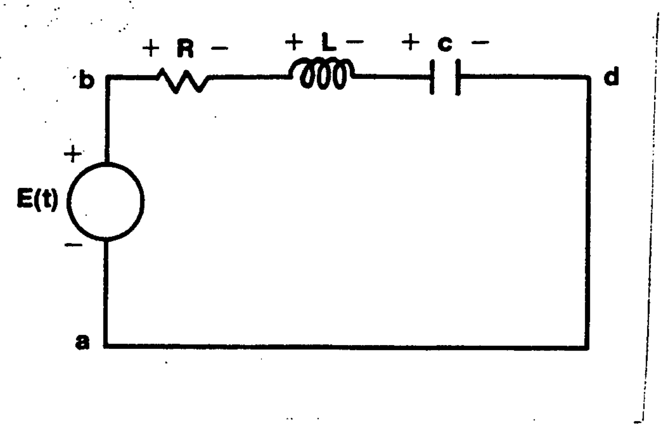

FIGURE 3.15. SERIES ELECTRICAL CIRCUIT

where

L = inductance

R = resistance

C = capacitance

q = charge

i = current

Assume $q(0) = \dot{q}(0) = 0$, then Kirchhoff's voltage law gives

$$\Sigma V_{abd} = 0$$

or

$$E(t) - V_R - V_L - V_C = 0$$

$$E(t) - iR - L\frac{di}{dt} - \frac{1}{C}\int_0^t i\,dt = 0$$

Since

$$i = \frac{dq}{dt}$$

$$E(t) = L\ddot{q} + R\dot{q} + \frac{q}{c} \tag{3.106}$$

The following parameters can now be defined

$$\omega_n \equiv \sqrt{\frac{1}{LC}} \tag{3.107}$$

$$\zeta \equiv \frac{R}{2\sqrt{L/C}} \tag{3.108}$$

and

$$2\zeta\omega_n = \frac{R}{L} \tag{3.109}$$

Using these parameters, Equation 3.106 can be written

$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = \frac{E(t)}{L} \tag{3.110}$$

3.6.3 Servomechanisms

For linear control systems work in Chapter 13, the applicable second order equation is

$$I\ddot{\theta}_0 + f\dot{\theta}_0 + \mu\theta_0 = \mu\theta_i \tag{3.111}$$

where

I = inertia

f = friction

μ = gain

θ_i = input

θ_0 = output

Rearranging Equation 3.111

$$\ddot{\theta}_0 + \frac{f}{I}\dot{\theta}_0 + \frac{\mu}{I}\theta_0 = \frac{\mu}{I}\theta_i \tag{3.112}$$

or

$$\ddot{\theta}_0 + 2\zeta\omega_n\dot{\theta}_0 + \omega_n{}^2\theta_0 = \omega_n{}^2\theta_i \tag{3.113}$$

where the following parameters are defined

$$\omega_n \equiv \sqrt{\frac{\mu}{I}} \tag{3.114}$$

$$\zeta \equiv \frac{f}{2\sqrt{\mu I}} \tag{3.115}$$

Thus, in general, any second order differential equation can be written in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2x = f_1(t) \tag{3.116}$$

where each term has the same qualitative significance, but different physical significance.

3.7 LAPLACE TRANSFORMS

A technique has been presented for solving linear differential equations with constant coefficients, with and without inputs or forcing functions. The method has limitations. It is suited for differential equations with inputs of only certain forms. Further, solution procedures require looking for special cases which require careful handling. However, these procedures have the remarkable property of changing or "transforming" a problem of integration into a problem in algebra, that is, solving a quadratic equation in the case of linear second order differential equations. This is accomplished by making an assumption involving the number e.

Given the second order homogeneous equation

$$a\ddot{x} + b\dot{x} + cx = 0 \tag{3.117}$$

The following solution is assumed

$$x_t = e^{mt} \tag{3.118}$$

Substituting into Equation 3.117 gives

$$am^2e^{mt} + bme^{mt} + ce^{mt} = 0 \tag{3.119}$$

and, factoring the exponential term

$$e^{mt}(am^2 + bm + c) = 0 \tag{3.120}$$

leading to the assertion that Equation 3.118 will produce a solution to Equation 3.117 if m is a root of the characteristic equation

$$am^2 + bm + c = 0 \tag{3.121}$$

Introducing operator notation, p = d/dt, the characteristic equation can be written by inspection.

$$ap^2 + bp + c = 0 \tag{3.122}$$

Equation 3.122 can then be solved for p to give a solution of the form

$$x_t = c_1e^{p_1t} + c_2e^{p_2t} \tag{3.123}$$

Of course, the great shortcoming of this method is that it does not provide a solution to an equation of the form

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.124}$$

It works only for the homogeneous equation. Still, a solution to the equation can be found by obtaining a particular solution and adding it to the transient solution of the homogeneous equation. The technique used to obtain the particular solution, the method of undetermined coefficients, also provides a solution by algebraic manipulation.

However, there is a technique which exchanges (transforms) the whole differential equation, including the input and initial conditions into an algebra problem. Fortunately, the method applies to linear first and second order equations with constant coefficients.

In Equation 3.124, x is a function of t. For emphasis, Equation 3.124 can be rewritten

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t) \tag{3.125}$$

Multiplying each term of Equation 3.125 by the integrating factor e^{mt} gives

$$a\ddot{x}(t)e^{mt} + b\dot{x}(t)e^{mt} + cx(t)e^{mt} = f(t)e^{mt} \tag{3.126}$$

It is now possible that Equation 3.126 can be integrated term by term on both sides of the equation to produce an algebraic expression in m. The algebraic expression can then be manipulated to eventually obtain the solution of Equation 3.125.

The new integrating factor e^{mt} should be distinguished from the previous integrating factor used in developing the operator techniques for solving the homogeneous equation. In order to accomplish this, m will be replaced by -s. The reason for the minus sign will be apparent later. In order to integrate the terms in Equation 3.126, limits of integration are required. In most physical problems, events of interest take place subsequent to a given starting time which is called t = 0. To be sure to include the duration of all significant events, the composite of effects from time t = 0 to time t = ∞ will be included. Equation 3.126 now becomes

$$\int_0^{\infty} a\ddot{x}(t)\, e^{-st}\, dt + \int_0^{\infty} b\, \dot{x}(t)\, e^{-st}\, dt + \int_0^{\infty} c\, x(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-st}\, dt \qquad (3.127)$$

Equation 3.127 is called the Laplace transform of Equation 3.125. The problem now is to integrate the terms in the equation.

3.7.1 Finding the Laplace Transform of a Differential Equation

The integrals of the terms of Equation 3.127 must now be found. The Laplace transform is defined as

$$\int_0^\infty x(t)\, e^{-st}\, dt \equiv L\{x(t)\} \equiv X(s) \qquad (3.128)$$

where the letter L is used to signify a Laplace transform. X(s) must, for the present, remain an unknown. (m was carried along as an unknown until the characteristic equation evolved, at which time m was solved for explicitly.) Since Equation 3.128 transforms x(t) into a function of the variable, s,

then

$$\int_0^\infty c\, x(t)\, e^{-st}\, dt = c \int_0^\infty x(t)\, e^{-st}\, dt = cX(s) \qquad (3.129)$$

and X(s) will be carried along until such time that it can be solved for.

The transform for the second term, bx(t) is given by

$$\int_0^\infty b\, \dot{x}(t)\, e^{-st}\, dt = b \int_0^\infty \dot{x}(t)\, e^{-st}\, dt \qquad (3.130)$$

To solve Equation 3.130, a useful formula known as integration by parts is used

$$\int_a^b u\,dv = uv \Big|_a^b - \int_a^b v\,du \qquad (3.131)$$

Applying this formula to Equation 3.130, let

$$u = e^{-st}$$

and

$$dv = \dot{x}(t)\,dt$$

then

$$du = -se^{-st}\,dt$$

and

$$v = x(t)$$

Substituting these values into Equation 3.131 and integrating from $t = 0$ to $t = \infty$

$$\int_0^\infty \dot{x}(t)e^{-st}\,dt = x(t)e^{-st} \Big|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt$$

$$= x(t)e^{-st} \Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\,dt$$

$$= x(t)e^{-st}\Big|_0^{\infty} + sX(s) \qquad (3.132)$$

Now

$$x(t)e^{-st}\Big|_0^{\infty} = \lim_{t\to\infty} x(t)e^{-st} - x(0) \qquad (3.133)$$

and assume that the term e^{-st} "dominates" the term $x(t)$ as $t \to \infty$. The reason for using the minus sign in the exponent should now be apparent. Thus, $\lim_{t\to\infty} x(t)e^{-st} = 0$, and Equation 3.131 becomes

$$\int_0^{\infty} \dot{x}(t)e^{-st}\,dt = 0 - x(0) + sX(s) = sX(s) - x(0) \qquad (3.134)$$

Equations 3.129 and 3.134 can now be abbreviated to signify Laplace transformations.

$$L\{x(t)\} = X(s) \qquad (3.135)$$

$$L\{cx(t)\} = cX(s) \qquad (3.136)$$

$$L\{\dot{x}(t)\} = sX(s) - x(0) \qquad (3.137)$$

$$L\{b\dot{x}(t)\} = b\Big(sX(s) - x(0)\Big) \qquad (3.138)$$

Equation 3.138 can be extended to higher order derivatives. Such an extension gives

$$L\{a\ddot{x}(t)\} = a\,[s^2X(s) - sx(0) - \dot{x}(0)] \qquad (3.139)$$

Returning to Equation 3.127, note that the Laplace transforms of all the terms except the forcing function have been found. To solve this transform, the forcing function must be specified. A few typical forcing functions will be considered to illustrate the technique for finding Laplace transforms.

EXAMPLE

$$f(t) = A = \text{constant}$$

Then

$$L\{A\} = \int_0^\infty Ae^{-st}\,dt = -\frac{A}{s}\int_0^\infty e^{-st}(-sdt) = -\frac{A}{s}\,e^{-st}\Big|_0^\infty$$

or

$$L\{A\} = \frac{A}{s} \tag{3.140}$$

EXAMPLE

$$f(t) = t$$

Then

$$L\{t\} = \int_0^\infty te^{-st}\,dt$$

To integrate by parts, let

$$u = t$$

$$dv = e^{-st}\,dt$$

Then

$$du = dt$$

$$v = -\frac{1}{s}\,e^{-st}$$

Substituting into Equation 3.131

$$\int_0^\infty te^{-st}\,dt = \frac{-te^{-st}}{s}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= 0 - \frac{1}{s^2}e^{-st}\Bigg|_0^\infty = 0 + \frac{1}{s^2}$$

or

$$L\{t\} = \frac{1}{s^2} \qquad (3.141)$$

EXAMPLE

$$f(t) = e^{2t}$$

Then

$$L\{e^{2t}\} = \int_0^\infty e^{2t}e^{-st}\,dt = \int_0^\infty e^{(2-s)t}dt = \frac{1}{s-2}$$

or

$$L\{e^{2t}\} = \frac{1}{s-2} \qquad (3.142)$$

EXAMPLE

$$f(t) = \sin at$$

Then

$$L\{\sin at\} = \int_0^\infty \sin at\; e^{-st}\,dt$$

Integrate by parts, letting

$$u = \sin at$$

$$dv = e^{-st}\,dt$$

Then

$$du = a\,(\cos at)dt$$

$$v = -\frac{1}{s}\,e^{-st}$$

Substituting into Equation 3.131

$$\int_0^\infty (\sin t)\,e^{-st} = \left.\frac{-(\sin at)(e^{-st})}{s}\right|_0^\infty + \frac{a}{s}\int_0^\infty (\cos at)\,e^{-st}\,dt$$

or

$$\int_0^\infty (\sin at)\,e^{-st}\,dt = 0 + \frac{a}{s}\int_0^\infty (\cos at)\,e^{-st}\,dt \qquad (3.143)$$

The expression $(\cos at)\,e^{-st}$ can also be integrated by parts, letting

$$u = \cos at$$

$$dv = e^{-st}\,dt$$

and

$$du = -a\,(\sin at)dt$$

$$v = \frac{1}{s}\,e^{-st}$$

Giving

$$\int_0^\infty (\cos at)\, e^{-st}\, dt = \frac{-(\cos at)(e^{-st})}{s}\Bigg|_0^\infty - \frac{a}{s}\int_0^\infty (\sin at)\, e^{-st}\, dt$$

or

$$\int_0^\infty (\cos at)\, e^{-st}\, dt = \frac{1}{s} - \frac{a}{s} L\{\sin at\} \qquad (3.144)$$

Substituting Equation 3.144 into Equation 3.143

$$L\{\sin at\} = 0 + \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s} L\{\sin at\}\right) = \frac{a}{s^2} - \frac{a^2}{s^2} L\{\sin at\}$$

which "obviously" yields

$$L\{\sin at\} = \frac{a}{s^2 + a^2} \qquad (3.145)$$

Also note that Equation 3.143 may be written as

$$L\{\sin at\} = \frac{a}{s} L\{\cos at\}$$

which yields

$$L\{\cos at\} = \frac{s}{s^2 + a^2} \qquad (3.146)$$

The Laplace transforms of more complicated functions may be quite tedious to derive, but the procedure is similar to that above. Fortunately, it is not necessary to derive Laplace transforms each time they are needed. Extensive tables of transforms exist in most advanced mathematics and control system textbooks. All of the transforms needed for this course are listed in Table 3.2 Page 3.73.

The technique of using Laplace transforms to assist in the solution of a differential equation is best described by an example.

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \tag{3.147}$$

with initial conditions $x(0) = 1$, $\dot{x}(0) = -4$. Taking the Laplace transform of the equation gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\,[sX(s) - x(0)] + 4X(s) = \frac{4}{s-2}$$

or

$$[s^2 + 4s + 4]\,X(s) + [-s + 4 - 4] = \frac{4}{s-2}$$

Solving for X(s)

$$X(s) = \frac{s^2 - 2s + 4}{(s-2)(s+2)^2} \tag{3.148}$$

In order to continue with the solution, it is necessary to discuss partial fraction expansions.

3.7.2 Partial Fractions

The method of partial fractions enables the separation of a complicated rational proper fraction into a sum of simpler fractions. If the fraction is not proper (the degree of the numerator less than the degree of the denominator), it can be made proper by dividing the fraction and considering the remaining expression. Given a fraction of two polynomials in the variable s as shown in Equation 3.148 there occur several cases:

3.7.2.1 Case 1: Distinct Linear Factors. To each linear factor such as $(as + b)$, occurring once in the denominator, there corresponds a single partial fraction of the form, $A/(as + b)$.

EXAMPLE

$$\frac{7s - 4}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2} \tag{3.149}$$

where A, B, and C are constants to be determined.

3.7.2.2 Case 2: Repeated Linear Factors. To each linear factor, $(as + b)$, occurring n times in the denominator there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{s^2 - 9s + 17}{(s - 2)^2(s + 1)} = \frac{A}{(s + 1)} + \frac{B}{(s - 2)} + \frac{C}{(s - 2)^2} \tag{3.150}$$

where A, B, and C are constants to be determined.

3.7.2.3 Case 3: Distinct Quadratic Factors. To each irreducible quadratic factor, $as^2 + bs + c$, occurring once in the denominator, there corresponds a single partial fraction of the form, $(As + B)/(as^2 + bs + c)$.

EXAMPLE

$$\frac{3s^2 + 5s + 8}{(s + 2)(s^2 + 1)} = \frac{A}{s + 2} + \frac{Bs + C}{s^2 + 1} \tag{3.151}$$

where A, B, and C are constants to be determined.

3.7.2.4 Case 4: Repeated Quadratic Factors. To each irreducible quadratic factor, $as^2 + bs + c$, occurring n times in the denominator, there corresponds a set of n partial fractions.

EXAMPLE

$$\frac{10\,s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A}{s - 4} + \frac{Bs + C}{s^2 + 4} + \frac{Ds + E}{(s^2 + 4)^2} \tag{3.152}$$

where A, B, C, D, and E are constants to be determined.

The "brute-force" technique for finding the constants will be illustrated by solving Equation 3.152. Start by finding the common denominator on the right side of Equation 3.152

$$\frac{10\ s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = \frac{A(s^2 + 4)^2 + (Bs + C)(s - 4)(s^2 + 4) + (Ds + E)(s - 4)}{(s - 4)(s^2 + 4)^2} \tag{3.153}$$

Then the numerators are set equal to each other

$$10s^2 + s + 36 = A(s^2 + 4)^2 + (Bs + C)(s^2 + 4)(s - 4) + (Ds + E)(s - 4) \tag{3.154}$$

Since Equation 3.154 must hold for all values of s, enough values of s are substituted into Equation 3.154 to find the five constants.

1. Let $s = 4$, then Equation 3.154 becomes

$$(10)(16) + 4 + 36 = 400A$$

and

$$A = 1/2$$

2. Let $s = 2j$, then Equation 3.154 becomes

$$-40 + 2j + 36 = -4D + 2jE - 8jD - 4E$$

$$-4 + 2j = -4(D + E) + 2j\ (E - 4D)$$

The real and imaginary parts must be equal to their counterparts on the opposite side of the equal sign, thus

$$(D + E) = 1$$

and

$$E - 4D = 1$$

or

$$D = 0$$

and

$$E = 1$$

3. Now let $s = 0$, then Equation 3.154 becomes

$$36 = 16A - 16\ (C) - 4E$$

and from steps 1 and 2

$$A = 1/2,\ E = 1$$

hence

$$36 = 8 - 16C - 4$$

and

$$C = -2$$

4. Let $s = 1$, then Equation 3.154 becomes

$$47 = 25\ (1/2) + (B - 2)\ (-15) - 3$$

$$94 = 25 - 30B + 60 - 6$$

or

$$B = -\ 1/2$$

Now Equation 3.155 may be written by substituting the values of A, B, C, D, and E into Equation 3.152

$$\frac{10\ s^2 + s + 36}{(s - 4)(s^2 + 4)^2} = 1/2\ \frac{1}{s - 4} - 1/2\ \frac{s + 4}{s^2 + 4} + \frac{1}{(s^2 + 4)^2} \qquad (3.155)$$

Returning now to the example Laplace solution of the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \tag{3.147}$$

The Laplace transformed equation was

$$X(s) = \frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} \tag{3.148}$$

which can now be expanded by partial fractions

$$\frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} = \frac{A}{s - 2} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2} \tag{3.156}$$

Taking the common denominator, and setting numerators equal

$$s^2 - 2s + 4 = A(s + 2)^2 + B(s + 2)(s - 2) + C(s - 2) \tag{3.157}$$

The "brute-force" technique could again be used to solve for the constants A, B, and C by substituting different values of s into Equation 3.157. An alternate method exists for solving for the constants. Multiplying the right side of Equation 3.157 gives

$$s^2 - 2s + 4 = As^2 + 4As + 4A + Bs^2 - 4B + Cs - 2C$$

$$s^2 - 2s + 4 = (A + B)s^2 + (4A + C)s + (4A - 4B - 2C)$$

Now the coefficients of like powers of s on both sides of the equation must be equal (that is, the coefficient of s^2 on the left side equals the coefficient of s^2 on the right side, etc.). Equating gives

$$s^2 : 1 = A + B$$

$$s^1 : -2 = 4A + C$$

$$s^0 : 4 = 4A - 4B - 2C$$

Solving for the constants gives

$$A = 1/4$$

$$B = 3/4$$

$$C = -3$$

Substituting the constants into Equation 3.156 results in the expanded right side

$$X(s) = 1/4 \frac{1}{s-2} + 3/4 \frac{1}{s+2} - 3 \frac{1}{s+2}^{2} \tag{3.158}$$

Another expansion method called the Heaviside Expansion Theorem can be used to solve for the constants in the numerator of distinct linear factors. This method of expansion is used extensively in Chapter 13, Linear Control Theory. If the denominator of an expansion term has a distinct linear factor, $(s - a)$, the constant for that factor can be found by multiplying $X(s)$ by $(s - a)$ and evaluating the remainder of $X(s)$ at $s = a$.

Stated mathematically the Heaviside Expansion Theorem is

$$X(s) = \frac{A}{s-a} + \ldots$$

$$A = (s-a)\,X(s)\Big|_{s=a}$$

EXAMPLE

$$X(s) = \frac{7s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$A = sX(s)\Big|_{s=0} = \frac{7s-4}{(s-1)(s+2)}\Big|_{s=0} = \frac{-4}{(-1)(2)} = 2$$

$$B = (s - 1)X(s)\Big|_{s=1} = \frac{7s - 4}{s(s + 2)}\Big|_{s=1} = \frac{7 - 4}{(1)(3)} = 1$$

$$C = (s + 2)X(s)\Big|_{s=-2} = \frac{7s - 4}{s(s - 1)}\Big|_{s=-2} = \frac{-14 - 4}{(-2)(-3)} = -3$$

As another example, the constant A in the first term on the right side of Equation 3.156 can be evaluated using the Heaviside Expansion Theorem.

$$\frac{s^2 - 2s + 4}{(s - 2)(s + 2)^2} = \frac{A}{s - 2} + \frac{B}{s + 2} + \frac{C}{(s + 2)^2} \qquad (3.156)$$

$$A = (s - 2)X(s)\Big|_{s=2} = \frac{s^2 - 2s + 4}{(s + 2)^2}\Big|_{s=2} = \frac{4}{16} = \frac{1}{4}$$

which is the same result obtained earlier by equating like powers of s.

3.7.3 Finding the Inverse Laplace Transform

Now that methods to expand the right side of X(s) have been discussed in detail, all that remains is to transform the expanded terms back to the time domain. This is easily accomplished using any suitable transform table.

Returning to the Laplace transformed and expanded equation in the example

$$X(s) = 1/4 \frac{1}{s - 2} + 3/4 \frac{1}{s + 2} - 3 \frac{1}{(s + 2)^2} \qquad (3.158)$$

Using Table 3.2, it can be easily verified that Equation 3.158 can be transformed to

$$x(t) = 1/4\, e^{2t} + 3/4\, e^{-2t} - 3t\, e^{-2t} \qquad (3.159)$$

In summary, the strength of the Laplace transform is that it converts linear differential equations with constant coefficients into algebraic equations in the s-domain. All that remains to do is to take the inverse transform of the explicit solutions to return to the time domain. Although the applications here at the School will consider time as the independent variable, a linear differential equation with any independent variable may be solved by Laplace transforms.

3.7.4 Laplace Transform Properties

There are several important properties of the Laplace transform which should be included in this discussion.

In the general case

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - [s^{n-1} x(0) + s^{n-2}\frac{dx(0)}{dt} + \ldots + \frac{d^{n-1}x(0)}{dt^{n-1}}] \quad (3.160)$$

For quiescent systems

$$L\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) \quad (3.161)$$

This result enables transfer functions to be written by inspection.

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x = 4e^{2t} \quad (3.162)$$

with quiescent initial conditions, the Laplace transform can immediately be written by inspection as

$$X(s)(s^2 + 4s + 4) = \frac{4}{s-2} \quad (3.163)$$

In most cases, reference to Table 3.2 will probably be needed to transform the right side forcing function (input).

Another significant transform is that of an indefinite integral. In the general case

$$L\left\{\iiint \ldots x(t)dt^n\right\} = \frac{X(s)}{s^n} + \frac{\int x(t)dt_{t=0}}{s^n} + \frac{\iint x(t)dt_{t=0}}{s^{n-1}} + \ldots \quad (3.164)$$

Equation 3.164 allows the transformation of integro-differential equations such as those arising in electrical engineering.

For the case where all integrals of f(t) evaluated at zero are zero (quiescent system) the transform becomes

$$L\left\{\iiint \ldots x(t)dt^n\right\} = \frac{X(s)}{s^n} \quad (3.165)$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 4\dot{x} + 4x + \int xdt = 4e^{2t} \quad (3.166)$$

TABLE 3.2.
LAPLACE TRANSFORMS

	X(s)	x(t)
1.	$aX(s)$	$ax(t)$
2.	$a[sX(s) - x(0)]$	$a\dot{x}(t)$
3.	$a[s^2X(s) - sx(0) - \dot{x}(0)]$ (which can be extended to any necessary order)	$a\ddot{x}(t)$
4.	$\frac{1}{s}$	1
5.	$\frac{1}{s^2}$	t
6.	$\frac{n!}{s^{n+1}}$ $(n = 1, 2, \ldots)$	t^n
7.	$\frac{1}{s + a}$	e^{-at}
8.	$\frac{1}{(s + a)^2}$	te^{-at}
9.	$\frac{n!}{(s + a)^{n+1}}$ $(n = 1, 2, \ldots)$	$t^n e^{-at}$
10.	$\frac{1}{(s + a)(s + b)}$ $a \neq b$	$\frac{1}{b - a}(e^{-at} - e^{-bt})$
11.	$\frac{s}{(s + a)(s + b)}$ $a \neq b$	$\frac{1}{a - b}(ae^{-at} - be^{-bt})$
12.	$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{(b-c)e^{-at} - (a-c)e^{-bt} + (a-b)e^{-ct}}{(a-b)(b-c)(a-c)}$

TABLE 3.2
LAPLACE TRANSFORMS (continued)

	X(s)	x(t)
13.	$\frac{a}{s^2 + a^2}$	sin at
14.	$\frac{s}{s^2 + a^2}$	cos at
15.	$\frac{a^2}{s(s^2 + a^2)}$	1 - cos at
16.	$\frac{a^3}{s^2 (s^2 + a^2)}$	at - sin at
17.	$\frac{2a^3}{(s^2 + a^2)^2}$	sin at - at cos at
18.	$\frac{2as}{(s^2 + a^2)^2}$	t sin at
19.	$\frac{2as^2}{(s^2 + a^2)^2}$	sin at + at cos at
20.	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$	t cos at
21.	$\frac{(b^2 - a^2)s}{(s^2 + a^2)(s^2 + b^2)}$ $(a^2 \neq b^2)$	cos at - cos bt
22.	$\frac{b}{(s + a)^2 + b^2}$	e^{-at} sin bt
23.	$\frac{s + a}{(s + a)^2 + b^2}$	e^{-at} cos bt

with quiescent initial conditions, the Laplace transform can immediately be written by inspection as

$$X(s)\ (s^2 + 4s + 4) + \frac{X(s)}{s} = \frac{4}{s - 2}$$

The right side transform is the same as Equation 3.163. Factoring results in

$$X(s)\ \left(s^2 + 4s + 4 + \frac{1}{s}\right) = \frac{4}{s - 2} \qquad (3.167)$$

Multiplying Equation 3.167 by s gives

$$X(s)\ (s^3 + 4s^2 + 4s + 1) = \frac{4s}{s - 2}$$

which raises the order of the left side and acts to differentiate the right side (input).

The usefulness of the Laplace transform technique will be demonstrated by solving several example problems.

EXAMPLE

Solve the given equation for x(t),

$$\dot{x} + 2x = 1 \qquad (3.168)$$

when $x(0) = 1$.

By Laplace transform of Equation 3.168

$$L\{\dot{x}\} = sX(s) - x(0)$$

$$L\{2x\} = 2X(s)$$

$$L\{1\} = \frac{1}{s}$$

Thus

$$(s + 2)\,X(s) = \frac{1}{s} + 1$$

$$X(s) = \frac{s + 1}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2}$$

Solving,

$$A = 1/2$$

and

$$B = 1 - 1/2 = 1/2$$

$$X(s) = \frac{1/2}{s} + \frac{1/2}{s + 2}$$

Inverse Laplace transforming gives

$$x(t) = 1/2 + 1/2\, e^{-2t} \tag{3.169}$$

EXAMPLE

Given the differential equation

$$\dot{x} + 2x = \sin t, \quad x(0) = 5 \tag{3.170}$$

solve for x(t).

Taking the Laplace transform of Equation 3.170

$$sX(s) - x(0) + 2X(s) = \frac{1}{s^2 + 1}$$

and

$$X(s) = \frac{1}{(s^2 + 1)(s + 2)} + \frac{5}{s + 2} \tag{3.171}$$

Expanding the first term on the right side of the equation gives

$$\frac{1}{(s^2 + 1)(s + 2)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2} \qquad (3.172)$$

Taking the common denominator and equating numerators gives

$$1 = (As + B)(s + 2) + C(s^2 + 1)$$

Substituting values of s leads to

$$A = -1/5$$

$$B = 2/5$$

$$C = 1/5$$

and substituting back into Equation 3.171 gives

$$X(s) = \frac{-1/5\ s}{s^2 + 1} + \frac{2/5}{s^2 + 1} + \frac{1/5}{s + 2} + \frac{5}{s + 2}$$

Inverse Laplace transforming gives the solution

$$X(t) = -1/5 \cos t + 2/5 \sin t + 5\ 1/5\ e^{-2t} \qquad (3.173)$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 5\dot{x} + 6x = 3e^{-3t},\ x(0) = \dot{x}(0) = 1 \qquad (3.174)$$

solve for x(t).
Taking the Laplace transform of Equation 3.174

$$s^2X(s) - sx(0) - \dot{x}(0) + 5sX(s) - 5x(0) + 6X(s) = \frac{3}{s + 3} \qquad (3.175)$$

or

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)\ (s^2 + 5s + 6)} \tag{3.176}$$

Factoring the denominator,

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)(s + 2)(s + 3)} \tag{3.177}$$

$$X(s) = \frac{s^2 + 9s + 21}{(s + 3)^2 (s + 2)} \tag{3.178}$$

$$X(s) = \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{C}{s + 2} \tag{3.179}$$

Finding the common denominator of Equation 3.179, and setting the resultant numerator equal to the numerator of Equation 3.178.

$$s^2 + 9s + 21 = A(s + 3)(s + 2) + B(s + 2) + C(s + 3)^2$$

which can be solved easily for

$$A = -6$$

$$B = -3$$

$$C = 7$$

Now X(s) is given by

$$X(s) = \frac{-6}{s + 3} - \frac{3}{(s + 3)^2} + \frac{7}{s + 2}$$

which can be inverse Laplace transformed to

$$x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t} \tag{3.180}$$

EXAMPLE

Given the differential equation

$$\ddot{x} + 2\dot{x} + 10x = 3t + 6/10 \tag{3.181}$$

$$x(0) = 3$$

$$\dot{x}(0) = -27/10$$

solve for x(t).

Laplace transforming Equation 3.181 and solving for X(s) gives

$$X(s) = \frac{3s^3 + 3.3s^2 + 0.6s + 3}{s^2\ (s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 10} \tag{3.182}$$

where

$$A = 0$$

$$B = 0.3$$

$$C = 3$$

$$D = 3$$

Thus,

$$X(s) = \frac{0.3}{s^2} + \frac{3s + 3}{s^2 + 2s + 10} \tag{3.183}$$

To make the inverse Laplace transform easier, Equation 3.183 is rewritten as

$$X(s) = \frac{0.3}{s^2} + 3\left[\frac{(s + 1)}{(s + 1)^2 + 3^2}\right] \tag{3.184}$$

which is readily inverse transformable to

$$x(t) = 0.3t + 3e^{-t}\cos 3t \tag{3.185}$$

3.8 TRANSFER FUNCTIONS

A transfer function is defined in Chapter 13, Linear Control Theory as, "The ratio of the output to the input expressed in operator or Laplace notation with zero initial conditions." The term "transfer function" can be thought of as what is done to the input to produce the output. A transfer function is essentially a mathematical model of a system and embodies all the physical characteristics of the system. A linear system can be completely described by its transfer function. Consider the following quiescent system.

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.186}$$

$$x(0) = \dot{x}(0) = 0$$

Taking the Laplace transform of Equation 3.186 results in

$$as^2X(s) + bsX(s) + cX(s) = F(s) \tag{3.187}$$

factoring gives

$$X(s)(as^2 + bs + c) = F(s) \tag{3.188}$$

or

$$\frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \tag{3.189}$$

Since Equation 3.186 represents a system whose input is f(t) and whose output is x(t) the following transforms can be defined

$$X(s) \equiv \text{output transform}$$

$$F(s) \equiv \text{input transform}$$

The transfer function can then be given the symbol TF and defined as

$$TF \equiv \frac{X(s)}{F(s)} \tag{3.190}$$

In the example represented by Equation 3.189

$$TF = \frac{1}{as^2 + bs + c} \tag{3.191}$$

Note that the denominator of the transfer function is algebraically the same as the characteristic equation appearing in the Equation 3.186. The characteristic equation completely defines the transient solution, and the total solution is only altered by the effect of the particular solution due to the input (or forcing function). Thus, from a physical standpoint, the transfer function completely characterizes a linear system.

The transfer function has several properties that will be used in control system analysis. Suppose that two systems are characterized by the differential equations

$$a\ddot{x} + b\dot{x} + cx = f(t) \tag{3.192}$$

and

$$d\ddot{y} + e\dot{y} + gy = x(t) \tag{3.193}$$

From the equations it can be seen that the first system has an input f(t), and an output x(t). The second system has an input x(t) and an output y(t). Note that the input to the second system is the output of the first system. Taking the Laplace transform of these two equations gives

$$(as^2 + bs + c)\,X(s) = F(s) \tag{3.194}$$

and

$$(ds^2 + es + g)\,Y(s) = X(s) \tag{3.195}$$

Finding the transfer functions,

$$TF_1 = \frac{X(s)}{F(s)} = \frac{1}{as^2 + bs + c} \tag{3.196}$$

$$TF_2 = \frac{Y(s)}{X(s)} = \frac{1}{ds^2 + es + g} \tag{3.197}$$

Now, both of these systems can be represented schematically by the block diagrams shown in Figure 3.16.

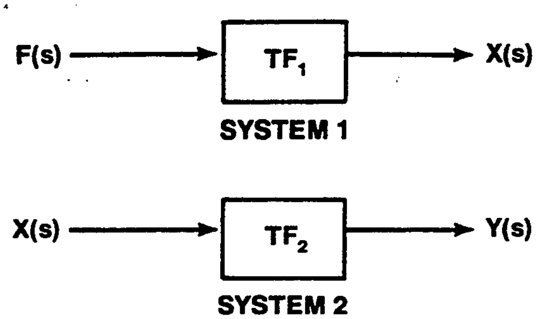

FIGURE 3.16. EXAMPLE BLOCK DIAGRAM NOTATION

If it is desired to find the output y(t) of System 2 due to the input f(t) of System 1, it is not necessary to find x(t) since the two systems can be linked using transfer functions as shown in Figure 3.17.

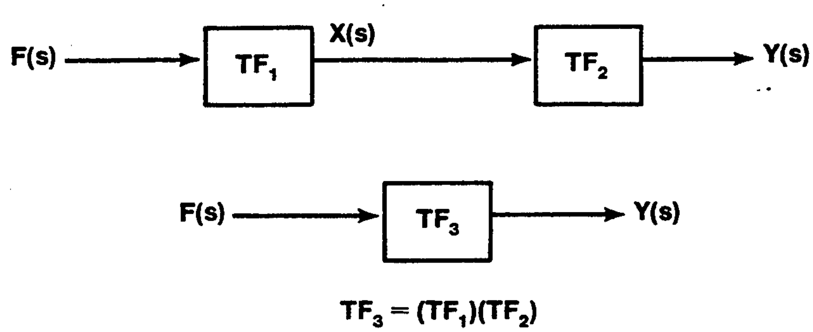

FIGURE 3.17. COMBINING TRANSFER FUNCTIONS

The solution y(t) is then given by the inverse transform of Y(s), or

$$Y(s) = [TF_3] \quad F(s) \tag{3.198}$$

or

$$Y(s) = [TF_1] [TF_2] \quad F(s) \tag{3.199}$$

This method of solution can be logically extended to include any desired number of systems.

3.9 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

In many physical problems the mathematical description of the system can most conveniently be written as simultaneous differential equations with constant coefficients. The basic procedure for solving a system of n ordinary differential equations in n dependent variables consists in obtaining a set of equations from which all but one of the dependent variables, say x, can be eliminated. The equation resulting from the elimination is then solved for the variable x. Each of the other dependent variables is then obtained in a similar manner.

A very effective means of handling simultaneous linear differential equations is to take the Laplace transform of the set of equations and reduce the problem to a set of algebraic equations that can be solved explicitly for the dependent variable in s. This method is demonstrated below.

Given the set of equations

$$3 \frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 3y = f(t) \tag{3.200}$$

$$2 \frac{d^2x}{dt^2} + x + \frac{d^2y}{dt^2} + 2y = g(t) \tag{3.201}$$

where $x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$, find x(t) and y(t). Taking the

Laplace transform of this system yields

$$(3s^2 + 1)\ X(s) + (s^2 + 3)\ Y(s) \quad = \quad F(s) \tag{3.202}$$

$$(2s^2 + 1)\ X(s) + (s^2 + 2)\ Y(s) \quad = \quad G(s) \tag{3.203}$$

Cramer's rule will now be used to solve this set of equations. Cramer's rule can be stated in its simplest form as, given the equations

$$P_1(s)\ X(s) + P_2(s)\ Y(s) \quad = \quad F_1(s) \tag{3.204}$$

$$Q_1(s)\ X(s) + Q_2(s)\ Y(s) \quad = \quad F_2(s) \tag{3.205}$$

then,

$$X(s) \quad = \quad \frac{\begin{vmatrix} F_1 & P_2 \\ F_2 & Q_2 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}} \tag{3.206}$$

for unknown X(s), and

$$Y(s) \quad = \quad \frac{\begin{vmatrix} P_1 & F_1 \\ Q_1 & F_2 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{vmatrix}} \tag{3.207}$$

for the unknown Y(s).

The X(s) unknown in Equations 3.202 and 3.203 can be solved for in this fashion by applying Cramer's rule

$$X(s) = \frac{\begin{vmatrix} F(s) & (s^2+3) \\ G(s) & (s^2+2) \end{vmatrix}}{\begin{vmatrix} (3s^2+1) & (s^2+3) \\ (2s^2+1) & (s^2+2) \end{vmatrix}} \tag{3.208}$$

In a similar manner,

$$Y(s) = \frac{\begin{vmatrix} (3s^2+1) & F(s) \\ (2s^2+1) & G(s) \end{vmatrix}}{\begin{vmatrix} (3s^2+1) & (s^2+3) \\ (2s^2+1) & (s^2+2) \end{vmatrix}} \tag{3.209}$$

For the particular inputs $f(t) = t$ and $g(t) = 1$,

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s^2} & (s^2+3) \\ \frac{1}{s} & (s^2+2) \end{vmatrix}}{(s^4-1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2\ (s^4-1)} \tag{3.210}$$

Expanded as a partial fraction

$$X(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{(s^2+1)} + \frac{E}{s-1} + \frac{F}{(s+1)} = \frac{-s^3 + s^2 - 3s + 2}{s^2\ (s^4-1)} \tag{3.211}$$

Solving for A, B, etc.,

$$X(s) = \frac{-2}{s^2} + \frac{3}{s} + \frac{1/2 - s}{s^2+1} - \frac{7/4}{s+1} - \frac{1/4}{s-1} \tag{3.212}$$

which yields a solution

$$x(t) = -2t + 3 - 7/4e^{-t} - 1/4e^{t} + 1/2 \sin t - \cos t \qquad (3.213)$$

A similar approach will obtain the solution for y(t).

In the case of three simultaneous differential equations, the application of Laplace transforms and use of Cramer's rule will yield the solution.

$$P_1(s)X(s) + P_2(s)Y(s) + P_3(s)Z(s) = F_1(s) \qquad (3.214)$$

$$Q_1(s)X(s) + Q_2(s)Y(s) + Q_3(s)Z(s) = F_2(s) \qquad (3.215)$$

$$R_1(s)X(s) + R_2(s)Y(s) + R_3(s)Z(s) = F_3(s) \qquad (3.216)$$

where

$$X(s) = \frac{\begin{vmatrix} F_1 & P_2 & P_3 \\ F_2 & Q_2 & Q_3 \\ F_3 & R_2 & R_3 \end{vmatrix}}{\begin{vmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{vmatrix}} \qquad (3.217)$$

Y(s) and Z(s) will have similar forms.

3.10 ROOT PLOTS

Some insight into the response of a second order system can be gained by examining the roots of the differential equation describing the system on a root plot. A root plot is a plot of the roots of the characteristic equation in the complex plane. Root plots are used in Chapter 8, Dynamics, to describe aircraft longitudinal and lateral directional modes of motion. These plots are also used extensively in Chapter 13, Linear Control Theory, for linear control system analysis.

It was shown earlier that a second order linear system can be put into the following form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{M} \tag{3.93}$$

whose roots can be written as

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

or

$$p_{1,2} = -\zeta\omega_n \pm j\omega_d \tag{3.218}$$

Figure 3.18 is a plot of the two roots of Equations 3.88 and 3.218 in the complex plane.

$$p_{1,2} = \underset{\uparrow\ \text{real}}{-\zeta\omega_n} \pm \underset{\uparrow\ \text{imaginary}}{j\omega_n\sqrt{1-\zeta^2}} \tag{3.88}$$

where the first term is plotted on the real axis and the second term plotted on the imaginary (j) axis.

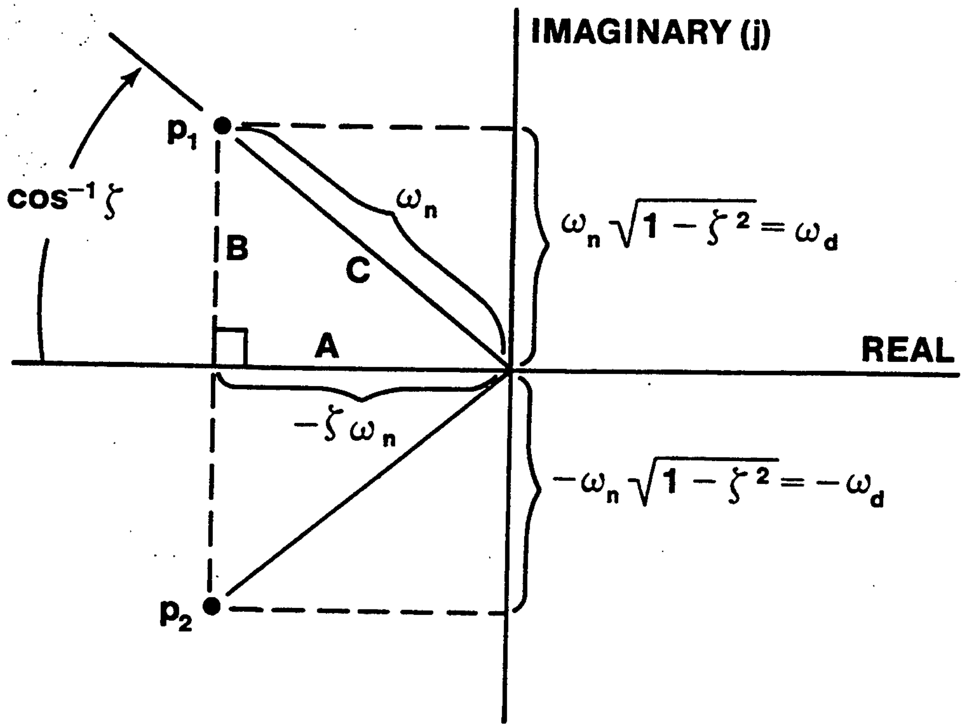

FIGURE 3.18. GENERAL ROOT PLOT IN THE COMPLEX PLANE

From the right triangle relationship shown in Figure 3.18, it can be easily shown that the length of the line from the origin to either point p_1 or p_2 is equal to ω_n.

$$A^2 + B^2 = C^2$$

$$(\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2 = C^2$$

$$\zeta^2\omega_n^{\ 2} + \omega_n^{\ 2}(1-\zeta^2) = C^2$$

$$\zeta^2\omega_n^{\ 2} + \omega_n^{\ 2} - \zeta^2\omega_n^2 = C^2$$

$$\omega_n^{\ 2} = C^2$$

$$C = \omega_n$$

The five distinct damping cases previously discussed can be examined on root plots through the use of Equation 3.88.

1. $\zeta = 0$, undamped (Figure 3.19)

$$p_{1,2} = -\zeta\,\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad (3.88)$$

$$p_{1,2} = 0 + j\omega_n$$

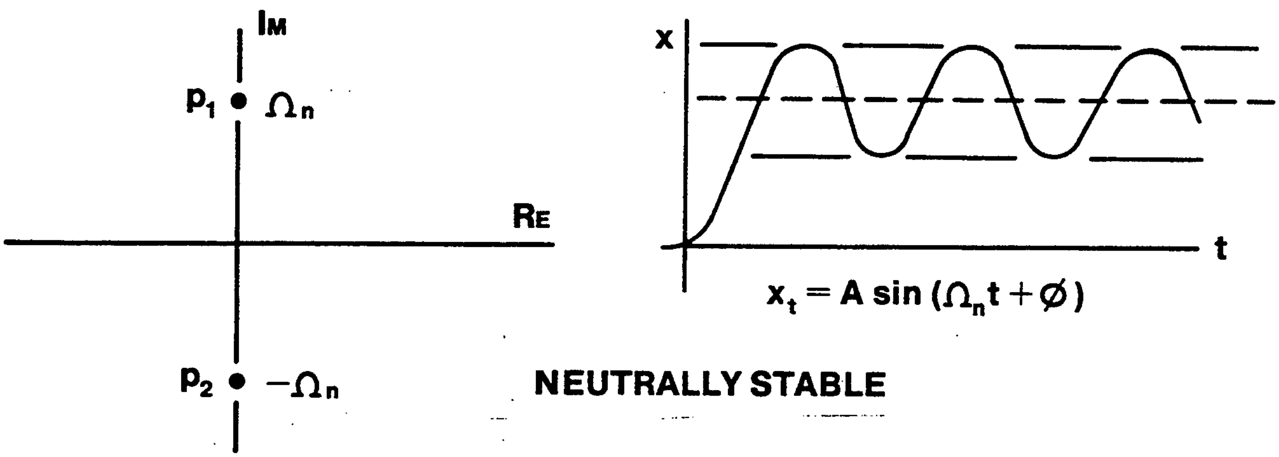

FIGURE 3.19. NEUTRALLY STABLE UNDAMPED RESPONSE

2. $0 < \zeta < 1.0$, Underdamped (Figure 3.20)

$$p_{1,2} = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \qquad (3.88)$$

$$p_{1,2} = (-) \pm j\,(+)$$

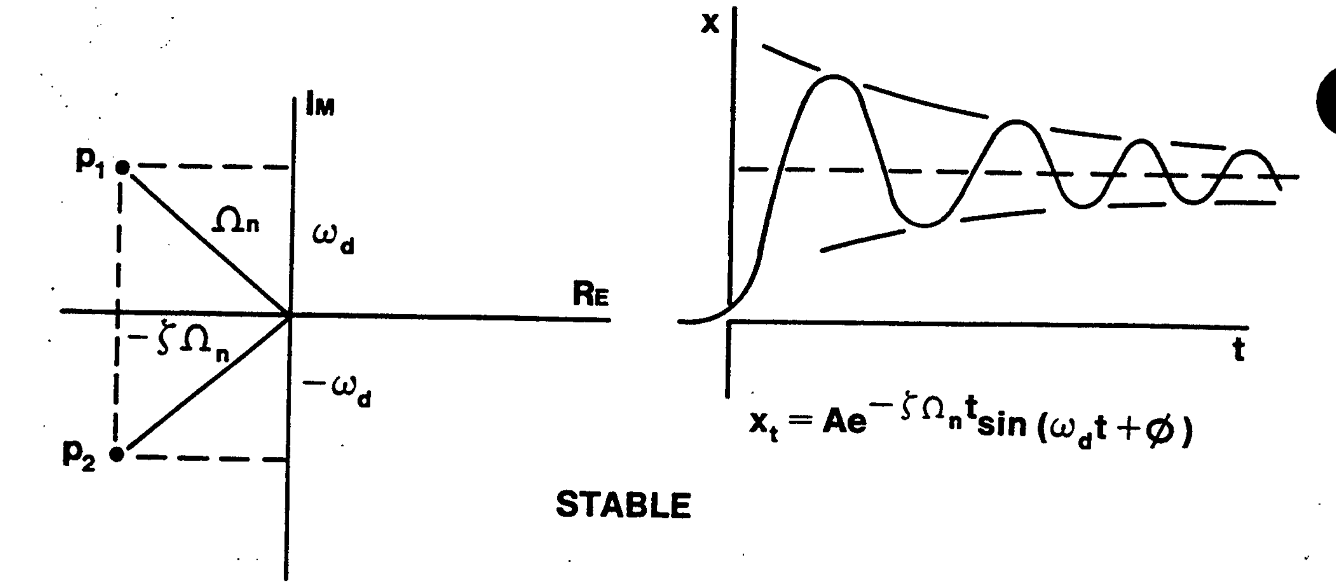

FIGURE 3.20. STABLE UNDERDAMPED RESPONSE

3. ζ = 1.0 Critically damped (Figure 3.21)

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = -\omega_n$$

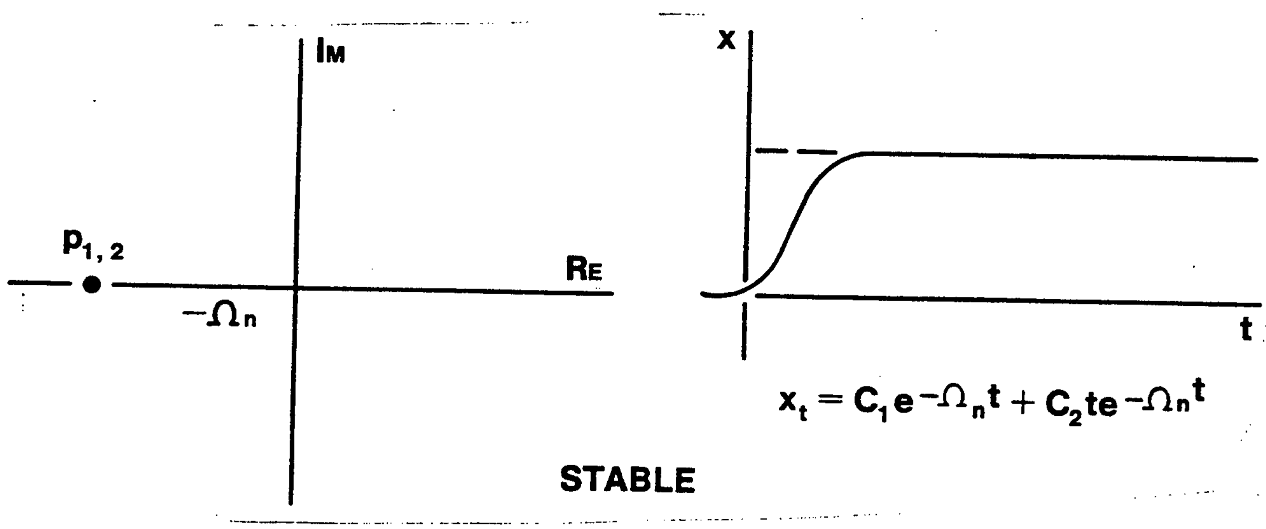

FIGURE 3.21. STABLE CRITICALLY DAMPED RESPONSE

4. $\zeta > 1.0$ Overdamped (Figure 3.22)

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = -\zeta\omega_n \pm\omega_n \sqrt{\zeta^2 - 1}$$

real

$$p_{1,2} = (-), (-)$$

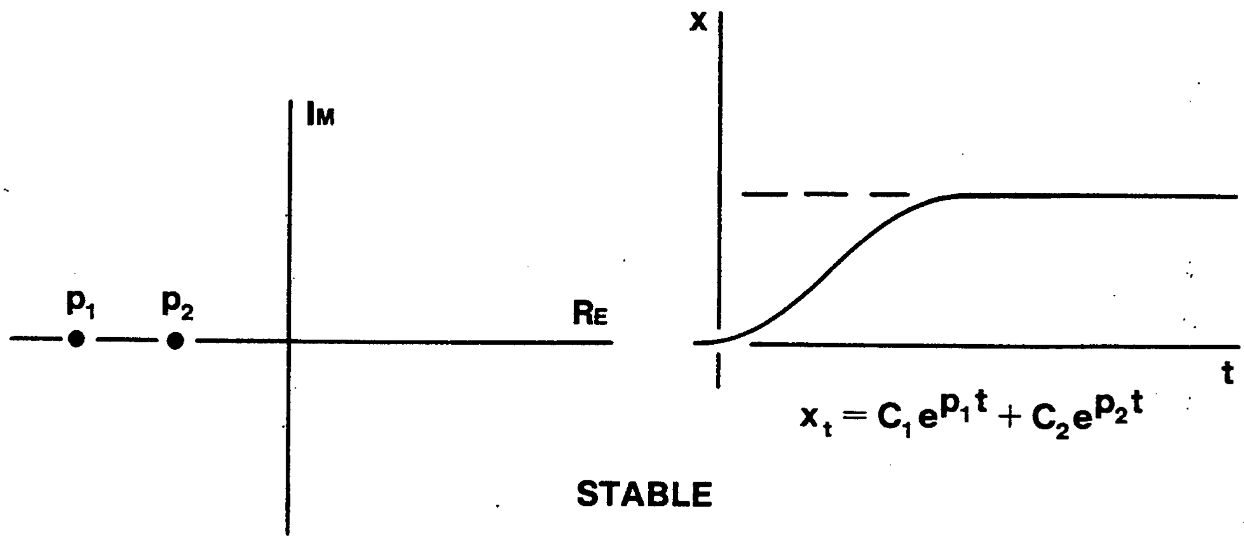

FIGURE 3.22. STABLE OVERDAMPED RESPONSE

5. $\zeta < 0$ Unstable

$\zeta = -1.0$ (Figure 3.23)

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = \omega_n$$

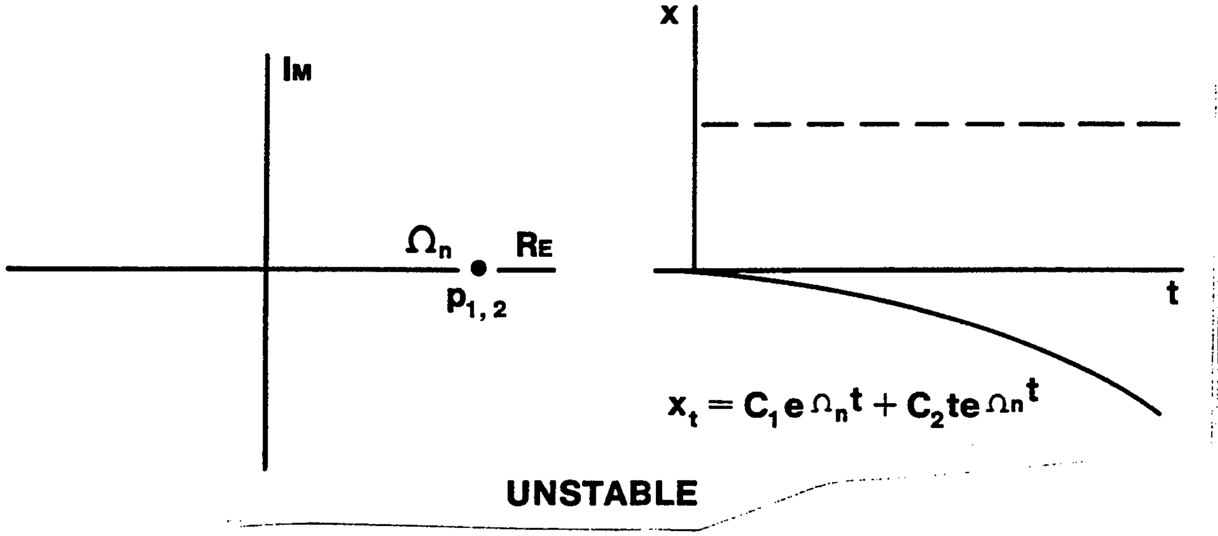

FIGURE 3.23. UNSTABLE RESPONSE

$-1.0 < \zeta < 0$ (Figure 3.24)

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{3.88}$$

$$p_{1,2} = (+) \pm j\ (+)$$

$$p_{1,2} = (+) \pm j\ (+)$$

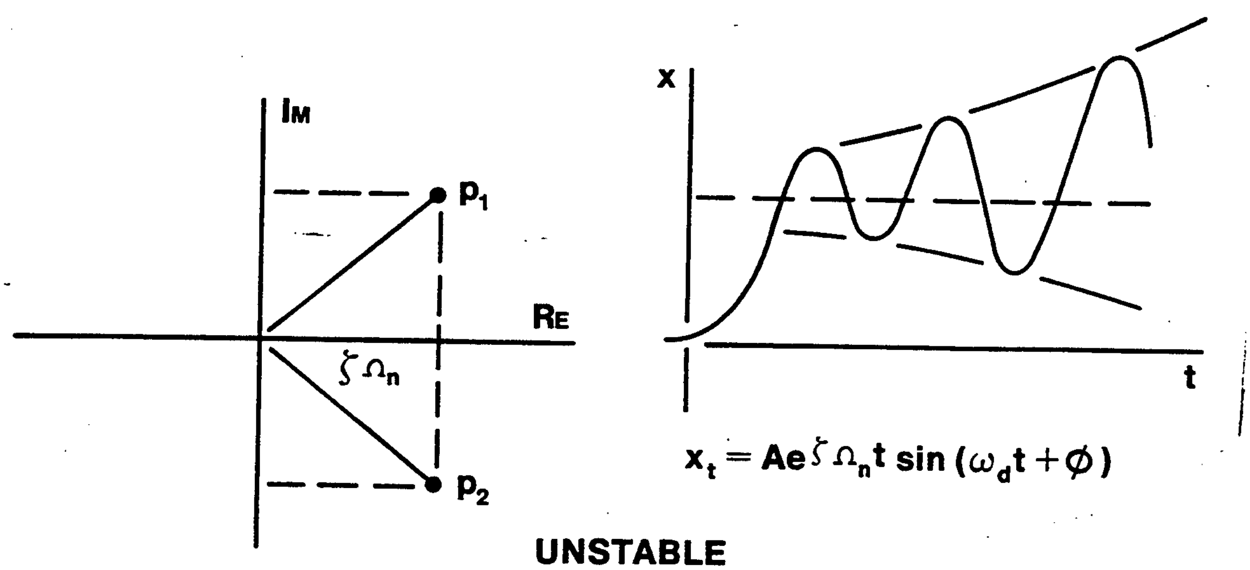

FIGURE 3.24 UNSTABLE RESPONSE

$\zeta < -1.0$ Both roots positive (Figure 3.25)

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \tag{3.88}$$

$$p_{1,2} = (+), (+)$$

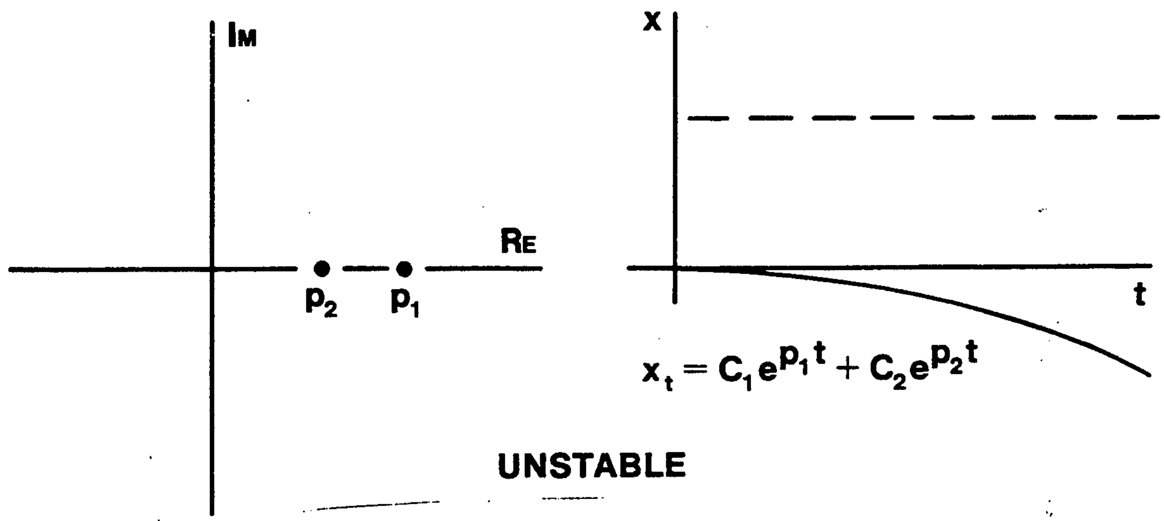

FIGURE 3.25. UNSTABLE RESPONSE

In summary, a second order system with both roots located to the left of the imaginary axis is stable. If both roots are on the imaginary axis the system is neutrally stable, and if one or more roots are located to the right of the imaginary axis the system is unstable. These conditions are shown in Figure 3.26.

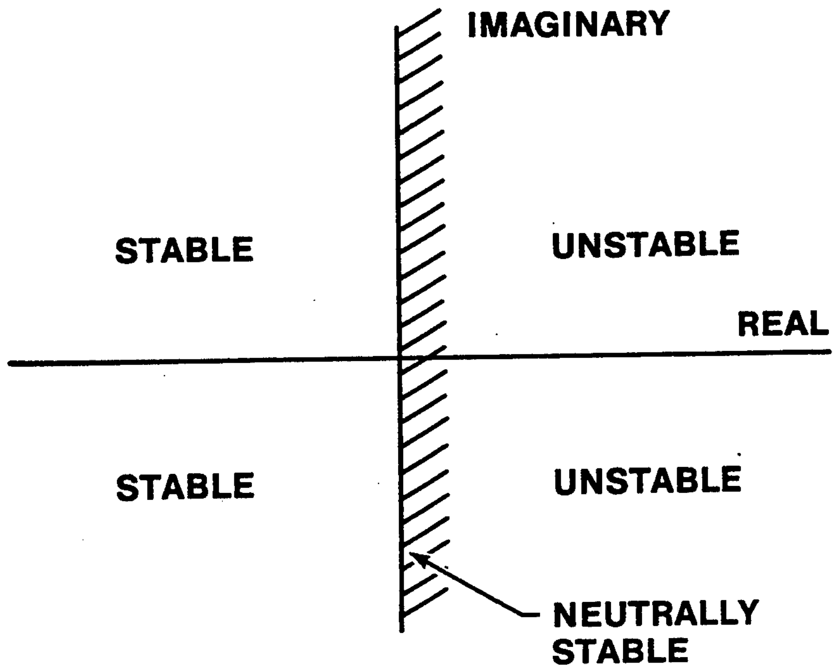

FIGURE 3.26. ROOT LOCUS STABILITY

Root plots can be used for analysis of the aircraft modes of motion. For example, the longitudinal static statiblity of an aircraft is greatly influenced by center of gravity (cg) position. Figure 3.27 shows how the roots of the characteristic equation describing one of the longitudinal motion modes change position as the cg is moved aft. This plot is called a root locus plot.

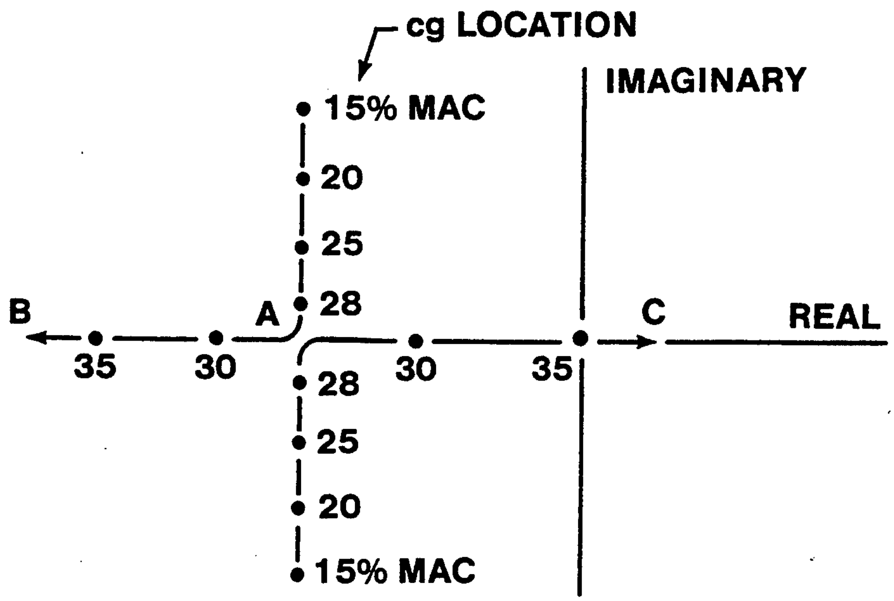

FIGURE 3.27 EFFECT OF CG SHIFT ON LONGITUDINAL STATIC STABILITY OF A TYPICAL AIRCRAFT

Note that as the cg is moved aft of its initial location at 15% MAC, damping of this mode of motion (short period) increases while the frequency decreases. Zero frequency is reached between a cg location of 28% and 30% MAC. The root locus then splits into a pair of real roots, branches AB and AC of the locus. These branches represent damped aperiodic (nonoscillatory) motion. The short period mode of motion goes unstable at a cg location of 35% MAC. The location of the cg where this instability occurs (35% MAC in this example) is known as the maneuver point and it is discussed in detail in Chapter 6, Maneuvering Flight.

PROBLEMS

Solve for y.

3.1. $\frac{dy}{dx} = x^4 + 4x + \sin 6x$

3.2. $\frac{d^2y}{dx^2} = e^{-x} + \sin \omega x$

3.3 $\frac{d^3y}{dx^3} = x^5$

3.4. $y \frac{dy}{dx} + 3x^2 = 0$

3.5. $(x-1)^2\, y dx + x^2 (y-1) dy = 0$
Just find a solution. Solving for y is tough.

Test for exactness and solve if exact.

3.6. $(y^2 - x)\, dx + (x^2 - y)\, dy = 0$

3.7. $(2x^3 + 3y)\, dx + (3x + y - 1)\, dy = 0$

3.8. $(2xy^4 e^y + 2xy^3 + y)\, dx + (x^2y^4e^y - x^2y^2 - 3x)\, dy = 0$

3.9. Multiply Problem 3.8 by $1/y^4$ and solve for y. Note this assumes that $y \neq 0$.

Solve for y_t

3.10. $5y' + 6y = 0$

3.11. $y''' - 5y'' - 24y' = 0$

3.12. $y'' + 12y' + 36y = 0$

3.13. $y'' + 4y' + 13y = 0$

Solve for y_t and y_p in Problems 3.14 - 3.17, then solve for the general solution.

3.14. $\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$ $\quad y(0) = 1, \dot{y}(0) = 6$

3.15. $\ddot{y} + 4\dot{y} + 4y = \cos t$ $\quad y(0) = \frac{28}{25}, \dot{y}(0) = -\frac{104}{25}$

3.16. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$, $\quad x(0) = 3, \dot{x}(0) = -\frac{27}{10}$

3.17. $3\dot{x} + 2x = -4e^{-2t}$, $\quad x(3) = -0.14$

3.18. Find ω_n, ω_d, ζ, and describe system damping (i.e., underdamped, overdamped, etc.) where applicable.

$$\ddot{y} + 5\dot{y} + 6y = 3e^{-3t}$$

3.19. $\ddot{y} + 4\dot{y} + 4y = \cos t$

3.20. $2\ddot{x} + 4\dot{x} + 20x = 6t + \frac{6}{5}$

3.21. $3\dot{x} + 2x = -4e^{-2t}$

In problems 3.22 - 3.24 find x(s), <u>do not</u> find the inverse transform.

3.22. $3\ddot{x} + \dot{x} + 6x = \sin 6t, \qquad x(0) = \dot{x}(0) = 0$

3.23. $\ddot{x} - 2\dot{x} + 5x = e^{-t} \sin 3t, \qquad x(0) = -1, \dot{x}(0) = 9$

3.24. $4\ddot{x} + 3\dot{x} - x = t^3 - t \sin 2t, \qquad x(0) = 3, \dot{x}(0) = -2$

In Problems 3.25 - 3.27, expand X(s) by partial fractions and find the inverse transforms.

3.25. $X(s) = \dfrac{5s + 29s + 36}{(s + 2)(s^2 + 4s + 3)}$

3.26. $X(s) = \dfrac{2s^2 + 6s + 5}{(s^2 + 3s + 2)(s + 1)}$

3.27. $X(s) = \dfrac{2s^4 + 7s^3 + 27s^2 + 51s + 27}{(s^3 + 9s)(s^2 + 3s + 3)}$

Solve the following problems by Laplace transform techniques.

3.28. $\dot{x} + 2x = \sin t, \qquad x(0) = 5$

3.29. $\ddot{x} + 5\dot{x} + 6x = 3e^{-3t}, \qquad x(0) = \dot{x}(0) = 1$

Solve using Laplace Transforms

3.30. $\dot{x} + 3x - y = 1 \qquad x(0) = y(0) = 0$

$\dot{x} + 8x + y = 2$

3.31. Read the question and circle the correct answer, true (T) or false (F):

T F The particular solution to a second order differential equation contains two arbitrary constants that are solved for using initial conditions and the transient solution too.

T F Solutions to linear differential equations are generally nonlinear functions.

T F Differential equation solutions are free of derivatives.

T F Direct integration will give solutions to some differential equations without the necessity of arbitrary constants.

T F In general, the number of arbitrary constants in the solution of a differential equation is equal to the order of the differential equation.

T F There is no known way to determine if a differential equation is exact.

T F The solution to a first order linear differential equation with constant coefficients is always of exponential form.

T F The Laplace variable s can be real, imaginary, or complex.

T F Inverse Laplace transforms are used to return from the s to the time domain.

T F Quiescent systems have zero initial conditions.

T F First order equation roots cannot be plotted on root plots.

T F A transfer function can be defined as input transform divided by output transform.

T F The characteristic equation completely describes the transient solution.

T F The method of undetermined coefficients is used to solve for the particular solution.

T F $\ddot{x} + 4\dot{x} + 13x = 3$, is a second degree equation.

T F $\ddot{x} + 4\dot{x} + 13x = 3$, is a second order equation.

T F $4\dot{x} + 13x = 3$, is a first order equation.

T F It is impossible to have a linear, second degree equation.

T F $13x = 3$, is a linear equation.

T F 13x = 3, is a differential equation.

T F Damping ratio and natural frequency have no physical significance.

T F The time constant and time to half amplitude for a first order system are equal.

T F The Laplace transform converts a differential equation from the time domain to the s domain.

T F The transient response is dependent on the input.

T F Laplace transforms are easy to derive.

T F In general, it is easier to check a candidate solution to see if it is a solution than to determine the candidate solution.

T F Superposition can be used for adding linear differential equation solutions.

T F The method of partial fractions is used to solve for the particular solution of a differential equation.

T F The number "e" is a variable.

T F The Laplace transform of the characteristic equation appears in the denominator of the transfer function.

T F There is a general technique which can be used to solve any linear differential equation.

T F Cramer's rule is in centimeters.

T F Cramer's rule is an outdated method of solving simultaneous equations.

T F The transfer function completely characterizes a linear system.

T F The Heaviside Expansion theorem is often cited by weight watchers.

T F A root plot is a short hand method of presenting transient time response.

T F The settling time is a measure of damping ratio of a system without regard for the damped frequency.

T F If y = f(x), then y is the dependent and x the independent variable.

3.32. The following terms are important. Define and provide symbols for those you are not sure of.

Differential Equation

Dependent Variable

Independent Variable

Ordinary Differential Equation

Partial Differential Equation

Exact Differential Equation

Linear Differential Equation

Degree of a Differential Equation

Order of a Differential Equation

General solution

Transient solution

Particular solution

Steady-state solution

Forcing function

Input to a system (related to the differential equation)

Output of a system (related to the differential equation)

Time Constant

Damping ratio

Damped natural frequency

Natural Frequency

Undamped response

Underdamped response

Overdamped response

Unstable system response

Critical Damping

Linear system

Laplace Transform

Inverse Laplace Transform

Unit Step

Unit Impulse

Ramp function

Transfer function

Pole

Zero

Root Plot

Root Locus

Rise time

Settling Time

Peak Overshoot

Time to peak overshoot

Steady-state error

3.33. Solve the following problems. Sketch root locus plots, and find ζ, ω_n, ω_d, and τ where appropriate.

A. $\dfrac{d^2y}{dx^2} = x^2 + 4x$

B. $dy = \dfrac{-2xydx}{(x^2 + \cos y)}$

C. $\dfrac{dx}{dt} + t^3x = 0$

D. $$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0$$

E. $$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$$

F. $$\frac{d^2x}{dt^2} + 4x = 0$$

G. $$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 22x = 0$$

H. Given: $y_t = 2 \sin 3x + 2 \cos 3x$

Find A and ϕ in the expression

$$y_t = A \sin(3x + \phi)$$

The following problems are the same as D thru G with forcing functions.

I. $$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 9$$

J. $$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = e^{2t}$$

K. $$\frac{d^2x}{dt^2} + 4x = \sin 3t$$

L. $$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 22x = t$$

The following problems are the same as D thru G with forcing function and initial conditions:

M. $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 9$ $\qquad x(0) = 3/2$, $\dot{x}(0) = 2$

N. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = e^{2t}$ $\qquad x(0) = 2$, $\dot{x}(0) = 4$

O. $\dfrac{d^2x}{dt^2} + 4x = \sin 3t$ $\qquad x(0) = 0$, $\dot{x}(0) = -3/10$

P. $\dfrac{d^2x}{dt^2} + 7\dfrac{dx}{dt} + 22x = t$ $\qquad x(0) = 0$, $\dot{x}(0) = 1/22$

3.34. Solve the following problems using Laplace techniques:

A. $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 9$ $\qquad x(0) = 3/2$, $\dot{x}(0) = 2$

B. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = e^{2t}$ $\qquad x(0) = 2$, $\dot{x}(0) = 4$

3.35. Given the set of equations

$$3\frac{dx}{dt} + \frac{dy}{dt} = t$$

$$2\frac{dx}{dt} + \frac{dy}{dt} = 1$$

where $x(0) = y(0) = 0$, find $y(t)$ using Laplace transform methods.

ANSWERS

3.1. $y = \frac{x^5}{5} + 2x^2 - \frac{\cos 6x}{6} + C$

3.2. $y = e^{-x} - \frac{\sin \omega x}{\omega^2} + C_1 x + C_2$

3.3. $y = \frac{x^8}{336} + \frac{C_1 \; x^2}{2} + C_2 x + C_3$

3.4. $y^2 = -2x^3 + C$

3.5. $y^{et} = Cx^2 \; \exp(\frac{1-x^2}{x})$

3.6. Not exact.

3.7. $f = \frac{x^4}{2} + 3xy + \frac{y^2}{2} - y + C$

3.8. Not exact.

3.9. $f = x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = C$

3.10. $y_t = Ce^{-6/5t}$

3.11. $y_t = C_1 + C_2 e^{-3t} + C_3 e^{8t}$

3.12. $y_t = C_1 e^{-6t} + C_2 te^{-6t}$

3.13. $y_t = C\,e^{-2t}\cos(3t + \phi)$

3.14. $y = -11e^{-3t} + 12e^{-2t} - 3te^{-3t}$

3.15. $y = e^{-2t} - \frac{58}{25}te^{-2t} + \frac{3}{25}\cos t + \frac{4}{25}\sin t$

3.16. $y = 3e^{-t}\cos 3t + 3/10\ t$

3.17. $y = -e^{-2/3t} + e^{-2t}$

3.18. $\omega_n = \sqrt{6}$

$\zeta = 1.02$

3.19. $\omega_n = 2$

$\zeta = 1$

$\omega_d = 0$

3.20. $\omega_n = \sqrt{10}$

$\zeta = 0.316$

$\omega_d \simeq 3.0$

3.21. $\tau = 3/2$

3.22. $$x(s) = \frac{\dfrac{6}{s^2 + 36}}{3s^2 + s + 6}$$

3.23. $$x(s) = \frac{\dfrac{3}{(s+1)^2 + 9} - s + 11}{s^2 - 2s + 5}$$

3.24. $$x(s) = \frac{\frac{6}{s^4} - \frac{4s}{(s^2+4)^2} + 12s + 1}{4s^2 + 3s - 1}$$

3.25. $y(t) = 2e^{-2t} - 3e^{-3t} + 6e^{-t}$

3.26. $y(t) = e^{-2t} + e^{-t} + te^{-t}$

3.27. $y(t) = 1 + 2/3 \sin 3t + e^{-3/2t} \cos \sqrt{3/4}\, t + \frac{1}{3}\, e^{-3/2t} \sin \sqrt{3/4}\, t$

3.28. $x(t) = 26/5\, e^{-2t} - 1/5 \cos t + 2/5 \sin t$

3.29. $x(t) = -6e^{-3t} - 3te^{-3t} + 7e^{-2t}$

3.30. $x(t) = 1/4\, (1 - e^{-2t} (\cos 2t - \sin 2t))$

$y(t) = 1/4\, (-1\, e^{-2t} (\cos 2t + 3 \sin 2t))$

3.33. A. $y = \frac{x^4}{12} + 2/3\, x^3 + Cx + C_1$

B. $x^2 y + \sin y = C$

C. $x = Ce^{-t^4}/4$

D. $x(t) = C_1 e^{2t} + C_2 e^{3t}$

E. $\omega_n = 2$
$\omega_d = 0$
$\zeta = -1$

F. $\zeta = 0$

$\omega_n = 2$

$\omega_d = 2$

G. $\omega_n = 4.69$

$\omega_d = 3.12$

$\zeta = 0.746$

H. $A = \sqrt{8}$

$\phi = \pi/4$

I. $x = C_1 e^{2t} + C_2 e^{3t} + 3/2$

J. $x = C_1 e^{2t} + C_2 te^{2t} + 1/2\, t^2 e^{2t}$

K. $x = C_1 \cos 2t + C_2 \sin 2t - 1/5 \sin 3t$

L. $x = e^{-7/2t} (C_1 \cos \frac{\sqrt{39}}{2} t + C_2 \sin \frac{\sqrt{39}}{2} t) + 1/22\, t - \frac{7}{484}$

M. $x = -2e^{2t} + 2e^{3t} + 3/2$

N. $x = 2e^{2t} + 1/2\, t^2 e^{2t}$

O. $x = 3/20 \sin 2t - 1/5 \sin 3t$

P. $x = e^{-7/2t} (\frac{7}{484} \cos \frac{\sqrt{39}}{2} t + .016 \sin \frac{\sqrt{39}}{2} t) + \frac{1}{22} t - \frac{7}{484}$

3.34. A. $x(t) = \frac{3}{2} - 2e^{2t} + 2e^{3t}$

B. $x(t) = 2e^{-2t} + 1/2\, t^2 e^{2t}$

3.35. $y(t) = 3t - t^2$

$x(t) = 1/2\, t^2 - t$