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CALCULATION OF THE DERIVATIVE OF THE PARABOLIC CYLINDER FUNCTION

Robert A. Shore, Thorkild B. Hansen

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Rome Laboratory
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APPROVED: Daniel J. Jacavanco

DANIEL J. JACAVANCO, Chief
Antennas and Components Division
Electromagnetics & Reliability Directorate

FOR THE COMMANDER: Robert V. McGahan

ROBERT V. McGAHAN
Acting Director
Electromagnetics & Reliability Directorate

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Robert A. Shore, Thorkild B. Hansen*

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The numerical evaluation of the derivative of the parabolic cylinder function, $D'_p(z)$, is described for the argument $z = x\exp(-i\pi/4)$ and order $p = 1/2 + iy$, with x and y real. The case is needed for calculating the currents excited on a parabolic cylinder by a transverse electric plane wave, and for calculating the scattered far field pattern. Numerically accurate and efficient expressions for $D'_p(z)$ are presented for each of five regions into which the xy plane is divided.

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Calculation of the Derivative of the Parabolic Cylinder Function

1 Introduction

This report describes the numerical evaluation of the derivative of the parabolic cylinder function, $D'_p(z)$, where $D_p(z)$ is the parabolic cylinder function defined by Whittaker and Watson [1, pp.347-349]. (See also [2, Ch.19], [3, Ch.46], and [4].) The numerical evaluation is for complex argument $z = xe^{-i\pi/4}$, and complex order $p = -1/2 + iy$, x and y real. This case is needed to calculate the currents excited on a perfectly conducting parabolic cylinder by a transverse electric (TE) plane wave, and the corresponding scattered far field pattern [5, Eqs. (7.47) and (7.48)]. This is the motivation for this report, designed to complement the valuable communication of Blanchard and Newman [6] which treats the numerical evaluation of the same case of the parabolic cylinder function itself, needed for the corresponding transverse magnetic (TM) scattering calculations [5, Eqs.(7.12) and (7.13)].

2 Numerical Evaluation of $D'_p(z)$

We present expansions for $D'_p(z)$ for $z = x\alpha$, $p = -1/2 + iy$, x and y real, and $\alpha = e^{-i\pi/4}$. The discussion closely parallels that of Blanchard and Newman [6]. We adopt their division of the xy plane into five regions, denoted Regions A-E, as shown in Figure 1, and give numerically useful expressions for calculating $D'_p(z)$ for each of these regions.

Region A: $|x| \leq 5$, $|y| \leq 4$.

In this region, the exact power series for $D'_p(z)$ can be used, obtained by differentiating the

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exact power series of $D_p(z)$ given in [6]:

$$D_p'(z) = -\frac{2^{-(p+5)/2}e^{-z^2/4}}{\Gamma(-p)}\Bigg\{\Gamma\left(\frac{1}{2}-\frac{p}{2}\right)\Bigg[4-p(\sqrt{2}z)^2 + \sum_{k=2}^{\infty}\frac{(\frac{1}{2}-\frac{p}{2})\cdots(k-2+\frac{1}{2}-\frac{p}{2})}{(2k)!}\left[2k-2(p+1)\right](\sqrt{2}z)^{2k}\Bigg]$$
$$-\Gamma\left(-\frac{p}{2}\right)\Bigg[(-2p-1)(\sqrt{2}z)+\sum_{k=1}^{\infty}\frac{(-\frac{p}{2})\cdots(k-1-\frac{p}{2})}{(2k+1)!}\left[2k+1-2(p+1)\right](\sqrt{2}z)^{2k+1}\Bigg]\Bigg\} \quad (1)$$

for $p \neq 0, 1, 2, ...$, and where Γ is the gamma function [2, Ch.6]. On a computer with 16 digit double precision, this expression gives at least 6 digit accuracy in Region A if the summations are continued until the terms decrease below $1 \cdot 10^{-10}$ in magnitude. The gamma function of complex argument is also computed in double precision, using the relations (2) and (3), derived using the recurrence, reflection, and duplication formulas for the gamma function [2, Ch.6], to obtain $\Gamma(-p)$ and $\Gamma(1/2 - p/2)$ from $\Gamma(-p/2)$:

$$\Gamma(-p) = \left(\frac{\pi}{2}\right)^{1/2} 2^{-(p+1/2)}\frac{1}{\cos\left(\frac{\pi p}{2}\right)}\frac{\Gamma\left(-\frac{p}{2}\right)}{\Gamma^*\left(-\frac{p}{2}\right)} \quad (2)$$

$$\Gamma\left(\frac{1}{2}-\frac{p}{2}\right) = \frac{\pi}{\cos\left(\frac{\pi p}{2}\right)}\frac{1}{\Gamma^*\left(-\frac{p}{2}\right)} \quad (3)$$

where $*$ denotes complex conjugation.

Although the exact power series (1) for $D_p'(z)$ is absolutely convergent for all x and y, the terms of the sums grow very large before they become small if $|x|$ or $|y|$ is large, just as happens for $D_p(z)$ itself [6]. The accuracy of (1) is then degraded by the process of subtracting large numbers of comparable magnitude, and also eventually by the generation of numbers that overflow the dynamic range of the computer.

Region B: $x \geq 5$, $|y| \leq 2$.

In this region, the asymptotic series

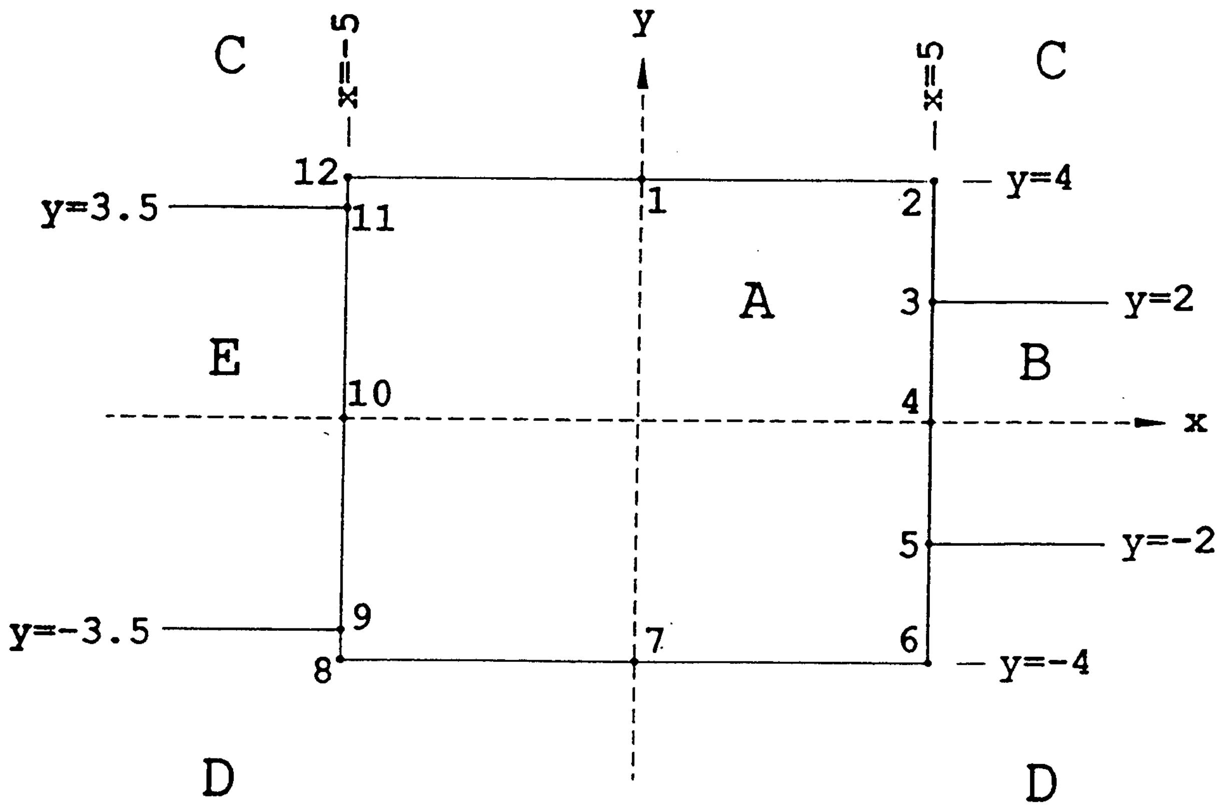

Figure 1: Definition of Regions A-E [6].

$$D'_p(z) \sim -\frac{e^{-z^2/4} z^{p+1}}{2} \left[1 - \frac{p(p+3)}{1!(2z^2)^1} + \frac{p(p-1)(p-2)(p+5)}{2!(2z^2)^2} - \frac{p(p-1)(p-2)(p-3)(p-4)(p+7)}{3!(2z^2)^3} + - \cdots \right] \quad (4)$$

is used. This series can be obtained by differentiating the asymptotic expansion for $D_p(z)$ given in [1, p.347] or, more easily, by using the recursion formula [2, Eq. 19.6.3]:

$$D'_p(z) = \frac{1}{2} \left[pD_{p-1}(z) - D_{p+1}(z) \right]. \quad (5)$$

The expansion (4) is valid for all p and for $|\arg(z)| < 3\pi/4$ when $p \ll z$ [1, p.347]. The summation in (4) is continued until the terms start to increase in magnitude. At least five significant figure accuracy can be obtained.

Region C: All x, and $y \geq 2$ except in regions A and E.

In this region we use a large order asymptotic expansion obtained by Olver [7]: [1]

$$D'_{-\frac{1}{2}+iy}(x\alpha) \sim -2^{-1/2}\mu' g(\mu'\alpha^*)(t'^2+1)^{1/4} e^{-i\mu'^2\xi} \sum_{s=0}^{\infty} \frac{i^s \tilde{v}_s(t)}{(R^3\mu'^2)^s} \tag{6}$$

where

$$\mu' = \sqrt{2y}, \quad \mu' > 0,$$

$$t' = \frac{x}{\sqrt{2}\mu'},$$

$$R = (t'^2+1)^{1/2},$$

$$g(w) = 2^{-(1+w^2)/4} e^{-w^2/4} w^{-(1-w^2)/2} \left[1 + \sum_{s=0}^{\infty} \frac{g_{2s+1}}{w^{4s+2}}\right]^{-1},$$

$$g_1 = \frac{1}{24}, \quad g_3 = -\frac{2021}{207360}, \quad g_5 = \frac{5149591}{418037760},$$

$$\xi = \frac{i\pi}{4} - \frac{\mathrm{sgn}(t')}{2}\left[|t'|R + \ln(|t'|+R)\right], \quad \mathrm{sgn}(t') = \frac{t'}{|t'|},$$

$$\tilde{v}_0 = 1, \quad \tilde{v}_1 = -\frac{t'^3 - 6t'}{24}, \quad \tilde{v}_2 = -\frac{15t'^4 + 327t'^2 - 143}{1152},$$

$$\tilde{v}_3 = \frac{4042t'^9 + 18189t'^7 + 36387t'^5 + 238425t'^3 - 259290t'}{414720}.$$

The terms given here give at least four significant figure accuracy, with the accuracy improving as y increases. Additional terms in the expansion can be obtained using algorithms given by Olver [7].

Region D: All x, and $y \leq -2$ except Regions A and E.

In this region we use the following large order asymptotic expansion obtained by Olver [7]: [2]

$$D'_{-\frac{1}{2}+iy}(x\alpha) \sim i\left(\frac{\pi}{2}\right)^{1/2} \mu'^{2/3} \frac{g(\mu'\alpha)}{\Phi} \left\{ \frac{\left[Ai(-\mu'^{4/3}\zeta) - iBi(-\mu'^{4/3}\zeta)\right]}{\mu'^{4/3}} \sum_{s=0}^{\infty} (-1)^s \frac{C_s}{\mu'^{4s}} - \left[Ai'(-\mu'^{4/3}\zeta) - iBi'(-\mu'^{4/3}\zeta)\right] \sum_{s=0}^{\infty} (-1)^s \frac{D_s}{\mu'^{4s}} \right\}, \quad x \geq 0 \tag{7}$$

[1]Equation (6) is obtained starting with [7, Eq.(4.13)], identifying μ with $\sqrt{2y}e^{i\pi/4}$, t with $-ix/(2\sqrt{y})$, and using [7, Eqs.(3.4), (11.6), (11.7), (11.8), (6.1), (4.12), (4.14), (4.16) , and Fig. (2b)]. See also [6, Eq.(6)] and [8, Sect. 12.6.5].

[2]Equation (7) is obtained starting with [7, Eq.(8.15)], identifying μ with $2\sqrt{|y|}e^{-i\pi/4}$, t with $x/(2\sqrt{|y|})$, using [7, Eqs. (8.16),(7.4), (11.21), (8.18), (8.19), (4.14), (4.16), (8.13)], and [2, Eq. (10.4.9]. Equation (8) is similarly obtained starting again with [7, Eq.(8.15)], letting $\mu \to \mu^{i\pi}$, and using [7, Eq.(5.24)]. See also [6, Eqs.(7) and (8)]. For general background on these expansions see [9, Ch.11].

$$D'_{-\frac{1}{2}+iy}(x\alpha) \sim (2\pi)^{1/2}\mu'^{2/3}e^{\pi\mu'^2/2}\frac{g(\mu'\alpha)}{\Phi}\left\{\frac{-Ai(-\mu'^{4/3}\zeta)}{\mu'^{4/3}}\sum_{s=0}^{\infty}(-1)^s\frac{C_s}{\mu'^{4s}}\right.$$
$$\left. + Ai'(-\mu'^{4/3}\zeta)\sum_{s=0}^{\infty}(-1)^s\frac{D_s}{\mu'^{4s}}\right\}, \quad x \le 0 \tag{8}$$

where Ai, Bi, Ai', and Bi' are the Airy functions and their derivatives [2, Sect. 10.4], and $g(\mu'\alpha)$ is defined as for Region C. The other quantities are defined by

$$\mu' = \sqrt{2|y|},$$

$$t = \frac{|x|}{\sqrt{2}\mu'},$$

$$\frac{4}{3}\zeta^{3/2} = t(t^2-1)^{1/2} - \ln[t + (t^2-1)^{1/2}], \quad t \ge 1, \quad (\zeta \ge 0),$$

$$\frac{4}{3}(-\zeta)^{3/2} = \arccos(t) - t(t^2-1)^{1/2}, \quad -1 < t \le 1, \quad (\zeta \le 0),$$

$$\Phi = \left(\frac{\zeta}{t^2-1}\right)^{1/4},$$

$$C_s = -\frac{1}{(t^2-1)^{3s+1}}\Phi^2\sum_{m=0}^{2s+1} b_m\Phi^{-6m}v_{2s-m+1}(t),$$

$$D_0 = 1, \quad D_s = -\frac{1}{(t^2-1)^{3s}}\sum_{m=0}^{2s} a_m\Phi^{-6m}v_{2s-m}(t),$$

$$v_0(t) = 1, \quad v_1(t) = \frac{t^3+6t}{24}, \quad v_2(t) = \frac{15t^4 - 327t^2 - 143}{1152},$$

$$v_3(t) = \frac{-4042t^9 + 18189t^7 - 36387t^5 + 238425t^3 + 259290t}{414720},$$

$$a_0 = 1, \quad a_m = \frac{(2m+1)(2m+3)\cdots(6m-1)}{m!(144)^m}, \quad b_m = -\frac{6m+1}{6m-1}a_m.$$

For $|t-1| < 0.25$, Φ and ζ should be computed by the expressions given in [6], while for $|t-1| < 0.05$, the following expansions, obtained using Mathematica, should be used for C_0, C_1, and D_1:

$$\delta = t - 1$$

$$C_0 = -\frac{\Phi^2}{t+1}\left(\frac{1}{5} + \frac{3}{25}\delta + \frac{97}{2250}\delta^2 - \frac{173}{385000}\delta^3 + \cdots\right),$$

$$C_1 = \frac{\Phi^2}{(t+1)^4}\left(\frac{88579}{462000} + \frac{88579}{220000}\delta + \frac{11626038701}{22702680000}\delta^2 + \frac{1480656484897}{4824319500000}\delta^3 + \cdots\right),$$

$$D_1 = \frac{1}{(t+1)^3}\left(\frac{2041}{25200} + \frac{48577}{2772000}\delta + \frac{279481}{840840000}\delta^2 - \frac{21114659}{227026800000}\delta^3 + \cdots\right),$$

The terms given here give at least four significant figure accuracy for $D'_{-1/2+iy}(x\alpha)$ in Region D.

Region E: $x \leq -5$, $-3.5 \leq y \leq 3.5$.

In this region we use the relation, valid for all z and p, obtained from [1, Sect. 16.52]:

$$D'_p(z) = -e^{p\pi i}D'_p(-z) - \frac{i\sqrt{2\pi}}{\Gamma(-p)}e^{(p+1)\pi i/2}D'_{-p-1}(-iz) \tag{9}$$

For $p = -1/2 + iy$ and $z = xe^{-i\pi/4}$,

$$D'_p(z) = -e^{p\pi i}D'_p(-z) - \frac{i\sqrt{2\pi}}{\Gamma(-p)}e^{(p+1)\pi i/2}D'^*_p(-z) \tag{10}$$

where $D'_p(-z)$ is calculated as above for Regions B, C, or D.

3 Numerical Results

To illustrate the accuracy of the expansions we have presented, in Table 1 we compare the results obtained with the expansions for Regions B-E with the Region A results for 12 points on the perimeter of Region A (see Figure 1), corresponding to the results given by Blanchard and Newman [6] for the parabolic cylinder function itself. Table 1 gives the values of x and y, the region whose expansion was used in the calculation, the calculated value of $D'_p(z)$, and the number of significant digits. The exact power series results for Region A have been checked with Mathematica and are correct to the number of figures shown.

4 Conclusion

In this report we have presented numerically useful expressions for calculating the derivative of the parabolic cylinder function, $D'_p(z)$, for the special case $p = -1/2 + iy$ and $z = xe^{-i\pi/4}$. This case is needed for calculating the currents excited on a perfectly conducting parabolic cylinder by a TE plane wave, and for calculating the scattered far field pattern.

Table 1: Values of $D'_p(z)$ at selected points in Figure 1: $z = xe^{-i\pi/4}$ and $p = -1/2 + iy$

Point	x	y	Region	$D'_p(z)$	Significant Digits
1	0	4	A	-0.214019E+02 - i0.867551E+01	6
1	0	4	C	-0.214038E+02 - i0.867624E+01	4.1
2	5	4	A	-0.292821E+02 - i0.586851E+00	6
2	5	4	C	-0.292820E+02 - i0.586872E+00	4.7
3	5	2	A	0.557884E+01 - i0.146969E+01	6
3	5	2	B	0.557885E+01 - i0.146966E+01	5.5
3	5	2	C	0.557886E+01 - i0.146962E+01	5.2
4	5	0	A	-0.103126E+01 + i0.437434E+00	6
4	5	0	B	-0.103126E+01 + i0.437433E+00	6
5	5	-2	A	0.197734E+00 - i0.775546E-01	6
5	5	-2	B	0.197733E+00 - i0.775537E-01	5
5	5	-2	D	0.197735E+00 - i0.775577E-01	4.5
6	5	-4	A	-0.384484E-01 + i0.378112E-02	6
6	5	-4	D	-0.384484E-01 + i0.378114E-02	5.7
7	0	-4	A	-0.214019E+02 + i0.867551E+01	6
7	0	-4	D	-0.214033E+02 + i0.867600E+01	4.3
8	-5	-4	A	0.581438E+04 - i0.235691E+04	6
8	-5	-4	D	0.581437E+04 - i0.235691E+04	6
9	-5	-3.5	A	0.424382E+04 - i0.236275E+03	6
9	-5	-3.5	D	0.424382E+04 - i0.236275E+03	6
9	-5	-3.5	E	0.424382E+04 - i0.236275E+03	6
10	-5	0	A	-0.190613E+01 - i0.437434E+00	6
10	-5	0	E	-0.190613E+01 - i0.437433E+00	6
11	-5	3.5	A	-0.112470E+02 - i0.159690E+02	6
11	-5	3.5	C	-0.112468E+02 - i0.159688E+02	5.7
11	-5	3.5	E	-0.112470E+02 - i0.159690E+02	6
12	-5	4	A	-0.214258E+02 - i0.199681E+02	6
12	-5	4	C	-0.214258E+02 - i0.199680E+02	6

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