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The proposed work for the grant was divided into several parts: shallow water, deep water and the qualitative behavior of solutions of nonlinear equations. The overall objective of the research in shallow water is to develop an accurate model of wave propagation in shallow water. The model we have developed is based on the Kadomtsev-Petviachvili equation which can derived as an asymptotic limit of Stokes' euations of water waves.

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Final Report
Office of Naval Research Grant
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Nonlinear Ocean Waves

Principal Investigator Harvey Segur
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CO-Principal Investigator James H. Curry

Our proposed work for this grant was divided into several parts: *shallow water*, *deep water* and the *qualitative behavior of solutions of nonlinear equations*.

Shallow Water:

The overall objective of our research in shallow water is to develop an accurate theoretical model of wave propagation in shallow water. The model that we have developed is based on the Kadomtsev-Petviashvili equation,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6\,u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{(KP)}$$

which can be derived as an asymptotic limit of Stokes' equations of water waves. The model has several interesting features:

(a) Nonlinear effects like wave-breaking are known to be important in shallow water. The KP model is inherently nonlinear.

(b) The water's surface is two-dimensional, as is the KP equation. It is *not* restricted to one-dimensional wave propagation.

(c) A set of equations that is well known in meteorology is called the "shallow water equations". This model ignores dispersive effects, as the KP equation does not. These features suggest that the KP model might describe actual waves in shallow water with reasonable accuracy. Our objective during this grant has been to test this hypothesis.

The KP equation admits huge families of exact solutions of the form,

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln \Theta_N\,, \qquad (*)$$

where Q_N is a Riemann theta function with N phases. In principle N can be any positive integer, but the theory of theta functions has been developed extensively only for small N. If N=1, (*) yields the *cnoidal wave*, discovered by Korteweg and de Vries 100 years ago. Having one-dimensional surface patterns, cnoidal waves lose the important two-dimensionality of the KP model. Genuinely two-dimensional surface patterns begin at N=2 (called "two-phase waves", or "waves of genus 2"). If a two-phase wave is genuinely two-dimensional, then it turns out that the wave pattern has permanent form and travels with constant speed. These waves are periodic in two spatial directions, and they are the simplest KP solutions that are both periodic and intrinsically two-dimensional.

Laboratory Work:
Experimental work done under this grant (HSS '89, HMSS '95) showed that in shallow water of uniform depth, there exist actual water waves with the properties of two-phase KP solutions: spatially periodic waves that propagate with nearly permanent form at nearly constant speed. (Small but nonzero viscous effects preclude waves of truly permanent form.) The basic template for each wave pattern is a hexagonal cell: six narrow wave crests surrounding a broad wave trough, with the hexagonal cell repeated to tile the plane. These waves are easy to generate experimentally, and they seem to be stable. In addition, the KP model describes the observed waves with reasonable accuracy over the range of parameter-values tested experimentally.

The KP equation approximates the water-wave problem in a particular limit (long, nearly one-dimensional waves of small amplitude), but hexagonal waves of nearly permanent form also exist well outside this limit. They seem to be stable wave patterns in shallow water, and the natural generalization of cnoidal waves from one spatial dimension to two.

That water waves corresponding to two-phase KP solutions occur in carefully controlled laboratory experiments does not guarantee that they occur naturally in the ocean. The objective of our most recent work (Curry *et al.*, enclosed) is to show that two-phase KP solutions also describe actual ocean waves, including large-amplitude waves that are observed during severe storms. In particular, the Field Research Facility (FRF) of the Army Corps of Engineers at Duck, NC regularly records waves that impinge on its beach. In the enclosed paper, we compare two-phase KP solutions with data from a gauge-array at FRF during the "Halloween storm of 1991" (associated with Hurricane Grace), including some of the biggest waves ever recorded at FRF. We find that the storm waves often arrange themselves into two-dimensional, spatially periodic surface patterns that propagate with nearly permanent form. This two-dimensional, self-organized pattern can dictate many of the observed features of waves in shallow water, including their "groupiness". Two-phase KP solutions often approximate these self-organized patterns with surprising accuracy.

Two-phase solutions are the simplest KP solutions that are not degenerate (*i.e.*, they are genuinely two-dimensional, periodic waves with finite amplitudes). Showing that these simple wave-patterns occur naturally among oceanic storm waves justifies development of the KP model and encourages its further development.

Several lines of development of the KP model are possible beyond the two-phase solutions discussed above. One choice is to develop three-phase solutions of the KP equation (*i.e.*, N = 3 in (*)). The physical meaning of three-phase solutions is as follows.

(*if N=1*) The simplest KP solutions have one phase; they are essentially one-dimensional cnoidal waves, with permanent form.
(*if N=2*) The simplest KP solutions that are genuinely two-dimensional have two phases; these are the 2-D waves of permanent form discussed above.
(*if N=3*) The simplest KP solutions that are genuinely two-dimensional and have nontrivial time-dependence have three phases.

Three-phase solutions are essential to describe any shallow-water process that is intrinsically time-dependent (*e.g.*, energy-transfer between nonlinear modes). The enclosed paper by Dubrovin *et al.* develops three-phase KP solutions into a computational tool. The end result is a computer program, available on the world-wide web, that allows the reader to input values for each of the 12 parameters that characterize a three-phase KP solution. For each such choice of parameters, the program computes and displays the two-dimensional KP solution as it evolves in time.

The computation is fast, and one can observe time-dependent wave evolution as it occurs. No other KP program permits such "real-time" observations. Historically, published pictures of nonlinear, genuinely two-dimensional waves by Segur & Finkel (SF '85) inspired the experimental program of Hammack & Scheffner (HSS '89, HSS '91, HMSS '95). Our hope is that easy access to our code will lead to new experimental studies of processes that are both two-dimensional and time-dependent.

Almost every three-phase KP solution is both two-dimensional and time-dependent. To our surprise, however, we also found a large class of solutions that are "nearly stationary". The time-evolution of one of these solutions is so slow and so weak that it appears to be nearly stationary. Discovery of this family of solutions was an unexpected bonus that came with the development of our time-dependent numerical code.

Finally, here are some things that we cannot yet do:
(a) Solve the initial-value problem for KP. An N-phase KP solution is a nonlinear generalization of an N-term Fourier series, so ideally one could simply measure two-dimensional, time-dependent, oceanic waves in shallow water, determine how many phases (N) are required to resolve those waves to some desired accuracy, and then find the N-phase KP solution that approximates the measured wave-pattern to that accuracy. How to do this is an outstanding problem in nonlinear partial differential equations. [Osborne *et al.* (OP '94) have developed such an algorithm for data that are known to be one-dimensional. However, ocean waves are rarely one-dimensional, and their algorithm never checks the validity of its assumption of one-dimensional waves.]
(b) Propagate a N-phase solution across variable topography, in the presence of small dissipation. Even for N=2 or N=3, the KP solutions currently available require an inviscid fluid of uniform depth. Generalized KP equations that account for variable depth and small dissipation (*e.g.*, CL '95) are known, but how these changes affect the parameters of a two-phase or three-phase solution has never been worked out.

(c) Develop a statistical theory of waves in shallow water, in which the underlying waves are genuinely nonlinear. Virtually all statistical theories use random collections of Fourier modes (*i.e.*, linear, infinitesimal waves). Some models like JONSWAP include weakly nonlinear corrections to the basic linear theory, but no theory exists that starts with genuinely nonlinear waves. To do so is an outstanding problem in statistical physics, not just in oceanography.
(d) Generalize the KP model to include wave-breaking. The KP model is intrinsically nonlinear, but it does not describe breaking waves. How to include this important physical process in a generalized KP model is an open question.

Accomplishments:

1. J.H. Curry, J.L. Hammack, C.E. Long, N.W. Scheffner, & H. Segur, "Oceanic Storm Waves near Shore", submitted for publication

2. B.A. Dubrovin, R. Flickinger, & H. Segur, "Three-Phase Solutions of the Kadomtsev-Petviahsvili Equation", to appear in *Stud. in Appl. Math.*

3. J. Hammack, D. McCallister, N. Scheffner, & H. Segur, *J Fluid Mech.*, **285**, pp. 95-122, 1995

Nonlinear Equations Exact Properties:
This work was concerned with several related issues whose focus was mathematical software for determining exact solution of nonlinear partial differential equations. The mathematical foundation of our approach and software rest on the ideas of S. Lie.

A goal of this research program was to develop robust production software that provides an excellent problem solving environment that supported computational science and engineering applications. Specifically, (1) apply the theory to several nonlinear partial differential equations that arise from the study of problems in nonlinear waves and (2) develop and incorporate new algorithms into future versions of our symbolic partial differential equation package, MathSym. In the early stages of his research Werckman, in the article indicated below, used the MathSym software to generate additional symmetries of the Boussinesq equations.

We predict that as software such as MathSym becomes both more widely available and more robust it will become a standard tool used to explore and analyze solutions of partial differential equations. The utility of this type of software in scientific and engineering communities may prove to be analogous to the fast Fourier transform which greatly facilitates numerical computation

Accomplishments:

1. Werckman, T., " A note on an extra constraint in some multi- dimensional extensions of integrable equations," to appear in *Wave Motion, 1997*

Bifurcation of Solutions in Nonlinear Equations:

"Bairstow Methods" are best known as numerical tools that find roots of high degree polynomials. Leonard Bairstow, an aeronautical engineer, developed one such algorithm in 1916 \cite{Bairstow}. Because of the primitive nature of the computational mathematics at the turn of the century, it was extremely economical to decompose polynomials into products of quadratic factors, thus finding solutions without using complex numbers. While complex arithmetic no longer poses such great difficulties, Bairstow's legacy of two-dimensional noninvertible maps generated by Newton's method has some current interest. There is no general proof of convergence to these fixed points exists, but there is value in exploring such mappings because of their richness as dynamical systems and because they present new behaviors for iterative methods which must be addressed when such algorithms are used to solve nonlinear equations.

A portion of our research focuses on families of noninvertible maps and the role singularities play in their dynamics. Such maps always present themselves whenever Newton's method, for example, is used to find the stationary solutions to an evolution equation.

In the specific case of Bairstow's methods there is an ``eruption" that produces a large locally attracting set that is an attractor in the sense of Milnor. An eruption in a noninvertible mapping is a bifurcation involving the merger of an attracting periodic orbit or fixed point with points on a singular curve. This bifurcation appears to be a transfer of stability from the attracting periodic orbit to another invariant set. The map restricted to the invariant set is a strange attractor. And some of the periodic points in the invariant set are saddles. For a more complete look at these eruptions, see our recent publication, ``On noninvertible mappings of the plane: Eruptions". The creation of such an invariant set would cause an iterative method to fail to converge because iterates would become tangled in the local dynamics of an invariant set.

We have introduced a class of noninvertible dynamical systems that arise naturally when attempting to factor polynomials. These mappings have intrinsic interest because in certain cases they reduce to one dimensional Newton's method and the one dimensional dynamics, certainly influences, but may also ``govern" critical transitions. Understanding the role played by singular sets of such functions seems to be important.

Further, there are few examples in dynamical systems theory which lend themselves to exact computations of macroscopic variables of interest. One such variable is the transverse Lyapunov exponent which measures the average attraction of an invariant set. In a just submitted article we presents a family of noninvertible transformations of the plane for which such computations are possible. This model sheds additional insight into the notion of what it can mean for an attracting invariant set to have a riddled basin of attraction.

Specifically, we presented two examples of a riddling bifurcation caused by singularities. And as noted above, we know of few problems that themselves to the exact determination of their Lyapunov exponents. We employed this property to accentuate the unexpected riddling of a basin of attraction in the presence of negative transverse Lyapunov exponents after the bifurcation. For noninvertible maps, other important elements contributing to the riddling include focal points and "bow ties." We also conjecture that the new basin has positive measure in the plane. Riddling is a complicated phenomenon and by no means fully understood and clearly deserving of additional study.

Accomplishments:

1. L. Billings, & J. H. Curry, "On noninvertible mappings of the plane: Eruptions," *Chaos* Vol. 6, pp108-120, 1996.

2. L. Billings, & J. H. Curry, "Bifurcations in a Class of Noninvertible Mappings of the Plane," Proceedings of the CESA' 96 IMACS Multiconference: Computational Engineering in Systems Applications (Lille, France) Vol. 2 (1996), pp 625-629

3. L. Billings, J. H. Curry, & E. Phipps, " Lyapunov Exponents, Singularities, and a Riddling Bifurcation," submitted for publication.

Deep Water:

Computational and Effective Chaos

We have been investigating the computational simulations of certain nonlinear equations analyzable by the inverse scattering transform (IST). We use these equations as prototypes since they are physically interesting systems about whose solutions and properties we have concrete analytical understanding. On the other hand, for the periodic boundary value problems we are considering, the analytical solutions are extremely complicated. Computationally speaking, these equations provide a vehicle by which: i) computational schemes can be compared and ii) errors in the schemes can be detected. We have demonstrated that in certain circumstances computational chaos results. Since we are interested in the long time numerical integration of nonlinear systems, there is no existing theory of error analysis which describes the phenomena.

We have been aided by the fact that we have obtained exact IST based discrete analogues of many of the continuous systems which, in practice, have proven to be excellent computational schemes. IST based schemes are difference equations which easily allows a range of boundary conditions to be applied.

Many researchers investigate nonlinear systems by computational means. Our work is essentially the only studies of the computational chaos associated with physically interesting partial differential equations. The work involves both analysis and computation.

The paradigm is the focusing NLS equation with periodic boundary conditions: $$iu_t +u_{xx}+2|u|^2u=0,$$ $$u(x,t)=u(x+L,t).$$ As mentioned above, we have a corresponding integrable discrete scheme, $$iu_{nt}+\displaystyle\frac{u_{n+1}+u_{n-1}-2u_n}{h^2} +|u_n|^2(u_{n+1}+u_{n-1})=0,$$ where h is the mesh size. For focusing NLS, there is a class of initial conditions which are linearly unstable. By changing a parameter in the initial condition, we can excite a number of linearly unstable modes; we call the number of unstable modes M.

In our work we have shown that a) for small values of M, standard numerical schemes (non-IST based schemes) break down and computational chaos results from truncation errors; b) for large values of M, the numerical chaos can even be induced by roundoff errors; c) for even initial data the chaos is explained by the demonstration of continual but temporally irregular crossings of unperturbed homoclinic manifolds (i.e.\ crossing of the NLS homoclinic manifolds). These manifolds are complicated. They have 2^n ``sides" and allow for an extremely rich dynamical evolution; d) recently we have studied the case when the initial data is not even. In this latter situation, which is the generic case, the phase space is no longer foliated and we find that the solution to the perturbed NLS system can evolve from one ``side" of the homoclinic manifold to another without crossing an unperturbed homoclinic manifold. The chaos we have observed in the latter case is depicted by irregular and continual changes of the velocity of the underlying periodic waves. The case of even initial data is typified by the periodic waves being essentially standing waves (no left/right velocity).

We believe that the computational chaos we have observed is, in fact, a manifestation of an important physical effect which should be observable in laboratory experiments. The NLS equation is well-known to govern the modulation of water waves in moderate-deep water, and modulational instability in nonlinear optics. When the waves are exeited in a periodic manner, with small modulation, then the NLS equation with periodic initial data described above is the relevant equation. The Benjamin-Feir/modulational instability only says that there are M unstable modes in the linearized version of the NLS equation. The NLS equation governs the long time evolution of the instability process. As mentioned above, our results regarding NLS, based on the extensive numerical and associated analysis, show that there is a significant difference in the long time dynamics depending on whether one excites a small number of unstable modes M (e.g M=1,2) or a large number (e.g M=5,6). In the former case, the dynamical evolution is explainable (and repeatable) in the context of NLS theory. In the latter situation, where round off error induces numerical chaos, the NLS equation is itself highly unstable in much the same way as coupled pendula nearly in the ``up-position" are highly unstable. In the latter case, we believe that small physical perturbations should be capable of causing dynamical chaos in the evolution. In effect we are experimentally and dynamically close to the homoclinic manifold which induces the chaotic dynamics. We have discussed our results in detail with J. Hammack who is a well-known water wave experimentalist. He is planning on carrying out the experiments. We believe that these experiments will lead to interesting and important conclusions about the long time behavior of the Benjamin-Feir/modulational instabilities when the underlying waves are generated in a modulated periodic manner.

Accomplishments:

1. Numerical Chaos, Roundoff Errors and Homoclinic Manifolds, M.J. Ablowitz, C. Schober, and B.M. Herbst, Phys.\ Rev.\ Lett., 71 (1993) 2683-2686.

2. Homoclinic Manifolds and Numerical Chaos in the Nonlinear Schr\"{o}dinger Equation, M.J. Ablowitz and C. Schober, Mathematics and Computers in Simulation, 37 (1994) 249-264.

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4. Integrability, Computation and Applications, M.J. Ablowitz, S. Chakravarty and B.M. Herbst, Acta Applicande Mathematicae, 39 (1995) 5-37.

5. Numerical Simulation of Quasi-Periodic Solutions of the Sine-Gordon Equation, M.J. Ablowitz, B.M. Herbst and C.M. Schober, Physica D, 87, (1995) 37-47.

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8. The Nonlinear Schr\"odinger Equation: Asymmetric Perturbations, Traveling Waves and Chaotic Structures, M.J. Ablowitz, B.M. Herbst and C.M. Schober, (to be published, Mathematics and Computers in Simulation

9. On the Numerical Solution of the Sine-Gordon Equation II. Performance of Numerical Schemes, M.J. Ablowitz, B.M. Herbst and C.M. Schober, to be published, J. Comp. Phys.