THEORY AND DESIGN OF OPTIMUM FIR COMPACTION FILTERS‡

Ahmet Kirac, Student Member, IEEE, and P. P. Vaidyanathan, Fellow, IEEE

Dept. of Electrical Engineering, 136-93

California Institute of Technology

Pasadena, CA 91125

Contact author: P. P. Vaidyanathan, ppvnath@sys.caltech.edu

Fax : (818) 795 8649 Phone : (818) 395 4681

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Abstract. Energy compaction filters have attracted considerable attention due in part, to the fact that they are the building blocks of optimal orthonormal (paraunitary) filter banks. In this paper we introduce some new design techniques for optimum $M-$channel FIR compaction filters for a given input power spectrum. Some properties of the optimum FIR compaction filters and the corresponding maximum compaction gains are also derived. For the design part, a modification of the well-known linear programming technique is considered. We also consider multistage (IFIR) designs of compaction filters. A new, efficient design method called the **window method** is then introduced. The method generates $M-$channel FIR compaction filters for any given power spectrum. Although it is suboptimal, no optimization tools of any kind are involved and the algorithm terminates in a finite number of elementary steps. As the filter order increases, the window method produces compaction gains that are very close to the optimal ones. We give a necessary condition for a compaction filter to be optimum and provide some bounds on the maximum compaction gains. Finally we propose an analyical method for the two-channel case which finds the optimum FIR compaction filters for a class of random processes.

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I. INTRODUCTION

Energy compaction filters have attracted a great deal of attention due in part, to the fact that they are the building blocks of optimal orthonormal (paraunitary) filter banks [1, 2, 3, 4]. This connection is made for the case where the filters are allowed to be ideal. More recently a number of authors have considered the FIR energy compaction problem for the two-channel case [1, 5, 6, 7, 8, 9, 10] and for the $M-$channel case [11]. The FIR compaction filters have applications in many areas including data compression, signal analysis, signal modeling, and data transmission [1, 3, 12]. An $M-$channel FIR compaction filter can be considered as one of M filters of a maximally decimated $M-$channel FIR orthonormal filter bank. Hence, one can view the problem as compaction of most of the signal energy into one channel of an orthonormal filter bank.

In this paper we consider some new design techniques for optimum FIR compaction filters. We give analytical solutions in the two-channel case for a class of random processes. Some properties of optimum FIR compaction filters and corresponding gains are also considered. Detailed outline is provided in Sec. 1.4.

1.1. Notations and Terminology

1. Bold faced upper and lower case letters represent matrices and vectors respectively.

2. $X(z)$ and $X(e^{j\omega})$ stand for $z-$transform and Fourier transform, respectively of a sequence $x(n)$. The notation $\tilde{X}(z)$ denotes the $z-$transform of $x^*(-n)$ where $*$ stands for complex conjugation. If $x(n)$ is real, then $\tilde{X}(z) = X(z^{-1})$. Notice that $\tilde{X}(z) = X^*(1/z^*)$, and the Fourier transform of $x^*(-n)$ is $X^*(e^{j\omega})$.

3. The symbols $\downarrow M$ and $\uparrow M$ denote $M-$fold decimation and expansion as defined in [13]. The notation $X(z)|_{\downarrow M}$ denotes the $z-$transform of the decimated sequence $x(Mn)$.

4. **Nyquist(M) property.** A sequence $x(n)$ is said to be Nyquist(M) if $x(Mn) = \delta(n)$ or equivalently $X(z)|_{\downarrow M} = 1$. This can be rewritten in the form [13]:
$$\sum_{k=0}^{M-1} X(zW^k) = M \tag{1}$$
where $W = e^{-j2\pi/M}$. In an $M-$channel orthonormal filter bank $\{H_k(z)\}$, each $|H_k(e^{j\omega})|^2$ is Nyquist(M). So the integer M is often referred to as the number of channels.

5. The notation $x_L(n)$ stands for a periodic sequence with periodicity L. If there is a reference to a finite sequence $x(n)$ as well, then it is to be understood that $x_L(n)$ is the periodical expansion of $x(n)$ with period L, i.e., $x_L(n) = \sum_{i=-\infty}^{\infty} x(n + Li)$. The Fourier series coefficients of $x_L(n)$ is denoted by $X_L(k)$.

6. For L a multiple of M, a periodic sequence $x_L(n)$ is said to be Nyquist(M) if
$$x_L(Mn) = \delta_K(n) \triangleq \sum_{i=-\infty}^{\infty} \delta(n + Ki) \tag{2}$$

where $K = L/M$. The equivalent form of this property in terms of the Fourier series coefficients $X_L(k)$ is

$$\sum_{i=0}^{M-1} X_L(k+iK) = M, \quad k = 0, \ldots, K-1 \tag{3}$$

(see Lemma 2 in Sec. III).

7. **Positive definite sequences.** Let a sequence $\{x(n), n = 0, \ldots, N\}$ be given and let $\mathbf{P}$ be the Hermitian Toeplitz matrix whose first row is $[x(0)\ x(1)\ \ldots\ x(N)]$. The sequence $\{x(n), n = 0, \ldots, N\}$ is called positive (negative) definite or semidefinite if $\mathbf{P}$ is positive (negative) definite or semidefinite respectively. Let $[a(0)\ a(1)\ \ldots\ a(N)]^T$ denote the corresponding eigenvector. Then the filter $A(z) = \sum_{n=0}^{N} a(n)z^{-n}$ will be called a **maximal eigenfilter** of $\mathbf{P}$. If we consider the minimum eigenvalue instead, we will call the corresponding filter a **minimal eigenfilter** of $\mathbf{P}$.

1.2. The FIR energy compaction problem

A filter $H(z)$ of order N will be called a valid **compaction filter** for the pair (M, N) if the product $G(z) = H(z)\tilde{H}(z)$ is Nyquist(M). We will refer to $G(z)$ as the **product filter** corresponding to $H(z)$. Conversely, $G(z)$ is the product filter of a valid compaction filter for the pair (M, N) if it is of symmetric order N, that is $G(z) = \sum_{n=-N}^{N} g(n)z^{-n}$ and it satisfies the following conditions:

$$g(Mn) = \delta(n) \quad \text{and} \quad G(e^{j\omega}) \geq 0. \tag{4}$$

Now consider Fig. 1 where $H(z)$ is an FIR filter of order N applied to an input $x(n)$ which is a zero-mean WSS random process with the power spectral density $S_{xx}(e^{j\omega})$. The output of the filter is decimated by M to produce $y(n)$. The optimum FIR energy compaction problem is to find a valid compaction filter $H(z)$ for the pair (M, N) such that the variance σ_y^2 of $y(n)$ is maximized. Since decimation of a WSS process does not alter its variance, we have

$$\sigma_y^2 = \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi} \tag{5}$$

We will consistently use the notation $G(e^{j\omega}) = |H(e^{j\omega})|^2$. We define the **compaction gain** as

$$G_{comp}(M, N) = \frac{\sigma_y^2}{\sigma_x^2} = \frac{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi}}{\int_{-\pi}^{\pi} S_{xx}(e^{j\omega}) \frac{d\omega}{2\pi}} \tag{6}$$

where σ_x^2 is the variance of $x(n)$. The aim therefore is to maximize the compaction gain.

Special cases. There are two extreme cases worth examining: the case where $N < M$ and the case where ideal filters are allowed. In the first case, the condition $g(Mn) = \delta(n)$ is the same as $g(0) = 1$. This is equivalent to saying that $H(e^{j\omega})$ has unit energy. Hence, by Rayleigh's principle [14], the best filter is the **maximal eigenfilter** of the Hermitian Toeplitz matrix $\mathbf{P}$ whose first row is $[r(0)\ r(1)\ \ldots\ r(N)]$ where $r(n) = r^*(-n)$ is the autocorrelation sequence of $x(n)$. The corresponding compaction gain is the maximum eigenvalue of $\mathbf{P}$ and will be called the **KLT gain**. The second case has been studied in [4] where the author describes a constructive solution that involves comparison of a set of values of the power spectral density at each frequency ω and assignment of certain values to $H(e^{j\omega})$ accordingly.

In the FIR energy compaction problem, we do not have the flexibility of assigning values to $H(e^{j\omega})$ independently for each ω. This is because $H(e^{j\omega})$ is determined by its $N+1$ frequency samples. For $N > M$, the problem is not an eigenfilter problem either, as the condition $g(Mn) = \delta(n)$ implies more than the simple unit-energy condition. In Sec. III we will introduce a suboptimal method called the window method for design of such filters. Interestingly enough, this design method involves two stages that can be associated with the above two extreme cases. While the method is not optimal, it produces compaction gains very close to the optimum ones especially for high filter orders.

1.3. Previous work

The major motivation for studying compaction filters is their connection to optimal subband coding problem which has been widely studied [1, 2, 4, 15, 16, 17, 18]. When the ideal filters are allowed, the optimization of a maximally decimated $M-$channel orthonormal filter bank for a given input statistics has been solved [2, 3, 4], and the biorthogonal case is addressed in [18]. Optimization of FIR orthonormal filter banks is analytically more difficult, and some numerical methods have been developed for the two-channel case [1, 5, 9, 10] and for the $M-$channel case [11]. Lattice parametrization is utilized in [1] and the parameters are iteratively optimized. An optimum $M-$channel FIR compaction filter is designed by linear programming in [11] and then an orthonormal filter bank is constructed in some optimal sense using the remaining degrees of freedom. The FIR energy compaction problem has been considered by several authors. There are different approaches to the problem summarized below.

1. **Eigenfilter method.** In [19], the authors design one filter of an $M-$channel orthonormal filter bank using the so-called eigenfilter method. The objective in their design is to have a good frequency response. However, one can modify the technique to incorporate the input statistics. This can be done by using the psd $S_{xx}(e^{j\omega})$ as a weighting function in the optimization. The paper also discusses how to design a good orthonormal filter bank using the remaining degrees of freedom. In [11] the authors show how to use this idea for the statistical optimization of orthonormal filter banks.

2. **Linear programming.** If one considers the problem of finding the product filter $G(z)$ corresponding to the optimum compaction filter $H(z)$, then it can be formulated as a linear programming problem. This has been done recently by Moulin [9, 11]. For a brief description of the formulation, the reader is referred to Sec. 2.1. The compaction filter $H(z)$ is obtained from $G(z)$ by spectral factorization.

3. **Quadratic constrained optimization method.** If one formulates the FIR energy compaction problem in terms of the compaction filter impulse response $h(n)$, rather than the product filter $g(n)$, then it becomes a quadratic constrained optimization problem with a quadratic objective. In this method no

spectral factorization is involved. This method has been applied for the two-channel case by Caglar et al. [5]. The general $M-$channel case has been considered in a completely different context in the communications literature by Chevillat and Ungerboeck [12] where an explicit algorithm flow-chart is provided. In [20], a filter with good frequency response is designed using a similar approach. This is a valid compaction filter according to our definition (Sec. 1.2), and the extension to the optimal compaction filter design is straightforward.

4. **Analytical methods.** Aas et al. [21] worked on a closely related problem for the two-channel case even though they have not explicitly considered the energy compaction problem. These authors have optimized a real-coefficient FIR filter $H(z)$ such that $H(z)H(z^{-1})$ is Nyquist(2) (that is, it is a valid real-coefficient compaction filter), and
$$\int_{-\pi/2}^{\pi/2} |H(e^{j\omega})|^2 \frac{d\omega}{2\pi} \tag{7}$$
is maximum. The elegance of the method in [21] lies in the fact that no iterative numerical optimization is involved. Based on the fundamentals of Gaussian quadrature, the authors were able to obtain an analytical method to identify the unit-circle zeros of $H(z)$ which uniquely determine it. In our paper, this method will be referred to as **analytical method**. In Sec. V we present extensions of the analytical method. While the original method primarily addresses conventional lowpass filter design, we will show how to adapt the idea for the case of FIR compaction filter design for a given power spectrum. Interestingly enough, we shall show that the analytical method is related to the well-known line-spectral theory in signal processing society [22]. We also mention here the work of Usevitch and Orchard [8] where an analytical expression for the compaction gain is presented for $N = 3$ and $M = 2$.

1.4. Summary of the new results and outline of the paper

1. **Modification of linear programming method.** In Sec. 2.2 we give a simple procedure to ensure that linear programming solutions guarantee the nonnegativity requirement on $G(e^{j\omega})$. Since linear programming normally guarantees the nonnegativity only at a selected number of frequencies, it is important to make a modification to be able to obtain a solution so that $G(e^{j\omega}) \geq 0, \ \forall\omega$.

2. **Multistage (IFIR) extensions.** In Sec. 2.3 we present the so-called IFIR extension of the linear programming technique to design FIR compaction filters in two stages. That is, the filter $H(z)$ is represented in the form $H_0(z)H_1(z^{M_0})$ where M_0 is a factor of M and the much smaller filters $H_0(z)$ and $H_1(z)$ are optimized. While theoretically suboptimal, the system $H_0(z)H_1(z^{M_0})$ offers a much higher filter order for a fixed number of coefficients. This makes the design as well as implementation very efficient, similar to the case of IFIR filters in filter design practice [13, 23]. In Sec. 2.4 we consider a slight variation of the configuration where the Nyquist(M) property is theoretically assured and any compaction filter design technique can be used to design individual filters. The multistage method is also motivated by the fact

that in the case of ideal filters (i.e., when there is no order constraint), this approach does not result in a loss of compaction gain as proved in [24].

3. **Window method.** A new and efficient design technique is introduced in Sec. III, called the **window method** for the design of FIR compaction filters. For a given input power spectrum, this method yields a suboptimal solution which is very close to the optimal solution especially for large filter orders. The window method has the advantage that no optimization tools or iterative numerical techniques are necessary. The solution is generated in a finite number of elementary steps, the crucial step being a simple comparison operation on a **finite frequency grid**. Combined with the fact that the solution is close to optimal, the method offers an attractive alternative to linear programming. In fact, we will show in Sec. 3.2 that there is a connection between the two methods.

4. **Properties of optimum FIR compaction filters and gains.** In Sec. IV we examine some properties of optimum FIR compaction filters and corresponding maximum compaction gains. For example, we give a necessary condition for a compaction filter to be optimum for a given power spectrum. It is shown that an optimum FIR compaction filter continues to be optimum if the power spectrum is modified in certain ways. The behaviour of the optimum compaction gain, denoted by $G_{opt}(M,N)$, as a function of M and N for a given power spectrum is investigated. We give some useful lower and upper bounds on $G_{opt}(M,N)$.

5. **Analytical method.** We consider the FIR energy compaction problem analytically for the two-channel case (Sec. V). This is possible for a certain class of random processes only. It can be regarded as a generalization of the technique in [21]. For the algorithm to be applicable, a certain sequence derived from the odd autocorrelation sequence has to be a positive or negative definite sequence. We characterize classes of random processes for which this is the case and therefore the method is applicable. As examples, we give analytical solutions for MA(1) and AR(1) processes.

1.5. Connection between energy compaction filters and optimum orthonormal filter banks

For the two-channel orthonormal subband coder shown in Fig. 2, the coding gain expression takes the form [4]

$$G = \frac{\sigma_x^2}{\sqrt{\sigma_{x_0}^2\sigma_{x_1}^2}} = \frac{\sigma_x^2}{\sqrt{\sigma_{x_0}^2(2\sigma_x^2 - \sigma_{x_0}^2)}} \tag{8}$$

where the second line follows from the condition $2\sigma_x^2 = \sigma_{x_0}^2 + \sigma_{x_1}^2$ imposed by orthonormality. Here $\sigma_{x_i}^2$ is the variance at the output of $H_i(z)$. Maximizing the coding gain is therefore equivalent to maximizing (or minimizing) $\sigma_{x_0}^2$ under the Nyquist(2) constraint

$$H_0(z)\tilde{H}_0(z)\Big|_{\downarrow 2} = 1. \tag{9}$$

Notice that the above argument holds even if $H_0(z)$ is constrained to be a finite order (IIR or FIR) transfer function. In the $M-$channel case, the coding gain is still the AM/GM ratio of subband variances, but it cannot be expressed in terms of a single subband variance $\sigma^2_{x_0}$. It can however be shown that if the filter orders are unconstrained, then the analysis filters are optimal compaction filters for appropriate power spectra derived from the input [4]. For the finite order case and arbitrary M, optimal compaction filters are still of interest beacuse of the large coding gain obtainable from them.

II. LINEAR PROGRAMMING METHOD AND IFIR DESIGNS

The use of linear programming method in compaction filter design was proposed in [9], and is reviewed in Sec. 2.1. The technique yields a solution $G(e^{j\omega})$ that is Nyquist(M) and is nonnegative on a certain discrete grid of specified frequencies. Note that after finding $G(z)$, one needs to spectrally factorize it to find the compaction filter $H(z)$. This step will succeed only if $G(e^{j\omega}) \geq 0,\ \forall\omega$. In Sec. 2.2, we propose a simple procedure to guarantee the nonnegativity of the resulting solution for all frequencies. We then give extensions of the technique for the case of multistage (IFIR) compaction filters in Sec. 2.3. We will also consider a special IFIR configuration in Sec. 2.4 which has certain advantages.

2.1. Review of the linear programming method

Assume that the input process $x(n)$ is real. The output variance can be written as

$$\sigma_y^2 = r(0) + 2\sum_{n=1}^{N} g(n)r(n) \tag{10}$$

Let $\mathbf{g}_d$ and $\mathbf{r}_d$ be the vectors formed by the nonzero components of $g(n)$ and $r(n)$ for $n = 1, \ldots N$. That is,

$$\mathbf{g}_d = [g(1)\ g(2)\ \ldots\ g(M-1)\ g(M+1)\ \ldots\ g(N)]^T, \quad \mathbf{r}_d = [r(1)\ r(2)\ \ldots\ r(M-1)\ r(M+1)\ \ldots\ r(N)]^T \tag{11}$$

Then (10) can be written as $\sigma_y^2 = r(0) + 2\mathbf{r}_d^T\mathbf{g}_d$. This incorporates the Nyquist(M) condition but not the nonnegativity constraint in (4). Let $\mathbf{c}_d(\omega) \triangleq [\cos(\omega)\ \cos(2\omega)\ \ldots\ \cos((M-1)\omega)\ \cos((M+1)\omega)\ \ldots\ \cos(N\omega)]^T$. Then $G(e^{j\omega}) = 1 + 2\mathbf{c}_d^T(\omega)\mathbf{g}_d$. Hence the problem is equivalent to the following:

$$\text{maximize } \mathbf{r}_d^T\mathbf{g}_d \quad \text{subject to} \quad \mathbf{c}_d^T(\omega)\mathbf{g}_d \geq -0.5,\ \forall\omega \in [0, \pi] \tag{12}$$

This type of problem is typically classified as semiinfinite linear programming [9]. By discretizing the frequency, one reduces this to a well known standard linear programming problem. This discretization however needs to be done with care. The resulting $G(e^{j\omega})$ can go negative in between the discrete frequencies. To avoid this, one can put some tolerance in the inequality (12). Our experience however was that even with a high tolerance, the resulting $G(z)$ had single (rather than double) unit-circle zeros and $G(e^{j\omega})$ was not nonnegative. One way to overcome the difficulty is to numerically determine the zeros of $G(e^{j\omega})$ and to merge the pairs of zeros that are very close to the unit-circle into double unit-circle zeros. This requires determining the roots of $G(z)$ that are in the vicinity of the unit-circle. This can be done by looking at the Fourier transform $G(e^{j\omega})$ using a large number of frequency points. Yet another way is to "lift" $G(e^{j\omega})$ by increasing $g(0)$ relative to the other coefficients (since $g(0)$ has to be 1, in effect we scale $g(n)$ for $n \neq 0$ by a constant that is slightly less than 1). In the next section we propose a third technique to overcome the difficulty without having to locate any zeros or the minimum of $G(e^{j\omega})$.

2.2. Windowing of the linear programming solution

Consider the periodical expansion $g_L(n)$ of the linear programming solution where L is the number of discrete uniform frequencies $\{\omega_k\}$ used in the design process. Assume that $L > 2N$. Linear programming assures that $G(e^{j\omega})$ is nonnegative at the frequencies $\{\omega_k\}$. Hence the Fourier series coefficients $G_L(k)$ of $g_L(n)$ are nonnegative. Now if we consider the product

$$w(n)g_L(n) \tag{13}$$

where $w(n)$ is a symmetric window of order $K < L - N$ (length $2K + 1$), the resulting Fourier transform is nonnegative provided $w(n)$ has nonnegative Fourier transform $W(e^{j\omega})$. This is depicted in Fig. 3. The reason follows from the fact that the Fourier transform of $w(n)g_L(n)$ is a weighted sum of shifted versions of $W(e^{j\omega})$ with nonnegative weights. For maximum compaction gain, the symmetric order of $w(n)$ is chosen to be maximum, namely $K = L - N - 1$. Note that when $L = 2N$, we have $g_L(N) = 2g(N)$. One can use a fixed window like a triangular window as depicted in the figure and get a satisfactory compaction gain. However one can always optimize the window. The optimization of the window given the periodic sequence $g_L(n)$ is discussed in Sec. III where we show that the optimum $w(n)$ is the product filter of the maximal eigenfilter of the $K \times K$ Hermitian Toeplitz matrix formed by the product $r(n)g_L(n)$. Since the symmetric window order K is very high in linear programming designs, we suggest to use a triangular window rather than optimizing the window. The performance loss is negligibly small.

Example 1. Let the input be psd be as in Fig. 4 and let $N = 65$ and $M = 2$. In the same figure, we plot the magnitude square $|H(e^{j\omega})|^2$ of the compaction filter $H(z)$ designed by the linear programming method. The number of frequencies used in the design process was $L = 512$. We have used triangular window of symmetric order $K = L - N - 1 = 446$ and found that the resulting compaction gain is $G_{comp}(2, 65) = 1.8698$. If we optimize the window the compaction gain becomes 1.8744. One can verify that the compaction gain of the ideal (infinite order) compaction filter is 1.8754.

2.3. IFIR designs using linear programming

In the design of narrowband FIR lowpass filters for conventional applications, it is possible to decompose the transfer function $H(z)$ into the form $H(z) = H_0(z)H_1(z^{M_0})$, where $H_0(z)$ and $H_1(z)$ have significantly smaller lengths than $H(z)$. While $H(z)$ is theoretically suboptimal compared to, e.g., an equiripple solution, it has the advantage of actually requiring fewer active multiplier elements (because of the zero-valued coefficients in $H_1(z^{M_0})$). This technique, called the **IFIR technique** [23] has also been extended to bandpass and multiband filters in the past, and is in fact related to multistage design of interpolation filters [25]. The method offers significant economy both in terms of design time and implementation complexity. For the case of compaction filter design, a similar decomposition proves to be valuable as we shall now demonstrate.

Assume $M = M_0 M_1$. Consider Fig. 5(a) where $H_0(z)$ and $H_1(z)$ are such that the equivalent filter in Fig. 5(b) $H(z) = H_0(z)H_1(z^{M_0})$ is a valid compaction filter for the pair (M, N). The problem is to optimize the pair of filters $H_0(z)$ and $H_1(z)$ for maximum compaction gain. If we fix one of the filters, it will be shown that the design of the other can be formulated as a linear programming problem. We will describe the details of how to find $H_1(z)$ for a fixed $H_0(z)$ and vice versa, in an iterative manner.

Let $G_0(z) = H_0(z)H_0(z^{-1})$, $G_1(z) = H_1(z)H_1(z^{-1})$, and $G(z) = H(z)H(z^{-1})$ with impulse responses $g_0(n)$, $g_1(n)$, and $g(n)$ respectively. Denote the orders of $H_0(z)$, $H_1(z)$, and $H(z)$ by N_0, N_1, and N respectively. Note that $N = M_0 N_1 + N_0$. Define

$$\mathbf{g}_0 = [g_0(0)\ g_0(1)\ \ldots\ g_0(N_0)]^T, \quad \mathbf{g}_1 = [g_1(0)\ g_1(1)\ \ldots\ g_1(N_1)]^T, \quad \mathbf{g} = [g(0)\ g(1)\ \ldots\ g(N)]^T. \tag{14}$$

Optimization of $H_1(z)$ for a given $H_0(z)$: We have $G(z) = G_0(z)G_1(z^{M_0})$. Let $\mathbf{G}_0$ be the $(2N+1)\times(2M_0N_1+1)$ convolution matrix formed by $g_0(n)$. Taking into account the symmetries and the fact that $G_1(z^{M_0})$ has nonzero components only for multiples of M_0, we can write $\mathbf{g} = \mathbf{A}_0\mathbf{g}_1$, where $\mathbf{A}_0$ is an $(N+1)\times(N_1+1)$ matrix that is obtained from $\mathbf{G}_0$. Now, the Nyquist(M) constraint requires that if we decimate $\mathbf{g}$ by M we should get $\mathbf{e}_0 = [1\ 0\ \ldots\ 0]^T$. Let $\mathbf{B}_0$ denote the matrix that is obtained by taking every Mth row of $\mathbf{A}_0$. Then we should have $\mathbf{B}_0\mathbf{g}_1 = \mathbf{e}_0$. To force the nonnegativity constraint on $G_1(e^{j\omega})$, let $\mathbf{c}_0(\omega) \triangleq [1\ 2\cos(\omega)\ 2\cos(2\omega)\ \ldots\ 2\cos(N_1\omega)]^T$. Then the constraint $G_1(e^{j\omega}) \geq 0$ becomes $\mathbf{c}_0^T(\omega)\mathbf{g}_1 \geq 0,\ \forall\omega\in[0,\pi]$. If $\mathbf{r} = [r(0)\ 2r(1)\ \ldots\ 2r(N)]^T$, the objective is to maximize $\mathbf{r}^T\mathbf{g} = \mathbf{r}^T\mathbf{A}_0\mathbf{g}_1$. Hence we have reduced the problem to the following:

$$\text{maximize}\quad \mathbf{r}_0^T\mathbf{g}_1, \quad \text{subject to}\quad \mathbf{B}_0\mathbf{g}_1 = \mathbf{e}_0, \quad \text{and}\quad \mathbf{c}_0^T(\omega)\mathbf{g}_1 \geq 0,\ \forall\omega\in[0,\pi] \tag{15}$$

where $\mathbf{r}_0 = \mathbf{A}_0^T\mathbf{r}$. Hence a standard linear programming algorithm can be applied once a set of freqencies is chosen for the inequality constraint.

Optimization of $H_0(z)$ for a given $H_1(z)$: Similarly, one can reduce the problem of finding the best $H_0(z)$ for a given $H_1(z)$ to the following linear programming problem:

$$\text{maximize}\quad \mathbf{r}_1^T\mathbf{g}_0, \quad \text{subject to}\quad \mathbf{B}_1\mathbf{g}_0 = \mathbf{e}_0, \quad \text{and}\quad \mathbf{c}_1^T(\omega)\mathbf{g}_0 \geq 0,\ \forall\omega\in[0,\pi] \tag{16}$$

where $\mathbf{r}_1 = \mathbf{A}_1^T\mathbf{r}$, $\mathbf{c}_1(\omega) = [1\ 2\cos(\omega)\ 2\cos(2\omega)\ \ldots\ 2\cos(N_0\omega)]^T$. The $(N+1)\times(N_0+1)$ matrix $\mathbf{A}_1$ is obtained from the $(2N+1)\times(2N_0+1)$ convolution matrix formed by $g_1(n)$ by taking the symmetries into account and the matrix $\mathbf{B}_1$ is obtained by taking every Mth row of $\mathbf{A}_1$.

One can iterate between the above two optimization steps until there is no significant change in the compaction gain. The initial choice of $g_0(n)$ can significantly affect the resulting compaction gain. According to our design experience if $g_0(n)$ is chosen to be a triangular sequence, the compaction gain at the end of the iteration is very good. The filters $g_0(n)$ and $g_1(n)$ which result from the iteration should spectrally be factorized to identify $H_0(z)$ and $H_1(z)$. This step will be successful only if the solutions are such that $G_0(e^{j\omega}) \geq 0$ and $G_1(e^{j\omega}) \geq 0$ for all ω. If this is not the case, we can force it by use of windowing on $g_0(n)$ and $g_1(n)$ as described in Sec. 2.2. If this is done then the product filter $G_0(z)G_1(z^{M_0})$ will not be exactly Nyquist(M). In the next subsection we show how to overcome this problem.

Example 2. Let us design IFIR compaction filters for the pair $(M, N) = (36, 65)$, and for the input process whose psd is given in Fig. 4. Let $M_0 = 9$ and $M_1 = 4$, and let $N_0 = 11$ so that $N_1 = 6$. The number of frequencies used in the designs is $L = 1024$. Starting with a triangular sequence for $g_0(n)$, the algorithm converges in a few steps. We windowed the resulting solutions $g_0(n)$ and $g_1(n)$ with triangular windows of symmetric orders $L - N_0 - 1$ and $L - N_1 - 1$ respectively. The final product filter was not exacly Nyquist(M) because it was found that $g(36) \simeq -0.0018 \neq 0$. The final compaction gain was 5.1444. If we design a compaction filter of order 18 directly (i.e., not using IFIR technique), the compaction gain is 4.4225. This corresponds to a compaction filter with the same number of active multipliers, namely 19. If we design a compaction filter of order 65 directly (66 active multipliers), then the resulting compaction gain is 7.2337.

2.4. A Particular IFIR configuration

In Fig. 5, if $G_0(z)$ is Nyquist(M_0) and $G_1(z)$ is Nyquist(M_1), it can be verified that $G(z)$ given by $G_0(z)G_1(z^{M_0})$ is Nyquist(M). Now, let us fix $H_0(z)$ to be a valid compaction filter for the pair (N_0, M_0). Referring to Fig. 6(a), the best $H_1(z)$ is the optimum compaction filter for (N_1, M_1), and for the input $x_0(n)$ which has the psd $S_{x_0x_0}(z) = \left(G_0(z)S_{xx}(z)\right)\Big|_{\downarrow M_0}$. Similarly, if $H_1(z)$ is a fixed compaction filter for the pair (N_1, M_1), then we can redraw the configuration as in Fig. 6(b) and therefore the best $H_0(z)$ is the optimum compaction filter for (N_0, M_0), and for the input $x_1(n)$ which has the psd $S_{x_1x_1}(z) = G_1(z^{M_0})S_{xx}(z)$. One can design the compaction filters $H_0(z)$ and $H_1(z)$ iteratively using any of the known techniques. Hence, one can use the linear programming technique as well as any other technique like the window method to be introduced in Sec. III. Also note that if the ideal filters are allowed, this multistage configuration has no loss of generality as shown in [24].

Example 3. Let the setup be the same as in Example 2. We have designed the compaction filters $H_0(z)$ and $H_1(z)$ iteratively using the standard linear programming procedure as in Example 1. We have started with $H_1(z) = 1$. The first compaction filter $H_0(z)$ is therefore the optimal compaction filter for the pair $(M_0, N_0) = (9, 11)$ for the original autocorrelation sequence. We have windowed the final product filters as we did in Example 1 to guarantee the nonnegativity. The resulting overall compaction gain is 4.9432. This is slightly smaller than the overall compaction gain 5.1444 in Example 1. However, the resulting overall filter here is exactly Nyquist(M) unlike the case of Example 2.

III. WINDOW METHOD

In this section we will describe a new method to design compaction filters for general M. The input process might be real or complex and its psd may have any shape. The technique is quite simple while the resulting compaction gains are very close to the optimum ones especially for relatively high filter orders.

3.1. Derivation of the window method

The idea behind the method is to represent the impulse response of the product filter $G(z)$ in the form

$$g(n) = w(n) f_L(n), \tag{17}$$

where $w(n)$ and $f_L(n)$ are conjugate symmetric (i.e., $w(n) = w^*(-n)$, $f_L(n) = f_L^*(-n)$), and $w(0) = f_L(0) = 1$ (see Fig. 7). The window $w(n)$ has the same length as $g(n)$, namely $2N+1$ and the sequence $f_L(n)$ is periodic with period $L = KM \geq 2N$ for some K. Evidently, only one period of $f_L(n)$ matters in (17). Define the Fourier series coefficients of $f_L(n)$ as

$$F_L(k) = \sum_{n=0}^{L-1} f_L(n) W_L^{kn}, \quad W_L = e^{-j2\pi/L} \tag{18}$$

This is periodic with the same period L (the first period $\{F_L(k),\ k = 0, \ldots, L-1\}$ being just the DFT of the sequence $\{f_L(n), n = 0, \ldots, L-1\}$). We have the following observation:

Lemma 1. Consider the representation (17) for $g(n)$ where $w(n)$ and $f_L(n)$ are as explained above. If the Fourier transform $W(e^{j\omega})$ of $w(n)$ is nonnegative for all ω, if the Fourier series coefficients $F_L(k)$ of $f_L(n)$ are nonnegative for all k, and if $f_L(n)$ is Nyquist(M) then $G(z)$ is product filter of a valid compaction filter for the pair (M, N). That is, $g(Mn) = \delta(n)$ and $G(e^{j\omega}) \geq 0$.

Proof. It is readily verified that $G(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} F_L(k) W(e^{j(\omega - \frac{2\pi}{L}k)})$. Since $F_L(k) \geq 0$ and $W(e^{j\omega}) \geq 0$, it follows that $G(e^{j\omega}) \geq 0$. If $f_L(n)$ is Nyquist(M) then so is $g(n)$ because $L \geq 2N$. ■

The choice of L will be discussed later. Assume the conditions of the lemma hold so that $G(z)$ is product filter of a valid compaction filter. If $w(n)$ is fixed, what is the best $f_L(n)$ that maximizes the compaction gain? To answer the question we first note the following:

Lemma 2. A periodic sequence $f_L(n)$ with period $L = KM$ is Nyquist(M), that is, $f_L(Mn) = \delta_K(n)$, if and only if its Fourier series coefficients $F_L(k)$ satisfy the following:

$$\sum_{i=0}^{M-1} F_L(k + iK) = M, \quad k = 0, \ldots, K-1 \tag{19}$$

A proof of this can be found in Appendix A.

To obtain the best $f_L(n)$ let us write the objective (5) in terms of $w(n)$ and $f_L(n)$:

$$\sigma_y^2 = \sum_{n=-N}^{N} g(n) r^*(n) = \sum_{n=-N}^{N} w(n) f_L(n) r^*(n) \tag{20}$$

Let $\hat{r}(n) = w^*(n) r(n)$ and let $\hat{S}_L(k)$ be the Fourier series coefficients of its periodic expansion $\hat{r}_L(n)$. For simplicity assume that $L > 2N$. Then the objective can be written as

$$\sigma_y^2 = \sum_{n=0}^{L-1} f_L(n) \hat{r}_L^*(n) = \frac{1}{L}\sum_{k=0}^{L-1} F_L(k) \hat{S}_L(k) \tag{21}$$

Notice that both $F_L(k)$ and $S_L(k)$ are real. Now to incorporate the Nyquist(M) constraint we write the preceding as

$$\frac{1}{L}\sum_{k=0}^{K-1}\sum_{i=0}^{M-1} F_L(k+iK)\hat{S}_L(k+iK) \tag{22}$$

For a fixed k, let $S_L(k+i_0K)$ be the maximum of the set $\{\hat{S}_L(k+iK),\ i=0,\ldots,M-1\}$. Then by (19), and noting that $F_L(k)\geq 0$, the objective (22) is maximized if we assign

$$F_L(k+i_0K)=M, \text{ and } \quad F_L(k+i_lK)=0,\ l=1,\ldots,M-1. \tag{23}$$

The procedure is illustrated in Fig. 8. By repeating the process for each $k=0,\ldots,K-1$, the Fourier series coefficients of the best $f_L(n)$ is determined. The sequence $f_L(n)$ can now be calculated by the inverse relation:

$$f_L(n)=\frac{1}{L}\sum_{k=0}^{L-1}F_L(k)W_L^{-nk}. \tag{24}$$

Summary of the window algorithm

Assume a window $w(n)$ of the same symmetric order as $g(n)$ with nonnegative Fourier transform has been chosen. Let $L=KM>2N$. Then the algorithm steps are

1. Calculate $\hat{S}_L(k)$, the DFT coefficients of $\hat{r}_L(n), n=0,\ldots,L-1$, where $\hat{r}_L(n)$ is the periodical expansion of $\hat{r}(n)=w^*(n)r(n)$.
2. For each $k=0,\ldots,K-1$, determine the index i_0 for which $\hat{S}_L(k+i_0K)$ is maximum, and assign $F_L(k+i_0K)=M$ and $F_L(k+i_lK)=0,\ l=1,\ldots,M-1$.
3. Calculate $f_L(n)$ by the inverse relation (24). We need only to determine $f_L(n)$ for $n=1,\ldots,N$.
4. Form the product filter impulse response $g(n)=w(n)f_L(n)$ and spectrally factorize $G(z)$ to find the compaction filter $H(z)$.

Real case. Before proceeding to a design example, consider the case of real inputs. In this case, the above algorithm can be modified to produce real-coefficient compaction filters. Let us consider the set of values

$$\{\hat{S}_L(k+iK),\ i=0,\ldots,M-1\} \tag{25}$$

for each $k=0,\ldots,K-1$. Since $\hat{S}_L(k)=\hat{S}_L(L-k)$ if the process is real, the above set is equivalent to

$$\{\hat{S}_L(L-k-iK)=\hat{S}_L(K-k+K(M-1-i)),\ i=0,\ldots,M-1\} \tag{26}$$

Hence in the comparison, we need to consider only $k=0,\ldots,P$ where $P=\frac{K}{2}$ if K is even, and $P=\frac{K-1}{2}$ if it is odd. Let $\hat{S}_L(k+i_0K)$ be the maximum of the set (25) for each $k=0,\ldots,P$. Because of the symmetry requirement, we need to be careful in the assignments. There are two cases to consider:

1. The index $L-k-i_0K$ is among the set $\{k+iK,\ i=0,\ldots,M-1\}$,

2. It is not.

The first case happens if and only if $2k \bmod K = 0$. This happens if $k = 0$ or $k = \frac{K}{2}$. We assign $F_L(k+i_0K) = F_L(L-k-i_0K) = \frac{M}{2}$ if $k + i_0K \neq \frac{L}{2}$, and $F_L(k+i_0K) = M$ if $k + i_0K = \frac{L}{2}$. In the second case, we assign $F_L(k+i_0K) = F_L(L-k-i_0K) = M$ if $k+i_0K \neq 0$ and $F_L(k+i_0K) = M$ if $k + i_0K = 0$. In either case, we set the remaining values in the set $\{F_L(k+iM)\}$ to zeros. This will maximize the objective (22), and $f_L(n)$ calculated by the inverse relation (24) is the best sequence and it is real.

Summary of the window algorithm for the real case

Assume a real symmetric window $w(n)$ of order N, with nonnegative Fourier transform is given. Let $L = KM > 2N$ as before. Let P be as explained above. Then the algorithm for the real processes has the following steps:

1. Calculate $\hat{S}_L(k)$, the DFT coefficients of $\hat{r}_L(n), n = 0, \ldots, L-1$, where $\hat{r}_L(n)$ is the periodical expansion of $\hat{r}(n) = w(n)r(n)$.

2. For each $k = 0, \ldots, P$, determine the index i_0 for which $\hat{S}_L(k+i_0K)$ is maximum,

3. If $k + i_0K = 0$ or $k + i_0K = \frac{L}{2}$ then set $F_L(k+i_0K) = M$, else if $k = 0$ or $k = \frac{K}{2}$, then set $F(k+i_0K) = F(L-k-i_0K) = \frac{M}{2}$, else, set $F(k+i_0K) = F(L-k-i_0K) = M$. Set all the remaining values to zeros.

4. Calculate $f_L(n)$ by the inverse relation (24). We need only to determine $f_L(n)$ for $n = 1, \ldots, N$.

5. Form the product filter $g(n) = w(n)f_L(n)$ and spectrally factorize $G(z)$ to find the compaction filter $H(z)$.

Optimization of the window. The algorithm produces very good compaction gains especially when the filter order is high as we shall demonstrate shortly. However, one can get better compaction gains by optimizing the window $w(n)$. Consider the representation (17) again and let $w(n)$ and $f_L(n)$ satisfy the conditions of Lemma 1. If we fix $f_L(n)$, what is the best window $w(n)$? The objective (5) can be written as

$$\sigma_y^2 = \int_{-\pi}^{\pi} \hat{S}_{xx}(e^{j\omega}) W(e^{j\omega}) \frac{d\omega}{2\pi} \tag{27}$$

where $W(e^{j\omega})$ is the Fourier transform of $w(n)$ and $\hat{S}_{xx}(e^{j\omega})$ is the Fourier transform of $f^*(n)r(n)$ where $f(n)$ is one period of $f_L(n)$ centered at $n = 0$. Let $W(e^{j\omega}) = |A(e^{j\omega})|^2$, where $A(z) = \sum_{n=0}^{N} a(n)z^{-n}$ is the spectral factor of $W(e^{j\omega})$. The only constraint on $A(e^{j\omega})$ is that it has to have unit energy in view of $w(0) = \int_{-\pi}^{\pi} |A(e^{j\omega})|^2 \frac{d\omega}{2\pi} = 1$. Hence, by Rayleigh's principle [14], (27) is maximized if $A(z)$ is the maximal eigenfilter of $\mathbf{P}$. The corresponding compaction gain is the maximum eigenvalue of $\mathbf{P}$.

We have described how to optimize $w(n)$ given $f_L(n)$, and vice versa. It is reasonable to expect that one can iterate and obtain better compaction gains at each stage. We have observed that two stages of iterations were sufficient to get near-optimal compaction gains. We started with a triangular window and found that $f_L(n)$ did not change after the reoptimization of the window. Notice that, the use of an initial window is not necessary if one is willing to use a window after finding $f_L(n)$. However, in most of the design examples we considered, we have observed that using an initial window with nonnegative Fourier transform (in particular, the triangular window) and then reoptimizing the window resulted in better compaction gains.

Example 4: MA(1) process. Let $N = 5,\ M = 4,\ r(0) = 1,\ r(1) = \rho$, and $r(n) = 0,\ n > 1$. Assume the process is real so that $r(-n) = r(n)$. Let the window be triangular, i.e.,

$$w(n) = \begin{cases} 1 - \frac{|n|}{6}, & n = 0,\ \pm1,\ \ldots,\ \pm5 \\ 0, & \text{elsewhere.} \end{cases} \tag{28}$$

The Fourier transform of $\hat{r}(n) = w(n)r(n)$ is $\hat{S}(e^{j\omega}) = 1 + \frac{5}{3}\rho\cos\omega$. Hence, the DFT coefficients $\hat{S}_L(k)$ of $\hat{r}(n)$ in step 1 are

$$\hat{S}_L(k) = 1 + \frac{5}{3}\rho\cos\frac{2\pi}{L}k, \quad k = 0, \ldots, L-1. \tag{29}$$

Now, assume $L = 12 > 10$, so that $K = 3$ and $P = 1$. So we have the following sets to consider in step 2:

$$\{\hat{S}_L(0), \hat{S}_L(3), \hat{S}_L(6), \hat{S}_L(9)\}, \quad \{\hat{S}_L(1), \hat{S}_L(4), \hat{S}_L(7), \hat{S}_L(10)\} \tag{30}$$

which are evaluated below respectively:

$$\{1 + \frac{5}{3}\rho, 1, 1 - \frac{5}{3}\rho, 1\}, \quad \{1 + \frac{5\sqrt{3}}{6}\rho, 1 - \frac{5}{6}\rho, 1 - \frac{5\sqrt{3}}{6}\rho, 1 + \frac{5}{6}\rho\}. \tag{31}$$

First assume $\rho > 0$. The maximum of the first set is $\hat{S}_L(0)$ and the maximum of the second set is $\hat{S}_L(1)$. Hence applying step 3 of the algorithm we have

$$\{F_L(k),\ k = 0, \ldots, L-1\} = \{4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4\} \tag{32}$$

By the inverse relation (24) we calculate in step 4:

$$\{f_L(n), n = 0, \ldots, N\} = \{1, \frac{1+\sqrt{3}}{3}, \frac{2}{3}, \frac{1}{3}, 0, \frac{1-\sqrt{3}}{3}\} \tag{33}$$

Hence the product filter $g(n) = w(n)f_L(n)$ has been found, and

$$G(z) = \frac{1-\sqrt{3}}{18}z^5 + \frac{1}{6}z^3 + \frac{4}{9}z^2 + \frac{5(1+\sqrt{3})}{18}z + 1 + \frac{5(1+\sqrt{3})}{18}z^{-1} + \frac{4}{9}z^{-2} + \frac{1}{6}z^{-3} + \frac{1-\sqrt{3}}{18}z^{-5} \tag{34}$$

The corresponding compaction gain is $1 + \frac{5(1+\sqrt{3})}{9}\rho \simeq 1 + 1.5178\rho$. An optimum compaction filter $H(z)$ is obtained by spectrally factorizing $G(z)$. Next consider the case $\rho < 0$. Let $\hat{f}_L(n)$ be the corresponding solution with the Fourier series coefficients $\hat{F}_L(k)$. Referring to (31), $\hat{S}_L(6)$ in the first set and $\hat{S}_L(7)$ in the second set is maximum. Hence,

$$\{\hat{F}_L(k),\ k = 0, \ldots, L-1\} = \{0, 0, 0, 0, 0, 4, 4, 4, 0, 0, 0, 0\} \tag{35}$$

which is equal to $F_L(k-6)$ where $F_L(k)$ is the previous solution. Hence $\hat{f}_L(n) = (-1)^n f_L(n)$ and therefore $\hat{G}(z) = G(-z)$, with the corresponding compaction gain $1 - \frac{5(1+\sqrt{3})}{9}\rho$. An optimal compaction filter is $H(-z)$, where $H(z)$ is a solution for the previous case.

For comparison, we have also designed an optimum compaction filter using the linear programming technique. The corresponding compaction gain is approximately $1+1.6657|\rho|$. This is achieved by using $L=512$ and a triangular window of symmetric order $L-N-1$. The compaction gain of the window method is only slightly lower. Let us find the improvement we can get by optimizing the window when we fix $f_L(n)$. The compaction gain is the maximum eigenvalue of the 6×6 symmetric Toeplitz matrix with the first row $[1\ f_L(1)\ \rho\ 0\ 0\ 0\ 0]$. This eigenvalue is $1+1.8019 f_L(1)|\rho|$. Using $f_L(1)$ given in (33), the improved compaction gain is $1+1.6410|\rho|$ which is very close to the linear programming compaction gain $1+1.6657|\rho|$.

Can we improve the compaction gain further given this optimal window by reoptimizing $f_L(n)$? In this and all the other design examples we considered, we used the triangular window and then found the optimum $f_L(n)$, and then reoptimized $w(n)$ for $f_L(n)$. Interestingly enough, the reoptimization of $f_L(n)$ did not change it!

Choice of the periodicity L

Increasing L does not necessarily increase the resulting compaction gain. For example using $L=\infty$ which corresponds to using optimum ideal filter $f_L(n)$ for the autocorrelation sequence $\hat{r}_L(n)$ does not result in the best achievable compaction gain using the algorithm. This is true even if no initial window $w(n)$ is used. For the above example, we increased L to 16 and found that the compaction gain decreased! When we used the ideal filter for $f_L(n)$ which corresponds to $L=\infty$, the compaction gain was better than that of the case $L=16$ but worse than that of the case $L=12$.

The best period L. Until this point we assumed that $L>2N$. With this choice $\hat{r}_L(n)$ is the periodical expansion of $\hat{r}(n)$ with no aliasing (see Fig. 9(a)). The first period of $\hat{r}_L(n)$ is

$$\{\hat{r}_L(n),\ n=0,\ldots,L-1\}=\{\hat{r}(0),\hat{r}(1),\ldots,\hat{r}(N),0,\ldots,0,\hat{r}^*(N),\ldots,\hat{r}^*(1)\}. \tag{36}$$

However, for the algorithm to successfully find a valid compaction filter, it is only necessary to use a period L which is a multiple of M and is greater than N. This will ensure that $f_L(n)$ will be Nyquist(M). The smallest such period is $M\lceil N/M\rceil$. This choice however leads to an additional symmetry in $f_L(n)$ and according to our experience, the corresponding compaction gains are not good. If we use a period L that is the smallest multiple of M such that $L\geq 2N$, then we obtain very good compaction gains. This choice can be compactly written as

$$L=M\lceil 2N/M\rceil \tag{37}$$

If $L=2N$, the sequence $\hat{r}_L(n)$ has the following first period:

$$\{\hat{r}_L(n),\ n=0,\ldots,L-1\}=\{\hat{r}(0),\hat{r}(1),\ldots,\hat{r}(N)+\hat{r}^*(N),\ldots,\hat{r}^*(1)\}. \tag{38}$$

This is illustrated in Fig. 9(b) for a real process. In this case, we have $\hat{r}_L(N)=2\hat{r}(N)$. This will always be the case if $M=2$, since $L=2N$ is a multiple of M.

3.2. Connection between the linear programming and window methods

In both the linear programming and window methods, we use windows to assure the nonnegativity of $G(e^{j\omega})$. Consider the equations (13) and (17). When L is a multiple of M, a periodic sequence $g_L(n)$ in the linear programming method, and a periodic sequence $f_L(n)$ in the window method are found such that they are Nyquist(M) and their Fourier series coefficients are all nonnegative. For $L > 2N$, two problems are not the same because $g_L(n)$ is necessarily zero for some n, while $f_L(n)$ can be nonzero for all n (except $n = kM$, of course). If however $L = 2N$, then the two problems are exactly the same! If windowing is done in the same way in both methods, then we see that the resulting compaction gains should be the same. Hence, one can view the window method as an efficient and noniterative technique to solve a linear programming problem when $L = 2N$. If L is inreased, we saw that the window method does not necessarily yield better gains whereas this is the case for the linear programming method provided the window order is increased as well. However, optimization of the window becomes costly as the order increases. If one uses a fixed triangular window (with a high order) in the linear programming, and if the windows are optimized in the window method, then window method is very close and sometimes superior to the linear programming method as we demonstrate in the following example.

Example 5: Comparison of linear programming and window methods. Let the input power spectrum be as in Fig. 4. In Fig. 10(a) we plot for a fixed number of channels of $M = 2$, the compaction gains of both the linear programming and the window method versus the filter order. The number of frequencies used in the linear programming method is $L = 512$ while the periodicity used in the window method is $L = 2N$. The windows used in the linear programming are triangular windows with symmetric order $L-N-1$. In the window method, the autocorrelation sequence is first windowed by a triangular window of symmetric order N to find $f_L(n)$ and then the window is reoptimized.

From the figure we observe that if the order is high, one has slightly better compaction gains using the window method. This implies that, if one optimizes the window, there is no need to use large number of frequencies in the linear programming method! More importantly, there is no need to use the linear programming technique for high filter orders. However, it should be emphasized that if the windows are optimized in the linear programming method, one can get slightly better compaction gains than the window method. In Fig. 10(b), we show the plots of the compaction gains of the two methods for various values of M for a fixed filter order of 65. We observe that the window method performs very close to the linear programming method especially for low values of M. We show the upper bounds on compaction gains in both plots. The upper bound in the first plot is achieved by an ideal compaction filter and that in the second plot is achieved by a maximal eigenfilter as discussed in Sec. 1.2.

IV. PROPERTIES OF FIR COMPACTION FILTERS AND BOUNDS ON COMPACTION GAINS

In this section we will give a number of results pertaining to the properties of optimum FIR compaction filters and the corresponding compaction gains for a given filter order N and the number of channels M.

1. A necessary condition on the compaction filter for optimality. For an FIR compaction filter $H(z)$

to be optimum it is necessary that the Fourier transform of the sequence $r(n)g^*(n)$ attains a maximum at the frequency $w = 0$, where $g(n)$ is the impulse response of the product filter of $H(z)$. To see this, consider the Fourier transform of the product $r(n)g^*(n)$ at a frequency ω_0:

$$\sum_n r(n)g^*(n)e^{-j\omega_0 n} = \int_{-\pi}^{\pi} S_{xx}(e^{j\omega})G(e^{j(\omega-\omega_0)})\frac{d\omega}{2\pi} \tag{39}$$

But the LHS is the output variance of a valid compaction filter whose product filter is $g(n)e^{j\omega_0 n}$. Since $g(n)$ is optimum for the given autocorrelation sequence $r(n)$, it follows that the above RHS attains a maximum when $\omega_0 = 0$. If the process is real, then by considering $\hat{g}(n) = g(n)\cos\omega_0 n$ one can arrive at the same result with real coefficient filters.

2. Class of random processes that have the same optimum compaction filters for a pair (M, N). Let us consider the objective in the time domain:

$$\sigma_y^2 = r(0) + \sum_{n \neq kM} r(n)g^*(n) \tag{40}$$

From this, one can deduce that an optimal compaction filter does not depend on $r(kM)$ and it continues to be optimal for a modified autocorrelation sequence of the form $\hat{r}(n) = c\, r(n),\ n \neq kM$, where $c > 0$. We will see later (Sec. V) that in the two-channel case, if $c < 0$, then $H(-z)$ is the optimum solution for $\hat{r}(n)$.

3. Monotonic behaviour of the optimum FIR compaction gain. Let $G_{opt}(M, N)$ denote the optimum compaction gain for a given number of channels M and FIR filter order N. It is then clear that

$$G_{opt}(kM, N) \leq G_{opt}((k+1)M, N),\ k = 1, 2, \ldots, \quad \text{and} \quad G_{opt}(M, N) \leq G_{opt}(M, N+1). \tag{41}$$

4. Bounds in terms of eigenvalues. Let $f_L(n)$ be any Nyquist(M) with nonnegative Fourier series coefficients. Assume $L > N$. Then,

$$\lambda_{max}\Big\{r(n)f_L^*(n)\Big\}_0^N \leq G_{opt}(M, N) \leq \lambda_{max}\Big\{r(n)\Big\}_0^N. \tag{42}$$

Here the notation $\lambda_{max}\Big\{r(n)\Big\}_0^N$ stands for the maximum eigenvalue of the Hermitian Toeplitz matrix whose first row is $[r(0)\ r(1)\ \ldots\ r(N)]$. The inequality on the left follows because the product filter $g(n) = w(n)f_L(n)$ achieves that bound by choosing $w(n)$ as the product filter of the maximal eigenfilter of the Hermitian Toeplitz matrix formed by the sequence $r(n)f_L^*(n)$. For the inequality on the right, note that there exists an integer $k \geq 1$ such that $kM > N$. From Sec. 1.2 we have $G_{opt}(kM, N) = \lambda_{max}\Big\{r(n)\Big\}_0^N$ which is called the KLT gain. From (41), it follows that $G_{opt}(M, N) \leq G_{opt}(kM, N) = \lambda_{max}\Big\{r(n)\Big\}_0^N$.

If we replace $f_L(n)$ by a positive definite Nyquist(M) sequence $f(n)$ of order N, the inequality on the left continues to be valid because $w(n)f(n)$ is still a product filter of a valid compaction filter. To see this, note that the sequence $f(n)$ can be extended to an infinite sequence (e.g., using autoregressive extrapolation) such that its Fourier transform is nonnegative. Hence the product $w(n)f(n)$ has nonnegative Fourier transform. The Nyquist(M) property of the product follows from that of $f(n)$.

5. Upper bound by M. For all FIR compaction filters we have

$$G_{opt}(M,N) \leq M \tag{43}$$

with strict inequality as long as $S_{xx}(e^{j\omega})$ is not a line-spectral process. To see this, first observe that $G(e^{j\omega}) \leq M$. Hence, $\sigma_y^2 = \int_{-\pi}^{\pi} G(e^{j\omega})S_{xx}(e^{j\omega})\frac{d\omega}{2\pi} \leq M \int_{-\pi}^{\pi} S_{xx}(e^{j\omega})\frac{d\omega}{2\pi} = M\sigma_x^2$. The equality holds if and only if $G(e^{j\omega}) = M$ for all ω for which $S_{xx}(e^{j\omega}) \neq 0$. If $S_{xx}(e^{j\omega})$ is not line-spectral, this requires $G(e^{j\omega})$ to be identically zero for some region of frequency which is impossible since the order is assumed to be finite. For $M = 2$, we will derive another upper bound for $G_{opt}(2,N)$ in Sec. 5.1 (see (53)). This bound will be applicable under certain conditions on the input power spectrum, to be made more explicit in that section. Under some further conditions that we will describe, the stated bound can in fact be achieved.

V. ANALYTICAL METHOD

In this section we consider the special case of two channels ($M = 2$) and assume that the input $x(n)$ is real so that the compaction filter coefficients $h(n)$ can be assumed to be real. For this two-channel case we will show that the optimal product filter $G(e^{j\omega})$ can sometimes be obtained using an analytical method instead of going through a numerical optimization procedure. We will also present a number of examples which demonstrate that the method is applicable to a large class of input power spectra. Also presented are examples where the analytical method can be shown to fail.

The analytical method is motivated by the fact that, under some conditions to be explained, the objective function (5), which can be written as $\sigma_y^2 = \int_{-\pi}^{\pi} G(e^{j\omega})S_{xx}(e^{j\omega})\frac{d\omega}{2\pi}$ can be conveniently expressed as a finite summation. The summation involves a modified polyhase component of $G(e^{j\omega})$ at a discrete set of frequencies ω_k determined by the psd $S_{xx}(e^{j\omega})$. This will allow us to optimize the modified polyhase component, and hence $G(z)$, essentially *by inspection.*

The inspiration for our work in this section comes from the recent contribution by Aas et al. [21] where the Gaussian quadrature technique is cleverly used to adress the problem of optimizing the frequency responses of filters. Our work in this section differs in a number of respects. First we do not use Gaussian quadrature, but take advantage of an elegant representation for positive definite sequences which results from the theory of line-spectral processes. Second, we take into account the knowledge of the input psd in the optimization process. Finally we present several design examples demonstrating the usefulness as well as the limitations of the proposed method.

Suppose the product filter $G(z) = \sum_{n=-N}^{N} g(n)z^{-n}$ is expressed in the traditional polyphase form [13] $G(z) = E_0(z^2) + z^{-1}E_1(z^2)$. For the real coefficient case we have $g(n) = g(-n)$, and it follows that the coefficients of the FIR filter $E_1(z)$ have the symmetry demonstrated in Fig. 11. This implies, in particular, that $E_1(z) = 0$ for $z = -1$. By factoring the zero at $z = -1$ we can write $E_1(z) = (1+z)G_1(z)$ where $G_1(z)$ is FIR and has symmetric real coefficients. We therefore obtain the modified polyphase representation

$$G(z) = 1 + (z + z^{-1})G_1(z^2), \quad \text{i.e.,} \quad G(e^{j\omega}) = 1 + 2\cos\omega G_1(e^{j2\omega}) \tag{44}$$

where we have also used the Nyquist(2) property of $G(z)$. Since Nyquist condition and nonnegativity of $G(e^{j\omega})$ together imply $0 \leq G(e^{j\omega}) \leq 2$, we have the following bound on the modified polyphase component $G_1(e^{j\omega})$:

$$-\frac{1}{2\cos(\omega/2)} \leq G_1(e^{j\omega}) \leq \frac{1}{2\cos(\omega/2)}, \quad -\pi < \omega < \pi \tag{45}$$

Notice that $G(z)$ and $G_1(z)$ can be determined from each other uniquely. We shall express the output variance σ_y^2 in terms of $G_1(e^{j\omega})$ so that we can see how to optimize the coefficients of $G_1(z)$. For this, write the input psd in the traditional polyphase form as $S_{xx}(z) = S_0(z^2) + z^{-1}S_1(z^2)$. Then σ_y^2 can be simplified into the form $\sigma_y^2 = r(0) + \int_{-\pi}^{\pi} G_1(e^{j\omega})\Psi_x(e^{j\omega})\frac{d\omega}{2\pi}$ where $\Psi_x(z) = (1 + z^{-1})S_1(z)$ or equivalently

$$\Psi_x(e^{j\omega}) = \cos(\omega/2)\Big(S_{xx}(e^{j(\omega/2)}) - S_{xx}(e^{j(\pi-\omega/2)})\Big) \tag{46}$$

Using Parseval's relation the objective can be written as

$$\sigma_y^2 = r(0) + \sum_{n=-(N-1)/2}^{(N-1)/2} g_1(n)\psi_x(n) \tag{47}$$

where $\psi_x(n)$ is the inverse transform of $\Psi_x(z)$ which is produced below explicitly for convenience.

$$\psi_x(0) = 2r(1),\ \psi_x(1) = r(1) + r(3),\ \ldots,\ \psi_x(\frac{N-1}{2}) = r(N-2) + r(N). \tag{48}$$

Note that $\psi_x(n) = \psi_x(-n)$. Our aim is to maximize the second term in the expression (47) for fixed $\psi_x(n)$ (i.e., fixed input) by choosing $g_1(n)$ under the constraint (45) and the usual filter-order constraint. Under the assumption that the input-dependent sequence $\psi_x(n)$ is positive or negative definite (see Sec. 1.1 for definition) we will show how this can be done analytically. (The significance of this assumption on $\psi_x(n)$ is explained in Sec. 5.3). We will need the following representation theorem for positive definite sequences:

Theorem: Representation of positive definite sequences. Given a positive definite sequence of $m+1$ complex numbers $\{\phi(n),\ n = 0, \ldots, m\}$, there exists a representation of the form

$$\phi(n) = \sum_{k=0}^{m} \alpha_k e^{j\omega_k n}, \quad n = 0, \ldots, m \tag{49}$$

where α_k's are all positive and ω_k's are all distinct.

Proof. Let $\mathbf{P}$ be the $(m+1) \times (m+1)$ Hermitian Toeplitz matrix whose first row is $\Phi^T = [\phi(0)\ \phi(1) \ldots \phi(m)]$. Consider the extension of $\mathbf{P}$ into a singular $(m+2) \times (m+2)$ Hermitian Toeplitz matrix $\hat{\mathbf{P}}$ such that its $(m+1) \times (m+1)$ principal submatrix is $\mathbf{P}$. This extension is merely augmenting an extra element $\phi(m+1)$ to the end of Φ and forming the corresponding Hermitian Toeplitz matrix. The number $\phi(m+1)$ is chosen to make $\hat{\mathbf{P}}$ singular. This can always be done because of the following reason: for the matrix $\mathbf{P}$, one can run the well-known Levinson recursion procedure [26] to obtain the optimal mth order predictor polynomial $A_m(z)$. If one now considers the following continuation of the recursion $P_c(z) = A_m(z) + cz^{-(m+1)}\tilde{A}_m(z)$ with $|c| = 1$, then this corresponds to the singular predictor polynomial of a random process with singular autocorrelation matrix $\hat{\mathbf{P}}$. The result now follows from a well established fact [26, 27] that states that a WSS process is line spectral with exactly $m+1$ lines if and only if its $(m+1) \times (m+1)$ autocorrelation matrix is nonsingular and $(m+2) \times (m+2)$ autocorrelation matrix is singular. ∎

Remarks. It is clear that $P_c(z)$ is also the minimal eigenfilter of $\hat{\mathbf{P}}$. The zeros of $P_c(z)$ are all on the unit circle and distinct. Let $\{e^{j\omega_k},\ k = 0, \ldots, m\}$ be these zeros. The distinct frequencies $\{\omega_k, k = 0, \ldots, m\}$ are referred to as the line-spectral frequencies. The representation (49) is not unique because of the nonuniqueness of the unit magnitude constant c in the proof.

Real case. The predictor polynomial $A_m(z)$ and the constant c are real. Hence we have two cases: $c = \pm 1$. The case $c = 1$ leads to a symmetric singular predictor polynomial $P_1(z)$, while the case $c = -1$ leads to an antisymmetric polynomial $P_{-1}(z)$. It is a well-known fact that the distinct unit-circle zeros of these two polynomials are interleaved [22]. For simplicity assume that m is odd. Then $P_{-1}(z)$ has both of the zeros $z = 1$ and $z = -1$ and $P_1(z)$ has none of them. Using $P_1(z)$, we have the following representation for a real positive definite sequence $\phi(n)$:

$$\phi(n) = \sum_{k=0}^{(m-1)/2} \beta_k \cos \omega_k n, \quad n = 0, \ldots, m \tag{50}$$

where β_k's are all positive and ω_k's are all distinct and different from 0 and π.

5.1. Derivation of the analytical method

Assume for simplicity $(N-1)/2$ is odd and let $\{\psi_x(n),\ n = 0, \ldots, (N-1)/2\}$ be positive definite. Applying the real form of the representation we have

$$\psi_x(n) = \sum_{k=0}^{(N-3)/4} \beta_k \cos \omega_k n, \quad n = 0, \ldots, \frac{N-1}{2} \tag{51}$$

The objective (47) can therefore be written as

$$\sigma_y^2 = r(0) + \sum_{k=0}^{(N-3)/4} \beta_k \sum_{n=-(N-1)/2}^{(N-1)/2} g_1(n) \cos \omega_k n = r(0) + \sum_{k=0}^{(N-3)/4} \beta_k G_1(e^{j\omega_k}) \tag{52}$$

From (45), the output variance (52) is maximized if $G_1(e^{j\omega_k}) = \frac{1}{2\cos(\omega_k/2)}$, $k = 0, \ldots, (N-3)/4$. This implies $G(e^{j\omega_k/2}) = 2$, and by Nyquist(2) property $G(e^{j(\pi-\omega_k/2)}) = 0, \quad k = 0, \ldots, (N-3)/4$. Notice that these zeros are all located in the region $(\pi/2, \pi)$. Since $0 \leq G(e^{j\omega}) \leq 2$, the derivatives of $G(e^{j\omega})$ should vanish at the above frequencies. Hence we should have $G'(e^{j\omega_k/2}) = 0,\ k = 0, \ldots, (N-3)/4$. In view of (44), this in turn implies $G_1'(e^{j\omega_k}) = \frac{\sin(\omega_k/2)}{4\cos^2(\omega_k/2)}$, $k = 0, \ldots, (N-3)/4$. The total number of constraints on $G_1(e^{j\omega})$ and $G_1'(e^{j\omega})$ is $\frac{N+1}{2}$. Since $G_1(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} g_1(n) z^{-n}$ with $g_1(n) = g_1(-n)$, it is determined uniquely. If these $\frac{N+1}{2}$ constraints are satisfied, the solution $G(e^{j\omega})$ so found is necessarily nonnegative in the frequency region $[\pi/2, \pi]$ (Appendix B). If it is nonnegative in the region $[0, \pi/2)$ as well, then it is the optimum compaction filter with the corresponding compaction gain

$$G_{opt}(2,N) = 1 + \frac{\sum_{k=0}^{(N-3)/4} \frac{\beta_k}{2\cos(\omega_k/2)}}{r(0)} \tag{53}$$

If however, $G(e^{j\omega})$ turns out to be negative at some frequency in $[0, \pi/2)$, then it is not a valid solution and the above RHS is only an upper-bound for $G_{opt}(2, N)$.

Uniqueness of the solution. Assume that $G(e^{j\omega})$ obtained by the method is nonnegative. Then it is the unique solution! To see this, assume there is another optimal product filter $K(z)$. Assume $K_1(z)$ is its modified polyphase component. Then, there exists a frequency ω_k among the line-spectral frequencies such that $K_1(e^{j\omega_k}) < \frac{1}{2\cos(\omega_k/2)}$. Hence the summation (52) for $K_1(e^{j\omega})$ is necessarily less than that for $G_1(e^{j\omega})$, resulting in contradiction. Notice that it is the product filter $G(z)$ that is unique, not the compaction filter $H(z)$. However, if the zeros of $H(z)$ are constrained to be on and inside the unit-circle, then $H(z)$ is unique as well.

5.2. Construction of optimal $G(z)$

Consider the following factorization of $G(z)$:

$$G(z) = \hat{G}_0(z)\hat{G}_1(z) \tag{54}$$

where $\hat{G}_0(z)$ contains the unit-circle zeros determined by the above procedure. Hence we have

$$\hat{G}_0(z) = \prod_{k=0}^{\frac{N-3}{4}} (z + 2\cos(\omega_k/2) + z^{-1})^2 \tag{55}$$

Using the Nyquist(2) property, it is possible to determine $\hat{G}_1(z)$ and hence $G(z)$. Let $\hat{g}_0(n)$ and $\hat{g}_1(n)$ be the impulse responses of $\hat{G}_0(z)$ and $\hat{G}_1(z)$ respectively. The product (54) in $z-$domain is equivalent to the convolution in time domain. Using the convolution matrix and taking into account the symmetries we get

$$\mathbf{g} = \mathbf{A}\hat{\mathbf{g}}_1 \tag{56}$$

where the vectors $\mathbf{g}, \hat{\mathbf{g}}_1$ have the components $g_n = g(2n), \hat{g}_{1n} = g_1(n), \quad n = 0, \ldots, (N-1)/2$, and $\mathbf{A}$ is obtained from the impulse response $g_0(n)$. From the Nyquist(2) property, it is clear that $\mathbf{g} = [1\ 0\ \ldots 0]^T$. Hence $\hat{g}_1(n)$ is determined by inverting the above equation. The invertibility of the matrix $\mathbf{A}$ follows from the fact that $G_1(z)$ is uniquely defined and hence $G(z)$ is uniquely defined. Therefore a unique solution to $\hat{\mathbf{g}}_1$ must exist, implying that $\mathbf{A}$ is nonsingular.

Efficient determination of $\hat{G}_0(z)$: We will show that we can obtain $\hat{G}_0(z)$ from the singular predictor polynomial $P_1(z)$ without having to find its roots. For this, let us write $P_1(z)$ explicitly:

$$P_1(z) = cz^{-\frac{N+1}{2}} \prod_{k=0}^{\frac{N-3}{4}} (z - e^{j\omega_k})(z - e^{-j\omega_k}) = cz^{-\frac{N+1}{4}} \prod_{k=0}^{\frac{N-3}{4}} (z - 2\cos\omega_k + z^{-1}) \tag{57}$$

Now, consider the upsampled polynomial $P_1(z^2)$. This can be written in the form $P_1(z^2) = P_0(z)P_0(-z)$, where

$P_0(z)$ is a polynomial in z^{-1} of order $\frac{N+1}{2}$ with all its zeros in the left half plane. To be explicit:

$$P_0(z) = z^{-\frac{N+1}{4}} \prod_{k=0}^{\frac{N-3}{4}} (z + 2\cos(\omega_k/2) + z^{-1}) \tag{58}$$

Hence we can write $\hat{G}_0(z) = z^{\frac{N+1}{2}} P_0^2(z)$. Therefore, given the singular predictor polynomial $P_1(z)$, one can apply a continuous-time spectral factorization algorithm [28] to $P_1(z^2)$ to obtain $P_0(z)$ and therefore $\hat{G}_0(z)$. Since $G(z)$ can be determined from $\hat{G}_0(z)$ as we explained before, we observe that there is no need to find the roots of $P_1(z)$!

Spectral factorization. To find the compaction filter $H(z)$, we need to spectrally factorize $G(z)$. It is clear that we can write $H(z)$ as

$$H(z) = H_0(z)H_1(z) \tag{59}$$

where $H_0(z)$ and $H_1(z)$ are the spectral factors of $\hat{G}_0(z)$ and $\hat{G}_1(z)$ respectively. We can deduce $H_0(z)$ immediately: $H_0(z) = P_0(z)$. Hence all we need to do is to determine $H_1(z)$ which is of order $\frac{N-1}{2}$. This can be done by a discrete-time spectral factorization of $\hat{G}_1(z)$ [29].

The case where $\frac{N-1}{2}$ is even can be treated in a very similar manner. In this case, we use the singular polynomial $P_{-1}(z)$ corresponding to $c = -1$ and one of the line-spectral frequencies is 0, that is, $z = 1$ is a root of $P_{-1}(z)$. The resulting product filter $G(e^{j\omega})$ continues to be nonnegative in $[\pi/2, \pi]$. We skip the details and give the summary of the algorithm for both cases.

Summary of the analytical method

Given the autocorrelation sequence $r(n),\ n = 0, \ldots, N$, where N is odd, we would like to find an optimum compaction filter $H(z)$ of order N. We first obtain the sequence $\psi_x(n),\ n = 0, \ldots, (N-1)/2$ using the relations (48). **If this sequence is positive definite**, then we do the following

1. Calculate $A_{\frac{N-1}{2}}(z)$, the optimum predictor polynomial of order $\frac{N-1}{2}$, corresponding to the positive definite sequence $\psi_x(n)$ and obtain $P_c(z)$ from

   $$P_c(z) = A_{\frac{N-1}{2}}(z) + cz^{-\frac{N+1}{2}} A_{\frac{N-1}{2}}(z^{-1}) \tag{60}$$

   where $c = 1$ if $\frac{N-1}{2}$ is odd, and $c = -1$ otherwise.

2. Obtain the spectral factor, $P_0(z)$ of $P_c(z^2)$ using a continuous time spectral factorization algorithm and determine $\hat{G}_0(z) = z^{\frac{N+1}{2}} P_0^2(z)$.

3. Calculate $\hat{G}_1(z)$ using (56) and find its spectral factor $H_1(z)$.

4. The optimum compaction filter is $H(z) = P_0(z)H_1(z)$.

Case where $\psi_x(n)$ is negative definite. From our developments for the positive definite case, and using the sequence $-\psi_x(n)$, it can be proven that the optimum compaction filter is $H(z) = \hat{H}(-z)$ where $\hat{H}(z)$ is the optimum compaction filter for the positive definite sequence $\hat{\psi}_x(n) = -\psi_x(n)$. However, it is easier to see this directly by looking at the objective in time domain (40). First note that $\hat{\psi}_x(n)$ corresponds to the autocorrelation sequence $\hat{r}(n) = -r(n),\ n \neq 0$. Let $g(n)$ and $\hat{g}(n)$ be the product filter impulse responses for $H(z)$ and $\hat{H}(z)$ respectively. The objective is then to maximize $\sum_{n=1}^{N} -g(n)\hat{r}(n)$, This has the solution $-g(n) = \hat{g}(n),\ n \neq 0$. Hence we have $G(z) = \hat{G}(-z)$ and therefore $H(z) = \hat{H}(-z)$.

Remark. The observation made above clearly extends to the case where $\hat{r}(n) = c\, r(n),\ c < 0$, as scaling does not change the optimal solution as stated in Sec. IV.

Example 6: AR(1) process. Let $N = 3$, and $r(n) = \rho^n$ where $0 < \rho < 1$. Then, $\psi_x(0) = 2\rho$ and $\psi_x(1) = \rho(1+\rho^2)$. The Hermitian Toeplitz matrix corresponding to $\{\psi_x(n),\ n = 0, 1\}$ is

$$\mathbf{P} = \rho \begin{bmatrix} 2 & 1+\rho^2 \\ 1+\rho^2 & 2 \end{bmatrix} \tag{61}$$

which is positive definite. Hence we can apply the above algorithm. Running the Levinson recursion, we have: $A_1(z) = 1 - \frac{1+\rho^2}{2} z^{-1}$ and using $c = 1$ we have: $P_1(z) = 1 - (1+\rho^2)z^{-1} + z^{-2}$. By straightforward calculation $P_0(z) = 1 + \sqrt{3+\rho^2}z^{-1} + z^{-2}$ from which it follows that $\hat{G}_0(z) = (z + \sqrt{3+\rho^2} + z^{-1})^2$ and $\hat{G}_1(z) = -\frac{1}{2(3+\rho^2)^{3/2}}(z - 2\sqrt{3+\rho^2} + z^{-1})$. For all values of ρ, $\hat{G}_1(z)$ does not have unit-circle zeros and therefore it is nonnegative. The spectral factor of $\hat{G}_1(z)$ turns out to be $\frac{1}{\sqrt{2}(3+\rho^2)^{3/4}}(a + bz^{-1})$ where $a = \sqrt{\sqrt{3+\rho^2} + \sqrt{2+\rho^2}}$ and $b = -\sqrt{\sqrt{3+\rho^2} - \sqrt{2+\rho^2}}$. Finally we obtain

$$H(z) = P_0(z)H_1(z) = \frac{1}{\sqrt{2}(3+\rho^2)^{3/4}} \left(a + (b + a\sqrt{3+\rho^2})z^{-1} + (a + b\sqrt{3+\rho^2})z^{-2} + bz^{-3}\right) \tag{62}$$

The product filter is $G(z) = \frac{1}{2(3+\rho^2)^{3/2}}(-z^3 + 3(2+\rho^2)z + 2(3+\rho^2)^{3/2} + 3(2+\rho^2)z^{-1} - z^{-3})$, with the corresponding optimum compaction gain $G_{opt}(2,3) = 1 + \frac{2\rho}{\sqrt{3+\rho^2}}$. For comparison purposes we have also designed compaction filters using the linear programming and the window methods for several values of ρ. Table 1 shows the filter coefficients and compaction gains using the linear programming and the window methods together with those obtained by the above analytical method.

Linear phase constraint is a loss of generality. Notice that the compaction filters obtained by the analytical method in Example 6 cannot have linear phase. This is observed by considering the zeros of $G(z)$ which consist of a single reciprocal pair and two double unit-circle zeros. For $H(z)$ to have a linear phase, the multiplicity of the zeros of the reciprocal pair should be two. Since the solution $G(z)$ is unique, we conclude that the linear-phase constraint on the compaction filter $H(z)$ is a loss of generality.

Example 7: MA(1) process. Let $N = 3$, $r(0) = 1,\ r(1) = \rho > 0$, and $r(n) = 0,\ n > 1$. The sequence $\psi_x(n)$ is therefore $\psi_x(0) = 2\rho,\ \psi_x(1) = \rho$. The corresponding Hermitian Toeplitz matrix is

$$\mathbf{P} = \rho \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{63}$$

which is positive definite. Hence applying the algorithm, $A_1(z) = 1 - \frac{1}{2}z^{-1}$ and therefore using $c = 1$, the singular predictor polynomial is $P_1(z) = 1 - z^{-1} + z^{-2}$. The continuous spectral factor $P_0(z)$ of $P_1(z^2)$ is

$\hat{P}_0(z) = 1 + 2\cos(\pi/6)z^{-1} + z^{-2} = 1 + \sqrt{3}z^{-1} + z^{-2}$ and therefore $\hat{G}_0(z) = (z + \sqrt{3} + z^{-1})^2$. Hence $\hat{G}_1(z)$ is calculated to be $\hat{G}_1(z) = -\frac{\sqrt{3}}{18}(z - 2\sqrt{3} + z^{-1})$. The spectral factor $H_1(z)$ of $G_1(z)$ is found to be

$$H_1(z) = 3^{-3/4}2^{-1/2}(\sqrt{\sqrt{3}+\sqrt{2}} - \sqrt{\sqrt{3}-\sqrt{2}}z^{-1}) \tag{64}$$

Hence the compaction filter $H(z)$ is

$$3^{-3/4}2^{-1/2}\left(\sqrt{\sqrt{3}+\sqrt{2}} + (\sqrt{3+\sqrt{6}} - \sqrt{\sqrt{3}-\sqrt{2}})z^{-1} + (\sqrt{\sqrt{3}+\sqrt{2}} - \sqrt{3-\sqrt{6}})z^{-2} - \sqrt{\sqrt{3}-\sqrt{2}}z^{-3}\right) \tag{65}$$

The product filter is $G(z) = -\frac{\sqrt{3}}{18}z^3 + \frac{\sqrt{3}}{3}z + 1 + \frac{\sqrt{3}}{3}z^{-1} - \frac{\sqrt{3}}{18}z^{-3}$, and the corresponding optimum compaction gain is $G_{opt}(2,3) = 1 + \frac{2}{\sqrt{3}}\rho$.

Example 8: MA(1) process, general order N. We will analytically find the optimum compaction filters of arbitrary (odd) order N for MA(1) processes. Note from Sec. IV that, if we find $H(z)$ corresponding to a MA(1) process with $\rho > 0$ then this is the solution for all MA(1) processes with $\rho > 0$ since the autocorrelation sequences are the scaled versions of each other in the sense explained in that section. From the arguments given for the case where $\psi_x(n)$ is negative definite, $H(-z)$ is the solution for all MA(1) processes with $\rho < 0$.

Now, following the steps of the algorithm we have

$$P_c(z) = 1 - z^{-1} + z^{-2} - \ldots + (-1)^{\frac{N+1}{2}}z^{-\frac{N+1}{2}} \tag{66}$$

If $\frac{N-1}{2}$ is odd, then the zeros of $P_1(z)$ are $e^{\pm j\omega_k}$, $w_k = (2k-1)\frac{2\pi}{N+3}$, $k = 1, \ldots, (N+1)/4$. Therefore the roots of $P_0(z)$, hence the unit-circle zeros of the optimum compaction filter $H(z)$ are

$$e^{\pm j\Omega_k}, \quad \Omega_k = \pi - (2k-1)\frac{\pi}{N+3}, \quad k = 1, \ldots, \frac{N+1}{4} \tag{67}$$

Similarly, if $\frac{N-1}{2}$ is even, the roots of $P_{-1}(z)$ are 0, $e^{\pm j\omega_k}$, $w_k = 2k\frac{2\pi}{N+3}$, $k = 1, \ldots, (N-1)/4$. Therefore the roots of $P_0(z)$, hence the unit-circle zeros of the optimum compaction filter $H(z)$ are

$$\pi,\ e^{\pm j\Omega_k}, \quad \Omega_k = \pi - 2k\frac{\pi}{N+3}, \quad k = 1, \ldots, \frac{N-1}{4} \tag{68}$$

The rest of the procedure involves spectral factorization and it is not easy to see what $H_1(z)$ will be in closed form. However we note that the algorithm successfully finds the optimum compaction filter for any odd order N. Table 2 shows the compaction filters and the corresponding compaction gains for various filter orders.

Example 9: KLT. If $N = 1$, then we have $A_0(z) = 1$, and $P_{-1}(z) = 1 - z^{-1}$. and therefore $P_0(z) = 1 + z^{-1}$, $\hat{G}_0(z) = z + 2 + z^{-1}$, and $\hat{G}_1(z) = \frac{1}{2}$. Hence the optimum compaction filter of first order is $H(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$ if $r(1) > 0$ and it is $H(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$ if $r(1) < 0$. Notice that these correspond to the two-channel KLT subband coder which is known to be fixed. The corresponding compaction gain is $G_{opt}(2,1) = 1 + \frac{|r(1)|}{r(0)}$. It should be noted that the above filters and the corresponding compaction gains are optimal for any power spectrum and for any number of channels. Hence we have:

$$G_{opt}(M,1) = 1 + \frac{|r(1)|}{r(0)}, \quad M \geq 2 \tag{69}$$

If $r(m)$ is maximum of all $r(n)$ where n is not a multiple of M, then one can achieve the compaction gain of $1+\frac{|r(m)|}{r(0)}$ by using the filter $\frac{1}{\sqrt{2}}(1+z^{-m})$ if $r(m)>0$ and the filter $\frac{1}{\sqrt{2}}(1-z^{-m})$ if $r(m)<0$.

Case where $\psi_x(n)$ is semidefinite. Assume that $\psi_x(n)$ is positive semidefinite. Then there exists an integer $P<(N-1)/2$ such that $\{\psi_x(n),\ n=0,1,...,P\}$ is positive definite and $\{\psi_x(n),\ n=0,1,...,P+1\}$ is only positive semidefinite. Then we can replace $(N-1)/2$ in the above arguments with P and write the objective (52) in terms of $P+1$ corresponding line-spectral frequencies. This enables us to determine a product filter of symmetric order $2P+1<N$. If this resulting filter is nonnegative, then we have found the unique minimum symmetric order product filter that is optimum among the filters of symmetric order less than or equal to N! The case where $\psi_x(n)$ is negative semidefinite is similar, the details are omitted.

Example 10: Case where $\psi_x(n)$ is positive semidefinite. Let $N=3$, $r(0)=1$ and $r(1)=r(3)=\rho>0$. Then, $\psi_x(0)=\psi_x(1)=2\rho$. The associated Toeplitz matrix is

$$2\rho\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \tag{70}$$

which is positive semidefinite and singular. The number P is 0 in this case and the objective (52) is $1+2\rho G_1(e^{j0})$. By letting $G_1(e^{j0})=\frac{1}{2}$, the product filter $G(z)$ of symmetric order 1 can easily be seen to be $\frac{1}{2}z+1+\frac{1}{2}z^{-1}$, and it is readily verified that $G(e^{j\omega})\geq 0$. In fact this is the KLT solution with the compaction filter $H(z)=\frac{1}{\sqrt{2}}(1+z^{-1})$. The corresponding optimum compaction gain is $1+\rho$. No 3rd order solution can achieve better gain than this.

5.3. Characterization of processes for which the analytical method is applicable.

For the technique of Sec. 5.1. to be applicable for a particular order N, we need to have the sequence $\psi_x(n)$ to be positive or negative definite for that N. However, although the sequence $\psi_x(n)$ is positive or negative definite and the algorithm is applicable, it may happen that the resulting $G(e^{j\omega})$ is not nonnegative. This will be the case if $\hat{G}_1(e^{j\omega})$, obtained from nonnegative $\hat{G}_0(e^{j\omega})$, is not nonnegative. In what follows we will describe a class of processes for which the sequence $\psi_x(n)$ is positive definite for all N and therefore the analytical method is applicable for all orders N.

The sequence $\psi_x(n)$ is positive definite for all N, if and only if $\Psi_x(e^{j\omega})$ is not a line-spectrum and $\Psi_x(e^{j\omega})\geq 0$. Using (46), this is true if and only if $S_{xx}(e^{j\omega})$ is not a line-spectrum and

$$\left(S_{xx}(e^{j\omega})-S_{xx}(e^{j(\pi-\omega)})\right)\geq 0,\ \omega\in[0,\pi/2] \tag{71}$$

We will say that the process is 'low-pass' if its psd satifies the above condition. Notice that, in the ideal case, the optimum compaction filter for that type of process is the ideal half-band low-pass filter [13] (see Fig. 12(a)). For the case where $\psi_x(n)$ is negative definite for all N, the preceding is replaced with

$$\left(S_{xx}(e^{j(\omega)})-S_{xx}(e^{j(\pi-\omega)})\right)\leq 0,\ \omega\in[0,\pi/2] \tag{72}$$

For this type of input process, the optimum ideal compaction filter is the ideal half-band high-pass filter [13]. Hence we will say that the process is 'high-pass' if its psd satisfies (72) (see Fig. 12(b)).

Cases where the algorithm fails. Assume that the process is such that the sequence $\{\psi_x(n),\ n = 0, \ldots, (N-1)/2\}$ is positive definite and therefore the algorithm is applicable for the filter order N. Assume however that one of the line-spectral frequencies ω_k is close to π. The algorithm will require $e^{j(\pi-\omega_k/2)}$ to be a zero of $G(z)$. Hence $G(e^{j\omega})$ will have a zero close to $\pi/2$. But this may be impossible if the order N is low. To see this, note that $G(e^{j\pi/2}) = 1$ from the Nyquist(2) property and therefore requiring $G(e^{j\omega})$ to have a zero close to the frequency $\pi/2$ is the same as requiring a narrow transition band for $G(e^{j\omega})$ which is impossible if the order is not sufficiently high. One can however, increase the filter order to overcome the problem.

Example 11. Let $N = 3$, and $r(n) = \cos\omega_1 n,\ \omega_1 \in [0, \pi/2)$. Hence $\psi_x(0) = 2\cos\omega_1,\ \psi_x(1) = \cos 3\omega_1 + \cos\omega_1$, and $\psi_x(n)$ is positive definite. Using the procedure of Sec. 5.2, we find $\hat{G}_0(z) = (z + 2\cos\omega_1 + z^{-1})^2$ from which it follows that $\hat{G}_1(z) = -\frac{1}{16\cos^3\omega_1}(z - 4\cos\omega_1 + z^{-1})$. This has a unit-circle zero if $\omega_1 \in (\pi/3, \pi/2)$ and therefore $\hat{G}_1(e^{j\omega})$ is not nonnegative. Hence the algorithm fails if the impulse is within $\pi/6$ neighborhood of $\pi/2$. We have designed optimum compaction filters for the above autocorrelation sequence using the linear programming method for various values of ω_1. We have observed that the optimum compaction filters agree with the above analytical solution if $\omega_1 \in (0, \pi/3]$. For the complementary case of $\omega_1 \in (\pi/3, \pi/2)$ where the analytical method fails for $N = 3$, linear programming yields the solution $G(z) = -\frac{1}{2}z^3 + 1 - \frac{1}{2}z^{-3}$, *regardless of the exact value of* ω_1. The factors $\hat{G}_0(z)$ and $\hat{G}_1(z)$ of $G(z)$ are $\hat{G}_0(z) = (z + 2\cos(\pi/3) + z^{-1})^2$, and $\hat{G}_1(z) = -\frac{1}{16\cos^3(\pi/3)}(z - 4\cos(\pi/3) + z^{-1})$. This is the same as the previous solution except that ω_1 in the previous solution is replaced with a constant value equal to $\pi/3$.

As another example, let us fix $\omega_1 = 2\pi/5 > \pi/3$, and find the optimal FIR compaction filter of order 5. The corresponding product filter is $G(z) = \frac{1}{2}z^5 + 1 + \frac{1}{2}z^{-5}$ and the compaction gain is $G_{opt}(2, 5) = 2$ which is the largest possible gain for $M = 2$! Since the process is line-spectral, this is not surprising. The important point here is that while the algorithm is not successful for the filter order 3, it is successful for a higher order 5.

Example 12: Case where the process is multiband. Finally we will consider an example in which the input is not "low-pass" or "high-pass" but rather is of multiband nature. Let $r(0) = 1, r(1) = \frac{1}{10}, r(2) = 0$, and $r(3) = -\frac{1}{4}$. The sequence $\psi_x(n)$ is positive definite for $N = 3$ so that the algorithm is applicable. There is more than one way to extrapolate this sequence and find the corresponding psd. For example, one can consider MA(3), AR(3), or line-spectra(4). In all three cases, we have verified that the psd is neither 'low-pass' nor 'high-pass'. Rather it is of multi-band nature. Applying the algorithm steps we have $\hat{G}_0(z) = (z + \frac{1}{\sqrt{2}} + z^{-1})^2$ from which it follows that $\hat{G}_1(z) = -\sqrt{2}(z - \sqrt{2} + z^{-1})$. This has a complex conjugate pair of unit-circle zeros! Hence $\hat{G}_1(e^{j\omega})$ is not nonnegative and therefore $G(e^{j\omega}) = \hat{G}_0(e^{j\omega})\hat{G}_1(e^{j\omega})$ is not nonnegative either. The algorithm halts because $\hat{G}_1(e^{j\omega})$ cannot be spectrally factorized.

VI. CONCLUDING REMARKS

In this paper we have developed some design techniques for optimal FIR compaction filters and elaborated some of their properties. We have proposed a procedure to guarantee the nonnegativity of the linear programming solutions. Multistage (IFIR) designs are considered and some design examples are provided to demonstrate the usefulness. We have developed a new and efficient design method that we called the window method which does not involve any of the iterative optimization techniques such as linear programming. We have given its relation to the linear programming technique. Finally we have given an analytical method for the two-channel case that works for a broad class of random processes. In all the techniques we have concentrated on finding the product filter $G(z) = H(z)\tilde{H}(z)$ of the compaction filter $H(z)$. Hence the final stage of every algorithm is a spectral factorization to find the compaction filter $H(z)$. However, we can first calculate the compaction gains for different filter orders using the product filter coefficients $g(n)$. This enables one to decide what filter order to use. The bounds we gave on the maximum compaction gain are also useful for this purpose.

APPENDIX A

Fact A1. Let $x_L(n)$ be a periodic sequence with period $L = KM$ and consider the decimated sequence $y_K(n) = x_L(Mn)$ which is periodic with period K. Let $X_L(k)$ and $Y_K(k)$ be the respective Fourier series coefficients. Then we have $Y_K(k) = \frac{1}{M}\sum_{i=0}^{M-1} X_L(k + iK)$.

Proof. The Fourier series coefficients of $y_K(n)$ is

$$\begin{aligned} Y_K(k) &= \sum_{n=0}^{K-1} x_L(Mn)W_K^{kn} = \sum_{n=0}^{K-1}\frac{1}{L}\sum_{l=0}^{L-1} X_L(l)W_L^{-Mln}W_K^{kn} = \frac{1}{M}\sum_{l=0}^{L-1} X_L(l)\frac{1}{K}\sum_{n=0}^{K-1} W_K^{(k-l)n} \\ &= \frac{1}{M}\sum_{i=0}^{M-1}\sum_{j=0}^{K-1} X_L(j+iK)\frac{1}{K}\sum_{n=0}^{K-1} W_K^{(k-j)n} = \frac{1}{M}\sum_{i=0}^{M-1} X_L(k+iK) \end{aligned} \tag{73}$$

In the last step we have used the fact that $\frac{1}{K}\sum_{n=0}^{K-1} W_K^{mn} = \delta_K(m)$. ■

Proof of Lemma 2. The Fourier series coefficients of $\delta_K(n)$ are all 1. Hence from Fact A1, it follows that $y_K(n) = x_L(Mn) = \delta_K(n)$ if and only if $Y_K(k) = \frac{1}{M}\sum_{i=0}^{M-1} X_L(k + iK) = 1$. ■

APPENDIX B

Proof of nonnegativity. We will show that $G(e^{j\omega})$ obtained by the procedure in Sec. V is necessarily nonnegative in the region $[\pi/2, \pi]$. The Nyquist(2) property of $G(e^{j\omega})$ implies $G'(e^{j\omega}) = G'(e^{j(\pi-\omega)})$. We therefore have $G'(e^{j\omega_k/2}) = G'(e^{j(\pi-\omega_k/2)}) = 0, \quad k = 0, \ldots, (N-3)/4$. Now, by the mean value theorem in calculus, we also have $G'(e^{j\hat{\omega}_k/2}) = G'(e^{j(\pi-\hat{\omega}_k/2)}) = 0, \quad k = 0, \ldots, (N-7)/4$, for some $\hat{\omega}_k \in (\omega_k, \omega_{k+1})$. Notice that since ω_k's are all distinct and lie in the open region $(0, \pi)$, all of the above zeros are distinct. The total number of such zeros is therefore $N - 1$. Since $G(e^{j\omega})$ is a cosine polynomial of order N, $G'(e^{j\omega})$ is a sine polynomial of order N and therefore it can be written in the form $G'(e^{j\omega}) = \sin\omega\ T(\cos\omega)$, where $T(x)$ is a polynomial of order $N - 1$. Excluding the zeros at 0 and π, the total number of zeros $G'(e^{j\omega})$ can have in $[0, \pi]$ is $N - 1$. Hence $G'(e^{j\omega})$ cannot have any other zero on the unit-circle. If $G(z)$ has a zero at $\pi - \omega_k/2$ with multiplicity greater than 2, then, $G'(e^{j\omega})$ has at least double zero at that frequency implying that the total number of its zeros is more than $N-1$ which is a contradiction. If $G(z)$ has a single zero in the region $(\pi/2, \pi)$ which is different from all ω_k's, then, by applying the mean value theorem once more, $G'(e^{j\omega})$ has to have another zero which is again a contradiction. Hence we have proved that $G(e^{j\omega})$ has double zeros at $\pi - \omega_k/2,\ k = 0, \ldots, (N-3)/4$, and that it does not have any other unit-circle zeros in $[\pi/2, \pi]$. This in particular implies $G(e^{j\omega}) \geq 0$ for $\omega \in [\pi/2, \pi]$. The proof for the case of even $\frac{N-1}{2}$ is similar, the details are omitted. ∎

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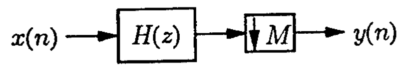

Figure 1: Pertaining to the FIR energy compaction problem.

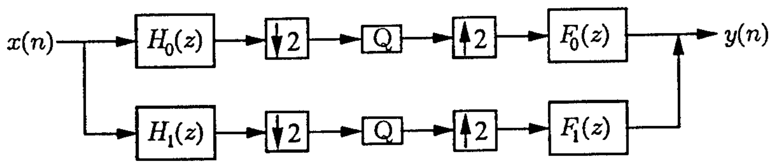

Figure 2: Two-channel orthonormal filter bank. The filter $H_0(z)$ determines all the other filters.

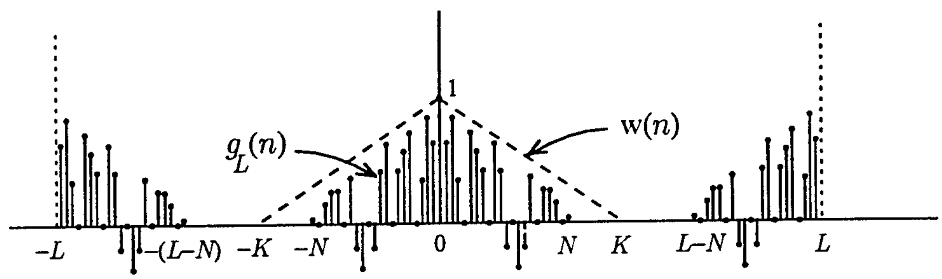

Figure 3: Windowing of the linear programming solution. The window $w(n)$ has nonnegative Fourier transform. For illustration purposes the window is chosen to be triangular.

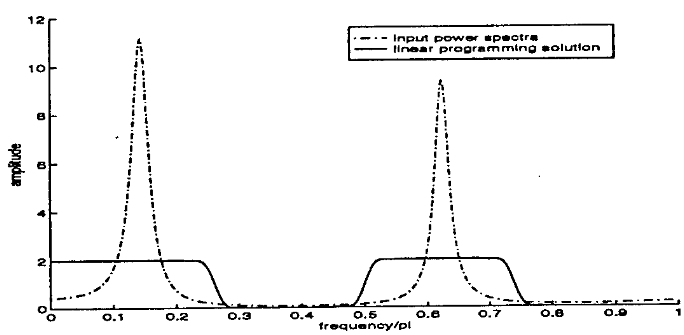

Figure 4: The psd of an AR(5) process, and the magnitude square of an optimal compaction filter for $N = 65$ and $M = 2$, designed by linear programming. The parameter L is 1024 and a triangular window is used.

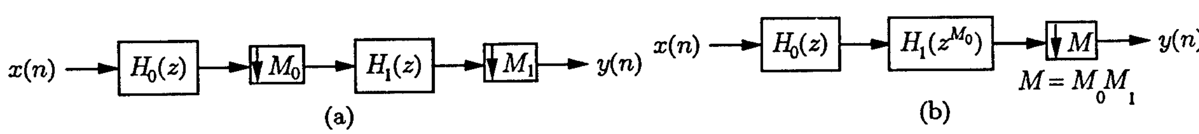

Figure 5: IFIR compaction filter design. (a) Basic configuration, (b) Equivalent system.

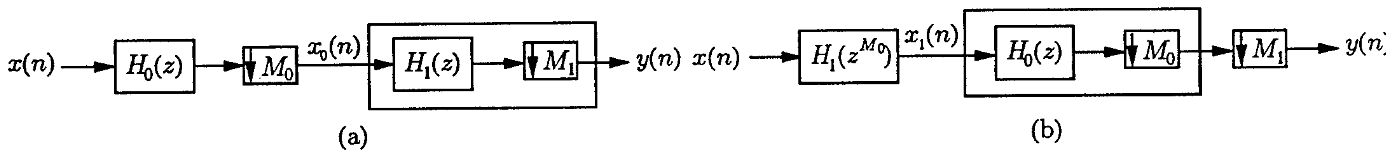

Figure 6: Special IFIR design configuration where $H_0(z)$ is a valid compaction filter for (M_0, N_0) and $H_1(z)$ is a valid compaction filter for (M_1, N_1). (a) $H_0(z)$ is fixed, $H_1(z)$ is optimum compaction filter for $x_0(n)$, (b) $H_1(z)$ is fixed, $H_0(z)$ is optimum compaction filter for $x_1(n)$.

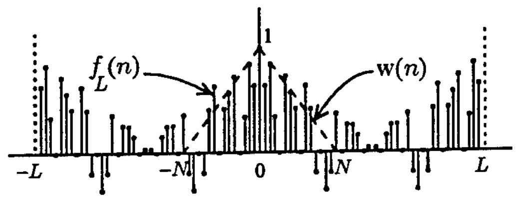

Figure 7: Decomposition of $g(n)$ as $w(n)f_L(n)$ where $W(e^{j\omega}) \geq 0$ and $F_L(k) \geq 0$.

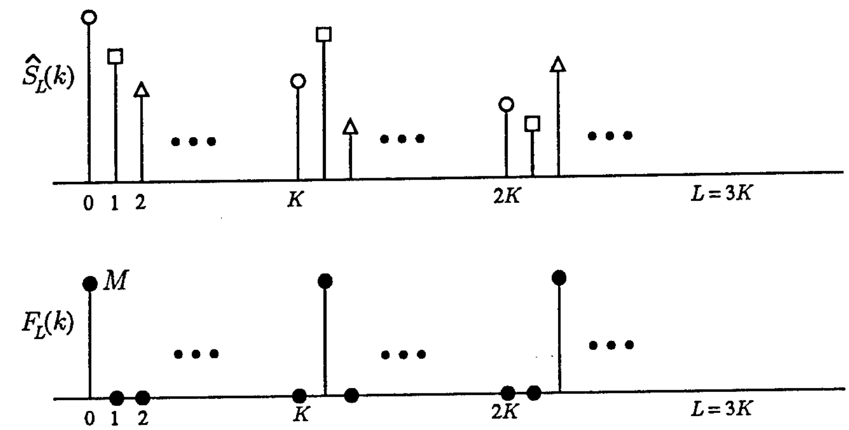

Figure 8: The procedure to find $F_L(k)$: $\hat{S}_L(0)$ is maximum among $\{\hat{S}_L(iK)\}$, hence $F_L(0) = M$, $F_L(lK) = 0, l \neq 0$. $\hat{S}_L(1+K)$ is maximum among $\{\hat{S}_L(1+iK)\}$, hence $F_L(1+K) = M$, $F_L(1+lK) = 0, l \neq 1$, and so on.

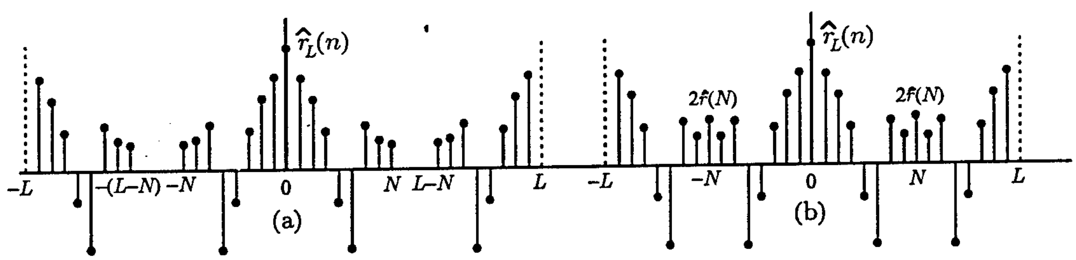

Figure 9: Periodical expansion of $\hat{r}(n)$. (a) $L > 2N$, (b) $L = 2N$.

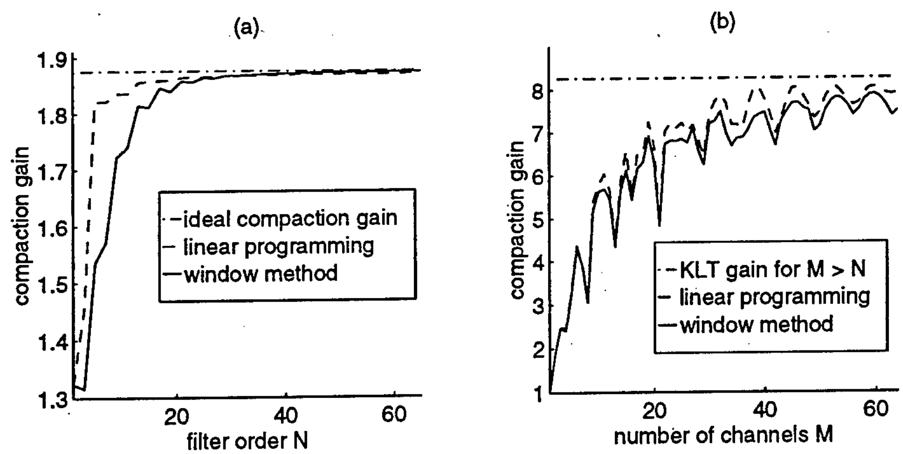

Figure 10: Comparison of the window and linear programming methods. The input power spectrum is as shown in Fig. 4. (a) Compaction gain versus filter order, $M = 2$, (b) Compaction gain versus number of channels, $N = 65$.

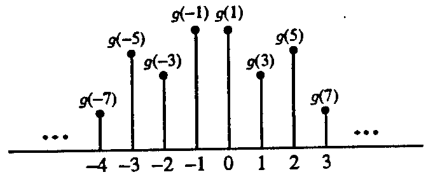

Figure 11: Coefficients of the polyphase component $E_1(z)$ of the product filter $G(z)$. Because of the symmetry $g(n) = g(-n)$, we have $E_1(-1) = 0$.

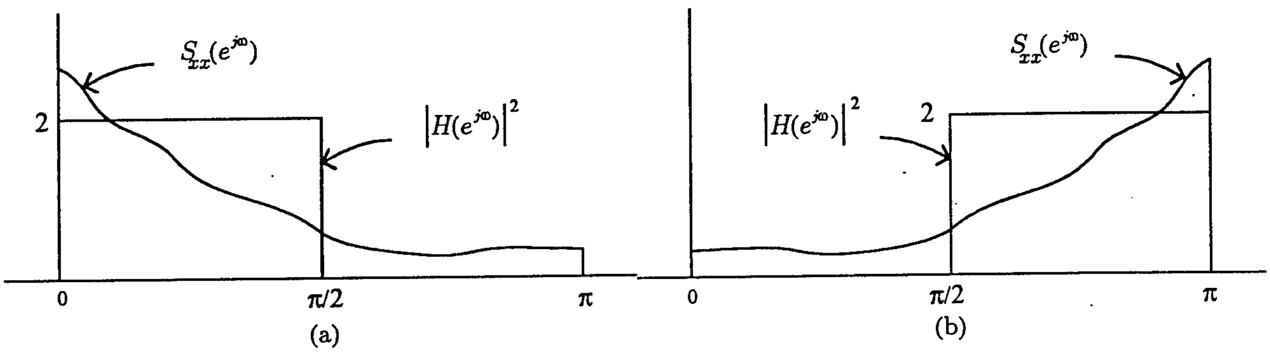

Figure 12: Illustration of low-pass and high-pass power spectra. (a) A low-pass power spectral density and the corresponding optimum ideal compaction filter response, (b) A high-pass power spectral density and the corresponding optimum ideal compaction filter response.

$\rho = 0.1$

n	analytical method	window method	linear programming
0	0.5494144350	0.6940928372	0.5839818982
1	0.7789293967	0.7136056607	0.7658293099
2	0.2470689810	0.0680132766	0.2140666953
3	-0.1742690225	-0.0661535225	-0.1632362113
compaction gain	1.1153	1.1078	1.1151

$\rho = 0.5$

n	analytical method	window method	linear programming
0	0.5308991349	0.6817974052	0.5693221037
1	0.7963487023	0.7258587819	0.7821354608
2	0.2411149862	0.0663296736	0.2047520302
3	-0.1607433241	-0.0623033026	-0.1490404954
compaction gain	1.5547	1.5283	1.5537

$\rho = 0.9$

n	analytical method	window method	linear programming
0	0.4938994371	0.6550553981	0.5605331011
1	0.8279263239	0.7510864372	0.8017336546
2	0.2281902949	0.0620169861	0.1699982390
3	-0.1361269173	-0.0540877314	-0.1188544843
compaction gain	1.9222	1.9118	1.9207

Table 1: Compaction filter coefficients $h(n)$ and corresponding gains for AR(1) processes with various ρ values. The filter order is $N = 3$ and the number of channels is $M = 2$.

n	N = 3	N = 9	N = 15	N = 21
0	0.5502267080	0.3472380509	0.2619442448	0.2135948251
1	0.7781380728	0.7212669193	0.6444985282	0.5837095513
2	0.2473212614	0.5313628729	0.6197371546	0.6522949264
3	-0.1748825411	-0.0301418144	0.1178983343	0.2237822545
4		-0.2357012104	-0.2498547909	-0.2185003959
5		0.0008621669	-0.1127984531	-0.1849242462
6		0.1250275869	0.1367316336	0.1009864921
7		-0.0141611881	0.0800586128	0.1357184335
8		-0.0608205190	-0.0879123348	-0.0574793351
9		0.0292806975	-0.0512638394	-0.0996883819
10			0.0616834351	0.0386787054
11			0.0272577935	0.0732876341
12			-0.0441106561	-0.0298852666
13			-0.0065141401	-0.0529392251
14			0.0275486991	0.0255464898
15			-0.0111966480	0.0362662187
16				-0.0230508869
17				-0.0216340483
18				0.0206081274
19				0.0077883397
20				-0.0156868986
21				0.0057402527
$\rho = 0.1$	1.1155	1.1244	1.1260	1.1266
compaction gains $\rho = 0.3$	1.3464	1.3732	1.3781	1.3798
$\rho = 0.5$	1.5774	1.6220	1.6301	1.6330

Table 2: Compaction filter coefficients and corresponding gains for MA(1) processes. The number of channels is 2.

FOOTNOTES

1. Manuscript received ______________________________

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3. The authors are with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125.