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Learning in Embedded Systems

by

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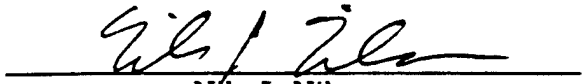
June 1990

A dissertation submitted to the Department of Computer Science and the Committee on Graduate Studies of Stanford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.



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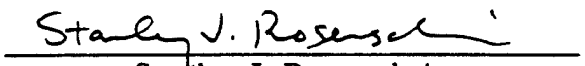

Stanley J. Rosenschein

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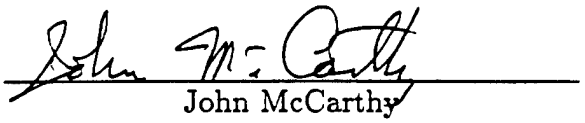

John McCarthy

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Approved for the University Committee on Graduate Studies:

Dean of Graduate Studies

Abstract

This dissertation addresses the problem of designing algorithms for learning in embedded systems. This problem differs from the traditional supervised learning problem. An agent, finding itself in a particular input situation must generate an action. It then receives a reinforcement value from the environment, indicating how valuable the current state of the environment is for the agent. The agent cannot, however, deduce the reinforcement value that would have resulted from executing any of its other actions. A number of algorithms for learning action strategies from reinforcement values are presented and compared empirically with existing reinforcement-learning algorithms.

The interval-estimation algorithm uses the statistical notion of confidence intervals to guide its generation of actions in the world, trading off acting to gain information against acting to gain reinforcement. It performs well in simple domains but does not exhibit any generalization and is computationally complex.

The cascade algorithm is a structural credit-assignment method that allows an action strategy with many output bits to be learned by a collection of reinforcement-learning modules that learn Boolean functions. This method represents an improvement in computational complexity and often in learning rate.

Two algorithms for learning Boolean functions in k-DNF are described. Both are based on Valiant's algorithm for learning such functions from input-output instances. The first uses Sutton's techniques for linear association and reinforcement comparison, while the second uses techniques from the interval estimation algorithm. They both perform well and have tractable complexity.

A generate-and-test reinforcement-learning algorithm is presented. It allows symbolic representations of Boolean functions to be constructed incrementally and tested in the environment. It is highly parametrized and can be tuned to learn a broad range of function classes. Low-complexity functions can be learned very efficiently even in the presence of large numbers of irrelevant input bits. This algorithm is extended to construct simple sequential networks using a set-reset operator, which allows the agent to learn action strategies with state.

These algorithms, in addition to being studied in simulation, were implemented and tested on a physical mobile robot.

Acknowledgements

My first thanks go to Stan Rosenschein. He has been a colleague and mentor, providing me with lots of ideas, criticism and inspiration, and with two excellent environments for carrying out research. Thanks to John McCarthy for getting me interested in AI and to Nils Nilsson for making me go back to school before I forgot how. Rich Sutton helped me to ground my work in the context of existing literatures I had never heard of and suggested interesting extensions of my basic ideas. Jeff Kerr and Stanley Reifel have provided robots to play with and lots of help and good advice. David Chapman, Stan Rosenschein, Rich Sutton and Nils Nilsson provided insightful comments on drafts of this dissertation; they have helped the clarity of the exposition considerably. Thanks to Ann Reid for baking the scones and being a distraction. Many other colleagues at Stanford, CSLI, SRI and Teleos have helped in indirect but important ways. David Kaelbling has been kind and patient through all this and was a big help with running the experiments.

Finally, I would like to dedicate this dissertation to the memory of my parents, who taught me I could do anything if I tried.

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Chapter 1

Introduction

Embedded systems, such as autonomous robots and process controllers, must be able to learn about and adapt to their environments. This dissertation addresses the problem of designing algorithms for learning in embedded systems. It provides a formal framework in which this problem can be explored, discusses previous work in this area, and then goes on to present novel algorithms for efficient and effective learning in embedded systems. These algorithms are explored theoretically and are validated empirically, both in simulation and in use on a mobile robot.

1.1 Why Learn?

Why should we build learning agents? A program that "learns" is not intrinsically better than one that does not.

One reason to build learning agents is that it is very difficult for humans to write explicit programs for agents that must work in complex, uncertain environments. In programming robots, for instance, it is common for a human programmer to learn a great deal about the operation of the robot's sensors and effectors in the course of debugging programs for the robot. It would be much easier and less time-consuming if the programmer were able to articulate only general principles about the environment, allowing the robot to experiment and learn about the details.

Another reason for building agents that learn to act is that we would like to have agents that are flexible enough to work in a variety of environments, adapting their perception and action strategies to the worlds in which they find themselves. Even if a human could completely specify the program for an agent operating in a particular environment, the agent's program would have to be respecified if the agent were moved to a new environment.

1.2 Reinforcement Learning

When building learning agents, the goal of the agent's designer is for the agent to learn what actions it should perform in which situations in order to maximize an external measure of success. All of the information the agent has about the external world is contained in a series of inputs that it receives from the environment. These inputs may encode information ranging from the output of a vision system to a robot's current battery voltage. The agent can be in many different states of information about the environment, and it must map each of these information states, or situations, to a particular action that it can perform in the world. The agent's mapping from situations to actions is referred to as an *action map*. Part of the agent's input from the world encodes the agent's *reinforcement*, which is a scalar measure of how well the agent is performing in the world. The agent should learn to act in such a way as to maximize the total reinforcement it gains over its lifetime.

As a concrete example, consider a simple robot with two wheels and two photosensors. It can execute five different actions: stop, go forward, go backward, turn left, and turn right. It can sense three different states of the world: the light in the left eye is brighter than that in the right eye, the light in the right eye is brighter than that in the left eye, and the light in both eyes is roughly equally bright. Additionally, the robot is given high values of reinforcement when the average value of light in the two eyes is increased from the previous instant. In order to maximize its reinforcement, this robot should turn left when the light in its left eye is brighter, turn right when the light in its right eye is brighter, and move forward when the

light in both eyes is equal. The problem of learning to act is to discover such a mapping from information states to actions.

Thus, the problem of learning to act can be cast as a function-learning problem: the agent must learn a mapping from the situations in which it finds itself, represented by streams of input values, to the actions it can perform. In the simplest case, the mapping will be a pure function of the current input value, but in general it can have state, allowing the action taken at a particular time to depend on the entire stream of previous input values.

In the past few years there has been a great deal of work in the artificial intelligence (AI) and theoretical computer science communities on the problem of learning pure Boolean-valued functions [31,43,50,55,76]. Unfortunately, this work is not directly relevant to the problem of learning action maps because of the different settings of the problem. In the traditional function-learning work, often referred to in the AI community as "concept learning," a learning algorithm is presented with a set or series of input-output pairs that specify the correct output to be generated for that particular input. This setting allows for effective function learning, but differs from the situation of an agent trying to learn an action map. The agent, finding itself in a particular input situation, must generate an action. It then receives a reinforcement value from the environment, indicating how valuable the current world state is for the agent. The agent cannot, however, deduce the reinforcement value that would have resulted from executing any of its other actions. Also, if the environment is noisy, as it will be in general, just one instance of performing an action in a situation may not give an accurate picture of the reinforcement value of that action.

This learning scenario reduces to concept learning when the agent has only two possible actions, the world generates Boolean reinforcement that depends only on the most recently taken action, there is exactly one action that generates the high reinforcement value in each situation, and there is no noise. In this case, from performing a particular action in a situation, the agent can deduce that it was the correct action if it was positively reinforced; otherwise it can infer that the other action would have been correct.

The problem of learning action maps by trial and error is often referred to as *reinforcement learning* because of its similarity to models used in psychological studies of behavior-learning in humans and animals [22]. It is also referred to as "learning with a critic," in contrast with the "learning with a teacher" of traditional supervised concept learning [81]. One of the most interesting facets of the reinforcement-learning problem is the tension between performing actions that are not well understood in order to gain information about their reinforcement value and performing actions that are expected to be good in order to increase overall reinforcement. If an agent knows that a particular action works well in a certain situation, it must trade off performing that action against performing another one that it knows nothing about, in case the second action is even better than the first. Or, as Ashby [6] put it,

> The process of trial and error can thus be viewed from two very different points of view. On the one hand it can be regarded as simply an attempt at success; so that when it fails we give zero marks for success. From this point of view it is merely a second-rate way of getting to success. There is, however, the other point of view that gives it an altogether higher status, for the process may be playing the invaluable part of *gathering information*, information that is absolutely necessary if adaptation is to be successfully achieved.

The longer the time span over which the agent will be acting, the more important it is for the agent to be acting on the basis of correct information. Acting to gain information may improve the expected long-term performance while causing short-term performance to decline.

Another important aspect of the reinforcement-learning problem is that the actions that an agent performs influence the input situations in which it will find itself in the future. Rather than receiving an independently chosen set of input-output pairs, the agent has some control over what inputs it will receive and complete control over what outputs will be generated in response. In addition to making it difficult to make distributional statements about the inputs to the agent, this

degree of control makes it possible for what seem like small "experiments" to cause the agent to discover an entirely new part of its environment.

1.3 Models versus Action Maps

One way for an agent to learn an action map is first to learn a state-transition model of the world and the expected reinforcement value gained from being in each world state, and then to apply standard dynamic programming techniques to choose the best action from any given world state. Although this method will work in the general case, the internal structures that the agent must build up will tend to be quite complex.

When the target action-map is state-free, it can be represented much more compactly and executed much more directly as a simple function, rather than as a world model with a procedure for choosing the optimal action. Sutton [72] and Whitehead and Ballard [80] have found that in cases in which the reinforcement from the world is delayed, learning may be sped up by a kind of compilation from a world model. However, this opens up the new problem of learning world models, which has been addressed by a number of people, including Sutton and Pinette [73], Drescher [18], Mason, Christiansen, and Mitchell [40], Mel [42], and Shen [68].

This dissertation will focus on methods for learning action maps without using models. Even those methods that do use models have this simpler form of reinforcement learning as a component, so improved algorithms for learning action maps will benefit both approaches.

1.4 Statistical versus Symbolic Learning

Most previous learning work can be divided into statistical and symbolic methods.

Statistical learning encompasses much of the early learning work in pattern recognition [54] and adaptive control [25], as well as current work in artificial neural networks (also known as connectionist systems) [9]. The internal representations used are typically numeric and the correctness of algorithms is demonstrated using

statistical methods. These systems tend to be highly noise-tolerant and robust. However, the internal states are difficult for humans to interpret and the algorithms often perform poorly on complex problems.

More symbolic approaches to learning, such as those standardly pursued in the artificial intelligence community, attempt to address these issues of understandability and complexity. They have resulted in algorithms, such as Mitchell's version spaces [49] and Michalski's STAR [43], that use easily-interpretable symbolic representations and whose correctness hinges on arguments from logic rather than from statistics. These algorithms tend to suffer from noise-intolerance and high computational complexity, more so than statistical algorithms do.

One of the aims of the work in this dissertation is to blend the statistical and the symbolic in algorithms for reinforcement learning in embedded systems. An important characteristic of most embedded systems is that they operate in environments that are not (to them) completely predictable. In order to work effectively in such environments, a system must be able to summarize general tendencies of its environment. The well-understood methods of statistics are most appropriate for this task. This does not, however, mean we must abandon all of the benefits of symbolic AI methods. Rather, these two approaches can be synthesized to make learning systems that are robust and noise-tolerant as well as being easy to understand and capable of working in complex environments. A good example of this kind of synthesis is Quinlan's successful concept-learning method, ID3 [55]. Within the combined approach, complexity issues can be addressed by explicitly considering limited classes of functions to be learned.

Many researchers use symbolic representations because, as Michie [45] puts it, "In AI-type learning, explainability is all." That is not the motivation for this work, which simply seeks the most effective algorithms for building embedded systems. There is, however, an important benefit of using symbolic representations of concepts and strategies being learned by an agent: it may allow the learned knowledge to be more easily integrated with knowledge that is provided by humans at design time. Although such integration is not explored in this dissertation, it is an important direction in which learning research should be pursued.

1.5 Organization

The next chapter addresses the formal foundations of reinforcement learning. These arise largely from previous work in statistics, dynamic programming, and learning-automata theory. These foundations are important to AI because they allow widely disparate algorithms to be compared in common, objective terms. Chapter 3 goes on to present previous work on algorithms for reinforcement learning from a variety of different literatures. This previous work is the direct basis of many of the new algorithms and results presented in this dissertation.

Chapter 4 describes a novel statistical algorithm for reinforcement learning. It empirically shows this algorithm to be more effective than a variety of other reinforcement-learning algorithms. Finally, it discusses weaknesses of this algorithm and other related algorithms, due to high computational complexity and lack of generalization across input instances.

Chapter 5 describes a problem reduction and an algorithm that can be used to implement it. The problem of learning an action map with many output bits can be reduced to the problem of learning many action maps, each with a single output bit. This will allow us to restrict our attention to learning action maps that can be described as Boolean functions, knowing they can be recombined to form more complex systems.

Chapters 6 and 7 each present a novel algorithm for learning Boolean functions from reinforcement; these algorithms represent points on a generality-efficiency tradeoff. The algorithm in Chapter 6 is restricted to learning Boolean functions describable as propositional formulae in the class k-DNF, but it learns these functions more efficiently than the algorithms of Chapters 3 and 4. The algorithm in Chapter 7 is more flexible—according to the settings of internal parameters, it can be made more or less restricted and, hence, more or less efficient.

All of the discussion up to this point has been of environments that present reinforcement immediately and of action maps that are pure, state-free functions. Chapter 8 presents an extended version of the algorithm of Chapter 7 that can learn

simple action maps with state. Chapter 9 addresses the problem of delayed reinforcement. It presents two existing methods and shows how they may be combined with the statistical method developed in Chapter 4.

The algorithms presented in this dissertation are finally validated through their application to moderately complex domains, including a real mobile robot. Chapter 10 describes these experiments, documenting their successes and failures. Finally, Chapter 11 summarizes the work presented in the previous chapters. It notes problems and points out important directions for future research.

Chapter 2

Foundations

This chapter focuses on building formal foundations for the problem of learning in embedded systems. These foundations must allow a clear statement of the problem and provide a basis for evaluating and comparing learning algorithms. It is important to establish such a basis: there are many instances in the machine learning literature of researchers doing interesting work on learning systems, but reporting the results using evaluation metrics that make it difficult to compare their results with the results of others. The foundational ideas presented in this chapter are a synthesis of previous work in statistics [12], dynamic programming [57], the theory of learning automata [53], and previous work on the foundations of reinforcement learning [8,70,71,78,83,84].

2.1 Acting in a Complex World

An embedded system, or agent, can be seen as acting in a world, continually executing a procedure that maps the agent's perceptual inputs to its effector outputs. Its world, or environment, is everything that is outside the agent itself, possibly including other robotic agents or humans. The agent operates in a cycle, receiving an input from the world, performing some computation, then generating an output that affects the world. The mapping that it uses may have state or memory, allowing its action at any time to depend, potentially, on the entire stream of inputs that

it has received until that time. Such a mapping from an input stream to an output stream is referred to as a *behavior*.

In order to study the effectiveness of particular behaviors, whether or not they involve learning, we must model the connection between agent and world, understanding how an agent's actions affect its world and, hence, its own input stream.

2.1.1 Modeling an Agent's Interaction with the World

The world can be modeled as a deterministic finite automaton whose state transitions depend on the actions of an agent [41]. From the agent's perspective, the world is everything that is not itself, including other agents and processes. This model will be extended to include non-deterministic worlds in the next section. A world can be formally modeled as the triple $(S, \mathcal{A}, W)$, in which S is the set of possible states of the world, $\mathcal{A}$ is the set of possible outputs from the agent to the world (or actions that can be performed by the agent), and W is the state transition function, mapping $S \times \mathcal{A}$ into S. Once the world has been fixed, the agent can be modeled as the 4-tuple $(\mathcal{I}, I, R, B)$ where $\mathcal{I}$ is the set of possible inputs from the world to the agent, I is a mapping from S to $\mathcal{I}$ that determines which input the agent will receive when the world is in a given state, R is the reinforcement function of the agent that maps S into real numbers (it may also be useful to consider more limited models in which the output of the reinforcement function is Boolean-valued), and B is the behavior of the agent, mapping $\mathcal{I}^*$ (streams of inputs) into $\mathcal{A}$. The expressions $i(t)$ and $a(t)$ will denote the input received by the agent at time t and the action taken by the agent at time t, respectively.

The process of an agent's interaction with the world is depicted in Figure 1. The world is in some internal state, s, which is projected into i and r by the input and reinforcement functions I and R. These values serve as inputs to the agent's behavior, B, which generates an action a as output. Once per synchronous cycle of this system, the value of a, together with the old value of world state s, is transformed into a new value of world state s by the world's transition function W.

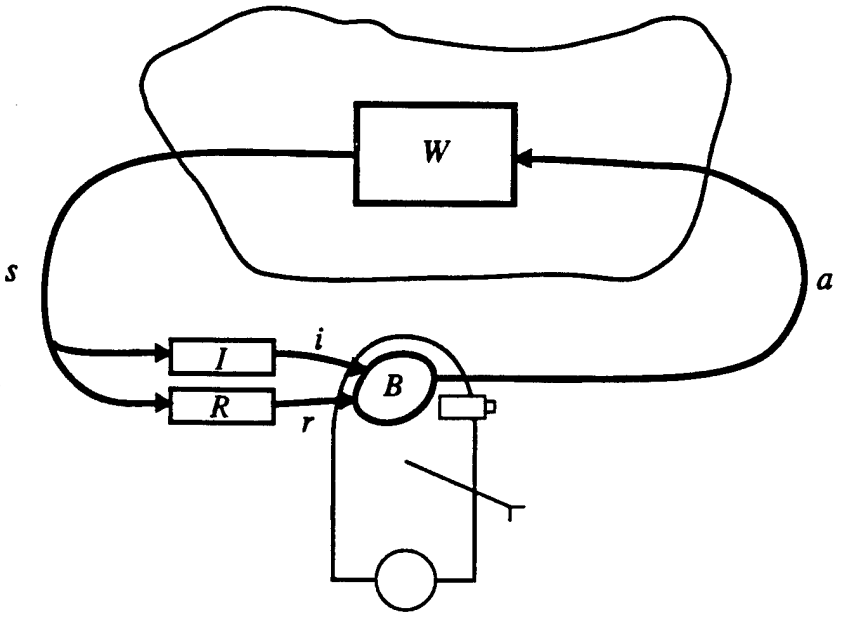


Figure 1: An agent's interaction with its world.

Note that if the agent does not have a simple stimulus-response behavior, but has some internal state, then the action taken by the behavior can be a function of both its input and its internal state. This internal state may allow the agent to discriminate among more states of the world and, hence, to obtain higher reinforcement values by performing more appropriate actions. To simplify the following discussion, actions will be conditioned only on the input, but the treatment can be extended to the case in which the action depends on the agent's internal state as well.

2.1.2 Inconsistent Worlds

One of the most difficult problems that a learning agent must contend with is apparent inconsistency. A world is said to be ***apparently inconsistent*** for an agent if it is possible that, on two different occasions in which the agent receives the same input and generates the same action, the next states of the world differ in their reinforcement or the world changes state in such a way that the same string of future actions will have different reinforcement results.

There are many different phenomena that can account for apparent inconsistency:

- *The agent does not have the ability to discriminate among all world states.* If the agent's input function I is not one-to-one, which will be the case in general, then an individual input could have arisen from many world states. When some of those states respond differently to different actions, the world will appear inconsistent to the agent.

- *The agent has "faulty" sensors.* Some percentage of the time, the world is in a state s, which should cause the agent to receive $I(s)$ as input, but it appears that the world is in some other state s', causing the agent to receive $I(s')$ as input instead. Along with the probability of error, the nature of the errors must be specified: are the erroneously perceived states chosen maliciously, or according to some distribution over the state space, or contingently upon what was to have been the correct input?

- *The agent has "faulty" effectors.* Some percentage of the time, the agent generates action a, but the world actually changes state as if the agent had generated a different action a'. As above, both the probability and nature of the errors must be specified.

- *The world has a probabilistic transition function.* In this case, the world is a stochastic automaton whose transition function, W', actually maps $S \times A$ into a probability distribution over S (a mapping from S into the interval $[0,1]$) that describes the probability that each of the states in S will be the next state of the world.

Some specific cases of noise phenomena above have been studied in the formal function-learning literature. Valiant [76] has explored a model of noise in which, with some small probability, the entire input instance to the agent can be chosen maliciously. This corresponds, roughly, to having simultaneous faults in sensing and action that can be chosen in a way that is maximally bad for the learning algorithm. This model is overly pessimistic and is hard to justify in practical situations.

Angluin [5] works with a model of noise in which input instances are misclassified with some probability; that is, the output part of an input-output pair is specified incorrectly. This is a more realistic model of noise, but is not directly applicable to the action-learning problem under consideration here.

If the behavior of faulty sensors and effectors is not malicious, the inconsistency they cause can be described by transforming the original world model into one in which the set of world states, S, is identical to the set of agent inputs, $\mathcal{I}$, and in which the world has a probabilistic transition function. Inconsistency due to inability to discriminate among world states can also be modeled in this way, but such a model is correct only for the one-step transition probabilities of the system. Reducing each of these phenomena to probabilistic world-transition functions allows the rest of the discussion of embedded behaviors to ignore the other possible modes of inconsistency. The remainder of this section shows how to transform worlds with each type of inconsistency into worlds with state set $\mathcal{I}$ and probabilistic transition functions.

Consider an agent, embedded in a world with deterministic transition function W, whose effectors are faulty with probability ρ, so that when the intended action is a, the actual action is $\nu(a)$. This agent's situation can be described by a probabilistic world transition function $W'(s,a)$ that maps the value of $W(s,a)$ to the probability value $1-\rho$, the value of $W(s,\nu(a))$ to the probability value ρ and all other states to probability value 0. That is,

$$\begin{aligned} W'(s,a)(W(s,a)) &= 1-\rho \\ W'(s,a)(W(s,\nu(a)) &= \rho \end{aligned}$$

The result of performing action a in state s will be $W(s,a)$ with probability $1-\rho$, and $W(s,\nu(a))$ with probability ρ. Figure 2 depicts this transition function. First, a deterministic transition is made based on the action of the agent; then, a probabilistic transition is made by the world. This model can be easily extended to the case in which ν is a mapping from actions to probability distributions over actions. For all a' not equal to a, the value of $W(s,a')$ is mapped to the probability value $\rho\ \nu(a)(a')$, which is the probability, ρ, of an error times the probability that

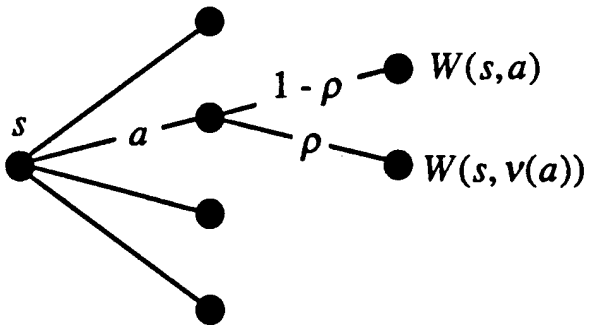


Figure 2: Modeling faulty effectors as a probabilistic world transition function.

action a' will be executed given that the agent intended to execute the action a. The value of $W(s,a)$ is mapped to the probability value $1-\rho+\rho\ \nu(a)(a)$, which is the probability that there is no error, plus the probability that the error actually maps back to the correct action.

Faulty input sensors are somewhat more difficult to model. Let the agent's sensors be faulty with probability ρ, yielding a value $I(\nu(s))$ rather than $I(s)$. We can construct a new model with a probabilistic world-transition function in which the states of the world are those that the agent *thinks* it is in. The model can be most simply viewed if the world makes more than one probabilistic transition, as shown in Figure 3. If it appears that the world is in state s, then with probability ρ_s, it actually is, and the first transition is to the same state. The rest of the probability mass is distributed over the other states in the inverse image of s under ν, $\nu^{-1}(s)$, causing a transition to some world state s' with probability $\rho_{s'}$. Next, there is a transition to a new state on the basis of the agent's action according to the original transition function W. Finally, with probability ρ, the world makes a transition to the state $\nu(W(s',a))$, allowing for the chance that this result will be misperceived on the next tick. In Figure 4, the diagram of Figure 3 is converted into a more standard form, in which the agent performs an action, and then the world makes a probabilistic transition. This construction can also be extended to the cases in which $\nu(s)$ is a probability distribution over S and in which the initial world-transition function is probabilistic.

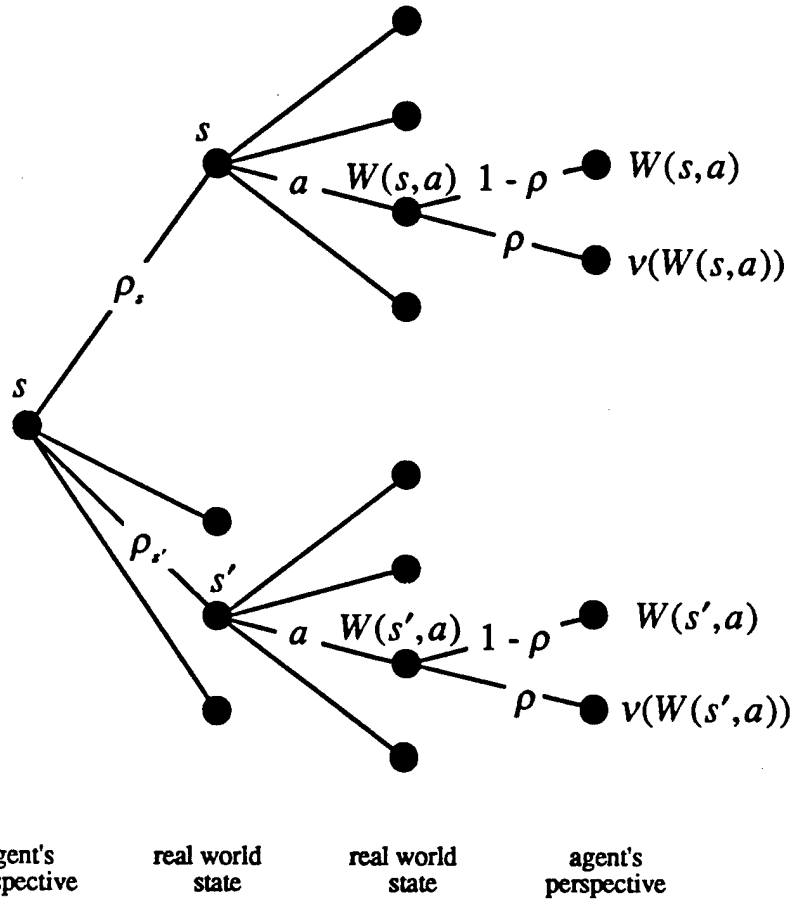


Figure 3: Modeling faulty sensors with multiple probabilistic transitions.

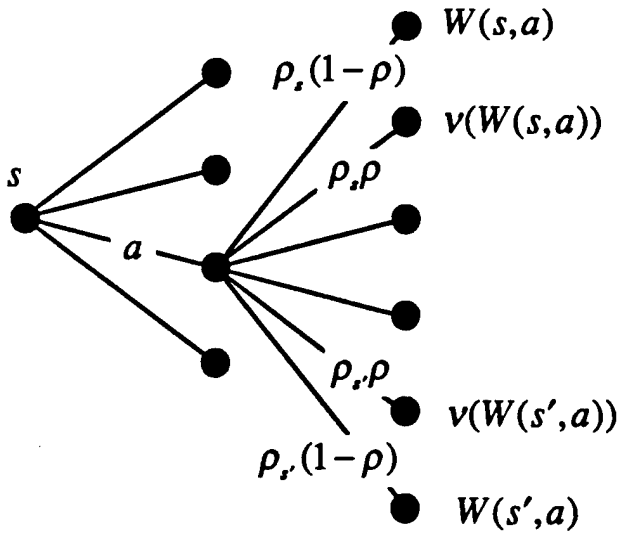


Figure 4: Modeling faulty sensors as a probabilistic world transition function.

We can construct an approximate model of an agent's inability to discriminate among world states by creating a new model of the world in which the elements of $\mathcal{I}$ are the states, standing for equivalence classes of the states in the old model. Let $\{s_1, ..., s_n\}$ be the inverse image of i under I. There is a probabilistic transition to each of the s_j, based on the probability, ρ_j, that the world is in state s_j given that the agent received the input i. From each of these states, the world makes a transition on the basis of the agent's action, a, to the state $W(s_j, a)$, which is finally mapped back down to the new state space by the function I. This process is depicted in Figure 5 and the resulting transition function is shown in Figure 6. The new transition function gives a correct 1-step model of the transition probabilities, but will not generate the same distribution of sequences of two or more states.

In the construction for faulty sensors, it is necessary to evaluate the probability that the world is in some state s_k, given that it appears to the agent to be in another state s. This probability depends on the unconditional probability that the world is in the state s_k, as well as the unconditional probability that the world appears to be in the state s. These unconditional probabilities depend, in the general case, on the behavior that the agent is executing, so the construction cannot be carried out before the behavior is fixed. A similar problem exists for the case of lack of

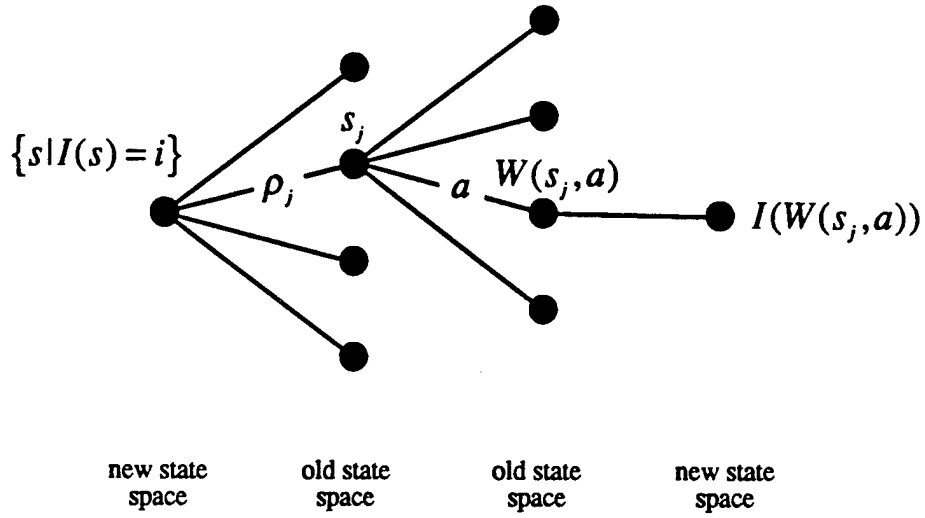


Figure 5: Modeling inability to discriminate among worlds.

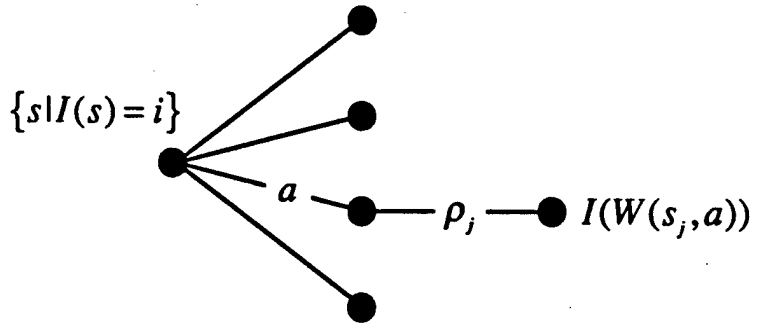


Figure 6: Modeling inability to discriminate among worlds as a probabilistic world transition function.

discrimination: it is necessary to evaluate the probability that the world is in each of the individual states in the inverse image of input i under I given that the agent received input i. These probabilities also depend on the behavior that is being executed by the agent. This leads to a very complex optimization problem that is, in its general form, beyond the scope of this work.

This dissertation will mainly address learning in worlds that are globally consistent for the learning agent. A world is *globally consistent* for an agent if and only if for all inputs $i \in \mathcal{I}$ and actions $a \in A$, the *expected* value of the reinforcement given i and a is constant. Global consistency allows for variations in the result of performing an action in a situation, as long as the expected, or average, result is the same. It simply requires that there not be variations in the world that are undetectable by the agent and that affect its choice of action. Important hidden state in the world can cause such variations; methods for learning to act in such worlds are discussed in Chapter 8. If the transformation described above has been carried out so that the sets $\mathcal{I}$ and S are the same, the requirement for global consistency is tantamount to requiring that the resulting world be a Markov decision process with stationary transition and output probabilities [35]. In addition, the following discussion will assume that the world is consistent over changes in the agent's behavior.

2.1.3 Learning Behaviors

The problem of programming an agent to behave correctly in a world is to choose some behavior B, given that the rest of the parameters of the agent and world are fixed. If the programmer does not know everything about the world, or if he wishes the agent he is designing to be able to operate in a variety of different worlds, he must program an agent that will *learn to behave correctly.* That is, he must find a behavior B' that, through changing parts of its internal state on the basis of its perceptual stream, eventually converges to some behavior B'' that is appropriate for the world that gave rise to its perceptions. Of course, to say that a program learns is just to take a particular perspective on a program with internal state. A behavior with state can be seen as "learning" if parts of its state eventually converge to some

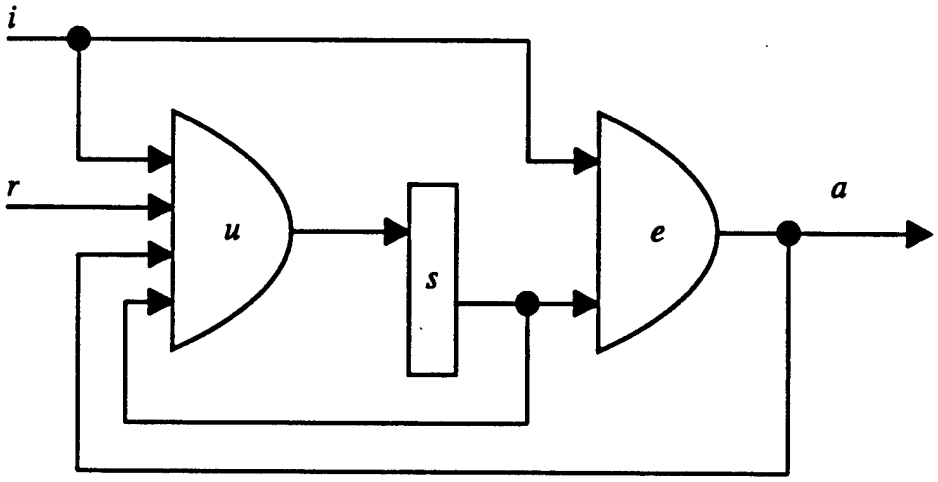


Figure 7: Decomposition of a learning behavior.

fixed or slowly-varying values. The behavior that results from those parameters having been fixed in that way can be called the "learned behavior."[1]

A *learning behavior* is an algorithm that learns an appropriate behavior for an agent in a world. It is itself a behavior, mapping elements of $\mathcal{I}$ to elements of $\mathcal{A}$, but it requires the additional input r, which designates the reinforcement value of the world state for the agent. A learning behavior consists of three parts: an initial state s_0, an update function u, and an evaluation function e.[2] At any moment, the internal state, s, encodes whatever information the learner has chosen to save about its interactions with the world. The update function maps an internal state of the learner, an input, an action, and a reinforcement value into a new internal state, adjusting the current state based on the reinforcement resulting from performing that action in that input situation. The evaluation function maps an internal state

[1] In general, it is very difficult to formally differentiate between processes to which we would apply the natural language term "perception" and those to which we would apply the term "learning." In common usage, "perception" tends to refer to gaining information that is specific, transient, or at a low level of abstraction, whereas "learning" tends to refer to more general information that is true over longer time spans. This issue is addressed in more detail in a paper comparing different views of the nature of knowledge [34].

[2] From this point on, the variable s will refer to an internal state of the learning behavior. Because we have assumed the transformations described in the previous section, it is no longer important to name the different states of the world.

```
s := s0
loop
     i := input
     a := e(s,i)
     output a
     r := reinforcement
     s := u(s,i,a,r)
end loop
```

Figure 8: General algorithm for learning behaviors.

and an input into an action, choosing the action that seems most useful for the agent in that situation, based on the information about the world stored in the internal state. Recall that an action can be useful for an agent either because it has a high reinforcement value or because the agent knows little about its outcome. Figure 7 shows a schematic view of the internal structure of a learning behavior. The register s has initial value s_0 and can be thought of as programming the evaluation function e to act as a particular action map. The update function, u, updates the value of s on each clock tick.

A general algorithm for learning behaviors, based on these three components, is shown in Figure 8. The internal state is initialized to s_0, and then the algorithm loops forever. An input is read from the world and the evaluation function is applied to the internal state and the input, resulting in an action, which is then output. At this point, the world makes a transition to a new state. The program next determines the reinforcement associated with the new world state, uses that information, together with the last input and action, to update the internal state, and then goes back to the top of its loop. Formulating learning behaviors in terms of s_0, e, and u facilitates building experimental frameworks that allow testing of different learning behaviors in a wide variety of real and simulated worlds.

2.2 Performance Criteria

In order to compare algorithms for learning behaviors, we must fix the criteria on which they are to be judged. There are three major considerations: correctness, convergence, and time-space complexity. First, we must determine the correct behavior for an agent in a domain. Then we can measure to what degree a learned behavior approximates the correct behavior and the speed, in terms of the number of interactions with the world, with which it converges. We must also be concerned with the amount of time and space needed for computing the update and evaluation functions and with the size of the internal state of the algorithm.

2.2.1 Correctness

When shall we say that a behavior is correct for an agent in an environment? There are many possible answers that will lead to different learning algorithms and analyses. An important quantity is the expected reinforcement that the agent will receive in the next instant, given that the current input is $i(t)$ and the current action is $a(t)$, which can be expressed as

$$\begin{aligned} er(i(t), a(t)) &= E(R(i(t+1)) \mid i(t), a(t)) \\ &= \sum_{i' \in \mathcal{I}} R(i') W'(i(t), a(t))(i') \, . \end{aligned}$$

It is the sum, over all possible next world states, of the probability that the world will make a transition to that state times its reinforcement value. This formulation assumes that the inputs directly correspond to the states of the world and that W' is a probabilistic transition function. If the world is globally consistent for the agent, the process is Markov and the times are irrelevant in the above definition, allowing it to be restated as

$$er(i, a) = \sum_{i' \in \mathcal{I}} R(i') W(i, a)(i') \, .$$

One of the simplest criteria is that a behavior is correct if, at each step, it performs the action that is expected to cause the highest reinforcement value to be

received on the next step. A correct behavior, in this case, is one that generates actions that are optimal under the following definition:

$$\forall i \in \mathcal{I}, a \in A.\ \mathrm{Opt}(i,a) \leftrightarrow \forall a' \in A.\ er(i,a) \geq er(i,a') .$$

Optimal behavior is defined as a relation on inputs and actions rather than as a function, because there may be many actions that are equally good for a given input. However, it can be made into a function by breaking ties arbitrarily. This is a local criterion that may cause the agent to sacrifice future reinforcement for immediately attainable current reinforcement.

The concept of expected reinforcement can be made more global by considering the total expected reinforcement for a finite future interval, or *horizon*, given that an action was taken in a particular input situation. This is often termed the *value* of an action, and it is computed with respect to a particular behavior (because the value of the next action taken depends crucially on how the agent will behave after that). In the following, expected reinforcement is computed under the assumption that the agent will act according to the optimal policy the rest of the time. The expected reinforcement, with horizon k, of doing action a in input situation i at time t is defined as

$$er_k(i(t),a(t)) = E(\sum_{j=1}^{k} R(i(t+j)) \mid i(t),a(t),\forall h < k.\ \mathrm{Opt}_{k-h}(i(t+h),a(t+h))) .$$

This expression can be simplified to a recursive, time-independent formulation, in which the k-step value of an action in a state is just the one-step value of the action in the state plus the expected $k-1$-step value of the optimal action for horizon $k-1$ in the following state:

$$er_k(i,t) = er(i,a) + \sum_{i' \in \mathcal{I}} W'(i,a)(i') er_{k-1}(i', \mathrm{Opt}_{k-1}(i')) .$$

This definition is recursively dependent on the definition of optimality k steps into the future, Opt_k:

$$\forall i \in \mathcal{I}, a \in A.\ \mathrm{Opt}_k(i,a) \leftrightarrow \forall a' \in A.\ er_k(i,a) \geq er_k(i,a') .$$

The values of er_1 and Opt_1 are just er and Opt given above. The k-step value of action a in situation i at time t, $er_k(i,a)$, can be computed by dynamic programming [12]. First, the Opt_1 relation is computed; this allows the er_2 function to be calculated for all i and a. Proceeding for k steps will generate the value for er_k. Because of the assumption that the world is Markov, these values are not dependent on the time. However, if k is large, the computational expense of this method is prohibitive.

Another way to define global optimality is to consider an infinite sum of future reinforcement values in which near term values are weighted more heavily than values to be received in the distant future. This is referred to as a *discounted* sum, depending on the parameter γ to specify the rate of discounting. *Expected discounted reinforcement* at time t is defined as

$$er_\gamma(i(t),a(t)) = E(\sum_{j=1}^{\infty} \gamma^{j-1} R(i(t+j)) \mid i(t),a(t),\forall h > 0.\ \text{Opt}_\gamma(i(t+h),a(t+h)))\ .$$

Properties of the exponential allow us to reduce this expression to

$$er(i(t),a(t)) + \gamma er_\gamma(i(t+1),a(t+1))\ ,$$

which can be expressed independent of time as

$$er_\gamma(i,a) = er(i,a) + \gamma \sum_{i' \in \mathcal{I}} W'(i,a)(i') er_\gamma(i', \text{Opt}_\gamma(i'))\ .$$

The related definition of γ-discounted optimality is given by

$$\forall i \in \mathcal{I}, a \in A.\ \text{Opt}_\gamma(i,a) \leftrightarrow \forall a' \in A.\ er_\gamma(i,a) \geq er_\gamma(i,a')\ .$$

For a given value of γ and a proposed definition of Opt_γ, er_γ can be found by solving a system of equations, one for each possible instantiation of its arguments. A dynamic programming method called *policy iteration* [57] can be used in conjunction with that solution method to adjust policy Opt_γ until it is truly the optimal behavior. This definition of optimality is more widely used than finite-horizon optimality because its exponential form makes it more computationally tractable. It is also an intuitively satisfying model, with slowly diminishing importance attached to events in the distant future.

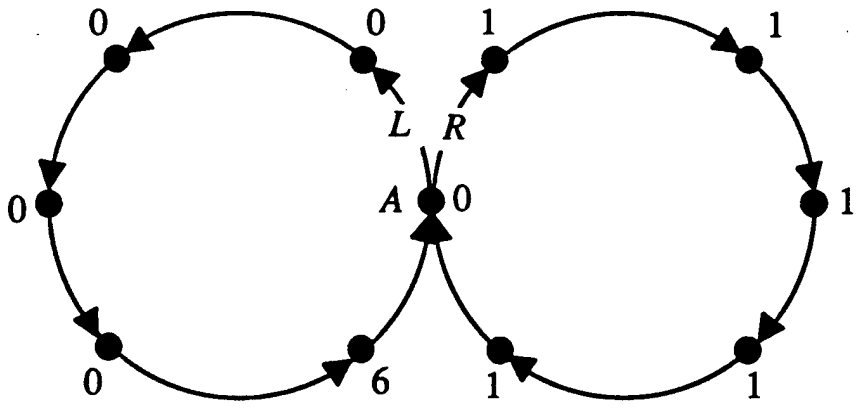


Figure 9: A sample deterministic world. The numbers represent the immediate reinforcement values that the agent will receive when it is in each of the states. The only choice of action is in state A.

As an illustration of these different measures of optimality, consider the world depicted in Figure 9. In state A, the agent has a choice as to whether to go right or left; in all other states the world transition is the same no matter what the agent does. In the left loop, the only reinforcement comes at the last state before state A, but it has value 6. In the right loop, each state has reinforcement value 1. Thus, the average reinforcement is higher around the left loop, but it comes sooner around the right loop. The agent must decide what action to take in state A. Different definitions of optimality lead to different choices of optimal action.

Under the local definition of optimality, we have $er(A, L) = 0$ and $er(A, R) = 1$. The expected return of going left is 0 and of going right is 1, so the optimal action would be to go right.

Using the finite-horizon definition of optimality, which action is optimal depends on the horizon. For very short horizons, it is clearly better to go right. When the horizon, k, is 5, it becomes better to go left. A general rule for optimal behavior is that when in state A, if the horizon is 5 or more, go left, otherwise go right. Figure 10 shows a plot of the values of going left (solid line) and going right (dashed line) initially, assuming that all choices are made optimally thereafter. We can see that going right is initially best, but it is dominated by going left for all $k \geq 5$.

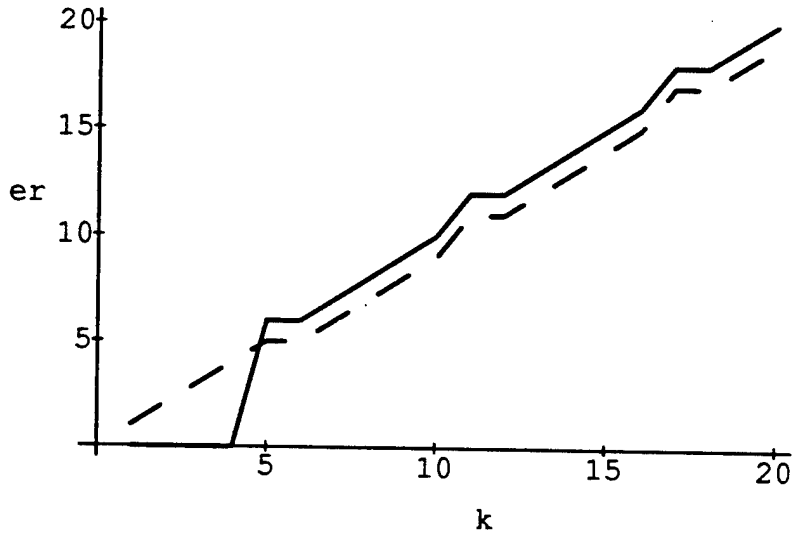


Figure 10: Plot of expected return against horizon k. Solid line indicates strategy of going left first, then behaving optimally. Dashed line indicates strategy of going right first, then behaving optimally.

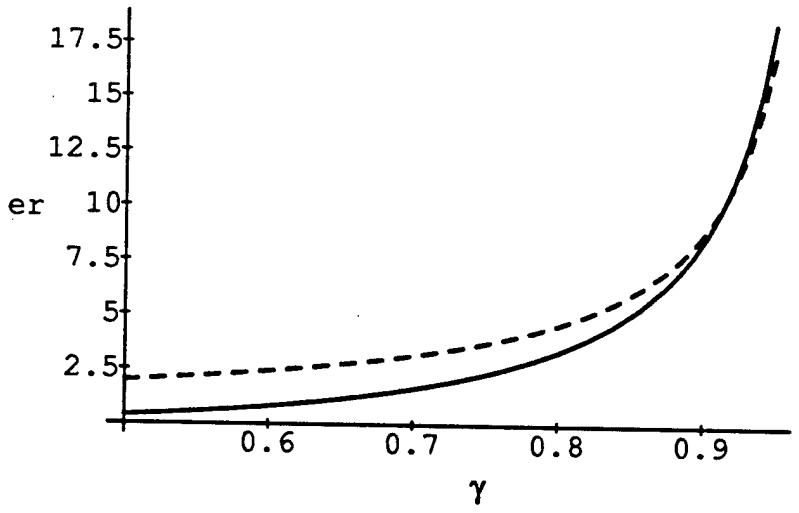


Figure 11: Plot of expected return against discount factor γ. Solid line indicates strategy of always going left. Dashed line indicates strategy of always going right.

Finally, we can consider discounted expected value. Figure 11 shows a plot of the values of the strategies of always going left at state A (solid line) and always going right at state A (dashed line) plotted as a function of γ. When there is a great deal of discounting (γ is small), it is best to go right because the reward happens sooner. As γ increases, going left becomes better, and at approximately $\gamma = 0.915$, going left dominates going right.

Using a global optimality criterion can require agents to learn that chains of actions will result in states with high reinforcement value. In such situations, the agent takes actions not because they directly result in good states, but because they result in states that are closer to the states with high payoff. One way to design learning behaviors that attempt to achieve these difficult kinds of global optimality is to divide the problem into two parts: transducing the global reinforcement signal into a local reinforcement signal and learning to perform the locally best action. The global reinforcement signal is the stream of values of $R(i(t))$ that come from the environment. The optimal local reinforcement signal, $\hat{R}(i(t))$, can be defined as $R(i(t)) + \gamma er_\gamma(i(t), \mathrm{Opt}\gamma(\mathrm{i(t)})$. It is the value of the state $i(t)$ assuming that the agent acts optimally. As shown by Sutton [70], this signal can be approximated by the value of the state $i(t)$ given that the agent follows the policy it is currently executing. Sutton's adaptive heuristic critic (AHC) algorithm, an instance of the general class of temporal difference methods, provides a way of learning to generate the local reinforcement signal from the global reinforcement signal in such a way that, if combined with a correct local learning algorithm, it will converge to the true optimal local reinforcement values [70,71]. A complication introduced by this method is that, from the local behavior-learner's point of view, the world is not stationary. This is because it takes time for the AHC algorithm to converge and because changes in the behavior cause changes in the values of states and therefore in the local reinforcement function. This and related methods will be explored further in Chapter 9.

The following discussion will be in terms of some definition of the optimality of an action for a situation, $\mathrm{Opt}(i, a)$, which can be defined in any of the three ways

above, or in some novel way that is more appropriate for the domain in which a particular agent is working.

2.2.2 Convergence

Correctness is a binary criterion: either a behavior is or is not correct for its world. Since correctness requires that the behavior perform the optimal actions from the outset, it is unlikely that any "learning" behavior will ever be correct. Using a definition of correctness as a reference, however, it is possible to develop other measures of how close particular behaviors come to the optimal behavior. This section will consider two different classes of methods for characterizing how good or useful a behavior is in terms of its relation to the optimal behavior.

Classical Convergence Measures

Early work in the theory of machine learning was largely concerned with *learning in the limit* [13,27]. A behavior converges to the optimal behavior in the limit if there is some time after which every action taken by the behavior is the same as the action that would have been taken by the optimal behavior.

Work in learning-automata theory has relaxed the requirements of learning in the limit by applying different definitions of probabilistic convergence to the sequence of internal states of a learning automaton. Following Narendra and Thathachar [53], the definitions are presented here. A learning automaton is said to be *expedient* if

$$\lim_{n\to\infty} E[M(n)] < M_0 \ ,$$

where $M(n)$ is the average penalty (they are trying to minimize "penalty" rather than maximize "reinforcement"—merely a terminological difference) for the internal state at time step n and M_0 is $M(n)$ for the pure-chance automaton that selects each action randomly with a uniform distribution. A learning automaton is said to be *optimal* if

$$\lim_{n\to\infty} E[M(n)] = c_l \ ,$$

where $c_l = min_i\{c_i\}$ and c_i is the expected penalty of executing action i. A learning automaton is said to be *ϵ-optimal* if

$$\lim_{n\to\infty} E[M(n)] < c_l + \epsilon$$

can be obtained for any arbitrary $\epsilon > 0$ by a proper choice of the parameters of the automaton. Finally, a learning automaton is said to be ***absolutely expedient*** if

$$E[M(n+1) \mid s(n)] < M(n)$$

for all legal internal states of the algorithm $s(n)$ and for all possible sets $\{c_i\}(i = 1, 2, \ldots, r)$ (under the assumption that environments with all expected penalties equal are excluded).

An important recent theoretical development is a model of Boolean-function learning algorithms that are ***probably approximately correct*** (PAC) [5,76], that is, that have a high probability of converging to a function that closely approximates the optimal function. The correctness of a function is measured with respect to a fixed probability distribution on the input instances—a function is said to approximate another function to degree ϵ if the probability that they will disagree on any instance chosen according to the given probability distribution is less than ϵ. This model requires that there be a fixed distribution over the input instances and that each input to the algorithm be drawn according to that distribution.

For an agent to act effectively in the world, its inputs must provide some information about the state that the world is in. In general, when the agent performs an action it will bring about a change in the state of the world and, hence, a change in the information the agent receives about the world. Thus, it will be very unlikely that such an agent's inputs could be modeled as being drawn from a fixed distribution, making PAC-convergence an inappropriate model for autonomous agents.

In addition, the PAC-learning model is distribution-independent—it seeks to make statements about the performance of algorithms no matter how the input instances are distributed. As Buntine has pointed out [14], its predictions are often overly conservative for situations in which there is *a priori* information about the distribution of the input instances, or even in which certain properties of the actual sample, such as how many distinct elements it contains, are known.

Measuring Error over an Agent's Lifetime

None of the classical convergence measures take into account the behavior of the agent during the period in which it converges. Instead, they make what is, for an agent embedded in the world, an artificial distinction between a learning phase and an acting phase. Autonomous agents that have extended run times will be expected to learn for their entire lifetime. Because they may not encounter certain parts or aspects of their environments until arbitrarily late in the run, it is inappropriate to require all mistakes to be made before some fixed deadline.

Another way of characterizing the performance of a function-learning algorithm is to count the divergences it makes from the optimal function. Littlestone [37] has investigated this model extensively, characterizing the optimal number of 'mistakes' for a Boolean-function learner and presenting algorithms that perform very well, under this measure, on certain classes of Boolean functions. This model is intuitively pleasing, making no restrictive division into learning and acting phases, but it is not presented as being suited to noisy or inconsistent domains. However, by assimilating the inconsistency of the domain into the definition of the target function, as in the requirement for optimal behavior, Opt, we can make use of mistake bounds in inconsistent domains. A behavior is said to make an *avoidable mistake* if, given some input instance i, it generates action a and $\mathrm{Opt}(i, a)$ does not hold; that is, there was some other action that would have had a higher expected reinforcement.

Avoidable mistake bounds take into account the fact that many mistakes cannot be avoided by an agent with limited sensory abilities and unreliable effectors. However, this measure is not entirely appropriate, because every non-optimal choice of action is considered to be a mistake of the same magnitude. The expected error of an action a given an input i, $err(a, i)$, is defined to be

$$err(a, i) = er(a', i) - er(a, i) \ ,$$

in which a' is any action such that $opt(a', i)$. The expected error associated with an optimal action is 0; for a non-optimal action, it is just the decrease in expected reinforcement due to having executed that action rather than an optimal one. The error of a behavior, either in the limit, or for runs of finite length, can be measured

by summing the errors of the actions it generates. This value, referred to in the statistics literature as the *regret* of a strategy [12], represents the expected amount of reinforcement lost due to executing this behavior rather than an optimal one. This is an appropriate performance metric for agents embedded in inconsistent environments because it measures expected loss of reinforcement, which is precisely what we would like to minimize in our agents.

In many situations, the optimal behavior is unknown or difficult to compute, which makes it difficult to calculate the error of a given behavior. It is still possible to use this measure to compare two different behaviors for the same agent and environment. The expected reinforcement for an algorithm over some time period can be estimated by running it several times and averaging the resulting total reinforcements. Because expectations are additive, the difference between the expected errors of two algorithms is the same as the difference between their expected total reinforcement values. Thus, the difference between average reinforcements is a valid measure of a behavior's correctness that is independent of the internal architecture of the algorithm and that can be used to compare results across a wide variety of techniques.

2.2.3 Time and Space Complexity

Autonomous agents must operate in the real world, continually receiving inputs from and performing actions on their environments. Because the world changes dynamically, an autonomous agent must be *reactive*—always aware of and reacting to changes in its environment. To ensure reactivity, an agent must operate in *real-time*; that is, its sense-compute-act cycle must keep pace with the unfolding of important events in the environment. The exact constraints on the reaction time of an agent are often difficult to articulate, but it is clear that, in general, unbounded computation must never take place.

A convenient way to guarantee real-time performance is to require that the behavior spend only a constant amount of time, referred to as a 'tick,' generating an action in response to each input. If the behavior is a learning behavior, the learning process must also spend only a constant amount of time on each input

instance. There are two strategies for designing such a learning system: incremental and batch.

An incremental system processes each new data set or learning instance as it arrives as input. The processing must be efficient enough that the system is always ready for new data when it arrives. If new relevant data can arrive every tick, the learning algorithm must spend only one constant tick's worth of time on each instance. The requirement for incrementality can, theoretically, be relaxed to yield a batch system, in which a number of learning instances are collected, then processed for many ticks. As long as the learning system adheres to the tick discipline, this process need not interfere with the reactiveness of the rest of the system. Working in batch mode may limit the usefulness of the learning system to some degree, however, because the system will be working with old data that may not reflect the current situation and it will force the data that arrive during the computation phase to be ignored. When using this method, the input data must be sampled with care, in order to avoid statistical distributions of inputs that do not reflect those of the external world.

An algorithm can be said to be *strictly incremental*[3] if it uses a bounded amount of time and space throughout its entire lifetime. This is in contrast with such approaches as Kibler and Aha's instance-based learning [1], which is incremental in that it processes one instance at a time, but is not strictly incremental because instances are stored in a memory whose size may increase without bound. For an incremental system that processes one instance per tick to perform in real time, the system must be strictly incremental.

By definition, the amount of time a strictly incremental behavior spends on each input does not vary as a function of the number of inputs that have been received. It will, however, depend on the size of the input and the output, but that is fixed at design time. This allows the programmer to know how long each tick of the learning behavior will take to compute on the available hardware and to compare that rate with the pace of events in the world.

[3]This terminology was suggested by R. Sutton.

Any formalization of the interaction between an agent and its world will depend on the rate of the interaction; behaviors that work at different rates will essentially be working in different environments. The expected values of optimal behaviors for different reaction rates will be quite different. In general, up to some minimum value, the faster an agent can interact with the world, the better (otherwise the agent does not have time to avert impending bad events), so we should strive for the most efficient algorithms possible, though a slow algorithm with better convergence properties might be preferable to a fast algorithm that is far from optimal.

Complex agents, such as mobile robots with a wide variety of sensors and effectors, will have a huge number of possible inputs and outputs. If algorithms for these agents are to be practical, they must have time and space complexity that is at worst polynomial in the number of input bits, $\lg(|\mathcal{I}|)$, and the number of output bits, $\lg(|A|)$, rather than the number of inputs and outputs. As we shall see in Section 4.6, this will only be achievable, in general, by limiting the class of behaviors that can be learned by the agent.

2.3 Related Foundational Work

The problem of learning the structure of a finite-state automaton from examples has been studied by many theoreticians, including Moore [51], Gold [28] and, more recently, Rivest and Schapire [56]. This is a very difficult problem that has only been studied in the case of deterministic automata. If the entire structure of the world can be learned, it is conceptually straightforward to compute the optimal behavior. It is important to note, however, that learning an action-map that maximizes reinforcement is likely to be much less complex than learning the world's transition function.

Watkins [78] presents a clear discussion of different types of optimality from an operations-research perspective and characterizes possible algorithms for learning optimal behavior from delayed rewards. Sutton [70,71] shows how to divide the problem of learning from delayed reinforcement into the problems of locally optimal behavior learning and secondary reinforcement-signal learning. The implications

of these ideas for learning from delayed reinforcement will be explored further in Chapter 9.

Williams has done important work on the foundations of reinforcement learning, which is considerably different than the framework provided in this chapter [83,84]. He has developed a general form for expressing reinforcement algorithms in which a wide variety of existing reinforcement learning algorithms may be described. In addition, he has shown that the algorithms expressed in this form are performing a gradient ascent search, in which the average update of the internal parameters of the algorithm is in the direction of steepest ascent for expected reinforcement.

Chapter 3

Previous Approaches

The problem of learning from reinforcement has been studied by a variety of researchers: statisticians studying the "two-armed bandit" problem, psychologists working on mathematical learning theory, learning-automata theorists, and AI researchers. This chapter explores the differing frameworks in which these groups have studied reinforcement learning and presents a few important algorithms and results from each area. It presents previous approaches only to the simple reinforcement-learning scenario in which all reinforcement is instantaneous (the goal is to optimize local, immediate reinforcement) and the action maps to be learned are pure functions. As these assumptions are relaxed, later in the dissertation, other relevant work pertaining to the more complex situations will be discussed.

3.1 Bandit Problems

The reinforcement learning problem is addressed within the statistics community as the "two-armed bandit" problem: given a machine with two levers that pays some amount of money each time a lever is pulled, develop a strategy that gains the maximum payoff over time by choosing which lever to pull based on the previous experience of lever-pulling and payoffs. Among the early results was that the "stick with a winner but switch on a loser" strategy is expedient (better than random), but not optimal [12].

Algorithm 1 (BANDIT) *The initial state, s_0, consists of 3 components: c, an array with two integer elements, and integers d and l. Initially, c contains zeros, $d = -1$, and $l = 0$.*

```
u(s,a,r) =  if d = -1 then
                c[a] := c[a] + 1
e(s) =      if d = -1 then
                if c[0] - c[1] > k then begin
                    d := 0; return 0; end
                else if c[1] - c[0] > k then begin
                    d := 1; return 1; end
                else if l = 0 then begin
                    l := 1; return 1; end
                else begin
                    l := 0; return 0; end
            else return d
```

Figure 12: Formal description of the BANDIT algorithm.

Most of the technical results in this area make very strict assumptions about the *a priori* information the player has about the probabilistic models underlying the payoff processes of the two arms. These results may be useful in restricted situations, but are not applicable to the general problem of building learning agents.

There has been some consideration, however, of the *minimax* case, in which it is assumed that the events of arm-pulling are independent, that they pay off either nothing or a fixed amount, that the probability of each arm paying off remains constant for the entire game, and that the world will choose the probabilities in the way that is worst for the player. It has been shown [12] that the best possible strategy for such a domain has regret proportional to $(1-\gamma)^{-1/2}$ for discounting factor γ and to $n^{1/2}$ for finite horizon n.

An example algorithm satisfying these requirements is formally described in Figure 12.[1] The algorithm alternates between the two arms, keeping track of the

[1]There is no input argument, i, to the update and evaluation functions. This algorithm, as well as most of the others in the first part of the chapter, makes a choice about what action to perform

number of successes of each. When the number of successes of one arm exceeds the number of successes of the other by a number k, it chooses the winning arm forever into the future. The array c contains counts of the number of successes of each arm; d encodes the decision about future actions; if it has value -1, the decision has not yet been made; l encodes the last action taken so that the algorithm can alternate between actions in the pre-decision phase. If reinforcement is to be optimized over a fixed horizon n, k should be chosen to be $n^{1/2}$. If reinforcement with discounting factor γ is to be optimized, k should be chosen to be $(1-\gamma)^{-1/2}$. This is a simple algorithm with an upper bound on regret of $(1-\gamma)^{-1/2}(1+\frac{1}{2e})$ in the discounted case or $(1-n^{-1})^{-(n-1)}n^{1/2}(1+\frac{1}{2e})$ in the finite horizon case. This value is itself bounded above by $n^{1/2}(e+1/2)$. In both cases, the upper bound on regret is within a constant factor of optimal. However, as we will see in Section 4.4, this algorithm is outperformed by many others in empirical tests.

3.2 Learning Automata

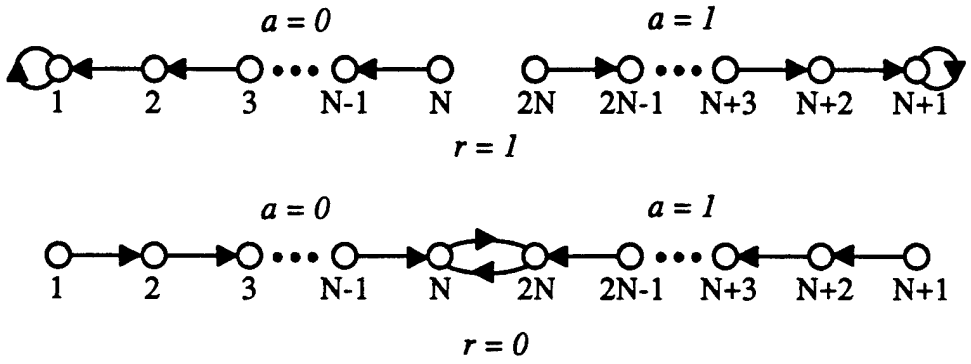
Another closely related field is that of learning automata. The phrase "learning automata" means, in this case, automata that learn to act in the world, as opposed to automata that learn the state-transition structures of other automata (as in Moore [51]).

3.2.1 Early Work

The first work in this area took place in the Soviet Union. An example of early learning-automaton work is the *Tsetlin automaton*, designed by M. L. Tsetlin [75]. The input set of the automaton is $\{0,1\}$, with 1 corresponding to the case when the agent receives reinforcement and 0 corresponding to the case when it does not. As in the BANDIT algorithm, there is no input corresponding to i, the information about the state of the world. The automaton has two possible actions, or outputs: 0 and 1. The operation of the Tsetlin automaton is described in Figure 13.

for every future time step, with only reinforcement as input.

Algorithm 2 (TSETLIN)



The initial state can be any of the states, but would most reasonably be chosen to be state N or state $2N$. All of the states on the left half of the graph evaluate to action 0 and on the right half of the graph to action 1. The state update operation consists of making one of the labeled transitions: when reinforcement has value 1, a transition to the left is taken if the action was 0 and to the right if the action was 1; when the reinforcement has value 0, a right transition is taken if the action was 0 and a left transition if the action was 1. Zero reinforcement values move the state toward the center and positive reinforcement values move the state toward the end corresponding to the action that was taken.

Figure 13: The Tsetlin automaton

The Tsetlin automaton is parametrizable by the number, N, of states between the center state and the ends of the chains going to the right and left. It can be shown that, if one of the actions has success probability greater than .5, then, as the value N approaches infinity, the average reinforcement approaches the maximum success probability [53].

There are many other similar learning automata, some with better convergence properties than this one. The BANDIT algorithm can also be easily modeled as a finite-state machine.

3.2.2 Probability-Vector Approaches

As it is difficult to conceive of complex algorithms in terms of finite-state transition diagrams, the learning automata community moved to a new model, in which the internal state of the learning algorithm is a vector of non-negative numbers that sum to 1. The length of the vector corresponds to the number of possible actions of the agent. The agent chooses an action probabilistically, with the probability that it chooses the nth action equal to the nth element of the state vector. The problem, then, is one of updating the values in the state vector depending on the most recent action and its outcome.

These and similar, related models were also independently developed by the mathematical psychology community [15] as models for human and animal learning.

The most common of these approaches, called the *linear reward-penalty* algorithm, is shown in Figure 14. Whenever an action is chosen and succeeds, the probability of performing that action is increased in proportion to 1 minus its current probability; when an action is chosen and fails, the probability of performing the other action is increased in proportion to its current probability. The parameters a and b govern the amount of adjustment upon success and failure, respectively. An important specialization is the *linear reward-inaction* algorithm, also described in Figure 14, in which no adjustment is made to the probability vector when reinforcement value 0 is received.

Algorithm 3 (L_{RP}) *The initial state, s_0, consists of p_1 and p_2, two positive real numbers such that $p_1 + p_2 = 1$.*

$$
\begin{aligned}
u(s,a,r) = \; & \textit{if } a = 0 \textit{ then} \\
& \quad \textit{if } r = 0 \textit{ then} \\
& \quad\quad p_0 := (1-b)p_0 \\
& \quad \textit{else } p_0 := p_0 + ap_1 \\
& \textit{else} \\
& \quad \textit{if } r = 0 \textit{ then} \\
& \quad\quad p_0 := p_0 + bp_1 \\
& \quad \textit{else } p_0 := (1-a)p_0 \\
& p_1 := 1 - p_0
\end{aligned}
$$

$$
e(s) = \begin{cases} 0 \textit{ with probability } p_0 \\ 1 \textit{ with probability } p_1 \end{cases}
$$

Algorithm 4 (L_{RI}) Any instance of Algorithm L_{RP} in which $b = 0$.

Figure 14: The linear reward-penalty (L_{RP}) and linear reward-inaction (L_{RI}) algorithms.

The linear reward-penalty algorithm has asymptotic performance that is better than random (that is, it is expedient), but it is not optimal. It has no absorbing states, so it always executes the wrong action with some non-zero probability. The linear reward-inaction algorithm, on the other hand, has the absorbing states [1,0] and [0,1], because a probability is only ever increased if the corresponding action is taken and it succeeds. Once one of the probabilities goes to 0, that action will never be taken, so its probability can never be increased. The linear reward-inaction algorithm is ϵ-optimal; that is, the parameter a can be chosen in order to make the probability of converging to the wrong absorbing state as small as desired. As the value of a is decreased, the probability of converging to the wrong state is decreased; however, the rate of convergence is also decreased. Theoreticians have been unable to derive a general formula that describes the probability of convergence to the wrong state as a function of a and the initial value of p_1. This would be necessary in order to choose a to optimize reinforcement for runs of a certain length or with a certain discounting factor, as we did with k in the BANDIT algorithm above.

Algorithm 5 (TS) *The initial state, s_0, consists of the following 6 components: p_0 and p_1, which are positive real numbers such that $p_0 + p_1 = 1$, and $R_0 = R_1 = Z_0 = Z_1 = 0$.*

$$
\begin{aligned}
u(s,a,r) = \quad & \hat{d}_0 := R_0/Z_0;\ \hat{d}_1 := R_1/Z_1 \\
& \textit{if } a = 0 \textit{ then begin} \\
& \quad \textit{if } \hat{d}_0 > \hat{d}_1 \textit{ then} \\
& \quad\quad p_0 := p_0 + \lambda(\hat{d}_0 - \hat{d}_1)p_1 \\
& \quad \textit{else } p_0 := p_0 + \lambda(\hat{d}_0 - \hat{d}_1)p_0^2 \\
& \quad p_1 := 1 - p_0 \\
& \quad R_0 := R_0 + r \\
& \quad Z_0 := Z_0 + 1 \\
& \textit{end else begin} \\
& \quad \textit{if } \hat{d}_1 > \hat{d}_0 \textit{ then} \\
& \quad\quad p_1 := p_1 + \lambda(\hat{d}_1 - \hat{d}_0)p_0 \\
& \quad \textit{else } p_1 := p_1 + \lambda(\hat{d}_1 - \hat{d}_0)p_1^2 \\
& \quad p_0 := 1 - p_1 \\
& \quad R_1 := R_1 + r \\
& \quad Z_1 := Z_1 + 1 \\
& \textit{end} \\
e(s) = \quad & \begin{cases} 0 \textit{ with probability } p_0 \\ 1 \textit{ with probability } p_1 \end{cases}
\end{aligned}
$$

where $0 < \lambda < 1$ is a positive constant.

Figure 15: The TS algorithm

In addition to these linear approaches, a wide range of non-linear approaches have been proposed. One of the most promising is Thathachar and Sastry's method [74]. It is slightly divergent in form from the previous algorithms in that it keeps more state than simply the vector **p** of action probabilities. In addition, there is a vector $\hat{\mathbf{d}}$ of estimates of the expected reinforcements of executing each action. Reinforcement values are assumed to be real values in the interval [0,1]. A simple two-action version of this algorithm is shown in Figure 15.

The R_j are the summed reinforcement values for each action, the Z_j are the number of times each action has been tried, and the $\hat{d}_j$ are the average reinforcement

values for each action. The adjustment to the probability vector depends on the values of the $\hat{d}_j$ rather than on the direct results of recent actions. This introduces a damping effect, because as long as, for instance, $\hat{d}_0 > \hat{d}_1$, p_0 will be increased, even if it has a few negative-reinforcement results during that time.

The TS algorithm converges much faster than the linear algorithms L_{RP} and L_{RI}. One of the reasons may be that it naturally takes big steps in the parameter space when the actions are well differentiated (the difference between $\hat{d}_0$ and $\hat{d}_1$ is large) and small steps when they are not. It has been shown that, for any stationary random environment, there is some value of λ such that $p_l(n) \rightarrow 1$ in probability[2] as $n \rightarrow \infty$, where $p_l(n)$ is the probability of executing the action that has the highest expected reinforcement [74].

3.3 Reinforcement-Comparison Methods

One drawback of most of the algorithms that have been presented so far is that reinforcement values of 0 and 1 cause the same sized adjustment to the internal state independent of the *expected* reinforcement value. Sutton [70] addressed this problem with a new class of algorithms, called *reinforcement-comparison* methods. These methods work by estimating the expected reinforcement, then adjusting the internal parameters of the algorithm proportional to the difference between the actual and estimated reinforcement values. Thus, in an environment that tends to generate reinforcement value 1 quite frequently, receiving the value 1 will cause less adjustment that will be caused by receiving the value 0.

An instance of the reward-comparison method, taken from Sutton's thesis [70], is shown in Figure 16. The internal state consists of the "weight" w, which is initialized to 0, and the predicted expected reinforcement, p, which is initialized to the first reinforcement value received. The output, $e(s)$, has value 1 or 0 depending on the values of w and the random variable ν. The addition of the random value causes the algorithm to "experiment" by occasionally performing actions that it would not

[2] According to Narendra and Thathachar [53], "The sequence $\{X_n\}$ of random variables converges in probability to the random variable X if for every $\epsilon > 0$, $\lim_{n\rightarrow\infty} \Pr\{|\ X_n - X\ | \geq \epsilon\} = 0$."

Algorithm 6 (RC) *The internal state, s_0, consists of the values $w = 0$ and p, which will be initialized to the first reinforcement value received.*

$$u(s,a,r) = \quad w := w + \alpha(r-p)(a-1/2)$$
$$\qquad\qquad\quad p := p + \beta(r-p)$$
$$e(s) = \quad \begin{cases} 1 & \text{if } w + \nu > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$, $0 < \beta < 1$, and ν is a normally distributed random variable of mean 0 and standard deviation δ_y.

Figure 16: A reward-comparison (RC) algorithm.

otherwise have taken. The state component w is incremented by a value with three terms. The first term, α, is a constant that represents the learning rate. The next term, $r - p$, represents the difference between the actual reinforcement received and the predicted reinforcement, p. This serves to normalize the reinforcement values: the absolute value of the reinforcement signal is not as important as its value relative to the average reinforcement that the agent has been receiving. The third term in the update function for w is $a - 1/2$; it has constant absolute value and the sign is used to encode which action was taken. The predicted reinforcement, p, is a weighted running average of the reinforcement values that have been received.

3.4 Associative Methods

The algorithms presented so far have addressed the case of reinforcement learning in environments that present only reinforcement values as input to the agent. A more general setting of the problem, called *associative reinforcement learning*, requires the agent to learn the best action for each of a possibly large number of input states. This section will describe three general approaches for converting simple reinforcement-learning algorithms to work in associative environments. The first is a simple copying strategy, and the second two are instances of a large class of

Algorithm 7 (COPY) *Let (s_0, u, e) be a learning behavior that has only reinforcement as input. We can construct a new learning behavior (s'_0, u', e') with 2^M inputs as follows:*

$$\begin{aligned} s'_0 &= \textit{array } [1..2^M] \textit{ of } s_0 \\ u'(s', i, a, r) &= u(s'[i], a, r) \\ e'(s', i) &= e(s'[i], a) \end{aligned}$$

Figure 17: Constructing an associative algorithm by making copies of a non-associative algorithm.

associative reinforcement-learning methods developed by researchers working in the connectionist learning paradigm. Other approaches not described here include those of Minsky [48] and Widrow, Gupta, and Maitra [81]. Barto [9] gives a good overview of connectionist learning for control, including learning from reinforcement.

3.4.1 Copying

The simplest method for constructing an associative reinforcement-learner, shown in Figure 17, consists of making a copy of the state of the no-input version of the algorithm for each possible input and training each copy separately. It requires 2^M (the number of different input states) times the storage of the original algorithm.

In addition to being very computationally complex, the copying method does not allow for any generalization between input instances: that is, the agent cannot take advantage of the intuition that "similar" situations require "similar" responses.

3.4.2 Linear Associators

In his thesis [70], Sutton gives methods for converting standard reinforcement-learning algorithms to work in an associative setting in a way that allows an agent to learn efficiently and to generalize across input states. He uses a version of the Widrow-Hoff or Adaline [82] weight-update algorithm to associate different internal state values with different input situations. This approach is illustrated by the LARC

Algorithm 8 (LARC) *The input is represented as an M-dimensional vector i. The internal state, s_0, consists of two M-dimensional vectors, $\mathbf{v}$ and $\mathbf{w}$.*

$$
\begin{aligned}
u(s,i,a,r) = \quad & \textit{let } p := \mathbf{v} \cdot i \\
& \textit{for } j = 1 \textit{ to } M \textit{ do begin} \\
& \quad w_j := w_j + \alpha(r-p)(a-1/2)i_j \\
& \quad v_j := v_j + \beta(r-p)i_j \\
& \textit{end} \\
e(s,i) = \quad & \begin{cases} 1 & \textit{if } \mathbf{w} \cdot i + \nu > 0 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}
$$

where $\alpha > 0$, $0 < \beta < 1$, and ν is a normally distributed random variable of mean 0 and standard deviation δ_ν.

Figure 18: The linear-associator reinforcement-comparison (LARC) algorithm.

(linear-associator reinforcement-comparison) algorithm shown in Figure 18. It is an extension of the RC algorithm to work in environments with multiple input states.

The inputs to the algorithm are represented as M-dimensional vectors The output, $e(s,i)$, has value 1 or 0 depending on the inner product of the weight vector $\mathbf{w}$ and i and the value of the random variable ν. The updating of the vector $\mathbf{w}$ is somewhat complicated: each component is incremented by a value with four terms. The first term, α, is a constant that represents the learning rate. The next term, $r - p$, represents the difference between the actual reinforcement received and the predicted reinforcement, p. The predicted reinforcement, p, is generated using a standard linear associator that learns to associate input vectors with reinforcement values by setting the weights in vector $\mathbf{v}$. The third term in the update function for $\mathbf{w}$ is $a - 1/2$: it has constant absolute value and the sign is used to encode which action was taken. The final term is i_j, which causes the jth component of the weight vector to be adjusted in proportion to the jth value of the input.

Another instance of the linear-associator approach is Barto and Anandan's *associative reward-penalty* (A_{RP}) algorithm [7]. It is a hybrid of the linear reward-penalty and linear-associator methods and was shown (under a number of restrictions, including the restriction that the set of input vectors be linearly independent) to be ϵ-optimal.

The linear-associator approach can be applied to any of the learning algorithms whose internal state consists of one or a small number of independently-interpretable numbers for each input. If the input set is encoded by bit strings, the linear-associator approach can achieve an exponential improvement in space over the copy approach, because the size of the state of the linear-associator is proportional to the number of input bits rather than to the number of inputs. This algorithm works well on simple problems, but algorithms of this type are incapable of learning functions that are not linearly separable [47].

3.4.3 Error Backpropagation

To remedy the limitations of the linear-associator approach, multi-layer connectionist learning methods have been adapted to reinforcement learning. Anderson [3], Werbos [79], and Munro [52], among others, have used error back-propagation methods[3] with hidden units in order to allow reinforcement-learning systems to learn more complex action mappings. Williams [85] presents an analysis of the use of backpropagation in associative reinforcement-learning systems. He shows that a class of reinforcement-learning algorithms that use back-propagation (an instance of which is given below) perform gradient ascent search in the direction of maximal expected reinforcement. This technique is effective and allows considerably more generalization across input states, but it requires many more presentations of the data in order for the internal units to converge to the features that they need to detect in order to compute the overall function correctly. Barto and Jordan [10] demonstrate the use of a multi-layer version of the associative reward-penalty algorithm to learn non-linear functions. This method is argued to be more biologically

[3]A good description of error back-propagation for supervised learning is given by Rumelhart, Hinton, and Williams [58].

plausible than back-propagation, but requires considerably more presentations of the data.

As an example of the application of error backpropagation methods to reinforcement learning, Anderson's method [3] will be examined in more detail. It uses two networks: one for learning to predict reinforcement and one for learning which action to take. The weights in the action network are updated in proportion to the difference between actual and predicted reinforcement, making this an instance of the reinforcement-comparison method (discussed in Section 3.3 above). Each of the networks has two layers, with all of the hidden units connected to all of the inputs and all of the inputs and hidden units connected to the outputs. The system was designed to work in worlds with delayed reinforcement (which are discussed at greater length in Chapter 9), but it is easily simplified to work in our simpler domain.

The BP algorithm is shown in Figures 19 and 20 and is explained in detail by Anderson [3]. The presentation here is simplified in a number of respects, however. In this version, there is no use of momentum and the term $(a - 1/2)$ is used to indicate the choice of action rather than the more complex expression used by Anderson. Also, Anderson uses a different distribution for the random variable ν.

This method is theoretically able to learn very complex functions, but tends to require many training instances before it converges. The time and space complexity for this algorithm is $O(MH)$, where M is the number of input bits and H is the number of hidden units. Also, this method is somewhat less robust than the more standard version of error back-propagation that learns from I/O pairs, because the error signal generated by the reinforcement-learning system is not always correct.

3.5 Genetic Algorithms

Genetic algorithms constitute a considerably different approach to the design and implementation of reinforcement-learning systems. This section will briefly describe the general approach and point to some representative applications of these methods

Algorithm 9 (BP) *The input is represented as an $M+1$-dimensional vector i, in which the last element contains a constant value. The internal state, s_0, consists of*

W_{EH} : *Weights of the hidden units in the evaluation network, an H by $M+1$ element array initialized to small random values.*

W_{EO} : *Weights of the output unit in the evaluation network, an $H+M+1$ element array initialized to small random values.*

W_{AH} : *Weights of the hidden units in the action network, an H by $M+1$ element array initialized to small random values.*

W_{AO} : *Weights of the output unit in the action network, an $H+M+1$ element array initialized to small random values.*

In addition, the algorithm makes use of the following local variables

O_{EH} : *Outputs of the hidden units in the evaluation network, an H element array.*

O_{AH} : *Outputs of the hidden units in the action network, an H element array.*

p : *Output of the output unit in the evaluation network.*

Figure 19: An application of error backpropagation to reinforcement learning: data structures.

Algorithm 9 (BP) *(continued)*

$u(s,i,a,r) =$ *for* $j = 1$ *to* H *do*
$O_{EH}[j] := f(i \cdot W_{EH}[j])$
$p := W_{EO} \cdot concat(i, O_{EH})$
for $j = 1$ *to* $M+1$ *do*
$W_{EO}[j] := W_{EO}[j] + \beta\,(r-p)\,i[j]$
for $j = 1$ *to* H *do*
$W_{EO}[j+M+1] := W_{EO}[j+M+1] + \beta\,(r-p)\,O_{EH}[j]$
for $j = 1$ *to* H *do begin*
$d := (r-p)\,\mathrm{sign}(W_{EO}[j+M+1])\,O_{EH}[j]\,(1-O_{EH}[j])$
for $k = 1$ *to* $M+1$ *do*
$W_{EH}[j,k] := \beta_h\,d\,i[k]$
end
for $j = 1$ *to* $M+1$ *do*
$W_{AO}[j] := W_{AO}[j] + \rho\,(r-p)\,(a-1/2)\,i[j]$
for $j = 1$ *to* H *do*
$W_{AO}[j+M+1] := W_{AO}[j+M+1] + \rho\,(r-p)\,(a-1/2)\,O_{AH}[j]$
for $j = 1$ *to* H *do begin*
$d := (r-p)\,(a-1/2)\,\mathrm{sign}(W_{AO}[j+M+1])\,O_{AH}[j]\,(1-O_{AH}[j])$
for $k = 1$ *to* $M+1$ *do*
$W_{AH}[j,k] := \rho_h\,d\,i[k]$
end

$e(s,i) =$ *for* $j = 1$ *to* H *do*
$O_{AH}[j] := F(i \cdot W_{AH}[j])$
$\begin{cases} 1 & \textit{if } (W_{AO} \cdot \mathrm{concat}(i, O_{AH})) + \nu > 0 \\ 0 & \textit{otherwise} \end{cases}$

where $\beta, \beta_h, \rho, \rho_h > 0$, $f(x) = 1/(1+e^{-x})$, *and* ν *is a normally distributed random variable of mean 0 and standard deviation* δ_y.

Figure 20: An application of error-backpropagation to reinforcement learning: update and evaluation functions

to reinforcement learning. An excellent introduction to and survey of this field is given in Goldberg's book [29].

In their purest form, genetic algorithms (GA's) can be seen as a technique for solving optimization problems in which the elements of the solution space are coded as binary strings and in which there is a scalar objective function that can be used to compute the "fitness" of the solution represented by any string. The GA maintains a "population" of strings, which are initially chosen randomly. The fitness of each member of the population is calculated. Those with low fitness values are eliminated and members with high fitness values are reproduced in order to keep the population at a constant size. After the reproduction phase, operators are applied to introduce variation in the population. Common operators are crossover and mutation. In crossover, two population elements are chosen, at random, as operands. They are recombined by randomly choosing an index into the string and making two new strings, one that consists of the first part of the first string and the second part of the second string and one that consists of the first part of the second string and the second part of the first string. Mutation simply changes bits in population elements, with very low probability.

A more complex type of GA is the *classifier system* [33]. Developed by Holland, it consists of a population of production rules, which are encoded as strings. The rules can be executed to implement an action function that maps external inputs to external actions. When the rules chain forward to cause an external action, a reinforcement value is received from the world. Holland developed a method, called the Bucket Brigade, for propagating reinforcement back along the chain of production rules that caused the action. This method is an instance of the class of temporal difference methods, which will be discussed further in Chapter 9. As a set of rules is run, each rule comes to have a relatively stable value which is used as its fitness. The standard genetic operations of reproduction, crossover, mutation, etc., are used to generate new populations of rules from old ones.

Although classifier systems are reinforcement-learners, they are not well-suited for use in embedded systems. As with most production systems, there is no bound

on the number of rule-firings that will be required to generate an output in response to an input, preventing the algorithm's operation from being real-time.

Grefenstette [30] has applied GA methods directly to the time-constrained problem of learning action strategies from reinforcement. The elements of the population of his system are symbolic representations of action maps. The fitness of an element is determined by executing it in the world for a number of ticks and measuring the average reinforcement. Action maps that perform well are reproduced and recombined to generate new action maps.

The GA approach works well on problems that can be effectively coded as syntactic objects in which the interpretation of individual elements is relatively context-independent and for which there are useful recombination operators. It is not yet clear what classes of problems can be so specified. An interesting extension of the research carried out in this dissertation would be to implement genetic algorithms for the problems considered and compare their performance with that of the algorithms tested herein.

3.6 Extensions to the Model

The algorithms of the previous sections have been presented in their simplest possible forms, with only Boolean reinforcement as input and with two possible actions. It is a relatively simple matter to extend all of the algorithms except RC, LARC, and BP to the case of multiple actions. Because the details differ for each one, however, they shall be omitted from this discussion. The algorithms that choose an action by comparing an internal value plus noise to a threshold are more difficult to generalize in this way.

The rest of this section will briefly detail extensions of these algorithms to work in domains with non-Boolean and nonstationary reinforcement.

3.6.1 Non-Boolean reinforcement

Algorithms BANDIT and TSETLIN have no obvious extensions to the case of non-Boolean reinforcement.

The learning-automata community considers three models of reinforcement: P, Q, and S. The P-model of reinforcement is the Boolean-reinforcement model we have already explored. In the Q-model, reinforcement is one of a finite number of possible values that are known ahead of time. These reinforcement values can always be scaled into values in the interval $[0, 1]$. Finally, the S-model allows real-valued reinforcement in the interval $[0, 1]$. The notions of expediency and optimality can be extended to apply to the Q- and S-models.

Algorithms designed for P-model environments, such as the L_{RP} and L_{RI} algorithms, can be adjusted to work in Q- and S-models as follows. Let $\Delta_{i,0}$ be the change made to action-probability i when reinforcement 0 is received and let $\Delta_{i,1}$ be the change made when reinforcement value 1 is received. We can define, for the new models, $\Delta_{i,r}$, the change made when reinforcement value r is received as

$$\Delta_{i,r} = r\Delta_{i,1} + (1 - r)\Delta_{i,0} \ ,$$

a simple linear combination of the updates for the old reinforcement cases [53].

Algorithm TS was designed to work in an S-model of reinforcement and can be used in such environments without change. Algorithm RC, as well as the associative reinforcement-comparison algorithms LARC and BP, work in the more general case of real-valued reinforcement that is not necessarily scaled to fall in the interval $[0, 1]$.

3.6.2 Nonstationary environments

A world is *nonstationary* if $er(i, a)$ (the expected reinforcement of performing action a in input situation i) varies over time. It is very difficult to prove formal results about the performance of learning algorithms in nonstationary environments, but several observations can be made about which algorithms are likely to perform better in such environments. For instance, algorithms with absorbing states, such as BANDIT and L_{RI}, are inappropriate for nonstationary environments: if the world

changes after the algorithm has converged, it will never sample the other actions and adjust its behavior to the changed environment. On the other hand, algorithms that are less effective in stationary environments, such as TSETLIN and L_{RP}, continue to sample all of the actions and will adapt to changes in the environment.

3.7 Conclusions

A number of effective reinforcement-learning algorithms have been developed by different research communities. The work in this dissertation seeks to extend and improve upon the previous work by developing more effective learning methods and by finding approaches to associative reinforcement learning that are capable of learning a broader class of functions than the linear approaches can, but doing so more space-efficiently than the copy method and with fewer input instances than are required by the error backpropagation method. In addition, this dissertation will extend previous work on the problem of learning from delayed reinforcement.

Chapter 4

Interval Estimation Method

The interval estimation method is a simple statistical algorithm for reinforcement learning. It is a logical extension of the statistical algorithms presented in the previous chapter. By allowing the state of the algorithm to encode not only estimates of the relative merits of the various actions, but also the degree of confidence that we have in those estimates, the interval estimation method builds on previous approaches by making it easier to control the tradeoff between acting to gain information and acting to gain reinforcement in a careful way. The interval estimation algorithm performs well on a variety of tasks and its basis in standard statistical methods makes it an illustrative example for formal analysis.

This chapter presents the algorithm, together with an estimate of its expected error and experimental comparisons with many of the algorithms of Chapter 3. Next, it explores ways of extending the basic algorithm to deal with the more general learning models presented in Section 3.6. Finally, this chapter discusses the computational complexity of the interval-estimation algorithm and argues that it, along with other existing reinforcement-learning algorithms to which the linear-association or backpropagation methods cannot be directly applied, is too computationally expensive for use in embedded systems.

4.1 Description of the Algorithm

The interval estimation method can be applied in a wide variety of environments; the simplest form will be presented first, and extensions to the basic algorithm will be described in Section 4.5. The basic interval estimation algorithm is formally described in Figure 21. The state consists of simple statistics: for each action a, n_a and x_a are the number of times that the action has been executed and the number of those times that have resulted in reinforcement value 1, respectively. The evaluation function uses these statistics to compute, for each action, a confidence interval [1] on the underlying probability, p_a, of receiving reinforcement value 1 given that action a is executed. If n is the number of trials and x the number of successes arising from a series of Bernoulli trials [2] with probability p, the upper bound of a $100(1-\alpha)$ percent confidence interval for p can be approximated by $ub(x,n)$. [3] The evaluation function generates the action with the highest upper bound on expected reinforcement.

Initially, each of the actions will have an upper bound of 1, and action 0 will be chosen arbitrarily. As more trials take place, the bounds will tighten. The interval estimation method balances acting to gain information with acting to gain reinforcement by taking advantage of the fact that there are two reasons that the upper bound for an action might be high: because there is little information about that action, causing the confidence interval to be large or because there is information that the action is good, causing the whole confidence interval to be high. The parameter $z_{\alpha/2}$ is the value that will be exceeded by the value of a standard normal variable with probability $\alpha/2$. [4] It controls the size of the confidence intervals and, thus, the relative weights given to acting to gain information and acting to gain reinforcement. As α increases, more instances of reinforcement value 0 are required

[1] A $100(1-\alpha)$ percent confidence interval for a quantity is a range of values that, with probability $1-\alpha$, contains that quantity.

[2] Bernoulli trials are a series of statistically independent events with binary outcomes that are generated by some fixed underlying probability.

[3] This is a somewhat more complex form than usual, designed to give good results for small values of n [36].

[4] Tables of this relationship can be found in most probability and statistics texts [36].

Algorithm 10 (IE) *The initial state, s_0, consists of the integer variables x_0, n_0, x_1, and n_1, each initialized to 0.*

$u(s,a,r) =$ *if* $a = 0$ *then begin*
$x_0 := x_0 + r$
$n_0 := n_0 + 1$
end else begin
$x_1 := x_1 + r$
$n_1 := n_1 + 1$
end

$e(s) =$ *if* $ub(x_0, n_0) > ub(x_1, n_1)$ *then*
return 0
else
return 1

where

$$ub(x,n) = \frac{\frac{x}{n} + \frac{z^2_{\alpha/2}}{2n} + \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\left(\frac{x}{n}\right)\left(1 - \frac{x}{n}\right) + \frac{z^2_{\alpha/2}}{4n}}}{1 + \frac{z^2_{\alpha/2}}{n}}$$

and $z_{\alpha/2} > 0$.

Figure 21: The interval-estimation (IE) algorithm.

```
  a0s    a0t    a0b       a1s    a1t    a1b
-------------------------------------------
(  14 /   19) .88194    (   0 /    1) .79346
(  81 /  138) .66567    (   0 /    2) .65763
(  85 /  147) .65507    (   0 /    3) .56151
```

Figure 22: A sample run with $p_0 = .55$, $p_1 = .45$, and $z_{\alpha/2} = 1.96$. In this case, it converges very quickly.

to drive down the upper bound of the confidence intervals, causing more weight to be placed on acting to gain information. By the DeMoivre-Laplace theorem [36], these bounds will converge, in the limit, to the true underlying probability values, and, hence, if each action is continually attempted, this algorithm will converge to a function that satisfies Opt.

In order to provide intuition about the workings of this algorithm, Figures 22 and 23 show output from two sample runs in a simulated environment in which the actions a_0 and a_1 succeed with probabilities p_0 and p_1. The listings show the number of success and trials of a_0 (the columns headed `a0s` and `a0t`), the upper bound on the confidence interval of p_0 (the column headed `a0b`) and the same for a_1 and p_1 (columns headed `a1s`, `a1t`, and `a1b`). These statistics are just shown at interesting points during the run of the algorithm. In Figure 22, the first few trials of a_1 fail, causing the estimate of p_1 to be quite low; it will be executed a few more times, once the upper bound for p_0 is driven near .56. The run shown in Figure 23 is somewhat more characteristic. The two actions have similar probabilities of success, so it takes a long time for one to establish dominance.

4.2 Analysis

In order to analytically compare this algorithm with other algorithms, we would like to know the expected error of executing this algorithm in an environment specified by the action-success probabilities p_0 and p_1. This section informally derives an approximate expression for the expected error in terms of p_0, p_1, and $z_{\alpha/2}$.

```
  a0s     a0t     a0b       a1s     a1t     a1b
-------------------------------------------------
(   4 /     7) .84178    (   1 /     3) .79235
(  39 /    75) .62931    (  22 /    45) .62996
( 226 /   394) .62150    (  22 /    46) .61863
( 358 /   631) .60549    (  31 /    59) .64734
( 963 /  1789) .56128    (  52 /   111) .56080
(5548 /  9888) .57084    (  52 /   112) .55630
```

Figure 23: Another sample run with $p_0 = .55$, $p_1 = .45$, and $z_{\alpha/2} = 1.96$. This time, the two actions battle for a long time, but a_0 is clearly winning after 10,000 trials.

Regular Error

For concreteness, let us assume that $p_0 > p_1$. An error occurs every time a_1 is executed, and we expect it to be executed a number of times that is sufficient to drive the upper bound of p_1 below the actual value of p_0. We can compute this expected number of errors by setting the expected value of the upper bound on p_1 equal to p_0 and solving for n_1. The expected value of the upper bound on p_1 is approximately[5] the upper bound with the number of successes set to $n_1 p_1$. This allows us to solve the equation $ub(n_1 p_1, n_1) = p_0$ for n_1, yielding

$$n_1 = \frac{z^2_{\alpha/2} p_0 (1 - p_0)}{(p_0 - p_1)^2} .$$

As p_0 and p_1 grow close, n_1 goes to infinity. This is as it should be—it becomes infinitely hard to tell which of the two actions is better. We can simplify this expression further by abstracting away from the actual values of p_0 and p_1 and considering their difference, δ, instead. For probabilities with a fixed difference, n_1 is maximized by setting p_1 to .5 and p_0 to $.5 + \delta$. Making this simplification, we can bound n_1 above by

$$\frac{z^2}{4\delta^2} .$$

This is an approximate upper bound on the expected number of errors that will be made on a run of infinite length. The *amount* of error can be obtained simply by

[5]This is only an approximation because n_1 occurs inside a square-root, which does not commute with the expectation operator.

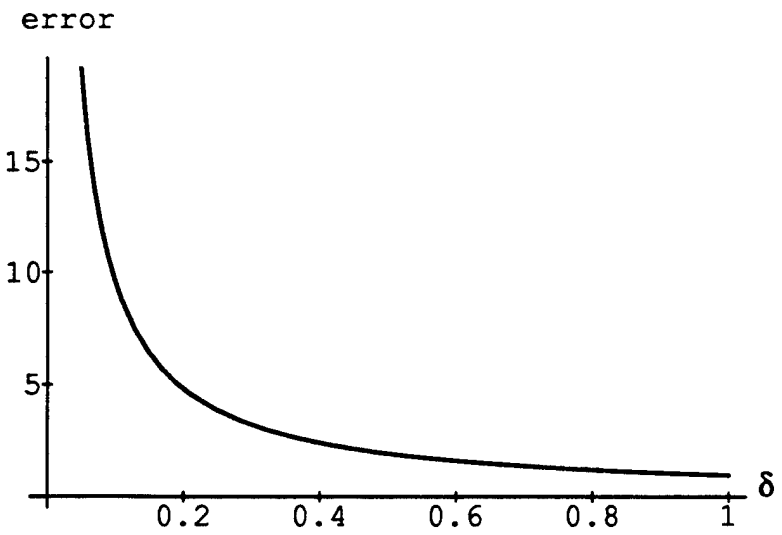


Figure 24: Expected regular error on an infinite run as a function of δ, with $z_{\alpha/2} = 1.96$.

multiplying by δ, the magnitude of the error, yielding

$$\frac{z^2}{4\delta},$$

which is plotted as a function of δ in Figure 24.

This result is somewhat disturbing, because the amount of error on an infinitely long run can be made arbitrarily large by making δ arbitrarily small. However, it is possible to bound the amount of error on a finite run of length m. The maximum expected number of errors that could be made on such a run is $m/2$ (when the two probabilities are equal, we expect to perform the actions equal numbers of times). The number of errors is monotonically decreasing in δ, so we can easily find the largest value of δ that could cause this many errors by solving the equation

$$\frac{m}{2} = \frac{z^2}{4\delta^2}$$

for δ, getting $\frac{z}{\sqrt{2m}}$. Thus, the maximum expected regular error on a run of length m would be

$$\frac{z\sqrt{m}}{2\sqrt{2}},$$

a0s	a0t	a0b	a1s	a1t	a1b
(0 /	2)	.65763	(4 /	8)	.78479
(0 /	3)	.56151	(67 /	137)	.57191
(1 /	4)	.69936	(70 /	146)	.55997
(16 /	34)	.63264	(78 /	176)	.51701

Figure 25: A sample run with $p_0 = .55$, $p_1 = .45$, and $z_{\alpha/2} = 1.96$. The first action almost gets stuck.

obtained by multiplying the maximum number of errors, $m/2$, by the maximum magnitude of the error. This maximum regular error is $O(m^{1/2})$, which means that the interval estimation algorithm, like the BANDIT algorithm, performs within a constant factor of optimal when the environment is as hostile as possible.

Error Due to Sticking

The analysis of the previous section was all carried out under the assumption that the action a_0 would be executed an infinite number of times during an infinite run. Unfortunately, this is not always the case—it is possible for a_0 to get *stuck* below a_1 in the following way. If there is a statistically unlikely series of trials of a_0 that cause the upper bound on p_0 to go *below* the actual value of p_1, then it is very likely that a_0 will never be executed again. When this happens, we shall say that a_0 is stuck. A consequence of a_0 being stuck is that errors will be made for the remainder of the run. The process of sticking is illustrated by two sample runs. In Figure 25, there is an early series of failures for a_0, causing a_1 to be dominant. However, because the upper bound on p_0 was not driven *below* p_1, the upper bound on p_1 eventually goes down far enough to cause more trials of a_0, which bring its upper bound back up. The run shown in Figure 26 is a case of permanent sticking. After 0 successes in 5 trials, the upper bound on the confidence interval for p_0 is less than p_1, causing a_1 to be executed for the remainder of the run.

By assuming that once a_0 becomes stuck below a_1 it never becomes unstuck, we can bound expected error due to sticking on a run in which a_0 would be executed

a0s	a0t	a0b	a1s	a1t	a1b
(0 /	2) .65763	(0 /	1) .79346		
(0 /	3) .56151	(11 /	24) .64925		
(0 /	4) .48990	(57 /	121) .55953		
(0 /	5) .43449	(108 /	253) .48847		
(0 /	5) .43449	(132 /	300) .49658		

Figure 26: A sample run with $p_0 = .55$, $p_1 = .45$, and $z_{\alpha/2} = 1.96$. Here, the first action really does get stuck below the second.

T times, if unstuck, by

$$\sum_{t=1}^{T} \Pr(ub(x_0, t) \text{ first goes below } p_1 \text{ at time } t)(T-t)(p_0 - p_1) \ .$$

It is the sum, over all time steps t on which a_0 is executed, of the probability that a_0 first gets stuck at time t times the number of time steps that remain, $(T-t)$, times the magnitude of the error, $(p_0 - p_1)$. By solving for x_0, we can transform the constraint that $ub(x_0, t) < p_1$ into

$$x_0 < tp_1 - z_{\alpha/2}\sqrt{tp_1(1-p_1)}$$

Now we must compute the probability that x_0 first goes below some function $f(t)$ at time t. The sequence of values taken on by x_0 over time can be modeled as a 0-1 random walk, with $x_0(t)$ the value taken on by the walk at time t. Figure 27 depicts the function f and process x_0. Letting $k = \lfloor f(t) \rfloor$, the probability that x_0 first goes below f at time t is the product of the probabilities that $x_0(t) = k$ and that x_0 never goes below f before time t. The first probability is simply

$$\binom{t}{k} p_0^k (1-p_0)^{t-k} \ .$$

We can approximate the probability that x_0 never goes below f before time t by substituting for f the line l that goes through the point $\langle t, k \rangle$ with slope $f'(t)$. This line is approximately tangent to $f(t)$. The probability that x_0 never goes below l before time t can be approximated by constructing a new random walk problem as

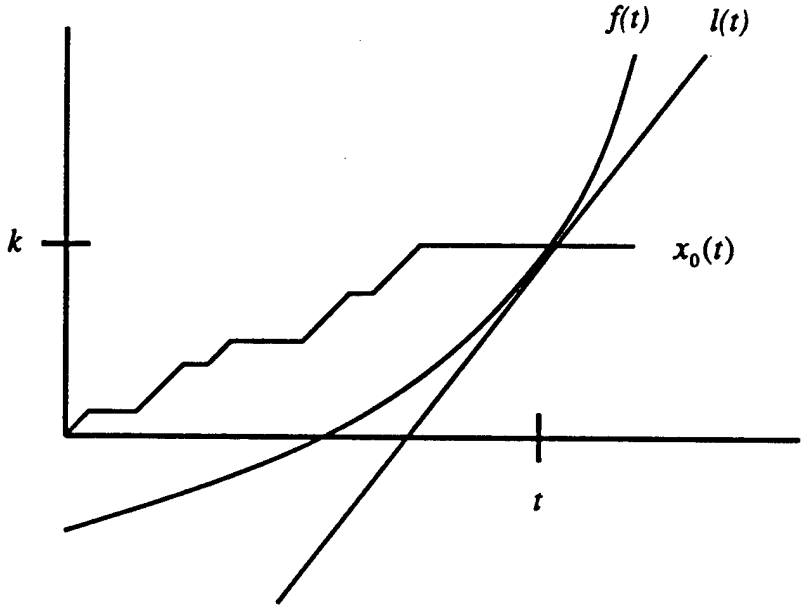


Figure 27: The random walk $x_0(t)$ and function $f(t)$.

shown in Figure 28. The origin is the point $\langle t, k \rangle$ and the coordinates run backward in each direction. The process x_0^* is a 0-1 random walk with probability k/t of getting a 1, and the line l is the same as before. The probability that a 0-1 random walk ever hits a line through the origin is approximately p/m where p is the probability of getting a 1 in the random walk and m is the slope of the line [38]. Thus, the probability that x_0^* never hits the line is $1 - k/(tf'(t))$.

So, our final (approximate) answer for the probability that x_0 first goes below $tp_1 - z\sqrt{tp_1(1-p_1)}$ at time t (called $sp(t)$ for *sticking probability* at time t) is

$$sp(t) = \left(1 - \frac{k}{t(p_1 - \frac{1}{2}z_{\alpha/2}\sqrt{p_1(1-p_1)/t})}\right)\binom{t}{k}p_0^k(1-p_0)^{t-k} ,$$

where $k = \lfloor tp_1 - z_{\alpha/2}\sqrt{tp_1(1-p_1)} \rfloor$.

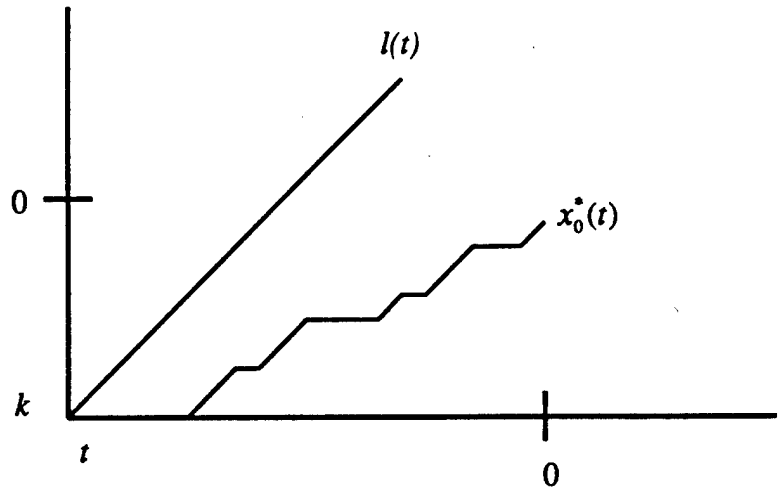


Figure 28: New random walk in inverted coordinate system.

Total Error

An approximate upper bound on the total expected error on a run of length T can finally be expressed as the sum of the regular and sticking error:

$$\frac{z_{\alpha/2}^2}{4(p_0 - p_1)} + \sum_{t=1}^{T'} sp(t)(N-t)(p_0 - p_1) \ .$$

The sticking error is summed to T', the expected number of times a_0 will be executed, which is $T - \frac{z_{\alpha/2}^2}{4(p_0-p_1)}$. There has not yet been any discussion of appropriate values for $z_{\alpha/2}$ to take on. It determines the size of the confidence interval and, therefore, the number of trials it takes to drive an upper bound below a certain value. Thus, regular error *increases* as $z_{\alpha/2}$ increases and the interval gets larger. As $z_{\alpha/2}$ increases, the height of $f(t)$ decreases, making it less likely that x_0 will go below. Thus, error due to sticking *decreases* as $z_{\alpha/2}$ increases. This tradeoff is illustrated in Figure 29, which plots regular error and error due to sticking as functions of $z_{\alpha/2}$.

If we had any *a priori* expectations (and had some idea how to usefully approximate the monstrous form for expected error as a closed form) about the underlying values of p_0 and p_1, we could choose $z_{\alpha/2}$ to minimize expected error.

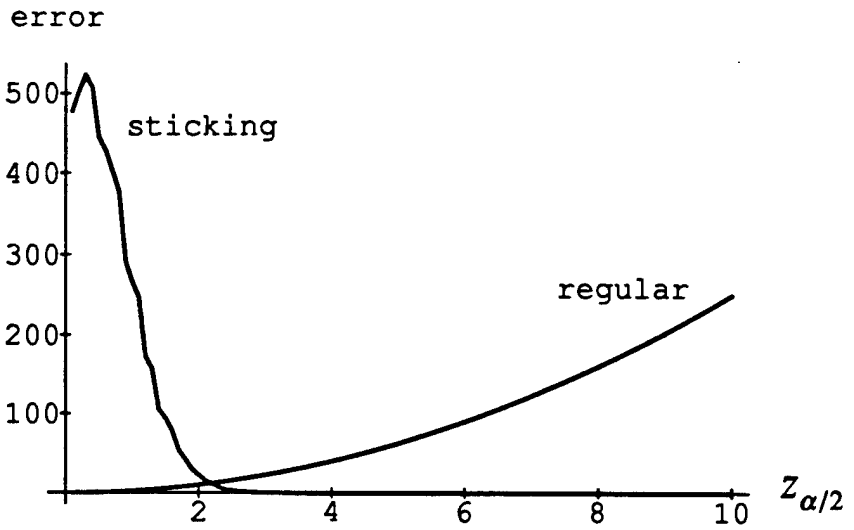


Figure 29: Expected regular error and sticking error plotted as a function of $z_{\alpha/2}$.

4.3 Empirical Results

The approximations of the previous section were tested by comparing predicted results against actual results of the interval estimation algorithm in a simulated world. The algorithm was executed for δ ranging, in increments of .05, from .05 to .6, with p_1 and p_2 equally spaced about .5 (for $\delta = .1$, $p_1 = .55$ and $p_2 = .45$.) For each value of δ, 1079 runs of length 10,000 were conducted. The variable $z_{\alpha/2}$ had value 1.96 throughout. Figure 30 contains a plot, for each δ, of the mean error of the runs that did not stick, together with the predicted error. The predictions seem to be fairly accurate for regular error. Figure 31 shows the mean error due to sticking for each δ, along with the predicted values. This prediction is somewhat less accurate. Nonetheless, these results are encouraging, because we can see that, in these cases, the total expected error is quite small—less than 50 fewer instants of reinforcement value 1 than expected from the optimal algorithm for runs of length 10,000.

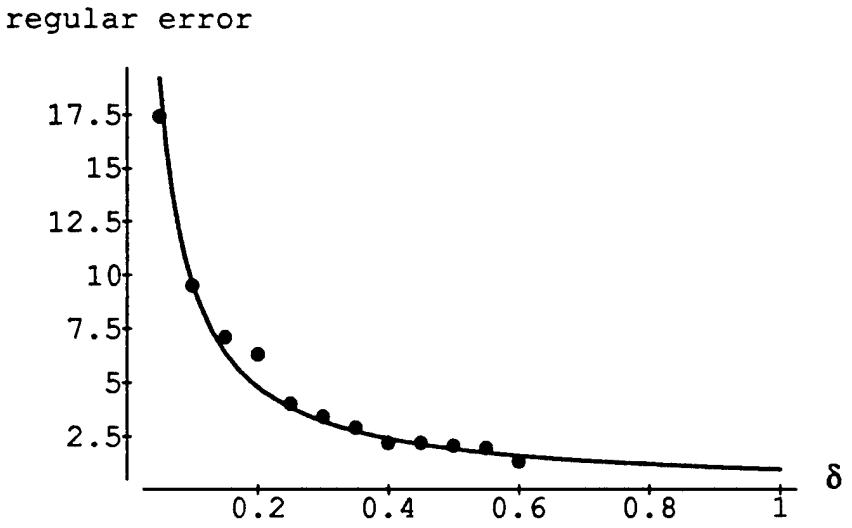


Figure 30: Regular error as a function of δ; dots indicate the mean regular error on 1079 runs of length 10,000; the curve is predicted error.

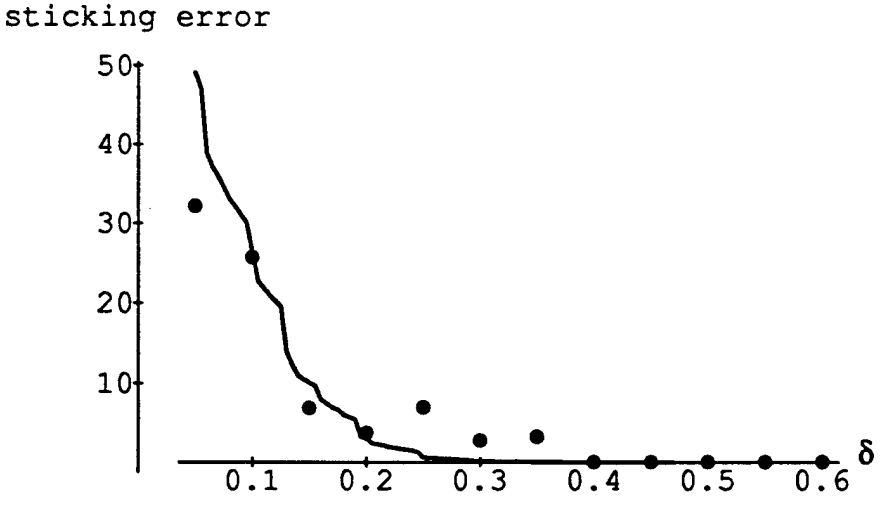


Figure 31: Error due to sticking as a function of δ; dots indicate the mean error due to sticking on 1079 runs of length 10,000; the curve is predicted error.

Task	p_0	p_1
1	.9	.1
2	.6	.4
3	.9	.8
4	.2	.1

Table 1: Parameters of test environments.

4.4 Experimental Comparisons

This section reports the results of a set of experiments designed to compare the performance of the interval estimation algorithm with a number of the most promising reinforcement-learning algorithms.

4.4.1 Algorithms and Environments

The following algorithms were compared in these experiments:

- BANDIT (described in Figure 12)
- L_{RP} (described in Figure 14)
- L_{RI} (described in Figure 14)
- TS (described in Figure 15)
- RC (described in Figure 16)
- IE (described in Figure 21)

Each of the algorithms was tested in four different environments. The environments generate Boolean reinforcement, with positive reinforcement resulting with probability p_0 after doing action a_0 and with probability p_1 after doing action a_1. Table 1 shows the values of p_0 and p_1 for each environment.

ALG-TASK	1	2	3	4
BANDIT(k)	1	12	10	10
L_{RP} (α)	.60	.60	.30	.40
L_{RI} (α)	.55	.1	.05	.15
TS (λ)	.30	.20	.20	.35
RC (α)	.40	.30	.15	.50
IE ($z_{\alpha/2}$)	3.0	2.0	3.0	2.0

Table 2: Best parameter value for each algorithm in each environment.

4.4.2 Parameter Tuning

Each of the algorithms has a single parameter that can be chosen to make the algorithm more or less conservative;[6] the best choice of value for these parameters typically depends on the length of the run, because it is more important to insure that an absorbing algorithm converges to the correct action on a long run. For each algorithm and environment, a series of 100 trials of length 1000 were run with different values of the parameter. Table 2 shows the best parameter value found for each algorithm and environment pair.

Although these experiments are illuminating, in actual applications we will typically want to apply these algorithms to situations in which the underlying probabilities are not known or there is not enough time to make many runs with different parameter values. In such situations, an algorithm that performs well over a wide range of problems with the same parameter value is to be preferred over one that performs well when the parameter is chosen exactly appropriately for the problem, but poorly otherwise. As we can see in Table 2, the interval estimation algorithm operates at its best in all of these problems with a $z_{\alpha/2}$ value between 2 and 3—this roughly corresponds to using 95 or 99 percent confidence intervals, values that, interestingly, are often used by human decision-makers.

[6] Actually, RC also has parameters β and σ, but following the author [70], these parameters were held constant at .1 and .3, respectively.

ALG-TASK	1	2	3	4
BANDIT	.8982	.5856	.8892	.1888
L_{RP}	.8172	.5190	.8665	.1521
L_{RI}	.8911	.5872	.8780	.1934
TS	.8979	.5893	.8941	.1870
RC	.8988	.5890	.8897	.1930
IE	.9004	.5953	.8937	.1972
random	.5000	.5000	.8500	.1500
optimal	.9000	.6000	.9000	.2000

Table 3: Average reinforcement over 100 runs of length 1000.

4.4.3 Results

After choosing the best parameter value for each algorithm and environment, the performance of the algorithms was compared on runs of length 1000. The performance metric was average reinforcement per tick, averaged over the entire run. The results are shown in Table 3. These results do not tell the entire story, however. It is important to test for statistical significance to be relatively sure that the ordering of one algorithm over another did not arise by chance. Figure 32 shows, for each task, a pictorial representation of the results of a 1-sided t-test applied to each pair of experimental results. The graphs encode a partial order of significant dominance, with solid lines representing significance at the .95 level and dashed lines representing significance at the .85 level. We can see that the interval-estimation algorithm dominates in nearly every task. On Task 3 its average reinforcement value was slightly lower than that of the TS algorithm, but this difference was not significant. The L_{RP} algorithm is, as expected, uniformly sub-optimal, and the rest of the algorithms perform about the same at quite a high level.

Another view of the relative performance of the algorithms is given by examining their learning curves. A learning curve is a plot of expected reinforcement values versus time, which shows the rate of performance improvement. Figures 33, 34, 35, and 36 contain, for each task, the superimposed learning curves of each algorithm for that task. Each point represents the average reinforcement received over a sequence of 50 ticks, averaged over 100 runs of length 1000. For Tasks 1 and 2, the curves

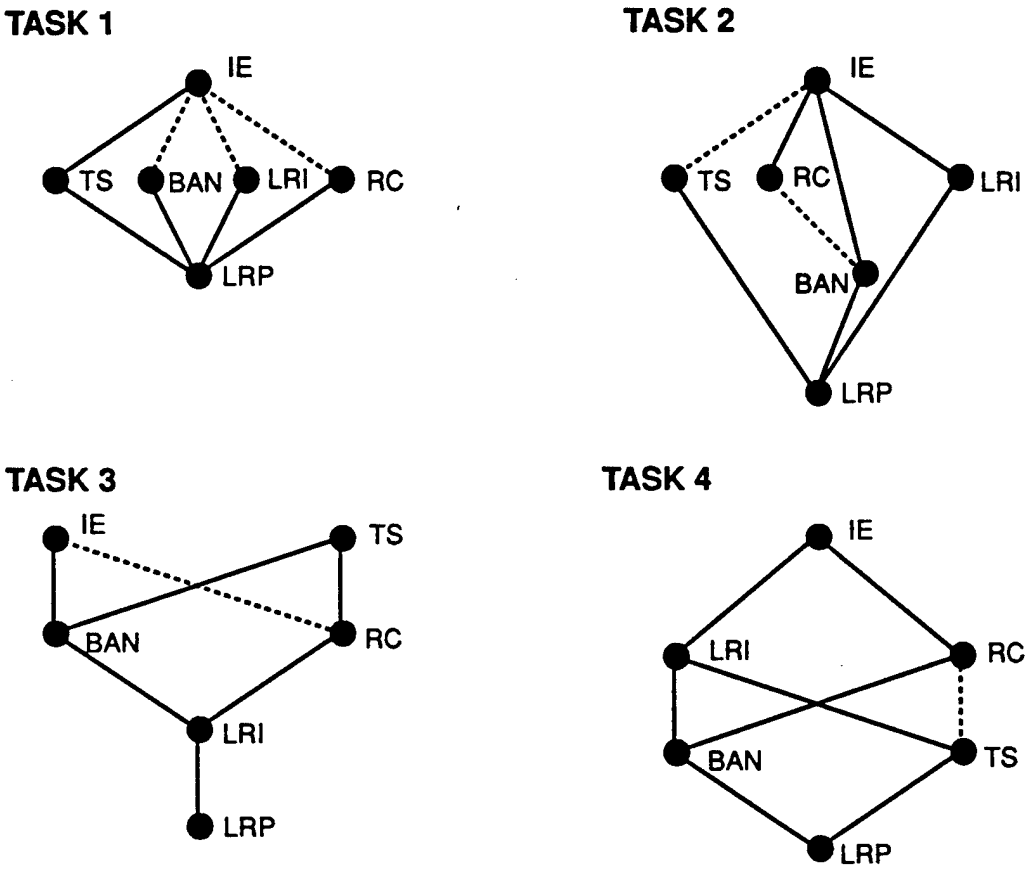


Figure 32: Significant dominance partial order among algorithms for each task.

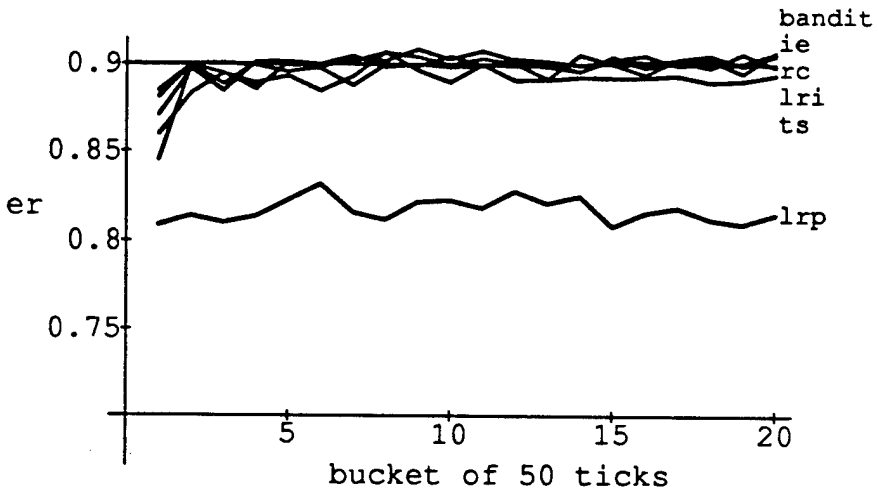


Figure 33: Learning curves for Task 1.

are hard to differentiate; the labels on the right hand sides of the graphs indicates the average relative performance of the algorithms on the first sample of 50 ticks.

4.5 Extensions

As with the algorithms of Chapter 3, the interval estimation algorithm can be extended to work in more complex environments. All of the extensions described in this section have been implemented and tested in simulated environments.

4.5.1 Multiple Inputs and Actions

The interval estimation algorithm is directly generalizable to multiple actions. Statistics are collected for each action and are used to construct upper bounds. The action with the highest upper bound is chosen to be executed at each tick.

There is no specific way to tailor the interval estimation algorithm to work in situations where there are multiple input states. The method of making a copy of the internal state for each possible input situation can be applied to the interval estimation algorithm, but because there is more than a single number associated

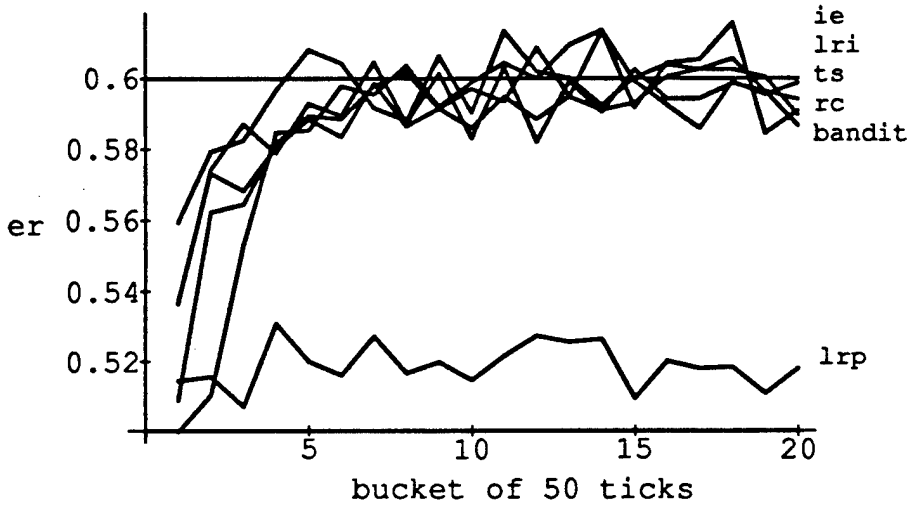


Figure 34: Learning curves for Task 2.

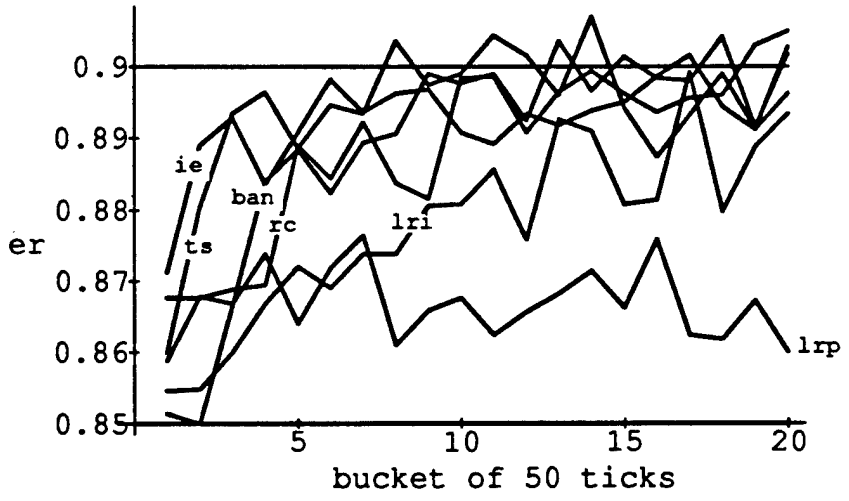


Figure 35: Learning curves for Task 3.

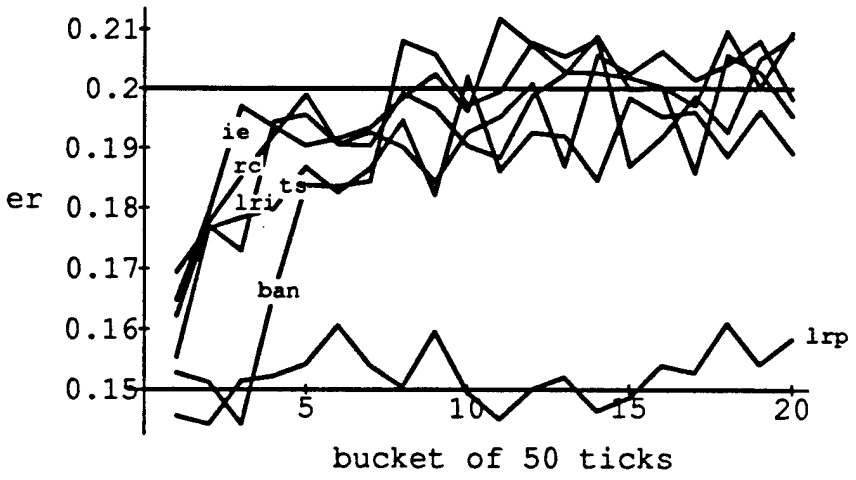


Figure 36: Learning curves for Task 4.

with each input state, it would be difficult to apply the linear association or error backpropagation methods.

4.5.2 Real-valued Reinforcement

Rather than thinking of choosing the action with the highest probability of succeeding, we can think of choosing the action with the highest expected reinforcement. Under this view, the interval estimation process can be applied to the expected value of reinforcement given that the action a is executed in situation i. If the reinforcement for each tick is binomially distributed with parameter p, this is exactly what is taking place in the version of the algorithm presented in Section 4.1.

Simple extensions can be made if a different probabilistic distribution underlies the reinforcement associated with taking action. In order to handle real-valued reinforcement, for example, we can apply the following two methods: assume the normal distribution or use non-parametric statistics.

If the reinforcement values are normally distributed, we can use standard statistical methods to construct a confidence interval for the expected value. In order to do this, we must keep the following statistics: n, the number of trials, $\sum x$, the sum of the reinforcement received so far, and $\sum x^2$, the sum of squares of the individual

reinforcement values. The upper bound of a $100(1-\alpha)$ % confidence interval for the mean of the distribution can be computed by

$$nub(n, \sum x, \sum x^2) = \bar{y} + t_{\alpha/2}^{(n-1)} \frac{s}{\sqrt{n}}$$

where $\bar{y} = x/n$ is the sample mean,

$$s = \sqrt{\frac{n\sum x^2 - (\sum x)^2}{n(n-1)}}$$

is the sample standard deviation, and $t_{\alpha/2}^{(n)}$ is Student's t function with $n-1$ degrees of freedom [69]. Other than using a different statistical method to compute the upper bound of the expected reinforcement, the algorithm remains the same.

Even when the reinforcement values cannot be assumed to be normally distributed, the interval estimation algorithm can be implemented using simple non-parametric statistics.[7] In this case, it is not possible to derive an upper bound on expected value from summary statistics, so we must keep the individual reinforcement values. Obviously, it is impossible to store them all, so only the data in a sliding window are kept. The non-parametric version of the interval estimation algorithm requires another parameter, w, that determines the size of the window of data. The data are kept sorted by value as well as by time received. The upper bound of a $100(1-\alpha)$ % confidence interval for the center of the underlying distribution (whatever it may be) can be calculated, using the ordinary sign test [26], to be the $(n-u)$th element of the sorted data, if they are labelled, starting at 1, from smallest to largest, where n is minimum of w and the number of instances received. The value u is chosen to be the largest value such that

$$\sum_{k=0}^{u} \binom{n}{k} .5^n \leq \alpha/2 \ .$$

For large values of n, u can be approximated using the normal distribution.

[7] Non-parametric methods tend to work poorly when there are a small number of discrete values with very different magnitudes. Practical results have been obtained in such cases by using methods for the normal distribution with the modification that each action is performed at least a certain fixed number of times. This prevents the sample variance from going to 0 on small samples with identical values.

4.5.3 Non-stationary environments

The basic version of the interval estimation algorithm can converge to absorbing states and, as noted in Section 3.6.2, that makes it inappropriate for use in non-stationary environments. One way to modify the algorithm in order to fix this problem is to decay all of the statistics associated with a particular input value by some value d less than, but typically near, 1, whenever that input value is received. This decaying will have the effect that the recorded number of trials of an action that is not being executed decreases over time, causing the confidence interval to grow, the upper bound to increase, and the neglected action to be executed again. If its underlying expected value has increased, that will be revealed when the action is executed and it may come to be the dominant action.

This technique may be similarly applied when using statistical methods for normally-distributed reinforcement values. The non-parametric method described above is already partially suited to non-stationary environments because old data only has a finite period of influence (of length w) on the choices of the algorithm. It can be made more responsive to environmental changes by occasionally dropping a data point from the list of an action that is not being executed. This will cause the upper bound to increase, eventually forcing the action to be executed again.

Another method of changing an algorithm to work in non-stationary environments is to choose the "wrong action" (one that would not have been chosen by the algorithm) with probability $1/n$, where n is the number of trials that have taken place so far. As time passes, it becomes less and less likely to do an action that is not prescribed by the current learned policy, but executing these "wrong" actions ensures that if they have become "right" due to changes in the environment, the algorithm will adapt. This method is more suited to situations in which environmental changes are expected to be more likely to happen early in a run, rather than later.

4.6 Applicability of this Algorithm

The interval estimation algorithm is of theoretical interest because of its simplicity and its direct ties to standard statistical methods. It also performs slightly better than many proposed reinforcement-learning algorithms. However, this algorithm, as well as other reinforcement-learning algorithms that require copies of the state for each possible input, is fundamentally unsuitable for learning in embedded systems because of its high computational complexity and lack of generalization.

Except for the linear-association and error-backpropagation algorithms, all of the other algorithms we have examined require time at least proportional to the number of possible actions, and space proportional to the product of the number of inputs and the number of actions. As we begin to apply these algorithms to real-world problems, their time and space requirements will make them impractically slow. A driving factor in the rest of this dissertation is the need for reinforcement-learning algorithms with lower time and space complexity, ideally proportional to the *logarithms* of the numbers of inputs and actions.

In addition, the interval estimation algorithm completely compartmentalizes the information it has about individual input situations. If it learns to perform a particular action in one input situation, that has no influence on what it will do in similar input situations. In realistic environments, an agent cannot expect ever to encounter all of the input situations, let alone have enough experience with each one to learn the appropriate response. Thus, it is important to develop algorithms that will generalize across input situations. Generalization is a dangerous thing, however; too much generalization defeats the learning of very complex action functions.

It is possible to modify the interval-estimation algorithm in order to support some degree of generalization across input situations. Instead of simply using the upper bound on expected value of an action a in a situation i, it is possible, instead, to compute a kind of average based on the results of performing action a in situations similar to i, with "nearer" situations weighted more heavily than those farther away. This technique requires a measure on the nearness of input situations to one another and is no longer directly grounded in statistical theory. By addressing the

generalization issue in this way, however, we increase the computation time of the algorithm, for now it requires evaluating action a in a number of input situations. This only adds a constant factor that depends on the number of neighbors that are used, but it just makes a bad situation worse.

The interval-estimation method might be made both more computationally efficient and able to generalize across situations by using associative methods (such as linear association or backpropagation) to store each of the components of the state for an input-action pair. The statistical foundations of such an approach would be weak, potentially causing a number of problems.

It is important to note, however, that in order to find more efficient algorithms, we must give up something. What we will be giving up is the possibility of learning any arbitrary action mapping. In the worst case, the only way to represent a mapping is as a complete look-up table, which is what the multiple-input version of the interval-estimation algorithm does. There are many useful and interesting functions that can be represented much more efficiently, and the remainder of this work will rest on the hope and expectation that an agent can learn to act effectively in interesting environments without needing action maps of pathological complexity.

Chapter 5

Divide and Conquer

Because we wish to reduce the complexity of learning algorithms to be proportional to the logarithms of the numbers of inputs and outputs, it is useful to think of the inputs and outputs as being encoded in some binary code. The problem, then, is one of constructing a function that maps a number of input bits to a number of output bits. If we can construct algorithms that effectively learn interesting classes of functions with time and space complexity that is polynomial in the number of input and output bits, we will have improved upon the previous group of algorithms.

Having decided to view the problem as one of learning a mapping from many input bits to many output bits, we can reduce this problem to the problem of learning a mapping from many input bits to one output bit. This chapter discusses such a problem reduction, first describing it informally, then proving its correctness. It concludes with an application of the reduction method to a complex learning problem.

5.1 Boolean-Function Learners

A Boolean-function learner (BFL) is a reinforcement-learning behavior that learns a mapping from many input bits to one output bit. It has the same input-output structure as any of the algorithms discussed so far, but is limited to having only two actions. We can describe a BFL with k input bits in the general form of a

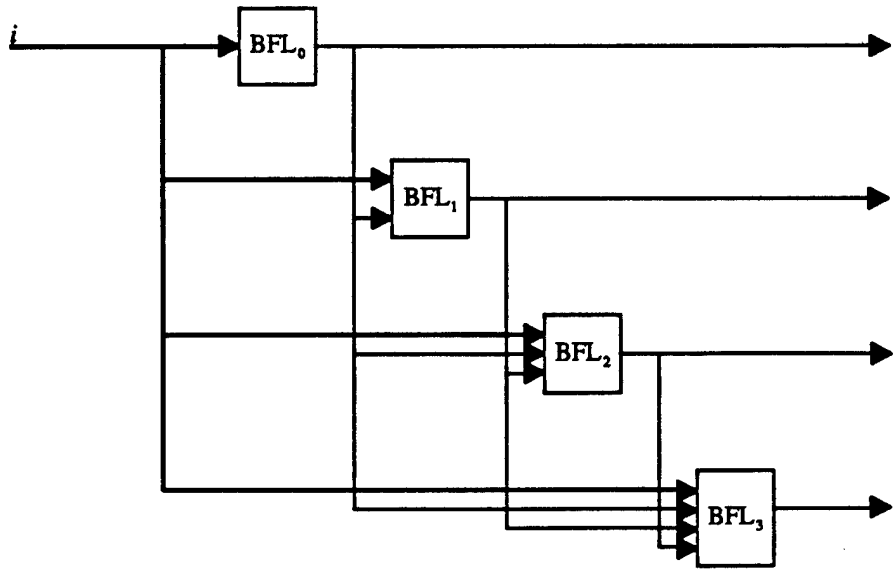


Figure 37: A cascaded learner constructed from BFL's.

learning behavior where $\hat{s}_{0,k}$ is the initial state, $\hat{u}_k$ is the update function and $\hat{e}_k$ is the evaluation function.

A BFL is *correct* if and only if whenever it chooses an action a in situation i, $er(i,a) \geq er(i,\neg a)$. That is, it always chooses the action that has the higher expected reinforcement.

5.2 Cascade Algorithm

We can construct an algorithm that learns an action map with N output bits by using N copies of a Boolean-function learning algorithm, one dedicated to learning the function corresponding to each individual output bit. If we do this in the simplest way, it will not work correctly: when the collection of BFL's generates an output pattern that does not result in positive reinforcement, it is difficult to know whose fault it was. Perhaps only one of the bits was "wrong." To avoid this problem, often referred to the as "structural credit assignment" problem, we construct a learning algorithm (as shown in Figure 37) from N cascaded BFL's. The BFL dedicated to learning to generate the first output bit (referred to as BFL_0) has

Algorithm 11 *(CASCADE)*

$$
\begin{aligned}
s_0 &= \textit{array of } \hat{s}_{0,M+j} \textit{ where } j \textit{ goes from } 0 \textit{ to } N-1 \\
u(s,i,a,r) &= \textit{for } j := 0 \textit{ to } N-1 \\
&\qquad \hat{u}_{M+j}(s[j], \mathit{concat}(i, a[0..j-1]), a[j], r) \\
e(h,i) &= \textit{for } j := 0 \textit{ to } N-1 \\
&\qquad a[j] := \hat{e}_{M+j}(s[j], \mathit{concat}(i, a[0..j-1])) \\
&\quad \textit{return } a
\end{aligned}
$$

Figure 38: The CASCADE algorithm.

the M real input bits as input. The next one, BFL_1, has the M real inputs as well as the output of BFL_0 as input. In general, BFL_k will have $M+k$ bits of input, corresponding to the real inputs and the outputs of the k lower-numbered BFL's. Each one learns what its output bit should be, given the input situation and the values of the output bits of the lower-numbered BFL's.

The cascade algorithm can be described as a learning behavior as shown in Figure 38. The complexity of this algorithm can be expressed as a function of the complexity of the component BFL's, letting $S(\hat{s}_{0,k})$ be the size of the initial state of a BFL with k inputs, $T(\hat{u}_k)$ be the time for the BFL update function on k input bits, and $T(\hat{e}_k)$ be the time for the BFL evaluation function with k input bits. For the entire cascade algorithm with M input bits and N output bits, the size of the state is

$$O(\sum_{j=0}^{N-1} S(\hat{s}_{0,M+j})) \ ,$$

which reduces to

$$O(N\, S(\hat{s}_{0,M+N})) \ ;$$

the time for an update is

$$O(N\, T(\hat{u}_{M+N})) \ ;$$

and the time for an evaluation is

$$O(N\, T(\hat{e}_{M+N})) \ .$$

Given efficient algorithms for implementing the BFL's, the cascade method can construct an efficient algorithm for learning functions with any number of output bits.[1]

This efficiency comes at a price, however. Even if there is no noise in the environment, a mistake made on bit j will cause the reinforcement information for bits 0 through $j-1$ to be in error. To see this, consider the case of two output bits. Given input instance i, bit 0 is generated to have the value 1; then, bit 1 is generated, as a function of both i and the value of bit 0, to have the value 0. If the correct response in this case was $\{1,1\}$, then each of the bits will be given low reinforcement values, even though bit 0 was correct. This brings to light another requirement of the BFLs: they must work correctly in nonstationary environments. As the higher-numbered BFL's are in the process of converging, the lower-numbered ones will be getting reinforcement values that are not necessarily indicative of how well they are performing. Once the higher-numbered BFL's have converged, the lower-numbered BFL's must be able to disregard their earlier training and learn to act correctly given the functions that the higher-numbered BFL's are now implementing.

5.3 Correctness and Convergence

In order to show that this algorithm works, we must demonstrate two points. First, that if the component BFL's converge to correct behavior then the behavior of the entire construction will be correct. Second, that the component BFL's are trained in a way that guarantees that they will converge to correct behavior. These requirements will be referred to as correctness and convergence.

5.3.1 Correctness

This section presents a proof that the cascade construction is correct for the case of two output bits. A similar proof can be constructed for cases with any number of bits. Assume that the two BFL's have already converged, the first one to the

[1]This assumes that $S(\hat{s}_{0,k})$, $T(\hat{u}_k)$, and $T(\hat{e}_k)$ are all monotonically non-decreasing in k.

function f_0, and the second to the function f_1. The following formula asserts that the function f_0 is correct, given the choice of f_1:

$$\forall i.\ er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), f_1(i, \neg f_0(i)))\ ; \tag{1}$$

that is, that for any value of the input i, it is better for the first bit to have the value $f_0(i)$ than its opposite. Similarly, we can assert that the function f_1 is correct:

$$\forall i, b.\ er(b, f_1(i, b)) \geq er(b, \neg f_1(i, b))\ ; \tag{2}$$

that is, that for any value of input i and first bit b (b is the output of f_0 in the cascade), it is better that the second bit have the value $f_1(i, b)$ than its opposite.

We would like to show that the composite output of the cascade algorithm is correct: that is, that for any input, no two-bit output has higher expected reinforcement than the one that is actually chosen by f_0 and f_1. This can be stated formally as the following conjunction:

$$\forall i.\ er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), f_1(i, f_0(i))) \wedge \tag{3}$$

$$\forall i.\ er(f_0(i), f_1(i, f_0(i))) \geq er(f_0(i), \neg f_1(i, f_0(i))) \wedge \tag{4}$$

$$\forall i.\ er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), \neg f_1(i, f_0(i)))\ . \tag{5}$$

The first conjunct, 3, can be shown with a proof by cases. In the first case, given input i, function f_1 is insensitive to its second argument: that is, $f_1(i, x) = f_1(i, \neg x)$. In this case,

$$er(\neg f_0(i), f_1(i, f_0(i))) = er(\neg f_0(i), f_1(i, \neg f_0(i)))\ ; \tag{6}$$

from 6 and assumption 1 we can conclude that

$$er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), f_1(i, f_0(i)))\ .$$

In the second case, function f_1 is sensitive to its second argument when the first argument has value i; that is, $f_1(i, x) = \neg f_1(i, \neg x)$. In this case,

$$er(\neg f_0(i), f_1(i, f_0(i))) = er(\neg f_0(i), \neg f_1(i, \neg f_0(i)))\ . \tag{7}$$

Combining assumptions 1 and 2, we can derive

$$er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), \neg f_1(i, \neg f_0(i)))\ . \tag{8}$$

From 7 and 8, we have our desired conclusion, that

$$er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), f_1(i, f_0(i))) \ .$$

The second conjunct, 4, follows directly from assumption 2.

The third conjunct, 5, also requires a proof based on the same cases used in the proof of the first conjunct. In the first case, $f_1(i, x) = f_1(i, \neg x)$, so

$$er(\neg f_0(i), \neg f_1(i, f_0(i))) = er(\neg f_0(i), \neg f_1(i, \neg f_0(i))) \ . \tag{9}$$

From 9 and result 8 above, we can derive

$$er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), \neg f_1(i, f_0(i))) \ .$$

In the second case, $f_1(i, x) = \neg f_1(i, \neg x)$, so

$$er(\neg f_0(i), \neg f_1(i, f_0(i))) = er(\neg f_0(i), f_1(i, \neg f_0(i))) \ .$$

Combining this result with assumption 1, we get the desired result, that

$$er(f_0(i), f_1(i, f_0(i))) \geq er(\neg f_0(i), \neg f_1(i, f_0(i))) \ .$$

Thus, we can see that local assumptions of correctness for each BFL are sufficient to guarantee global correctness of the entire cascade algorithm.

5.3.2 Convergence

Now, we must show that the BFL's are trained in a way that justifies assumptions 1 and 2 above. It is difficult to make this argument precise without making very strong assumptions about the BFL's and the environment. Informally, the argument is as follows. The highest-numbered BFL (BFL_N) always gets correct reinforcement and so converges to the correct strategy; this is because, independent of what the lower-numbered BFL's are doing, it can learn always to make the best of a bad situation. Once this has happened, BFL_{N-1} will get correct reinforcement; because its internal learning algorithm works in non-stationary environments, it will converge to behave

in the best way it can in light of what BFL_N does (which now is correct). This argument can be made all the way up to BFL_0.

In general, the convergence process may work somewhat differently. Convergence happens on an input-by-input basis, because there is no guarantee that the whole input space will be explored during any finite prefix of a run of the agent. Rather, an input comes in from the world and all the BFL's except BFL_N generate their output bits. This constitutes a learning instance for BFL_N, which can gain information about what to do in this situation. After this situation has occurred a few times, BFL_N will converge *for that input situation* (including the bits generated by the lower-numbered BFL's). As the lower-numbered BFL's begin to change their behavior, they may generate output patterns that BFL_N has never seen, requiring BFL_N to learn what to do in that situation before the lower-numbered BFL's can continue their learning process.

5.4 Example

As a simple illustration of the cascade reduction method, this section outlines its use, in conjunction with the interval estimation algorithm, to solve a complex learning problem. As a baseline for comparison, we also consider the use of the interval estimation algorithm in conjunction with the method of adding extra copies of the basic statistical algorithm to handle multiple actions. These two methods will be compared in terms of computational complexity and performance on the learning problem.

5.4.1 Complexity

If there are M input bits and N output bits, the space complexity of an instance of the interval estimation algorithm with a copy of the basic algorithm for each input-action pair is $O(2^{M+N})$. The cascade method requires N copies of the algorithm, each with 1 output bit and up to $M+N-1$ input bits. The total space requirement

for the cascade algorithm would, in this case, be $O(N2^{M+N})$, which is worse than using the simple copying method.

The time complexity of an update operation (if indexing is ignored) is constant for the copying method; the cascade method requires each component BFL to be updated, using $O(N)$ time.

The time complexity of an evaluation using the simple copying method is $O(2^N)$, because each possible action must be evaluated. Using the cascade method, however, it is $O(N)$, because only 2 actions must be evaluated for each output bit.

Each cycle of a learning behavior requires one update and one evaluation: for the copying method this requires $O(1) + O(2^N) = O(2^N)$ time; for the cascade method it requires $O(N) + O(N) = O(N)$ time. Thus, the space complexity is somewhat greater using the cascade method, but computation time is considerably shorter.

5.4.2 Performance

A moderately complex reinforcement-learning problem is that of learning to be an n-bit adder: the learner has $2n$ input bits, representing the addends, and n output bits, representing the result. It is given reinforcement value 1 if the output bits are the binary sum of the first n input bits and the second n input bits, otherwise it is given reinforcement value 0. For this experiment, a 5-bit adder problem was used; it has fairly high complexity, with 1024 possible inputs and 32 possible outputs.

As we can see in Figure 39, which shows average reinforcement as a function of time (data points represent averages of 100 time steps), the cascade method has much better performance than the simple copying method. One reason for the superior performance of the cascade method over the copy method is that, in the cascade method, the output bits are being trained in parallel and the agent will not, in general, have to try all (or even half) of the 2^N possible actions in each input situation before finding the correct one. At first, it may seem that the algorithm is somehow taking advantage of the structure of the adder problem, because the general solution to the n-bit adder problem involves feeding intermediate results (carries) to later parts of the computation. Upon closer examination, however, it

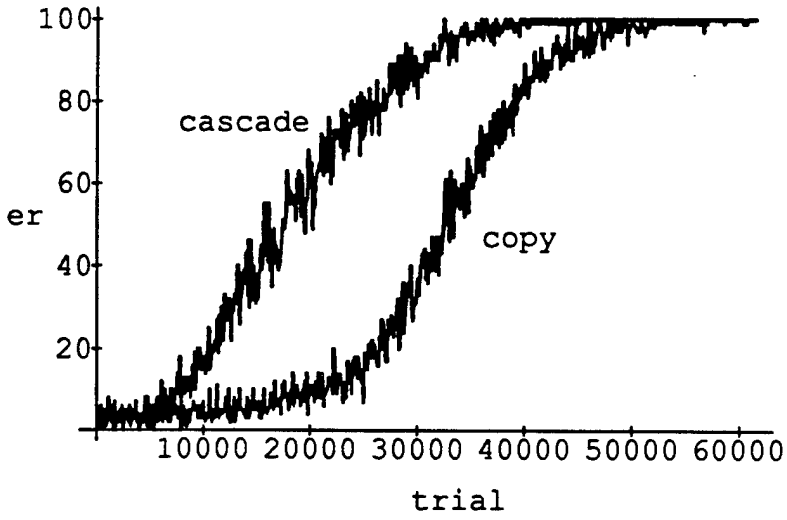


Figure 39: Performance of interval estimation algorithm on 5-bit adder problem using copying method and cascade method of generating multiple outputs.

is clear that the intermediate results are simply less-significant output bits, which are not related to the values of the carries and do not simplify the computation of the more-significant output bits. Thus, the performance of the CASCADE algorithm cannot be attributed to the special structure of the adder problem.

Chapter 6

Learning Boolean Functions in k-DNF

6.1 Background

In the previous chapter, we saw that the problem of learning an action map with many output bits can be reduced to the problem of learning a collection of action maps with single Boolean outputs. Such action maps can be described by formulae in propositional logic, in which the atoms are input bits. The formula $(i_1 \wedge i_2) \vee \neg i_0$ describes an action map that performs action 1 whenever input bits 1 and 2 are on or input bit 0 is off and performs action 0 otherwise.

As we saw in Section 4.6, any learning algorithm that is to be more efficient than methods like interval estimation will only be able to learn a restricted class of action maps. When there are only two possible actions, we can describe the class of action maps that are learnable by an algorithm in terms of syntactic restrictions on the corresponding class of propositional formulae. This method is widely used in the formal literature on concept learning.

A restriction that has proved useful to the concept-learning community is to the class of functions that can be expressed as propositional formulae in k-DNF. A formula is said to be in *disjunctive normal form* (DNF) if it is syntactically organized into a disjunction of purely conjunctive terms; there is a simple algorithmic method

for converting any formula into DNF [21]. A formula is in the class k-DNF if and only if its representation in DNF contains only conjunctive terms of length k or less. There is no restriction on the number of conjunctive terms—just their length. Whenever k is less than the number of atoms in the domain, the class k-DNF is a restriction on the class of functions.

The next section presents Valiant's algorithm for learning functions in k-DNF from input-output pairs. The following sections describe algorithms for learning action maps in k-DNF from reinforcement and present the results of an empirical comparison of their performance. For each reinforcement-learning algorithm, the inputs are bit-vectors of length M, plus a distinguished reinforcement bit; the outputs are single bits.

6.2 Learning k-DNF from Input-Output Pairs

Valiant was one of the first to consider the restriction to learning functions expressible in k-DNF [76,77]. He developed an algorithm, shown below, for learning functions in k-DNF from input-output pairs, which actually only uses the input-output pairs with output 0.

Algorithm 12 (VALIANT) *Let T be initialized to the set of conjunctive terms of length k over the set of atoms (corresponding to the input bits) and their negations, and let L be the number of learning instances required to learn the concept to the desired accuracy.*[1]

for $i := 1$ *to* L *do begin*
    $v :=$ *randomly drawn negative instance*
    $T := T-$ *any term that is satisfied by* v
end
return T

[1]This choice is not relevant to our reinforcement-learning scenario—the details are described in Valiant's papers [76,77].

Algorithm 13 (LARCKDNF) *Let F_T be a function mapping an M-bit input vector into a $2^k\binom{M}{k}$-bit vector, each of whose elements is the result of evaluating an element of T on the raw input vector.*
Let s_0 of this algorithm be the initial state, s_0, of an instance of the LARC *algorithm with $2^k\binom{M}{k}$ bits. The update function will be u of* LARC*, with the input $F_T(i)$, and, similarly, the evaluation will be e of* LARC*, with the input $F_T(i)$.*

Figure 40: The linear-association reinforcement-comparison algorithm for learning functions in k-DNF from reinforcement.

The VALIANT algorithm returns the set of terms remaining in T, with the interpretation that their disjunction is the concept that was learned by the algorithm. This method simply examines a fixed number of negative instances and removes any term from T that would have caused one of the negative instances to be satisfied.[2]

6.3 Combining the LARC and VALIANT Algorithms

Given our interest in restricted classes of functions, we can construct a hybrid algorithm for learning action maps in k-DNF. It hinges on the simple observation that any such function is a linear combination of terms in the set T, where T is the set of conjunctive terms of length k over the set of atoms (corresponding to the input bits) and their negations. It is possible to take the original M-bit input signal and transduce it to a wider signal that is the result of evaluating each member of T on the original inputs. We can use this new signal as input to a linear-associative reinforcement learning algorithm, such as Sutton's LARC algorithm (described in Figure 18. If there are M input bits, the set T has size $\binom{2M}{k}$ because we are choosing from the set of input bits and their negations. However, we can eliminate all elements that contain both an atom and its negation, yielding a set of size $2^k\binom{M}{k}$. The combined algorithm, called LARCKDNF, is described formally in Figure 40.

[2]Valiant's presentation of the algorithm defines T to be the set of conjunctive terms of length k *or less* over the set of atoms and their negations; however, because any term of length less than k can be represented as a disjunction of terms of length k, we use a smaller set T for simplicity in exposition and slightly more efficient computation time.

The space required by the LARCKDNF algorithm, as well as the time to update the internal state or to evaluate an input instance, is proportional to the size of T, and thus, $O(M^k)$.

6.4 Interval Estimation Algorithm for k-DNF

The interval estimation algorithm for k-DNF is, like the algorithm described in Section 6.3, based on Valiant's algorithm, but the interval estimation algorithm uses standard statistical estimation methods, like those used in the IE algorithm, rather than weight-adjustments.

The algorithm will first be described independent of particular statistical tests, which will be introduced later in the section. We shall need the following definitions, however. An input bit vector *satisfies* a term whenever all the bits mentioned positively in the term have value 1 in the input and all the bits mentioned negatively in the term have value 0 in the input. The quantity $er(t, a)$ is the expected value of the reinforcement that the agent will gain, per trial, if it generates action a whenever term t is satisfied by the input and action $\neg a$ otherwise. The quantity $ubr_\alpha(t, a)$ is the upper bound of a $100(1-\alpha)\%$ confidence interval on the expected reinforcement gained from performing action a whenever term t is satisfied by the input. The formal definition of the algorithm is given in Figure 41.

At any moment in the operation of this algorithm, we can extract a symbolic description of its current hypothesis. It is the disjunction of all terms t such that $ubr_\alpha(t, 1) > ubr_\alpha(t, 0)$ and $\Pr(er(t, 1) = er(t, 0)) < \beta$. This is the k-DNF expression according to which the agent is choosing its actions.

As in the regular interval-estimation algorithm, the evaluation criterion is chosen in such a way as to make the important trade-off between acting to gain information and acting to gain reinforcement. Thus, the first requirement for a term to cause a 1 to be emitted is that the upper bound on the expected reinforcement of emitting a 1 when this term is satisfied is higher than the upper bound on the expected reinforcement of emitting a 0 when the term is satisfied.

Algorithm 14 (IEKDNF)

$s_0 =$ *the set* T*, with a collection of statistics associated with each member of the set*

$e(s,i) =$ *for each* t *in* s
if i *satisfies* t *and*
$ubr_\alpha(t,1) > ubr_\alpha(t,0)$ *and*
$\Pr(er(t,1) = er(t,0)) < \beta$
then return 1
return 0

$u(s,i,a,r) =$ *for each* t *in* s
update_term_statistics(t,i,a,r)
return s

Figure 41: The interval estimation algorithm for learning concepts in k-DNF from reinforcement.

Let the *equivalence probability* of a term be the probability that the expected reinforcement is the same no matter what choice of action is made when the term is satisfied. The second requirement for a term to cause a 1 to be emitted is that the equivalence probability be small. Without this criterion, terms for which no action is better will, roughly, alternate between choosing action 1 and action 0. Because the output of the entire algorithm will be 1 whenever any term has value 1, this alternation of values can cause a large number of wrong answers. Thus, if we can convince ourselves that a term is irrelevant by showing that its choice of action makes no difference, we can safely ignore it.

In the simple Boolean reinforcement-learning scenario, the necessary statistical tests are quite simple. For each term, the following statistics are stored: n_0, the number of trials of action 0; s_0, the number of successes of action 0; n_1, the number of trials of action 1; and s_1, the number of successes of action 1. These are incremented only when the associated term is satisfied by the current input instance. Using the definition of $ub(x,n)$ from Figure 21, we can define $ubr_\alpha(t,0)$ as $ub(s_0,n_0)$ and

$ubr_\alpha(t,1)$ as $ub(s_1,n_1)$, where s_0, n_0, s_1, and n_1 are the statistics associated with term t and α is used in the computation of ub.

To test for equality of the underlying Bernoulli parameters, we use a two-sided test at the β level of significance that rejects the hypothesis that the parameters are equal whenever

$$\frac{\frac{s_0}{n_0} - \frac{s_1}{n_1}}{\sqrt{\frac{(\frac{s_0+s_1}{n_0+n_1})(1-\frac{s_0+s_1}{n_0+n_1})(n_0+n_1)}{n_0 n_1}}} \text{ is either } \begin{cases} \leq -z_{\beta/2} \\ \text{or} \\ \geq +z_{\beta/2} \end{cases},$$

where $z_{\beta/2}$ is a standard normal deviate [36]. Because sample size is important for this test, the algorithm is slightly modified to ensure that, at the beginning of a run, each action is chosen a minimum number of times. This parameter will be referred to as β_{min}.

As for the interval-estimation algorithm, real-valued reinforcement can be handled in IEKDNF using statistical tests appropriate for normally-distributed values or for non-parametric models. In nonstationary environments, statistics can be scaled in order to ensure that the algorithm does not stay converged to a non-optimal strategy.

The order complexity of this algorithm is the same as that of the LARCKDNF algorithm of Section 6.3, namely $O(M^k)$.

6.5 Empirical Comparison

This section reports the results of a set of experiments designed to compare the performance of the algorithms discussed in this chapter with one another, as well as with some other standard methods.

6.5.1 Algorithms and Environments

The following algorithms were tested in these experiments:

- LARC (Defined in Figure 18)
- LARC+ (LARC with an extra input wired to have a constant value)
- LARCKDNF (Defined in Figure 40)
- IEKDNF (Defined in Figure 41)
- BP (Defined in Figures 19 and 20)
- IE (Defined in Figure 21)

The regular interval-estimation algorithm IE is included as a yardstick; it is computationally much more complex than the other algorithms and should be expected to out-perform them.

Each of the algorithms was tested in three different environments. The environments are called *binomial Boolean expression worlds* and can be characterized by the parameters M, *expr*, p_{1s}, p_{1n}, p_{0s}, and p_{0n}. The parameter M is the number of input bits; *expr* is a Boolean expression over the input bits; p_{1s} is the probability of receiving reinforcement value 1 given that action 1 is taken when the input instance satisfies *expr*; p_{1n} is the probability of receiving reinforcement value 1 given that action 1 is taken when the input instance does not satisfy *expr*; p_{0s} is the probability of receiving reinforcement value 1 given that action 0 is taken when the input instance satisfies *expr*; p_{0n} is the probability of receiving reinforcement value 1 given that action 0 is taken when the input instance does not satisfy *expr*. Input vectors are chosen by the world according to a uniform probability distribution.

Table 4 shows the values of these parameters for each task. The first task has a simple, linearly separable function; what makes it difficult is the small separation between the reinforcement probabilities. Task 6 has highly differentiated reinforcement probabilities, but the function to be learned is a complex exclusive-or. Finally, Task 7 is a simple conjunctive function, but all of the reinforcement probabilities are high and it has twice as many input bits as the other two tasks.

Task	M	$expr$	p_{1s}	p_{1n}	p_{0s}	p_{0n}
5	3	$(i_0 \wedge i_1) \vee (i_1 \wedge i_2)$	.6	.4	.4	.6
6	3	$(i_0 \wedge \neg i_1) \vee (i_1 \wedge \neg i_2) \vee (i_2 \wedge \neg i_0)$	.9	.1	.1	.9
7	6	$i_2 \wedge \neg i_5$	.9	.5	.6	.8

Table 4: Parameters of test environments for k-DNF experiments.

6.5.2 Parameter Tuning

Each of the algorithms has a set of parameters. For both IEKDNF and LARCKDNF, $k = 2$. Algorithms LARC, LARC+, and LARCKDNF have parameters α, β, and σ. Following Sutton [70], parameters β and σ in LARCKDNF, LARC, and LARC+ are fixed to have values .1 and .3, respectively. The IEKDNF algorithm has two confidence-interval parameters, $z_{\alpha/2}$ and $z_{\beta/2}$, and a minimum age for the equality test β_{min}, while the IE algorithm has only $z_{\alpha/2}$. Finally, the BP algorithm has a large set of parameters: β, learning rate of the evaluation output units, β_h, learning rate of the evaluation hidden units, ρ, learning rate of the action output units, and ρ_h, learning rate of the action hidden units. All of the parameters for each algorithm are chosen to optimize the behavior of that algorithm on the chosen task. The success of an algorithm is measured by the average reinforcement received per tick, averaged over the entire run.

For each algorithm and environment, a series of 100 trials of length 3000 were run with different parameter values. Table 5 shows the best set of parameter values found for each algorithm-environment pair.

6.5.3 Results

Using the best parameter values for each algorithm and environment, the performance of the algorithms was compared on runs of length 3000. The performance metric was average reinforcement per tick, averaged over the entire run. The results are shown in Table 6, together with the expected reinforcement of executing a completely random behavior (choosing actions 0 and 1 with equal probability) and of executing the optimal behavior.

ALG-TASK	1	2	3
LARC			
α	.0625	.125	.125
LARC+			
α	.125	.0625	.25
LARCKDNF			
α	.125	.25	.03125
IEKDNF			
$z_{\alpha/2}$	3	3.5	2.5
$z_{\beta/2}$	1	2.5	3.5
β_{min}	15	5	25
BP			
β	.1	.25	.1
β_h	.2	.3	.05
ρ	.15	.15	.35
ρ_h	.2	.05	.1
IE			
$z_{\alpha/2}$	3.0	1.5	2.5

Table 5: Best parameter values for each k-DNF algorithm in each environment.

ALG-TASK	1	2	3
LARC	.5329	.7418	.7769
LARC+	.5456	.7459	.7722
LARCKDNF	.5783	.8903	.7825
IEKDNF	.5789	.8900	.7993
BP	.5456	.7406	.7852
IE	.5827	.8966	.7872
random	.5000	.5000	.6750
optimal	.6000	.9000	.8250

Table 6: Average reinforcement for k-DNF problems over 100 runs of length 3000.

As in the set of experiments described in Chapter 4, we must examine the relationships of statistically significant dominance among the algorithms for each task. Figure 42 shows, for each task, a pictorial representation of the results of a 1-sided t-test applied to each pair of experimental results. The graphs encode a partial order of significant dominance, with solid lines representing significance at the .95 level and dashed lines representing significance at the .85 level.

With the best parameter values for each algorithm, it is also instructive to compare the rate at which performance improves as a function of the number of training instances. Figures 43, 44, and 45 show superimposed plots of the learning curves for each of the algorithms. Each point represents the average reinforcement received over a sequence of 100 steps, averaged over 100 runs of length 3000.

6.5.4 Discussion

On Tasks 5 and 6, the basic interval-estimation algorithm, IE, performed significantly better than any of the other algorithms. The magnitude of its superiority, however, is not extremely great—Figures 43 and 44 reveal that the IEKDNF and LARCKDNF algorithms have similar performance characteristics both to each other and to IE. On these two tasks, the overall performance of IEKDNF and LARCKDNF were not found to be significantly different.

The backpropagation algorithm, BP, performed considerably worse than expected on Tasks 5 and 6. It is very difficult to tune the parameters for this algorithm, so its bad performance may be explained by a sub-optimal setting of parameters.[3] However, it is possible to see in the learning curves of Figures 43 and 44 that the performance of BP was still increasing at the ends of the runs. This may indicate that with more training instances it would eventually converge to optimal performance.

[3]In the parameter tuning phase, the parameters were varied independently—it may well be necessary to perform gradient-ascent search in the parameter space, but that is a computationally difficult task, especially when the evaluation of any point in parameter space may have a high degree of noise.

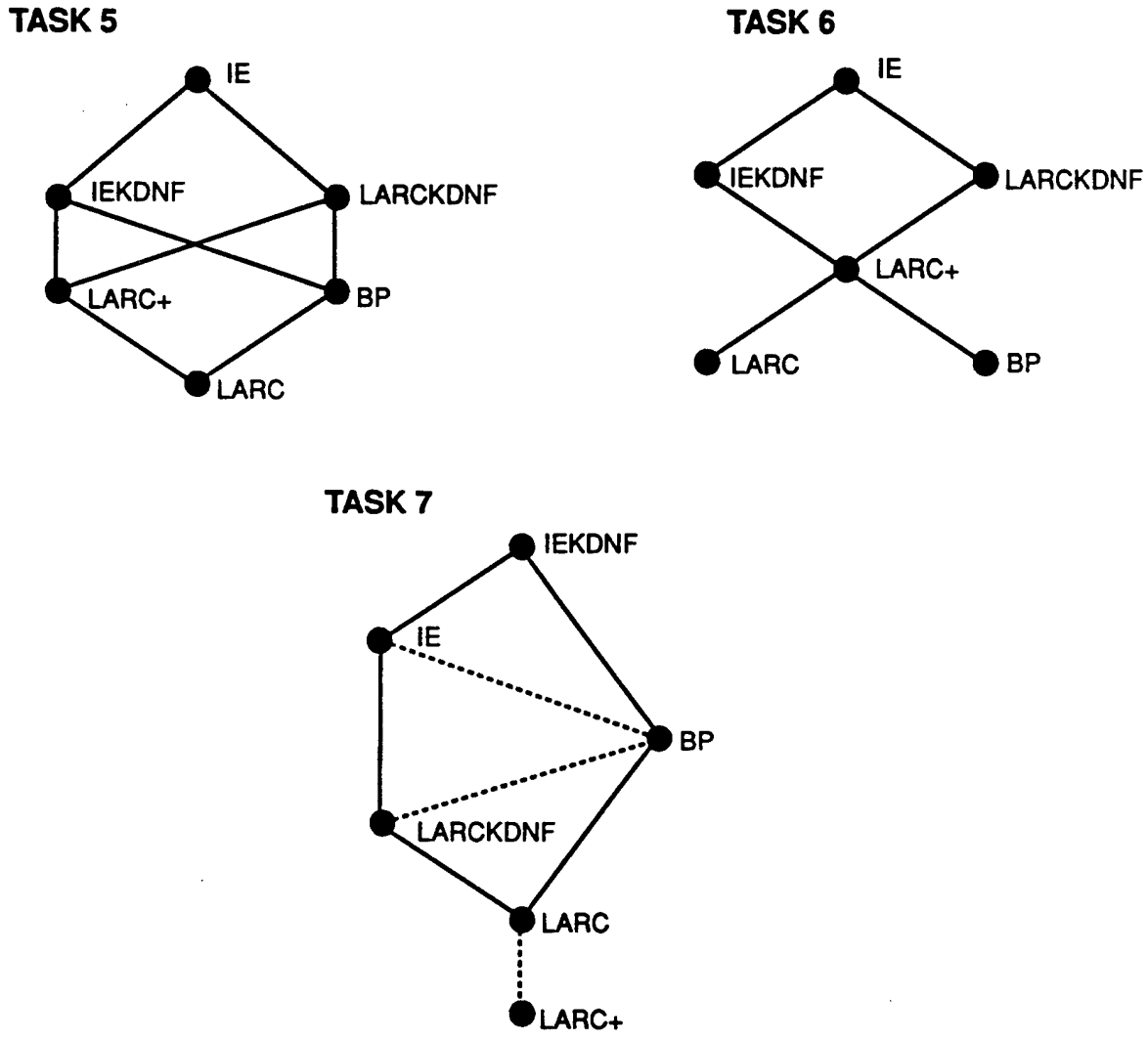


Figure 42: Significant dominance partial order among k-DNF algorithms for each task.

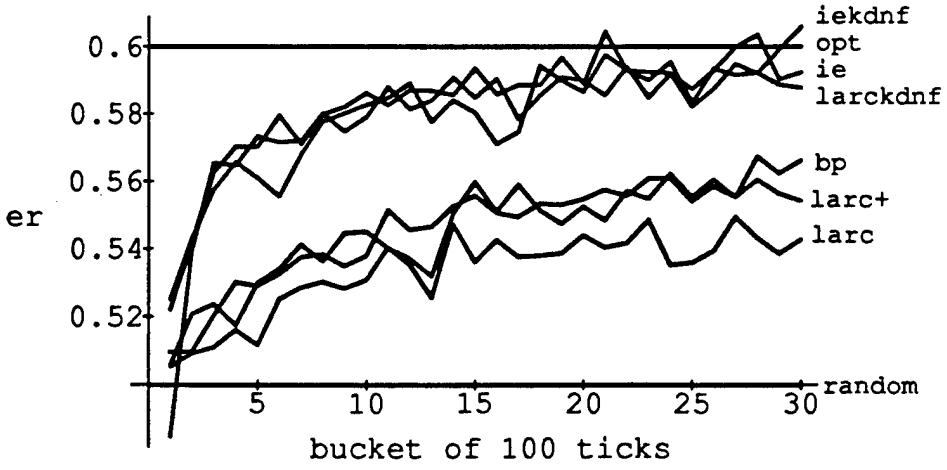


Figure 43: Learning curves for Task 5.

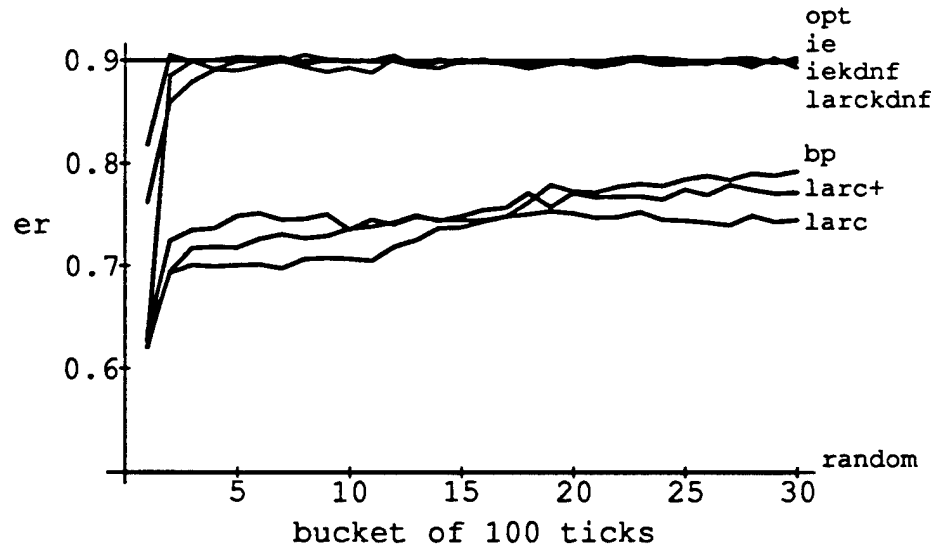


Figure 44: Learning curves for Task 6.

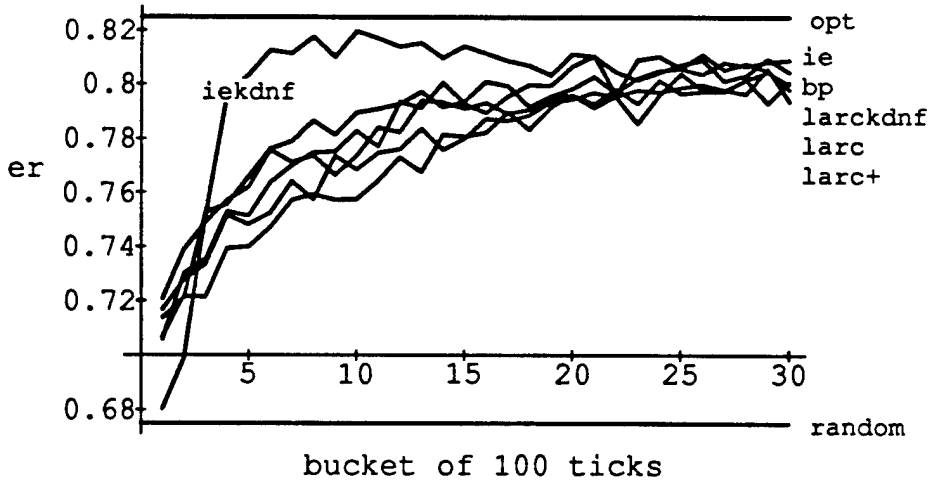


Figure 45: Learning curves for Task 7.

The linear-association algorithms performed poorly on both Tasks 5 and 6. This poor performance was expected on Task 6, because such algorithms are known to be unable to learn non-linearly-separable functions [47]. Task 5 is difficult for these algorithms because, during the execution of the algorithm, the evaluation function is often too complex to be learned by the simple linear associator. Adding a constant input value to the LARC algorithm made a significant improvement in performance; this is not surprising, because it allows the algorithm to find discrimination hyperplanes that do not pass through the origin of the space.

Task 7 reveals many interesting strengths and weaknesses of the algorithms. One of the most interesting is that IE is no longer the best performer. Because the target function is simple and there is a larger number of input bits, the ability to generalize across input instances becomes important. The IEKDNF algorithm is able to find the correct hypothesis early during the run (this is apparent in the learning curve of Figure 45). However, because the reinforcement values are not highly differentiated and because the size of the set T is quite large, it begins to include extraneous terms due to statistical fluctuations in the environment, causing slightly degraded performance. The IE, BP, and LARCKDNF algorithms all have very

similar performance on Task 7, with the linear-associator algorithms performing slightly worse, but still reasonably well.

6.6 Conclusion

From this study, we can see that it is useful to design algorithms that are tailored to learning certain restricted classes of functions. The two specially-designed algorithms far out-performed standard methods of comparable complexity. The LARCKDNF and IEKDNF algorithms each have their strengths and weaknesses. It is possible that LARCKDNF may outperform IEKDNF to some extent because in LARCKDNF each term gets to contribute to the answer with different degrees. This avoids errors that occur in IEKDNF when a single term is barely over the threshold for generating a 1. On the other hand, the state of IEKDNF has internal semantics that are clear and directly interpretable in the language of classical statistics. This simplifies the process of extending the algorithm to apply to other types of worlds in a principled manner.

Chapter 7

A Generate-and-Test Algorithm

This chapter describes GTRL, a highly parametrized generate-and-test algorithm for learning Boolean functions from reinforcement. Some parameter settings make it highly time- and space-efficient, but allow it to learn only a restricted class of functions; other parameter settings allow arbitrarily complex functions to be learned, but at a cost in time and space.

7.1 Introduction

The generate-and-test reinforcement-learning algorithm, GTRL, performs a bounded, real-time beam-search in the space of Boolean formulae, searching for a formula that represents an action function that exhibits high performance in the environment. This algorithm adheres to the strict synchronous tick discipline of the learning-behavior formulation of Chapter 2, performing its search incrementally, while using the best available solution to generate actions for the inputs with which it is presented.

The algorithm has, at any time, a set of hypotheses that it is considering. A hypothesis has as its main component a Boolean formula whose atoms are input bits or their negations. Negations can occur only at the lowest level in the formulae.[1] Each formula represents a potential action-map for the behavior, generating action 1

[1]Any Boolean formula can be put in this form using DeMorgan's laws.

whenever the current input satisfies the formula and action 0 when it does not. The GTRL algorithm generates new hypotheses by combining the formulae of existing hypotheses using syntactic conjunction and disjunction operators.[2] This generation of new hypotheses represents a search through Boolean-formula space; statistics related to the performance of the hypotheses in the domain are used to guide the search, choosing appropriate formulae to be combined.

This search is quite constrained, however. There is a limit on the number of hypotheses with formulae at each level of Boolean complexity (depth of nesting of Boolean operators), making the process very much like a beam search in which the entire beam is retained in memory. As time passes, old elements may be deleted from and new elements added to the beam, as long as the size is kept constant. This guarantees that the algorithm will operate in constant time per input instance and that the space requirement will not grow without bound over time. [3]

This search method is inspired by Schlimmer's STAGGER system [65,66,64,63,62] for learning Boolean functions from input-output pairs. STAGGER makes use of a number of techniques, including a Bayesian weight-updating component, that are inappropriate for the reinforcement-learning problem. In addition, it is not strictly limited in time- or space-complexity. The GTRL algorithm exploits STAGGER's idea of performing incremental search in the space of Boolean formulae, using statistical estimates of the "necessity" and "sufficiency" (these notions will be made concrete in the following discussion) to guide the search.

The presentation of the GTRL algorithm will be independent of any distributional assumptions about the reinforcement values generated by the environment; it will, however, assume that the environment is consistent (see Section 2.1.2 for the definition) for the agent. The process of tailoring the algorithm to work for particular kinds of reinforcement will be described in Section 7.3.

[2]Other choices of syntactic search operators are possible. Conjunction and disjunction are used here because of the availability of good heuristics for guiding their application. These heuristics will be discussed in Section 7.5.1.

[3]An alternative would be to simply limit the total number of hypotheses, without sorting them into levels. This approach would give added flexibility, but would also cause some increase in computational complexity. In addition, it is often beneficial to retain hypotheses at low levels of complexity because of their usefulness as building blocks.

7.2 High-Level Description

As with other learning behaviors, we will view the GTRL algorithm in terms of initial state, update function, and evaluation function, as shown in Figure 46. The internal state of the GTRL algorithm consists of a set of hypotheses organized into levels. Along with a Boolean formula, each hypothesis contains a set of statistics that reflect different aspects of the performance of the formula as an action map in the domain. Each level contains hypotheses whose formulae are of a given Boolean complexity. Figure 47 shows an example GTRL internal state. Level 0 consists of hypotheses whose formulae are individual atoms corresponding to the input bits and to their negations, as well as the hypotheses whose formulae are the logical constants *true* and *false*.[4] Hypotheses at level 1 have formulae that are conjunctions and disjunctions of the formulae of the hypotheses at level 0. In general, the hypotheses at level n have formulae that consist of conjunctions or disjunctions of two formulae: one from level $n-1$ and one from any level, from 0 to $n-1$. The hypotheses at each level are divided into working and candidate hypotheses; the reasons for this distinction will be made clear during the detailed explanation of the algorithm.

The update function of the GTRL algorithm consists of two phases: first, updating the statistics of the individual hypotheses and, second, adding and deleting hypotheses.

The evaluation function also works in two phases. The first step is to find the working hypothesis at any level that has the best performance at choosing actions. If the chosen working hypothesis is satisfied by the input to be evaluated, action 1 is generated; if it is not satisfied, action 0 is generated.

The following sections will examine these processes in greater detail.

[4]It is necessary to include *true* and *false* in case either of those is the optimal hypothesis. Hypotheses at higher levels are simplified, so even if $a \wedge \neg a$ or $a \vee \neg a$ were to be constructed, it would not be retained.

Algorithm 15 (GTRL)

```
s_0 =          array[0..L] of
                   record
                      working-hypoths: array[0..H] of hypoth
                      candidate-hypoths: array[0..C] of hypoth
                   end
u(s,i,a,r) =   update-hypotheses (s,i,a,r)
               for each level in s do begin
                   add-hypotheses (level, s)
                   promote-hypotheses (level)
                   prune-hypotheses (level)
               end
e(s,i) =       h := best-predictor (s)
               if satisfies (i, h) then
                   return 1
               else return 0
```

Figure 46: High-level description of the GTRL algorithm.

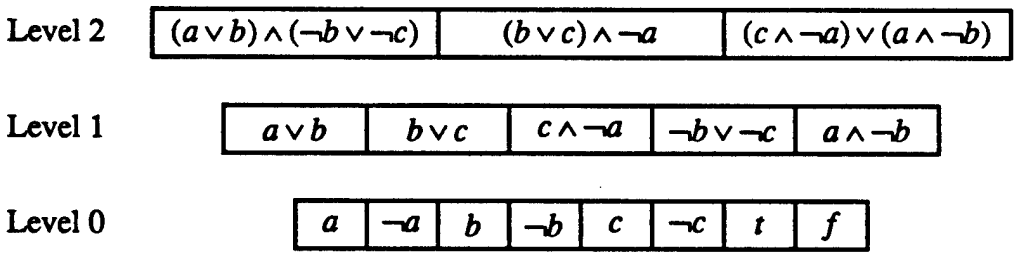


Figure 47: Example GTRL internal state.

7.3 Statistics

Associated with each working and candidate hypothesis is a set of statistics; these statistics are used to choose working hypotheses for generating actions and for combination into new candidate hypotheses at higher levels. The algorithms for updating the statistical information and computing statistical quantities are modularly separated from the rest of the GTRL algorithm. The choice of statistical module will depend on the kind and distribution of reinforcement values received from the environment. Appendix A provides the detailed definitions of statistics modules for cases in which the reinforcement values are binomially or normally distributed; in addition, it contains a non-parametric statistics module for use when there is no known model of the distribution of reinforcement values. A statistics module supplies the following functions:

age(h): The number of times the behavior, as a whole, has taken the action that would have been taken had hypothesis h been used to generate the action.

er(h): A point estimate of the expected reinforcement received given that the action taken by the behavior agrees with the one that would have been generated had hypothesis h been used to generate the action.

er-ub(h): The upper bound of a $100(1-\alpha)\%$ confidence interval estimate of the quantity estimated by *er*(h).

erp(h): A point estimate of the expected reinforcement received given that hypothesis h was used to generate the action that resulted in the reinforcement.

erp-ub(h): The upper bound of a $100(1-\alpha)\%$ confidence interval estimate of the quantity estimated by *erp*(h).

$N(h)$: A point estimate of the expected reinforcement received given that the action taken by the behavior was 0 and hypothesis h would have generated action 0 as well.

$S(h)$: A point estimate of the expected reinforcement received given that the action taken by the behavior was 1 and hypothesis h would have generated action 1 as well.

7.4 Evaluating Inputs

Each time the evaluation function is called, the most predictive working hypothesis is chosen, by taking the one with the highest value of pv, defined as

$$pv(h) = \lfloor \kappa \ er(h) \rfloor + erp\text{-}ub(h) \ .$$

This definition has the effect of sorting first on the criterion of er, then breaking ties based on the value of $erp\text{-}ub$. The constant multiplier κ can be adjusted to make this criterion more or less sensitive to low-order digits of the value of $er(h)$.[5]

What makes this an appropriate criterion for choosing the hypothesis with the best performance? The quantity that most clearly represents the predictive value of the hypothesis is $erp(h)$, which is a point estimate of the expected reinforcement given that actions are chosen according to hypothesis h. Unfortunately, this quantity only has a useful value after the hypothesis has been chosen to generate actions a number of times. Thus, as in the interval estimation algorithm, we make use of $erp\text{-}ub(h)$, the upper bound of a confidence interval estimate of the expected reinforcement of acting according to hypothesis h.

So, why not simply choose the working hypothesis with the highest value of $erp\text{-}ub(h)$, similar to what would be done in the interval estimation algorithm? The reason lies in the fact that in the GTRL algorithm, new hypotheses are continually being created. If it always chooses hypotheses with high values of $erp\text{-}ub(h)$, it will be in danger of spending nearly all of its time choosing hypotheses because little is known about them, rather than because they are known to perform well. The value of $er(h)$ serves as a filter on hypotheses that will prevent most of this fruitless exploration. The quantity $er(h)$ is not a completely accurate estimator of $erp(h)$, because the distribution of instances over which it is defined may be different than

[5] In all of the experiments described in this chapter, κ had the value 1000.

the distribution of input instances presented to the entire algorithm,[6] but it serves as a useful approximation. We can use $er(h)$ rather than $er\text{-}ub(h)$ because the statistics used to compute $er(h)$ get updated even when h is not used to generate actions, so that statistic becomes valid eventually without having to do any special work. Thus, hypotheses that look good on the basis of the value of $er(h)$ tend to get chosen to act; as they do, the value of $erp\text{-}ub(h)$ begins to reflect their true predictive value. This method still spends some time acting according to untested hypotheses, but that is necessary in order to allow the algorithm to discover the correct hypothesis initially and to adjust to a dynamically changing world. The amount of exploration that actually takes place can be controlled by changing the rate at which new hypotheses will be generated, as will be discussed in Section 7.7.

Once a working hypothesis is chosen, it is used to evaluate the input instance. An input vector i satisfies hypothesis h if h's formula evaluates to *true* under the valuation of the atoms supplied by input i. If the input instance satisfies the chosen hypothesis, action 1 is generated; otherwise, action 0 is generated.

7.5 Managing Hypotheses

The process by which hypotheses are managed in the GTRL algorithm can be divided into three parts: adding, promoting, and pruning. On each call to the update function, the statistics of all working and candidate hypotheses are updated. Then, if it is time to do so, a new hypothesis may be constructed and added to the candidate list of some level. Candidate hypotheses that satisfy the appropriate requirements are "promoted" to be working hypotheses. Finally, any level that has more working hypotheses than the constant number allotted to it will have its working hypothesis list pruned.

[6]This difference in distributions depends on the fact that $er(h)$ is conditioned on the agreement between hypothesis h and whatever hypotheses are actually being used to generate actions.

7.5.1 Adding Hypotheses

Search in the GTRL algorithm is carried out through the addition of hypotheses. Each new hypothesis is a conjunction or disjunction of hypotheses from lower levels.[7] On each update cycle, a candidate hypothesis is added to a level if the level is not yet fully populated (the total number of working and candidate hypotheses is less than the maximum number of working hypotheses) or if it has been a certain length of time since a candidate hypothesis was last generated for this level and there is room for a new candidate.

If it is time to generate a new hypothesis, it is randomly decided whether to make a conjunctive or disjunctive hypothesis.[8] Once the combining operator is determined, operands must be chosen.

The following search heuristic is used to guide the selection of operands:

> *When making a conjunction, use operands that have a high value of* necessity*; when making a disjunction, use operands that have a high value of* sufficiency.

The terms *necessity* and *sufficiency* have a standard logical interpretation: P is sufficient for Q if P implies Q; P is necessary for Q if $\neg P$ implies $\neg Q$ (that is, Q implies P). Schlimmer follows Duda, Hart, and Nilsson [19,20], defining the logical sufficiency of evidence E for hypothesis H as

$$\mathrm{LS}(E,H) = \frac{\Pr(E \mid H)}{\Pr(E \mid \bar{H})}$$

and the logical necessity of E for H as

$$\mathrm{LN}(E,H) = \frac{\Pr(\bar{E} \mid H)}{\Pr(\bar{E} \mid \bar{H})} \ .$$

[7] Terminology is being abused here in order to simplify the presentation. Rather than conjoining hypotheses, the algorithm actually creates a new hypothesis whose formula is the conjunction of the formulae of the operand hypotheses. This use of terminology should not cause any confusion.

[8] Schlimmer's STAGGER system generates new hypotheses in response to errors, using the nature of the error (false positive vs. true negative) to determine whether the new hypothesis should be a conjunction or a disjunction. This method cannot be applied in the general reinforcement-learning scenario, in which the algorithm is never told what the "correct" answer is, making it unable to know whether or not it just made an "error."

If E is truly logically sufficient for H, then E implies H, so $\Pr(E \mid \bar{H}) = 0$, making $\mathrm{LS}(E,H) = \infty$. If E and H are statistically independent, then $\mathrm{LS}(E,H) = 1$. Similarly, if E is logically necessary for H, then $\bar{E}$ implies $\bar{H}$, so $\Pr(\bar{E} \mid H) = 0$, making $\mathrm{LN}(E,H) = 0$. As before, if E and H are independent, $\mathrm{LN}(E,H) = 1$.

What makes functions like these useful for our purposes is that they encode the notions of "degree of implication" and "degree of implication by."[9] Let $h^*(i)$ be the optimal hypothesis, defined by

$$\forall i.h^*(i) \leftrightarrow \mathrm{Opt}(i,1) \ ,$$

where Opt is defined as in Chapter 2. We would like to use these same notions of necessity and sufficiency to guide our search, estimating the necessity and sufficiency of hypotheses in the GTRL algorithm state for h^*, the Boolean function that encodes the optimal action policy for the environment. But, because of the reinforcement-learning setting of our problem, we have no access to or direct information about h^*—the environment never tells the agent which action it *should* have taken.

If we define the sufficiency of hypothesis h for the optimal policy, $S(h)$ as

$$S(h) = er(i,1 \mid \mathit{satisfies}(i,h)) \ ,$$

we have a function with the desired properties. If h implies h^*, then

$$S(h) = er(i,1 \mid \mathit{satisfies}(i,h^*)) \ ,$$

which is the best that can be done on this set of inputs, because whenever action 1 would be taken by h, it would also be taken by h^*. In all other cases, $S(h) < S(h^*)$, with $S(h)$ roughly encoding the degree to which h implies h^*. If h and h^* are completely uncorrelated, $S(h)$ is the expected reinforcement of acting according to a random policy. Similarly, we define the necessity of a hypothesis h for the optimal policy, $N(h)$, as

$$N(h) = er(i,0 \mid \mathit{satisfies}(i,h)) \ .$$

[9]The LS and LN functions were designed for combining evidence in a human-intuitive way; their quantitative properties are crucial to their correctness and usefulness for this purpose. The S and N operators that will be proposed do not have the appropriate quantitative properties for such uses.

If $\neg h$ implies $\neg h^*$, then

$$N(h) = er(i, 0 \mid satisfies(i, h^*)) \ ,$$

because whenever action 0 would be taken by h it would be taken by h^*. In all other cases, $N(h) < N(h^*)$, with N roughly encoding the degree to which h is implied by h^*.

Now we understand the definition and purpose of the necessity and sufficiency operators, but what makes them appropriate for use as search-control heuristics? In general, if we have a hypothesis that is highly sufficient, it can be best improved by making it highly necessary as well; this can be achieved by making the hypothesis more general by disjoining it with another sufficient hypothesis. Similarly, given a highly necessary hypothesis, we would like to make it more sufficient; we can achieve this through specialization by conjoining it with another necessary hypothesis. As a simple example, consider the case in which $h^* = a \vee b$. In this case, the hypothesis a is logically sufficient for h^*, so the heuristic will have us try to improve it by disjoining it with another sufficient hypothesis. If $h^* = a \wedge b$, the hypothesis a is logically necessary for h^*, so the heuristic would give preference to conjoining it with another necessary hypothesis.

Having decided, for instance, to create a new disjunctive hypothesis at level n, the algorithm uses sufficiency as a criterion for choosing operands. This is done by creating two sorted lists of hypotheses: the first list consists of the hypotheses of level $n-1$, sorted from highest to lowest sufficiency; the second list contains all of the hypotheses from levels 0 to $n-1$, also sorted by sufficiency. The first list is limited in order to allow complete coverage of the search space without duplication of hypotheses at different levels. Thus, for example, a hypothesis of depth 2 can be constructed at level 2, but one of depth 1 cannot.

Given the two sorted lists (another sorting criterion could easily be substituted for necessity or sufficiency at this point), a new disjunctive hypothesis is constructed by syntactically disjoining the formulae associated with the hypotheses at the top of each list. This new formula is then simplified and put into a canonical form.[10]

[10]The choice of canonicalization and simplification procedures represents a tradeoff between computation time and space used in canonicalization against the likelihood that duplicate hypotheses

```
index-1 := 0
index-2 := 0
index-sum := 0
loop
    try-hypoth(list-1[index-1],list-2[index-2]);

    index-1 := index-1 + 1;
    index-2 := index-2 - 1;
    if index-2 = -1 then begin
        index-sum := index-sum + 1
        index-1 := 0
        index-2 := index-sum
    end
end
```

Figure 48: Code to generate the best new hypothesis.

If the simplified formula is of depth less than n it is discarded, because if it is important, it will occur at a lower level and we wish to avoid duplication. If it is of depth n, it is tested for syntactic equality against all other hypotheses at level n. If the hypothesis is not a syntactic duplicate, it is added to the candidate list of level n and its statistics are initialized. If the new hypothesis is too simple or is a duplicate, two new indices into the sorted lists are chosen and the process is repeated. The new indices are chosen so that the algorithm finds the non-duplicate disjunction made from a pair of hypotheses whose sum of indices is least. This is achieved by the code shown in Figure 48. The complexity of this process can be controlled by limiting the total number of new hypotheses that can be tried before giving up. In addition, given such a limit, it is possible to generate only prefixes of the sorted operand-lists that are long enough to support the desired number of attempts.

---

will not be detected. Any process for putting Boolean formulae into a normal form that reduces semantic equivalence to syntactic equivalence has exponential worst-case time and space complexity in the original size of the formula. The GTRL algorithm currently uses a very simple simplification process whose complexity is linear in the original size of the formula and that seems, empirically, to work well. This simplification process is described in detail in Appendix B.

7.5.2 Promoting Hypotheses

On each update phase, the candidate hypotheses are considered for promotion. The reason for dividing the candidate hypotheses from the working hypotheses is to be sure that they have gathered enough statistics for their values of N, S, and er to be fairly accurate before they enter the pool from which operands and the action-generating hypothesis are chosen. Thus, the criterion for promotion is simply the *age* of the hypothesis, which reflects the accuracy of its statistics. Any candidate that is old enough is moved, on this phase, to the working hypothesis list.

7.5.3 Pruning Hypotheses

After candidates have been promoted, the total number of working hypotheses in a level may exceed the preset limit. If this happens, the working hypothesis list for the level is pruned. An hypothesis can play an important role in the GTRL algorithm for three reasons: its prediction value is high, making it useful for choosing actions; its sufficiency is high, making it useful for combining into disjunctions; or its necessity is high, making it useful for combining into conjunctions. For these reasons, we adopt the following pruning strategy:

> *To prune down to n hypotheses, first choose the $n/3$ hypotheses with the highest predictive value; of the remaining hypotheses, choose the $n/3$ with the highest necessity; and, finally, of the remaining hypotheses, choose the $n/3$ with the highest sufficiency.*

This pruning criterion is applied to all but the bottom-most and top-most levels. Level 0, which contains the atomic hypotheses and their negations, must never be pruned, or the capability of generating the whole space of fixed-size Boolean formulae will be lost. Because its hypotheses will not undergo further recombination, the top level is pruned so as to retain the n most predictive hypotheses.

7.6 Parameters of the Algorithm

The GTRL algorithm is highly configurable, with its complexity and learning ability controlled by the following parameters:

L: The number of levels of hypotheses.

$z_{\alpha/2}$: The size of the confidence interval used to generate *erp-ub*.

$H(l)$: The maximum number of working hypotheses per level; can be a function of level number, l.

$C(l)$: The maximum number of candidate hypotheses per level; can be a function of level number, l.

PA: The age at which candidate hypotheses are promoted to be working hypotheses.

R: The rate at which new hypotheses are generated; every R ticks, for each level, l, if there are not more than $C(l)$ candidate hypotheses, a new one is generated.

T: The maximum number of new hypotheses that are tried, in a tick, to find a non-duplicate hypothesis.

M: The number of input bits.

Because level 0 is fixed, we have $H(0) = 2M + 2$.

7.7 Computational Complexity

The space complexity of the GTRL algorithm is

$$O(\sum_{j=0}^{L}(H(j) + C(j))2^j) \ ;$$

for each level j of the L levels, there are $H(j) + C(j)$ working and candidate hypotheses, each of which has size at most 2^j for the Boolean expression, plus a

constant amount of space for storing the statistics associated with the hypothesis. This expression can be simplified, if H and C are independent of level, to

$$O(L(H+C)(2^{L+1}-1)) \ .$$

which is

$$O(L(H+C)2^L) \ .$$

The time complexity for the evaluation function is

$$O(\sum_{j=0}^{L} H(j) + 2^L) \ ;$$

the first term accounts for spending a constant amount of time examining each working hypothesis to see which one has the highest predictive value. Once the most predictive working hypothesis is chosen, it must be tested for satisfaction by the input instance; this process takes time proportional to the size of the expression, the maximum possible value of which is 2^L. If H is independent of level, this simplifies to

$$O(LH + 2^L) \ .$$

The expression for computation time of the update function is considerably more complex. It is the sum of the time taken to update the statistics of all the working and candidate hypotheses plus, for each level, the time to add hypotheses, promote hypotheses, and prune hypotheses for the level.

The time to update the hypotheses is the sum of the times to update the individual hypotheses. The update phase requires that each hypothesis be tested to see if it is satisfied by the input. This testing requires time proportional to the size of the hypothesis. Thus we have a time complexity of

$$O(\sum_{j=0}^{L}(H(j)+C(j))2^j)$$

which simplifies to

$$O(L(H+C)2^L) \ .$$

The time to add hypotheses consists of the time to create the two sorted lists (assumed to be done in $n \log n$ time in the length of the list) plus the number of new

hypotheses tried times the amount of time to construct and test a new hypothesis for duplication. This time is, for level j,

$$O(H(j-1)\log H(j-1) + (\sum_{k=0}^{j-1} H(k))\log(\sum_{k=0}^{j-1} H(k)) + T2^j(H(j)+C(j))) \ .$$

The last term is the time for testing new hypotheses against old ones at the same level to be sure there are no duplicates. Testing for syntactic equality takes time proportional to the size of the hypothesis and must be done against all working and candidate hypotheses in level j. There is no explicit term for simplification of newly created hypotheses because GTRL uses a procedure that is linear in the size of the hypothesis.

The time to promote hypotheses is simply proportional to the number of candidates, $C(j)$.

Finally, the time to prune hypotheses is 3 times the time to choose the $H(j)/3$ best hypotheses which, for the purpose of developing upper bounds, is $H(j)\log H(j)$.

Summing these expressions for each level and making the simplifying assumption that H and C do not vary with level yields a time complexity of

$$O(L(H\log H + LH\log(LH) + T2^L(H+C) + C + H\log H)) \ ,$$

which can be further simplified to

$$O(L^2 H\log(LH) + T2^L L(H+C)) \ . \tag{10}$$

The time complexity of the statistical update component, $O(L(H+C)2^L)$, is dominated by the second term above, making expression 10 above the time complexity of the entire update function. This is the complexity of the longest possible tick. The addition and pruning of hypotheses, which are the most time-consuming steps, will happen only once every R ticks. Taking this into account, we get a kind of "average worst-case" total complexity (the average is guaranteed when taken over a number of ticks, rather than being a kind of expected complexity based on assumptions about the distribution of inputs) of

$$O(L(H+C)2^L + \frac{1}{R}L^2 H\log(LH) + \frac{T}{R}2^L L(H+C)) \ .$$

The complexity in the individual parameters is $O(2^L)$, $O(H \log H)$, $O(1/R)$, $O(T)$, $O(C)$. Clearly, the number of levels and the number of hypotheses per level have the greatest effect on total algorithmic complexity.[11]

7.8 Choosing Parameter Values

This section will explore the relationship between the settings of parameter values and the learning abilities of the GTRL algorithm.

7.8.1 Number of Levels

Any Boolean function can be written with a wide variety of syntactic expressions. Consider the set of Boolean formulae with the negations driven in as far as possible, using DeMorgan's laws. The *depth* of such a formula is the maximum nesting depth of binary conjunction and disjunction operators within the formula. The *depth* of a Boolean function is defined to be the depth of the shallowest Boolean formula that expresses the function.

An instance of the GTRL algorithm with L levels of combination is unable to learn functions with depth greater than L. Whether it can learn all functions of depth L or less depends on the settings of other parameters in the algorithm. The time and space complexities of the algorithm are, technically, most sensitive to this parameter, both being exponential in the number of levels. However, in practical applications of this algorithm, H is usually considerably larger than 2^L.

7.8.2 Number of Working and Candidate Hypotheses

The choice of the size of the hypothesis lists at each level also has a great effect on the overall complexity of the algorithm. The working hypothesis list needs to be at least big enough to hold all of the subexpressions of some formula that describes the target function. Thus, in order to learn the function described by

[11] This complexity is not as bad as it may look, because 2^L is just the length of the longest formula that can be constructed by the algorithm. The time and space complexities are linear in this length.

$i_0 \wedge (i_1 \vee i_2) \wedge (i_3 \vee \neg i_4)$, level 1 must have room for at least two working hypotheses, $i_1 \vee i_2$ and $i_2 \vee \neg i_4$, and levels 2 and 3 must have room for at least one working hypothesis each.

This amount of space will rarely be sufficient, however. There must also be room for newly generated hypotheses to stay until they are tested and proven or disproven by their performance in the environment. Exactly how much room this is depends on the rate, R, at which new hypotheses are generated and on the size, $z_{\alpha/2}$, of the confidence intervals used to generate *erp-ub*. To see this, consider the case in which a representation of the optimal hypothesis, h^*, has already been constructed. The algorithm continues to generate new hypotheses, one every R ticks, with each new hypothesis requiring an average of j ticks to be proven to be worse than h^*. That means there must be an average of R/j slots for extra hypotheses at this level. Of course, it is likely that during the course of a run, certain non-optimal hypotheses will take more than j ticks to disprove. This can cause h^* to be driven out of the hypothesis list altogether during the pruning phase. Thus, a more conservative strategy is to prevent this by increasing the size of the hypothesis lists, but at a penalty in computation time.

Even when there is enough space for all subexpressions and their competitors at each level, it is possible for the size of the hypothesis lists to affect the speed at which the optimal hypothesis is generated by the algorithm. This can be easily understood in the context of the *difficulty* of a function for the algorithm. Intuitively, functions whose subexpressions are not naturally preferred by the necessity and sufficiency search heuristics are difficult for the GTRL algorithm to construct. In such cases, the algorithm is reduced to randomly choosing expressions at each level.

Consider the case in which $h^* = (i_0 \wedge \neg i_1) \vee (\neg i_0 \wedge i_1)$, an exclusive-or function. Because h^* neither implies nor is implied by any of the input bits, the atoms will all have similar, average values of N and S. Due to random fluctuations in the environment, different atoms will have higher values of N and S at different times during a run. Thus, the conjunctions and disjunctions at level 1 will represent a sort of random search through expression space. This random search will eventually generate one of the following expressions: $i_0 \wedge \neg i_1$, $\neg i_0 \wedge i_1$, $i_0 \vee i_1$, $\neg i_0 \vee \neg i_1$. When

one of these is generated, it will be retained in the level 1 hypothesis list because of its high necessity or sufficiency. We need only wait until the random combination process generates its companion subexpression, and they will be combined into a representation of h^* at level 2.

Even with very small hypothesis lists, the correct answer will eventually be generated. However, as problems become more difficult, the probability that the random process will, on any given tick, generate the appropriate operands becomes very small, making the algorithm arbitrarily slow to converge to the correct answer. This process can be made to take fewer ticks by increasing the size of the hypothesis list. In the limit, the hypothesis list will be large enough to hold all conjunctions and disjunctions of atoms at the previous level and as soon as it is filled, the correct building blocks for the next level will be available and apparent.

We can measure the overall difficulty of a function for the GTRL algorithm in the context of a particular distribution of input instances by measuring the degree to which the individual input bits are necessary or sufficient for the function. We can define the difficulty of function f, $D(f)$, as

$$D(f) = \sum_{j \leq M} \left(\min(\frac{1}{LS(i_j, f)}, \frac{1}{LS(\neg i_j, f)}) + \min(LN(i_j, f), LN(\neg i_j, f)) \right) .$$

For each positive atom, the lack of sufficiency or necessity makes the problem more difficult; the term $\min(\frac{1}{LS(i_j,f)}, \frac{1}{LS(\neg i_j,f)})$ measures the degree to which the atom and its negation are insufficient; the term $\min(LN(i_j, f), LN(\neg i_j, f))$ measures the degree to which the atom and its negation are unnecessary (recall that high values of LS indicate sufficiency and low values of LN indicate necessity). Given that $LS(a, b) = LN(\neg a, b)$, we can simplify the definition to

$$D(f) = \sum_{j \leq M} \left(\min(\frac{1}{LS(i_j, f)}, \frac{1}{LS(\neg i_j, f)}) + \min(LS(i_j, f), LS(\neg i_j, f)) \right) .$$

In this form, the difficulty of the function *true* would be $2M$, where M is the number of input bits, because each of the bits is unnecessary and insufficient for the function. We can correct for irrelevant input bits by subtracting 2 for every bit that has no

effect on the value of f, yielding

$$D(f) = \sum_{j \leq M} \left(\min(\frac{1}{LS(i_j, f)}, \frac{1}{LS(\neg i_j, f)}) + \min(LS(i_j, f), LS(\neg i_j, f)) \right) - 2C \ ,$$

where C is the number of input bits that have no effect on the value of f.

The definition uses LS and LN rather than S and N, because LS and LN have well-understood ranges, with values of 1 indicating lack of necessity and sufficiency. Because S and N are monotonic in LS and LN, distinctions that are apparent when using LS and LN, which is what are measured by D, will also be apparent when using S and N. When the input bits all have an effect on the value of f, but are completely unnecessary and insufficient for f, its difficulty will be $2M$.

The values of LS and LN depend on being able to evaluate the probability of a particular input vector arriving; thus, this measure assumes that there is some fixed distribution on the input vectors. If there is no such fixed distribution (as we have argued may not be the case in many embedded learning scenarios), the difficulty could be defined to be the supremum over all possible distributions.

This difficulty measure can be illustrated by considering the space of possible Boolean functions on three input bits, in which the individual input vectors are assumed to be uniformly distributed. Following Schlimmer [62], the set of 3-input Boolean functions can be divided into 19 classes, which are equivalence classes under permutation and negation of the input bits. Table 7 uses Schlimmer's numbering system, giving a representative function from each class and its D measure. The classes, going from easy to difficult are ordered as follows:

$$\{0, 4e, 8\}, \{2c, 6c\}, \{1, 7\}, \{4d\}, \{3b, 5b\}, \{4c\}, \{3a, 5a\}, \{4a\}, \{2b, 6b\}, \{4b\}, \{2a, 6a\} \ .$$

Interestingly, all functions with difficulty less than 3 are linearly separable and those with difficulty greater than 3 are not. Also, D seems to measure the difficulty of problems for STAGGER more accurately, in many cases, than the measure used by Schlimmer.[12]

[12]Schlimmer used a measure of the dependence of the concept on the input bits which is based on Fisher's [23] work on category utility.

CLASS	f	$D(f)$
0.	**false**	0.00
1.	$a \wedge b \wedge c$	1.28
2a.	$(a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$	6.00
2b.	$a \wedge ((b \wedge c) \vee (\neg b \wedge \neg c))$	4.33
2c.	$a \wedge b$	0.67
3a.	$(a \wedge b) \vee (\neg a \wedge \neg b \wedge \neg c)$	3.47
3b.	$a \wedge (b \vee c)$	2.50
4a.	$(a \wedge \neg b) \vee (\neg a \wedge b)$	4.00
4b.	$(a \wedge (b \vee c)) \vee (\neg a \wedge \neg b \wedge \neg c)$	4.67
4c.	$(a \wedge c) \vee (\neg b \wedge \neg c)$	3.33
4d.	$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$	2.00
4e.	a	0.00
5a.	$(a \vee b) \wedge (\neg a \vee \neg b \vee \neg c)$	3.47
5b.	$a \vee (b \wedge c)$	2.50
6a.	$(a \vee b \vee c) \wedge (\neg a \vee \neg b \vee \neg c)$	6.00
6b.	$a \vee ((b \vee c) \wedge (\neg b \vee \neg c))$	4.33
6c.	$a \vee b$	0.67
7.	$a \vee b \vee c$	1.28
8.	**true**	0.00

Table 7: Difficulties of classes of 3-input Boolean functions.

7.8.3 Promotion Age

The choice of values for the age parameter depends on how long it takes for the *er*, N, and S statistics to come to be a good indication of the values they are estimating. If reinforcement has a high variance, for instance, it may take more examples to get a true statistical picture of the underlying processes. If the value of R is large, causing new combinations to be made infrequently, it is often important for promotion age to be large, ensuring that the data that guides the combinations is accurate. If R is small, the effect of occasional bad combinations is not so great and may be outweighed by the advantage of moving candidate hypotheses more quickly to the working hypothesis list.

7.8.4 Rate of Generating Hypotheses

The more frequently new hypotheses are generated, the sooner the algorithm will construct important subexpressions and the more closely it will track a changing environment. However, each new hypothesis that has a promising value of *er* will be executed a number of times to see if its value of *erp* is as high as that of the current best hypothesis. In general, most of these hypotheses will not be as good as the best existing one, so using them to choose actions will decrease the algorithm's overall performance significantly.

7.8.5 Maximum New Hypothesis Tries

The attempt to make a new hypothesis can fail for two reasons. Either the newly-created hypothesis already exists in the working or candidate hypothesis list of the level for which it was created or the expression associated with the hypothesis was subject to one of the reductions of Appendix B, causing it to be inappropriate for this level. It is possible, but very unlikely, to have more than $H + C$ failures of the first type. The number of failures of the second type is harder to quantify.

7.9 Empirical Results

This section describes a set of experiments with the GTRL algorithm. First, the operation of the GTRL algorithm is illustrated by discussing a sample run. Then, the dependence of the algorithm's performance on the settings of its parameters is explored. Finally, the performance of the GTRL algorithm is compared with the algorithms of the previous chapter on Tasks 5, 6, and 7.

7.9.1 Sample Run

Figure 49 shows the trace of a sample run of the GTRL algorithm. It is executed on Task 8, a binomial Boolean-expression world[13] with 3 input bits, in which the expression is $(b_0 \vee b_1) \wedge (b_1 \vee b_2)$, $p_{1s} = .9$, $p_{1n} = .1$, $p_{0s} = .1$, and $p_{0n} = .9$. The figure shows the state of the algorithm at ticks 50, 100, and 250. The report for each tick shows the working hypotheses for each level, together with their statistics. [14] In order to save space in the figure, only the four most predictive working hypotheses are shown at each level. At tick 50, the two component hypotheses, $b_0 \wedge b_1$ and $b_1 \wedge b_2$, have been constructed. They both have high levels of sufficiency, which makes them good operands for disjunction. By tick 100, the correct disjunction has been made, and the most predictive hypothesis is the optimal hypothesis $(b_0 \wedge b_1) \vee (b_1 \wedge b_2)$. At tick 250, the optimal hypothesis is still winning and the average reinforcement is approaching optimal.

7.9.2 Effects of Parameter Settings on Performance

The section describes a set of experiments that illustrate how learning performance varies as a function of the values of the parameters PA, R, and H on Task 8, which was described in the previous section. The parameter L was set to 3, $z_{\alpha/2}$ to 2, C to be equal to H, and T to 100. Figures 50, 51, and 52 show the results,

[13] Binomial Boolean-expression worlds are defined in Section 6.5.1.

[14] The **age** statistic reported in the trace is the number of times the hypothesis has been chosen to generate actions, rather than the value of *age*, which is the number of times this hypothesis has agreed with the ones that have been chosen to generate actions.

```
******   Tick    50 Summary   ******
----Level 0----
PV = 850.9243 EPPUB = 0.92 EP = 0.85 H = 0.87 S = 0.84 AGE =  14 H: 1
PV = 834.0000 EPPUB = 1.00 EP = 0.83 H = 0.69 S = 0.94 AGE =   0 H: 2
PV = 770.0000 EPPUB = 1.00 EP = 0.77 H = 0.75 S = 0.78 AGE =   0 H: 0
PV = 751.0000 EPPUB = 1.00 EP = 0.75 H = 0.75 S = **** AGE =   0 H: f
----Level 1----
PV = 904.9776 EPPUB = 0.98 EP = 0.90 H = 0.85 S = 1.00 AGE =   8 H: (and 1 2)
PV = 894.9699 EPPUB = 0.97 EP = 0.89 H = 0.80 S = 1.00 AGE =   6 H: (and 0 1)
PV = 882.8500 EPPUB = 0.85 EP = 0.88 H = 1.00 S = 0.87 AGE =   4 H: (or 0 2)
PV = 847.0000 EPPUB = 1.00 EP = 0.85 H = 0.67 S = 0.90 AGE =   0 H: (or 1 (not 0))
----Level 2----
PV = 866.9055 EPPUB = 0.91 EP = 0.87 H = 0.75 S = 0.91 AGE =   2 H: (or (and 1 2) (or 1 2))
PV = 819.0000 EPPUB = 1.00 EP = 0.82 H = 1.00 S = 0.78 AGE =   0 H: (or 0 (and 1 2))
PV = 728.0000 EPPUB = 1.00 EP = 0.73 H = **** S = 0.73 AGE =   0 H: (or 0 (or 1 2))
*** Reinf = (   37 /    50)   74.00%  Long term = (   37 /    50)   74.00%  ***

******   Tick   100 Summary   ******
----Level 0----
PV = 898.9243 EPPUB = 0.92 EP = 0.90 H = 0.90 S = 0.90 AGE =  14 H: 1
PV = 876.0000 EPPUB = 1.00 EP = 0.87 H = 0.81 S = 0.94 AGE =   0 H: 2
PV = 850.0000 EPPUB = 1.00 EP = 0.85 H = 0.85 S = **** AGE =   0 H: f
PV = 844.0000 EPPUB = 1.00 EP = 0.84 H = 0.88 S = 0.81 AGE =   0 H: 0
----Level 1----
PV = 931.9699 EPPUB = 0.97 EP = 0.93 H = 0.90 S = 1.00 AGE =   6 H: (and 0 1)
PV = 927.9801 EPPUB = 0.98 EP = 0.93 H = 0.89 S = 1.00 AGE =   9 H: (and 1 2)
PV = 914.0000 EPPUB = 1.00 EP = 0.91 H = 0.91 S = 0.93 AGE =   3 H: (and 2 (not 0))
PV = 911.8500 EPPUB = 0.85 EP = 0.91 H = 1.00 S = 0.88 AGE =   4 H: (or 0 2)
----Level 2----
PV = 962.9706 EPPUB = 0.97 EP = 0.96 H = 0.94 S = 1.00 AGE =  19 H: (or (and 0 1) (and 1 2))
PV = 947.9055 EPPUB = 0.91 EP = 0.95 H = 0.96 S = 0.92 AGE =   2 H: (and (or 0 2) (or 1 (not 2)))
PV = 945.7935 EPPUB = 0.79 EP = 0.95 H = 0.96 S = 0.92 AGE =   1 H: (or (and 0 1) (and 2 (not 0)))
PV = 940.0000 EPPUB = 1.00 EP = 0.94 H = 0.94 S = 0.94 AGE =   0 H: (and (or 0 1) (or 0 2))
*** Reinf = (   45 /    50)   90.00%  Long term = (   82 /   100)   82.00%  ***

******   Tick   250 Summary   ******
----Level 0----
PV = 925.9243 EPPUB = 0.92 EP = 0.93 H = 0.93 S = 0.92 AGE =  14 H: 1
PV = 891.0000 EPPUB = 1.00 EP = 0.89 H = 0.89 S = 0.89 AGE =   0 H: 0
PV = 886.0000 EPPUB = 1.00 EP = 0.89 H = 0.95 S = 0.80 AGE =   0 H: (not 2)
PV = 886.0000 EPPUB = 1.00 EP = 0.89 H = 0.89 S = **** AGE =   0 H: f
----Level 1----
PV = 927.0000 EPPUB = 1.00 EP = 0.93 H = 0.96 S = 0.91 AGE =   0 H: (or 1 (not 2))
PV = 922.9699 EPPUB = 0.97 EP = 0.92 H = 0.91 S = 0.95 AGE =   6 H: (and 0 1)
PV = 921.0000 EPPUB = 1.00 EP = 0.92 H = 0.92 S = 0.92 AGE =   0 H: (or 1 (not 0))
PV = 917.0000 EPPUB = 1.00 EP = 0.92 H = 0.95 S = 0.90 AGE =   0 H: (or 0 1)
----Level 2----
PV = 931.9491 EPPUB = 0.95 EP = 0.93 H = 0.92 S = 0.95 AGE = 166 H: (or (and 0 1) (and 1 2))
PV = 928.9055 EPPUB = 0.91 EP = 0.93 H = 0.93 S = 0.93 AGE =   2 H: (and (or 0 2) (or 1 (not 2)))
PV = 921.0000 EPPUB = 1.00 EP = 0.92 H = 0.90 S = 0.94 AGE =   0 H: (and (or 0 1) (or 0 2))
PV = 916.8500 EPPUB = 0.85 EP = 0.92 H = 0.91 S = 0.92 AGE =   4 H: (or (and 0 1) (and 2 (not 0)))
*** Reinf = (   46 /    50)   92.00%  Long term = (  219 /   250)   87.60%  ***
```

Figure 49: A sample run of the GTRL algorithm.

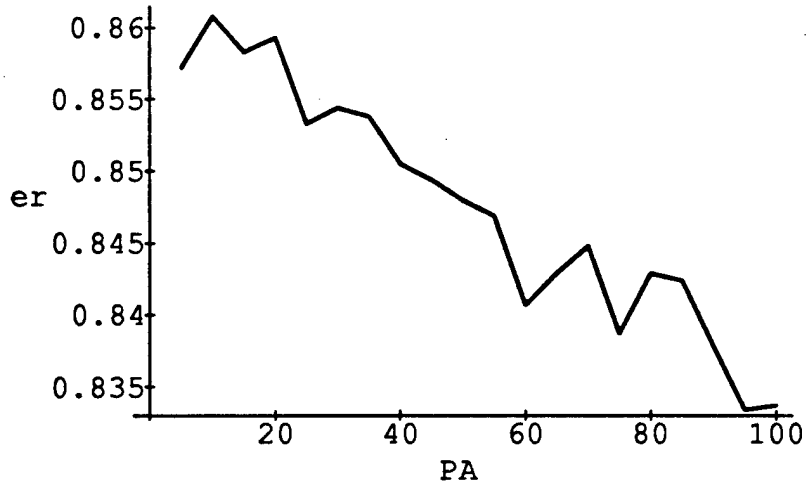


Figure 50: Performance versus parameter value *PA* for Task 8.

plotting average reinforcement per tick on 100 runs of length 3000 against each of the remaining parameters, *PA*, *R*, and *H*.

The expected reinforcement is maximized at a low value of *PA*, the promotion age of candidate hypotheses, because it is relatively easy to discriminate between good and bad actions in Task 8. When the probabilities of receiving reinforcement value 1 are closer to one another, as they are in the tasks discussed in the next section, it becomes necessary to use higher values of *PA*. Because this task (and all of the others discussed in this chapter) is stationary, the only reason to have a low value of *R*, the inverse of the rate at which new hypotheses are generated, is if the function is very difficult and hypothesis list is too small to hold all subexpressions at once. This is not the case for Task 8, so high values of *R* are desirable. Finally, performance increases with the length of the hypothesis lists, *H*, in every task. Because this task is relatively easy, however, the correct answer is usually found fairly quickly with even small values of *H*, so the increase is not dramatic (this is evidenced by the small range of *er* in Figure 52.)

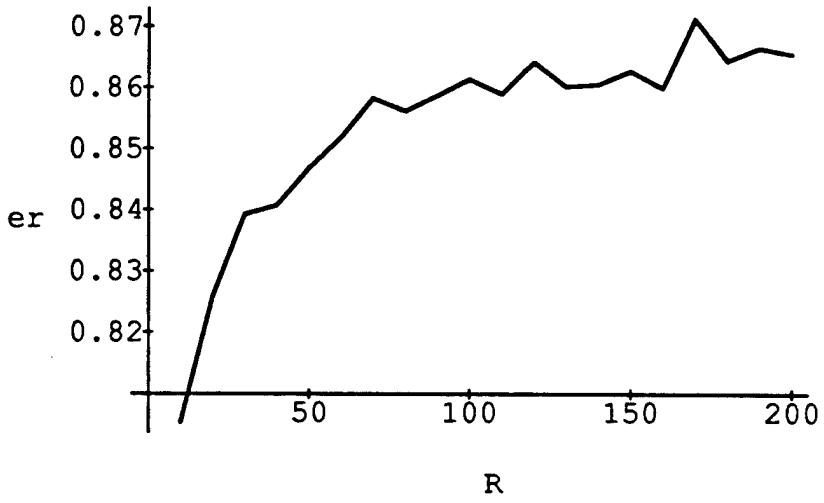


Figure 51: Performance versus parameter value R for Task 8.

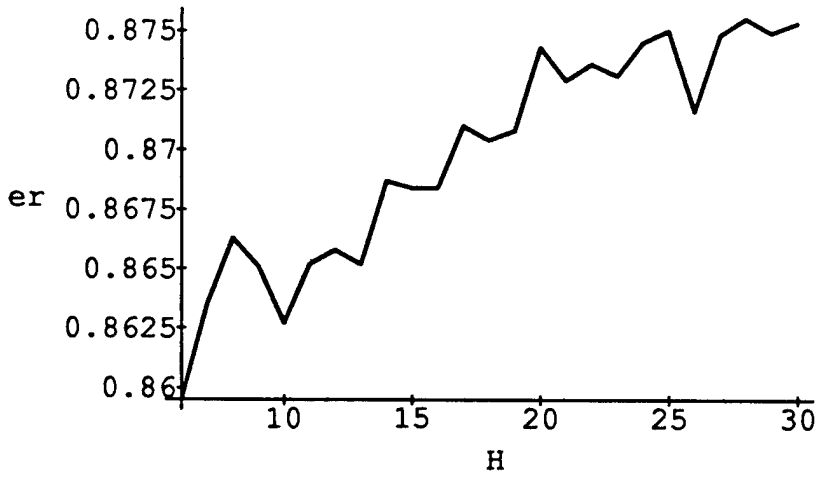


Figure 52: Performance versus parameter value H for Task 8.

Task/Param	*PA*	*R*	*H*	Results
5	35	200	30	.5648
6	10	100	30	.7879
7	45	110	20	.7877

Table 8: Best parameter values for GTRL on Tasks 5, 6, and 7 from Chapter 6.

7.9.3 Comparison with Other Algorithms

The GTRL algorithm was tested on Tasks 5, 6, and 7 from Chapter 6. The best values of the parameters for each task were determined through extensive testing, and are shown in Table 8. Some of the values are arbitrarily cut off where the parameter testing stopped. For instance, performance on Task 5 might be improved with higher values of *PA* and performance on Task 6 would be improved with higher values of *H*. The average reinforcement per tick of executing GTRL at these parameter settings on 100 runs of length 3000 are shown in the final column of the table.

Figure 53 is a modified version of Figure 42, with the results of the GTRL algorithm included with those of the algorithms of Chapter 6 for Tasks 5, 6, and 7. On Tasks 5 and 6, the GTRL algorithm performs significantly better than the LARC, LARC+, and BP algorithms, but not as well as IE, IEKDNF, or LARCKDNF. Finally, on Task 7, the real advantage of GTRL is illustrated. On a task with a large number of inputs, GTRL works efficiently and is significantly outperformed only by IEKDNF.

The learning curves of GTRL on each of the tasks are shown in Figures 54, 55 and 56. They are superimposed on the learning curves of the algorithms tested in Chapter 6; the GTRL curves are drawn in bold lines.

This comparison is, to some degree, unfair, because the GTRL algorithm is designed for nonstationary environments. We can see in the learning curves that, although it improves quickly early in run, it does not reach as high a steady-state level of performance as the other algorithms. It does not converge to a fixed state, because it is always entertaining new competing hypotheses. This flexibility causes a large decrease in performance. If the GTRL algorithm is to be applied in a domain in which changes, if any, are expected to take place near the beginning of a run,

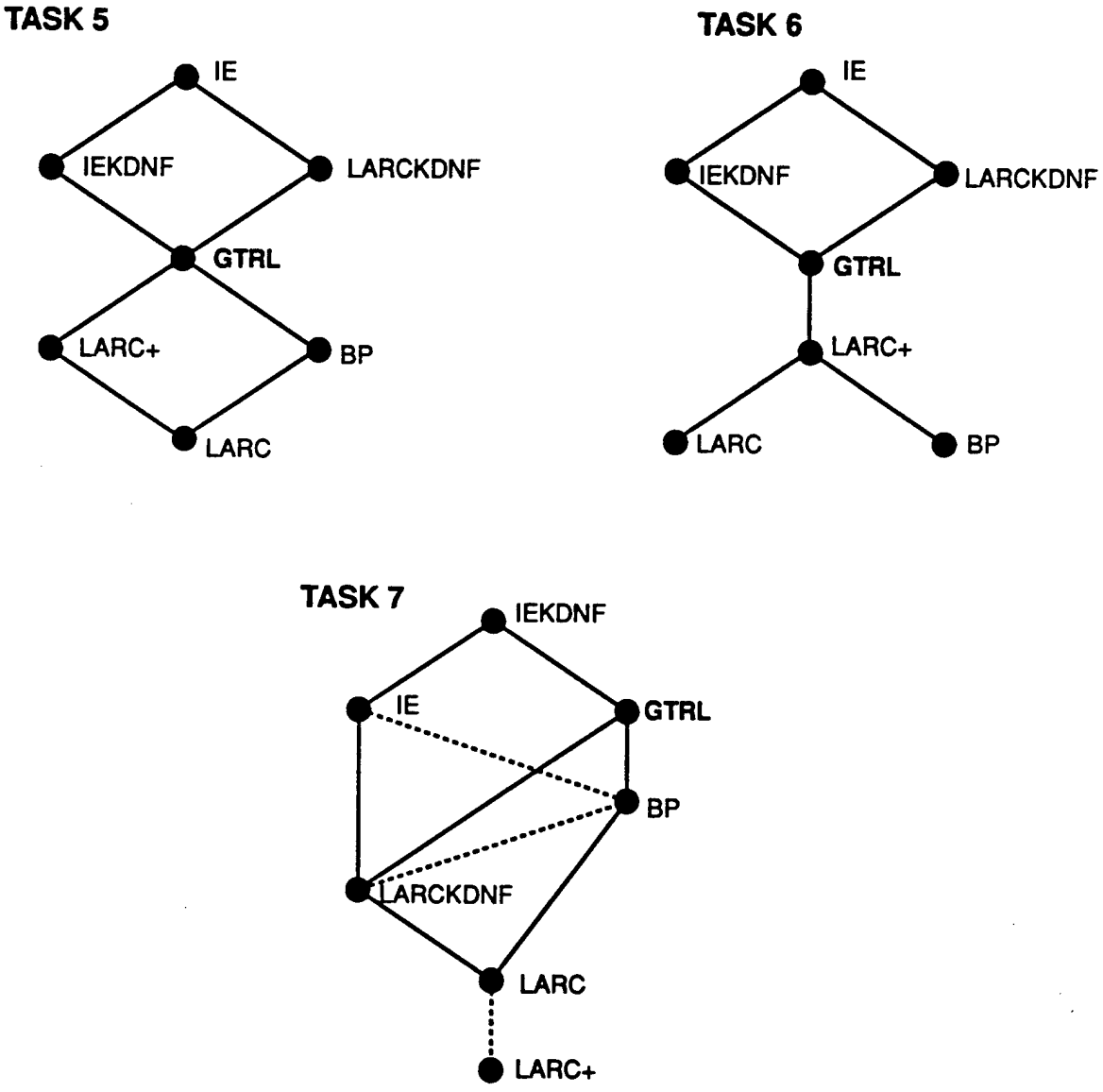


Figure 53: Significance of GTRL results on Tasks 5, 6, and 7, compared with the results of the algorithms of Chapter 6.

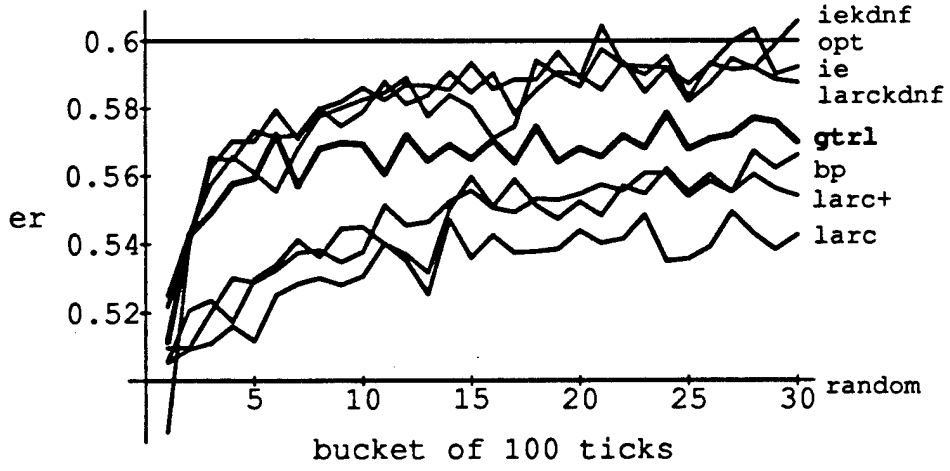


Figure 54: GTRL learning curve for Task 5 (bold) compared with the algorithms of Chapter 6.

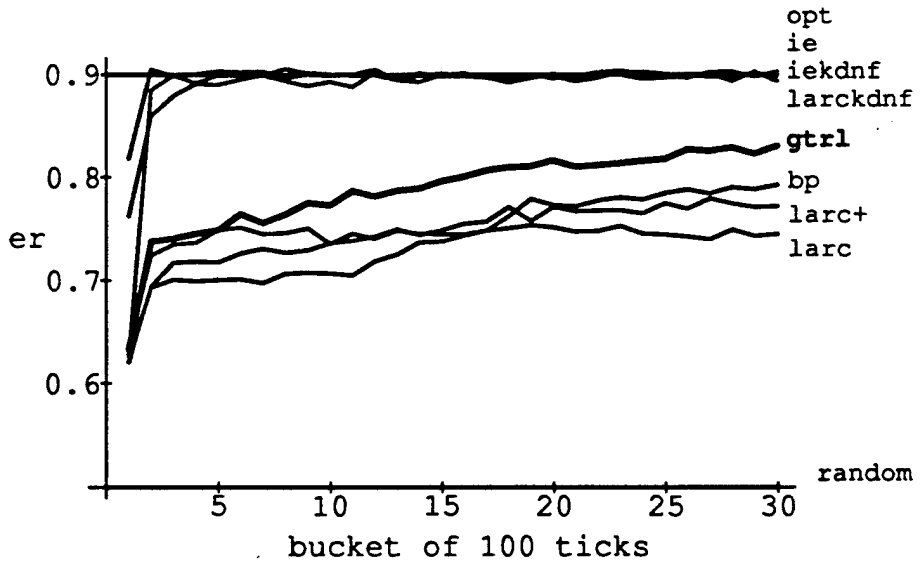


Figure 55: GTRL learning curve for Task 6 (bold) compared with the algorithms of Chapter 6.

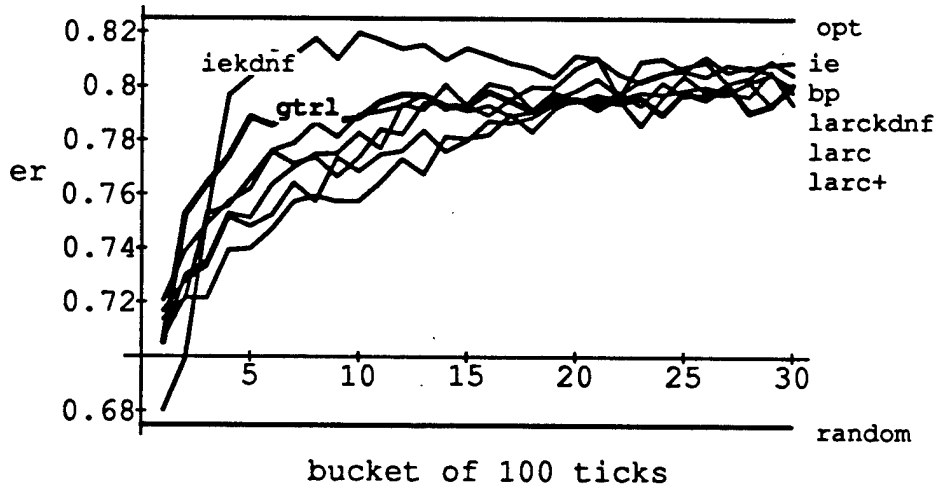


Figure 56: GTRL learning curve for Task 7 (bold) compared with the algorithms of Chapter 6.

performance can be improved by decreasing over time the rate at which new candidate hypotheses are generated. This will cause the algorithm to spend less time experimenting and more time acting on the basis of known good hypotheses.

7.10 Conclusions and Extensions

We have seen that the GTRL algorithm can be used to learn a variety of Boolean function classes with varying degrees of effectiveness and efficiency. This chapter describes only a particular instance of a general, dynamic generate-and-test method—there are a number of other possible variations.

The algorithm is designed so that other search heuristics may be easily accommodated. An example of another, potentially useful, heuristic is to combine hypotheses that are highly correlated with the optimal hypothesis. One way to implement this heuristic would be to run a linear-association algorithm, such as LARC, over the input bits and the outputs of the newly-created hypotheses, then make combinations of those hypotheses that evolve large weights. It is not immediately apparent how this would compare to using the N and S heuristics.

Another possible extension would be to add genetically-motivated operators, such as crossover and mutation, to the set of search operators. Many genetic methods are concerned only with the performance of the final result so this extension would have to be made carefully in order to preserve good on-line performance.

Chapter 8

Learning Action Maps with State

All of the algorithms that we have considered thus far are capable of learning only actions maps that are pure, instantaneous functions of their inputs. It is more generally the case, however, that an agent's actions must depend on the past history of input values in order to be effective. By storing information about past inputs, the agent is able to induce a finer partition on the set of world states, allowing it to make more discriminations and to tailor its actions more appropriately to the state of the world.

Perhaps the simplest way to achieve this finer-grained historical view of the world is to simply remember all input instances from the last k ticks and present them in parallel to the behavior-learning algorithm. This method has two drawbacks: it is not possible for actions to depend on conditions that reach back arbitrarily far in history and the algorithmic complexity increases considerably as the length of the available history is increased.

This chapter will present an alternative approach, based on the GTRL algorithm, which can efficiently learn simple action maps with temporal dependencies that go arbitrarily far back in history.

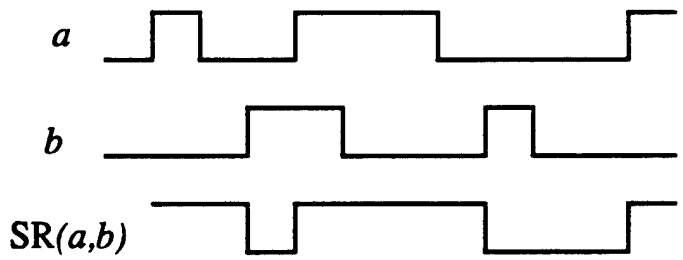


Figure 57: Timing diagram for a set-reset flip-flop.

8.1 Set-Reset

A common component in hardware logic design is a set-reset (SR) flip-flop.[1] It has two input lines, designated *set* and *reset*, a clock, and an output line. Whenever the clock is triggered, if the *set* line is high, then the output of the unit is high; else, if the *reset* line is high, the output of the unit is low; finally, if both input lines are low, the output of the unit remains the same as it was during the previous clock cycle. The value of the output is held in the determined state until the next clock tick.

The behavior of an SR flip-flop can be described logically in terms of the following binary Boolean operator

$$\mathrm{SR}(a,b) \equiv a \vee (\neg b \wedge \bullet \mathrm{SR}(a,b)) \ ,$$

where $\bullet$ is the temporal operator "last." Figure 57 shows a timing diagram, in which the top two lines represent a time-history of the values of wires a and b and the bottom line represents the time history of the values of $\mathrm{SR}(a,b)$, the output of a set-reset flip-flop whose inputs are wires a and b.

In the logical definition of SR as a Boolean operator, no initial value is specified. This problem is dealt with by adding a third logical value, $\perp$, which means, intuitively, "undefined." When an expression of the form $\mathrm{SR}(a,b)$ is to be evaluated for the first time, it is assumed that the value of $\bullet\mathrm{SR}(a,b)$ is $\perp$. The value $\perp$ combines

[1] Components of this kind are also commonly referred to as RS (reset-set) flip-flops in the logic-design literature.

with the other logical values as follows:

$$\begin{aligned}
\textbf{true} \vee \perp &\equiv \textbf{true} \\
\textbf{false} \vee \perp &\equiv \perp \\
\perp \vee \perp &\equiv \perp \\
\textbf{true} \wedge \perp &\equiv \perp \\
\textbf{false} \wedge \perp &\equiv \textbf{false} \\
\perp \wedge \perp &\equiv \perp \\
\neg\perp &\equiv \perp
\end{aligned}$$

Thus, the expression $\mathrm{SR}(a, b)$ will have value $\perp$ until either $a \equiv \textbf{true}$, in which case $\mathrm{SR}(a, b) \equiv \textbf{true} \vee \ldots \equiv \textbf{true}$, or $a \equiv \textbf{false}$ and $b \equiv \textbf{true}$, in which case $\mathrm{SR}(a, b) \equiv \textbf{false} \vee (\textbf{false} \wedge \perp) \equiv \textbf{false}$.

8.2 Using SR in GTRL

In the original version of the GTRL algorithm, the hypotheses were pure Boolean functions of the input bits. This section describes an extended version of that algorithm, called GTRL-S, which has simple sequential networks as hypotheses.

8.2.1 Hypotheses

The GTRL-S algorithm is structured in exactly the same way as the GTRL algorithm. The main difference is that SR is added as another binary hypothesis-combination operator. This allows hypotheses such as

$$\mathrm{SR}(\neg b_0, b_1 \wedge b_2) \wedge (b_1 \vee \mathrm{SR}(\mathrm{SR}(b_0, b_1), \neg b_2) \ ,$$

which represents the sequential network shown in Figure 58, to be constructed.

This operator does not allow every possible sequential circuit to be generated, however. In the pure-function case it was not necessary to have a negation operator because DeMorgan's laws guarantee that having access to the negated atoms is

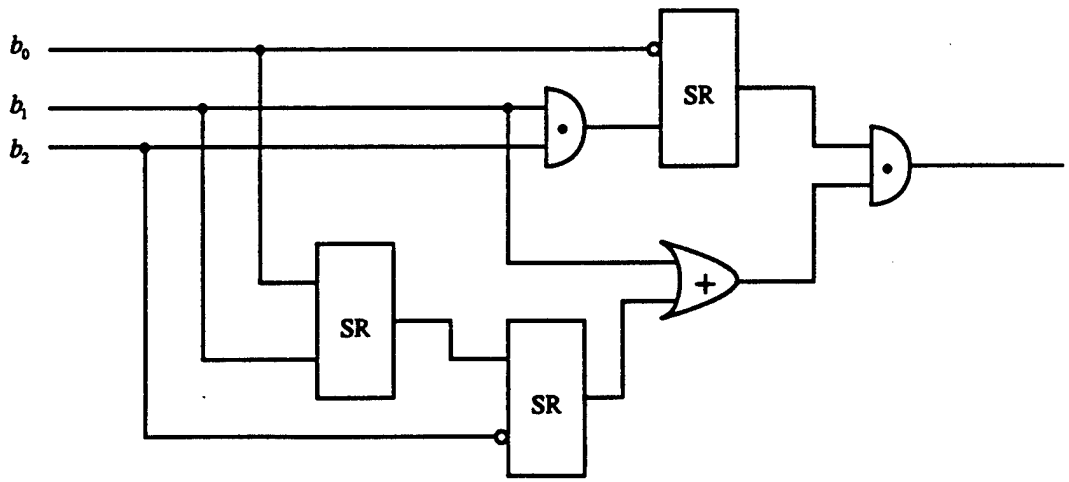


Figure 58: A sample sequential network, described by $\mathrm{SR}(\neg b_0, b_1 \wedge b_2) \wedge (b_1 \vee \mathrm{SR}(\mathrm{SR}(b_0, b_1), \neg b_2)$

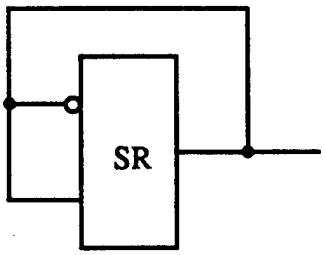


Figure 59: This circuit generates the sequence $0, 1, 0, 1, \ldots$; because it has feedback, it cannot be constructed by the GTRL-S algorithm.

sufficient to generate any Boolean function. Unfortunately, negation cannot be moved past the SR operator in any general way, so, for instance, a sequential circuit equivalent to $\neg\mathrm{SR}(i_0, i_1)$ cannot be generated by applications of the SR operator to atoms and their negations. This deficiency can be simply remedied by adding a unary negation operator or by adding an operator NSR, which is defined as

$$\mathrm{NSR}(a, b) \equiv \neg\mathrm{SR}(a, b) \ .$$

Another deficiency is that the construction of sequential networks with feedback is not allowed. Thus, the circuit shown in Figure 59, which generates the sequence $0, 1, 0, 1, \ldots$, cannot be constructed. For agents embedded in realistic environments, this limitation may not be too great in practice. We would not, in general, expect

such agents to have to make state changes that are not a function of changes in the world that are reflected in the agent's input vector. There is one additional limitation that is both more serious and more easily corrected. With the semantics of SR defined as they are, it is not possible to construct an expression equivalent to $\bullet a$. One way to solve this problem would be to redefine $\mathrm{SR}(a, b)$ as $\bullet a \vee (\bullet \neg b \wedge \bullet \mathrm{SR}(a, b))$. In that case, $\bullet a$ could be expressed as $\mathrm{SR}(a, \neg a)$, but the search heuristics to be used in GTRL-S (described in Section 8.2.3) would no longer be applicable. Another option would be to add $\bullet$ as a unary operator, along with negation. This is a reasonable course of action; it is not followed in this chapter, however, both because it would complicate the exposition and because no appropriate search heuristics for the last and negation operators are known.

In addition to the syntactic expression describing the network and the necessary statistics (discussed in Section 7.3), a hypothesis also contains the *state* of each of its SR components. When a new hypothesis is created with SR as the top-level operator, that component's state is set to $\perp$. The state of SR components occurring in the operands is copied from the operand hypotheses. In order to keep all state values up to date, a new state-update phase is added to the update function. In the state-update phase, the new state of each SR component of each hypothesis is calculated as a function of the input vector and the old state, then stored back into the hypothesis. The result of this calculation may be 1, 0, or $\perp$.

Expressions containing SR operators may be partially simplified using an extension of the simplification procedure used for standard Boolean expressions. This extended simplifier is also described in Appendix B.

8.2.2 Statistics

The statistical modules for GTRL-S differ from GTRL only when $satisfies(i, h)$ returns the value $\perp$. In that case, none of the statistics is updated. Once $satisfies(i, h)$ becomes defined for any input i, it will remain defined for every input, so this has no effect on the distribution of the instances for which statistics are collected, just on when the collection of statistics begins.

8.2.3 Search Heuristics

The problem of guiding the search for generating sequential networks is considerably more difficult than for pure functional networks. Statistics collected about the performance of expressions as generators of actions in the world are not necessarily a strong indication of their performance as the *set* or *reset* signal of an SR component. They can still provide some guidance, however.

Recall the logical definition of SR as

$$\mathrm{SR}(a, b) \equiv a \vee (\neg b \wedge \bullet \mathrm{SR}(a, b)) \ .$$

First, we can see that $a \rightarrow \mathrm{SR}(a, b)$ and that $\mathrm{SR}(a, b) \rightarrow (a \vee \neg b)$. The first observation should guide us to choose *set* operands that are sufficient for the target hypothesis. The second observation is slightly more complex, due to the fact that *set* takes precedence over *reset*, but it makes it reasonable to choose *reset* operands whose negations are necessary for the target hypothesis. From these observations we can derive the following heuristic:

> When making a set-reset hypothesis, use a *set* operand that has a high value of sufficiency and a *reset* operand whose negation has a high value of necessity.

8.2.4 Complexity

The computational complexity of the GTRL-S algorithm is the same as that of GTRL, which is discussed in Section 7.7. The only additional work performed by GTRL-S is the state-update computation. It has complexity $O(L(H+C)2^L)$ (assuming that H and C are independent of level), which is of the same order as the statistical updating phase that occurs in both algorithms.

8.3 Experiments with GTRL-S

This section documents experiments with GTRL-S in some simple domains that require action mappings with state. There are no direct comparisons with other

algorithms because no other comparable algorithms that learn action mappings with state from reinforcement are known.

8.3.1 Lights and Buttons

The first domain of experimentation is very simple. It can be thought of as consisting of two light-bulbs and two buttons. The input to the agent is a vector of two bits, the first having value 1 if the first light bulb is on and the second having value 1 if the second light bulb is on. The agent can generate two actions: action 0 causes the first button to be pressed and action 1 causes the second button to be pressed. One or no lights will be on at each instance. The optimal action map is to push the button corresponding to the light that is on if, in fact, a light is on. If no lights are on, the optimal action is to push the button associated with the light that was last on. A light is turned on on a given tick with probability p_l—the particular light is chosen randomly and equiprobably. Thus, the optimal hypothesis is simply $\mathrm{SR}(b_1, b_0)$.

Figure 60 shows parts of the trace of a sample run of the GTRL-S algorithm in the simple lights and buttons domain, in which the correct action (as discussed above) yields reinforcement value 1 with probability .9 and the incorrect action yields reinforcement value 1 with probability .1. A light comes on each tick with probability .1. The first section of the trace shows the state of the algorithm after 100 ticks. We can see that the correct hypothesis, $\mathrm{SR}(b_1, b_0)$,[2] has just been found and appears to be the best. After 200 ticks, we can see two recently-created hypotheses being tested. They are found wanting, however. By tick 500, the original winning hypothesis is still near the top of the list, surpassed only by another equivalent expression, $\mathrm{SR}(\neg b_0, \neg b_1)$. The GTRL-S algorithm works quite reliably on this problem because the search heuristics provide good guidance. In the statistics of the atomic hypotheses at level 0, it is easy to see that b_1 is the most sufficient hypothesis and $\neg b_0$ is the most necessary.

[2]The third value in the SR expressions of the printout indicates the stored value of the unit: `t` for 1, `nil` for 0, and `bottom` for $\perp$ (which does not happen to occur in this trace.)

```
******   Tick   100 Summary   ******
----Level 0----
PV =   750.7848 EPPUB = 0.78 EP = 0.75 I =  0.74 S =  0.80 AGE =    8 H: 1
PV =   734.8910 EPPUB = 0.89 EP = 0.73 I =  0.73 S = -1.00 AGE =   15 H: f
PV =   700.8619 EPPUB = 0.86 EP = 0.70 I =  1.00 S =  0.67 AGE =   21 H: (not 0)
PV =   641.0000 EPPUB = 1.00 EP = 0.64 I =  0.71 S =  0.00 AGE =    0 H: 0
PV =   608.0000 EPPUB = 1.00 EP = 0.61 I = -1.00 S =  0.61 AGE =    1 H: t
PV =   584.0000 EPPUB = 1.00 EP = 0.58 I =  0.50 S =  0.59 AGE =    0 H: (not 1)
----Level 1----
PV =   793.9776 EPPUB = 0.98 EP = 0.79 I =  0.79 S =  1.00 AGE =    8 H: (sr 1 0 nil)
PV =   724.9285 EPPUB = 0.93 EP = 0.72 I = -1.00 S =  0.72 AGE =    8 H: (sr (not 0) 1 t)
PV =   715.0000 EPPUB = 1.00 EP = 0.71 I = -1.00 S =  0.71 AGE =    0 H: (or (not 0) (not 1))
PV =   710.0000 EPPUB = 1.00 EP = 0.71 I =  0.72 S =  0.50 AGE =    0 H: (sr 1 (not 0) nil)
PV =   704.0000 EPPUB = 1.00 EP = 0.70 I = -1.00 S =  0.70 AGE =    0 H: (sr (not 1) 0 t)
PV =   654.0000 EPPUB = 1.00 EP = 0.65 I =  0.67 S =  0.50 AGE =    0 H: (or 0 1)
*** Reinf = (   67 /   100)   67.00%  Long term = (   67 /   100)   67.00%  ***

******   Tick   200 Summary   ******
----Level 0----
PV =   815.8619 EPPUB = 0.86 EP = 0.82 I =  0.89 S =  0.81 AGE =   21 H: (not 0)
PV =   774.7848 EPPUB = 0.78 EP = 0.77 I =  0.77 S =  0.82 AGE =    8 H: 1
PV =   774.0000 EPPUB = 1.00 EP = 0.77 I = -1.00 S =  0.77 AGE =    1 H: t
PV =   758.0000 EPPUB = 1.00 EP = 0.76 I =  0.33 S =  0.77 AGE =    0 H: (not 1)
PV =   752.8910 EPPUB = 0.89 EP = 0.75 I =  0.75 S = -1.00 AGE =   15 H: f
PV =   692.0000 EPPUB = 1.00 EP = 0.69 I =  0.74 S =  0.00 AGE =    0 H: 0
----Level 1----
PV =   902.9778 EPPUB = 0.98 EP = 0.90 I =  0.86 S =  0.91 AGE =   25 H: (or 1 (not 0))
PV =   898.9615 EPPUB = 0.96 EP = 0.90 I =  0.00 S =  0.91 AGE =   27 H: (or 0 (not 1))
PV =   870.9275 EPPUB = 0.93 EP = 0.87 I =  0.82 S =  0.92 AGE =   48 H: (sr 1 0 t)
PV =   849.9285 EPPUB = 0.93 EP = 0.85 I = -1.00 S =  0.85 AGE =    8 H: (sr (not 0) 1 t)
PV =   848.0000 EPPUB = 1.00 EP = 0.85 I = -1.00 S =  0.85 AGE =    0 H: (or (not 0) (not 1))
PV =   812.8632 EPPUB = 0.86 EP = 0.81 I =  0.81 S =  0.83 AGE =    8 H: (and 1 (not 0))
*** Reinf = (   86 /   100)   86.00%  Long term = (  153 /   200)   76.50%  ***

******   Tick   500 Summary   ******
----Level 0----
PV =   859.8619 EPPUB = 0.86 EP = 0.86 I =  0.89 S =  0.86 AGE =   21 H: (not 0)
PV =   844.7848 EPPUB = 0.78 EP = 0.84 I =  0.84 S =  0.90 AGE =    8 H: 1
PV =   844.0000 EPPUB = 1.00 EP = 0.84 I = -1.00 S =  0.84 AGE =    1 H: t
PV =   830.0000 EPPUB = 1.00 EP = 0.83 I =  0.25 S =  0.84 AGE =    0 H: (not 1)
PV =   825.8910 EPPUB = 0.89 EP = 0.82 I =  0.82 S = -1.00 AGE =   15 H: f
PV =   797.0000 EPPUB = 1.00 EP = 0.80 I =  0.82 S =  0.17 AGE =    0 H: 0
----Level 1----
PV =   913.9344 EPPUB = 0.93 EP = 0.91 I =  0.67 S =  0.92 AGE =   43 H: (sr (not 0) (not 1) t)
PV =   898.9231 EPPUB = 0.92 EP = 0.90 I =  0.86 S =  0.92 AGE =  245 H: (sr 1 0 t)
PV =   890.9468 EPPUB = 0.95 EP = 0.89 I =  0.88 S =  0.89 AGE =   41 H: (or 1 (not 0))
PV =   890.9335 EPPUB = 0.93 EP = 0.89 I = -1.00 S =  0.89 AGE =   33 H: (sr (not 1) 0 t)
PV =   886.9838 EPPUB = 0.96 EP = 0.89 I =  0.00 S =  0.89 AGE =   11 H: (sr (not 1) (not 0) t)
PV =   881.9615 EPPUB = 0.96 EP = 0.88 I =  0.00 S =  0.89 AGE =   27 H: (or 0 (not 1))
*** Reinf = (   89 /   100)   89.00%  Long term = (  418 /   500)   83.60%  ***
```

Figure 60: A sample run of the GTRL-S algorithm on the simple lights and buttons problem

8.3.2 Many Lights and Buttons

The lights-and-buttons domain described in Section 8.3.1 can be easily extended to have an arbitrary number, M, of lights and buttons. If we let each input bit correspond to a light and each output bit correspond to the pressing of a button, we have an environment with M input and M output bits. The agent is never rewarded for pressing more than one button at once.

The more complex lights-and-buttons problem can be solved by using the CASCADE method in conjunction with GTRL-S, with one copy of the GTRL-S algorithm for each bit of output (corresponding to each button.) Figure 61 shows excerpts from a sample run with two lights and two buttons (this differs from the domain described in the previous section in that there are two output bits rather than only one.) The first two levels belong to the instance of GTRL-S for the first output bit and the second two levels belong to the second instance of GTRL-S. After the first 100 ticks, neither instance has found the correct hypothesis and the performance is quite poor. By tick 200, however, the best hypothesis for the first bit is $\mathrm{SR}(\neg b_1, \neg b_0)$, which is equivalent to $\mathrm{SR}(b_0, b_1)$, the correct function. The best hypothesis for the second bit is $\mathrm{SR}(b_1, b_0)$, which is also correct. Again, it is easy to verify that the necessity and sufficiency heuristics are a good guide for the search.

The search heuristics for SR fail us when we wish to extend this problem to a larger number of lights and buttons using a cascade of 3-level instances of GTRL-S. When there are three lights and buttons, the optimal function for the first bit can be most simply expressed as $\mathrm{SR}(b_0, b_1 \vee b_2)$. In order to synthesize this expression, the expression $\neg b_1 \wedge \neg b_2$ must be available at the previous level. For that to happen, $\neg b_1$ and $\neg b_2$ must be highly sufficient, which is false, in general. Thus, the only way to learn this function is to generate all sub-expressions exhaustively, which is computationally prohibitive.

```
******   Tick   100 Summary   ******
----Level 0----
PV =   329.7308 EPPUB = 0.73 EP = 0.33 N =  0.29 S =  1.00 AGE =   22 H: 0
PV =   276.0000 EPPUB = 1.00 EP = 0.28 N =  0.28 S = -1.00 AGE =    3 H: f
PV =   260.0000 EPPUB = 1.00 EP = 0.26 N =  0.27 S =  0.00 AGE =    1 H: 1
----Level 1----
PV =   344.4019 EPPUB = 0.40 EP = 0.34 N =  0.29 S =  0.83 AGE =   40 H: (or 0 1)
PV =   328.0000 EPPUB = 1.00 EP = 0.33 N =  0.26 S =  1.00 AGE =    0 H: (sr 0 (not 1) nil)
PV =   291.0000 EPPUB = 1.00 EP = 0.29 N =  0.29 S = -1.00 AGE =    0 H: (and 0 1)
PV =   260.0000 EPPUB = 1.00 EP = 0.26 N =  0.26 S =  0.00 AGE =    0 H: (sr 1 (not 0) nil)
----Level 0----
PV =   428.5673 EPPUB = 0.57 EP = 0.43 N =  0.86 S =  0.33 AGE =   51 H: (not 2)
PV =   419.5131 EPPUB = 0.51 EP = 0.42 N =  0.42 S = -1.00 AGE =    7 H: f
PV =   411.6576 EPPUB = 0.66 EP = 0.41 N =  0.42 S =  0.33 AGE =    2 H: 1
PV =   370.0000 EPPUB = 1.00 EP = 0.37 N =  0.83 S =  0.33 AGE =    0 H: (not 0)
PV =   319.0000 EPPUB = 1.00 EP = 0.32 N = -1.00 S =  0.32 AGE =    0 H: t
PV =   319.0000 EPPUB = 1.00 EP = 0.32 N = -1.00 S =  0.32 AGE =    0 H: (not 1)
PV =   297.0000 EPPUB = 1.00 EP = 0.30 N =  0.32 S =  0.00 AGE =    0 H: 0
PV =    87.0000 EPPUB = 1.00 EP = 0.09 N =  0.06 S =  0.17 AGE =    0 H: 2
----Level 1----
PV =   372.7693 EPPUB = 0.77 EP = 0.37 N =  1.00 S =  0.30 AGE =    5 H: (and (not 0) (not 2))
PV =   353.0000 EPPUB = 1.00 EP = 0.35 N =  1.00 S =  0.28 AGE =    0 H: (sr (not 0) 2 t)
PV =   341.0000 EPPUB = 1.00 EP = 0.34 N =  1.00 S =  0.27 AGE =    0 H: (or (not 0) (not 2))
PV =   327.0000 EPPUB = 1.00 EP = 0.33 N =  1.00 S =  0.25 AGE =    0 H: (sr (not 2) 0 t)
*** Reinf = (   35 /   100)   35.00%  Long term = (   35 /   100)   35.00%  ***

******   Tick   200 Summary   ******
----Level 0----
PV =   563.5973 EPPUB = 0.60 EP = 0.56 N =  1.00 S =  0.55 AGE =   63 H: (not 1)
PV =   538.0000 EPPUB = 1.00 EP = 0.54 N = -1.00 S =  0.54 AGE =    1 H: t
PV =   495.0000 EPPUB = 1.00 EP = 0.49 N =  0.00 S =  0.51 AGE =    0 H: (not 0)
PV =   356.7308 EPPUB = 0.73 EP = 0.36 N =  0.30 S =  0.75 AGE =   22 H: 0
PV =   294.0000 EPPUB = 1.00 EP = 0.29 N =  0.29 S = -1.00 AGE =    3 H: f
PV =   256.0000 EPPUB = 1.00 EP = 0.26 N =  0.26 S =  0.00 AGE =    1 H: 1
----Level 1----
PV =   511.9743 EPPUB = 0.97 EP = 0.51 N =  1.00 S =  0.49 AGE =    7 H: (sr (not 1) (not 0) t)
PV =   506.7267 EPPUB = 0.73 EP = 0.51 N =  1.00 S =  0.49 AGE =   19 H: (and (not 0) (not 1))
PV =   475.0000 EPPUB = 1.00 EP = 0.47 N = -1.00 S =  0.47 AGE =    0 H: (sr (not 0) 1 t)
PV =   468.0000 EPPUB = 1.00 EP = 0.47 N = -1.00 S =  0.47 AGE =    0 H: (sr (not 1) 0 t)
----Level 0----
PV =   555.6576 EPPUB = 0.66 EP = 0.56 N =  0.55 S =  0.67 AGE =    2 H: 1
PV =   549.6988 EPPUB = 0.70 EP = 0.55 N =  0.55 S = -1.00 AGE =   15 H: f
PV =   517.0000 EPPUB = 1.00 EP = 0.52 N =  0.53 S =  0.00 AGE =    0 H: 0
PV =   506.5251 EPPUB = 0.53 EP = 0.51 N =  0.65 S =  0.34 AGE =   58 H: (not 2)
PV =   349.0000 EPPUB = 1.00 EP = 0.35 N =  0.69 S =  0.30 AGE =    0 H: (not 0)
PV =   296.0000 EPPUB = 1.00 EP = 0.30 N = -1.00 S =  0.30 AGE =    0 H: t
PV =   272.0000 EPPUB = 1.00 EP = 0.27 N = -1.00 S =  0.27 AGE =    0 H: (not 1)
PV =   137.0000 EPPUB = 1.00 EP = 0.14 N =  0.11 S =  0.16 AGE =    0 H: 2
----Level 1----
PV =   604.5066 EPPUB = 0.51 EP = 0.60 N =  0.73 S =  0.29 AGE =   28 H: (sr 1 0 nil)
PV =   597.7138 EPPUB = 0.71 EP = 0.60 N =  0.59 S =  0.80 AGE =   57 H: (sr 1 2 nil)
PV =   594.0000 EPPUB = 1.00 EP = 0.59 N =  0.58 S =  1.00 AGE =    0 H: (and 1 (not 2))
PV =   587.0000 EPPUB = 1.00 EP = 0.59 N =  0.58 S =  0.75 AGE =    0 H: (and 1 (not 0))
*** Reinf = (   50 /   100)   50.00%  Long term = (   85 /   200)   42.50%  ***
```

Figure 61: A sample run of the GTRL-S algorithm on the two-bit lights and buttons problem. Only the 4 most predictive hypotheses are shown at each non-atomic level.

8.4 Conclusion

Although the approach embodied in GTRL-S is capable of learning some simple action maps with state, it does not hold much promise for more complex cases. In such cases, it may, in fact, be necessary to learn a state-transition model of the world and values of the world states, using a combination of Rivest and Schapire's [56] method for learning models with hidden state and Sutton's [72] or Whitehead and Ballard's [80] method for "compiling" transition models into action maps. This will be a difficult job—currently available methods for learning models with hidden state only work in deterministic worlds. Even if they did work in non-deterministic worlds, they attempt to model every aspect of the world's state transitions. In realistic environments, there will be many more aspects of the world state than the agent can track, and its choice of which world states to represent must be guided by reinforcement, so that it can learn to make only the "important" distinctions. Drescher's work on generating "synthetic items" [18] is a promising step in this direction. His "schema mechanism" attempts to learn models of the world that will enable problem solving. When it is unsuccessful at discovering which preconditions will cause a particular action to have a particular result, it "reifies" that set of preconditions as an "item" and attempts to discover tests for its truth or falsity. In many cases the reified item turns out to be a particular aspect of the state of the world that is hidden from the agent.

Chapter 9

Delayed Reinforcement

Until now, we have only considered algorithms for learning to act in environments in which local reinforcement is generated each tick, giving the agent all of the information it will ever get about the success or failure of the action it just took. This is a simple instance of the more general case, in which actions taken at a particular time may not be rewarded or punished until some time in the future. This chapter surveys some existing approaches to the problem of learning from delayed reinforcement, focusing on the use of *temporal difference* methods [71], such as Sutton's *adaptive heuristic critic* method [70] and Watkins' *Q-learning* method [78]. It will be shown how these methods can be combined with the pure function-learning algorithms presented in previous chapters to create a variety of systems that can learn from delayed reinforcement.

9.1 Q Learning

There are well-known dynamic programming methods, such as *policy improvement* [57] that can be used for computing the optimal action mapping for an agent, given a complete state-transition model of the world. Watkins has developed a method for learning from delayed reinforcement that he describes [78] as "incremental dynamic programming by a Monte Carlo method: the agent's experience—the state-

Algorithm 16 (Q) *The initial state s_0 is an array indexed by the set of input states and the set of actions, whose elements are initialized to some constant value.*

$$\begin{array}{ll} u(s,i,a,r) = & s[i',a'] = (1-\alpha)s[i',a'] + \alpha(r + \gamma U(i)) \\ e(s,i) = & a \text{ such that } s[i,a] \text{ is maximized} \end{array}$$

where i' and a' are the input and action values from tick $t-1$, $0 < \alpha < 1$, $0 < \gamma < 1$, and $U(i) = \max_a\{s[i,a]\}$.

Figure 62: The Q-learning algorithm.

transitions and the rewards that the agent observes—are used in place of transition and reward models."

Watkins' method is referred to as *Q-learning* because it is concerned with learning values of $Q(i,a)$, where i is an input, a is an action, and $Q(i,a)$ is the expected discounted reward of taking action a in input state i then continuing by following the optimal policy. The agent's policy is always to execute, in input state i, the action a for which its estimate of $Q(i,a)$ is maximized. The Q algorithm is described formally in Figure 62.

The initial state of the Q algorithm is simply the array of estimated Q values, indexed by the input and action sets. To evaluate an input instance, i, the action, a, that maximizes $Q(i,a)$ is generated. The update function adjusts the estimated Q value of the *previous* input and action in the direction of

$$r + \gamma U(i) \ ,$$

which is the actual reinforcement received, r, plus a discounted estimate of the value of the next state, $\gamma U(i)$. The function $U(i)$ estimates the value of an input i by returning the estimated Q value of the best action that can be taken from that state. This update rule illustrates the concept of *temporal difference* learning, which was formulated by Sutton [71]. Rather than waiting until a reinforcement value is received and then propagating it back along the path of states that lead up to it, each state is updated as it is encountered by using the discounted estimated value of the next state as a component of the reinforcement. Initially, these estimated

values are meaningless, but as the agent experiences the world, they soon begin to converge to the true values of the states.

Watkins does not specify what the initial estimated Q values should be. If the value 0 is used and the optimal action values are positive, the algorithm will almost certainly fail, because it always chooses the action with the highest Q value. As soon as one action has positive value associated with it, it will be chosen forever more, to the exclusion of the other actions. There are two simple solutions to this problem. One is to perform random actions with a certain small probability. This guarantees that the whole space will eventually be explored, but can take a long time. Also, even if the best states are eventually reached, if they occur only rarely, it may not have a significant effect on the Q values. Another solution is to set the initial Q values to be higher than any of the actual Q values. This causes a process similar to the operation of the IE algorithm, in which the actions are chosen alternately until the Q values are driven down to the actual action values. If the initial Q values are much too high, however, this process can take a long time; it is effective only if a relatively tight upper bound on the action values is known *a priori*.

As the agent gains experience in the world, the Q values begin to become true reflections of the action-values of the states in the world, given that the optimal policy is being executed. Watkins proved that, in fact, the Q values will converge to the values of the actions under the optimal policy given, among other conditions, that each input-action pair is experienced an infinite number of times.

9.2 Q-Learning and Interval Estimation

The Q algorithm, as presented above, does not guarantee that each input-action pair will be sampled an infinite number of times. It is often the case that a particular action has a high Q value in a given state early on and other actions in that state are rarely, if ever, tried again. One approach to solving this problem (although it still does not guarantee convergence) is to apply the basic idea of interval estimation,

Algorithm 17 (IEQ) *The initial state is an array indexed by the set of input states and the set of actions, whose elements are initial states of a normal or non-parametric central-value estimator.*

$$\begin{array}{ll} u(s,i,a,r) = & s[i',a'] := \textit{update-stats}(s[i',a'], r + \gamma U(i)) \\ e(s,i) = & a \textit{ such that } ub_\alpha(s[i,a]) \textit{ is maximized} \end{array}$$

where $0 < \alpha < 1$, $0 \leq \gamma < 1$, *and* $U(i) = \max_a\{er(s[i,a])\}$ *(er is the expected reinforcement of performing action a in state i).*

Figure 63: The IEQ algorithm.

choosing the action with the highest upper bound on the underlying Q value. This approach is embodied in the IEQ algorithm, shown in Figure 63.

This algorithm can use either a normal or non-parametric model to estimate the expected action values. Using the normal distribution as a model can be dangerous, however, because at the beginning of this process, the sample variance is often 0, which causes the confidence intervals to be degenerate. The normal and non-parametric methods for generating confidence intervals were informally discussed in Section 4.5.2 and are presented in detail in Appendix A.

The function U changes over time, making early reinforcement values no longer representative of the current value of a particular action. This problem is already dealt with, in part, by the nature of the bounded-space non-parametric techniques, because only a sliding window of data is kept and used to generate upper bounds. However, this does not guarantee that poor-looking actions will be taken periodically in order to see if they have improved. One way of doing this is to decay the statistics, periodically dropping old measurements out of the sliding windows, making them smaller. A similar decay process can be used in the normal statistical model, as well. Decaying the statistics will have the effect of increasing upper bounds, eventually forcing the action to be re-executed. This method will keep the algorithm from absolutely converging to the optimal policy, but the optimal policy can be closely approximated by decreasing the decay rate over time. The IEQ algorithm has three

parameters: γ, the discount factor, α, the size of the confidence intervals, and δ, the decay rate.

The biggest practical improvement of IEQ over Q is that it is no longer necessary to estimate the values of the states in order to generate appropriate initial values. In the context of the Dyna architecture [72], Sutton has recently developed a similar extension to Q-learning, called Dyna-Q+, in which a factor measuring uncertainty about the results of actions is added to the Q values, giving a bonus to exploring actions about which little is known.

9.3 Adaptive Heuristic Critic Method

Sutton [70,71] has developed a different approach of applying the temporal difference method to learning from delayed reinforcement. Rather than learning the value of every action in every input state, the *adaptive heuristic critic* (AHC) method learns an evaluation function that maps input states into their expected discounted future reinforcement values given that the agent executes the policy it has been executing. One way of viewing this method is that the AHC module is learning to transduce the delayed reinforcement signal into a local reinforcement signal that can be used by any of the algorithms of the previous chapters. The algorithm used to learn from the local reinforcement signal need only optimize the reinforcement received on the next tick; such an algorithm is referred to as a local (as opposed to global) learning algorithm. It is a requirement, however, that the local learning algorithm be capable of learning in nonstationary environments, because the AHC module will be learning a transduction that changes as the agent's policy changes.

The AHC method, in combined operation with an algorithm for learning from local reinforcement, is formally described in Figure 64. There are two components to the state of the AHC algorithm: the vectors v and c. The v vector contains, at every tick, the current best estimate of the discounted future value of each state with discount rate γ, given that the agent is executing the behavior that it is currently executing. The c vector values represent the "activation" values of the states. States that have been visited recently have high activation values and those that have not

Algorithm 18 (AHC) *The initial state, s_0, consists of three parts: two n-dimensional vectors, c and v, and s_l, the initial state of the local learning algorithm.*

$$
\begin{aligned}
u(s,i,a,r) = \quad & \textit{for } j := 0 \textit{ to } n \textit{ do} \\
& \quad c[j] := \gamma \lambda c[j] \\
& c[i'] := c[i'] + 1 \\
& vi := v[i]; \quad vi' := v[i'] \\
& \textit{for } j := 0 \textit{ to } n \textit{ do} \\
& \quad v[j] := v[j] + \alpha\, c[j]\,(r'' + \gamma\, vi - vi') \\
& s_l := u_l(s_l, i'', a'', v[i']) \\
e(s,i) = \quad & e_l(s_l, i)
\end{aligned}
$$

where i' and a' are the input and action values from tick $t-1$; i'', a'', and r'' are from tick $t-2$; n is the size of the input set; s_l, u_l and e_l are the internal state, the update function, and the evaluation function of the local learner; $0 \leq \lambda \leq 1$; $0 \leq \gamma < 1$; and $0 \leq \alpha < 1$.

Figure 64: The AHC algorithm.

been visited recently have low values. Each of these vectors is initialized to contain 0 values.

The update function first updates the activation values. Each element's activation is multiplied by $\lambda\gamma$, where γ is the discounting rate and λ is an independent factor that controls the degree to which activation is spread backward from the currently active state. Then, the activation of the state whose value is being updated on this tick, state i', is increased by 1. The values of states are adjusted in proportion to their activations, so for $\lambda = 0$, only the currently active state's value is updated on each tick.

Next, the state values in vector v are updated. Each value $v[j]$ is incremented by the product of its activation, $c[j]$, the learning rate, α, and the prediction difference, $r'' - \gamma v[i] - v[i']$. The quantity $v[i']$ is the estimated value of state i'. The quantity $r'' + \gamma v[i]$ is a one-step lookahead value of state i', computed as the sum of the global value of state i' (as indicated by the reinforcement value r'' of the previous tick) and the discounted value of the next state, $\gamma v[i]$. Since the one-step lookahead value is

a better estimate than the stored value, the difference between the two values can be used as an error signal for updating the stored value. This updating method efficiently propagates global reinforcement values back along the chain of actions that lead to them, making the AHC algorithm another instance of the temporal difference method.

Finally, the update function feeds a learning instance to the update function of the local learning algorithm. The reason for updating the local learner two ticks behind is that if a large reinforcement value is received, we would like it to be reflected in the function learner as soon as possible. However, if a large r is received at time t, it takes two more ticks to receive the data that will allow its effect on v to be calculated. The algorithm would not be incorrect if it performed $s_l := u_l(s_l, i', a', v[i])$ instead, but it would not respond to good or bad results the first time they were encountered.

The AHC algorithm has no effect on the evaluation process and simply calls the evaluation method of the local learning algorithm.

Sutton has shown [71] that, for the non-discounted case, the expected values of the predictions found by the temporal difference method converge to the ideal predictions if the data sequences are generated by Markov processes and the value of parameter λ equals 0. When $\lambda = 1$, the temporal difference method generates the same weight adjustments as the Widrow-Hoff rule. Of course, when the agent is choosing actions that change the state of the world, the distributions of input data change and these results do not necessarily hold.

Sutton's presentation of the AHC algorithm was combined with a version of the LARC algorithm for local learning. The AHC method is presented here independent of assumptions about the local learning algorithm. This way of breaking down the problem is very useful, because it allows us to independently choose a local reinforcement-learning algorithm that is appropriate for the sorts of environments in which it will be run for use in combination with the AHC algorithm. In addition, Sutton used linear association methods to store the values of v and c more efficiently. In this version, the activation and state values are simply stored in a table, but it

is easy to see how a variety of more efficient (if less precise) associative storage methods could be applied.

There have been a number of implementations of temporal difference algorithms similar to AHC, but none have had a correct analysis of convergence results. The AHC work grew out of the adaptive critic element (ACE) used by Barto, Sutton, and Anderson [11].

Witten's [86] adaptive optimal controller algorithm computes state values as in the AHC algorithm, but differs from Sutton's work in the way it is combined with the local learner. This difference causes its performance to be significantly inferior [70].

One of AI's most striking early successes was Samuel's checkers-playing program [60,61]. In one of its learning modes, it learned an evaluation function for board positions from reinforcement. Although Samuel's learning procedure is very complex, it can be closely approximated by the AHC algorithm with $\gamma = 1$.

Another, more distantly related, learning method is Holland's bucket brigade method for assigning credit to chains of rules firing in a production system [33]. It differs significantly in the details, but shares the temporal-difference notion of assigning credit along a sequence based on the local predicted improvement rather than waiting for global reinforcement.

9.4 Other approaches

There have been a number of other approaches to learning from delayed reinforcement. They can be divided into those that learn a world model (generally assuming, unlike Rivest and Schapire [56], that there is no hidden state) and those that do not.

Drescher [18] presents a theory and implementation of learning based on the developmental psychology of Piaget. The agent learns precondition-action-result schemata that allow it to achieve dynamically presented goals. Drescher's methods have been demonstrated in a simple deterministic world with hidden state. There have been a number of other efforts to learn world models. These include the work

of Sutton and Pinette [73], Mason, Christiansen, and Mitchell [40], Mel [42], and Shen [68].

There has been a series of attempts to solve the pole-balancing problem using reinforcement. The problem is motivated by a physical system in which a pole is flexibly mounted on a cart. The pole can rotate about its connection to the cart in one dimension, and the cart can move along a one-dimensional track (in the same dimension as the plane in which the pole moves). The goal is to control the cart in such a way as to keep the pole from falling over and to keep the cart from reaching either end of its track. The system is given an encoding of the positions and velocities of the angle of the pole with respect to the cart and the offset of the cart with respect to the midpoint of the track, and the system chooses between applying a fixed-magnitude force on the cart in either a positive or negative direction. Negative reinforcement is received whenever the pole falls over or the cart reaches the end of its track. The system must learn a "bang-bang" control law that maximizes reinforcement by keeping the pole up and the cart within limits for as long as possible.

The first learning solution to this problem was the BOXES system of Michie and Chambers [44]. It was so named because of the quantization of the four-dimensional continuous-valued parameter space into a set of 255 regions or "boxes." Each box was viewed as making a separate decision about whether to generate a "left" or "right" action when the system was in that state, based on the expected run length given each choice of action. Learning only took place after a failure, and each policy was tested for an entire run. The details of the method are complex and somewhat *ad hoc*, but it recognizes the interesting issues of the problem setting, including temporal credit assignment and the tradeoff between acting to gain information and acting to gain reinforcement.

Connell and Utgoff's CART system [17] takes advantage of the continuity of the parameter space, using an algorithm that does not make an *a priori* division of the space into discrete boxes. Points in the state space are determined from experience to be either desirable or not desirable—interpolation is used to determine the desirability of states that have not yet been visited. The system has considerably

better performance than either the BOXES system or the application of the AHC algorithm to this problem by Selfridge and Sutton [67] or by Anderson [3,4]. The difference in performance seems principally to depend on differences in the encodings of the inputs, however.

9.5 Complexity Issues

Whether we are learning action values or an evaluation function, we are confronted again with the problem of high computational complexity.

With the Q and IEQ algorithms, we are back again to the kinds of exponential complexity in the size of the input and output that we have been trying to avoid. Watkins addresses this issue for Q-learning by using Albus' CMAC method [2] for associating Q values with input-action pairs for its "computational speed and simplicity, rather than accuracy or storage economy." It is possible to use a CMAC that is very space efficient, but at a potentially great cost in accuracy.

Another method of improving computational complexity at the expense of accuracy is to use a linear associator to store the values being learned. The Q values could be stored as a function of a bit vector constructed by concatenating the bit-vector encodings of the input state and the action. Sutton uses this method in his implementation of AHC, storing the evaluations of input states as functions of bit-vector encodings of those states. It is difficult to quantify exactly how much expressive power is lost by using such methods and how that loss in expressiveness will impact the performance of the learning methods as a whole. A related method, used by Anderson [3], is to store predictions in a multi-layer network trained using the error-backpropagation method (Section 3.4.3 describes this method in more detail).

Algorithms, such as IEQ, that must associate a whole collection of data with an input-action pair are harder to make more efficient in this way.

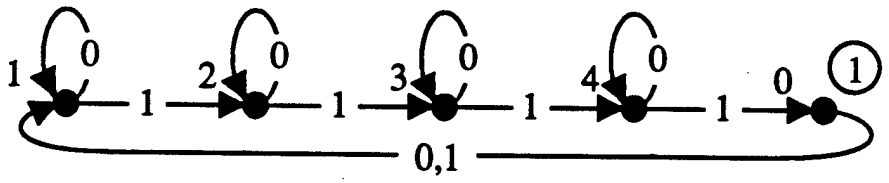


Figure 65: Environment D1: a very simple delayed-reinforcement environment.

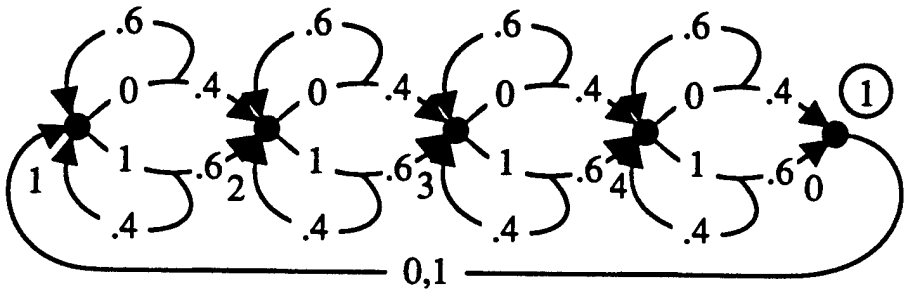


Figure 66: Environment D2: a more difficult delayed-reinforcement environment.

9.6 Empirical Comparison

This section describes the results of three different methods of learning from delayed reinforcement in three simple simulated environments.

9.6.1 Environments

The first two environments are taken from Sutton's thesis [70]. Figures 65 and 66 show their state-transition diagrams. The circled numbers are the reinforcement values of the states; most of the states have reinforcement value 0 (which is omitted from the figure). The first is a very easy deterministic environment. The second is a considerably more difficult non-deterministic environment, with little differentiation between "good" and "bad" actions. The third environment, from Watkins [78], is shown in Figure 67. It was constructed to be misleading, because, although the correct action in state 0 is 0, if the agent is executing a random policy, the action 1 will have a higher value. Before we apply the learning algorithms to these domains,

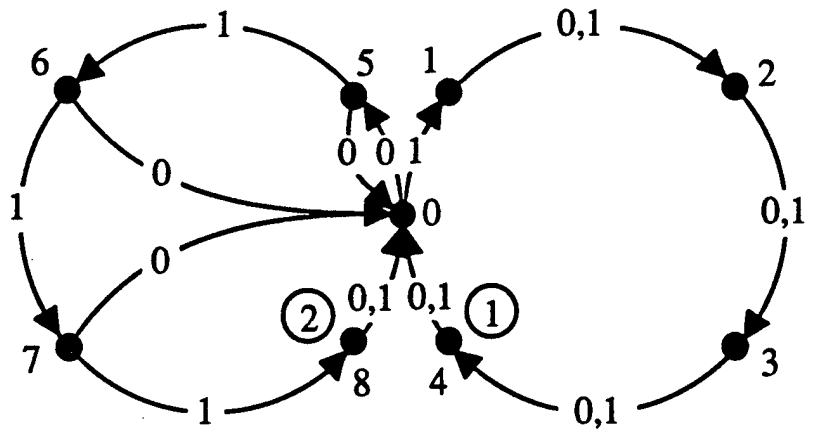


Figure 67: Environment D3: a highly misleading delayed-reinforcement environment.

it is interesting to consider the values of the states and the expected reinforcement of acting optimally in each case.

The optimal strategy for environment D1 is, obviously, always to execute action 1. Because the world is deterministic, it will take five steps to get payoff 1, so the average reinforcement of the optimal policy is 0.2. The values of the states can be calculated by solving the following set of equations, which specify the value of each state in terms of its global value and the discounted value of its successor under the optimal policy:

$$\begin{aligned} v_0 &= 1 + \gamma v_1 \\ v_1 &= \gamma v_2 \\ v_2 &= \gamma v_3 \\ v_3 &= \gamma v_4 \\ v_4 &= \gamma v_0 \end{aligned}$$

The solution to the equations is

$$\begin{aligned} v_0 &= 1/(1-\gamma^5) \\ v_1 &= \gamma^4/(1-\gamma^5) \\ v_2 &= \gamma^3/(1-\gamma^5) \end{aligned}$$

$$v_3 = \gamma^2/(1-\gamma^5)$$
$$v_4 = \gamma/(1-\gamma^5)$$

which, for $\gamma = .9$, yields the following values: $v_0 = 2.44, v_1 = 1.60, v_2 = 1.78, v_3 = 1.98, v_4 = 2.20$.

The second automaton, D2, is non-deterministic. In this case, the optimal strategy is also always to execute action 1. The expected number of failures preceding the first success in a sequence of Bernoulli trials with probability p is $(1-p)/p$, so we expect to remain in each of states 1 through 4 for an average of $1+0.4/0.6 = 1.67$ steps when executing the optimal policy. Thus, the total expected round-trip time is $4 \times 1.67 + 1 = 7.67$, making the expected reinforcement per tick approximately equal to 0.13. The action values are the solution to the equations

$$v_0 = 1 + \gamma v_1$$
$$v_1 = \gamma(.4v_1 + .6v_2)$$
$$v_2 = \gamma(.4v_2 + .6v_3)$$
$$v_3 = \gamma(.4v_3 + .6v_4)$$
$$v_4 = \gamma(.4v_4 + .6v_0)$$

which, for $\gamma = .9$, is $v_0 = 1.84, v_1 = 0.93, v_2 = 1.10, v_3 = 1.31, v_4 = 1.55$.

Finally, for the complex automaton D3, the optimal strategy is to take action 0 in state 0 and action 1 in states 5, 6 and 7. This path through the transition graph takes 5 steps to gain reinforcement value 2, yielding an average reinforcement per tick of 0.4. The values of the states under the optimal strategy can be expressed as

$$v_0 = \gamma v_5$$
$$v_1 = \gamma v_2$$
$$v_2 = \gamma v_3$$
$$v_3 = \gamma v_4$$
$$v_4 = 1 + \gamma v_0$$
$$v_5 = \gamma v_6$$

$$
\begin{aligned}
v_6 &= \gamma v_7 \\
v_7 &= \gamma v_8 \\
v_8 &= 2 + \gamma v_0
\end{aligned}
$$

Solving these equations with $\gamma = .9$ yields the state values $v_0 = 3.20, v_1 = 2.83, v_2 = 3.15, v_3 = 3.50, v_4 = 3.88, v_5 = 3.56, v_6 = 3.96, v_7 = 4.40, v_8 = 4.88$.

9.6.2 Algorithms

The following three algorithms for learning from delayed reinforcement were tested on each of these problems:

- Q (described in Figure 62)
- IEQ (described in Figure 63)
- AHC (described in Figure 64) in combination with a version of the IE algorithm (described in Figure 21) that uses normal statistics and is modified for use in non-stationary environments.

It would have been appropriate to compare Anderson's combined back-propagation and AHC method with these algorithms, but the parameter tuning problem for that algorithm seems computationally impractical.

9.6.3 Parameter Tuning

Each of these algorithms has a number of parameters. Algorithm Q has parameters α and γ; IEQ has parameters α_{ie},[1] γ, and δ; AHC has parameters α, γ, and λ; and IE with normal nonstationary statistics has parameters α_{ie} and δ. The parameter γ is part of the specification of the correctness criterion, and it will be set to 0.9 for each algorithm and task. To illustrate the dependence of the Q algorithm on its initial value, two versions of Q will be tested: one with initial values equal to 0

[1]Because we are using statistics for the normal distribution, it is easier to express the size of the confidence intervals in terms of α rather than $z_{\alpha/2}$; these are simply two ways of specifying the same parameter.

ALG-TASK	D1	D2	D3
Q			
α	.95	.95	.95
IEQ			
α_{ie}	.01	.05	.001
δ	.999	.9999	.99
AHC + IE			
α	.15	.1	.5
λ	.1	.2	1.0
α_{ie}	.05	.05	.001
δ	.9999	.99	.99

Table 9: Best parameter values for each algorithm in environments D1, D2, and D3.

(which is below the action values in all cases) and one with initial values equal to 20 (which is well above the action values in all cases). These two algorithms will be referred to as Q0 and Q20.

For each algorithm and environment, a series of 100 trials of length 3000 were run with different parameter values. Table 9 shows the best set of parameter values found for each algorithm-environment pair. The parameter α for the Q algorithms is largely irrelevant: if the initial value is too small, no value of α will result good performance; if the initial value is large, α should be as large as possible.

9.6.4 Results

Using the best parameter values for each algorithm and environment, the performance of the algorithms was compared on 100 runs of length 3000. The performance metric was average reinforcement per tick, averaged over the entire run. The results are shown in Table 10, together with the expected reinforcement of executing a completely random behavior (choosing actions 0 and 1 with equal probability) and of executing the optimal behavior.

As in the previous sets of experiments, we must examine the relationships of statistically significant dominance among the algorithms for each task. Figure 68 shows, for each task, a pictorial representation of the results of a 1-sided t-test

ALG-TASK	D1	D2	D3
Q0	.0000	.0910	.0000
Q20	.1907	.1222	.3780
IEQ	.1959	.1222	.2315
AHC + IE	.1988	.1153	.2923
random	.1100	.1100	.1250
optimal	.2000	.1300	.4000

Table 10: Average reinforcement for tasks D1, D2, and D3 over 100 runs of length 3000.

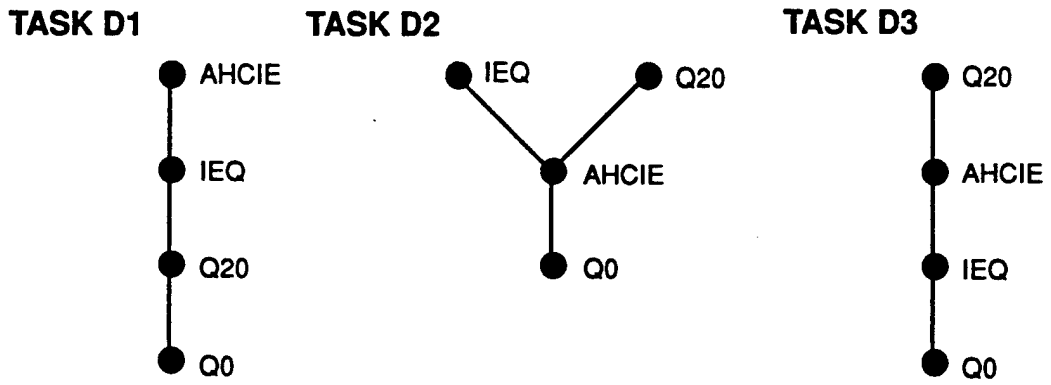


Figure 68: Significant dominance partial order among delayed-reinforcement algorithms for each task.

applied to each pair of experimental results. The graphs encode a partial order of significant dominance, with solid lines representing significance at the .95 level.

With the best parameter values for each algorithm, it is also instructive to compare the rate at which performance improves as a function of the number of training instances. Figures 69, 70, and 71 show superimposed plots of the learning curves for each of the algorithms. Each point represents the average reinforcement received over a sequence of 100 steps, averaged over 100 runs of length 3000.

9.6.5 Discussion

There are no clear winners among this set of algorithms. On the simple deterministic task D1, all of the algorithms approach the optimal performance level very closely.

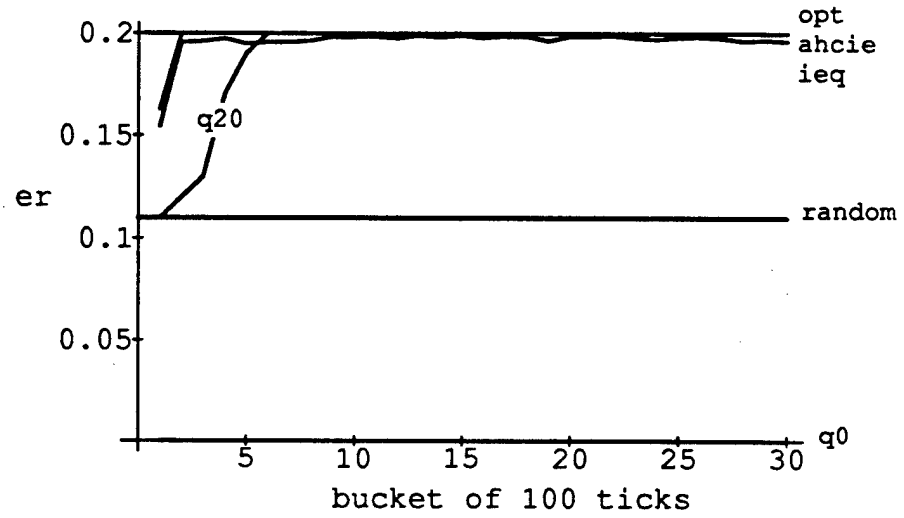


Figure 69: Learning curves for Task D1.

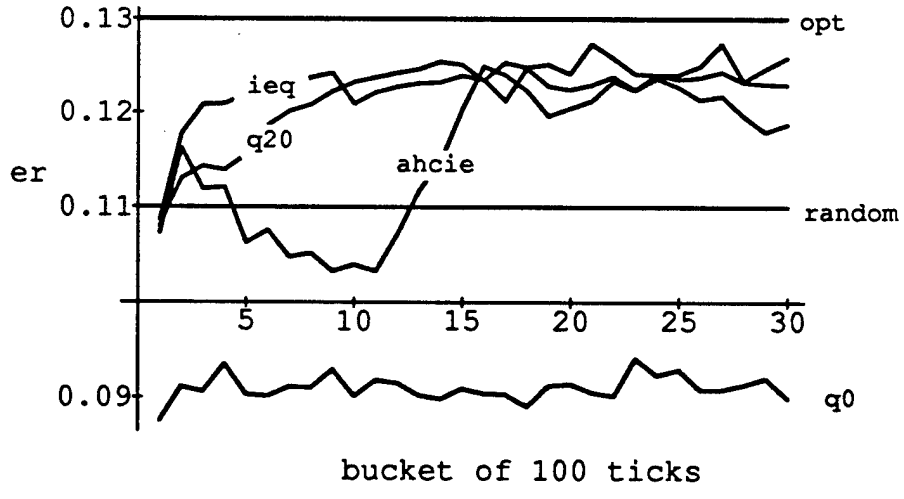


Figure 70: Learning curves for Task D2.

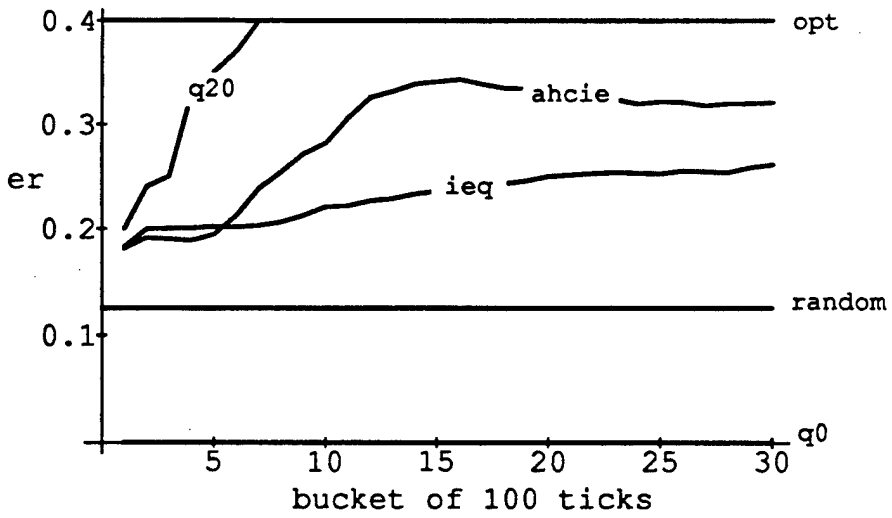


Figure 71: Learning curves for Task D3.

It takes the longest for Q20 to improve; if the initial values were smaller it would converge faster. When the initial value is too small, as in Q0, the algorithm performs significantly worse than random.

The non-deterministic task D2 is very difficult because of the similarity in transition probabilities between the two actions in each state. On this task, algorithms Q20 and IEQ perform essentially equivalently, approaching but not achieving optimal performance. The AHC+IE algorithm performs very poorly at first, but suddenly "realizes" the right course of action (perhaps when the AHC component has seen the higher-numbered states enough to realize that they are significantly better and the old statistics have decayed sufficiently in the IE component) and begins to perform as well as the other two algorithms. As usual, Q0 performs far worse than the random strategy.

Performance on the difficult problem of task D3 hinges on persistently trying, for a while, courses of action that appear bad. This persistence is necessary to discover that the left loop of the graph is better if the proper action strategy is known. The Q20 algorithm does a good job of this, and is the only one of the algorithms to achieve optimal performance during the course of a 3000-tick run. The other algorithms improve over time, but not nearly as fast. The fact that their

performance rises above the .2 level (which is achieved by going around the right loop of the graph) indicates that they are discovering the left loop of the graph. The Q0 algorithm performs as badly as possible, probably by looping between states 0 and 5.

More extensive experiments will be required before it is possible to formulate general rules of applicability of these algorithms to specific learning tasks.

Chapter 10

Experiments in Complex Domains

This chapter reports on three experiments comparing algorithms introduced in previous chapters on more complex domains. The first domain is a simulated one with a large number of input and output bits, but with a fairly low-complexity function defining the dependence of each output bit on the input bits. The second domain is a mobile-robot domain in which the agent learns from local reinforcement. The third domain is an extension of the mobile robot domain in which the agent learns from delayed reinforcement. The settings of the experiments will emulate, as much as possible, the deployment of these learning algorithms in realistic domains.

10.1 Simple, Large, Random Environment

This task, in its general form, has M input and M output bits. The optimal action mapping is generated randomly as follows: each output bit is the conjunction or disjunction of two input bits or their negations. If the agent chooses an action in agreement with this mapping, it receives reinforcement value 1 with probability p_1 and 0 otherwise; if the agent's action disagrees with the optimal mapping, it receives reinforcement value 1 with probability p_2 and 0 otherwise.

10.1.1 Algorithms

The following algorithms were tested in this domain:

- IE
- CASCADE + IE
- CASCADE + GTRL

The second and third algorithms consist of a set of Boolean-function learners combined using the CASCADE method. It is expected that the cascade of GTRL algorithms will be both more computationally efficient and learn more quickly than the other three algorithms because the functions are not too complex and the opportunity for generalization is great.

10.1.2 Task

The algorithms were tested on an instance of the general family of large random environments with $M = 8$, $p_1 = .8$, and $p_2 = .1$. It would have been desirable to use an even larger task, but the size of the data structures for $M = 8$ exhausted the available computational power. Each run of each algorithm was on a newly generated random task with the parameters described above.

10.1.3 Parameter Settings

When we wish to use a learning algorithm in a new setting, we will rarely have the luxury of performing extensive parameter-tuning runs to be sure that we get the best possible performance out of our algorithms. In this experiment, as well as in the other two described in this chapter, parameters for the algorithms will be chosen as well as possible to optimize performance within reasonable complexity constraints based on intuitions gained from the results of previous experiments that we have carried out. The parameter settings were:

IE: $z_{\alpha/2} = 3.0$

CASCADE + IE: $z_{\alpha/2} = 3.0, \delta = .9999$

CASCADE + GTRL: $z_{\alpha/2} = 3.0, \delta = .9999, H = 3M, PA = 20, R = 100$

All of the confidence-interval parameters are set to 3.0 and the decays are .9999. The size of the hypothesis lists, H, in the GTRL algorithm varies linearly as a function of the number of input bits. The number of input instances required for promotion was 20 and new candidates were generated once every 100 ticks.

10.1.4 Results

Each of the algorithms was run for 10 trials of length 10,000 each. This is is a small fraction of the number of trials that would be required for the agent to try all 512 possible actions in each of 512 possible input situations. The average reinforcement for each algorithm on this task is

IE	: .1019
CASCADE + IE	: .1050
CASCADE + GTRL	: .1634

The cascaded generate-and-test algorithm significantly outperforms either of the other algorithms, due to its ability to generalize both over the input and output sets. The learning curves for the algorithms are shown in Figure 72. As we can see, the GTRL algorithm improves in performance significantly more quickly than the others.

10.2 Mobile Robot Domain

This section describes the application of algorithms from this dissertation to a mobile-robot learning scenario. There have been very few implementations of reinforcement-learning algorithms on real robotic hardware. A notable example

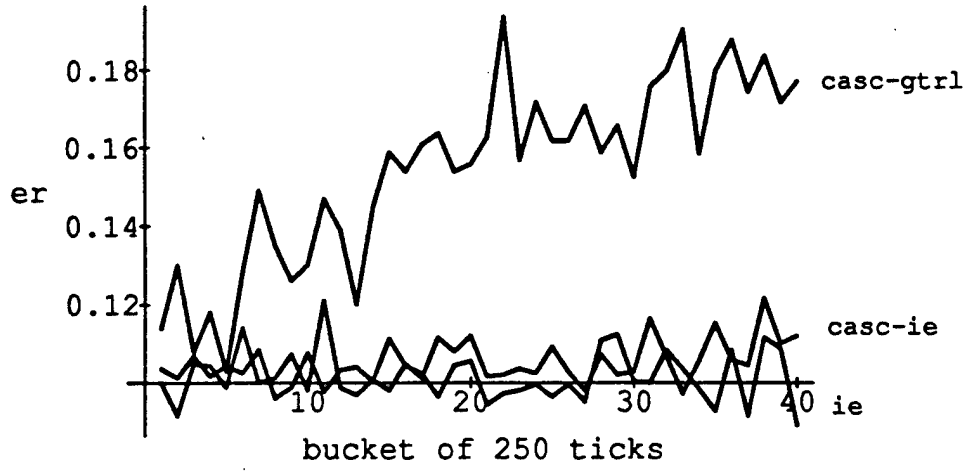


Figure 72: Learning curves for large, random environment.

is Maes and Brooks' [39] use of a simple algorithm to learn to coordinate predefined behaviors on a walking robot. A number of researchers have applied reinforcement-learning algorithms to simulated robotic domains, such as the cart-pole problem described in Chapter 3. Franklin [24] used learning-automata techniques and the A_{RP} algorithm to learn to adjust the outputs of an existing controller to compensate for externally applied torques on a simulated robot arm. In addition, there has been work on learning world models, such as Clocksin and Moore's [16], Miller's [46], and Mel's [42] work on learning a mapping from joint positions to visual coordinates in the workspace of a robotic arm [42] and Mason, Christiansen, and Mitchell's [40] work on learning the results of using a robotic arm to tip a tray of objects in various ways.

The robot pictured in Figure 73 was used to validate a variety of reinforcement-learning algorithms. It has two drive wheels, one on each side, which allow it to move forward and backward along circular arcs. A set of five "feelers" allow it to detect obstacles to its front and sides, the round bumper detects contact anywhere on its perimeter, and four photosensors, facing forward, backward, left, and right, measure the light levels in each direction.

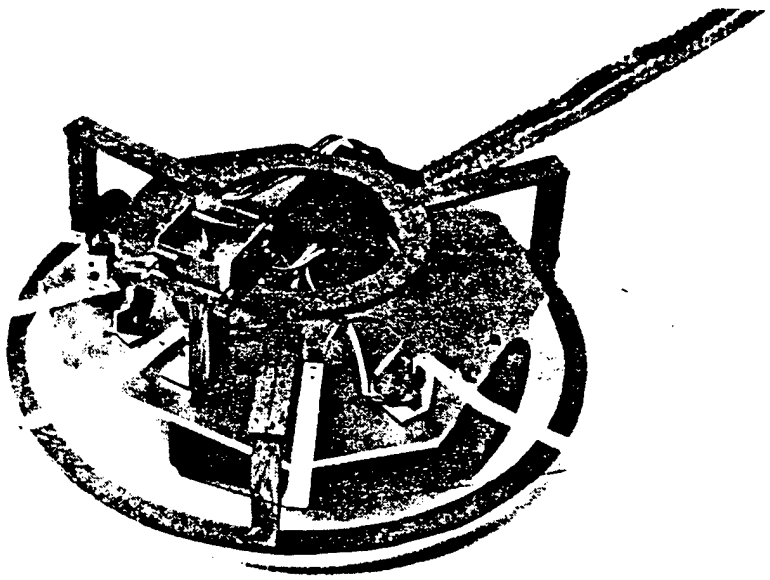


Figure 73: Spanky, a mobile robot.

10.2.1 Algorithms

The same algorithms and parameter settings were used in this experiment as in the previous one.

10.2.2 Task

In this task, the robot is given negative reinforcement, normally distributed with mean -2 and standard deviation 0.5 whenever the round bumper makes contact with any physical object. If the bumper is not engaged, the robot is given positive reinforcement, normally distributed with mean 1 and standard deviation 0.2, whenever the light in its front sensor gets brighter. If the bumper has not engaged and the brightness has not increased, it is given "zero" reinforcement, normally distributed with mean 0 and standard deviation 0.2.

The robot interacts with the world by making fixed-length motions, either forward or rotating in place to the left or right. The agent gets the following five bits of input:

Bits 0 and 1: Which direction is currently the brightest? 0 = front, 1 = left, 2 = right, 3 = back.

Bit 2: Is the rightmost feeler engaged?

Bit 3: Is the leftmost feeler engaged?

Bit 4: Is (at least) one of the middle three feelers engaged?

The agent must learn a mapping from this input space to its three actions that maximizes its local reinforcement. It develops a behavior that avoids bumping into obstacles and tends to move toward the light.

10.2.3 Results

All of the algorithms were run in the real robotic domain, with varying degrees of success. Ideally, this section would describe a long series of trials of each algorithm

ALG	*er*
IE	.6439
CASCADE + IE	.6203
CASCADE + GTRL	.4930
random	.3074
optimal	.6695

Table 11: Average reinforcement for simulated mobile robot environment over 100 runs of length 2000.

on the real mobile robot. Unfortunately, it is difficult to conduct such trials fairly in the physical system. The first problem is that a human must intervene whenever the robot approaches the light source and move the robot to a new location. The second problem is that it takes a long time to conduct the experiments. The time that it takes the robot to move greatly dominates the computation time of the learning algorithms. So, instead of trials on the real robot, we must substitute a simulation of the robot and its domain described above. The simulation is not of high fidelity, which causes this to be a substantially different problem than that of running on the actual robot. Still, it serves as an interesting and slightly complex domain for testing reinforcement-learning algorithms. Also, the results in the simulated domain mirror informal impressions of the relative performance of the algorithms on the actual robot.

In the robot simulation, noise is added to the action and perception of the robot. Each action of the simulated robot is, with probability .1, changed to a randomly chosen action; each perception of the state of the world is, with probability .1, changed to a randomly chosen world state. Whenever the robot reaches the light source in the simulated world, the light is "teleported" to a new randomly-chosen location.

The results of running each algorithm for 100 runs of length 2000 are shown in Table 11. The optimal expected reinforcement value was estimated by running a hand-crafted non-learning behavior in the environment under the same conditions as the experimental algorithms. Similarly, the expected reinforcement of a random strategy was estimated by running a random strategy in the world. All of the

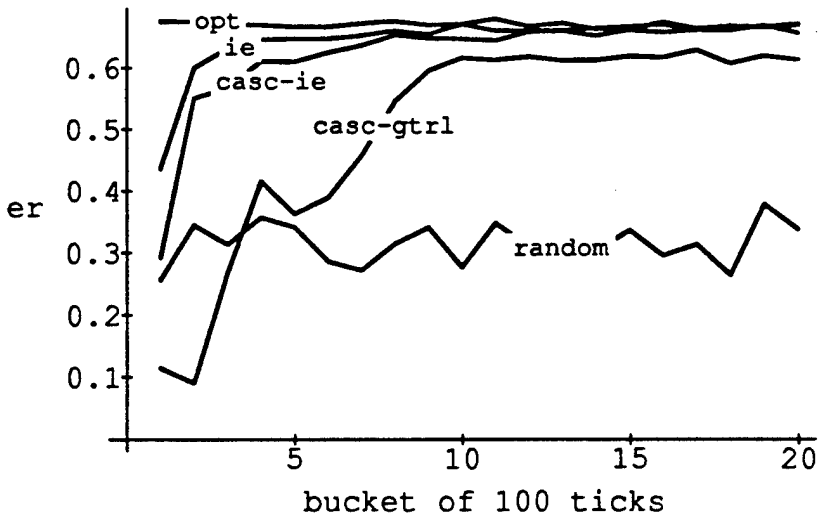


Figure 74: Learning curves for the simulated mobile robot task.

differences in expected reinforcement are significant. There is only a small difference in performance between the pure IE algorithm and the cascaded version, but the GTRL algorithm performs markedly worse than either of them. As we can see in the learning curves, shown in Figure 74, the GTRL algorithm takes longer to converge to its maximum performance, which is lower than optimal because it is continually trying new hypotheses.

10.3 Robot Domain with Delayed Reinforcement

The previous mobile robot domain can be complicated by giving the robot a large reinforcement only when it reaches the light source. This problem is considerably more difficult than other domains used for delayed reinforcement, such as the cart-pole domain. In the cart-pole domain, the robot receives a large negative reinforcement value whenever the pole falls over. In the absence of a good control strategy, the pole will fall over quite readily, giving the learner a lot of good data early on. In this robot domain, the robot may execute its initial random strategy for a very long time before it accidentally encounters the light source. Informal experiments with the real mobile robot were only successful if a human took an active role near the

beginning of the run, putting the robot in situations from which it was relatively easy to reach the light and, therefore, get useful reinforcement data.[1]

This section will report formal experiments carried out in a simulated version of the robotic domain with delayed reinforcement.

10.3.1 Algorithms

This experiment compares the same algorithms as were compared in the experiment described in Section 9.6: Q, IEQ, and AHC + IE. The parameter settings were

Q: $\alpha = .95$, *init*= 20

IEQ: $\alpha_{ie} = .01$, $\delta = .9999$

AHC + IE: $\alpha = .1$, $\lambda = .2$, $\delta = .9999$, $\alpha_{ie} = .05$

10.3.2 Task

The inputs and outputs available to the agent remain the same as in the local reinforcement task. The reinforcement generated by the world is, in this domain, global rather than local. When the agent comes very close to the light source, it is given reinforcement that is normally distributed with mean 10 and standard deviation 2.0; when it bumps into an obstacle, it is given reinforcement normally distributed with mean -2 and standard deviation 0.25; finally, if it neither bumps into the wall or comes near the light, it is given reinforcement normally distributed with mean 0 and standard deviation 0.25. When the light is reached by the robot, it is randomly moved to a new location.

10.3.3 Results

The results of running each algorithm for 10 runs of length 10,000 are shown in Table 12. As before, the optimal expected reinforcement value was estimated by

[1]This process is an instance of a class of methods for expediting learning that are referred to by psychologists [32] as "shaping." Its use in the robot domain described here was suggested by R. Sutton.

ALG	*er*
Q	.1634
IEQ	.1828
AHC + IE	.3651
random	.0000
optimal	.8269

Table 12: Average reinforcement for simulated robot domain with delayed reinforcement over 10 runs of length 10,000.

running a hand-crafted non-learning behavior in the environment under the same conditions as the experimental algorithms. Similarly, the expected reinforcement of a random strategy was estimated by running a random strategy in the world. The performance of AHC + IE was significantly better than that of Q or IEQ, which were not significantly different from one another. The learning curves for this domain are shown in Figure 75. The poor performance of the algorithms in this domain may be somewhat deceiving. In many cases, the learning strategies learned quickly to perform at near-optimal levels. However, in many other cases, the robot never, or only late in the run, acquired enough experience with the light source to learn an appropriate strategy. It is likely that if the runs were another order of magnitude longer than those reported here, the asymptotic performance of each of the algorithms would be very high. For this reason, a "shaping" process used early in the runs would allow the agent to get more useful information and hence improve its performance. An interesting area for future research would be to formally specify such shaping processes and characterize their role in expediting learning.

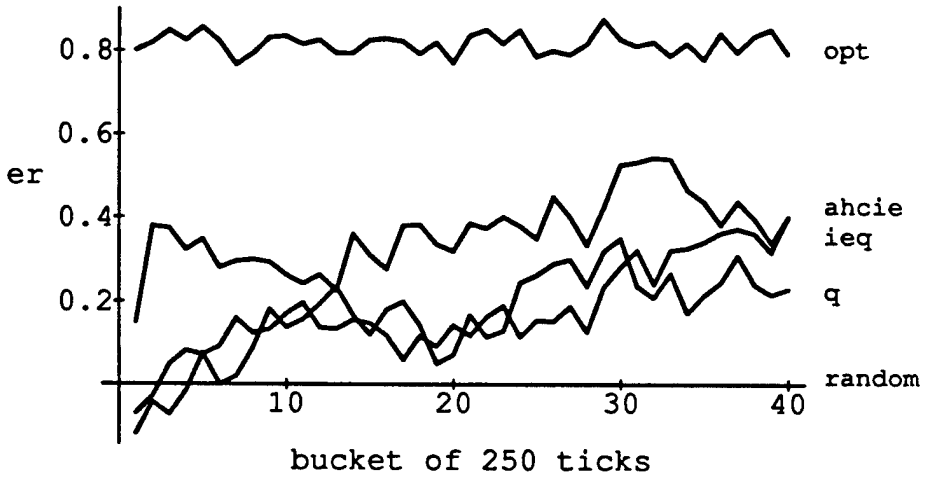


Figure 75: Learning curves for the simulated delayed-reinforcement mobile robot task.

Chapter 11

Conclusion

Simple reinforcement-learning problems can be effectively solved using the interval-estimation algorithm. It has two serious limitations, however. First, its computational complexity increases exponentially in the size of the input and output spaces. Second, it exhibits no generalization across input and output instances.

These problems have been addressed by the use of linear-association and error back-propagation methods for associative reinforcement-learning. Each of these methods has its own problems. The linear-association method can only learn action maps that are in the class of linearly-separable functions. Error backpropagation methods can, theoretically, learn functions of arbitrarily complexity, but it generally requires a large number of presentations of the learning data and is very sensitive to internal parameter values.

This dissertation has addressed the problem of finding new algorithms for efficiently learning limited classes of action maps from reinforcement.

The first step was to simplify the job of the algorithm designer by reducing the problem of learning action maps with many output bits to the problem of learning action maps with single output bits. The CASCADE method implements this problem reduction, providing decreased time complexity and improved learning rates, as well.

Valiant's algorithm for learning Boolean functions in k-DNF provided a useful foundation for creating new reinforcement-learning algorithms. The LARCKDNF and

IEKDNF algorithms integrate the ideas of linear-associative reinforcement-comparison and of interval-estimation with Valiant's methods. These new algorithms efficiently learn action maps in k-DNF: they are both more time-efficient than the raw IE algorithm, require fewer presentations of data than the BP algorithm, and can learn a large class of functions than linear-associative approaches.

The GTRL algorithm is also an algorithm for learning Boolean functions from reinforcement. Its main advantage is that it can learn low-complexity functions very efficiently; however, by changing internal parameter values, it can be configured to learn a variety of different classes of functions with different computational complexities. In addition, its use of internal symbolic representations allows it to be extended to learn simple sequential networks.

All of this work has only addressed the problem of local learning from immediate reinforcement. Existing work on temporal difference methods can also be seen as a problem reduction. It reduces the problem of global learning from delayed reinforcement to the problem of local learning from non-stationary immediate reinforcement. This perspective allows TD methods to be integrated with any available local learning method.

All of these methods can be integrated in various ways, such as using the CASCADE and TD problem reductions together with the GTRL, LARCKDNF, or IEKDNF algorithms to construct an algorithm that learns an action mapping with many output bits from delayed reinforcement. These combined methods have been tested and shown to work robustly on a physical mobile robot, demonstrating their applicability to embedded systems in the real world.

The rest of this chapter consists of two sections. The first briefly lists the novel contributions of the work described in this dissertation. The second discusses directions for extending this research.

11.1 Contributions

The work described in this dissertation has made a number of contributions to solving the problem of learning in embedded systems. They are summarized below,

organized in the order in which they were presented.

Foundations: The description of the foundations of reinforcement learning integrates existing work in dynamic programming, learning-automata theory, statistics, and previous work on the foundations of reinforcement learning into a general framework for describing learning behaviors and measuring their success. In addition to making the existing work more accessible to AI researchers, this formulation makes it easier for researchers to compare their results directly and to share implementations of learning behaviors and of simulated environments.

Interval Estimation Algorithm: The interval estimation algorithm is a novel extension of existing methods for reinforcement learning that is grounded directly in statistical theory. In empirical tests, it learns more effectively than other algorithms of its kind. However, its computational complexity makes it impractical for use on large problems.

Cascade Method: The cascade method of building a reinforcement learner with many output bits from a collection of reinforcement learners with one output bit is new. It has been shown that if each of the individual components has learned to perform the behavior that is correct for it, the entire system will perform the behavior that is correct overall. The cascade method works well in empirical tests, often resulting in improved convergence rates as well as lower time complexity.

Reinforcement Learning of k-DNF: Two algorithms are presented that learn Boolean functions from reinforcement, based on Valiant's concept learning algorithm for concepts expressible in k-DNF. One uses the techniques of the interval estimation algorithm, while the other is derived from Sutton's linear-association reinforcement-comparison algorithm. They are both computationally much more efficient than standard methods and perform nearly as well as standard methods on a variety of tasks.

Generate-and-Test Reinforcement Learner: The GTRL algorithm is a novel reinforcement-learning method that uses syntactic search through the space of Boolean function descriptions to learn single-bit output functions from reinforcement. It is based on, but has diverged significantly from, Schlimmer's STAGGER system, using statistical measures of necessity and sufficiency to guide its search. It is highly configurable and can learn low-complexity functions very efficiently, even in the presence of a large number of irrelevant attributes.

Action Maps with State: The generate-and-test reinforcement learner was extended by adding *set-reset* as an additional binary operator. This extension allows simple action maps whose output depends on input values from arbitrarily far back in history to be constructed. Although the method cannot generate all possible sequential networks, it does represent a first effort at learning action maps with state directly from reinforcement.

Delayed Reinforcement: Two existing approaches to learning from delayed reinforcement were combined with the interval-estimation method to yield robust algorithms. Watkins' Q-learning method was extended to use the techniques of the interval-estimation method to keep the algorithm from converging prematurely to suboptimal solutions. In addition, Sutton's AHC method of learning to generate a local reinforcement signal was tested with the IE algorithm as the local learning component.

Mobile Robot Experiments: Many of the algorithms described here were tested on a mobile robot in a moderately complex and noisy physical environment. In these experiments, the algorithms were successfully used to learn control strategies and exhibited considerable robustness.

11.2 Future Work

There is a long list of interesting variations and extensions that could be made to the work described in this dissertation. Many of them are suggested at the ends of

the relevant chapters. As well as considering local improvements, it is important to understand the global setting of this work.

Tabula rasa learning, as described in this dissertation, may not be a sufficient method for creating intelligent embedded agents. However, the methods of reinforcement learning may be used in concert with other knowledge provided in different forms by a human programmer, in order to construct agents that start with a useful base of knowledge and can improve upon it. Knowledge might be provided by programmers in a number of different forms.

One of the simplest kinds of information that would improve the performance of reinforcement-learning algorithms is the expected reinforcement of the optimal policy. An agent that has this information can use it to make more informed tradeoffs between acting to gain information and acting to gain reinforcement. The agent will be able to tell when it has found the best policy and need not experiment further.

Russell [59] has introduced the idea of using *determinations* to bias learning. Determinations are, essentially, descriptions of which input values the outputs depend on. Such information would be of great help in constraining the search done by the GTRL algorithm or in limiting the size of the set of conjunctive terms in the k-DNF algorithms.

Finally, we might start from a complete or partial program specified in terms of condition-action rules. An interesting research direction would be to develop representations of programs that are amenable to adjustment using reinforcement-learning methods.

Appendix A

Statistics in GTRL

This appendix describes three statistical modules that can be used with the GTRL algorithm. They can be applied when reinforcement is binomially or normally distributed, as well as in cases for which there is no model. Each module implements the statistical functions described in Section 7.3.

A.1 Binomial Statistics

Each hypothesis has the following set of statistics associated with it:

b_0 The number of times this hypothesis has agreed with the action 0 (not necessarily chosen by it) and received reinforcement value 0 (mnemonically "bad 0").

b_1 The number of times this hypothesis has agreed with the action 1 and received reinforcement value 0.

g_0 The number of times this hypothesis has agreed with the action 0 and received reinforcement value 1 (mnemonically, "good 0").

g_1 The number of times this hypothesis has agreed with the action 1 and received reinforcement value 1.

pb_0 The number of times this hypothesis has chosen the action 0 and received reinforcement value 0 (mnemonically, "predicted bad 0").

pb_1 The number of times this hypothesis has chosen the action 1 and received reinforcement value 0.

pg_0 The number of times this hypothesis has chosen the action 0 and received reinforcement value 1 (mnemonically, predicted good 0).

pg_1 The number of times this hypothesis has chosen the action 1 and received reinforcement value 1.

The procedure for updating these statistics should be apparent from the descriptions given above.

Given this data structure, we can define the statistical functions as follows:

$$\begin{aligned}
age(h) &= b_0 + b_1 + g_0 + g_1 \\
er(h) &= \frac{g_0 + g_1}{b_0 + b_1 + g_0 + g_1} \\
er\text{-}ub(h) &= ub(g_0 + g_1, b_0 + b_1 + g_0 + g_1) \\
erp(h) &= \frac{pg_0 + pg_1}{pb_0 + pb_1 + pg_0 + pg_1} \\
erp\text{-}ub(h) &= ub(pg_0 + pg_1, pb_0 + pb_1 + pg_0 + pg_1) \\
N(h) &= \frac{g_0}{g_0 + b_0} \\
S(h) &= \frac{g_1}{g_1 + b_1}
\end{aligned}$$

where the upper-bound function, ub, is defined [36] as

$$ub(x,n) = \frac{\frac{x}{n} + \frac{z^2_{\alpha/2}}{2n} + \frac{z_{\alpha/2}}{\sqrt{n}}\sqrt{\left(\frac{x}{n}\right)\left(1 - \frac{x}{n}\right) + \frac{z^2_{\alpha/2}}{4n}}}{1 + \frac{z^2_{\alpha/2}}{n}} .$$

The parameter $z_{\alpha/2}$ is used to determine the size of the confidence interval for computing ub.

A.2 Normal Statistics

Each hypothesis has the following set of statistics associated with it:

n_0 The number of times this hypothesis has agreed with the action 0 (not necessarily chosen by it).

n_1 The number of times this hypothesis has agreed with the action 1.

s_0 The sum of the reinforcement values received when the hypothesis has agreed with the action 0.

s_1 The sum of the reinforcement values received when the hypothesis has agreed with action 1.

ss The sum of the squares of the reinforcement values received when the hypothesis has agreed with the action taken.

n_p The number of times this hypothesis has chosen an action.

s_p The sum of reinforcement values received when the hypothesis has chosen an action.

ss_p The sum of the squares of the reinforcement values received when the hypothesis has chosen an action.

The procedure for updating these statistics should be apparent from the descriptions given above.

Given this data structure, we can define the statistical functions as follows:

$$\begin{aligned} age(h) &= n_0 + n_1 \\ er(h) &= \frac{s_0 + s_1}{n_0 + n_1} \\ er\text{-}ub(h) &= nub(n_0 + n_1, s_0 + s_1, ss) \\ erp(h) &= \frac{s_p}{n_p} \\ er\text{-}ub(h) &= nub(n_p, s_p, ss_p) \end{aligned}$$

$$N(h) = \frac{s_0}{n_0}$$
$$S(h) = \frac{s_1}{n_1}$$

where the normal upper-bound function, *nub*, is defined as

$$nub(n, \sum x, \sum x^2) = \bar{y} + t_{\alpha/2}^{(n-1)} \frac{s}{\sqrt{n}}$$

where $\bar{y} = x/n$ is the sample mean,

$$s = \sqrt{\frac{n\sum x^2 - (\sum x)^2}{n(n-1)}}$$

is the sample standard deviation, and $t_{\alpha/2}^{(n)}$ is Student's t function with $n-1$ degrees of freedom [69]. The parameter $z_{\alpha/2}$ is used to determine the size of the confidence interval for computing *nub*.

A.3 Non-parametric Statistics

This statistical module is parametrized by w, the window size, as well as by the confidence-interval parameter $z_{\alpha/2}$. The parameter w controls the size of the data buffers kept by the module. Because this method employs no summary statistics, all of the data for the last w ticks are stored in this module. Each hypothesis has the following set of statistics associated with it:

n The number of times this hypothesis has agreed with the action taken.

r_t A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action taken, sorted increasing by time received.

r_v A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action taken, sorted increasing by value.

n_0 The number of times this hypothesis has agreed with the action 0.

r_{t0} A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action 0, sorted increasing by time received.

r_{v0} A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action 0, sorted increasing by value.

n_1 The number of times this hypothesis has agreed with the action 1.

r_{t1} A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action 1, sorted increasing by time received.

r_{v1} A list of the reinforcement values of the last w ticks on which this hypothesis agreed with the action 1, sorted increasing by value.

n_p The number of times this hypothesis has chosen the action.

r_{tp} A list of the reinforcement values of the last w ticks on which this hypothesis chose the action, sorted increasing by time received.

r_{vp} A list of the reinforcement values of the last w ticks on which this hypothesis chose the action, sorted increasing by value.

Updating these statistics is slightly more complex that in the previous cases. The n's are simply incremented appropriately. As long as the n value is less than or equal to w, new data are simply inserted into the appropriate places in the lists. Once n is greater than w, on each tick, the first element of r_t is removed from both r_t and r_v, and the new reinforcement value is inserted into the resulting r_v and put on the end of the resulting r_t. This keeps the window of data sliding along. We need r_t in order to know which element to remove from r_v before we can add a new element.

Given this data structure, we can define the statistical functions, using the ordinary sign test [26], as follows:

$$\begin{aligned}
age(h) &= n \\
er(h) &= r_v[\lfloor \min(w,n)/2 \rfloor] \\
er\text{-}ub(h) &= r_v[\min(w,n) - u] \\
erp(h) &= r_{vp}[\lfloor \min(w,n_p)/2 \rfloor] \\
er\text{-}ub(h) &= r_{vp}[\min(w,n_p) - u] \\
N(h) &= r_{v0}[\lfloor \min(w,n_0)/2 \rfloor] \\
S(h) &= r_{v1}[\lfloor \min(w,n_1)/2 \rfloor]
\end{aligned}$$

where value u is chosen to be the largest value such that

$$\begin{pmatrix} \sum_{k=0}^{u} \\ nk.5^n \leq \alpha/2 \ . \end{pmatrix}$$

For large values of n, u can be approximated using the normal distribution.

Appendix B

Simplifying Boolean Expressions in GTRL

This appendix describes the Boolean canonicalization and simplification rules that are used in the GTRL algorithm. It is assumed that simplification happens when a conjunction, disjunction, or set-reset expression is being constructed and that the arguments have already been simplified and canonicalized. The algorithm is described as first constructing the combined hypothesis, then testing to see if has depth appropriate to the level of the algorithm for which it was constructed. In fact, the procedures for constructing composite hypotheses simply return `nil` if any applicable simplification rules can be found.

The disjunctive hypothesis $e_1 \vee e_2$ can be simplified to a lower level of complexity if any of the following statements is true (e stands for any expression):

$$e_1 = e_2$$
$$e_1 = \textbf{false}$$
$$e_1 = \textbf{true}$$
$$e_2 = \textbf{false}$$
$$e_2 = \textbf{true}$$
$$e_1 = \neg e_2$$
$$e_2 = \neg e_1$$

$$
\begin{aligned}
e_1 &= e_2 \vee e \\
e_1 &= e \vee e_2 \\
e_2 &= e_1 \vee e \\
e_2 &= e \vee e_1 \\
e_1 &= e_2 \wedge e \\
e_1 &= e \wedge e_2 \\
e_2 &= e_1 \wedge e \\
e_2 &= e \wedge e_1
\end{aligned}
$$

The conjunctive hypothesis $e_1 \wedge e_2$ can also be simplified in any of the situations described above. The set-reset hypothesis $\mathrm{SR}(e_1, e_2)$ can be simplified in all of the situations described above, except the ones in which $e_1 = e_2 \wedge e$ or $e_1 = e \wedge e_2$. To see this, note that $\mathrm{SR}(a, a \wedge b) = SR(a, b)$ because setting takes priority, but $\mathrm{SR}(a \wedge b, a)$ cannot be reduced.

Canonicalization consists of ordering the two top-level subexpressions, because they are assumed to have already been canonicalized. An arbitrary ordering is defined on operators; atomic expressions referring to input bits are ordered according to their index into the input vector. The expression e_1 is less than expression e_2 if and only if

- e_1 and e_2 are both atoms and $e_1 < e_2$;
- e_1 is an atom and e_2 is not;
- neither e_1 nor e_2 is an atom and the top level operator of e_1 is less than the top level operator of e_2;
- neither e_1 nor e_2 is an atom, they both have the same top-level operator, and the first subexpression of e_1 is less than (under this definition) the first subexpression of e_2; or
- neither e_1 nor e_2 is an atom, they both have the same top-level operator, they both have the same first subexpression, and the second subexpression of e_1 is less than (under this definition) the second subexpression of e_2.

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