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BASIC CONCEPTS OF SPECTRAL ESTIMATION USING A UNIFORM LINEAR PHASED ARRAY

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Abstract

Understanding the concepts of spectral estimation may become increasingly important in future radar system design. The purpose of this technical memo is to introduce these concepts. A simple model is formulated using a uniformly spaced linear phased array, and the classical Fourier, Maximum Entropy (ME), and Multiple Signal Classification (MUSIC) techniques are described. The performance of these three techniques is discussed through qualitative comparison.

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I. Introduction

Many spectral estimation methods have been developed to determine the spatial distribution of sources in the object field of an antenna system. These techniques are useful for both resolving the presence of closely spaced sources, and detecting individual point scatterers on a single source for imaging applications. The intent of this technical memo is to provide the necessary background to understand these techniques. The presentation begins with the development of a simple model using a uniform linear phased array as a basis for the concepts. Three different spectral estimation techniques are discussed: the classical Fourier method, the maximum entropy(ME) method, and the multiple signal classification(MUSIC) method. The Fourier method is often termed conventional and non-parametric, while the ME and MUSIC methods are parametric and typically referred to as superresolution techniques. The performances of these techniques are compared qualitatively with respect to their underlying concepts, with reference to the open literature. The presentation is restricted to the narrowband case with all signals having the same carrier frequency.

II. Model Formulation

Consider the linear phased array shown in figure 1. The arriving wavefront at the aperture is the resultant sum of M radiating sources, and is sampled by each of the N elements. The complex amplitude of the noiseless signal s(n) at each element may be expressed as

(1)
$$s(n) = \sum_{m=1}^{M} A_m \exp\left[jkd\left(n - \frac{N+1}{2} \right) \sin(\varphi_m) \right]$$

where N is the total number of elements, M is the total number of arriving signals, $k = 2\pi/\lambda$ is the wave number, d is the element spacing, A_m is the complex amplitude of the m^{th} signal, and is the angle of arrival of the m^{th} signal. The array element locations are referenced to the center of the array as depicted in figure 1.

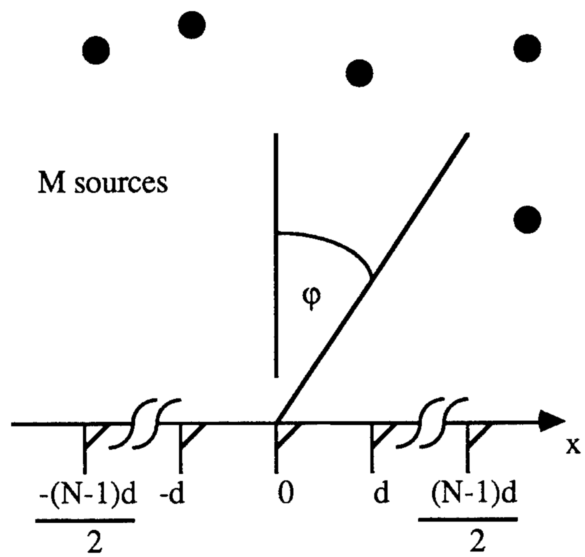


figure 1.

The signal measured at each array element is perturbed by additive noise. The noise is principally introduced by the antenna element and associated receive module, and is normally characterized by their combined noise figure. In practice, the noise is modeled as a white, stationary, ergodic random process [2],[3]. For the narrow band case, the complex amplitude of the noise signal present at each element is

$$w(n) = B_n \exp[-j\beta_n] \tag{2}$$

where the amplitude B_n is Rayleigh distributed, and the phase β_n is uniformly distributed over $(0,2\pi)$. The complex signal w(n) is Gaussian distributed, normally with zero mean. Combining equations (1) and (2), the observed signal at each element is

$$x(n) = s(n) + w(n) \tag{3}$$

The following sections are based on the simple model presented in this section.

III. Classical Fourier Approach to Spectral Estimation

Consider the noise free expression for the signal at each element given in equation (1). An interesting base-line approach may be taken to estimate the relative strength and angle of arrival of the m[th] signal. Recalling a common approach used in antenna system design, the far field expression of the aperture radiation pattern may be determined by the Fourier transform of the current distribution across the aperture. In this case, the far field source distribution may be determined from the inverse Fourier transform of the aperture excitation current. That is,

$$f(u) = \int_{-L/2}^{L/2} i(x) \exp[-jkxu]\, dx \tag{4}$$

where f(u) is the normalized complex far field source distribution, i(x) is the normalized complex aperture excitation current, L is the total length of the aperture, and $u = \sin(\varphi)$. For a uniformly spaced linear array, the aperture excitation may be expressed as a uniformly sampled continuous aperture excitation. Thus,

$$i(x) = i_c(x) \sum_{n=1}^{N} \delta\left(x - d\left(n - \frac{N+1}{2}\right)\right) \tag{5}$$

where $i_c(x)$ is the continuous current excitation. Combining equations (4) and (5) results in

$$f(u) = \sum_{n=1}^{N} i(n) \exp\left[-jkd\left(n - \frac{N+1}{2}\right)u\right] \tag{6}$$

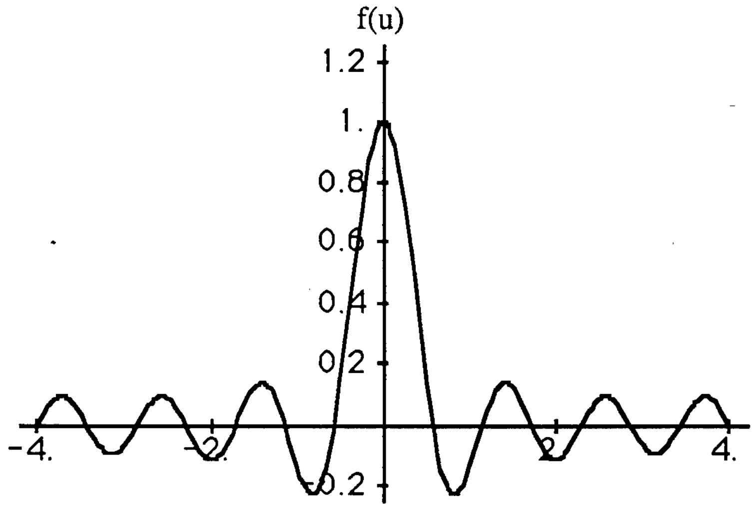
Observing equations (1) and (6), it is apparent that s(n) is proportional to i(n). Substituting s(n) into equation (6), and considering only the m[th] signal gives

$$f(u) = A_m \sum_{n=1}^{N} \exp\left[-jkd\left(n - \frac{N+1}{2}\right)(u - u_m)\right] \tag{7}$$

where $u_m = \sin(\varphi_m)$. Note that equation (7) is a geometric series, and may be simplified to

$$f(u) = A_m \frac{\sin\left(N\frac{(u-u_m)}{2}\right)}{\sin\left(\frac{u-u_m}{2}\right)} = A_m f_o(u - u_m) \tag{8}$$

where $f_o(u-u_m)$ represents the radiation pattern of a uniformly illuminated linear array, whose beam may be steered to a position φ_m by introducing a linear phase shift across the aperture. A plot of $f_o(u-u_m)$ with u_m=0, is shown in figure 2.



u in radians (N=11)

figure 2

The first sidelobe is -13dB relative to the main lobe, and successive sidelobes fall off at 6dB per octive. This pattern illustrates many of the problems associated with spatial processing, and in particular those pertaining to the performance of the Fourier technique. It is apparent that a main lobe will occur in the direction of a defined angle of arrival, but ambiguities exist in the directions of the sidelobes. In addition, the resolving power is determined by the width of the main lobe. When two or more sources are present, superposition may be applied to express the angle spectrum of the total field present at the aperture.

(9)
$$F(u) = \sum_{m=1}^{M} A_m \frac{\sin\left(N\frac{(u-u_m)}{2}\right)}{\sin\left(\frac{u-u_m}{2}\right)}$$

Equation (9) represents M complex weighted radiation patterns positioned throughout the angle spectrum corresponding to M discrete angles of arrival. Consider the simple case of two returns whose radiation patterns overlap. It is possible that the presence of both returns may not be detected, depending upon their relative strength and phase alignment. Thus, in the case of resolving multiple returns having overlapping radiation patterns, the Fourier approach appears to have considerable limitations. One way to improve the performance is to taper the aperture distribution[3]. This may reduce the sidelobe level considerably, at the expense of widening the main lobe. There are a variety of aperture tapers presented in the literature that realize this exchange[3]. The effective performance may be extended further by adaptively controlling the aperture taper. Thus, the trade-off between the sidelobe level and the width of the main lobe could adapt to the situation. The resolving power of the classical Fourier approach is at best on the order of a standard 3dB beamwidth(λ/L), depending upon the relative source strength and phase alignment, and the SNR[3].

Despite the limitations presented, there are some notable advantages to the Fourier approach. First, it is non-parametric in the sense that no known information of the received signal is necessary for implementation. Second, it is robust because it is relatively insensitive to parameter changes. And third, it is easy to implement making it a practical approach.

IV. Parametric Spectral Estimation

Consider the principles behind the basic Fourier approach just described. It is clear that its limitations result from the assumption that the field is zero beyond the aperture. The principal idea behind parametric spectral estimation (superresolution) is to not make this assumption. Alternatively, the signal information available at the aperture is used to estimate additional signal information beyond the aperture. The pioneering work in this area was performed by Burg [5] in 1967, and is commonly known as the Maximum Entropy Method(MEM). The maximum entropy(ME) approach appears throughout the literature as the basis for solving many spectral estimation problems. Since its inception, a number of superresolution techniques have been developed. Perhaps the most popular being the Multiple Signal Classification (MUSIC) algorithm developed by Schmidt[9] in 1979. As with most parametric spectral estimation techniques, both MUSIC and ME operate on the spatial correlation or covariance matrix of the observed signal. The covariance matrix may be developed by first summing equation (3) over N, and expressing it in a more compact form using matrix notation. That is,

(10) $$\bar{\mathbf{x}} = \bar{\mathbf{s}} + \bar{\mathbf{w}} = \mathbf{B}\bar{\mathbf{a}} + \bar{\mathbf{w}}$$

(11) $$\bar{\mathbf{x}} = \begin{bmatrix} x(1) \\ x(2) \\ | \\ x(N) \end{bmatrix} = \text{the observed signal vector,}$$

(12) $$\bar{\mathbf{s}} = \begin{bmatrix} s(1) \\ s(2) \\ | \\ s(N) \end{bmatrix} = \text{the received signal vector,}$$

(13) $$\bar{\mathbf{a}} = \begin{bmatrix} a(1) \\ a(2) \\ | \\ a(M) \end{bmatrix} = \text{the signal in space vector,}$$

(14) $$\mathbf{B} = \begin{bmatrix} \exp\left[j\varphi_1 \frac{1-N}{2}\right] & \exp\left[j\varphi_2 \frac{1-N}{2}\right] & \text{——} & \exp\left[j\varphi_M \frac{1-N}{2}\right] \\ \exp\left[j\varphi_1 \frac{3-N}{2}\right] & \exp\left[j\varphi_2 \frac{3-N}{2}\right] & \text{——} & \exp\left[j\varphi_M \frac{3-N}{2}\right] \\ | & | & & | \\ \exp\left[j\varphi_1 \frac{N-1}{2}\right] & \exp\left[j\varphi_2 \frac{N-1}{2}\right] & \text{——} & \exp\left[j\varphi_M \frac{N-1}{2}\right] \end{bmatrix}$$

= the direction matrix of dimension $N \times M$, and

(15) $$\bar{\mathbf{v}} = \begin{bmatrix} v(1) \\ v(2) \\ | \\ v(N) \end{bmatrix} = \text{the noise vector.}$$

The observed signal vector $\bar{\mathbf{x}}$ represents a single snapshot of data corresponding to a particular instant of time. In general, data is collected over a short period of time such that $\bar{\mathbf{x}}$ consists of a number of independent snapshots. Therefore the measured data normally available for processing is

(16) $$\bar{\mathbf{x}}(p) = \mathbf{B}\bar{\mathbf{a}}(p) + \bar{\mathbf{w}}(p) \qquad p = 1,2,...P$$

where P is the total number of snapshots. The angles of arrival are assumed not to change significantly in the processing interval, which is shown in equation (16) where the direction matrix **B** is not a function of p. The signal in space vector $\bar{\mathbf{a}}$ however, is a function of p because sources responsible for its generation are generally unpredictable, and therefore, modeled as stochastic processes[1]. Under this assumption, temporal averaging may be performed to improve the spectral estimation.

The spatial correlation of the observed signal vector $\bar{\mathbf{x}}$ is defined as

(17) $$\mathbf{R} = E[\mathbf{x}(p)\mathbf{x}^{+}(p)]$$

where E[-] is the expectation operator, and + denotes complex conjugate transpose. Substituting equation (16) into (17) and noting that the noise and signal are statistically independent results in

(18) $$\begin{aligned}\mathbf{R} &= E[\mathbf{B}\mathbf{a}\mathbf{a}^{+}\mathbf{B}^{+}] + E[\mathbf{w}\mathbf{w}^{+}] \\ &= \mathbf{B}\mathbf{R}_{a}\mathbf{B}^{+} + \mathbf{R}_{w} = \mathbf{R}_{s} + \mathbf{R}_{w}\end{aligned}$$

where $\mathbf{R}_a$, $\mathbf{R}_w$ and $\mathbf{R}_s$ are the spatial correlation matrices of the signal in space vector, noise vector, and received signal vector, respectively. If the noise is white with zero mean and variance σ^2, the spatial correlation matrix of the noise vector may be expressed as

(19) $$\mathbf{R}_{w} = \sigma^{2}\mathbf{I}$$

and equation (18) becomes

(20) $$\mathbf{R} = \mathbf{R}_{s} + \sigma^{2}\mathbf{I}$$

where **R** is a square matrix whose rank corresponds to the number of elements, and the total number of sources that may be detected is N-1. The mathematics are also valid for the single snapshot case.

A. Maximum Entropy (ME) Method

The principle behind the ME approach is to choose the spectrum f(u) whose inverse Fourier transform is i(x) for $|x| < L/2$, while making the fewest assumptions regarding unmeasured information in the region $|x| > L/2$. This condition is the information theory concept for maximizing entropy. From the work of Shannon in 1948, entropy was found to be proportional to the logarithm of the power spectrum, and is commonly expressed as

$$H = \int_{-\infty}^{\infty} \ln[S(u)]du \qquad (21)$$

where $S(u) = |F(u)|^2$. An extension of the classical Fourier approach described in the previous section may be thought of in terms of the Wiener-Khintchine theorem. This theorem states that the power spectral density and the autocorrelation function are Fourier transform pairs. Using the notation of equation (21), this may be stated as

$$R(\pm x_n) = \int_{-\infty}^{\infty} S(u) \exp[\pm jkx_n u]\, du \qquad n=0,1,\ldots,(N-1) \qquad (22)$$

where $R(x_n)$ is the autocorrelation function of the field across the array. Mathematically, the ME method consists of maximizing equation (21), while satisfying equation (22). For the case of the periodic array, the ME estimate of the power spectrum may be expressed in closed form [4] as

$$S_{MEM}(u) = \frac{d_{11}}{\left|[\mathbf{R}^{-1}\cdot\delta]^T \mathbf{W}(u)\right|^2} \qquad (23)$$

where d_{11} = (1,1) element of $\mathbf{R}^{-1}$, $\delta = [1,0,0,\ldots,0]^T$, and $\mathbf{W}(u)$ is the direction of look vector referenced from the center of the array. That is,

$$\mathbf{W}(u) = \begin{bmatrix} \exp[jkdu(1-N/2] \\ \exp[jkdu(3-n)/2] \\ \vdots \\ \exp[jkdu(N-1)/2] \end{bmatrix} \qquad (24)$$

The resolution performance of the ME approach when the true autocorrelation function is known is noise limited, and is supperior to the Fourier approach for both coherent and incoherent sources. It is shown in reference [8] that the resolution limit expressed in fractional beamwidths is proportional to $(SNR)^{-1/3.26}$ where $0.1\lambda/L$ resolution requires $\approx$ 33dB of SNR. This limitation has been found to be a result of uncompensated noise bias present in the diagonal elements of the autocorrelation matrix[6] (see equations (18) thru (20)). If the effect of the noise bias is compensated, angular resolution approaching $0.1\lambda/L$ may be achieved at much lower values of SNR[6],[7]. Thus, the ME approach, with or without noise bias compensation, will out-perform the classical Fourier approach for the case of incoherent sources, where higher SNR's are required in the absence of noise bias compensation.

B. Multiple Signal Classification (MUSIC) Method

The MUSIC method is a second generation spectral estimation technique that primarily consists of performing eigenstructure analysis on the spatial correlation matrix **R**. Specifically, there are two basic concepts behind the MUSIC method resulting directly from eigenstructure analysis. First, the number of sources present is determined from the multiplicity of the smallest eigenvalue of $\mathbf{R}_s$. Second, the directions of arrival are determined as those whose direction vectors are orthogonal to the eigenvectors corresponding to the smallest eigenvalue of $\mathbf{R}_s$. Thus, the first step of the MUSIC method is to determine the complex eigenvalues and corresponding eigenvectors of **R**. The eigenvectors are then arranged as columns of an array in accordance to increasing eigenvalues, and the sum of the outer products is computed as,

$$\mathbf{C}_{EV} = \sum_{i=1}^{L-q} \mathbf{V}_i \mathbf{V}_i^{+} \tag{25}$$

where $\mathbf{V}_i$ is the array of eigenvectors, L is the order of the eigenvector array, and q is the maximum number of sources that can be detected as determined by the multiplicity of the smallest eigenvalue of **R**. Examining equation (25) and noting the arrangement of $\mathbf{V}_i$, it is apparent that the summation is performed over only those eigenvectors that are expected to correspond to signals. Consequently, the signals are separated from the noise bias that remains in the standard ME method. Now the orthogonality principles may be applied to $\mathbf{C}_{EV}$, and the power spectrum resulting from the MUSIC method is determined to be

(26) $$S_{MU} = [\mathbf{W}^+(u)\mathbf{C}^+_{EV}\mathbf{C}_{EV}\mathbf{W}(u)]^{-1}$$

where **W**(u) is the direction of look vector (equation (24)). Schmidt[9] has compared the MUSIC technique to the standard ME method and the results indicate that the MUSIC method is superior. This is expected since the noise bias is compensated for in the procedure. Nevertheless, the MUSIC approach is not without limitations. Consider the covariance or spatial correlation matrix described by equation (20). In the absence of noise, $\mathbf{R} = \mathbf{R}_s$ is diagonal when the sources are uncorrelated, non-diagonal and non-singular when the sources are partially correlated, and non-diagonal and singular when some of the sources are fully correlated or coherent. From basic linear algebra it is apparent that this approach will not perform well in the presence of coherent sources. Consequently, matrix techniques that perform spatial smoothing on **R** have been developed to remove the singularities[10]. These techniques are successful in that when the MUSIC method is applied to the smoothed spatial correlation matrix, the performance in the coherent case approaches that of the incoherent case. There is however a penalty paid for having to perform smoothing. Spatial smoothing consists of diagonal sub-array averaging resulting in a reduction in the dimensions of **R**, and consequently a reduction in the number of sources that may be detected.

C. Performance of Autocorrelation Based Techniques

Up until now the performance of the ME and MUSIC methods have been discussed with respect to a priori knowledge of the true autocorrelation function. The performance of all autocorrelation based techniques such as these, may be characterized by recognizing that the autocorrelation function is not directly measured[6]. In practice, it is estimated from the measured field. Referring to equations (1) and (3), the normalized autocorrelation function between any two elements spaced $\mathbf{i}$d apart is estimated by

(27) $$\hat{R}(\mathbf{i}d) = \frac{1}{N-\mathbf{i}}\sum_{n=1}^{N-\mathbf{i}} x^*(n)x(n+\mathbf{i}) \qquad \mathbf{i}=0,1,2,....$$

For a two source case, the sampled values of the true autocorrelation are

(28) $$R(\mathbf{i}d) = \exp[jk\mathbf{i}du_1] + \exp[jk\mathbf{i}du_2] \qquad \mathbf{i}=0,1,2,....$$

where u_1 and u_2 are the angular positions of the point sources in the far field. Using equations (27) and (28), references [6] and [7] have thoroughly evaluated the ME method in the presence of two sources. The following discussion extends their results to all autocorrelation based techniques.

For the noise free case of two equal strength, unit amplitude, coherent point sources in the far field, the magnitude of the error in the estimated autocorrelation function is

$$|e(\mathbf{i}d)| = |\hat{R}(\mathbf{i}d) - R(\mathbf{i}d)| = \frac{2}{N-\mathbf{i}} \left| \frac{\sin\left(kd(N-\mathbf{i})\frac{u_1-u_2}{2}\right)}{\sin\left(kd\frac{u_1-u_2}{2}\right)} \right| |\cos(\phi)| \tag{29}$$

where ϕ is the phase difference referenced to the center of the array. Upon examining equation (29), some interesting conclusions can be made. Considering only the cosine term, the error is zero for ϕ equal to odd multiples of $\pi/2$, and maximum for ϕ equal to even multiples of $\pi/2$. Considering only the ratio of the sine terms, as the angular separation of the sources approaches zero, the error approaches $2\cos(\phi)$. Thus, the overall worst case error will be $e(\mathbf{i}d)=2$, which is on the order of R(0). Note that no improvement is achieved through temporal averaging in the noiseless coherent case; the performance is identical regardless of the number of snapshots used.

If noise is included in equation (29) the error will become worse. Temporal averaging can help reduce the effects of noise, but cannot eliminate it (this will be discussed further when incoherent sources are considered). Consequently, if the true autocorrelation is unknown, it appears that autocorrelation based methods do not afford overall better performance than that of the classical Fourier method in the case of coherent sources .

For the incoherent case, where the phase differences between the sources varies with time over $(0,2\pi)$, temporal averaging may be performed to reduce the error described by equation (29) to zero. Thus, if the duration of the averaging interval is long enough, it is possible to obtain the true values of the autocorrelation function. In this case the performance of the autocorrelation based methods are not all identical, and for the ME and MUSIC approaches, have been previously discussed in sections A. and B., respectively.

V. Conclusions

The basic concepts behind spectral estimation have been presented. The classical Fourier approach exhibits some implementation advantages that make it a practical choice for many low performance applications. Alternatively, the superresolution techniques are computationally intensive, but can offer better performance provided the limitations are understood. For the case of incoherent sources, the ME method with noise bias correction and the MUSIC method seem to be comparable, and both offer better performance than the classical Fourier approach. For the case of coherent sources, direct comparison is a bit more involved. Provided a priori knowledge of the true autocorrelation function is available, the ME method is comparable to the classical Fourier approach, and both are inferior to the MUSIC method when it is applied to a smoothed covariance matrix . If the true autocorrelation function is unknown, autocorrelation based methods such as ME and MUSIC, do not appear to afford overall better performance than that of the classical Fourier method for the case of coherent sources.

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