DTIC

7190-MA-01 ENV

MATHEMATICAL MODELS OF PHASE-CHANGE
IN SATURATED AND UNSATURATED POROUS MEDIA

Final Technical Report

by

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(April 1996)

United States Army
EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY
London England

R&D 7190-MA-01
Contract n. DAJA 45-93-C-0046
Contractor: European Consortium for Mathematics in Industry

TABLE OF CONTESTS

1. GENERAL OBJECTIVES

The research under contract DAJA45-93-C-0046 and its modifications was aimed at the study of several mathematical models for ground freezing processes.

The research started in September 1993 with Prof. Mario Primicerio as the principal investigator. During 1995 Prof. Primicerio had to quit the team, following his election to Mayor of Florence and he was replaced by Prof. Antonio Fasano.

The research has been carried out in close contact with Dr. Y. Nakano at CRREL and included two visits at CRREL by Dr. Federico Talamucci. The last one took place during one week in March/April 1996 and was very useful in drawing some conclusions and in pointing out further subjects which deserve further investigation.

We can summarize the specific objects of the research under the following statements:

(a) reviewing the extant models on ground freezing according to the classification in (i) models with a sharp interface between frozen and unfrozen region, (ii) models with a frozen fringe

(b) examining the possibility of describing heaving with or without ice lens formation by means of models of class (i)

(c) studying models of class (ii) under various degrees of approximation.

Models of type (i) will be also referred to as "primary frost heave models", while ground freezing with frozen fringes (class (ii)) will be called "secondary frost heave".

In the first stage it also included a study on thawing [1]. The work on classification of models has been very large and is incorporated in Talamucci's Doctoral Thesis [2]. It will be used to produce a survey paper to be submitted to Surveys of Mathematics for Industry.

2. PRIMARY FROST HEAVE MODELS

The study of models with a sharp interface between frozen and unfrozen region has been undertaken under an innovative point of view. It has to be stressed that not all the models of this kind presented in the literature can be considered thermodynamically consistent.

In [3] a correct model has been selected and treated under the assumption of quasi-steady temperature field, showing that it can account both for frost penetration and for lensing. The complete evolution problem with ice lens formation has been studied in [4], while the frost penetration case has been considered in [5]. In [6] a numerical work has been developed dealing with the situation described in [3] and showing the possible occurrence of frost penetration and of ice lensing.

3. SECONDARY FROST HEAVE MODELS

Then the research has been oriented towards the models with a frozen fringe, including the purely thermal-induced water flow mechanism, whose existence was experimentally pointed out at CRREL by Y. Nakano.

The amount of work done in this direction is quite remarkable, in particular considering the extreme complexity of the mathematical structure of the problem. We quote the study of quasi-steady solutions [7], [8], and a massive series of results concerning the cases of constant and time varying boundary date. It is particularly remarkable that it has been possible to give criteria to predict whether ice lensing or frost penetration is going to take place.

Part of these results have been comunicated by Y. Nakano and F. Talamucci at the EUROMECH Conference 333, held in Montecatini, near Florence, in June 1995 and chaired by A. Fasano.

The results achieved not only provide a sound mathematical basis to the frost heave model in which the water flow is driven by the complex thermodynamical evolution of the frozen fringe, but are also a useful tool to investigate the qualitative properties of the phenomenon.

The enclosed report by F. Talamucci entitled "A quasi-steady model for secondary frost heave in freezing soils" contains a detailed description of the frost heave model with a frozen fringe, encompassing the fundamental law stating that the flux intensity of migrating water is the sum of a Darcy-like term (i.e. the usual pressure driven flow) and of a term whose contribution is a flow in the direction opposite to the thermal gradient.

This very large work (almost 100 pages) is organized in four sections. Sections 1 and 2 resume the main results already communicated in the previous interim reports and presented at the EUROMECH Conference, namely: the complete description of the mathematical model and the study of the case in which a constant heat flux is prescribed at the boundaries of the porous layer. Sections 3 and 4 present some new material, i.e. the analysis of the time dependent data, prescribed either as heat flux or as boundary temperatures.

In all cases existence and uniqueness of the corresponding problem is proved and particular emphasis is given to the possible occurrence of the ice lensing phenomenon. From the mathematical point of view we have a set of partial differential equations for pressure, temperature, water flux in each of the three phases: the frozen and the unfrozen region and the intermediate layer in which ice and water coexist, i.e. the frozen fringe. The interfaces between these regions are free boundaries (i.e. they are unknown) and the peculiar behaviour of the problem is mainly due to the fact that the quantities listed above have to satisfy some constraints. The constraints are unilateral and they are responsible for discriminating between frost penetration and ice lens growth.

All this material will be reorganized in a series of scientific papers for publication in the next future.

4. CONCLUSIONS AND FURTHER PERSPECTIVES

As a conclusion, we can say that the work performed under the contract has produced a significant step forward in understanding and modelling the frost heave phenomenon, taking into account some new factors like e.g. the thermal driven water flux in the frozen fringe and the possible rearrangements of the grains of the porous matrix during freezing, generating a change of porosity.

From the mathematical point of view, a series of very difficult free boundary problems has been solved, providing in some cases also numerical results. Particularly valuable is the qualitative analysis, leading to the prediction of the type of evolution of the freezing process.

Among the new problems to be investigated, if some cooperation can be continued, is a finer analysis of the distribution of thermal energy within the frozen fringe. This research area can be still a source of very interesting material both from the experimental and the theoretical point of view.

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DTIC 7190-MA-01 ENV
DAJA 45-93-C-0046

A QUASI-STEADY MODEL FOR SECONDARY FROST HEAVE IN FREEZING SOILS

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Abstract: The subject of this paper is the mathematical investigation of a one-dimensional secondary frost heave model for a saturated soil with a frozen fringe. The water flux q_w is driven by both thermal and hydraulic gradients; the volumetric ice content ν is a known decreasing function of the temperature; the ice in the frozen fringe is at rest and at the atmospherice pressure; the occurring of the ice segregation process is determined by the value of the water pressure in the frozen fringe.

The mathematical analysis of the model has been developed assuming that the temperature profile is linear in each region of the soil (quasi-steady approach) and that the porous matrix of the frozen fringe is undeformable. Besides the formation of a pure ice layer, we consider the possibility that the freezing front moves through the soil without a macroscopic break of the porous matrix (frost penetration).

In sect. 1 and 2 we outline and discuss the equations of the model and we examine the case in which constant thermal fluxes α_0, α_1 at the base and on the top of the soil are prescribed. The main result is concerned with the possibility of predicting the kind of process (lens formation, frost penetration or melting) which will take place on the basis of the knowledge of the data α_0, α_1 and of the properties of the soil (thermal conductivities, densities, functions K_1,K_2, ν). A suitable way to present the result is to display in a (α_0,α_1)-plane the regions where lens formation or frost penetration occur.

In sect. 3 we consider the general case when the assigned thermal fluxes α_0, α_1 depend on time. The analysis of the model leads us to conclude that the change in time of the boundary fluxes may give rise to the formation of several lenses, in agreement with experimental observations. Indeed, if α_0, α_1 verify certain conditions involving the soil properties the process of lens formation can alternate with frost penetration.

In sect. 4 we examine the model when the temperatures are assigned at the boundaries of the soil. Conditions involving the boundary data and the known properties of the soil in order to discriminate the kind of process have been obtained also in this case. Moreover, taking boundary temperatures depending on time, the phenomenon of "banding" (formation of several lenses) is correctly simulated by the model.

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1. The mathematical model

1.1 *Basic assumptions*

Let us consider a sample of soil saturated with water (in both phases) and subjected to imposed gradients or temperatures on the top and on the bottom of the soil.

We observe three different regions: an unfrozen part, from the bottom up to the freezing front, a frozen fringe with grains of soil, water and ice, and a frozen part, with ice and grains.

In the frozen part pure segregated layers of ice can form (*ice lences*): in this case, we have that the upper boundary of the frozen fringe (base of the lens) is at rest. On the other hand, if a macroscopic break in the soil does not occur, we can observe the penetration of the freezing front and the movement of the frozen fringe towards the base of the soil (*frost penetration*).

Many models have been proposed in order to describe the phenomenon.

We are going to give a mathematical description of the freezing process following the model proposed in [3].

Let us take a vertical coordinate z, positive upwards, and choose the fixed base of the soil as $z = 0$.

We consider

$z_F(t)$ as the boundary between the unfrozen soil and the frozen fringe;

$z_S(t)$ as the boundary between the frozen fringe and the frozen soil;

$z_L(t)$ as the upper boundary of the forming (or just formed) ice layer, that is immediately over the frozen fringe

$z_T(t)$ as the top of the soil.

In order to take into account the coexistence of water and ice in the frozen fringe, we introduce the unknown ν, such that $\varepsilon\nu$ is the volumetric ice content in a unit volume of mixture ice-water-soil.

We have $\nu = 0$ in the unfrozen soil, $0 \le \nu < 1$ in the frozen fringe, $\nu = 1$ in the frozen soil.

We base our discussion on the following statements (see [3] and [4] for more details):

(H_1) The process can be described by a one-dimensional model.

(H_2) The porous matrix in the unfrozen soil is undeformable as well as the skeleton of the frozen fringe (porosity ε is constant in $0 \leq z \leq z_S(t)$).

(H_3) In the heat balance equations terms containing derivative of the temperature with respect to time and convective terms are neglected.

(H_4) Pore ice in the frozen fringe is at rest with respect to the porous matrix (no regelation); moreover, there are not capillary interfacial effects between ice and water in the frozen fringe.

(H_5) The water flux q_w is given by the Darcy's law in the unfrozen soil:

$$q_w = -K_0 \frac{\partial p_w(z,t)}{\partial z} \qquad\qquad 0 \leq z \leq z_F(t)$$

with a constant hydraulic conductivity K_0, while in the frozen fringe the water is driven by a couple of gradients [5]:

$$q_w^{ff} = -K_1(T)\frac{\partial p_w(z,t)}{\partial z} - K_2(T)\frac{\partial T(z,t)}{\partial z} \qquad z_F(t) \leq z \leq z_S(t)$$

where the empirical functions K_1 and K_2 are properties of the soil.

(H_6) The water pressure p_w at $z = z_S(t)$ attains in any case the value $\sigma \geq 0$, that takes into account the overburden pressure loaded over the soil. At the base of the soil the water pressure is the atmospheric one and p_w is continous through the boundary $z = z_S(t)$.

(H_7) The boundary $z_F(t)$ is the isotherm $T = 0$ ($T(z_F(t),t) = 0$); moreover, the temperature T is continuous with respect to the spatial coordinate z throughout the whole soil.

(H_8) The ice volumetric content $\nu = \nu(T)$ (volume of ice in a unit volume of mixture) is a given decreasing function of the temperature T, such that $\nu(0) = 0$ (this means that at $z = z_F(t)$ there is no change of phase).

(H_9) The quantities k_u ,k_f, k_i (thermal conductivities), ρ_w, ρ_i, ρ_s (specific densities) and L (latent heat per unit volume) are constant. Suffixes u, ff, f, w, i and s refer respectively to unfrozen soil, frozen fringe, frozen soil, water, ice and soil grains. On the other hand, the termal conductivity in the frozen fringe k_{ff} depends on the temperature T. Empirically, k_{ff} is determined as a function of ν, that is in turn a function of T alone by virtue of hypothesis (H_8).

The set of equations which describe the freezing process, valid both in case of lens formation and in case of frost penetration, is the following:

$$\frac{\partial}{\partial z}\left(\frac{\partial T(z,t)}{\partial z}\right)=0 \tag{1.1}$$

$$q_w^u(t)=-K_0\,\frac{\partial p_w(z,t)}{\partial z} \tag{1.2}$$

$\left.\right\}$ $0<z<z_F(t)$ *(unfrozen soil)*

$$\frac{\partial}{\partial z}\left(k_{ff}(T)\frac{\partial T(z,t)}{\partial z}\right)=0 \tag{1.3}$$

$$q_w^{ff}(z,t)=-K_1(T(z,t))\frac{\partial p_w(z,t)}{\partial z}\ -K_2(T(z,t))\frac{\partial T(z,t)}{\partial z} \tag{1.4}$$

$\left.\right\}$ $z_F(t)<z<z_S(t)$ *(frozen fringe)*

$$\frac{\partial}{\partial z}\left(\frac{\partial T(z,t)}{\partial z}\right)=0 \tag{1.5}$$

$z_S(t)<z<z_T(t)$ *(frozen soil)*

with the boundary conditions

$$\frac{\partial T(0,t)}{\partial z}=-\alpha_0(t)\quad \textit{or, alternatively,} \tag{1.6}$$

$$T(0,t)=h(t) \tag{1.6a}$$

$$p_w(0,t)=0 \tag{1.7}$$

$$[\![T(z_F(t),t)]\!]^+_- = T(z_F(t),t)=0 \tag{1.8}$$

$$[\![-k\frac{\partial T(z_F(t),t)}{\partial z}]\!]^+_- =0 \tag{1.9}$$

$$q_w^{ff}(z_F,t)^+ - q_w^u(t)=0 \tag{1.10}$$

$$[\![p_w(z_F(t),t)]\!]^+_- =0 \tag{1.11}$$

$$[\![T(z_S(t),t)]\!]^+_- =0 \tag{1.12}$$

$$L\rho_w q_w^{ff}(z_S(t),t)^- = (1-\nu_S)\varepsilon_0\rho_w L\dot z_S(t)+[\![-k\frac{\partial T(z_S(t),t)}{\partial z}]\!]^+_- \tag{1.13}$$

$$p_w(z_S(t),t)=\sigma \tag{1.14}$$

$$\rho_i\dot z_T(t)=\rho_w q_w^{ff}(z_S(t),t)^- +\varepsilon_0(1-\nu_S)(\rho_i-\rho_w)\dot z_S(t) \tag{1.15}$$

(1.16) $$\frac{\partial T(z_T(t),t)}{\partial z} = -\alpha_1(t) \quad \textit{or, alternatively,}$$

(1.16a) $$T(z_T(t),t) = g(t)$$

with the initial conditions

(1.17) $$z_S(0) = b$$

(1.18) $$z_T(0) = H > b.$$

The symbol $[\![\chi]\!]^+_-$ in (1.8), (1.9), (1.11)-(1.13) denotes the "jump" of the quantity χ passing through the height $\hat{z}$:

$$[\![\chi]\!]^+_- = \chi^+ - \chi^- = \lim_{z\to\hat{z}^+} \chi(z) - \lim_{z\to\hat{z}^-} \chi(z)$$

(obviously, $z\to\hat{z}^+$ and $z\to\hat{z}^-$ respectively mean left-hand and right-hand limit).
In (1.9) and (1.13) the thermal conductivity k takes the appropriate index u, ff, i or f.
We comment briefly the set of equations (1.1)-(1.18). We refer to [9] for more details.
Equations (1.1), (1.3), (1.5) are the heat balances in the hypotheses (H_3) and (H_{10}). We remark that the latent heat L does not appear in the heat balance (1.3), since we eliminated the derivative of the temperature T with respect to time t. In pratice, we are assuming that the energetic contribution due to the change of phase in the frozen fringe can be neglected, while the most remarkable effects of latent heat release are on the boundary $z = z_S(t)$, where (1.13) holds.
Equations (1.2) and (1.4) come from (H_5). Equations (1.6) and (1.16) prescribe the thermal fluxes $-\alpha_0(t)$, $-\alpha_1(t)$ (where α_0, $\alpha_1 > 0$) at the base and on the top of the soil, respectively. Alternatively, the boundary temperatures $h(t) > 0$, $g(t) < 0$ at the extremities of the soil can be assigned (equations (1.6a) and (1.16a)). The initial conditions (1.17) and (1.18) give the initial position of the freezing front z_S and the initial height of the soil, respectively. Notice that the initial thickness of the frozen fringe is unknown, since the initial position of the isotherm z_F is not given.
Equations (1.7), (1.11) and (1.14) derive from (H_6), while (1.8), (1.9) (heat balance at z_F), (1.10) (mass balance at z_F) and (1.12) come from (H_7).

Equations (1.13) and (1.15) are respectively the heat and mass balances at z_S (we put $\nu_S = \nu^-(T(z_S(t),t))$).
The unknow quantities of the problem are the temperature $T(z,t)$, $0 < z < z_T(t)$, the water pressure $p_w(z,t)$, $0 < z < z_S(t)$, the hydraulic fluxes $q_w^u(t)$, $0 < z < z_F(t)$ and $q_w^{ff}(z,t)$, $z_F(t) < z < z_S(t)$, the free boundaries $z = z_F(t)$, $z = z_S(t)$ and $z = z_T(t)$.

1.2 *Temperature*

Integrating in each region of the soil (1.1), (1.3), (1.5) and taking into account of the boundary conditions (1.6), (1.8), (1.9), (1.12) e (1.16), or (1.6*a*), (1.8), (1.9), (1.12) and (1.16*a*) we get the following formulas for the temperature $T(z,t)$.
In the case of prescription of heat fluxes at the extremities of the soil (equations (1.6) and (1.16)) we have

$$(1.19)\qquad \frac{\partial T(z,t)}{\partial z} = \begin{cases} -\alpha_0(t) & 0 \le z \le z_F(t) \\ -\dfrac{k_{ff}(T(z,t))}{k_u \alpha_0(t)} & z_F(t) \le z \le z_S(t) \\ -\alpha_1(t) & z_S(t) \le z \le z_T(t). \end{cases}$$

From (1.19) one gets:

$$(1.20)\qquad \begin{cases} T(z,t) = -\alpha_0(t)((z - z_F(t)) & 0 \le z \le z_F(t) & (\textit{unfrozen soil}) \\ \displaystyle\int_0^{T(z,t)} k_{ff}(\eta)d\eta = -k_u\alpha_0(t)(z - z_F(t)) & z_F(t) \le z \le z_S(t) & (\textit{frozen fringe}) \\ T(z,t) = -\alpha_1(t)(z - z_S(t)) + T_S(t) & z_S(t) \le z \le z_T(t) & (\textit{frozen soil}) \end{cases}$$

where $T_S(t) = T(z_S(t),t)$ satisfies

$$(1.21)\qquad \int_0^{T_S(t)} k_{ff}(\eta)d\eta = -k_u\alpha_0(t)(z_S(t) - z_F(t)).$$

On the contrary, if we prescribe the temperatures at the base and on the top of the soil (equations $(1.6a)$ $(1.16a)$), we have the following equations for $T(z,t)$:

$$(1.20a)\quad \begin{cases} T(z,t)=\dfrac{z_F(t)-z}{z_F(t)}h(t) & 0\le z\le z_F(t) \qquad (unfrozen\ soil) \\ \displaystyle\int_0^{T(z,t)} k_{ff}(\eta)d\eta=\frac{z_F(t)-z}{z_F(t)}k_u h(t) & z_F(t)\le z\le z_S(t) \qquad (frozen\ fringe) \\ T(z,t)=\dfrac{z-z_T(t)}{z_S(t)-z_T(t)}T_S(t)+\dfrac{z-z_S(t)}{z_T(t)-z_S(t)}g(t) & z_S(t)\le z\le z_T(t) \qquad (frozen\ soil) \end{cases}$$

where $T_S(t)$ is such that

$$(1.21a)\qquad \int_0^{T_S(t)} k_{ff}(\eta)d\eta=\frac{z_F(t)-z_S(t)}{z_F(t)}k_u h(t).$$

We remark that, as a matter of facts, we would have to introduce in the frozen soil further boundaries separating the layers of pure ice form the layers of soil grains saturated with ice. In [9] we showed that, if we assume that $k_i\approx k_f$, the "diffractions" at those boundaries can be eliminated and the temperature T in the frozen soil is a straight line with respect to z, as in (1.20) and in (1.20a).

1.3 *Water flux and pressure*

Integrating (1.2) and using (1.7) we obtain

$$(1.22)\qquad p_w(z,t)=-\frac{q_w(t)}{K_0}z, \qquad\qquad 0\le z\le z_F(t).$$

On the other hand, if we neglect in the mass conservation equation in the frozen fringe, that is, under the hypotheses (H_2) and (H_8)

$$(1.23)\qquad \varepsilon_0(\rho_i-\rho_w)\frac{\partial\nu}{\partial T}\frac{\partial T}{\partial t}+\rho_w\frac{\partial}{\partial z}q_w^{ff}(z,t)=0,$$

we neglect the term with the time derivative, in the spirit of the quasi-steady approach (H_1), we get, taking also into account (1.10):

$$(1.24)\qquad q_w^{ff}=q_w^{ff}(t)=q_w^u(t), \qquad\qquad z_F(t)\le z\le z_S(t).$$

We call $q_w(t)$ the function that represents the hydraulic flux at each time t from the base of the soil $z=0$ up to the boundary $z=z_S(t)$.
From (1.13), (1.15), (1.19) and (1.24) we deduce the mass and heat balances at the front z_S in the following form, in the case when the thermal fluxes α_0 and α_1 are assigned:

$$(1.25) \qquad L\rho_w q_w(t) = (1-\nu_S)\varepsilon\rho_w L\dot{z}_S(t) + k_f\alpha_1(t) - k_u\alpha_0(t).$$

$$(1.26) \qquad \rho_i\dot{z}_T(t) = \rho_w q_w(t) + \varepsilon_0(1-\nu_S)(\rho_i-\rho_w)\dot{z}_S(t)$$

In a similar way, if the boundary temperatures h and g are prescribed, (1.26) still holds, while instead of (1.25) we have:

$$(1.25a) \quad L\rho_w q_w(t) = (1-\nu_S)\varepsilon\rho_w L\dot{z}_S(t) - k_f\frac{g(t)-T_S(t)}{z_T(t)-z_S(t)} - \frac{k_u h(t)}{z_F(t)}$$

By means of (1.19) and (1.4), one finds the following formulas for the water pressure gradient in the *frozen fringe*:

$$(1.27) \quad \frac{\partial p_w(z,t)}{\partial z} = \frac{1}{K_1(T(z,t))}\left(\frac{K_2(T(z,t))k_u\alpha_0(t)}{k_{ff}(T(z,t))} - q_w(t)\right) \qquad z_F(t) \le z \le z_S(t)$$

in the case (1.6), (1.16);

$$(1.27a) \quad \frac{\partial p_w(z,t)}{\partial z} = \frac{1}{K_1(T(z,t))}\left(\frac{K_2(T(z,t))k_u h(t)}{k_{ff}(T(z,t))z_F(t)} - q_w(t)\right) \qquad z_F(t) \le z \le z_S(t)$$

in the case (1.6*a*), (1.16*a*).
Let us assign now (1.6) and (1.16) and integrate (1.40), with the boundary condition (1.14); we get, making use also of the second equation of (1.19):

$$(1.28) \qquad p_w(z,t) = \sigma + \int_{T(z,t)}^{T_S(t)} \frac{K_2(\eta) - q_w(t)\dfrac{k_u\alpha_0(t)}{k_{ff}(\eta)}}{K_1(\eta)}d\eta, \qquad z_F(t) \le z \le z_S(t)$$

Imposing the continuity of p_w through $z=z_F(t)$ (condition (4.1.11)) and taking into account of (1.8), we find:

$$(1.29)\qquad \sigma+\int_0^{T_S(t)} \frac{K_2(\eta)-q_w(t)\frac{k_u\alpha_0(t)}{k_{ff}(\eta)}}{K_1(\eta)}d\eta=-\frac{q_w(t)}{K_0}\,z_F(t)$$

Analogously, if we prescribe (1.6a) and (4.1.16a), equations (1.28) and (1.29) becomes respectively:

$$(1.28a)\qquad p_w(z,t)=\sigma-\int_{T(z,t)}^{T_S(t)} \frac{K_2(\eta)-\frac{q_w(t)z_F(t)}{k_uh(t)}k_{ff}(\eta)}{K_1(\eta)}d\eta,\qquad z_F(t)\le z\le z_S(t)$$

$$(1.29a)\qquad \sigma+\int_0^{T_S(t)} \frac{K_2(\eta)-\frac{q_w(t)z_F(t)}{k_uh(t)}k_{ff}(\eta)}{K_1(\eta)}d\eta=-\frac{q_w(t)}{K_0}\,z_F(t)$$

Since the water flux depends only on time, we can achieve a formula for q_w by evaluating (1.4) on $z=z_S(t)$:

$$(1.30)\qquad q_w(t)=-K_1(T_S(t))\frac{\partial p_w(z_S(t),t)}{\partial z}+K_2(T_S(t))\frac{k_u\alpha_0(t)}{k_{ff}(T_S(t))}$$

or

$$(1.30a)\qquad q_w(t)=-K_1(T_S(t))\frac{\partial p_w(z_S(t),t)}{\partial z}+K_2(T_S(t))\frac{k_uh(t)}{z_F(t)k_{ff}(T_S(t))}\,.$$

1.4 *The mathematical problem*

We have obtained in the case (1.6) and (1.16) the set of equations (1.21), (1.25), (1.26), (1.29) and (1.30) with the initial conditions (1.17) and (1.18); let us write again, by the sake of convenience, the equations:

$$
(\mathbb{S}_{fl})\begin{cases}
z_F(t) - z_S(t) = \dfrac{1}{k_u\alpha_0(t)}\displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta \\[2ex]
q_w(t) = (1-\nu_S)\varepsilon\dot z_S(t) + \dfrac{k_f\alpha_1(t) - k_u\alpha_0(t)}{L\rho_w} \\[2ex]
\rho_i\dot z_T(t) = \rho_w q_w(t) + \varepsilon(1-\nu_S)(\rho_i-\rho_w)\dot z_S(t) \\[2ex]
\sigma + \displaystyle\int_0^{T_S(t)} \dfrac{1}{K_1(\eta)}\left(K_2(\eta) - \dfrac{k_{ff}(\eta)q_w(t)}{k_u\alpha_0(t)}\right)d\eta + \dfrac{q_w(t)z_F(t)}{K_0} = 0 \\[2ex]
q_w(t) = -K_1(T_S(t))\dfrac{\partial p_w(z_S(t),t)}{\partial z} + \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}k_u\alpha_0(t) \\[2ex]
z_S(0) = b, \quad z_T(0) = H > b.
\end{cases}
$$

On the other hand, when the boundary temperatures are assigned (case (1.6a) and (1.16a)), the set of equations is given by (1.21a), (1.25a), (1.26), (1.29a) and (4.1.30a), together with the same initial conditions:

$$
(\mathbb{S}_{tmp})\begin{cases}
\displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta = \dfrac{z_F(t)-z_S(t)}{z_F(t)}k_u h(t) \\[2ex]
L\rho_w q_w(t) = (1-\nu_S)\varepsilon\rho_w L\dot z_S(t) - k_f\dfrac{g(t)-T_S(t)}{z_T(t)-z_S(t)} - \dfrac{k_u h(t)}{z_F(t)} \\[2ex]
\rho_i\dot z_T(t) = \rho_w q_w(t) + \varepsilon(1-\nu_S)(\rho_i-\rho_w)\dot z_S(t) \\[2ex]
\sigma + \displaystyle\int_0^{T_S(t)} \dfrac{K_2(\eta) - \dfrac{q_w(t)z_F(t)}{k_u h(t)}k_{ff}(\eta)}{K_1(\eta)}d\eta = -\dfrac{q_w(t)}{K_0}z_F(t) \\[2ex]
q_w(t) = -K_1(T_S(t))\dfrac{\partial p_w(z_S(t),t)}{\partial z} + K_2(T_S(t))\dfrac{k_u h(t)}{z_F(t)k_{ff}(T_S(t))} \\[2ex]
z_S(0) = b, \quad z_T(0) = H > b
\end{cases}
$$

Remark 1.1. We will assume that at the boundary z_S holds

$$V_i(z_S^+,t) = V_s(z_S^+,t).$$

where V_i and V_s are the velocities of ice and soil grains, respectively.
In that case, the heave rate is given by

$$\dot{z}_T = V_i(z_S,t)^+ = V_s(z_S,t)^+$$

and the porosity $\varepsilon^+(z_S,t)$ can be calculated *a posteriori* by the formula (crf. [9])

$$\varepsilon(z_S,t)^+ = \frac{\dot{z}_T - \varepsilon_0 \dot{z}_S}{\dot{z}_T - \dot{z}_S}.$$

The five equations of ($\mathbb{S}_{fl}$) and ($\mathbb{S}_{tmp}$) contain the same six unknown quantities $z_F(t)$, $z_S(t)$, $z_T(t)$, $q_w(t)$, $T_S(t)$ e $p_w(z,t)$. Nevertheless, we have not yet introduced a criterion in order to discriminate the case of lens formation from frost penetration.
Before discussing such as question, we need to describe the propertis of the given functions $\nu(T)$, $K_1(T)$, $K_2(T)$ e $k_{ff}(T)$.

1.5 *The functions $\nu(T)$, $K_1(T)$, $K_2(T)$ e $k_{ff}(T)$ in the frozen fringe*

As for the volumetric ice content, we have to distinguish the two possibilities:

i) $\nu(T) < 1 \quad \forall\, T \leq 0$

ii) $\exists\, T < 0$ *such that* $\nu(T) \equiv 1$ *for* $T \leq T$.

We will restrict our analysis examinating only the first case, remarking that the results we will get can be easily formulated for the second case.
We assume for the function ν to have the same properties (cfr. hypothesis (H_6) and [7]):

$$(H_{10}) \qquad 0 \leq \nu(T) \leq 1\,,\ \frac{\partial \nu(T)}{\partial T} < 0\,,\ \ \nu(0) = 1, \quad \lim_{T \to -\infty} \nu(T) = 1$$

If we extend the definition of ν to the other regions of the soil by setting:

$$\begin{aligned} \nu(T(z,t)) &\equiv 0 \qquad \text{for} \quad 0 \leq z \leq z_F(t) \\ \nu(T(z,t)) &\equiv 1 \qquad \text{for} \quad z_S(t) < z \leq z_T(t), \end{aligned}$$

We remark that ν is continuous through the boundary $z = z_F(t)$, by virtue of (H_6), while it is generally discontinuous through $z = z_S(t)$.
The function $K_1(T)$ is positive and decreasing to zero for T decresing (see [8]):

$$(H_{11}) \qquad K_1(T) > 0 \ , \quad \frac{\partial K_1(T)}{\partial T} \geq 0 \ , \qquad \lim_{T \to -\infty} K_1(T) = 0$$

Furthermore, we assume that $K_2(T)$ is such that (see [6]):

$$(H_{12}) \qquad K_2(T) > 0 \ , \quad \frac{\partial K_2(T)}{\partial T} > 0 \ , \qquad \lim_{T \to -\infty} K_2(T) = 0$$

Let us comment the choise (H_{12}). On the ground of experimental data, K_2 is an increasing function of T. A question arises for the isotherm $T = 0$, that is the lower boundary of the frozen fringe: from one hand, K_2 should achieve its maximum value, since the temperature is decreasing with respect to z (see (1.20) and (1.20a)); on the other hand, from $\nu(0) = 0$ we deduce that only water is present at that boundary. Actually, supposing that the effects of the thermal gradient on the water flux are due to the simoltaneous presence of water and ice, it would be more appropriate to assume for $K_2(T)$ to have a maximum for some $\overline{T} < 0$, and to be decreasing for $T \geq \overline{T}$ up to $K_2(0) = 0$. However, the temperature $\overline{T}$ is so close to zero that a crescent profile for $K_2(T)$ can be considered as a good approximation.
The thermal conductivity in the frozen fringe can be written as (cfr. [1]):

$$k_{ff}(T) = (1 - \nu(T))\varepsilon k_w + \varepsilon \nu(T) k_i + (1 - \varepsilon) k_s$$

Since it is known that (see, for istance, [2]) $k_i > k_w$, we deduce from (H_{10}):

$$(1.31) \qquad \frac{\partial k_{ff}(T)}{\partial T} = \varepsilon_0 \big(k_i - k_w\big) \frac{\partial \nu(T)}{\partial T} < 0$$

The function $k_{ff}(T)$ has its minimum for $T = 0$, where it assumes the value k_u, which represents the thermal conductivity of the unfrozen soil:

$$k_u = \varepsilon k_w + (1 - \varepsilon) k_s$$

For T decreasing to $-\infty$, $k_{ff}(T)$ tends to k_f (thermal conductivity of the frozen soil):

$$k_f = \varepsilon k_i + (1-\varepsilon)k_s.$$

1.6 *Discriminating between lens formation and frost penetration*

The basic assumption we made about the water pressure is (1.14): at the boundary z_S p_w achieves, in any case, the positive value σ depending on the kind of soil and on the overburden pressure.
Let us assume that σ is a constant: this corresponds, essentially, to neglect the variation of weight of the frozen part that leans on z_S
Derivating (1.14) with respect to time, one gets

$$\frac{\partial p_w(z_S(t),t)}{\partial z}\,\dot z_S(t) + \frac{\partial p_w(z_S(t),t)}{\partial t} = 0. \tag{1.32}$$

In the spirit of the quasi-steady approach, we assume that the second term in previous fromula is negligible; so, (1.32) is approximated by

$$\frac{\partial p_w(z_S(t),t)}{\partial z}\,\dot z_S(t) = 0. \tag{1.33}$$

Therefore, whenever $\dot z_S(t) < 0$, which is a condition associated with the process of frost penetration, it must be:

$$\frac{\partial p_w(z_S(t),t)}{\partial z} = 0 \qquad \textit{frost penetration.} \tag{1.34}$$

On the other hand, during the process of lens formation the front $z = z_S$ is at rest (see [9]):

$$\dot z_S(t) = 0, \qquad \textit{lens formation.}$$

Taking into account of the initial condition (1.17), we get

$$z_S(t) \equiv b, \qquad \textit{lens formation.} \tag{1.35}$$

We show now that, during the lens formation, the water pressure gradient must verify at $z = z_S$:

$$(1.36) \qquad \frac{\partial p_w(b,t)}{\partial z} \geq 0, \qquad \textit{lens formation.}$$

indeed, if $\partial p_w(b,\bar{t})/\partial z < 0$ for some $\bar{t}$ during the process of lens formation, it must exist a point $\bar{z}$ in the frozen fringe so that $p_w(\bar{z},\bar{t}) > \sigma$. This is in contradiction with the fact that we assign to the condition $p_w = \sigma$ the property to separate the soil. Thus, (1.36) must hold.

Let us now examinate the case when both the quantities in (1.33) vanish.

When $\dot{z}_S = 0$ and $\dot{z}_T = V_s(z_S,t)$ is not zero, we have (see remark 1.1)

$$\varepsilon^+(z_S,t) = 1$$

that means that only ice is present immediately over the front $z = z_S$.

On the other hand, if it were $\dot{z}_S(\bar{t}) = \frac{\partial p_w(z_S(\bar{t}),\bar{t})}{\partial z} = \dot{z}_T(\bar{t}) = 0$ for some $\bar{t}$, we would get $q_w(\bar{t}) = 0$ from (1.15). Assume that we prescribe the thermal fluxes α_0 and α_1. From (1.4), we see that this is possible only if $\alpha_0 = 0$. In that case, α_1 is zero, too (see equation (1.25)) and the temperature $T(z,\bar{t})$ vanishes everywhere. Furthermore, from (1.22), (1.27) we would have $p_w(z,\bar{t}) \equiv 0$ for $0 \leq z < z_S(t)$. Hence,the water pressure p_w can not be continuous through$z = z_S$, owing to(1.14), unless $\sigma = 0$. At this point, it is clear that when both $\dot{z}_S$ and $\frac{\partial p_w(z_S,t)}{\partial z}$ vanish, $\dot{z}_T(\bar{t})$ can not be zero for a solution of (1.1)-(1.18). A similar argument holds in the case of precscriptions (1.6a), (1.16a).

Let us now make the following important comment. From (1.27) (or (4.1.27a)) and from the properties of the functions $K_2(T)$ and $k_{ff}(T)$ (H_{11}) and (4.1.31) we deduce that the water pressure gradient vanish at most in one point in the interval $z_F(t) \leq z \leq z_S(t)$. This entails that whenever

$$(1.37) \qquad \frac{\partial p_w(z_S(t),t)}{\partial z} \geq 0$$

holds, then the water pressure can not exceed the value σ in any point. Thus (1.37), that is a necessary condition in both the cases of lens formation and frost penetration and that requires the knowledge of quantities calculated only on the front $z = z_S(t)$, is also a sufficient condition in order to guarantee a consistent profile of the pressure p_w (in the sense that $p_w \leq \sigma$) in the whole *frozen fringe.*

We remark that if (1.37) is satisfies, not only $p_w(z,t) \leq \sigma$ in the frozen fringe, but p_w is a non decreasing function of z for each fixed time t.

On the ground of the previous analysis, we introduce the following criterion $\mathbb{C}$ of discriminating between lens formation and frost penetration:

$$(\mathbb{C}) \quad \begin{cases} \dot{z}_S(t) = 0 \text{ together with } \dfrac{\partial p_w(z_S,t)}{\partial z} \geq 0 \Leftrightarrow \textit{lens formation} \\ \\ \dot{z}_S(t) < 0 \text{ together with } \dfrac{\partial p_w(z_S,t)}{\partial z} = 0 \Leftrightarrow \textit{frost penetration} \end{cases}$$

Actually, if $\dot{z}_S(t) = 0$, then $\varepsilon^+(z_S,t) = 1$; moreover, the water pressure verifies $p_w(z,t) \leq \sigma$, $z \in [z_F(t), z_S(t)]$. On the other hand, if $\dot{z}_S(t) < 0$, the water pressure gradient at $z = z_S$ must vanish (see (1.33)).

1.7 *Further conditions*

We have finally to add the following constraints to the solutions of $(\mathbb{S}_{fl})$ or $(\mathbb{S}_{tmp})$, in order to eliminate solutions which can not be accepted from a physical point of view. Such as conditions, whose meaning is evident, are, for each time t:

$$(\mathbb{A}) \quad \begin{cases} (1.38) \qquad 0 \leq z_F(t) \leq z_S(t) \leq z_T(t) \\ \\ (1.39) \qquad q_w(t) \geq 0. \end{cases}$$

2. Constant thermal fluxes at the boundaries

In sections 2 and 3 we will discuss the solvability of system $(\mathbb{S}_{fl})$. We will investigate the following points:

- thermal fluxes given at the boundaries of the soil as costant, in the case of lens formation (par. 2.1) and of frost penetration (par. 2.2);

- thermal fluxes at the boundaries given as functions of time (sect. 3), with special attention to the transition processes from one phenomenon to the other (par. 3.2).

2.1 *Lens formation; α_0, α_1 constant*

Let us assign the thermal fluxes (1.6) and (1.16) by taking α_0 and α_1 as constant and consider the system $(\mathbb{S}_{fl})$ in the case of lens formation. Taking into occunt of (1.35), the set of equations to be solved is:

$$
(\mathbb{S}^L_{fl})\begin{cases}
(2.1) \qquad z_F(t) = \dfrac{1}{k_u\alpha_0}\displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta + b \\[2ex]
(2.2) \qquad q_w = \dfrac{k_f\alpha_1 - k_u\alpha_0}{L\rho_w} \\[2ex]
(2.3) \qquad \sigma + \dfrac{q_w z_F(t)}{K_0} + \displaystyle\int_0^{T_S(t)} \dfrac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)}d\eta = 0 \\[2ex]
(2.4) \qquad q_w = -K_1(T_S(t))\dfrac{\partial p_w(z_S(t),t)}{\partial z} + \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}k_u\alpha_0 \\[2ex]
(2.5) \qquad \rho_i \dot{z}_T(t) = \rho_w q_w
\end{cases}
$$

where

$$
(2.6) \qquad c(\alpha_0,\alpha_1) = \frac{k_f\alpha_1 - k_u\alpha_0}{L\rho_w k_u\alpha_0} = \frac{q_w}{k_u\alpha_0}
$$

The unknown quantities in $(\mathbb{S}^L_{fl})$ are the boundaries $z_F(t)$, $z_T(t)$, the temperature $T_S(t)$, the water flux $q_w(t)$ and the water pressure $p_w(z_S(t),t)$; the initial condition is (1.18):

$z_T(0) = H > b$.

Moreover, the criterion (C) introduced in par. 1.7 imposes

$$\text{(2.7)} \qquad \frac{K_2(T_S(t))}{k_{ff}(T_S(t))}\, k_u\alpha_0 - q_w \geq 0.$$

Conditions (A) (equations (1.38) and (1.39)) reduce to

$$\text{(2.8)} \qquad k_f\alpha_1 - k_u\alpha_0 \geq 0$$

$$\text{(2.9)} \qquad 0 \leq z_F(t) \leq b.$$

By using (2.1), we see that (2.9) is equivalent to

$$\text{(2.10)} \qquad -k_u\alpha_0 b \leq \int_0^{T_S} k_{ff}(\eta)d\eta \leq 0.$$

In particular:

$$\text{(2.11)} \qquad \begin{cases} z_F = 0 \quad \textit{if and only if} \quad \displaystyle\int_0^{T_S} k_{ff}(\eta)d\eta = -k_u\alpha_0 b; \\ z_F = b \quad \textit{if and only if} \quad T_S = 0. \end{cases}$$

Condition (2.7) can be written in the following way:

$$\text{(2.12)} \qquad ck_{ff}(T_S) \leq K_2(T_S), \quad c \geq 0.$$

In other words, we have to find T_S satisfying the equation

$$\text{(2.13)} \qquad G(T_S) = 0$$

where

$$\text{(2.14)} \qquad G(s) = \sigma + c\,\frac{k_u\alpha_0}{K_0}\left(b + \frac{1}{k_u\alpha_0}\int_0^s k_{ff}(\eta)d\eta\right) + \int_0^s \frac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)}d\eta$$

and such that (2.10) and (2.12) are satisfied.

It is immediately seen that a solution of (2.13) is constant, since no term depends on time; furthermore, the solution T_S can not be positive, otherwise condition (2.10) would be violated.

Consider now the function $K_2(s)-ck_{ff}(s)$, $s \leq 0$, $c \geq 0$.

By virtue of the properties of the functions $K_2(s)$ and $k_{ff}(s)$ (see (H_{11}) and (1.31) in par. 1.6), it is easlily seen that for any $c \in (0,K_2(0)/k_u)$ there exists exactly one temperature $T_S^*(c)$ such that

$$(2.15) \qquad K_2(T_S^*(c)) = ck_{ff}(T_S^*(c)).$$

Moreover, we have:

$$(2.16) \qquad \begin{cases} \lim\limits_{c \to 0^+} T_S^*(c) = -\infty \\ \\ \lim\limits_{c \to \left(\frac{K_2(0)}{k_u}\right)^-} T_S^*(c) = 0^- \end{cases}$$

Owing to (2.12), the solution T_S of (2.13) has to be searched in the interval $[T_S^*(c),0]$, with $c \in (0,K_2(0)/k_u)$. The lenght of that interval depends obviously on the data α_0, α_1 through c. Furthermore:

$$\frac{\partial p_w(b,t)}{\partial z} = 0 \textit{ if and only if } T_s = T_S^*(c).$$

If $c \geq K_2(0)/k_u$, there are not solutions of (2.13) so that (2.12) is fulfilled.

On the other hand, condition (2.10) means that the solution T_S of (2.13) must be in the interval $\left[T_S^b(\alpha_0),0\right]$, where $T_S^b(\alpha_0)$ is the one value such that

$$(2.17) \qquad -k_u\alpha_0 b = \int_0^{T_S^b} k_{ff}(\eta)d\eta.$$

Since $\int_0^s k_{ff}(\eta)d\eta$ is an increasing function of s, it is easy to verify that to every α_0 there corresponds a unique $T_S^b(\alpha_0)$; moreover:

$$(2.18) \qquad \lim_{\alpha_0 \to +\infty} T_S^b(\alpha_0) = -\infty.$$

We remark that $z_F = 0$ if and only if $T_S = T_S^b(\alpha_0)$.

In order to discuss the solvability of (2.13), we must consider the two possibilities:

i) $T^*_S(c(\alpha_0,\alpha_1)) \geq T^b_S(\alpha_0)$
ii) $T^*_S(c(\alpha_0,\alpha_1)) < T^b_S(\alpha_0)$

We call ($\mathbb{T}$) the set of equations (2.10), (2.12) and (2.13) and we state the following

LEMMA 2.1. *System* ($\mathbb{T}$) *has exactly one solution* T_S *if and only if*

(2.19) $\quad G(T^*_S(c(\alpha_0,\alpha_1)) \leq 0$ *in the case i),*

(2.20) $\quad G(T^b_S(\alpha_0)) \leq 0$ *in the case ii).*

Moreover, T_S *is constant.*
Dim.
By derivating (2.14) with respect to s one finds:

$$G'(s) = \frac{1}{K_0}\frac{(K_0(K_2 - ck_{ff}(s)) + ck_{ff}(s)K_1(s)}{K_1(s)}$$

We see that $G'(s) > 0$ if$T^*_S(c) \leq s \leq 0$. Furthermore, $G(0) = \sigma + c\,\frac{k_u\alpha_0}{K_0}b > 0$. The solution T_S of ($\mathbb{T}$) must lie in the interval $[T_\ell,0]$, where $T_\ell = T_\ell(\alpha_0,\alpha_1) = max\left\{T^b_S(\alpha_0),T^*_S(c(\alpha_0,\alpha_1))\right\}$. We deduce that there exists a unique solution of ($\mathbb{T}$) if and only if $G(T_\ell) \leq 0$, that is equivalent to (2.19) and (2.20).
Notice that in (2.19) equality holds (in that case the solution of (2.13) is $T_S = T^*_S(c(\alpha_0,\alpha_1)))$ if and only if the water pressure gradient computed at $z = b$ vanishes.
In (2.20) equality holds if and only if $z_F = 0$ (the base of the soil and the lower boundary of the frozen fringe are the same line). In that case, the solution of ($\mathbb{T}$) is obviously $T_S = T^b_S(\alpha_0)$. □

Our next goal is to locate the points (α_0,α_1) on the quarter of plane $\{(\alpha_0,\alpha_1)\colon \alpha_0 \geq 0,\alpha_1 \geq 0\}$ such that condition (2.19) or (2.20) of the previous lemma is verified. The region that we will single out and that we call $\mathcal{L}$, will correspond to all the pairs of thermal fluxes (α_0,α_1) and only those such that the process of lens formation is induced.
Since $c \in (0,K_2(0)/k_u)$, the region $\mathcal{L}$ is contained in the angle β:

$$(2.21) \qquad \mathcal{L} \subset \beta = \left\{ (\alpha_0,\alpha_1):\ \alpha_0 \geq 0,\ k_u\alpha_0 \leq k_f\alpha_1 \leq k_u\alpha_0\left(1 + \frac{\rho_w L}{k_u} K_2(0)\right)\right\}$$

We start by remarking that the distinction between case *i)* and case *ii)* is nothing but a partition in the angle β. Indeed, let us consider the set $T_S^*(c(\alpha_0,\alpha_1)) = T_S^b(\alpha_0)$, which we can write, owing to (2.17), as

$$(2.22) \quad (\mathfrak{C}) \qquad -k_u\alpha_0 b = \int_0^{T_S^*(c(\alpha_0,\alpha_1))} k_{ff}(\eta)d\eta$$

It is easy to check, by using (2.16), that the set ($\mathfrak{C}$) is a curve dividing β into two parts, corresponding to the sets $T_S^*(c(\alpha_0,\alpha_1)) \lesseqgtr T_S^b(\alpha_0)$.
As a matter af fact, let us take the half-straight lines starting from the origin

$$(2.23) \qquad r_c = \left\{ (\alpha_0,\alpha_1):\ \alpha_0 \geq 0,\ k_f\alpha_1 = k_u\alpha_0(1 + \rho_w Lc)\right\},$$

with $0 < c < K_2(0)/k_u$. Going along one of those lines and letting α_0 grow from zero to $+\infty$, we see that the quantity T_S^* is a constant negative value, while $T_S^b(\alpha_0)$ decreases monotonically from down to $-\infty$ (cfr. (2.17)). Hence, there exists a unique value α_0 such that the equality $T_S^*(c) = T_S^b(\alpha_0)$ holds; in other words, there is a unique point of intersection $P(c)$ between the straight line corresponding to c and the set $\mathfrak{C}$. By (2.16) and (2.18) we have

$$(2.24) \qquad \lim_{c \to 0^+} |P(c) - O| = \infty, \qquad \lim_{c \to \left(\frac{K_2(0)}{k_u}\right)^-} |P(c) - O| = 0$$

where the point O is the origin of axes in the (α_0,α_1)-plane.
Moreover, since $T_S^*(c)$ is an increasing function of c, we see that, as c increases from zero up to $K_2(0)/k_u$, the distance $|P(c) - O|$ decreases monotonically from ∞ to zero.
Let us denote now by

$$(2.25) \qquad \mathfrak{R}_1 = \left\{ (\alpha_0,\alpha_1) \in \beta:\ T_S^*(c(\alpha_0,\alpha_1)) < T_S^b(\alpha_0)\right\}$$

$$(2.26) \qquad \mathfrak{R}_2 = \left\{ (\alpha_0,\alpha_1) \in \beta:\ T_S^*(c(\alpha_0,\alpha_1)) \geq T_S^b(\alpha_0)\right\}$$

the two parts separated by the curve $\mathcal{C}$. The region $\mathcal{R}_1$ is bounded by the straight line $k_f\alpha_1 = k_u\alpha_0$ from below and by the curve $\mathcal{C}$ from above, while $\mathcal{R}_2$ is bounded by the curve $\mathcal{C}$ from below and by the straight line $k_f\alpha_1 = k_u\alpha_0(1+\rho_w LK_2(0)/k_u)$ from above.
By virtue of lemma 2.1 we can say that the process of lens formation occurs if and only if

$$(2.27)\qquad \sigma + \int_0^{T_S^b} \frac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)} d\eta \leq 0$$

whenever $(\alpha_0,\alpha_1) \in \mathcal{R}_1$,

$$(2.28)\qquad \sigma + \frac{K_2(T_S^*(c))}{K_0}\left(T_S^*(c) + \frac{k_u\alpha_0 b}{k_{ff}}\right) + \int_0^{T_S^*(c)} \frac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)} \; d\eta \leq 0$$

whenever $(\alpha_0,\alpha_1) \in \mathcal{R}_2$.
If we call

$$\mathcal{R}_{1,\ell} = \{(\alpha_0,\alpha_1) \in \mathcal{R}_1\text{: (4.2.27) } \textit{is verified}\}$$

$$\mathcal{R}_{2,\ell} = \{(\alpha_0,\alpha_1) \in \mathcal{R}_2\text{: (4.2.28) } \textit{is verified}\}$$

we may conclude that $\mathcal{L} = \mathcal{R}_{1,\ell} \cup \mathcal{R}_{2,\ell}$ is the part of the angle β where lens formation occurs.
We begin with the following result.

LEMMA 2.2. *Let $T_\sigma < 0$ be such that*

$$(2.29)\qquad \sigma + \int_0^{T_\sigma} \frac{K_2(\eta)}{K_1(\eta)} d\eta = 0;$$

then there exists a unique $c_l \in (0,\ K_2(0)/k_u)$ with the property that

$$(2.30)\qquad \sigma + \int_0^{T_S^*(c_l)} \frac{K_2(\eta) - c_l k_{ff}(\eta)}{K_1(\eta)} d\eta = 0.$$

Dim. We first remark that the function

$$f_1(s) = \sigma + \int_0^{s} \frac{K_2(\eta)}{K_1(\eta)} d\eta$$

is strictly increasing and $f_1(0) > 0$. Thus, the assumption (2.29) is equivalent to

(2.31) $$\lim_{s \to -\infty} f_1(s) = \ell < 0.$$

Indeed, if (2.31) holds, there is a unique T_σ such that (2.29) holds and viceversa.
Let us now consider the family of functions depending on the parameter c:

$$f_c(s) = \sigma + \int_0^s \frac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)} d\eta, \quad \forall s \leq 0, \qquad 0 \leq c \leq K_2(0)/k_u.$$

We have:

(2.32) $$f_c(s) \geq f_1(s), \qquad \lim_{c \to 0^+} f_c(s) = f_1(s) \quad \forall s \leq 0 \text{ (punctually)}.$$

Moreover, according to the properties of K_2 and k_{ff} and since

$$\frac{\partial f_c(s)}{\partial s} = \frac{K_2(s) - ck_{ff}(s)}{K_1(s)},$$

we see that for any c fixed in $[0, K_2(0)/k_u]$, the function $f_c(s)$ achieves its minimum value for $s = T_S^*(c)$, where $T_S^*(c)$ is defined by (2.15):

(2.33) $$f_c(s) \geq f_c(T_S^*(c)) \quad \forall s \leq 0, \; c \in [0, K_2(0)/k_u].$$

Suppose, contrary to our claims, that $f_c(T_S^*(c)) \geq 0 \;\; \forall c \in (0, \; K_2(0)/k_u]$ and consider the case $\ell > -\infty$.
By virtue of (2.31), once we have fixed $0 < \epsilon < -\ell/2$ we could find $\hat{s} < 0$ such that

$$f_1(\hat{s}) < \ell + \epsilon.$$

On the other hand, by (2.32) we could find $\hat{c}$ small enough that

$$0 \leq f_{\hat{c}}(\hat{s}) - f_1(\hat{s}) < \epsilon.$$

Therefore:

$$f_{\hat{c}}(\hat{s}) < \ell + 2\epsilon < 0.$$

On the contrary, from (2.33) we deduce:

$$f_{\widehat{c}}(\widehat{s}) \geq f_{\widehat{c}}(T^*_S(\widehat{c})) \geq 0,$$

and we obtail a contradiction. The case $\ell = -\infty$ can be proved with slight modifications. Thus, we may say that there exists $\overline{c} \in (0,\ K_2(0)/k_u)$ such that $f_{\overline{c}}(T^*_S(\overline{c})) < 0$.
Since the function

$$B(c) = \sigma + \int_0^{T^*_S(c)} \frac{K_2(\eta) \ - c k_{ff}(\eta)}{K_1(\eta)} d\eta, \qquad c \in (0,\ K_2(0)/k_u]$$

is continuous and verifies

$$B(K_2(0)/k_u) = \sigma > 0,\ B(\overline{c}) < 0,\ B'(c) = -\int_0^{T^*_S(c)} \frac{k_{ff}(\eta)}{K_1(\eta)} d\eta > 0 \text{ per } c \in (0,\ K_2(0)/k_u],$$

we conlcude that there exists a unique $c_l \in c \in (0,\ K_2(0)/k_u)$ with the property $B(c_l) = 0$, and the lemma is proved. □

The following result locates the region $\mathfrak{L}$ on the (α_0,α_1)-plane.

PROPOSITION 2.1.

i) If

$$(2.34) \qquad \sigma + \int_0^{\vartheta} \frac{K_2(\eta)}{K_1(\eta)} d\eta > 0 \qquad \forall \vartheta \in (-\infty, 0)\ ,$$

then $\mathfrak{L} = \emptyset$.

ii) If $\exists\ T_\sigma < 0$ *such that*

$$(2.29) \qquad \sigma + \int_0^{T_\sigma} \frac{K_2(\eta)}{K_1(\eta)} d\eta = 0,$$

then $\mathfrak{L}$ *is the non-empty set bounded by*

$$(2.35) \qquad \partial\mathfrak{L} = \left\{ (\alpha_0,\alpha_1) : k_f \alpha_1 = k_u \alpha_0\ ,\ \alpha_0 \geq \tilde{\alpha}_0 \right\} \cup \mathfrak{C}_1 \cup \mathfrak{C}_2$$

where $\widetilde{\alpha}_0$ *is defined by*

$$(2.36)\qquad -k_u\widetilde{\alpha}_0 b=\int_0^{T_\sigma}\frac{K_2(\eta)}{K_1(\eta)}d\eta$$

and

$$(2.37)\qquad \mathfrak{C}_1=\left\{(\alpha_0,\alpha_1):\ \sigma+\int_0^{T_S^b(\alpha_0)}\frac{K_2(\eta)\ -cK_2(\eta)}{K_1(\eta)}d\eta=0\ ,\ 0\le c\le c_l\right\}$$

$$(2.38)\qquad \mathfrak{C}_2=\left\{(\alpha_0,\alpha_1):\ \sigma+c\,\frac{k_u\alpha_0}{K_0}\left(b+\frac{1}{k_u\alpha_0}\int_0^{T_S^*(c)}k_{ff}(\eta)d\eta\right)+\int_0^{T_S^*(c)}\frac{K_2(\eta)-ck_{ff}(\eta)}{K_1(\eta)}d\eta=\right.$$
$$\left.=0,0<c\le c_l\right\}$$

are two curves contained in $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *respectively; in* (2.37) *and in* (2.38) c_l *is the value defined in lemma* 2.1 *and verifying* (2.30).

Dim. Case *i*) is immediately proved: if (2.34) holds, equation (2.13) can not have non-positive solutions and the condition (2.10) is violated. Hence $\mathcal{L}=\emptyset$.
Let us now pass to case *ii*).
We consider once again the straight lines (2.23). We know that there exists a unique point $P(c)$ whose coordinates $(\overline{\alpha}_0,\overline{\alpha}_1)$ are

$$(2.39)\qquad -k_u\overline{\alpha}_0 b=\int_0^{T_S^*(c(\overline{\alpha}_0,\overline{\alpha}_1))}k_{ff}(\eta)d\eta,\quad \overline{\alpha}_1=k_u\overline{\alpha}_0(1+\rho_w Lc)\ \ ,\ \text{per}\ \ 0<c\le\frac{K_2(0)}{k_u}.$$

Conventionally, we can assume that $\overline{\alpha}_0(0)=\overline{\alpha}_1(0)=+\infty$, recalling that the straight line $r_0: k_f\alpha_1=k_u\alpha_0$ is an asymptote for the curve $\mathfrak{C}$.
We examinate first the region $\mathfrak{R}_1$. Evaluating the quantity on the left-hand side in the inequality (2.27) along one of the lines r_c, where c is constant, we get:

$$(2.40)\qquad F(\alpha_0,c)=\sigma+\int_0^{T_S^b(\alpha_0)}\frac{K_2(\eta)\ -ck_{ff}(\eta)}{K_1(\eta)}d\eta\quad(\alpha_0,\alpha_1)\in\mathfrak{R}_1\ ,\ \ 0\le c\le\frac{K_2(0)}{k_u}.$$

By virtue of assumption (2.29), lemma 2.2 and remarking that $\frac{d}{dc}F(\overline{\alpha}_0(c),c)>0$, we have

$$(2.41)\qquad \lim_{\alpha_0\to 0}F(\alpha_0,c)=\ \sigma>0\qquad \forall c\in\left[0,K_2(0)/k_u\right]$$

$$(2.42)\qquad F(\overline{\alpha}_0(c),c)\begin{cases} <0 & \textit{if } 0<c<c_l \\ =0 & \textit{if } c=c_l \\ >0 & \textit{if } c_l<c\le \dfrac{K_2(0)}{k_u}. \end{cases}$$

Taking into account of (2.41), (2.42) and noticing that $F(\alpha_0,0)$ decreases along the lines r_c, that is

$$(2.43)\qquad \frac{\partial F}{\partial\alpha_0}(\alpha_0,c)<0\ ,\quad (\alpha_0,\alpha_1)\in\mathcal{R}_1$$

we conclude that whenever $(\alpha_0,\alpha_1)\in\mathcal{R}_1$, if $c_l<c\le K_2(0)/k_u$, there are not any points (α_0,α_1) on the straight line r_c such that (2.27) is verified. In other words:

$$(2.44)\qquad \mathcal{R}_{1,\ell}\cap r_c\ =\emptyset,\qquad c_l<c\le\frac{K_2(0)}{k_u}.$$

On the other hand, when $0<c\le c_l$, we have that (2.27) evaluated on each line r_c is valid if α_0 is such that

$$(2.45)\qquad \alpha_0\in[\overline{\overline{\alpha}}_0(c),\overline{\alpha}_0(c))\ ,$$

where $\overline{\overline{\alpha}}_0(c)$ is the value (which exists unique owing to the continuity of the function F with respect to α_0 and to the formulas (2.42)) with the property

$$(2.46)\qquad F(\overline{\overline{\alpha}}_0(c),c)=0,\quad 0<c\le c_l$$

This is equivalent to state that for $0<c\le c_l$ we have

$$(2.47)\qquad \mathcal{R}_{1,\ell}\cap r_c\ =\Big\{(\alpha_0,\alpha_1)\colon\ \overline{\overline{\alpha}}_0(c)\le\alpha_0<\overline{\alpha}_0(c)\ ,\ k_f\alpha_1=k_u\alpha_0(1+\rho_w Lc)\Big\}$$

When $c=0$, it is $\overline{\alpha}_0(0)=+\infty$, $\overline{\overline{\alpha}}_0(c)=\tilde{\alpha}_0$, where $\tilde{\alpha}_0$ satisfies (2.36). Thus:

$$\mathcal{R}_{1,\ell}\cap r_0=\Big\{(\alpha_0,\alpha_1)\colon\ \alpha_0\ge\tilde{\alpha}_0,\ k_f\alpha_1=k_u\alpha_0\Big\}.$$

Obviously

$$\bigcup_{0<c\leq c_l}\left(\Re_{1,\ell}\cap r_c\right)=\Re_{1,\ell}.$$

The set of points in $\Re_1$ satisfying (2.46), that is

$$\left\{(\alpha_0,\alpha_1)\in\Re_1:\ \alpha_0=\bar{\bar{\alpha}}_0(c)\ ,\ k_f\alpha_1=k_u\bar{\bar{\alpha}}_0(c)k_u\alpha_0(1+\rho_w Lc),\ 0\leq c\leq c_l\right\},$$

make a curve $\mathfrak{C}_1$ that is the boundary of the set $\Re_{1,\ell}$.
The curve $\mathfrak{C}_1$ matches $\mathfrak{C}$ defined by (2.22) only at the point $\hat{P}\equiv(\hat{\alpha}_0,\hat{\alpha}_1)$ with coordinates

$$(2.48)\qquad -k_u\hat{\alpha}_0 b=\int_0^{T_S^*(c_l)} k_{ff}(\eta)d\eta,\qquad k_f\hat{\alpha}_1=k_u\hat{\alpha}_0((1+\rho_w Lc_l)$$

and the straight line $k_f\alpha_1=k_u\alpha_0$ at $\alpha_0=\tilde{\alpha}_0$, where $\tilde{\alpha}_0$ is defined by (2.36). Moreover, the tangent to the curve $\mathfrak{C}_1$ at $\hat{P}$ alla curva $\mathfrak{C}_1$ is the straight line

$$r_{c_l}=\left\{(\alpha_0,\alpha_1):\ \alpha_0\geq 0,\ k_f\alpha_1=k_u\alpha_0(1+\rho_w Lc_l)\right\}.$$

We will examinate now the region $\Re_2$. In order to find the points (α_0,α_1) in $\Re_2$ which have the property (2.28), we check the values achieved by the auxiliary function H, that we are going to define, along each of the straight lines r_c, $0<c\leq K_2(0)/k_u$, where $T_S^*(c)$ is constant:

$$(2.49)\qquad H(\alpha_0,c)=\sigma+\frac{K_2(T_S^*(c))}{K_0}\left(T_S^*(c)+\frac{k_u\alpha_0 b}{k_{ff}}\right)+\int_0^{T_S^*(c)}\frac{K_2(\eta)-ck_{ff}(\eta)}{K_1(\eta)}d\eta,\ \ (\alpha_0,\alpha_1)\in\Re_2$$

We can exclude the value $c=0$, since the straight line $r_0:k_f\alpha_1=k_u\alpha_0$ verifies $\Re_2\cap r_0=\emptyset$.
We have to find the part of $\Re_2$ where $G(\alpha_0,c)\leq 0$, owing to (2.28).
It is immediately seen that:

$$(2.50)\qquad H(\bar{\alpha}_0,c)=F(\bar{\alpha}_0(c),c)\qquad\qquad 0<c\leq K(0)/k_u.$$

moreover, along one of the line r_c we have:

$$(2.51)\qquad \lim_{(\alpha_0,\alpha_1)\in r_c,\alpha_0\to+\infty}G(\alpha_0,c)=+\infty\qquad\qquad 0<c\leq K_2(0)/k_u.$$

Arguing as in the case i), we conclude that whenever $(\alpha_0,\alpha_1)\in\Re_2$:

$$(2.52)\quad \begin{cases} \textit{if } c_l < c \le K_2/k_u, \textit{ then } H(\alpha_0,c) > 0; \\ \textit{if } c = c_l, \textit{ then } H(\alpha_0,c) = 0; \\ \textit{if } 0 < c < c_l \textit{ then } H(\alpha_0,c) < 0 \textit{ when} \end{cases}$$

$$(2.53)\qquad \alpha_0 \in \left[\,\bar{\alpha}_0(c),\bar{\bar{\bar{\alpha}}}_0(c)\right],$$

where $\bar{\bar{\bar{\alpha}}}_0(c)$ *has the property*

$$(2.54)\qquad H(\bar{\bar{\bar{\alpha}}}_0(c),c) = 0, \qquad 0 < c < c_l.$$

The existence and the uniqueness of $\bar{\bar{\bar{\alpha}}}_0(c)$, $0 < c < c_l$ come form the continuity of the function H and from (2.51) and (2.52). Thus, in $\mathcal{R}_2$ we have:

$$\mathcal{R}_{2,\ell} \cap r_c \; = \emptyset, \quad \text{if } c_l < c \le K_2/k_u$$

$$\mathcal{R}_{2,\ell} \cap r_c = \left\{(\alpha_0,\alpha_1)\colon \bar{\bar{\alpha}}_0(c) \le \alpha_0 \le \bar{\bar{\bar{\alpha}}}_0(c)\ ,\ k_f\alpha_1 = k_u\alpha_0(1+\rho_w Lc)\ \right\}, \text{ if } 0 < c \le c_l.$$

Equation (2.54) defines a curve $(\mathfrak{C}_2)$

$$\left\{(\alpha_0,\alpha_1) \in \mathcal{R}_2\colon \alpha_0 = \bar{\bar{\bar{\alpha}}}_0(c)\ ,\ k_f\alpha_1 = k_u\bar{\bar{\bar{\alpha}}}_0(c)(1+\rho_w Lc),\ 0 < c \le c_l\right\}$$

that is the one we defined in (2.38). □

Remark 2.1. The point $\hat{P}$ (see (2.48)) lies on the curve $\mathfrak{C}_2$. Furthermore:

$$(2.55)\qquad \bar{\bar{\bar{\alpha}}}_0(c_l) = \bar{\bar{\alpha}}_0(c_l) = \bar{\alpha}_0(c_l) = \hat{\alpha}_0.$$

$$(2.56)\qquad \lim_{c\to 0^+} \bar{\bar{\bar{\alpha}}}_0(c) = +\infty.$$

If we denote by P, P_1 and P_2 the points

$$P(c) \equiv \left\{\bar{\alpha}_0(c),\ \frac{k_u}{k_f}\,\bar{\alpha}_0(c)(1+\rho_w Lc)\right\}$$

$$
(2.57) \qquad \begin{cases} P_1(c) \equiv \left\{ \bar{\bar{\alpha}}_0(c), \dfrac{k_u}{k_f}\, \bar{\bar{\alpha}}_0(c)(1+\rho_w Lc) \right\} \\ P_2(c) \equiv \left\{ \bar{\bar{\bar{\alpha}}}_0(c), \dfrac{k_u}{k_f}\, \bar{\bar{\bar{\alpha}}}_0(c)(1+\rho_w Lc) \right\} \end{cases}
$$

for each c in the range $0 \le c \le c_l$ (conventionallly, $P_2(0) = +\infty$), we see that $|P_1(c) - O|$ increasse if c increases, while $|P(c) - O|$, $|P_2(c) - O|$ decrease if c increases. besides that, we have:

$$
(2.58) \qquad |P_1(c) - O| \le |P(c) - O| \le |P_2(c) - O|
$$

and equality holds if and only if $c = c_l$.
The region $\mathcal{L}$ is drawn in fig. 4.1.
Remark 2.2. When the point $(\alpha_0, \alpha_1) \in \mathfrak{C}_1$, the solutions of $(\mathfrak{S}_{fl}^L)$ have the properties

$$
(2.59) \qquad z_F = 0,\ T_S = T_S^b(\alpha_0), \qquad\qquad (\alpha_0, \alpha_1) \in \mathfrak{C}_1.
$$

On the contrary, when $(\alpha_0, \alpha_1) \in \mathfrak{C}_2$ we find solutions of $(\mathfrak{S}_{fl}^L)$ such that

$$
(2.60) \qquad T_S = T_S^*(c(\alpha_0, \alpha_1)),\ \frac{\partial p_w(b,t)}{\partial z} = 0, \qquad (\alpha_0, \alpha_1) \in \mathfrak{C}_2.
$$

Once T_S has been calculated by means of system $(\mathbb{T})$, whenever $(\alpha_0, \alpha_1) \in \mathcal{L} \neq \emptyset$, we can compute the temperature in any point of the soil by using the formulas (1.20). We notice that the temperature T depends only on the spatial coordinate z but not on time t.
By (2.1) we find the (constant) thickness of the frozen fringe:

$$
b - z_F = \frac{1}{k_u \alpha_0} \int_0^{T_S} k_{ff}(\eta)\, d\eta
$$

while (2.2) and (2.5) allow us to calculate the upper boundary of the soil:

$$
z_T(t) = \frac{\rho_w}{\rho_i} \frac{k_f \alpha_1 - k_u \alpha_0}{L \rho_w}\, t + H
$$

Formulas (1.22) (1.28) give the water pressure in the unfrozen soil and in the frozen fringe, respectively:

$$p_w(z) = -\frac{k_f\alpha_1 - k_u\alpha_0}{L\rho_w K_0}\, z \qquad\qquad 0 \le z \le z_F(t)$$

$$p_w(z) = \sigma - \int_z^b \frac{1}{K_1(T(\xi))}\left(K_2(T(\xi))\frac{k_u\alpha_0}{k_{ff}(T(\xi))} - \frac{k_f\alpha_1 - k_u\alpha_0}{L\rho_w}\right) d\xi \qquad z_F(t) \le z \le z_S(t).$$

The water pressure p_w achieves the minimum value for $z = z_F$. Moreover, owing to (2.7) and to the properties of K_2 and k_{ff}, the pressure p_w increases with respect to z in the frozen fringe.

Recalling (1.2) and (1.27), we see that the z-derivative of p_w has a discontinuity at $z = z_F$:

$$[\![p'_w(z_F)]\!]_-^+ = \frac{1}{K_0} K_2(0)\alpha_0.$$

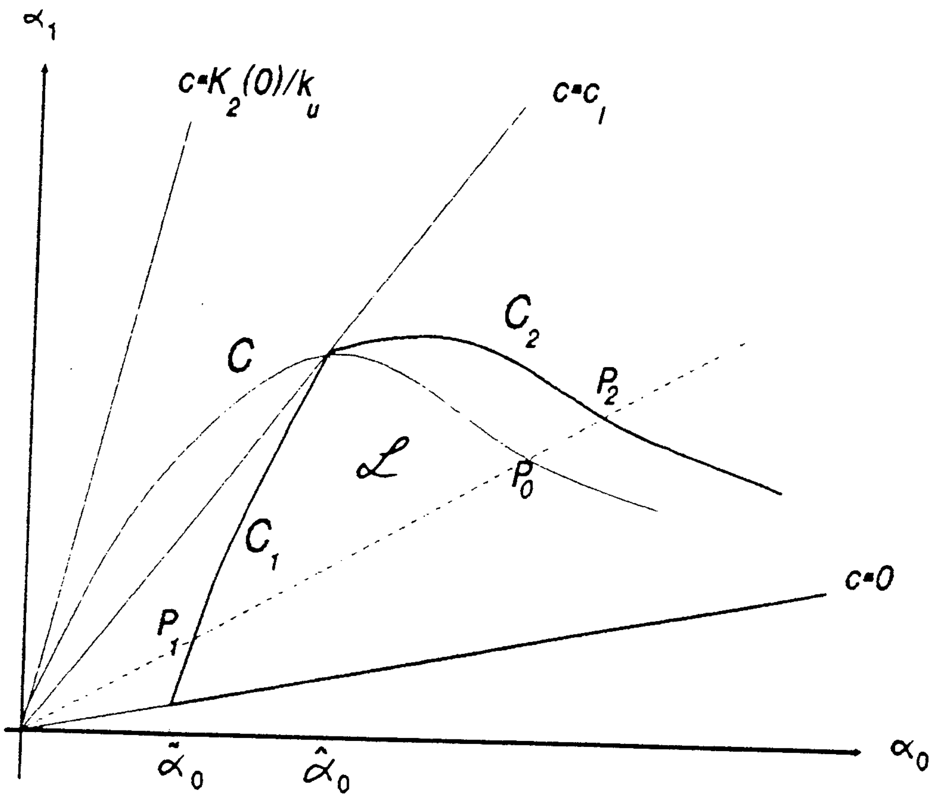

fig. 4.1: the region $\mathfrak{L}$ (lens formation) in the (α_0,α_1)-plane, bounded by the curves $\mathfrak{C}_1$, $\mathfrak{C}_2$ and by the straight line $c = 0$.

We investigate now the system $(\mathbb{S}_{fl})$ imposing the conditions that are peculiar of the process of *frost penetration.* We keep the assumption α_0, α_1 constant. Recalling the criterion($\mathbb{C}$) (see par. 1.7), we see that the set of equations $(\mathbb{S}_{fl})$ defined in par. 1.5 takes the form:

$$
(\mathbb{S}^F_{fl}) \quad \begin{cases}
(2.61) \quad z_F(t) - z_S(t) = \dfrac{1}{k_u\alpha_0}\displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta \\[2ex]
(2.62) \quad q_w(t) = (1-\nu_S)\varepsilon\dot{z}_S(t) + \dfrac{k_f\alpha_1 - k_u\alpha_0}{L\rho_w} \\[2ex]
(2.63) \quad \sigma + \displaystyle\int_0^{T_S(t)} \frac{1}{K_1(\eta)}\left(K_2(\eta) - \frac{k_{ff}(\eta)q_w(t)}{k_u\alpha_0}\right)d\eta + \frac{q_w(t)z_F(t)}{K_0} = 0 \\[2ex]
(2.64) \quad q_w(t) = \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}\,k_u\alpha_0 \\[2ex]
(2.65) \quad \rho_i\dot{z}_T(t) = \rho_w q_w(t) + \varepsilon(1-\nu_S)(\rho_i - \rho_w)\dot{z}_S(t)
\end{cases}
$$

with the initial conditions (1.17), (1.18):

$$z_S(0) = b, \quad z_T(0) = H > b.$$

The solution $(T_S(t), z_F(t), z_S(t), q_w(t))$ of $(\mathbb{S}^F_{fl})$ must fulfil the following conditions (cfr. criterion ($\mathbb{C}$) and conditions ($\mathbb{A}$) in parr. 1.7, 1.8):

$$(2.66) \qquad \dot{z}_S < 0$$

$$(2.67) \qquad 0 \le z_F(t) \le z_S(t) \le z_T(t)$$

$$(2.68) \qquad q_w(t) \ge 0.$$

Equation (2.64) gives the water flux in terms of the temperature $T_S(t)$ alone. Expressing also z_F also z_S in terms of T_S, one gets:

$$(2.69) \quad z_F(t) = -\frac{K_0 k_{ff}(T_S(t))}{k_u\alpha_0 K_2(T_S(t))}\left(\sigma + \int_0^{T_S(t)} \frac{K_2(\eta) - \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}k_{ff}(\eta)}{K_1(\eta)}\,d\eta\right)$$

$$(2.70)\quad z_S(t) = -\frac{K_0 k_{ff}(T_S(t))}{k_u \alpha_0 K_2(T_S(t))}\left(\sigma + \int_0^{T_S(t)} \frac{K_2(\eta) - \frac{K_2(T_S(t))}{k_{ff}(T_S(t))} k_{ff}(\eta)}{K_1(\eta)} d\eta\right) -$$

$$-\frac{1}{k_u \alpha_0} \int_0^{T_S(t)} k_{ff}(\eta) d\eta.$$

From (2.62) and (2.64) we find

$$(2.71)\qquad \dot{z}_S(t) = \frac{k_u \alpha_0}{(1-\nu_S)\varepsilon}\left(\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} - c\right)$$

where $c(\alpha_0,\alpha_1) = \dfrac{k_f \alpha_1 - k_u \alpha_0}{L \rho_w k_u \alpha_0}$.

Condition (2.66) imposes that the solution $T_S(t)$ must verify at any time t:

$$(2.72)\qquad K_2(T_S(t)) < c k_{ff}(T_S(t)).$$

Differentiating with respect to time (2.70):

$$(2.73)\qquad \dot{z}_S(t) = -\frac{K_0}{k_u \alpha_0}\left\{\frac{\partial}{\partial T_S}\left(\frac{k_{ff}(T_S(t))}{K_2(T_S(t))}\right)\left(\sigma + \int_0^{T_S(t)} \frac{K_2(\eta)}{K_1(\eta)} d\eta\right) + \frac{k_{ff}(T_S(t))}{K_0}\right\}\dot{T}_S(t)$$

and combining (2.71) with (2.73) we obtain the following ordinary differential equation for the temperature $T_S(t)$:

$$(2.74)\qquad \frac{K_0(1-\nu_S)\varepsilon}{(k_u \alpha_0)^2}\,\varphi(T_S(t))\dot{T}_S(t) = \frac{K_2(T_S(t))}{k_{ff}(T_S(t))} - c$$

where

$$(2.75)\qquad \varphi(s) = -\left\{\frac{\partial}{\partial s}\left(\frac{k_{ff}(s)}{K_2(s)}\right)\left(\sigma + \int_0^{s} \frac{K_2(\eta)}{K_1(\eta)} d\eta\right) + \frac{k_{ff}(s)}{K_0}\right\}$$

We assume that the given functions K_1, K_2, k_{ff} and ν are regular enough (say $C^1(-\infty,0])$ in order to guarantee local existence and uniqueness of a solution $T_S(t)$ of (2.74), supposing, for the moment, to know the initial value $T_S(0)$. The integration with respect to time of (2.74) yelds:

$$(2.76) \qquad \int_{T_S(0)}^{T_S(t)} \frac{K_0(1-\nu_S(y))k_{ff}(y)\varepsilon}{(k_u\alpha_0)^2(K_2(y)-ck_{ff}(y))}\,\varphi(y)dy = t.$$

Formally, we have found the solution of $(\mathcal{S}^F_{fl})$, since, once (2.76) has been inverted, the boundaries $z_F(t)$ and $z_S(t)$ are achieved by (2.69) and (2.70), respectively; the flux $q_w(t)$ and the surface $z_T(t)$ are calculated by means of (2.64) and (2.65), respectively.
However, we have to check in which cases, that is, for which values of (α_0,α_1), the solution is consistent with the conditions (2.66)-(2.68). For this purpose, we set

$$(2.77) \qquad \mathfrak{F} = \{(\alpha_0,\alpha_1)\colon \exists!\ \textit{solution of } (\mathcal{S}^F_{fl}) \textit{ such that } (2.66)\text{-}(2.68) \textit{ hold}\}$$

and we state the following

PROPOSITION 2.2.

i) If (2.34) *holds, then* $\mathfrak{F} = \emptyset$.

ii) If (2.29) *is valid, then* $\mathfrak{F}$ *is the set bounded by*

$$(2.78) \qquad \partial\mathfrak{F} = \{(\alpha_0,\alpha_1)\colon \alpha_0 = \hat{\alpha}_0,\ \alpha_1 \geq \hat{\alpha}_1\} \cup \mathcal{C}_2$$

where $\hat{\alpha}_0$ *and* $\hat{\alpha}_1$ *has been defined by* (2.48) *and the curv* $\mathcal{C}_2$ *by* (2.38).
In particular, if we define the angles

$$(2.79) \qquad \beta_0 = \{(\alpha_0,\alpha_1)\colon\ k_f\alpha_1 < k_u\alpha_0 \leq k_u\alpha_0(1+L\rho_w c_l)\}$$

$$(2.80) \qquad \beta_1 = \{(\alpha_0,\alpha_1)\colon\ k_f\alpha_1 > k_u\alpha_0(1+L\rho_w c_l)\},$$

we have:

ii)a if $(\alpha_0,\alpha_1) \in \mathfrak{F}_0 = \mathfrak{F} \cap \beta_0$, *then there exists a time* t_f *(finite or infinite) such that* $\lim_{t\to t_f} \dot{z}_S(t) = 0$;
ii)b if $(\alpha_0,\alpha_1) \in \mathfrak{F}_1 = \mathfrak{F} \cap \beta_1$, *then there exists a finite time* $\bar{t}$ *such that* $z_F(\bar{t}) = 0$.

Dim.

We begin by evaluating (2.69) and (2.70) when $t=0$:

$$(2.81)\quad z_F(0) = -\frac{K_0 k_{ff}(T_S(0))}{k_u\alpha_0 K_2(T_S(0))}\left(\sigma + \int_0^{T_S(0)} \frac{K_2(\eta) - \frac{K_2(T_S(0))}{k_{ff}(T_S(0))}k_{ff}(\eta)}{K_1(\eta)}\, d\eta\right)$$

$$(2.82)\quad z_S(0) = b = -\frac{K_0 k_{ff}(T_S(0))}{k_u\alpha_0 K_2(T_S(0))}\left(\sigma + \int_0^{T_S(0)} \frac{K_2(\eta) - \frac{K_2(T_S(0))}{k_{ff}(T_S(0))}k_{ff}(\eta)}{K_1(\eta)} d\eta\right) -$$

$$-\frac{1}{k_u\alpha_0}\int_0^{T_S(0)} k_{ff}(\eta)d\eta.$$

i) If (2.34) holds, no temperature $T_S(0)$ can be founded so that (2.69) could be satisfied. Actually, $T_S(0)$ has to be non-negative owing to (2.61) and (2.67); but by (2.81) and (2.34) it would result $z_F(0) < 0$, in contradiction with (2.67). Case *i*) is thus proved.

ii) In order to find solutions of (S_{fl}^F) consistent with the constraints, it is necessary to assume that the temperature T_σ verifying (2.29) exists; by virtue of lemma 2.2, there exists $c_l \in (0, K_2(0)/k_u)$ satisfying (2.30) (we can consider the open interval, since, if c_l exists, necessarily $c_l < K_2(0)/k_u$). So, we see that the angles (2.79) and (2.80) are well defined.

According to the properties of K_2 and k_{ff} (see assumption (H_{11}) and (1.31)), one has

$$(2.83)\qquad \frac{\partial z_F}{T_S} = -\frac{K_0}{k_u\alpha_0}\frac{\partial}{\partial T_S}\left(\frac{k_{ff}(T_S)}{K_2(T_S)}\right)\left(\sigma + \int_0^{T_S}\frac{K_2(\eta)}{K_1(\eta)}d\eta\right) < 0, \text{ for } T_S < T_\sigma.$$

Since $z_F(T_S^*(c_l)) = 0$ (cfr. (2.15) and (2.30)), from (2.83) we deduce that the condition $z_F \geq 0$ is satisfied if and only if $T_S \leq T_S^*(c_l)$.

Let us now examinate the two regions (2.79) and (2.80) separately.

ii)*a* $(\alpha_0, \alpha_1) \in \beta_0$.

We remark that whenever $(\alpha_0, \alpha_1) \in \beta_0$ the value defined immediately after the equation

(2.71) fixes univocally the temperature $T_S^*(c)$ such that (2.15) is verified.
We are going now to discuss the solvability of (2.82), that allows us to achieve the initial value of the freezing temperature $T_S(0)$, on the ground of the knowledge of the really assigned initial condition $z_S(0) = b$.
From (2.71) we deduce that $\dot{z}_S(t) < 0$ if and only if $T_S(t) < T_S^*(c)$, since $f(s) = K_2(s)/k_{ff}(s)$ is an increasing function of s and $f(T_S^*(c)) = c$ (c is fixed once α_0 and α_1 are prescribed).
Therefore, the initial temperature must verify

$$(2.84) \qquad T_S(0) < T_S^*(c).$$

On the other hand, because of (2.61) and (2.67) evaluated at $t = 0$ it must be:

$$(2.85) \qquad -b \le \frac{1}{k_u \alpha_0} \int_0^{T_S(0)} k_{ff}(\eta) d\eta.$$

Condition (2.85) is equivalent to

$$(2.86) \qquad T_S(0) \ge T_S^b(\alpha_0)$$

by virtue of (2.17).

From (2.84) and (2.86) we achieve the condition

$$(2.87) \qquad T_S^b(\alpha_0) < T_S^*(c).$$

Notice that (2.87) excludes the region $\mathfrak{R}_1$ defined by (2.25):

$$\mathfrak{F} \cap \mathfrak{R}_1 = \emptyset.$$

Call

$$(2.88) \qquad B(s) = -\frac{K_0 k_{ff}(s)}{k_u \alpha_0 K_2(s)} \left(\sigma + \int_0^s \frac{K_2(\eta) - \frac{K_2(s)}{k_{ff}(s)} k_{ff}(\eta)}{K_1(\eta)} d\eta \right) - \frac{1}{k_u \alpha_0} \int_0^s k_{ff}(\eta) d\eta.$$

Equation (2.82) is

$$(2.89) \qquad B(T_S(0)) = b.$$

It is easily seen that

$$(2.90) \qquad B(T_S^*(c)) > 0,\ B(T_S^b(\alpha_0)) > b,\ B'(s) < 0 \text{ for } s \le T_S^*(c).$$

We may conclude that (2.82) has a unique solution $T_S(0)$ in the interval $[T_S^b(\alpha_0), T_S^*(c)]$ provided that

$$(2.91) \qquad B(T_S^*(c)) < b.$$

It is evident that $B(T_S^*(c)) = b$ defines the curve $\mathfrak{C}_2$ (see (2.38)). The region outlined by (2.91) corresponds to the right-hand part with respect to the curve $\mathfrak{C}_2$ on the (α_0,α_1)-plane.

Remark 2.3. condition (2.87) is surely fulfilled if (2.91) holds, since the curve $\mathfrak{C}$ (defined by $T_S^b(\alpha_0) = T_S^*(c)$) is closer to the origin than $\mathfrak{C}_2$:

$$(2.92) \qquad \{(\alpha_0,\alpha_1) \in \beta_0 : B(T_S^*(c)) < b\} \subset \{(\alpha_0,\alpha_1) \in \beta_0 :\ T_S^b(\alpha_0) < T_S^*(c)\}.$$

Examinating conditions (2.66)-(2.68), we see that the imposed conditions are fulfilled for $t = 0$, provided that (2.91) holds. Moreover, $z_F(0) = 0$ if and only if $T_S(0) = T_S^b(\alpha_0) = T_S^*(c_l)$.

Once the initial condition $T_S(0)$ has been calculated by means of (2.89), the temperature $T_S(t)$ is achieved by integration from (2.76).

The function φ defined by (2.75) is positive for $s \le T_\sigma$ (cfr. (4.2.83)). Furthermore, since $T_S^*(c) \le T_S^*(c_l) < T_\sigma$ whenever $(\alpha_0,\alpha_1) \in \beta_0$, from (2.76) and from (2.84) we deduce that $\dot T_S(0) > 0$, so $\dot T_S(t)$ keeps positive provided that $T_S(t) < T_S^*(c)$. Moreover:

$$|\dot T_S(t)| \le \frac{\left(c - \dfrac{K_2(T_S(0))}{k_{ff}(T_S(0))}\right)(k_u\alpha_o)^2}{k_u(1-\nu(T_S(0)))}$$

It is easy to check that $T_S(t)$ can not have an asymptote in the interval $(T_S(0), T_S^*(c))$. Thus, the temperature $T_S(t)$ necesssarily reaches by increasing the value $T_S^*(c)$ in a finite or infinite time t_f. In the interval $[0,t_f)$ conditions (2.66)-(2.68) are verified (see (2.69), (2.70), (2.72), (2.73) and (2.83)). When $t = t_f$, we have:

$$z_F(t_f) = z_F^*(\alpha_0,\alpha_1) = -\frac{K_0 k_{ff}(T_S^*(c))}{k_u\alpha_0 K_2(T_S^*(c))}\left(\sigma + \int\limits_0^{T_S^*(c)} \frac{K_2(\eta) - \dfrac{K_2(T_S^*(c))}{k_{ff}(T_S^*(c))}k_{ff}(\eta)}{K_1(\eta)}\, d\eta\right)$$

$$z_S(t_f) = z_S^*(\alpha_0,\alpha_1) = -\frac{K_0 k_{ff}(T_S^*(c))}{k_u\alpha_0 K_2(T_S^*(c))}\left(\sigma + \int\limits_0^{T_S^*(c)} \frac{K_2(\eta) - \dfrac{K_2(T_S^*(c))}{k_{ff}(T_S^*(c))}k_{ff}(\eta)}{K_1(\eta)}\, d\eta\right) +$$

$$+ \frac{1}{k_u\alpha_0}\int\limits_0^{T_S^*(c)} k_{ff}(\eta)d\eta > z_F(t_f)$$

$$q_w(t_f) = k_u\alpha_0 c = \frac{k_{ff}(T_S^*(c))}{K_2(T_S^*(c))}\, k_u\alpha_0 > 0$$

where $T_S^*(c(\alpha_0,\alpha_1))$ is the stationary solution of (2.74) (that can not be accepted owing to the fact that condition (2.66) is not valid), with $T_S(t) \equiv T_S^*(c)$. We see that $\dot{z}_S(t_f) = 0$, that is when $t = t_f$ condition (2.63) fails. By (2.30) we have $z_F(t_f) \geq 0$ and equality holds if and only if $c = c_l$.

Remark 2.4. The function in the integral in (2.76) is positive for $T_S(0) \leq T_S(t) < T_S^*(c)$ and the denominator vanishes if and only if $y = T_S^*(c)$. Thus, the two possibilities $t_f < \infty$, $t_f = \infty$ occur according merely to

$$\text{(2.93)} \qquad \int\limits_{T_S(0)}^{T_S^*(c)} \frac{K_0(1-\nu_S(y))k_{ff}(y)\varepsilon}{(k_u\alpha_0)^2 K_2(y) - c k_{ff}(y)}\, \varphi(y)dy = \infty \;\Rightarrow t_f = \infty$$

$$\text{(2.94)} \qquad \int\limits_{T_S(0)}^{T_S^*(c)} \frac{K_0(1-\nu_S(y))k_{ff}(y)\varepsilon}{(k_u\alpha_0)^2 K_2(y) - c k_{ff}(y)}\, \varphi(y)dy < \infty \;\Rightarrow t_f < \infty$$

The integrals in (2.93), (2.94) can be computed once the functions $K_1(T)$, $K_2(T)$, $k_{ff}(T)$, $\nu(T)$ and the boundary values α_0, α_1 have been expressly prescribed.
We can also invert the equation (2.76) in order to find explicitly $T_S(t) = \Phi^{-1}(t)$, $0 \leq t < t_f$, where

$$\Phi(T_S(t)) = \int_{T_S(0)}^{T_S(t)} \frac{K_0(1-\nu_S(y))k_{ff}(y)\varepsilon}{(k_u\alpha_0)^2K_2(y) - ck_{ff}(y)}\,\varphi(y)dy \tag{2.95}$$

Let us now examinate the second region.

ii)b $(\alpha_0,\alpha_1) \in \beta_1$.

We first discuss, as in the previous case, the solvability of the equation (2.89) for the initial value $T_S(0)$. It must hold, as in case *ii)a*, condition (2.86), otherwise $z_F(0)$ would be negative.

On the other hand, according to (2.67) and (2.69) it must be

$$T_S(0) \le T_S^*(c_l). \tag{2.96}$$

From (2.95) and (2.96) we deduce

$$T_S^b(\alpha_0) \le T_S^*(c_l); \tag{2.97}$$

this condition is equivalent to (cfr. (2.48))

$$\alpha_0 \ge \hat{\alpha}_0 = -\frac{1}{k_u b}\int_0^{T_S^*(c_l)} k_{ff}(\eta)d\eta \tag{2.98}$$

Examinate now the equation (2.89). Formulas (4.2.90) are replaced by:

$$B(T_S^b(\alpha_0)) > b,\ B(T_S^*(c_l)) > 0,\ B'(s) < 0 \text{ for } s \le T_S^*(c_l) \tag{2.99}$$

where $B(s)$ is defined by (2.89).

Therefore, there exists a unique solution $s = T_S(0)$ of (2.88) if and only if $B(T_S^*(c_l)) < b$.

But by the definition of $T_S^b(\alpha_0)$ (2.17) we have

$$B(T_S^*(c_l)) = -\frac{1}{k_u\alpha_0}\int_0^{T_S^*(c_l)} k_{ff}(\eta)d\eta < -\frac{1}{k_u\alpha_0}\int_0^{T_S^b(\alpha_0)} k_{ff}(\eta)d\eta = b$$

Thus, provided that (2.97), there exists exactly one value $T_S(0)$ such that (2.82) is satisfied.

Whenever $(\alpha_0,\alpha_1) \in \beta_1$ and $\alpha_0 \geq \hat{\alpha}_0$ the initial situation is consistent with (2.66)-(2.68). Indeed, we have $\dot{z}_S(0) < 0$, since

$$(2.100) \qquad T_S^*(c) > T_S^*(c_l) \quad \text{for } c_l < c \leq K_2(0)/k_u.$$

If, on the contrary, $c > K_2(0)/k_u$ (that is there is not any $T_S^*(c)$ solution of (2.15)), it still holds $\dot{z}_S(0) < 0$. Moreover, we have $z_F(0) \geq 0$, and equality holds if and only if $T_S(0) = T_S^*(c_l)$.

From (2.74) and (2.75) we see that $\dot{T}_S(t) > 0$ whenever $T_S(0) \leq T_S(t) \leq T_S^*(c_l)$. The front speed $\dot{z}_S(t)$ does not vanid=sh as long as $T_S(t) \leq T_S^*(c_l)$ ($\dot{z}_S(t)$ would be zero if and only if $T_S(t) = T_S^*(c)$, but in β_1 (2.100) holds). The solution $T_S(t)$ of (2.74) can not have asymptotes $\ell < T_S^*(c)$: indeed, the quantity on the right-hand side in (2.74) would not vanish if $T_S(t)$ tended to ℓ. On the other hand, the derivative $\dot{T}_S(t)$ is bounded by:

$$\left|\dot{T}_S(t)\right| < \frac{(K_2(0) + ck_u)(k_u\alpha_o)^2}{k_u^2\,(1 - \nu(T_S(0)))} \quad \text{se } T_S(0) \leq T_S(t) \leq T_S^*(c_l).$$

Thus, the solution $T_S(t)$ of (2.74) must reach the value $T_S^*(c_l)$ in a finite time $\bar{t}$. At that time we have $z_F(\bar{t}) = 0$ and the unfrozen region of the soil becomes exhausted.

Once the temperature $T_S(t)$ has been calculated, by inverting (2.76), we can achieve the boundaries $z_F(t)$ and $z_S(t)$ by means of (2.69) and (4.2.70), respectively. Since $T_S(t) < T_S^*(c_l)$ for $0 \leq t < \bar{t}$, we see that the constraints (2.66) respected.

By (2.61) we find the final thickness of the frozen fringe:

$$z_S(\bar{t}) = -\frac{1}{k_u\alpha_0}\int_0^{T_S(\bar{t})} k_{ff}(\eta)d\eta > 0$$

Giving a geometrical interpretation on the (α_0,α_1)-plane, as we did for the set $\mathfrak{F}_0$, we can say that the left-side boundary of the region $\mathfrak{F}_1$ is the vertical straight line

$$(2.101) \qquad \alpha_0 = -\frac{1}{k_u z_S(t)}\int_0^{T_S^*(c_l)} k_{ff}(\eta)d\eta,$$

while on the left-hand side $\mathfrak{F}_1$ is bounded by the straight line r_{c_l}.

When $t = 0$ we get the condition (2.99). As t increases, the straight line moves on the right towards the point $T_S^*(\alpha_1)$) and when

$$\alpha_0 = -\frac{1}{k_u z_S(\bar{t})}\int_0 k_{ff}(\eta)d\eta,$$

the isotherm $T=0$ reaches the base of the soil. $\square$

On the ground of the analysis of cases *ii)a* and *ii)b* introduced in proposition 2.2, we see that it is more appropriate to denote the regions $\mathcal{F}_0$ and $\mathcal{F}_1$ by $\mathcal{F}_0(z_S(t))$ and by $\mathcal{F}_1(z_S(t))$. On the contrary, during the process of lens formation, the boundaries of $\mathcal{L}(b)$ keep at rest. This corresponds to the fact that the solution for lens formation is global, while the solution for frost penetration is local.

In order to understand better the difference between the teo cases $t_f<\infty$, $t_f=\infty$ when $(\alpha_0,\alpha_1)\in\mathcal{F}_0$ (case *ii)a* of proposition 2.2), we have to make some remarks about the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by (2.37) and (2.38), respectively.

Consider the two families of curves $\mathcal{C}_1(b)$ and $\mathcal{C}_2(b)$ depending on the parameter $b=z_S(0)$ and call

$$P_1^c(b)=r_c\cap\mathcal{C}_1,\ 0\le c\le c_l \qquad P_2^c(b)=r_c\cap\mathcal{C}_2,\ \ 0<c\le c_l$$

where the the straight lines r_c are defined by (2.23). The points $P_1^c(b)$ and $P_2^c(b)$ are well defined according to the properties of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ (see remark 2.1).

If $b_1<b_2$, it is easily checked, by examinating (2.37) and (2.38), that

$$(2.102)\qquad |P_1^c(b_2)-O|\le|P_1^c(b_1)-O|\le|P_2^c(b_2)-O|\le|P_2^c(b_1)-O|,\qquad 0<c\le c_l$$

and equality holds if and only if $c=c_l$.

We remark that the slope of the straight lines r_0 and r_c, which fix the boundaries of the angle β_0, does not depend on b. Hence, inequalities (2.102) mean that as b increases the curves $\mathcal{C}_1(b)$ and $\mathcal{C}_2(b)$ move towards the origin, but keeping inside the angle β_0.

The property (2.102) is showed in fig. 4.2.

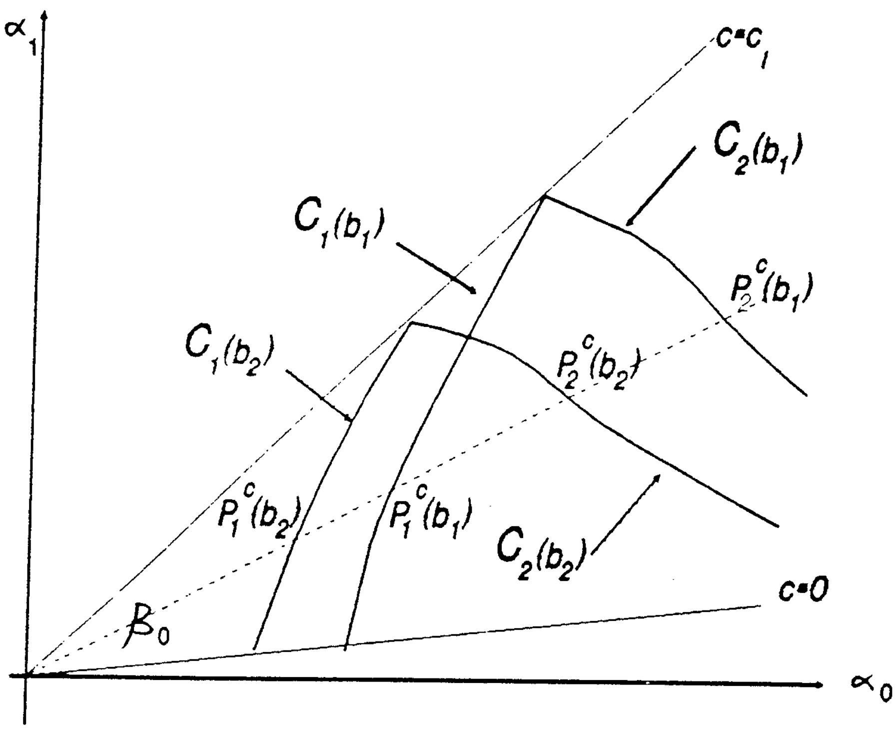

figura 4.2: the curves $\mathcal{C}_i(b_1)$ and $\mathcal{C}_i(b_2)$, $i = 1,2$, with $b_1 < b_2$.

Let us take now $(\alpha_0;\alpha_1) \in \mathcal{T}_0$ and the corresponding solution of $(\mathbf{S}^F_{fl})$ for $t \in [0,t_f)$; for each time $t \in [0,t_f)$ consider the curves

$$(2.103) \qquad \mathcal{C}_2(z_S(t)) = \Big\{(\alpha_0,\alpha_1): \sigma + c\frac{k_u\alpha_0}{K_0}\Big(z_S(t) + \frac{1}{k_u\alpha_0}\int_0^{T^*_S(c)} k_{ff}(\eta)d\eta\Big)$$

$$+ \int_0^{T^*_S(c)} \frac{K_2(\eta) - ck_{ff}(\eta)}{K_1(\eta)}d\eta = 0, 0 < c \leq c_l\Big\}.$$

According to (2.102), the distance between a point (α_0,α_1) fixed in $\mathcal{F}_0$ and the curve $\mathcal{C}_2(z_S(t))$ decreases as t increases.

Moreover, $\dot{z}_S(t)=0$ if and only if $(\alpha_0,\alpha_1)\in\mathcal{C}_2(z_S(t))$; in that case, $T_S(t)=T_S^*(c)$.

Now, if $t_f=\infty$, then $\dot{z}_S(t)$ never vanishes and

$$\lim_{t\to\infty}\left(z_F(t),z_S(t),T_S(t)\right)=\left(z_F^*(\alpha_0,\alpha_1),\ z_S^*(\alpha_0,\alpha_1),T_S^*(c(\alpha_0,\alpha_1))\right).$$

This is equivalent to state that the curves $\mathcal{C}_2(z_S(t))$ never cross the point (α_0,α_1) corresponding to the assigned boundary data.

If, on the contrary, $t_f<\infty$, we have $(\alpha_0,\alpha_1)\in\mathcal{C}_2(z_S(t_f))$ and $\dot{z}_S(t_f)=0$.

Consider the solution $T_S(t)$ of (2.74) for $t\ge t_f$. We have $\dot{T}_S(t_f)=0$. If $\dot{T}_S(t)>0$ in some right interval $t>t_f$, we get a contradition with (2.66), since $T_S(t)$ would be greater than $T_S^*(c)=T_S(t_f)$ (thus $ck_{ff}(T_S(t))>K_2(T_S(t))$), while $\varphi(T_S(t))$ defined by (2.75) is negative. On the other hand, if $\dot{T}_S(t)\le 0$ in some right interval of t_f, the solution of (2.74) is not consistent with (2.66), (2.73).

We conclude that it is not possible to get a solution of $(\mathcal{S}_{fl}^F)$ consistent with the prescribed conditions when $t>t_f$.

We wonder whether a process of lens formation may start at $t=t_f$. As a matter of fact, $T_S(t)\equiv T_S^*(c)$, $z_F(t)\equiv z_F^*(\alpha_0,\alpha_1)$ is solution of $(\mathcal{S}_{fl}^L)$ for $t\ge t_f$ with $b=z_S(t_f)=z_S^*(\alpha_0,\alpha_1)$.

The time $t=t_f$ is a transition time from the process of forst penetration to the other of lens formation. We remark that

$$\frac{\partial p_w(b,t)}{\partial z}=0,\ t\ge t_f$$

and that the point (α_0,α_1) given by the presribed data lies on the boundary $\mathcal{C}_2(z_S(t_f))$ of the region $\mathcal{L}$.

Summarizing the results got in parr. 2.1 and 2.2, e may conclude that for any assignment of the data (α_0,α_1) on the plane and of the initial value $z_S(0)=b<H$, we are in position to foresee if a process of lens formation, or frost penetration or none of them will take place: it is sufficient to check if the point (α_0,α_1) lies in $\mathcal{L}(b)$ or in $\mathcal{F}(b)=\mathcal{F}_0(b)\cup\mathcal{F}_1(b)$ or in none of these regions.

The following proposition sums up the developed analysis.

PROPOSITION 2.3.

i) *If* $\sigma + \int_0^{\vartheta} \frac{K_2(\eta)}{K_1(\eta)} d\eta > 0 \qquad \forall \vartheta \in (-\infty, 0)$, *then* $\mathcal{L}(b) = \mathcal{F}(b) = \emptyset$.

ii) *If* $\exists\, T_\sigma < 0$ *such that* $\sigma + \int_0^{T_\sigma} \frac{K_2(\eta)}{K_1(\eta)} d\eta = 0$,

then $\mathcal{L}$ *and* $\mathcal{F}$ *are two contiguous regions on the* (α_0,α_1)*-plane, having the curve* $\mathcal{C}_2$*, defined by* (2.38)*, as a boundary in common.*

Moreover, whenever $(\alpha_0,\alpha_1) \in \mathcal{L}(b)$*, the system* $(\mathcal{S}_{fl}^L)$ *has a unique solution for any time* $t \geq 0$ *such that the temperature* T_S *is constant, as well as the thickness of the frozen fringe; the growth of the upper boundary of the soil is linear with respect to time. When* $(\alpha_0,\alpha_1) \in \mathcal{F}_0$ *(case ii)a of proposition* 2.2)*, the system* $(\mathcal{S}_{fl}^F)$ *has a unique solution that attains in a finite time (in that case we have a transition to the process of lens formation) or infinite (in that case the solution is global) the stationary values introduced in the proof of thr proposition* 2.2.

When $(\alpha_0,\alpha_1) \in \mathcal{F}_1$ *(case ii)b of proposition* 2.2)*, the system* $(\mathcal{S}_{fl}^F)$ *has a unique solution whose main feature is that the isotherm* $z_F(t) = 0$ *reaches in a finite time the base of the soil.*

The regions $\mathcal{L}(b)$ and $\mathcal{F}(b)$ are outlined in fig. 4.3.

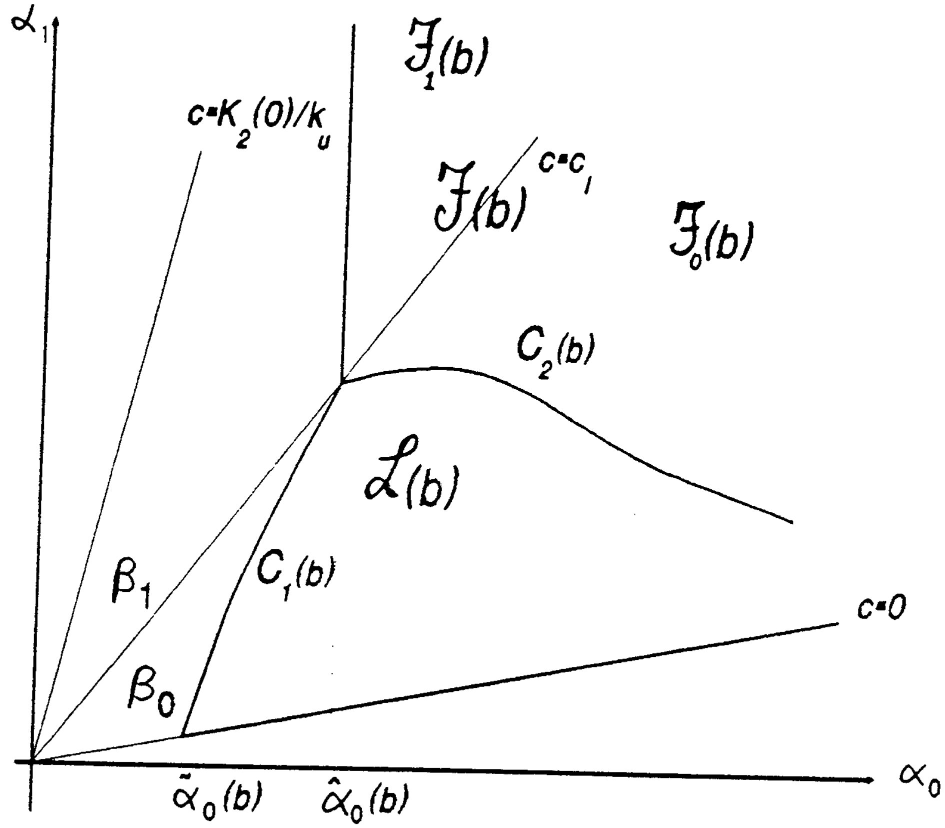

*fig.*4.3: the regions $\mathfrak{L}(b)$ (lens formation) and $\mathfrak{F}(b)$ (frost penetration) on the (α_0, α_1)-plane. The curve $\mathfrak{C}_2(b)$ is a common boundary to the two regions; the region $\mathfrak{F}(b)$ is bounded on the left-hand side bythe straight line $\alpha_0 = \hat{\alpha}_0(b)$. The two parts $\mathfrak{F}_0(b)$ and $\mathfrak{F}_1(b)$ are related respectively to the cases *ii)a* and *ii)b* of proposition 2.2.

3 Boundary thermal fluxes depending on time

We are going now to discuss the solvability of $(\mathbb{S}_{fl})$ (defined in par. 1.5), together with the conditions $(\mathbb{C})$ (par. 1.7) and the constraints $(\mathbb{A})$ (par. 1.8) removing the assumption α_0, α_1 constant.

Our main purpose is investigating the possibility to pass more than once from one process to the other, in order to simulate a process of penetration of the front z_S, which occasionally, under appropriate conditions, stops giving rise to the process of lens formation. In the previous section we have already dealt with one case of transition process form frost penetration to lens formation: more precisely, it occurred when $(\alpha_0,\alpha_1) \in \mathcal{F}_0$ and $t_f < \infty$. We find it convenient to write the set of equations $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ in the following way:

$$
(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A}) \begin{cases}
z_F(t) - z_S(t) = \dfrac{1}{k_u\alpha_0(t)} \displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta \\[2ex]
q_w(t) = (1-\nu_S)\varepsilon \dot{z}_S(t) + \dfrac{k_f\alpha_1(t) - k_u\alpha_0(t)}{L\rho_w} \\[2ex]
\rho_i \dot{z}_T(t) = \rho_w q_w(t) + \varepsilon(1-\nu_S)(\rho_i - \rho_w)\dot{z}_S(t) \\[2ex]
\sigma + \displaystyle\int_0^{T_S(t)} \frac{1}{K_1(\eta)}\left(K_2(\eta) - \frac{k_{ff}(\eta)q_w(t)}{k_u\alpha_0(t)}\right)d\eta + \frac{q_w(t)z_F(t)}{K_0} = 0 \\[2ex]
q_w(t) = -K_1(T_S(t))\dfrac{\partial p_w(z_S(t),t)}{\partial z} + \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}\,k_u\alpha_0(t) \\[2ex]
z_S(0) = b, \quad z_T(0) = H > b \\[2ex]
\dfrac{\partial p_w(z_S(t),t)}{\partial z}\,\dot{z}_S(t) = 0 \\[2ex]
\dot{z}_S(t) \le 0\ , \\[2ex]
\dfrac{\partial p_w(z_S,t)}{\partial z} \ge 0 \\[2ex]
0 \le z_F(t) \le z_S(t) \le z_T(t) \\[2ex]
q_w(t) \ge 0
\end{cases}
$$

Moreover, we know that when $\dot{z}_S(t) < 0$ (henc $\frac{\partial p_w(z_S(t),t)}{\partial z} = 0$), then $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ describes the process of frost penetration; when $\dot{z}_S(t) = 0$, the process is lens formation. We read the assigned data $\alpha_0(t)$, $\alpha_1(t)$ as a curve

$$\gamma \equiv (\alpha_0^\gamma(t), \alpha_1^\gamma(t)) \tag{3.1}$$

which is prescribed on the (α_0, α_1)-plane, with $\alpha_0^\gamma(t) > 0$, $\alpha_1^\gamma(t) > 0$, $t \in [0,t_0]$.
Using the same notations as in sect. 2, we will denote by $\mathcal{L}(b)$ and $\mathcal{F}(b)$ the regions on the plane where the initial conditions allow the occurrence of lens formation and frost penetration, respectively, according to the summary given in proposition 2.3. We will assume that the temperature T_σ verifying (2.29) exists, so that it is $\mathcal{L}(b) \neq \emptyset$, $\mathcal{F}(b) \neq \emptyset$.

3.1 *Preliminary results*

1) *If* $\gamma \subset \mathcal{L}(b)$ $\forall t \in [0,t_0]$, *then* $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ *has a unique solution describing a process of lens formation, such that the temperature* T_S, *the thickness of the frozen fringe* $b - z_F$ *and the hydraulic flux* q_w *depend on time.*
Actually, call

$$c^\gamma(\alpha_0^\gamma(t), \alpha_1^\gamma(t)) = \frac{k_f \alpha_1^\gamma(t) - k_u \alpha_0^\gamma(t)}{L \rho_w k_u \alpha_0^\gamma(t)} \tag{3.2}$$

or, more briefly, $c^\gamma(t)$ and assume, contraryto our claim, that a solution of $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ verifies $\dot{z}_S(\tau) < 0$ at some time $\tau \in [0,t_0]$. By virtue of (2.72), we would have

$$K_2(T_S(\tau)) < c^\gamma(\tau) k_{ff}(T_S(\tau)). \tag{3.3}$$

We denote by $T_S^*(\tau)$ the temperature satisfying $K_2(T_S^*(\tau)) = c^\gamma(\tau) k_{ff}(T_S^*(\tau))$ (it is univocally determined since $(\alpha_0^\gamma(\tau), \alpha_1^\gamma(\tau)) \in \mathcal{L}$, thus $c^\gamma(\tau) \leq c_l < K_2(0)/k_u$ and (2.15) has exactly one solution). From (3.3) we deduce:

$$T_S(\tau) < T_S^*(\tau) \tag{3.4}$$

Moreover, from $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ we easily find:

(3.5) $$b > z_S(\tau) = -\frac{K_0}{k_u\alpha_0^\gamma(\tau)}\, Z(T_S(\tau)) > -\frac{K_0}{k_u\alpha_0^\gamma(\tau)}\, Z(T_S^*(\tau))$$

where

$$Z(s) = \frac{k_{ff}(s)}{K_2(s)}\left(\sigma + \int_0^s \frac{K_2(\eta) - \frac{K_2(s)}{k_{ff}(s)}k_{ff}(\eta)}{K_1(\eta)}\, d\eta\right) + \frac{1}{K_0}\int_0^s k_{ff}(\eta)d\eta$$

(cfr. (2.70)).

Inequality (3.5) yelds that the point $P(\tau) \equiv (\alpha_0^\gamma(\tau), \alpha_1^\gamma(\tau))$ lies in $\mathfrak{F}(z_S(\tau))$ and that $P(\tau) \notin \mathfrak{C}_2(z_S(\tau))$. So, we get a contradiction.

We conclude that, whenever $(\alpha_0^\gamma(t), \alpha_1^\gamma(t)) \in \mathfrak{L}(b)$, it is possible to find a solution of $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ only if $\dot{z}_S(t) = 0$, $t \in [0, t_0]$.

On the other hand, taking $\dot{z}_S(t) \equiv 0$, $t \in [0, t_0]$, the solution of $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$ is for each time t

(3.6) $$q_w(t) = \frac{k_f\alpha_1^\gamma(t) - k_u\alpha_0^\gamma(t)}{L\rho_w}$$

(3.7) $$z_F(t) = \frac{1}{k_u\alpha_0^\gamma(t)}\int_0^{T_S(t)} k_{ff}(\eta)d\eta + b$$

where $T_S(t)$ is the uniqur solution (cfr. (2.13)) of

(3.8) $$\sigma + c^\gamma(t)\, \frac{k_u\alpha_0^\gamma(t)}{K_0}\left(b + \frac{1}{k_u\alpha_0^\gamma(t)}\int_0^{T_S(t)} k_{ff}(\eta)d\eta\right) + \int_0^{T_S(t)} \frac{K_2(\eta) - c^\gamma(t)k_{ff}(\eta)}{K_1(\eta)}d\eta = 0.$$

The speed $\dot{z}_T(t)$ and the height of the soil are given by

(3.9) $$\dot{z}_T(t) = \frac{\rho_w}{\rho_i}\, q_w(t)$$

(3.10) $$z_T(t) = H + \int_0^t \frac{k_f\alpha_1^\gamma(\tau) - k_u\alpha_0^\gamma(\tau)}{L\rho_I}\, d\tau$$

Finally, we achieve the temperature T and the pressure p_w in any point of the frozen fringe and at any time t by means of (1.20) and (1.28).

If we are interested in checking when the thickness of the frozen fringe $b - z_F(t)$ and the temperature $T_S(t)$ are increasing or decreasing with respect to time, we have to derivate

(3.7) and (3.8):

$$\dot{z}_F(t) = \frac{1}{k_u \alpha_0^\gamma(t)} \left(k_{ff}(T_S(t)) \dot{T}_S(t) - \frac{\dot{\alpha}^\gamma{}_0(t)}{\alpha_0^\gamma(t)} \int_0^{T_S(t)} k_{ff}(\eta) d\eta \right) \tag{3.11}$$

$$\dot{T}_S(t) = \frac{ -\frac{b}{K_0 L \rho_w} \left(k_f \dot{\alpha}_1^\gamma(t) - k_u \dot{\alpha}_0^\gamma(t) \right) - \dot{c}(t) \int_0^{T_S(t)} k_{ff}(\eta) \left(\frac{1}{K_0} - \frac{1}{K_1(\eta)} \right) d\eta }{ \frac{c^\gamma(t) k_{ff}(T_S(t))}{K_0} + \frac{K_2(T_S(t)) - c^\gamma(t) k_{ff}(T_S(t))}{K_1(T_S(t))} } \tag{3.12}$$

We make the following remarks (dropping the apex γ, for simplicity):

i) if $\dot{\alpha}_0(t) \le 0$, $\dot{\alpha}_1(t) \ge 0$, then $\dot{T}_S(t) \le 0$, $\dot{z}_F(t) \le 0$, $\dot{q}_w(t) \ge 0$. In particular, if $(\dot{\alpha}_0(t))^2 + (\dot{\alpha}_1(t))^2 > 0$, then $\dot{T}_S(t) < 0$, $\dot{z}_F(t) < 0$, $\dot{q}_w(t) > 0$ (curve γ_1 in fig. 4.4).

ii) If $\dot{\alpha}_0(t) \ge 0$, $\dot{\alpha}_1(t) \le 0$, then $\dot{T}_S(t) \ge 0$, $\dot{z}_F(t) \ge 0$, $\dot{q}_w(t) \le 0$. If $(\dot{\alpha}_0(t))^2 + (\dot{\alpha}_1(t))^2 > 0$, then the inequalities hold in the strict sense (curve γ_2 in fig. 4.4).
For a more general curve γ, we have $\dot{T}_S(t) \le 0$ if and only if

$$\frac{b}{K_0 L \rho_w} \left(k_f \dot{\alpha}_1(t) - k_u \dot{\alpha}_0(t) \right) + \dot{c}(t) \int_0^{T_S(t)} k_{ff}(\eta) \left(\frac{1}{K_0} - \frac{1}{K_1(\eta)} \right) d\eta \ge 0. \tag{3.13}$$

Inequality (3.13) is not easy to solve in terms of $\alpha_0(t)$, $\alpha_0(t)$, since $T_S(t)$ depends on $\alpha_0(t)$ and $\alpha_1(t)$ by means of the implicit relation (3.8).

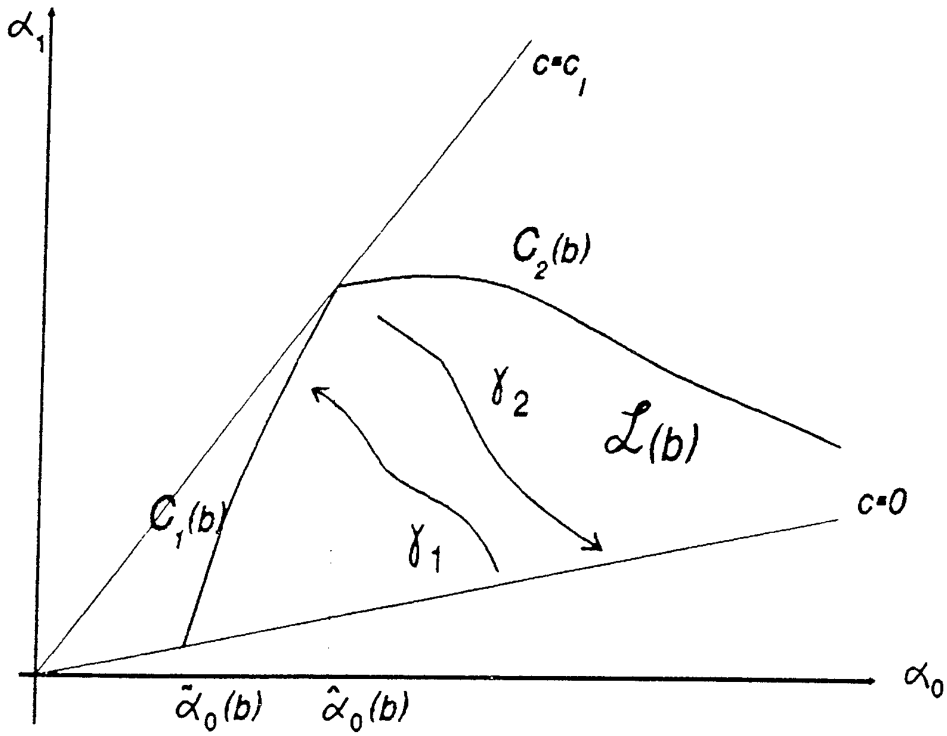

fig. 4.4: process of lens formation along the curves γ_1 and γ_2. Along γ_1 the temperature at the base of the lens T_S decreases, the thickness of the frozen fringe and the hydraulic flux are increasing with respect to time. Along γ_2 T_S is increasing, while the thickness of the frozen fringe and the hydraulic flux decrease.

2) *If the curve* γ *is such that* $(\alpha_0^\gamma(t),\alpha_1^\gamma(t)) \in \mathcal{F}(z_S(t))$ *for any time* $t \in [0,t_0]$, *where* $\mathcal{F}(z_S(t))$ *is the set bounded by the curve* $\mathfrak{C}_2(z_S(t))$ *and defined by* (2.38), *then* $(\mathbb{S}_{fl})+(\mathbb{C})+(\mathbb{A})$ *does not admit solutions describing lens formation (that is with* $\dot{z}_S(t) \equiv 0$).

In order to check 2), it is sufficient to argue as in point 1), by assuming *ab absurdo* that a solution $(\mathbb{S}_{fl})+(\mathbb{C})+(\mathbb{A})$ such that $\dot{z}_S(t) \equiv 0$ may exist. We easily find a contradiction with the assumption $(\alpha_0^\gamma(t),\alpha_1^\gamma(t)) \in \mathcal{F}(z_S(t))$.

Let us keep now the hypothesis $(\alpha_0^\gamma(t),\alpha_1^\gamma(t)) \in \mathcal{F}(z_S(t))$ and write formulas which generalize (2.71) and (2.73), in order to encompass the present case of thermal fluxes depending on time:

$$\dot{z}_S(t) = \frac{k_u \alpha_0^\gamma(t)}{(1-\nu_S)\varepsilon}\left(\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} - c^\gamma(t)\right) \tag{3.14}$$

$$\dot{z}_S(t) = -\frac{K_0}{k_u \alpha_0^\gamma(t)}\left(\frac{\partial}{\partial T_S}\left(\frac{k_{ff}(T_S(t))}{K_2(T_S(t))}\right)\left(\sigma + \int_0^{T_S(t)} \frac{K_2(\eta)}{K_1(\eta)}\, d\eta\right) + \frac{k_{ff}(T_S(t))}{K_0}\right)\dot{T}_S(t) + \tag{3.15}$$

$$+\frac{K_0 \dot{\alpha}^\gamma{}_0(t)}{k_u(\alpha_0^\gamma(t))^2}\left(\frac{k_{ff}(T_S(t))}{K_2(T_S(t))}\left(\sigma + \int_0^{T_S(t)} \frac{K_2(\eta) - \frac{K_2(T_S)}{k_{ff}(T_S)}k_{ff}(\eta)}{K_1(\eta)}\, d\eta\right) + \frac{1}{K0}\int_0^{T_S(t)} k_{ff}(\eta)d\eta\right)$$

By combining (3.14) and (3.15), we get the following ordinary differential equation:

$$\frac{K_0(1-\nu_S)\varepsilon}{(k_u\alpha_0^\gamma(t))^2}\left(\varphi(T_S(t))\dot{T}_S(t) + \frac{\dot{\alpha}^\gamma{}_0(t)}{\alpha_0^\gamma(t)}\psi(T_S(t))\right) = \left(\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} - c^\gamma(t)\right) \tag{3.16}$$

where $\varphi(s)$ has been defined by (2.75) and

$$\psi(s) = \frac{k_{ff}(s)}{K_2(s)}\left(\sigma + \int_0^s \frac{K_2(\eta) - \frac{K_2(s)}{k_{ff}(s)}k_{ff}(\eta)}{K_1(\eta)} d\eta\right) + \frac{1}{K0}\int_0^s k_{ff}(\eta)d\eta. \tag{3.17}$$

If we take constant α_0, α_1 in (3.15), we get the stationary equation (2.74).
The functions $\varphi(s)$ and $\psi(s)$ are surely negative for$s \le T_S^*(c_l)$.
Since we set $(\alpha_0^\gamma(t),\alpha_1^\gamma(t)) \in \mathcal{F}(z_S(t))$, we have, whenever $0 < c \le K_2(0)/k_u$:

$$\sigma + c^\gamma(t)\frac{k_u\alpha_0^\gamma(t)}{K_0}\left(z_S(t) + \frac{1}{k_u\alpha_0^\gamma(t)}\int_0^{T_S^*(c^\gamma(t))} k_{ff}(\eta)d\eta\right) + \tag{3.18}$$

$$+\int_0^{T_S^*(c^\gamma(t))} \frac{K_2(\eta) - c^\gamma(t)k_{ff}(\eta)}{K_1(\eta)} d\eta > 0$$

On the other hand, if $T_S(t)$, $z_S(t)$ verify $(\mathbb{S}_{fl}) + (\mathbb{C}) + (\mathbb{A})$, it must hold for any $t \in [0,t_0]$:

$$\sigma + \frac{K_2(T_S(t))}{k_{ff}(T_S(t))}\frac{k_u\alpha_0^\gamma(t)}{K_0}\left(z_S(t) + \frac{1}{k_u\alpha_0^\gamma(t)}\int\limits_0^{T_S(t)} k_{ff}(\eta)d\eta\right) + \tag{3.19}$$

$$+\int\limits_0^{T_S(t)} \frac{K_2(\eta) - \frac{K_2(T_S(t))}{k_{ff}(T_S(t))}k_{ff}(\eta)}{K_1(\eta)}d\eta = 0$$

From (3.18) and (3.19) and reminding that $c^\gamma(t) = \frac{K_2(T_S^*(c^\gamma(t)))}{k_{ff}(T_S^*(c^\gamma(t)))}$, we deduce:

$$T_S(t) < T_S^*(c^\gamma(t)) \tag{3.20}$$

and

$$\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} < c^\gamma(t). \tag{3.21}$$

If $c > K_2(0)/k_u$, then (3.21) obviously holds.

We may conclude that, when $\gamma \subset \mathfrak{F}(z_S(t))$ (that is γ is ever on the right side of the curve $\mathfrak{C}_2(z_S(t))$), the fornt $z_S(t)$ has the property $\dot{z}_S(t) < 0$. On the other hand, if we know the solution $T_S(t)$ of (3.16), we can achieve from (2.70) (which is valid even in the case α_0, α_1 depending on time) the boundary $z_S(t)$ and, by means of (2.103), the profiles of the curves $\mathfrak{C}_2(z_S(t))$ that delimitate $\mathfrak{F}(z_S(t))$ at each time t. By comparing the path γ assigned as in (3.1) with the regions $\mathfrak{F}(z_S(t))$, we are able to evaluate how long the process of frost penetration lasts.

If we add the hypothesis $\dot{\alpha}_0^\gamma(t) \leq 0$, then we get from (3.16):

$$\dot{T}_S(t) > 0. \tag{3.22}$$

3.2 *A transition process*

We are now in position to describe a process where the curve γ moves form the region $\mathfrak{F}$ to $\mathfrak{L}$ and viceversa.

Consider a path $\gamma \equiv (\alpha_0^\gamma(t), \alpha_1^\gamma(t))$, $t \in [0, t_0]$, such that $(\alpha_0^\gamma(0), \alpha_1^\gamma(0)) \in \mathfrak{F}$.

Assume, at least for the moment, that the additional hypothesis

$$\dot{\alpha}_0^\gamma(t) \leq 0 \tag{3.23}$$

holds.

Getting $\dot{T}_S$ from (3.16) yelds

$$(3.24)\qquad \dot{T}_S(t)=\frac{1}{\varphi(T_S(t))}\left(-\frac{\dot{\alpha}^{\gamma}{}_0(t)}{\alpha_0^{\gamma}(t)}\psi(T_S(t))+\frac{(k_u\alpha_0^{\gamma}(t))^2}{K_0(1-\nu_S)\varepsilon}\left(\frac{K_2(T_S(t))}{k_{ff}(T_S(t))}-c^{\gamma}(t)\right)\right)$$

Equation (3.24) shows that

$$(3.25)\qquad \dot{T}_S(t)>0\begin{cases} for\ \ T_S(t)<min\{T_S^*(c^{\gamma}(t)),T_S^*(c_l)\}, & 0<c^{\gamma}(t)<K_2(0)/k_u \\ for\ \ T_S(t)<T_S^*(c_l), & c\geq K_2(0)/k_u \end{cases}$$

Furthermore, the derivative $\dot{T}_S(t)$ is bounded by a constant if

$$\sup_{t\in[0,t_0]}\left|\dot{\alpha}_0^{\gamma}(t)\right|<\infty$$

and $T_S(t)$ can not have an asymptote $\ell\neq T_S^*(c^{\gamma}(t))$.

We may conclude that, whenever $\gamma\subset\mathcal{F}_1(z_S(t))$ for any $t\in[0,t_0]$ (curve γ_1 in fig. 4.5), then $T_S(t)$ reaches the temperature $T_S^*(c_l)$ in a finite time $\bar{t}$; in that case, the isotherm $T=0$ matches the base of the soil, that is $z_F(\bar{t})=0$. The final thickness of the frozen fringe is

$$(3.26)\qquad z_S(\bar{t})=\frac{1}{k_u\alpha_0^{\gamma}(\bar{t})}\int_0^{T_S(\bar{t})}k_{ff}(\eta)d\eta.$$

The second possibility for the path γ is when there exists a time $t_f<t_0$ such that $(\alpha_0^{\gamma}(t_f),\alpha_1^{\gamma}(t_f))\in\mathcal{C}_2(z_S(t_f))$ (fig. 4.6); hence $T_S(t_f)=T_S^*(c^{\gamma}(t_f))$ and $\dot{z}_S(t_f)=0$. We remark that in the present case of thermal fluxes depending on time, $\dot{T}_S(t_f)$ is not generally zero, as it occurs in the case α_0, α_1 constant.

It may happen, as we saw, that $T_S(t)$ tends to $T_S^*(c^{\gamma})$ in infinite time ($t_f=\infty$). In such as case, α_0^{γ} must verify $\lim_{t\to\infty}\dot{\alpha}_0^{\gamma}(t)=0$, otherwise it is impossible to have stationary solutions. Conversely, assume that $t_f<\infty$ and that $(\alpha_0^{\gamma}(t),\alpha_1^{\gamma}(t))\in\mathcal{L}(z_S(t_f))$ in some interval $t\geq t_f$ (curve γ_2 in fig. 4.6). In that case, a lens starts growing at the time t_f and at the height $z=z_S(t_f)$. As long as γ lies in $\mathcal{L}(z_S(t_f))$, which has fixed boundaries, the thickness of the lens goes on increasing; the temperature T_S and the thickness of the frozen fringe $z_S(t_f)-z_F(t)$ change with respect to time, according to the statement 1) in par. 3.1.

Essentially, the process of lens formation developes following two directions:

i) there exists a time $t_1 > t_f$ such that $(\alpha_0^\gamma(t_1),\alpha_1^\gamma(t_1)) \in \mathcal{C}_1(z_S(t_f))$: in that case the frozen fringe invades the whole unfrozen soil and $z_F(t_1) = 0$ (branch γ_2^a in fig. 4.6);

ii) there exists a time t_2 so that $k_u\alpha_0^\gamma(t_2) = k_f\alpha_1^\gamma(t_2)$: in that case, the water flux q_w vanishes at $t = t_2$ and a melting process will take place for $t > t_2$, if γ goes under the straight line $k_u\alpha_0 = k_f\alpha_1$ (branch γ_2^b in fig. 4.6).

For more general curves γ, which has not the property (3.23), further posiibilities occur, in addition to the ones just introduced.

First of all, $\dot T_S(t)$ may also be negative, if $\dot\alpha_0^\gamma(t) > 0$.

As long as $T_S(t) < min\{T_S^*(c^\gamma(t)),T_S^*(c_l)\}$, the velocity of the front $z_S(t)$ and the boundary $z = z_F(t)$ never vanish. If $T_S(t)$ decreases, it tends to a stationary value T_l if and only if

$$(3.27) \qquad \lim_{t\to\infty}\left(\psi(T_l)K_0(1-\nu(T_l))\ \varepsilon\ \frac{\dot\alpha^\gamma{}_0(t)}{(\alpha_0^\gamma(t))^3} + k_u^2 c^\gamma(t) \right) = k_u^2\ \frac{K_2(T_l)}{k_{ff}(T_l)}\ .$$

Conversely, if $T_S(t)$ increases up to the temperature $T_S^*(c^\gamma(t))$ when $t = t_f$ and $\gamma \subset \mathcal{L}(z_S(t_f))$ for $t > t_f$, then the process of lens formation starts.

In addition to the possibilities stated above as for the stop of the process of lens formation, a third case may occur, when the curve γ goes into the region $\mathcal{F}(z_S(t_f))$ once again, say when $t = t_3 > t_f$ and a second process of frost penetration takes place with the initial condition $z_S(t_f)$ for the boundary $z_S(t)$ (branch γ_2^d in fig. 4.6).

Iterating the described process, one can get a sequence oflenses and each ice layer is separated from the previous one (which is above) by a layer of frozen soil, corresponding to a process of frost penetration.

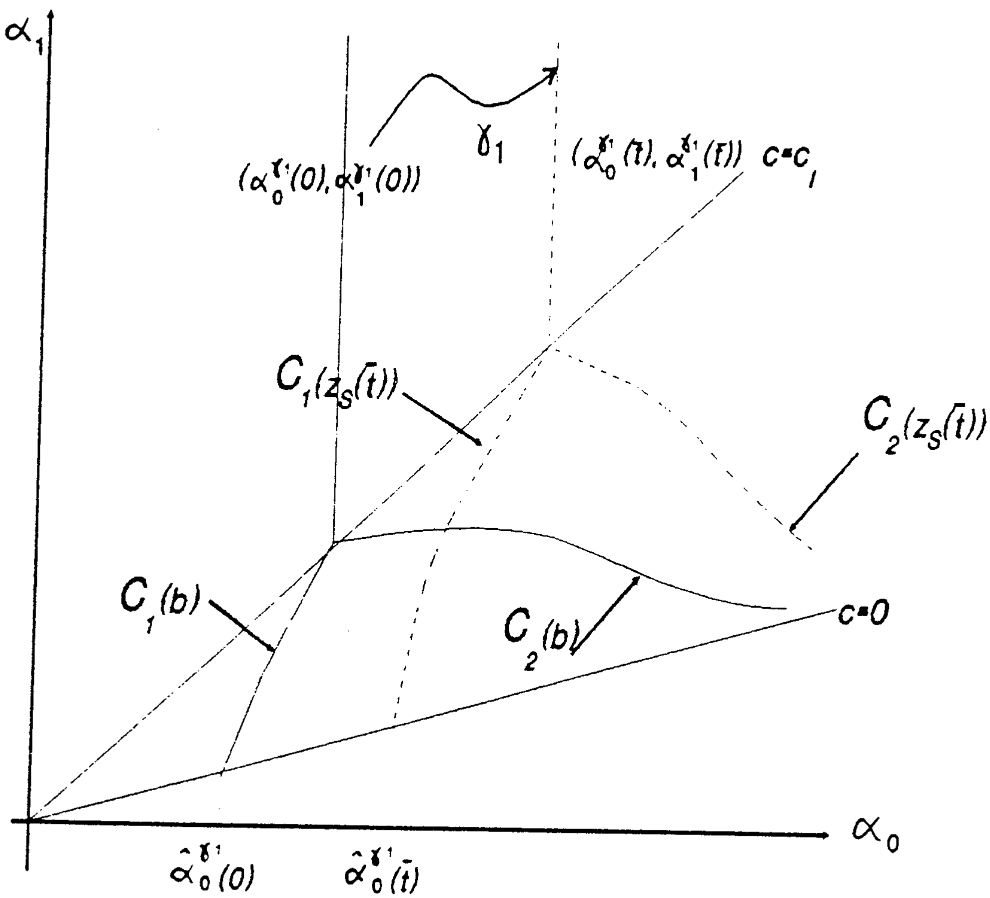

fig. 4.5: development of the process of frost penetration along the curve γ_1. For $t = \bar{t}$ the isotherm z_F reaches the base of the soil.

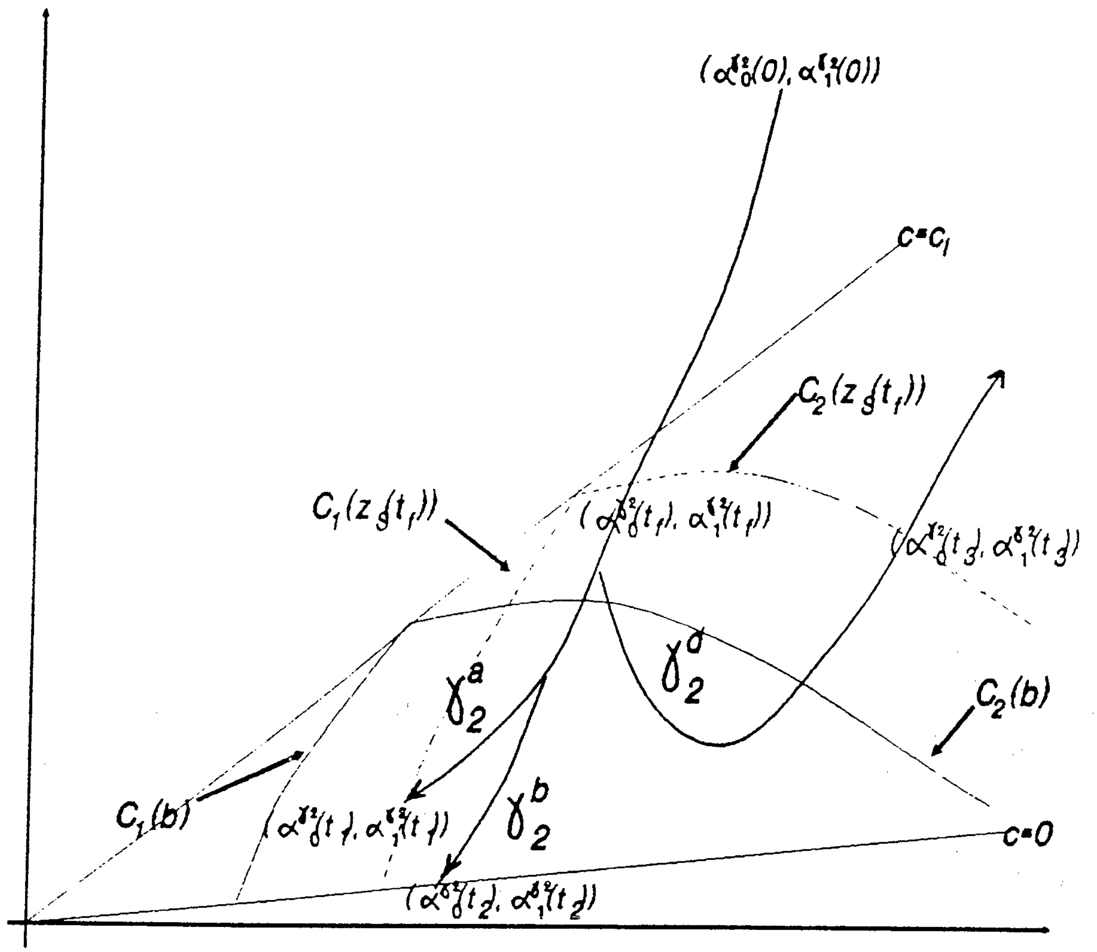

fig. 4.6: process of frost penetration starting from the boundary fluxes values $(\alpha_0^{\gamma}(0),\alpha_1^{\gamma}(0)) \in \mathcal{F}(z_S(0))$.

For $t = t_f$ the velocity of the front z_S vanishes and lens formation succeeds frost penetration. Along the branch γ_2^a the process stops since z_F reaches (by decreasing) the base of the soil when $t = t_1$; along γ_2^b the water flux q_w vanishes when $t = t_2$; along γ_2^d a second process of frost penetration takes place when $t = t_3$.

4 Temperatures specified at the boundaries

We write again the set of equations $(\mathbb{S}_{tmp})+(\mathbb{C})+(\mathbb{A})$, defined in sect. 2 that describes the freezing process in the case that we assign the temperatures at the extremities of the soil $h(t)=T(0,t)$ and $g(t)=T(z_T(t),t)$, $h(t)>0$, $g(t)<0$:

$$
(\mathbb{S}_{tmp})+(\mathbb{C})+(\mathbb{A})\left\{
\begin{array}{ll}
(4.1) & \displaystyle\int_0^{T_S(t)} k_{ff}(\eta)d\eta = \frac{z_F(t)-z_S(t)}{z_F(t)}k_u h(t) \\[2ex]
(4.2) & \displaystyle L\rho_w q_w(t) = (1-\nu_S)\varepsilon\rho_w L\dot{z}_S(t) - k_f\frac{g(t)-T_S(t)}{z_T(t)-z_S(t)} - \frac{k_u h(t)}{z_F(t)} \\[2ex]
(4.3) & \rho_i\dot{z}_T(t) = \rho_w q_w(t) + \varepsilon(1-\nu_S)(\rho_i-\rho_w)\dot{z}_S(t) \\[2ex]
(4.4) & \displaystyle \sigma + \int_0^{T_S(t)} \frac{K_2(\eta) - \frac{q_w(t)z_F(t)}{k_u h(t)}k_{ff}(\eta)}{K_1(\eta)}d\eta = -\frac{q_w(t)}{K_0}z_F(t) \\[2ex]
(4.5) & \displaystyle q_w(t) = -K_1(T_S(t))\frac{\partial p_w(z_S(t),t)}{\partial z} + K_2(T_S(t))\frac{k_u h(t)}{z_F(t)k_{ff}(T_S(t))} \\[2ex]
(4.6) & z_S(0)=b, \quad z_T(0)=H>b \\[2ex]
(4.7) & \displaystyle \frac{\partial p_w(z_S(t),t)}{\partial z}\,\dot{z}_S(t) = 0 \\[2ex]
(4.8) & \dot{z}_S(t)\le 0\ , \\[2ex]
(4.9) & \displaystyle \frac{\partial p_w(z_S,t)}{\partial z} \ge 0 \\[2ex]
(4.10) & 0\le z_F(t)\le z_S(t)\le z_T(t) \\[2ex]
(4.11) & q_w(t)\ge 0
\end{array}\right.
$$

The unknown quantities are the water flux $q_w(t)$, the boundaries $z_F(t)$, $z_S(t)$ and $z_T(t)$, the freezing temperature $T_S(t)$ and the water pressure $p_w(z_S(t),t)$. We recall again that when $\dot{z}_S(t)<0$ (in ttha case the water pressure gradient vanishes at $z=z_S$, see ...), we have a *frost penetration* process; when $\dot{z}_S(t)=0$, *lens formation* occurs.

Remark 4.1. In theory, we should make use of the results in sections 2 and 3, in order to solve the present case, basing ourselves on the relations

$$\alpha_0(t) = \frac{h(t)}{z_F(t)}, \quad \alpha_1(t) = \frac{T_S(t) - g(t)}{z_T(t) - z_S(t)}.$$

Considering, for istance, the case of lens formation, once $h(t)$ and $g(t)$ are given, we should find α_0 and α_1 such that the following equations, coming from (4.1), (3.8) e (3.10), are satisfied:

$$\alpha_0(t) = \frac{k_u b}{k_u h(t) - \int_0^{T_S(t)} k_{ff}(\eta) d\eta}$$

$$\alpha_1(t) = \frac{T_S(t) - g(t)}{H + \frac{1}{L\rho_i}\int_0^t (k_f \alpha_1(\tau) - k_u \alpha_0(\tau))d\tau - b}$$

$$\sigma + c(t)\,\frac{k_u \alpha_0(t)}{K_0}\left(b + \frac{1}{k_u \alpha_0(t)}\int_0^{T_S(t)} k_{ff}(\eta) d\eta\right) + \int_0^{T_S(t)} \frac{K_2(\eta) - c(t) k_{ff}(\eta)}{K_1(\eta)} d\eta \ = 0$$

where $c(t)$ is defined by (3.2).

The system we have just written is anything but simple, mainly owing to the implicit dependence of T_S on the fluxes α_0 and α_1 in the third equation.

Therefore, we believe it is more convenient and more interesting to solve directly the problem (4.1)-(4.11).

In paragraphes 4.1 and 4.2 we will discuss separately the cases of lens formation and frost penetration, respectively. Finally, we will investigate on the posssibility of a transition process (par. 4.3).

4.1 *Lens formation*

From (4.1) we get $z_F(t)$ in terms of $T_S(t)$:

(4.12) $$z_F(t) = \frac{k_u h(t) b}{k_u h(t) - \int_0^{T_S(t)} k_{ff}(\eta)d\eta}$$

Eliminating $q_w(t)$ and $z_F(t)$ from (4.2) e (4.4) by means of (4.3) and (4.12), we fnd the following problem for $T_S(t)$ and $z_T(t)$:

$$\begin{cases} (4.13) \qquad \dot z_T(t)\ \varphi_1(T_S(t),t) + \varphi_2(T_S(t)) = 0 \\[2ex] (4.14) \qquad \dot z_T(t) = -\dfrac{k_f}{L\rho_i}\dfrac{g(t) - T_S(t)}{z_T(t) - b} + \varphi_3(T_S(t),t) \end{cases}$$

where

(4.15) $$\varphi_1(s,t) = \frac{\rho_i b}{\rho_w K_0}\ \frac{k_u h(t) - K_0 \displaystyle\int_0^s \frac{k_{ff}(\eta)}{K_1(\eta)}d\eta}{k_u h(t) - \displaystyle\int_0^s k_{ff}(\eta)d\eta}$$

(4.16) $$\varphi_2(s) = \int_0^s \frac{K_2(\eta)}{K_1(\eta)}d\eta + \sigma$$

(4.17) $$\varphi_3(s,t) = \frac{1}{L\rho_i b}\left(\int_0^s k_{ff}(\eta)d\eta - k_u h(t)\right)$$

We remark that:

(4.18) $$\varphi_1(s,t) > 0,\ \varphi_1(s,t) < 0 \text{ per } s \le 0,\ t \ge 0.$$

From (4.13) and (4.14) one gets the following identity for $z_T(t)$ as a function of the temperature T_S:

(4.19) $$z_T(t) = b + \frac{k_f}{L\rho_i}\,\frac{\varphi_1(T_S(t),t)}{\varphi_2(T_S(t)) + \varphi_1(T_S(t),t)\varphi_3(T_S(t),t)}\,(g(t) - T_S(t)) = \varphi(T_S(t),t).$$

Derivating (4.19) with respect to time and comparing with (4.13), we get the following ordinary differential equation for $T_S(t)$:

(4.20) $$-\frac{\varphi_2(T_S(t))}{\varphi_1(T_S(t),t)} = \frac{\partial\varphi(T_S(t),t)}{\partial T_S}\,\dot T_S(t) + \frac{\partial\varphi(T_S(t),t)}{\partial t}.$$

We must add an initial condition for T_S, that is not esplicitly prescribed by the set of equations (4.1)-(4.11): $T_S(0)$ has to be computed, using the known initial values $h(0)$, $g(0)$, H e b, and solving the equation $\varphi(T_S(0),0) = H$, that is

$$(4.21) \qquad -\frac{\varphi_2(T_S(0))}{\varphi_1(T_S(0),0)} = -\frac{k_f}{L\rho_i}\frac{g(0)-T_S(0)}{H-b} + \varphi_3(T_S(0),0)$$

Our first purpose is to discuss the solvability of (4.21). To tis end, we rewrite the conditions (4.7)-(4.11), which in case of lens formation become

$$(4.22) \qquad \frac{\partial p_w(b,t)}{\partial z} \geq 0$$

$$(4.23) \qquad 0 \leq z_F(t) \leq b$$

$$(4.24) \qquad \dot{z}_T(t) \geq 0,$$

in terms of the temperature T_S. We will first concentrate our attention on the initial time $t=0$: we will look for suitable conditions on the initial data $h(0)$, $g(0)$, H and b, so that the initial situation is consistent with the conditions (4.22)-(4.24), computed for $t=0$.

We start with the following

LEMMA 4.1. *If there exists a temperature T_σ such that $\varphi_2(T_\sigma)=0$, then for all $t \geq 0$ there is exactly one value $T_p(h(t))$ satisfying the equation*

$$(4.25) \qquad \varphi_2(T_p(h(t)) = \rho_w L \frac{K_2(T_p(h(t))}{k_{ff}(T_p(h(t))} \varphi_1(T_p(h(t)),t)\varphi_3(T_p(h(t)),t)$$

Dim. We write (4.25) as follows:

$$(4.26) \qquad k_u h(t) = r(T_p(h(t)))$$

where

$$(4.27) \qquad r(s) = -K_0\frac{k_{ff}(s)}{K_2(s)}\left(\sigma + \int_0^s \frac{K_2(\eta) - \frac{K_2(s)}{k_{ff}(s)}k_{ff}(\eta)}{K_1(\eta)}d\eta\right).$$

Let us consider the function

$$F(s) = \sigma + \int_0^s \frac{K_2(\eta) - \frac{K_2(s)}{k_{ff}(s)} k_{ff}(\eta)}{K_1(\eta)} d\eta, \qquad s \le 0.$$

From the hypothesis (H_{11}) and from (1.31) one finds:

$$(4.28) \qquad F'(s) = -\left(\frac{K_2(s)}{k_{ff}(s)}\right)' \int_0^s \frac{K_2(\eta)}{K_1(\eta)} d\eta > 0,\ F(s) > \varphi_2(s) \quad \text{for } s < 0,\ \ F(0) = \varphi_2(0) = \sigma.$$

Assume *ab absurdo* that

$$(4.29) \qquad F(s) \ge 0 \quad \forall s \le 0.$$

By virtue of the properties of the functions K_2 e k_{ff} we have just recall, for any fixed $\bar{s} \le 0$ there exists exaclty one value $\bar{c} \in (0, K_2(0)/k_u]$ such that (see also (2.15))

$$K_2(\bar{s}) = \bar{c} k_{ff}(\bar{s}).$$

If (4.29) is valid, it would follow

$$(4.30) \qquad \sigma + \int_0^{T_S^*(\bar{c})} \frac{k_{ff}(\eta) - \bar{c} K_2(\eta)}{K_1(\eta)} d\eta \ge 0,$$

but (4.30) is in contradiction with Lemma 2.2 of sect. 2.1.
Therefore, there exists $s_1 < 0$ such that $F(s_1) < 0$.
Reminding (4.28) and (1.31), we find, for $s < s_1$:

$$r(s) = -K_0 \frac{k_{ff}(s)}{K_2(s)} F(s) > -K_0 \frac{k_{ff}(s)}{K_2(s)} F(s_1) > -K_0 \frac{k_u}{K_2(s)} F(s_1).$$

Since $K_2(s)$ tends to zero for s going to $-\infty$, we conclude that the function $r(s)$ reaches somewhere the value $k_u h(t)$: in other words, (4.26) has at least one solution for each $t \ge 0$ (notice that r does not depend explicitly on time t).
The uniqueness of the solution, that we call $T_p(h(t))$ to point out the dependence on time only through the boundary temperature h, is immediately achieved by observing

that $r(0) < 0$ and

$$(4.31)\qquad r'(s) \begin{cases} > 0 \text{ for } T_\sigma < s < 0 \\ = 0 \text{ for } s = T_\sigma \\ < 0 \text{ for } s < T_\sigma \end{cases}$$

Finally, taking into account that $r(s)$ achieves its negative minimum for $s = T_\sigma$, we have

$$(4.32)\qquad T_p(h(t)) < T_\sigma. \qquad \square$$

The following result allows us to estabilish when the equation (4.21) has soluctions consistent with the conditions (4.22)-(4.24), computed for $t = 0$.

PROPOSITION 4.1.

i) If

$$(4.33)\qquad \int_0^s \frac{K_2(\eta)}{K_1(\eta)} d\eta + \sigma > 0 \qquad \forall s \le 0,$$

then (4.13)-(4.14) *has not solutions consistent with* (4.22)-(4.24).

ii) Suppose that there exists one temperature T_σ *such that*

$$(4.34)\qquad \int_0^{T_\sigma} \frac{K_2(\eta)}{K_1(\eta)} d\eta + \sigma = 0,$$

then the following cases occur:

ii)a if

$$(4.35)\qquad k_f g(0) > k_f T_\sigma + (H - b) L \rho_i \varphi_3(T_\sigma, 0),$$

then (4.21) *has not solutions consistent with* (4.22)-(4.24) *computed for* $t = 0$.

ii)b If

$$(4.36) \qquad k_f T_p + (H-b)L\rho_i\varphi_3(T_p,0)\left(1+\rho_w L\frac{K_2(T_p)}{k_{ff}(T_p)}\right) \le k_f g(0) \le k_f T_\sigma + (H-b)L\rho_i\varphi_3(T_\sigma,0)$$

then the initial data are consistent with (4.22)-(4.24) and (4.21) is satisfied for exactly one value $T_S(0) \in [T_\rho(h(0),g(0),b,H),\ T_\sigma]$ *where*

$$(4.37) \qquad T_\rho = max\{T_p(h(0)), T_m(h(0),g(0),b,H)\},$$

while the temperature $T_p(h(0))$ *is defined by (4.26) (computed for* $t=0$*) and* T_m *is the one value such that*

$$(4.38) \qquad \frac{k_f(g(0)-T_m)}{L\rho_i\varphi_3(T_m,0)} = H-b$$

ii)c If

$$(4.39) \qquad k_f g(0) < k_f T_p + (H-b)L\rho_i\varphi_3(T_p,0)\left(1+\rho_w L\frac{K_2(T_p)}{k_{ff}(T_p)}\right),$$

then we conclude as in ii)a.

Dim. From (4.1) and (4.23) it follows that the temperature T_S cannot be positive. Hence, if $T_S(t)$ is the solution of our problem, we get from (4.18):

$$(4.40) \qquad \varphi_1(T_S(t),t) > 0, \quad \varphi_3(T_S(t),t) < 0$$

Assume that the hypothesis (4.33) holds: this means that $\varphi_2(s) > 0$ for all $s \le 0$. We see immediately that the equation (4.13) cannot be satisfied by a function $z_T(t)$ such that condition (4.24) is fulfilled. So, the case $i)$ is proved.

Let us now assume that there exists a temperature T_σ, which depends only on the functions K_1, K_2 and on σ (but not on the boundary temperatures h and g) such that (4.34) holds, that is $\varphi_2(T_\sigma) = 0$.

Since $\varphi_2(s)$ is increasing for $s \le 0$, we have:

$$(4.41) \qquad \varphi_2(s) < 0 \text{ if and only if } s < T_\sigma$$

Therefore, the solution $T_S(t)$ has to verify:

$$(4.42) \qquad T_S(t) \leq T_\sigma,$$

otherwise (4.13) is not consistent with (4.24).
On the other hand, from (4.21) and from (4.40) deduce that it must hold

$$(4.43) \qquad z_T(0) = H \leq b + \frac{k_f(g(0) - T_S(0))}{L\rho_i\varphi_3(T_S(0),0)} = F_1(T_S(0)) \ .$$

We remark that $F_1(s)$ is an increasing function for $s \leq 0$. So, if

$$(4.44) \qquad H > F_1(0) = b\left(1 - \frac{k_f g(0)}{k_u h(0)}\right),$$

no initial value $T_S(0)$ will satisfy the condition (4.43).
On the contrary, let be

$$(4.45) \qquad H \leq b\left(1 - \frac{k_f g(0)}{k_u h(0)}\right);$$

we easily see that there exists exctly one value T_m, $g(0) < T_m \leq 0$, depending on $h(0)$ and on $g(0)$, such that $F_1(T_m) = H$. We notice that

$$(4.46) \qquad T_m > g(0)$$

(indeed $F_1(g) = b < H = F_1(T_m)$).
Owing to the properties of F_1, we can conclude that (4.43) is verified if and only if

$$(4.47) \qquad T_S(0) \geq T_m.$$

By comparing (4.42) computed for $t = 0$ with (4.47) we see that it must hold:

$$(4.48) \qquad T_m \leq T_\sigma.$$

Condition (4.48) is equivalent to $F(T_\sigma) \geq H$, that is

(4.49) $\quad k_f(g(0)-T_\sigma) \le (H-b)L\rho_i\varphi_3(T_\sigma,0).$

Therefore, if (4.35) holds we see that no initial value $T_S(0)$ is such that (4.43) is satisfied. Thus, the case *ii)a* of proposition 4.1 is proved.
Assume now that (4.49) holds.
We have still to impose the condiotion on the water pressure gradient (4.22), that in terms of the temperature T_S becomes

(4.50) $$\varphi_2(T_S(t)) \ge \rho_w L \frac{K_2(T_S(t))}{k_{ff}(T_S(t))}\,\varphi_1(T_S(t),t)\varphi_3(T_S(t),t)$$

Condition (4.50) is equivalent to (see (4.27))

(4.51) $\quad r(T_S(t)) \le k_u h(t).$

By virtue of Lemma 4.1, whenever (4.34) is true there exists exactly one temperature $T_p(h(t))$ such that (4.51) vanishes.
Recalling (4.31), we see that (4.51) is true for a time $t \ge 0$ if and only if

(4.52) $\quad T_S(t) \ge T_p(h(t)).$

Now, the two posiibilites can occur (see (4.32)):

1) $\quad T_p(h(0)) \le T_m(h(0),g(0)) \le T_\sigma$

2) $\quad T_m(h(0),g(0)) < T_p(h(0)) < T_\sigma$

Let us write again (4.21) in the following way:

(4.53) $\quad \Psi(T_S(0)) = F_1(T_S(0)) - H$

where

(4.54) $$\Psi(s) = \psi(s,0) = (H-b)\,\frac{\varphi_2(s)}{\varphi_1(s,0)\varphi_3(s,0)}\,, \quad s \le 0$$

and $F_1(s)$ is defined by (4.43). We have:

(4.55) $\Psi(s) > 0 \; for \; s < T_\sigma \; , \; \Psi(T_\sigma) = 0$

$$(4.56) \qquad \Psi'(s) = -\frac{K_0\rho_w L}{K_1(s)} \frac{K_2(s)\left(k_u h(0) - K_0 \int\limits_0^s \frac{k_{ff}(\eta)}{K_1(\eta)}d\eta\right) + K_0 k_{ff}(s)\left(\sigma + \int\limits_0^s \frac{K_2(\eta)}{K_1(\eta)}d\eta\right)}{\left(k_u h(0) - K_0 \int\limits_0^s \frac{k_{ff}(\eta)}{K_1(\eta)}d\eta\right)^2} =$$

$$= -\frac{K_0\rho_w L K_2(s)}{K_1(s)} \frac{k_u h(0) - r(s)}{\left(k_u h(0) - K_0 \int\limits_0^s \frac{k_{ff}(\eta)}{K_1(\eta)}d\eta\right)^2}$$

Thus:

(4.57) $\Psi'(s) < 0$ if $T_p < s < 0$, $\Psi'(T_p) = 0$.

In the case 1, the equation $\Psi(s) = F_1(s) - H$ has exactly one solution $s = T_S(0)$, without further conditions, since $F_1(T_\sigma) \geq H$ (see (4.49)), $F_1(T_m) = H$, $F_1'(s) > 0$ for (at least) $T_m \leq s \leq 0$. Moreover, we have $T_m \leq T_S(0) \leq T_\sigma$ and $T_S(0) = T_\sigma$ if and only if $T_m = T_\sigma$. We point out that, by virtue of the properties of $F_1(s)$, one has

(4.58) $T_p < T_m$ *if and only if* $k_f g(0) > k_f T_p + (H - b)L\rho_i\varphi_3(T_p,0)$.

We conclude that in the case 1) we find exactly one initial temperature $T_S(0)$ consistent with the prescribed constraints if and only if

(4.59) $k_f T_p + (H - b)L\rho_i\varphi_3(T_p,0) \leq k_f g(0) \leq k_f T_\sigma + (H - b)L\rho_i\varphi_3(T_\sigma,0)$.

The inequality (4.59) is well defined by virtue of (4.52) and of the increasing profile of the function $\varphi_3(s,t)$ with respect to s.
In particular:

$$k_f g(0) = k_f T_\sigma + (H - b)L\rho_i\varphi_3(T_\sigma,0) \; \textit{if and only if} \; T_\sigma = T_m = T_S(0);$$
$$k_f g(0) = k_f T_p + (H - b)L\rho_i\varphi_3(T_p,0)k_f g(0) \;\; \textit{if and only if} \; T_p = T_m.$$

In the case 2), the equation $\Psi(s) = F(s) - H$ has exactly one solution if and only if

$$\Psi(T_p) \geq F_1(T_p) - H. \tag{4.60}$$

In that case, the solution is in the interval $[T_p, T_\sigma)$ and $T_S(0) = T_p$ if and only if in (4.60) is true the equality.
Taking into account that

$$\Psi(T_p) = (H-b)\rho_w L \frac{K_2(T_p)}{k_{ff}(T_p)}, \tag{4.61}$$

we see that (4.60) is equivalent to

$$k_f g(0) \geq k_f T_p + (H-b) L \rho_i \varphi_3(T_p, 0) \left(1 + \rho_w L \frac{K_2(T_p)}{k_{ff}(T_p)}\right). \tag{4.62}$$

Furthermore, kepping in mind (4.58) once again, we see the in the case 2) we find exactly one initial value for the freezing temperature T_S consistent with the imposed constraints if and only if

$$k_f T_p + (H-b) L \rho_i \varphi_3(T_p, 0) \left(1 + \rho_w L \frac{K_2(T_p)}{k_{ff}(T_p)}\right) \leq k_f g(0) < k_f T_p + (H-b) L \rho_i \varphi_3(T_p, 0). \tag{4.63}$$

Condition (4.63) is well defined by virtue of (4.18).
In particular, we find:

$$k_f T_p + (H-b) L \rho_i \varphi_3(T_p, 0) \left(1 + \rho_w L \frac{K_2(T_p)}{k_{ff}(T_p)}\right) = k_f g(0) \textit{ se e solo se } T_S(0) = T_p. \tag{4.64}$$

Putting together (4.59) and (4.63) and e defining T_ρ as in (4.37), we achieve (4.35) and we conclude the case *ii)b*.
We remark that, by the definition of T_ρ and by (4.46), we have

$$T_S(0) > g(0). \tag{4.65}$$

From (4.60) we deduce also that if (4.39) holds, no initial value $T_S(0)$ is suitable in order to have (4.12) (computed for $t = 0$) satisfied. Thus, also the case *ii)c* is proved.
□

Remark 4.2. Since $\partial r(T_p(h))/\partial h > 0$, we see that if $h(t)$ is increasing (decreasing), the temperature T_p decreases (increases).

Our next aim is to disuss the solvability of equation (4.20).
We will assume from now on that condition (4.36) is satisfied, otherwise the initial data do not allow to get solutions consistent with the prescribed constraints.
We consider at first the simper case in which the boundary temperatures are constant:

$$h(t) \equiv h > 0, \quad g(t) \equiv g < 0. \tag{4.66}$$

We remark that, in that case, the functions φ_1 and φ_3 do not depend explicitily on time and the equation (4.20) reduces to

$$-\frac{\varphi_2(T_S(t))}{\varphi_1(T_S(t))} = \varphi'(T_S(t))\dot{T}_S(t) \tag{4.67}$$

(for simplicity of notation, we omit the second argument of the functions φ_1, φ_3 e φ; moreover, the apex for φ denotes the derivative with respect to T_S).
Integrating (4.67), one gets

$$\int_{T_S(0)}^{T_S(t)} \frac{\varphi_1(y)}{\varphi_2(y)}\varphi'(y)dy = -t. \tag{4.68}$$

We compute explicitily $\varphi'(s)$ making use of (4.19):

$$\varphi'(s) = -\frac{k_f}{L\rho_i}\frac{H-b}{\varphi_3(s)(\Psi(s)+H-b)}\left(\left(\frac{\varphi_3'(s)}{\varphi_3(s)} + \frac{\Psi'(s)}{\Psi(s)+(H-b)}\right)(g-s)+1\right) \tag{4.69}$$

where Ψ is defined by (4.54).
From the definition of φ_3, we have:

$$\varphi_3(s) < 0, \; \varphi_3'(s) = \frac{k_{ff}(s)}{bL\rho_i} > 0 \quad \text{for } s \le 0$$

Furthermore:

$$\psi(s)+H-b>0 \ \text{ per } s\le T_\sigma, \quad \psi'(s)\le 0 \ \text{ for } T_p\le s<0.$$

Thus, since $T_m > g$ (see (4.46)), one finds

$$(4.70) \qquad \varphi'(s)>0 \quad \text{per } s\in[T_\rho,T_\sigma]$$

where $T_\rho = max\{T_p,T_m\}$.

By virtue of proposition 4.1, $T_S(0)$ belongs to the interval $[T_\rho,T_\sigma]$, if the initial data are consistent (that is if (4.36) holds). Thus, we have $\varphi'(T_S(0))\ge 0$, and the equality holds if and only if $T_S(0)=T_\sigma$.

Moreover, since $\varphi_1(s)>0$, $\varphi_2(s)>0$ for $T_\rho\le s<T_\sigma$, we see that the function inside the integral in (4.68) is strictly positive, then $T_S(t)$ is increasing, as long as $T_S(t)<T_\sigma$.

From (4.67) we deduce that $T_S(t)$ can not have an asymptotic value $\ell<T_\sigma$, because $\varphi_2(s)$ vanishes only for $s=T_\sigma$. Consequently, the solution $T_S(t)$ of (4.67) tends monotonically to T_σ, in a finite or infinite time t_∞, accordind to the fact that the integral in (4.68) tends to a finite or infinite value, respectively (notice that for $s=T_\sigma$ the denominator of the function in the integral vanishes, while the numerator achieves a finite negative value). If $t_\infty<\infty$, we expect a melting process for $t>t_\infty$.

If $t_\infty=\infty$, that is

$$(4.71) \qquad \int_{T_S(0)}^{-\infty} \frac{\varphi_2(y)}{\varphi_1(y)}\varphi'(y)dy = -\infty,$$

then the solution of $(\mathbb{S}_{tmp})+(\mathbb{C})+(\mathbb{A})$ (equations (4.1)-(4.10)) tends to stationary values ehich we are going to describe.

The final height of the soil H_∞ is achieved from (4.19):

$$(4.72) \qquad H_\infty = H + z_T^\infty - b$$

where $z_T^\infty = z_T(t_\infty) = b + \dfrac{k_f}{L\rho_i}\dfrac{g-T_\sigma}{\varphi_3(T_\sigma)} = F_1(T_\sigma)$.

The thickness of the *frozen fringe* changes with respect to time according to the formula (4.1):

$$(4.73) \qquad b - z_F(t) = \frac{\int_0^{T_S(t)} k_{ff}(\eta)d\eta}{L\rho_i\varphi_3(T_S(t))}\, b.$$

derivating (4.73), one finds:

$$(4.74) \qquad \dot z_F(t) = \frac{k_{ff}(T_S(t))k_u h}{(L\rho_i\varphi_3(T_S(t)))^2}\, \dot T_S(t) > 0.$$

Therefore, the thickness of the *frozen fringe* decrases, but the isotherm line $z = z_F(t)$ does not touch the base of the lens, since $b > z_F(t)$.

The final thickness of the *frozen fringe* is given by

$$(4.75) \qquad z_F^\infty = z_F(t_\infty) = b - \frac{\int_0^{T_\sigma} k_{ff}(\eta)d\eta}{L\rho_i\varphi_3(T_\sigma)}$$

Finally, the water flux q_w satisfies

$$\lim_{t\to\infty} q_w(t) = 0.$$

We conclude by remarking that the temperature T_p defined by (4.67) is constant, because of (4.66); so, $T_p \equiv T_p(0)$. Hence, condition (4.22) is surely verified at ant time $t \in [0,t_\infty)$, since $T_S(0) \geq T_p$ and $T_S(t)$ is increasing.

We examine now the general case in which h and g depend on time.

Let us write explicitily the differential equation (4.20), taking into account that the formula (4.69) can be used to compute the partial derivative $\dfrac{\partial\varphi(T_S(t),t)}{\partial T_S}$:

$$(4.76) \qquad -\frac{\varphi_2(T_S(t))}{\varphi_1(T_S(t),t)} = W_1(T_S(t),t)\dot T_S(t) + W_2(T_S(t),t)\dot g(t) + W_3(T_S(t),t)\dot h(t)$$

where

$$W_1(s,t) = -\frac{k_f(H-b)}{L\rho_i\varphi_3(s,t)(\psi(s,t)+H-b)}\left(\left(\frac{k_{ff}(s)}{bL\rho_i\varphi_3(s,t)} + \frac{\partial\psi(s,t)}{\partial s}\,\frac{1}{\psi(s,t)+(H-b)}\right)(g(t)-s)+1\right)$$

$$W_2(s,t) = \frac{k_f(H-b)}{L\rho_i}\,\frac{1}{\varphi_3(s,t)(\psi(s,t)+H-b)}$$

$$W_3(s,t) = \frac{g(t)-s}{(\varphi_3(s,t)(\psi(s,t)+H-b))^2}\left(\frac{\psi(s,t)+H-b}{bL\rho_i} - \frac{\psi(s,t)}{K_0\rho_w L\varphi_1(s,t)}\right)$$

The function $\psi(s,t)$ is defined as in (4.54):

$$\psi(s,t) = (H-b)\,\frac{\varphi_2(s)}{\varphi_1(s,t)\varphi_3(s,t)}. \tag{4.77}$$

Obviously, $W_1(s,t)$ coincide with $\varphi'(s)$ in the examinated case h and g constant.
The initial condition $T_S(0)$ for the differential equation (4.76) is achieved from (4.21), which has exactly one solution $T_S(0) \geq T_\rho$ (see (4.37)), by virtue of proposition 4.1.
Let us assume the validity of the following hypothesis which is absolutely not restrictive for the generality of our analysis:

$$\sup_{t\geq 0}\ h(t) = \bar{h} < \infty \tag{4.78}$$

We define

$$\bar{T}_p = T_p(\bar{h}) \tag{4.79}$$

where T_p satisfies the equation (4.67).

PROPOSITION 4.2. *Assume that* (4.78) *holds. If the functions* $k_{ff}(s)$, $K_1(s)$, $K_2(s) \in C^1(-\infty,0]$, $h(t)$, $g(t) \in C^1[0,\infty)$ *and if the initial data fulfil condition* (4.36), *then there exists an interval of time* $[0,t_f)$ *where* (4.4.76) *has a unique solution* $T_S(0)$ *in* $[T_p(\bar{h}),T_\sigma]$, *where* $T_p(\bar{h})$ *is defined by* (4.79) *and* T_σ *by* (4.34).
Dim.
From (4.36) we have (see remark 4.2):

$$T_\sigma \geq T_S(0) \geq T_p(h(0)) \geq T_p(\bar{h}).$$

We introduce now the function

$$W(s,t) = -\frac{1}{W_1(s,t)}\left(\frac{\varphi_2(s)}{\varphi_1(s,t)} + W_2(s,t)\dot{g}(t) + W_3(s,t)\dot{h}(t)\right). \tag{4.80}$$

Computing $\partial W(s,t)/\partial s$, one can easily check that $W(s,t) \in C^1[T_p(\bar{h}),T_\sigma]$, on the ground of the assumed hypotheses for $k_{ff}(s)$, $K_1(s)$, $K_2(s)$, $h(t)$ e $g(t)$ and of the estimates

$$(4.81)\qquad \frac{\rho_i b}{\rho_w K_0}\,\frac{-\displaystyle\int_0^{T_\sigma}\frac{k_{ff}(\eta)}{K_1(\eta)}d\eta}{k_u\bar{h}-\displaystyle\int_0^{T_p(\bar{h})}k_{ff}(\eta)d\eta}\le\varphi_1(s,t)\le\frac{\rho_i b}{\rho_w K_0}\,\frac{k_u\bar{h}-\displaystyle\int_0^{T_p(\bar{h})}\frac{k_{ff}(\eta)}{K_1(\eta)}d\eta}{-\displaystyle\int_0^{T_\sigma}k_{ff}(\eta)d\eta}$$

$$(4.82)\qquad \frac{1}{L\rho_i b}\left(-\int_0^{T_\sigma}k_{ff}(\eta)d\eta\right)\le\varphi_3(s,t)\le\frac{1}{L\rho_i b}\left(k_u h(\bar{t})-\int_0^{T_p(\bar{h})}k_{ff}(\eta)d\eta\right).$$

By virtue of the local existence and uniqueness theorem for ordinary differential equations, we conclude that there exists a finite time $t_f>0$ such that (4.76) has exactly one solution $T_S(t)$ with the initial datum $T_S(0)$ and whose graphic is contained in the compact set $[0,t_f]\times[T_p(\bar{h}),T_\sigma]$. □

Once (4.76) has been integrated, we can compute the boundaries $z_F(t)$ and $z_T(t)$ from (4.12) and (4.19), respectively. The water flux is given by

$$q_w(t)=-\frac{\rho_i\varphi_2(T_S(t))}{\rho_w\varphi_1(T_S(t),t)}.$$

The height of the soil $_T(t)$ verifies also the formula

$$(4.83)\qquad z_T(t)=H-\int_0^t\frac{\varphi_2(T_S(\tau))}{\varphi_1(T_S(\tau),\tau)}d\tau.$$

Finally, the water pressure p_w is given by (1.28*a*).

We are going now to check the sign of the coefficients W_i, $i=1,2,3$, in equation (4.76).

PROPOSITION 4.3. *We have:*

(4.84) $\quad W_1(s,t)>0\quad$ for $T_p(h(t))\le s<0,\ t\ge 0$

(4.85) $\quad W_2(s,t)<0\quad$ for $s\le T_\sigma,\ t\ge 0$

(4.86) $\quad W_3(s,t)<0\quad$ for $g(t)<s\le T_\sigma$

where $T_p(h(t))$ fulfils (4.25) *at each time t.*

Dim.

(4.85) comes from (4.18), (4.40) and (4.41).

Moreover, (4.84) is proved by remarking that

$$\frac{\partial \psi(s,t)}{\partial s} = \Psi'(s)$$

where$\Psi(s)$ is defined by (4.54) and by recalling (4.18), (4.55), (4.57).

Let us show now (4.86).

If we derivate (4.15) with respect to s, we find:

$$(4.87) \qquad \frac{\partial \varphi_1(s,t)}{\partial s} = \frac{\rho_i b k_{ff}(s)}{\rho_w K_0} \frac{k_u h(t)\left(1 - \frac{K_0}{K_1(s)}\right) + K_0 \int_0^s \left(\frac{1}{K_1(s)} - \frac{1}{K_1(\eta)}\right) k_{ff}(\eta) d\eta}{\left(k_u h(t) - \int_0^s k_{ff}(\eta) d\eta\right)^2}$$

Since $K_1(s)$ is increasing and $K_1(0) = K_0$, it follows that $\frac{\partial \varphi_1(s,t)}{\partial s} < 0$ for $s \leq 0,\ t \geq 0$.

Therefore:

$$\varphi_1(s,t) < \varphi_1(0,t) = \frac{\rho_i b}{\rho_w K_0} \text{ for } s < 0,\ t \geq 0$$

and consequently

$$\frac{\psi(s,t) + H - b}{bL\rho_i} - \frac{\psi(s,t)}{K_0 \rho_w L \varphi_1(s,t)} = \psi(s,t)\left(\frac{1}{b\rho_i} - \frac{1}{K_0 \rho_w \varphi_1(s,t)}\right) > 0 \text{ for } s < 0,\ t \geq 0$$

and (4.86) is thus proved □

PROPOSITION 4.4. *Assume that $T_S(t)$ is solution of* (4.76), $T_S(t) \in [T_p(\bar{h}), T_\sigma]$*; then, for each time* $t \in [0,t_f]$ *we have* $T_S(t) > g(t)$.

Dim. Call

$$t_a = inf\{t \in [0,t_f):\ T_S(t) \leq g(t)\}.$$

By (4.36) we see that $T_S(0) > g(0)$; hence $t_a > 0$. By the definition of *inf* we have $T_S(t_a) = g(t_a)$ and $T_S(t) > g(t)$ whenever $t \in [0,t_a)$. it follows that

(4.88) $\dot{T}_S(t_a) \le \dot{g}(t_a)$.

By the definition of the functions W_1, W_2 and W_3 one easily finds:

$$W_1(T_S(t_a),t_a) = -W_2(T_S(t_a),t_a), \qquad W_3(T_S(t_a),t_a) = 0.$$

Therefore:

$$(4.89) \qquad -\frac{\varphi_2(T_S(t_a))}{\varphi_1(T_S(t_a),t_a)} = -W_2(T_S(t_a),t_a)\Big(\dot{T}_S(t_a) - \dot{g}(t_a)\Big)$$

Suppose, contrary to our claim, that $T_S(t_a) < T_\sigma$. In that case, we would have that the left-hand quantity in (4.89) is strictly positive (see (4.40) and (4.41)), while the right-hand one is non positive, due to (4.88) and to proposition 4.3. Thus, we would obtain a contradiction. If $T_S(t_a) = T_\sigma$, the left-hand side of (4.89) vanishes, but we can conclude as in the previous case by considering times t close to t_a, $t \le t_a$ (in (4.88) holds the strict inequality, for such as times). $\square$

Let us consider now the maximal interval where the solution $T_S(t)$ of (4.76) is defined, in the sense of proposition 4.2; we keep on calling t_f the right boundary of the time interval.

PROPOSITION 4.5. *Under the same assumptions as in proposition* 4.2*, the solution* $T_S(t)$ *of* (4.76) *satisfies one and only one of the following possibilities:*

1) $T_S(t_f) = T_\sigma$ and $T_S(t) \ge T_p(h(t))$ $\forall t \in [0,t_f]$;

2) $\exists\ t_1 \in (0,t_f)$ such that $T_S(t_1) < T_p(h(t_1))$;

3) $T_p(t) \le T_S(t) < T_\sigma \quad \forall t \ge 0$.

Dim. It suffices to take into account propositions 4.3, 4.4 and recalling that the maximal solution can not be contained in any compact set which is properly contained in $[0,t_f] \times [T_p(\bar{h}),T_\sigma]$. $\square$

Let us comment the three possibilities.

In the case 1), the solution $T_S(t)$ reaches the value T_σ and the water flux q_w vanishes at that temperature. For $t > t_f$ condition (4.24) is violated and a melting process occurs.
In the case 2), by the continuity of $T_S(t)$ and by the estimate $T_p(\bar{h}) \le T_p(h(0)) \le T_S(0)$, there exists a time t_2 such that $T_S(t_2) = T_p(h(t_2))$. The solution $T_S(t)$ can be accepted only for $t \le t_2$, since for $t > t_2$ the water pressure gradient at the base of the ice lens becomes negative. It is to be expected that for $t > t_2$ a process *frost penetration* will take place.
The case 3) occurs when $t_f = \infty$ and the solution $T_S(t)$ remains bounded by denoted values. We have already met with this possibility, when the temperatures h and g are constant and the integral defined in (4.68) verifies (4.71). The solution $T_S(t)$ tends monotonically to T_σ. However, other asymptotic values for the solution $T_S(t)$ are possible, in the case $t_f = \infty$: as an istance, we can choose in (4.76) h constant (in that case W_1 and W_2 don't depend on time explicitily) and $g(0)$ such that (4.35) holds. Once the initial value $T_S(0)$ is calculated by (4.21), we define

$$(4.90) \qquad g(t) = g(0) - \frac{\varphi_2(T_S(0))}{\varphi_1(T_S(0)W_2(T_S(0))}t.$$

It is easily seen that the solution of (4.76) is $T_S(t) \equiv T_S(0)$.
Let us now deal with the following question: how the boundary temperatures $h(t)$, $g(t)$ has to be chosen in order to discriminate the three possibilities just described?
For this purpose, we state the following results (propositions 4.6 and 4.7), which allow us to answer the question just introduced at least in special cases.

PROPOSITION 4.6. *Let assumptions stated in proposition* 4.2 *hold. Then, if* $h(t)$ *and* $g(t)$ *are non decreasing, then the solution* $T_S(t)$ *of* (4.4.76) *is strictly increasing for* $0 \le t < t_f$ *and it reaches in a finite time or asymptotically the temperature* T_σ.
If $h(t)$ *and* $g(t)$ *are non increasing and*

$$\left(\dot{g}(t)\right)^2 + \left(\dot{h}(t)\right)^2 > 0, \quad 0 \le t \le t_f,$$

then $T_S(t) < T_\sigma$.

Dim. Assume that

(4.91) $\dot{g}(t) \geq 0,\ \dot{h}(t) \geq 0.$

By virtue of (4.91), the temperature $T_p(h(t))$ is non increasing (see remark 4.2).
On the other hand, by propositions 4.3 and 4.4 and by (4.40), (4.41), we see that the function W defined by (4.80) verifies at each time t such that the solution $T_S(t)$ of (4.76) exists

(4.92) $W(T_S(t),t) \geq 0, \quad 0 \leq t \leq t_f$

it follows that $T_S(t)$ is non decreasing. Since $T_S(0) \geq T_p(h(0))$ and $T_p(h(t))$ is non increasing, we obtain

(4.93) $T_S(t) \geq T_p(h(t)).$

Thus, it is not possible that the second possibility of proposition 4.5 occurs. Furthermore, from equation (4.76) we deduce that the solution $T_S(t)$ can not have asymptotic values lower than the temperature T_σ: it may be concluded that the temperature $T_S(t)$ reaches (in a finite or infinite time) the value T_σ.
We show now that $T_S(t)$ is strictly increasing for $0 \leq t < t_f$.
The equality in (4.92) holds if and only if $T_S(\tau) = T_\sigma$, $\dot{g}(\tau) = 0$, $\dot{h}(\tau) = 0$, for some $\tau \in [0,t_f]$.
Since $T_S(t)$ is the maximal solution, it must be $\tau = t_f$, hence

$$\dot{T}_S(t) = W(T_S(t),t) > 0, \quad \text{for } 0 \leq t < t_f.$$

Let us prove now that, keeping the assumption (4.91), the solution $T_S(t)$ has the following property:

$$T_S(t) \geq T_S^0(t), \qquad 0 \leq t \leq t_f$$

where $T_S^0(t)$ is the maximal solution of (4.76), defined in $[0,t_f^0)$, obtained by taking the boundary temperatures as

(4.94) $h^0(t) \equiv h(0),\ g^0(t) \equiv g(0).$

The solutions $T_S(t)$ and $T_S^0(t)$ satisfy, respectively:

$$\dot{T}_S(t) = W(T_S(t),t)$$

$$\dot{T}_S^0(t) = W_0(T_S^0(t))$$

where $W(s,t)$ is defined by (4.80) while $W_0(s) = -\varphi_2(s)/\varphi_1(s)\varphi'(s)$ (see (4.67)).
For the sake of clearness, we recall that the functions φ, φ_i with just one argument are related with the examinated case h and g constant.
We remark that the initial datum, that is the solution of (4.21), is the same for both the problems (so $T_S(0) = T_S^0(t)$), by virtue of (4.94).
Let us integrate both the differential equations in a interval where both the solution exist:

$$\int_{T_S(0)}^{T_S(t)} \frac{1}{W(y,t)} dy = \int_{T_S(0)}^{T_S^0(t)} \frac{1}{W^0(y)} dy = t.$$

and write explicitly W and W_0:

$$W(y,t) = -\frac{1}{W_1(y,t)}\left(\frac{\varphi_2(y)}{\varphi_1(y,t)} + W_2(y,t)\dot{g}(t) + W_3(y,t)\dot{h}(t)\right)$$

$$W^0(y) = -\frac{1}{W_1(y)}\frac{\varphi_2(y)}{\varphi_1(y)}.$$

By assumption (4.91), proposition 4.3 and (4.93), we get

$$(4.95) \qquad -W_2(y,t)\dot{g}(t) - W_3(y,t)\dot{h}(t) > 0$$

On the other hand, from the definition of W_i, $i = 1,2,3$ and keeping in mind (4.91) once again, it follows immediately that, for each time such that both the solutions of the two problems exist, it is

$$(4.96) \qquad \frac{\partial W(y,t)}{\partial t} \leq 0, \ \frac{\partial \varphi_1(y,t)}{\partial t} \leq 0$$

We conclude that $W(y,t) \geq W^0(y)$; therefore $T_S(t) \geq T_S^0(t)$.
Roughly speaking, we may say that, under the same initial condition $T_S(0)$, the increasing boundary temperatures "bring" the solution $T_S(t)$ to the value T_σ in a shorter time with respect to the case when the boundary temperatures are settled at the values $h(0)$, $g(0)$.

PROPOSITION 4.7. *If* $h(t)$ *and* $g(t)$ *satisfy the conditions*

$$(4.96) \qquad \dot{g}(t) \leq 0,\ \dot{h}(t) \leq 0,\ (\dot{g}(t))^2 + (\dot{h}(t))^2 > 0, \qquad 0 \leq t,$$

then $T_S(t) < T_\sigma$.

Dim. Assume that (4.96) holds. If there existed a $\tau \in [0,t_f]$ with the property $T_S(\tau) = T_\sigma$, we would have $\dot{T}_S(\tau) \geq 0$. By propositions 4.3, 4.4 and assumption (4.96), we would have $W(T_S(\tau),\tau) < 0$, where W is defined by (4.80) and we would obtain a contradiction. □
the example (4.90) shows that, even if the temperatures h and g are non increasing (in the example the temperature g is really decreasing in a linear way), the temperature $T_S(t)$ is not necessarily decreasing: the term in the left-hand side if equality (4.76) is positive, while in (4.95) it has the opposite sign. In qualitative terms, we can say that, if $g(t)$ decreases quickly, the temperature $T_S(t)$ reaches (by decreasing) the value $T_p(h(\tau))$ for some $\tau \in (0,t_f)$ only if $g(t)$ decreases rapidly enough. Moreover, when $h(t)$ decreases, the temperature $T_p(h(t))$ increases.
Let us give now the following example.
We choose $h(t) \equiv h_0$, where h_0 is a positive constant and $g(0)$ is taken such that (4.36) holds. By means of (4.21) we calculate $T_S(0)$ and we define:

$$f(t) = \alpha t + T_S(0), \qquad \alpha < 0.$$

Since h is constant, the temperature T_p which satisfies (4.25) is also constant; moreover, the functions W_2 and φ_1 don't depend explicitily on time.
It is a simple matter to find α and $g(t)$ such that

$$(4.97) \qquad \dot{f}(t) = \alpha > W(f(t),t) = -\frac{1}{W_1(f(t),t)}\left(\frac{\varphi_2(f(t))}{\varphi_1(f(t))} + W_2(f(t))\dot{g}(t)\right).$$

Inequality (4.97) means essentially that $g(t)$ must decrease rapidly enough.
The solution of (4.76) that is achieved by imposing the boundary temperatures h_0 and $g(t)$ (where $g(t)$ verifies (4.97)) keeps below the straight line $f(t)$. The temperature $T_S(t)$ goes necessarily down the vallue $T_p(h_0)$ and the water pressure gradient at the base of the lens becomes negative.
Owing to condition (4.22), the solution describing the process of lens formation can be accepted only up to the time t^* when $T_S(t^*) = T_p(h_0)$.

4.2 *Frost penetration*

In this section we wiil look for solutions of $(\mathbb{S}_{tmp}) + (\mathbb{C}) + (\mathbb{A})$ (eqq. (4.1)-(4.1)) which describe the process of *frost penetration*; since it must holds (see par. 4.1.7)

$$(4.98) \qquad \dot{z}_S(t) < 0,$$

from (4.4) and (4.7) we get

$$(4.99) \qquad q_w(t) = K_2(T_S(t))\frac{k_u h(t)}{z_F(t)k_{ff}(T_S(t))}.$$

From (4.1), (4.4) and (4.99) we deduce the equation for the temperature $T_S(t)$:

$$(4.100) \qquad \sigma + \frac{k_u h(t)}{K_0}\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} + \int_0^{T_S(t)} \frac{K_2(\eta) - \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}k_{ff}(\eta)}{K_1(\eta)} d\eta = 0.$$

Equation (4.100) is equivalent to

$$(4.101) \qquad \varphi_2(T_S(t)) = \rho_w L\,\frac{K_2(T_S(t))}{k_{ff}(T_S(t))}\,\varphi_1(T_S(t),t)\varphi_3(T_S(t),t),$$

(where φ_1,φ_2 and φ_3 are defined by (4.15)-(4.17)), or even to

$$(4.102) \qquad r(T_S(t)) = k_u h(t)$$

where $r(s)$ is defined by (4.27).
The solution of (4.102) is given by the temperature $T_p(h(t))$ defined by (4.25). Therefore,

we assume that the temperature T_σ verifying (4.34) exists, otherwise (4.102) has no solution (see lemma (4.25)).

Formally, once (4.102) has been solved, the freezing temperature T_S, which depends on time t only through the boundary temperature $h(t)$, is a known function.

Writing $q_w(t)$ and $z_F(t)$ in terms of $z_S(t)$ and $z_T(t)$ by means of the formulas (4.1), (4.4) and (4.99), one finds:

$$(4.103)\qquad q_w(t) = -\frac{K_2(T_S(t))}{k_{ff}(T_S(t))} bL\rho_i\varphi_3(T_S(t),t)\,\frac{1}{z_S(t)}$$

$$(4.104)\qquad z_F(t) = \gamma(t) z_S(t)$$

with

$$(4.105)\qquad \gamma(t) = \frac{k_u h(t)}{k_u h(t) - \int_0^{T_S(t)} k_{ff}(\eta)d\eta}.$$

we have $0 < \gamma < 1$, since if $T_S(t)$ is the solution (4.100), it must be $T_S(t) < 0$.

Substituting (4.103) and (4.104) in (4.2) and in (4.4.3), we get the following set of ordinary differential equations, where the unknown quantities are the boundaries $z_S(t)$ and $z_T(t)$:

$$\begin{cases} \dot{z}_S(t) = \dfrac{1}{(1-\nu_S)\varepsilon\rho_w L}\left\{k_f\dfrac{g(t)-T_S(t)}{z_T(t)-z_S(t)} - \dfrac{\rho_w L K_2(T_S(t)) + k_{ff}(T_S(t))}{k_{ff}(T_S(t))} bL\rho_i\varphi_3(T_S(t),t)\,\dfrac{1}{z_S(t)}\right\} \\ \dot{z}_T(t) = \dfrac{k_f(\rho_i-\rho_w)}{\rho_i\rho_w L}\,\dfrac{g(t)-T_S(t)}{z_T(t)-z_S(t)} - \dfrac{bL\rho_i\varphi_3(T_S(t),t)}{\rho_i\rho_w L k_{ff}(T_S(t))}\left\{\rho_i\rho_w L K_2(T_S(t)) + \right. \\ \qquad \left. + (\rho_i-\rho_w)k_{ff}(T_S(t))\right\}\dfrac{1}{z_S(t)} \end{cases}$$

The initial conditions are given by (4.6):

$$z_S(0) = b,\; z_T(0) = H > b.$$

By means of the definiton of the variables

$$x(t) = z_S(t),\ y(t) = z_T(t) - z_S(t)$$

we find the following equivalent differential system:

$$(\mathbf{S}_F) \quad \begin{cases} \dot{x} = \dfrac{A(t)}{x} + \dfrac{B(t)}{y} \\[2ex] \dot{y} = \dfrac{C(t)}{x} + \dfrac{D(t)}{y} \end{cases}$$

where

$$A(t) = -\frac{\rho_w L K_2(T_S(t)) + k_{ff}(T_S(t))}{(1-\nu_S)\varepsilon\rho_w k_{ff}(T_S(t))}\, b\rho_i\varphi_3(T_S(t),t)$$

$$B(t) = \frac{k_f(g(t) - T_S(t))}{(1-\nu_S)\varepsilon\rho_w L}$$

$$C(t) = -\frac{\left\{\rho_i\rho_w L K_2(T_S(t)) + (\rho_i - \rho_w)k_{ff}(T_S(t))\right\}(1-\nu_S)\varepsilon - \rho_i\left\{k_{ff}(T_S(t)) + \rho_w L K_2(T_S(t))\right\}}{\rho_i\rho_w\varepsilon(1-\nu_S)k_{ff}(T_S(t))}\times$$

$$\times\, b\rho_i\varphi_3(T_S(t),t) = A(t)((1-\nu_S)\varepsilon - 1) + b\varphi_3(T_S(t),t)$$

$$D(t) = \frac{k_f(g(t) - T_S(t))\left\{(1-\nu_S)\varepsilon(\rho_i - \rho_w) - \rho_i\right\}}{\rho_i\rho_w L\varepsilon(1-\nu_S)} = \frac{k_f(g(t) - T_S(t))(\rho_i - \rho_w)}{\rho_i\rho_w L(1-\nu_S)} - B(t)$$

The initial condition for $\mathbf{S}_F$ are

(4.106) $\quad x(0) = b,\ y(0) = H - b$

Let us now discuss the signs of the just defined coefficients. From (4.40) we obtain:

(4.107) $\quad A(t) > 0,\ C(t) < 0 \quad$ for $s < 0,\ t \geq 0.$

Moreover, noticing that $((1-\nu_S)\varepsilon(\rho_i - \rho_w) - \rho_i) < 0$ (this is true even in the case $\rho_w < \rho_i$), we have:

(4.108) $\quad B(t) < 0,\ D(t) > 0$ for $\quad g(t) < T_S(t)$.

Analogously eith the case of lens formation, we will deal first the problem to check in which cases the initial data are consistent with the prescribed conditions (4.10), (4.11), (4.98).

Under the assumption (4.34), which is assumed to be satisfied, we know that the equation (4.102) has one solution for each fixed time t.

Let us call to shorten notation $T_p(h(0)) = T_p(0)$ and state the following

PROPOSITION 4.8. *The initial temperature* $T_S(0) = T_p(0)$, *solution of* (4.102) *calculated for* $t = 0$ *is consistent with* (4.10), (4.11):

$$0 \le z_F(t) \le z_S(t) \le z_T(t),\ q_w(t) \ge 0$$

and (4.4.98) *if and only if*

$$k_f g(0) < k_f T_p(0) + (H-b)L\rho_i\varphi_3(T_p(0))\left(1 + \rho_w L\frac{K_2(T_p(0))}{k_{ff}(T_p(0))}\right). \tag{4.109}$$

Dim.

Assume that (4.109) holds.

From (S_F) we get

$$\dot{x}(0) = \frac{A(0)}{b} + \frac{B(0)}{H-b} = \frac{1}{(1-\nu(T_S(0)))\varepsilon\rho_w}\left(\frac{k_f(g(0)-T_S(0))}{L(H-b)} - \right.$$
$$\left. - \frac{\rho_w L K_2(T_S(0)) + k_{ff}(T_S(0))}{\varepsilon\rho_w k_{ff}(T_S(0))}\rho_i\varphi_3(T_S(0),0)\right) < 0. \tag{4.110}$$

Hence, condition (4.98) is fulfilled for $t = 0$.

Condition (4.10) is obviously satisfied for $t = 0$ (see (4.104)).

Furthermore, the solution of (4.102) must benegative; thus, from (4.103) and from (4.40) we see that $q_w(0) > 0$, that is (4.11) for $t = 0$.

On the other hand, if (4.109) were not true, from (4.110) we would find $\dot{z}_S\ (0) \ge 0$, contrary to condition (4.98). $\square$

Keeping in mind (4.98) and (4.10), we see that the solution $(x(t),\ y(t))$ of (S_F) must verify:

(4.111) $\dot{x}(t) < 0,$

(4.112) $x(t) \geq 0,\ y(t) \geq 0.$

Condition (4.11) is certainly fulfilled if (4.11) holds (see (4.40) and (4.103)).
Generalizing (4.110) to any time t and taking into account the equation (4.102), we see that (4.111) is equivalent to

(4.113) $$k_f g(t) < k_f T_p(h(t)) + (z_T(t) - z_S(t)) L \rho_i \varphi_3(T_p(h(t)), t) \left(1 + \rho_w L \frac{K_2(T_p(h(t)))}{k_{ff}(T_p(h(t)))}\right)$$

where $T_p(h(t)) = T_S(t)$,

or to

(4.114) $$\frac{y(t)}{x(t)} < -\frac{B(t)}{A(t)}.$$

If $T_S(t)$, solution of (4.102), satisfies (4.113), it must be

(4.115) $T_S(t) > g(t).$

Let us now calculate the quantity $AD - BC$; an easy computation shows that:

(4.116) $$A(t)D(t) - B(t)C(t) = L\rho_w b\, \frac{K_2(T_S(t))}{k_{ff}(T_S(t))}\, \varphi_3(T_S(t), t) B(t).$$

If $T_S(t)$ is solution of (4.102), we get from (4.115)

(4.117) $A(t)D(t) - B(t)C(t) > 0.$

In order to discuss the problem $(\mathcal{S}_F)$, let us investigate first the easier case h and g constant.

PROPOSITION 4.9. *Assume* $h>0$ *and* $g<0$ *costant and let be* (4.109) *verified. Then, there exists a time* $t_0<\infty$ *such that* ($\mathfrak{S}_F$) *has exactly one solution* $(x(t),y(t))$ *in the interval* $t\in[0,t_0)$, *so that* (4.111), (4.112) *are satisfied for* $t\in[0,t_0)$, *while* $\dot{x}(t_0)=0$.
Dim.
The system ($\mathfrak{S}_F$), in the present case of constant boundary temperatures, is autonomous (A,B,C,D are constant), since the temperature T_S solving (4.102) is constant.
Owing to condition (4.112), we are interested just in positive solutions $(x(t),y(t))$.
By the theorem of existence and uniqueness for ordinary differential equations, we can say that for each pair of initial values $x(0)>0$, $y(0)>0$ there exists a unique solution of ($\mathfrak{S}_F$) at least locally.
Consider now the projections of the solutions $(x(t),y(t))$ on the quarter of plane $Q=\{x>0,y>0\}$.
Let us write the orbits in Q in the form $y=y(x)$. We find the differential equation

$$(4.118)\qquad y'(x)=\frac{Cy+Dx}{Ay+Bx}$$

where the apex denotes the derivative with respect to x.
Conditions (4.111), (4.112) impose that the starting point $(x(0),y(0))$ has to lye in Q and that (4.114) evaluated for $t=0$:

$$(4.119)\qquad \frac{H-b}{b}<-\frac{B}{A}$$

must hold.
Integrating (4.117) and taking into account of the initial conditions (4.106), one finds the following formula:

$$(4.120)\qquad -\frac{1}{2}ln\left|-Au^2(t)+(C-B)u(t)+D\right|-c_1ln\left|\frac{2Au(t)+c_2}{2Au(t)+c_3}\right|=ln\left|\frac{x(t)}{b}\right|+c_4$$

where we set

$$u(t)=\frac{y(t)}{x(t)},$$

$$c_1=\frac{B+C}{2\sqrt{4AD+(C-B)^2}}<0,$$

$$c_2 = B - C - \sqrt{4AD + (B-C)^2},$$

$$c_3 = B - C + \sqrt{4AD + (B-C)^2},$$

$$c_4 = -\frac{1}{2} ln\big(- Au^2(0) + (C-B)u(0) + D\big) - c_1 ln \frac{-2Au(0) - c_2}{2Au(0) + c_3},$$

with $u(0) = \frac{H-b}{b}$.

We remark that $c_3 > 0$ and that (4.117) and (4.119) yeld

$$c_2 < 2B, \ -Au^2(0) + (C-B)u(0) + D > 0.$$

Therefore, the constant c_4 is well defined.
We see that $y' = 0$ if and only if the point (x,y) belongs the half-straight line

(4.121) $\quad s_0 = \{(x,y) \in Q : y = -Dx/C\};$

moreover, $y' = \infty$ if and only if (x,y) lies on the half-straight line

(4.122) $\quad s_\infty = \{(x,y) \in Q : y = -Bx/A\}.$

By virtue of (4.117), the slope of s_0 is greater that the slope of s_∞.
If we look for half-straight lines $y = mx$ in Q which are orbits for the system $(\mathbf{S}_F)$, we easily see that m must verify $Am^2 + (B-C)m - D = 0$. Such as equation has two solutions:

$$m_1 = -\frac{c_3}{2A} < 0, \ m_2 = -\frac{c_2}{2A} > 0.$$

Moreover: $-\frac{B}{A} < m_2 < -\frac{D}{C}$.

Thus, inside the angle bounded by s_0 and by s_∞ there exists one straight line-orbit.
Define the three angles

$$\Gamma_1 = \{(x,y) \in Q : 0 < y < -Bx/A\}$$

$$\Gamma_2 = \{(x,y) \in Q : -Bx/A < y < -Dx/C\}$$

$$\Gamma_3 = \{(x,y) \in Q : y > -Dx/C\}.$$

It is easily seen that $y' > 0$ if and only if $(x,y) \in \Gamma_2$. Moreover, $\dot{x} < 0$ if and only if $(x,y) \in \Gamma_1$, $\dot{y} > 0$ if and only if $(x,y) \in \Gamma_1 \cup \Gamma_2$.

Owing to condition (4.111), we have to consider only the orbits in the angle Γ_1, where $\dot{x} < 0$, $\dot{y} > 0$ hold. Choose an initial point $P_0 \equiv (x(0),y(0)) \in \Gamma_1$ and follow the orbit starting from P_0. As long as the orbit remains in Γ_1, we have $\dot{x}(t) < 0$, $\dot{y}(t) > 0$. The orbit meets necessarily the half-straight line $y = -Bx/A$ in a finite time $t = t_0$; at that time t_0, the boundary z_S is at the height:

$$z_S(t_0) = b e^{-c_4 \sqrt{\frac{A}{AD-BC}}\left(\frac{2B-c_2}{c_3-2B}\right)^{-c_1}} > 0.$$

We have $\dot{x}(t_0) = 0$ and for $t > t_0$ it is $\dot{x}(t) > 0$ and condition (4.111) is violated. We will check later if for $t = t_0$ a process of lens formation takes place. □

Once the system (S_F) has been integrated, the water flux q_w can be computed by (4.103):

$$q_w(t) = -\frac{K_2(T_p)}{k_{ff}(T_p)} bL\rho_i \varphi_3(T_p) \frac{1}{x(t)}$$

The function γ defined by (4.105) is constant:

$$z_F(t) = \gamma x(t) \tag{4.123}$$

with

$$\gamma = \frac{k_u h}{k_u h - \int_0^{T_p} k_{ff}(\eta) d\eta}, \quad 0 < \gamma < 1.$$

From (4.117) we deduce $\dot{y}(t) > 0$ for $t \in [0,t_0)$; taking also into account (4.123), we find that the condition (4.10) is also verified in the interval $t \in [0,t_0)$.

We remark that the thickness of the frozen soil $y(t) = z_T(t) - z_S(t)$ is increasing for

$0 \leq t < \bar{t}$ by virtue of (3.48).

The height of the soil at the time $t = t_0$ is given by

$$z_T(t_0) = y(t_0) + x(t_0) = \left(1 - \frac{B}{A}\right) z_S(t_0).$$

Let us consider now the system (S_F) in the general case when the boundary temperatures h and g depend on time. As in the previous case, we get the temperature T_S from (4.102), but in the present case T_S depend on time: in particular, if $h(t)$ increases (decreases), the temperature T_S decreases (increases).

Let us examine once again the quarter of plane Q and especially the angle $\Gamma_1(t)$ defined by

$$\Gamma_1(t) = \{(x, y) \in Q \,:\, 0 < y < -B(t)x/A(t)\};$$

Likewise the case h and g constant, (4.111) and (4.112) evaluated for $t = 0$ impose $P_0 \equiv (x(0), y(0)) \in \Gamma_1(0)$, that is

$$(4.124) \qquad \frac{H-b}{b} < -\frac{B(0)}{A(0)}.$$

Assuming that the prescribed functions $k_{ff}(s)$, $K_1(s)$, $K_2(s)$, $h(t)$ and $g(t)$ have the same regularity as stated in proposition 4.3 and applying the theorem of existence and uniqueness for ordinary differential equations, we have that one solution $(x(t), y(t))$ of (S_F) starting from P_0 is defined. Consider the projection of the solution on Q (obviously, in the present case the orbits are no longer of the form (4.119)). As long as the projection remains in the angle $\Gamma_1(t)$, whose width depends on time, the conditions (4.10), (4.11) and (4.98) are satisfied. Indeed, $\dot{z}_S(t) < 0$ if and only if $y/x < -B(t)/A(t)$; (4.11) is trivially verified (see (4.103)). By (4.117) and by (4.104) (4.10) is also fulfilled.

Let us write (4.110) in the following way:

$$(4.125) \qquad \frac{y(t)}{x(t)} < -\frac{B(t)}{A(t)} = \frac{k_f(g(t) - T_S(t))}{L\rho_i b \varphi_3(T_S(t), t)\left(1 + \rho_w L \dfrac{K_2(T_S(t))}{k_{ff}(T_S(t))}\right)} = F(h(t), g(t))$$

According to what we said above, the process of *frost penetration* stops when and only when in (4.125) holds the equality for some time τ:

$$(4.126) \qquad \frac{y(\tau)}{x(\tau)} = -\frac{B(\tau)}{A(\tau)} \text{ if and only if } \dot{z}_S(\tau) = 0.$$

From a geometrical point of view, (4.126) holds when the projection of the solution $(y(t),x(t))$ matches the straight line $y = -B(t)x/A(t)$, which is the upper side of the angle $\Gamma_1(t)$.

The ratio $y(t)/x(t)$ is certainly increasing if $(x(t),y(t))$ is solution of (S_F) with $P_0 \equiv (x(0),y(0)) \in \Gamma_1(0)$.

Nevertheless, the behaviour of the function F with respect to time is related to the profile of the boundary temperatures $h(t)$ and $g(t)$.

We state the following result.

PROPOSITION 4.10. *Assume that* (4.124) *is verified. If* $F(h(t),g(t))$ *is finite for* $t \geq 0$, *then there exists a* $\tau > 0$ *such that* (4.126) *holds.*

Dim.

By hypothesis we have

$$(4.127) \qquad \sup_{t \geq 0} \; -\frac{B(t)}{A(t)} < \infty$$

The projections on Q of the solutions $(x(t),y(t))$ of S_F satisfy the equation

$$(4.128) \qquad y'(x) = \frac{C(t)y + D(t)x}{A(t)y + B(t)x}.$$

The isocline straight line

$$s_\infty(t) = \{(x,y) \in Q : y = -B(t)x/A(t)\}$$

has a positive finite slope, depending on time; by the assumption (4.127), the slope of $s_\infty(t)$ is finite.

Condition (4.124) assures that $(x(0),y(0)) \in \Gamma_1(0)$. Moreover, from (4.128) we see that the projection $y = y(x)$ is decreasing with respect to x as long as $(x(t),y(t))$ remains in $\Gamma_1(t)$. On the other hand, $y = y(x)$ can not accumulate in any point in $\Gamma_1(t)$, since thera are no singular points, owing to (4.16). Thus, the projection of the solution must reach in a finite time $t = \tau$ the straight line $s_\infty(\tau)$ and $\dot{x}(\tau) = \dot{z}_S(\tau) = 0$. □

Remark 4.3. The result obtained by proposition 4.10 can be applied in the following special cases:

i) $h(t)$ and $g(t)$ are non decreasing (in particular constant): actually, the temperature $T_S(t)$ is in that case non increasing and the function F is bounded by

$$F(h(t),g(t)) \leq \frac{g(0)-T_S(0)}{\varphi_3(0)} \qquad t \geq 0;$$

ii) $g(t)$ is lower bounded: indeed, if $g(t) \geq -g_0 > -\infty$, $t \geq 0$, we have

$$F(h(t),g(t)) \leq -\frac{k_f g_0}{k_u \int_0^{T_\sigma} k_{ff}(\eta)d\eta} \qquad t \geq 0;$$

iii) $\frac{dF(h(t),g(t))}{dt} \leq 0$: in that case $F(h(t),g(t)) \leq F(h(0),g(0))$.

Once (S_F) has been solved, the water flux $q_w(t)$ and the isotherm $z = z_F(t)$ can be achieved by means of (4.103) and (4.104), respectively.

Arguing as in the case h and g constant, one can easily check that the conditions (4.10), (4.11) and (4.98) are satisfied for $0 \leq t < \tau$.

One may wonder how the boundary temperatures h and g has to be chosen in order to have a process of *frost penetration* for any time $t \geq 0$. Remarking that if g is bounded, $F(h(t),g(t))$ is bounded, too, for any value of $h(t)$, we may say that it is necessary that

$$\inf_{t \geq 0} g(t) = -\infty.$$

In qualitative terms, the just written condition corresponds to a rapid freezing process, as we expect in a process of solely *frost penetration.*

With regard to that possibility, let us give the following example. Consider, for the sake of convenience, $t = 1$ as initial time and precribe the boundary temperatures

$$h(t) \equiv h_0 > 0$$

$$g(t) = T_p(h_0) - \frac{1}{k_1}\left(\frac{BC-AD}{2bB}(t^2-1) + \frac{Db}{B}\left(\frac{1}{t}-1\right) + H - b\right)\left(\frac{b}{t^2} + \frac{At}{b}\right)$$

where

$$k_1 = \frac{(1-\nu_S)\varepsilon\rho_w L}{k_f} > 0.$$

Since the temperature h is constant, we have that T_p, A and C are constant (see (4.101) and the definition of the coefficients A, C). Furthermore, the ratio D/B does not depend on $g(t)$, hence neither $(BC-AD)/B$ does. We find $g(t)<0$ by virtue of (4.117). Condition (4.119) is verified anyhow the constant $h_0>0$ is chosen.
The system $\mathbf{S}_F$ has the form:

$$(\mathbf{S}_F) \quad \begin{cases} \dot{x} = \frac{A}{x} + \frac{k_1(g(t)-T_p)}{y} \\ \dot{y} = \frac{C}{x} + \frac{k_2(g(t)-T_p)}{y} \end{cases}$$

$$x(1) = b,\ y(1) = H-b,$$

where

$$k_2 = \frac{k_f\big((1-\nu_S)\varepsilon(\rho_i-\rho_w)-\rho_i\big)}{\rho_i\rho_w L\varepsilon(1-\nu_S)} < 0.$$

It easily seen that the solution of $\mathbf{S}_F$ is

$$x(t) = \frac{b}{t},\ y(t) = \frac{BC-AD}{2bB}(t^2-1) + \frac{Db}{B}\Big(\frac{1}{t}-1\Big) + H - b, \qquad t \geq 1.$$

The boundary $x(t) = z_S(t)$ tends asymptotically to the base of the soil $z=0$; the velocity $\dot{x}$ never vanishes for all $t\geq 1$: therefore, we get a process of *frost penetration* for each time $t\geq 1$ and the occurrence of the formation of an ice lens is precluded. The boundary $y(t)$ is in that example positive and increasing, since from (4.117) we get

$$\dot{y}(t) = \frac{k_1 C - k_2 A}{bk_1}t - \frac{k_2 b}{k_1 t^2} > 0.$$

4.3 *Transition from one process to the other*

Putting together the results we stated in propositions 4.1 and 4.8, we are able to foresee, on the ground of the knowledge of the initial boundary temperatures $h(0)$ e $g(0)$ and of the properties of the soil $K_1(T)$, $K_2(T)$ and $k_{ff}(T)$, whether a process of lens formation, or *frost penetration* or melting will occur. As a matter of fact, assuming that the temperature T_σ defined by (4.34) and solving the equation (4.26) evaluated for $t=0$ we find the temperature $T_p(h(0))$. At this point, if the condition

$$k_f g(0) > k_f T_\sigma + (H-b)L\rho_i\varphi_3(T_\sigma,0),$$

holds, where H is the known initial heigiht of the soil, b the known initial position of the freezing front z_S and φ_3 is defined by (4.17), then the temperatures h and g do not give rise to any freezing process.

On the contrary, if $T_p = T_p(h(0))$ is such that

$$k_f T_p + (H-b)L\rho_i\varphi_3(T_p,0)\left(1+\rho_w L\frac{K_2(T_p)}{k_{ff}(T_p)}\right) \le k_f g(0) \le k_f T_\sigma + (H-b)L\rho_i\varphi_3(T_\sigma,0), \tag{4.129}$$

then a process of lens formation will occur.

Finally, if

$$k_f g(0) < k_f T_p + (H-b)L\rho_i\varphi_3(T_p,0)\left(1+\rho_w L\frac{K_2(T_p)}{k_{ff}(T_p)}\right), \tag{4.130}$$

then a process of *frost penetration* will take place.

The diagram on figure 4.7, where we put the initial temperature $h(0)$ on the $x-$axis and the temperature $g(0)$ on the $y-$axis, exhibits the fact that for any pair of values $(h(0),g(0))$ chosen on the quarter of plane $h(0)\ge 0$, $g(0)\le 0$, the kind of process is discriminated.

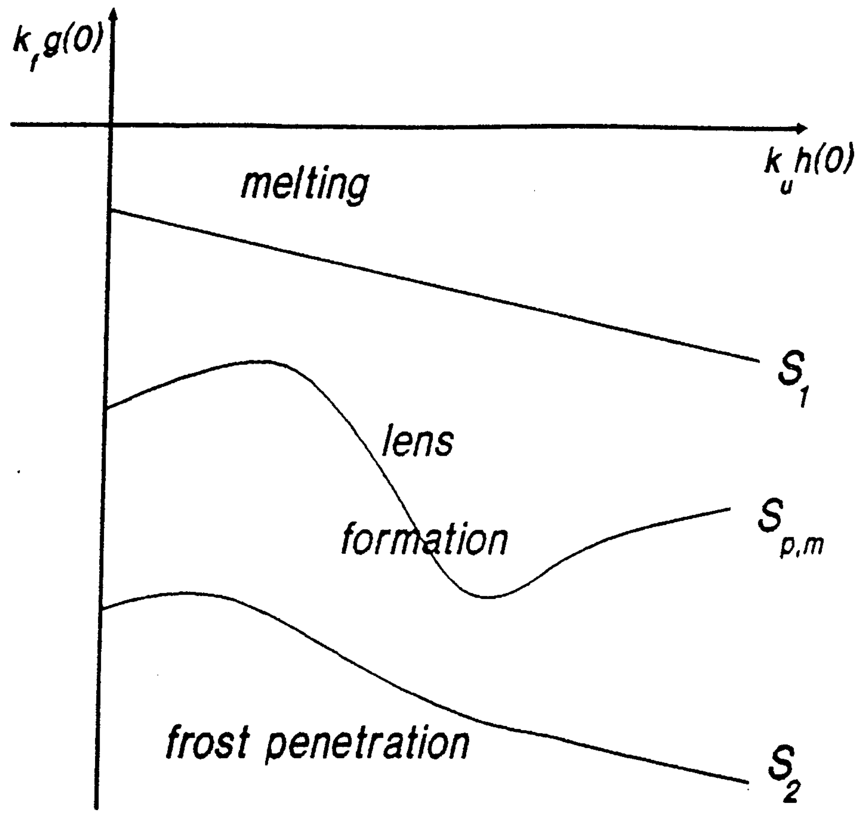

fig. 4.7: the straight line S_1 corresponds to $k_f g(0) = k_f T_\sigma + (H-b)L\rho_i\varphi_3(T_\sigma,0)$, the curve S_2 to

$k_f g(0) = k_f T_p(h(0)) + (H-b)L\rho_i\varphi_3(T_p(h(0)),0)\{1+\rho_w L K_2(T_p(h(0)))/k_{ff}(T_p(h(0)))\}$.

The $S_{p,m}$ curve given by the equation $k_f g(0) = k_f T_p + (H-b)L\rho_i\varphi_3(T_p,0)$ corresponds to $T_p = T_m$ (see eq. (4.4.58)). The two regions that form the region of lens formation correspond to the cases 1) and 2) introduced in the proof of proposition 4.1.

We are going now to give an example of transition from one process to the other. We choose non decreasing boundary temperatures

$$\dot{h}(t) \geq 0, \quad \dot{g}(t) \geq 0 \tag{4.131}$$

and we assume that for $t=0$ (4.130) is verified. The process of *frost penetration* goes on up to the time τ (see remark 4.3) when (4.126) holds:

$$k_f g(\tau) = k_f T_p(\tau) + (z_T(\tau) - z_S(\tau)) L\rho_i \frac{b}{z_S(\tau)} \varphi_3(T_p(\tau),\tau) \left(1 + \rho_w L \frac{K_2(T_p(\tau))}{k_{ff}(T_p(\tau))}\right)$$

where T_p is the solution of (4.102).

Define $b_1 = z_S(\tau)$, $H_1 = z_T(\tau)$. Consistently, the value b in the definition of the functions φ_1 and φ_3 has to be replaced by b_1:

$$\varphi_1(s,t) = \frac{\rho_i b_1}{\rho_w K_0} \frac{k_u h(t) - K_0 \displaystyle\int_0^s \frac{k_{ff}(\eta)}{K_1(\eta)} d\eta}{k_u h(t) - \displaystyle\int_0^s k_{ff}(\eta) d\eta}, \qquad \varphi_3(s,t) = \frac{1}{L\rho_i b_1}\left(\int_0^s k_{ff}(\eta)d\eta - k_u h(t)\right).$$

The temperature $T_p(h(t))$ does not depend on b, as we see from (4.26), (4.27).

Thinking of $t=\tau$ as the starting point of a second process, we notice that condition (4.129) holds: more precisely, it is

$$k_f g(\tau) = k_f T_p(h(\tau)) + (H_1 - b_1) L\rho_i \varphi_3(T_p(h(\tau)),\tau)\left(1 + \rho_w L \frac{K_2(T_p(h(\tau)))}{k_{ff}(T_p(h(\tau)))}\right).$$

By virtue of (4.64), the temperature $T_S(\tau)$ solution of (4.21) is exactly $T_p(h(\tau))$.

Choosing $z_S(t) \equiv z_S(\tau) = b_1$ and $z_T(\tau) = H_1$, we solve again the system $(\mathbf{S}_{tmp}) + (\mathbf{C}) + (\mathbf{A})$, in the same way as we described in section 4.1 and we get a first process of lens formation. Notice that

$$\frac{\partial p_w(z_S(\tau),\tau)}{\partial z} = 0, \quad \frac{\partial p_w(z_S(t),t)}{\partial z} > 0, \quad t > \tau. \tag{4.132}$$

Indeed, $T_S(\tau)$ verifies (4.67) and for $t \geq \tau$ the temperature $T_S(t)$ is strictly increasing by the assumption (4.131) (cfr. proposition 4.6), while $T_p(h(t))$ is non increasing (see remark 4.2).

The first equation of (4.132) guarantees that the water flux q_w is continuous for $t = \tau$, as we see from (4.5).

Owing to proposition 4.6 and to (4.131), the process of lens formation either stops in a finite time t_f or tends to a stationary profile; in the first case, for $t > t_f$ a process of melting will occur; in the second case, the solution tends to the asymptotic values defined by (4.72), (4.75).

Let us examine now the possibility of getting the inverse process, that is from lens formation to frost penetration. Obviously, we have to drop the assumption (4.131) and choose rapid freezing profile for the boundary temperatures. Actually, it is necessary for the temperature $T_S(t)$, solution of (4.20), to go under $T_p(h(t))$ (cfr. case 2) in proposition 4.5 and example (4.97)).

For $t > t_f$, the solution describing lens formation is meaningless: indeed, condition (4.9) is violated, since it would be (see remark 4.2):

$$(4.133) \qquad \frac{\partial p_w(b,t)}{\partial z} < 0, \text{ for } t > t_f$$

where b represents, as usual, the base of the lens.

Thus, the solution of (4.20) can not be accepted for $t > t_f$. Let us look for solutions describing a frost penetration process for $t \geq t_f$, taking $t = t_f$ as the starting time and imposing

$$(4.134) \qquad \frac{\partial p_w(z_S(t),t)}{\partial z} = 0, \text{ for } t \geq t_f.$$

The updated initial conditions are (see (4.83)):

$$z_S(t_f) = b, \quad z_T(t_f) = H - \int_0^{t_f} \frac{\varphi_2(T_S(\tau))}{\varphi_1(T_S(\tau),\tau)} d\tau.$$

In the definition of φ_1 and φ_3 the value for b has not to be changed, unlike the previous transition example, since during the lens formation the base of the ice layer keeps at rest.

Calling

$$(4.135) \qquad (q_w^{(f)}(t),\ z_F^{(f)}(t),\ z_S^{(f)}(t),\ z_T^{(f)}(t),\ T_S^{(f)}(t),\ p_w^{(f)}(z_S(t),t))$$

the solution of (4.1)-(4.5) obtained by imposing (4.134) (in particular, $T_S^{(f)}(t) = T_p(h(t))$, cfr. (4.102)), we are going to show that the boundary $z_S^{(f)}(t)$ can not be non-decreasing. Indeed, let us denote by

$$(4.136) \qquad (q_w^{(l)}(t),\ z_F^{(l)}(t),\ z_S^{(l)}(t),\ z_T^{(l)}(t),\ T_S^{(l)}(t),\ p_w^{(l)}(z_S(t),t))$$

the formal solution of (4.1)-(4.5) for $t > t_f$ obained by imposing $\dot{z}_S^{(l)}(t) \equiv 0$ (thus $z_S^{(l)}(t) \equiv b$) that is not a solution of $(\mathbb{S}_{tmp}) + (\mathbb{C}) + (\mathbb{A})$, because (4.9) is violated (see (4.133)).
Assume, contrary to our claim, that there exists an interval $[t_f,t_a)$ such that

$$(4.137) \qquad \dot{z}_S^{(f)}(t) \geq 0,\ t \in [t_f,t_a).$$

We have, according to (4.133):

$$(4.138) \qquad T_S^{(l)}(t) < T_S^{(f)}(t) = T_p(h(t)), \quad t \in (t_f,t_a).$$

Furthermore, (4.12), (4.99) and (4.137) yeld

$$(4.139) \qquad z_S^{(l)}(t) \equiv b \leq z_S^{(f)}(t),\ z_F^{(l)}(t) > z_F^{(f)}(t), \quad t \in (t_f,t_a).$$

From (4.5), (4.99) and (4.139) we get:

$$(4.140) \qquad q_w^{(l)}(t) > q_w^{(f)}(t), \quad t \in (t_f,t_a).$$

Using (4.3), (4.137), (4.140) and the comparison theorem for ordinary differential equations, one has (assume $\rho_i < \rho_w$):

$$(4.141) \qquad z_T^{(f)}(t) < z_T^{(l)}(t), \quad t \in (t_f,t_a).$$

But (4.2), (4.138), (4.139) and (4.141) entail:

$$q_w^{(l)}(t) > q_w^{(f)}(t), \quad t \in (t_f,t_a).$$

which contradicts (4.140). We deduce that $z_S^{(f)}(t)$ can not be non-decreasing.

Besides that, we have that $x(t) = z_S^{(f)}(t)$, $y(t) = z_T^{(f)}(t) - z_S^{(f)}(t)$ is the solution of system $(\mathbf{S}_F)$ defined in par. 4.2. From the equations of that system we easily get:

$$\frac{1}{2}\frac{d}{dt}(x^2(t)) = A(t) + B(t)\frac{x(t)}{y(t)}$$

$$\frac{d}{dt}\left(\frac{x(t)}{y(t)}\right) = \left(A(t)\frac{y(t)}{x(t)} + B(t) - C(t) - D(t)\frac{x(t)}{y(t)}\right)x^2(t).$$

Since (cfr. (4.117) and (4.126))

$$A(t_f)\frac{y(t_f)}{x(t_f)} + B(t_f) = 0,\; C(t_f) + D(t_f)\frac{x(t_f)}{y(t_f)} > 0,$$

we can state that the solution $x(t)$ can not oscillate infinite times so that the sign of x is not defined in the neighbours of t_f $t > t_f$, provided that the given temperatures h and g are, as it is natural, piece-wise increasing or decreasing.
According to the fact that $z_S^{(f)}(t)$ is not non-decreasing, as we proved, we conclude that

$$\dot{z}_S^{(f)}(t) < 0$$

in a suitable interval $(t_f, \tilde{t})$.
Hence, condition (4.98) is verified and the solution (4.135) describes a frost penetration process.

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