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Determination of the Stress State From Transverse Wave Speeds in Isotropic Inelastic Solids

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Abstract

For a transverse acceleration wave propagating along a principal axis of strain in a nonlinear isotropic elastic solid, a simple formula due to Ericksen relates the wave speed to the stress and strain state at the wave front. We derive the appropriate generalization of this result for finite deformation viscoplasticity models based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts. The inclusion of scalar internal state variables (e.g., to model damage) is also considered. The results may be used to obtain information on the stress state ahead of the wave if the strain state and wave speed are known. We discuss applications to the analysis of oblique plate impact tests, where the transverse wave propagates into uniaxially strained material.

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1 Introduction

For several classes of isotropic materials, we derive formulas for the transverse wave speed in oblique plate impact tests in which a flyer plate impacts a parallel target plate (the specimen under study) with both plates inclined relative to the direction of motion of the flyer; cf. [1]–[5]. The velocity imparted to the target face upon impact has nonzero components in directions normal and parallel to the target face; these are referred to as the longitudinal and transverse directions. A longitudinal wave brings the target into a state of compressive uniaxial strain in the longitudinal direction, and a slower transverse (or shear) wave propagates into this uniaxially strained material. The problem considered here is how to use the transverse wave speed to probe the stress state in the uniaxially strained region ahead of the transverse wave.

The Cauchy stress tensor is denoted by $\mathbf{T}$; σ_1, σ_2, σ_3 denote the principal stresses taken positive in compression; and p denotes the pressure:

$$-\mathbf{T} = \sum_{i=1}^{3} \sigma_i \mathbf{e}_i \otimes \mathbf{e}_i \,, \quad p \equiv -\tfrac{1}{3}\mathrm{tr}\,\mathbf{T} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\,. \tag{1}$$

The unit eigenvectors $\mathbf{e}_i$ of $\mathbf{T}$ lie along the principal axes of stress. We assume the target plate is undeformed and stress-free prior to impact; all measures of deformation are taken relative to this undeformed state. The principal stretches are denoted by λ_1, λ_2, λ_3. The densities in the deformed and undeformed states are denoted by ρ and ρ_0, respectively.

Consider the compressive uniaxial strain generated by the longitudinal wave:

$$\rho_0/\rho = \lambda_1 < 1 = \lambda_2 = \lambda_3\,. \tag{2}$$

For an isotropic material, we expect that the principal stresses satisfy

$$\sigma_1 > \sigma_2 = \sigma_3 > 0\,. \tag{3}$$

There is only one nonzero principal shear stress (to within a sign), and we denote it by τ:

$$\tau = \tfrac{1}{2}(\sigma_1 - \sigma_2) > 0\,. \tag{4}$$

Then by (1), (3), and (4), the longitudinal stress σ_1 is given in terms of the pressure p and shear stress τ by the well-known formula

$$\sigma_1 = p + \tfrac{4}{3}\tau\,. \tag{5}$$

Since σ_1 and λ_1 can usually be inferred from measurements of the longitudinal wave, we assume they are known. Then by (4) and (5), any one of the stresses σ_2, p, or τ determines the other two.

We expect that the transverse wave speed U_T is influenced by the stress and (uniaxial) strain state ahead of the transverse wave. If U_T depends on σ_2, p, or τ (and perhaps σ_1 and λ_1 as well), then by measuring U_T and using the known values of σ_1 and λ_1, we might be able to solve for one (and hence all three) of the stresses σ_2, p, and τ in the uniaxially strained region ahead of the transverse wave. This approach is due to Gupta [2, 3]; cf. also Aidun and Gupta [4] and Conner [5]. Our analysis differs from that in [2]–[5] in that we derive *exact* formulas for the transverse wave speed. The elastic case is treated in §2. In §3 we extend these results to viscoplasticity models with instantaneous elastic response based on a multiplicative decomposition of the deformation gradient into elastic and plastic parts. In §4 we discuss thermal effects and the incorporation of scalar internal variables that affect elastic response; possible applications include isotropic damage in ceramics.

Since the transverse wave may be structured, the wave speed U_T of interest here is that of the wave *front*, as this is the only part of the transverse wave traveling into a state of pure uniaxial strain. The results in the sequel are based on the assumption that the transverse wave front is an acceleration wave.[1] Across such a singular surface the particle velocity, stress, and strain are continuous, with jump discontinuities in the gradients and rates of the transverse component (only) of these quantities. As shown by Abou-Sayed and Clifton [1], these jumps need not be finite; e.g., the particle acceleration immediately behind the wave front may be infinite. For the materials considered in this report, such behavior does not affect the formulas for the wave speed.

2 Isotropic Elastic Response

Let $\mathbf{F}$ denote the deformation gradient, $\mathbf{R}$ and $\mathbf{V}$ the rotation and left stretch tensors in the left polar decomposition of $\mathbf{F}$, and $\mathbf{B}$ the left Cauchy-Green tensor: $\mathbf{F} = \mathbf{VR}$ and $\mathbf{B} = \mathbf{FF}^T = \mathbf{V}^2$. Then for isotropic elastic response, $\mathbf{T} = \hat{\mathcal{T}}(\mathbf{F}) = \hat{\mathcal{T}}(\mathbf{V}) = \mathcal{T}(\mathbf{B})$, where $\hat{\mathcal{T}}$ and $\mathcal{T}$ are isotropic functions; cf. [7, §47]. The principal stretches λ_i are the principal values of $\mathbf{V}$. The principal axes of $\mathbf{B}$ and $\mathbf{V}$ are the principal axes of strain in the deformed state. By isotropy, these are also the principal axes of stress, so that by $(1)_1$, $\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$ and $\mathbf{B} = \sum_{i=1}^{3} b_i \mathbf{e}_i \otimes \mathbf{e}_i$, where $b_i = \lambda_i^2$. Let J denote the Jacobian of the deformation and let $\tilde{\rho}$ denote the ratio of

[1] The other possibility is that the transverse wave front is a shock, across which there are jumps in the transverse components of the particle velocity, stress, and strain. Because of material nonlinearities, there may also be second-order jumps in the corresponding longitudinal components (cf. Davison [6] for the elastic case); for this reason such a wave front is often called a quasi-transverse shock. Stability arguments imply that the shock wave speed $\mathcal{U}_T$ should exceed the acceleration wave speed U_T. If the transverse wave front is a shock, then the exact results in the sequel may be converted to inequalities by using the inequality $U_T < \mathcal{U}_T$, or they may be regarded as approximations if U_T is replaced with $\mathcal{U}_T$.

the densities in the deformed and undeformed states:

$$\tilde{\rho} \equiv \rho/\rho_0 = 1/J\,, \quad J \equiv \det \mathbf{F} = \det \mathbf{V} = \sqrt{\det \mathbf{B}} = \sqrt{b_1 b_2 b_3} = \lambda_1 \lambda_2 \lambda_3\,. \tag{6}$$

Consider a transverse acceleration wave with direction of propagation $\mathbf{e}_1$ and amplitude (e.g., jump in acceleration) parallel to $\mathbf{e}_2$. If $b_1 \neq b_2$ (equivalently, $\lambda_1 \neq \lambda_2$), then the speed[2] U_T of this transverse acceleration wave is given by Ericksen's formula:[3]

$$\rho\, U_T^{\,2} = \frac{\sigma_1 - \sigma_2}{\dfrac{b_2}{b_1} - 1} = \frac{\sigma_1 - \sigma_2}{\left(\dfrac{\lambda_2}{\lambda_1}\right)^2 - 1}\,, \tag{7}$$

where all quantities are evaluated at the wave front.

By (2), (6), and the isotropy of $\mathcal{T}$, for compressive uniaxial strain we have

$$1 < \tilde{\rho} = 1/\sqrt{b_1} = 1/\lambda_1\,, \qquad \lambda_2 = \lambda_3 = b_2 = b_3 = 1\,, \qquad \sigma_2 = \sigma_3\,. \tag{8}$$

From (8), (7), and (4), we obtain the following exact formulas for the speed U_T of a transverse acceleration wave propagating in the longitudinal direction into a uniaxially strained isotropic elastic material:

$$\rho\, U_T^{\,2} = \frac{\sigma_1 - \sigma_2}{\tilde{\rho}^2 - 1} = \frac{2\tau}{\tilde{\rho}^2 - 1}\,, \tag{9}$$

where all quantities are evaluated at the wave front. From $(8)_1$ and (9) we see that $\sigma_1 > \sigma_2$ (equivalently $\tau > 0$) is necessary for the existence of a transverse acceleration wave. On solving (9) for τ and then using (5) and (4), we obtain exact formulas for the shear stress, pressure, and lateral stress in the uniaxially strained region ahead of the transverse wave (cf. Scheidler [11]):

$$\tau = \tfrac{1}{2}(\tilde{\rho}^2 - 1)\rho\, U_T^{\,2}\,, \tag{10}$$

$$p = \sigma_1 - \tfrac{4}{3}\tau = \sigma_1 - \tfrac{2}{3}(\tilde{\rho}^2 - 1)\rho\, U_T^{\,2}\,, \tag{11}$$

$$\sigma_2 = \sigma_1 - 2\tau = \sigma_1 - (\tilde{\rho}^2 - 1)\rho\, U_T^{\,2}\,. \tag{12}$$

For isotropic elastic response, this solves the problem posed in the introduction.

[2] U_T always denotes the speed relative to the deformed material. For elastic response, U_T is also the ultrasonic shear wave speed. For the cases covered by (7) and (9), the corresponding referential or Lagrangian wave speed is U_T/λ_1.

[3] Cf. Truesdell [8], Truesdell and Noll [7, §74], and Bowen and Wang [9, 10]. The universal relation (7) does not require the existence of a strain energy function.

3 Viscoplasticity Based on the Multiplicative Decomposition of the Deformation Gradient

In this section we consider viscoplasticity theories based on the multiplicative decomposition of the total deformation gradient $\mathbf{F}$ into an elastic part $\mathbf{F}^E$ and a plastic part $\mathbf{F}^P$: $\mathbf{F} = \mathbf{F}^E\mathbf{F}^P$; cf. [12, §4] and [13]. $\mathbf{F}^E$ and $\mathbf{F}^P$ have left and right polar decompostions $\mathbf{F}^E = \mathbf{V}^E\mathbf{R}^E$ and $\mathbf{F}^P = \mathbf{R}^P\mathbf{U}^P$, respectively, where $\mathbf{R}^E$ and $\mathbf{R}^P$ are rotations, and the elastic left stretch tensor $\mathbf{V}^E$ and plastic right stretch tensor $\mathbf{U}^P$ are symmetric positive-definite. Then

$$\mathbf{F} = \mathbf{F}^E\mathbf{F}^P = (\mathbf{V}^E\mathbf{R}^E)(\mathbf{R}^P\mathbf{U}^P) = \mathbf{V}^E\mathbf{R}^\sharp\mathbf{U}^P\,, \qquad \mathbf{R}^\sharp = \mathbf{R}^E\mathbf{R}^P\,. \tag{13}$$

The elastic left Cauchy-Green tensor $\mathbf{B}^E$ and the plastic right Cauchy-Green tensor $\mathbf{C}^P$ are defined as follows:

$$\mathbf{B}^E \equiv \mathbf{F}^E(\mathbf{F}^E)^T = (\mathbf{V}^E)^2\,, \qquad \mathbf{C}^P \equiv (\mathbf{F}^P)^T\mathbf{F}^P = (\mathbf{U}^P)^2\,. \tag{14}$$

Also note that from (13) and (14), we have

$$\mathbf{F}^E = \mathbf{F}(\mathbf{F}^P)^{-1}\,, \qquad \mathbf{B}^E = \mathbf{F}(\mathbf{C}^P)^{-1}\mathbf{F}^T\,, \qquad \mathbf{V}^E\mathbf{R}^\sharp = \mathbf{F}(\mathbf{U}^P)^{-1}\,. \tag{15}$$

We interpret $\mathbf{F}^P$ as the deformation gradient from the undeformed reference configuration to an intermediate plastically deformed configuration. For any deformations for which $\mathbf{F}^P$ remains fixed, the response is assumed to be elastic, so that the stress $\mathbf{T}$ depends only on the deformation gradient $\mathbf{F}^E$ from the intermediate configuration to the current configuration. We assume that this elastic response relative to the intermediate configuration is isotropic and unaffected by prior plastic deformation. Thus there are isotropic functions $\hat{\mathcal{T}}$ and $\mathcal{T}$ such that[4]

$$\mathbf{T} = \hat{\mathcal{T}}(\mathbf{F}^E) = \hat{\mathcal{T}}(\mathbf{V}^E) = \mathcal{T}(\mathbf{B}^E)\,. \tag{16}$$

Then $\mathbf{T}$, $\mathbf{V}^E$, and $\mathbf{B}^E$ are coaxial, so that by $(1)_1$, $\mathbf{V}^E = \sum_{i=1}^3 \lambda_i^E \mathbf{e}_i \otimes \mathbf{e}_i$ and $\mathbf{B}^E = \sum_{i=1}^3 b_i^E \mathbf{e}_i \otimes \mathbf{e}_i$, where $b_i^E = (\lambda_i^E)^2$, and the principal values λ_i^E of $\mathbf{V}^E$ are the principal elastic stretches. The spatial velocity gradient is $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ (a superposed dot denotes the material time derivative), so by (16) and $(15)_2$,

$$\dot{\mathbf{T}} = D\mathcal{T}(\mathbf{B}^E)[\dot{\mathbf{B}}^E]\,, \qquad \dot{\mathbf{B}}^E = \mathbf{L}\mathbf{B}^E + \mathbf{B}^E\mathbf{L}^T + \mathbf{F}\overline{(\mathbf{C}^P)^{-1}}^{\boldsymbol{\cdot}}\mathbf{F}^T\,, \tag{17}$$

[4]By $(15)_3$, $\mathbf{V}^E\mathbf{R}^\sharp$ is the left polar decomposition of $\mathbf{F}(\mathbf{U}^P)^{-1}$. Hence if $\mathbf{F}$ and $\mathbf{U}^P$ (or $\mathbf{C}^P$) are known, then $\mathbf{V}^E$ and $\mathbf{R}^\sharp$ (but not $\mathbf{F}^E$, $\mathbf{F}^P$, $\mathbf{R}^E$, or $\mathbf{R}^P$) are uniquely determined; cf. Nemat-Nasser [12, §4]. Since $\mathbf{T}$ is determined by $\mathbf{V}^E$, it follows that $\mathbf{T}$ is determined by $\mathbf{F}$ and $\mathbf{U}^P$ (or $\mathbf{C}^P$); this conclusion also follows from $(16)_3$ and $(15)_2$. Thus to complete the constitutive model, one need only provide evolution equations for $\mathbf{U}^P$ or $\mathbf{C}^P$. We do not consider specific evolution equations here, but instead postulate certain qualitative properties of these plastic deformation tensors.

where $D\mathcal{T}(\mathbf{B}^E)$, the derivative of $\mathcal{T}$ evaluated at $\mathbf{B}^E$, is a fourth-order tensor.

We assume that the material exhibits instantaneous elastic response in the sense that $\mathbf{C}^P$ and $\dot{\mathbf{C}}^P$ are continuous across an acceleration wave front. Thus

$$\left[\!\left[\mathbf{C}^P\right]\!\right] = \mathbf{0} \qquad \text{and} \qquad \left[\!\left[\dot{\mathbf{C}}^P\right]\!\right] = \mathbf{0}\,, \tag{18}$$

where $[\![\Phi]\!]$ denotes the jump in Φ across the wave front. These conditions are equivalent to $\left[\!\left[\mathbf{U}^P\right]\!\right] = \left[\!\left[\dot{\mathbf{U}}^P\right]\!\right] = \mathbf{0}$ and also to $\left[\!\left[(\mathbf{C}^P)^{-1}\right]\!\right] = \left[\!\left[\dot{\overline{(\mathbf{C}^P)^{-1}}}\right]\!\right] = \mathbf{0}$. Since $\mathbf{F}$ is also continuous across an acceleration wave, from $(15)_2$ and (16)–(18) it follows that $\mathbf{B}^E$ and $\mathbf{T}$ are continuous across an acceleration wave, whereas

$$\left[\!\left[\dot{\mathbf{T}}\right]\!\right] = D\mathcal{T}(\mathbf{B}^E)\left[\!\left[\dot{\mathbf{B}}^E\right]\!\right]\,, \qquad \left[\!\left[\dot{\mathbf{B}}^E\right]\!\right] = [\![\mathbf{L}]\!]\,\mathbf{B}^E + \mathbf{B}^E\,[\![\mathbf{L}]\!]^T\,. \tag{19}$$

These relations are analogous to the jump relations for the purely elastic case, where $\mathbf{B}^E = \mathbf{B}$. Then an analysis similar to that in [7]–[10] yields the following generalization of Ericksen's formula (7). Assume that $b_2^E \neq b_1^E$ (equivalently, $\lambda_2^E \neq \lambda_1^E$). Then the speed U_T of a transverse acceleration wave with direction of propagation $\mathbf{e}_1$ and amplitude parallel to $\mathbf{e}_2$ is given by

$$\rho\, U_T^2 = \frac{\sigma_1 - \sigma_2}{\dfrac{b_2^E}{b_1^E} - 1} = \frac{\sigma_1 - \sigma_2}{\left(\dfrac{\lambda_2^E}{\lambda_1^E}\right)^2 - 1}\,, \tag{20}$$

where all quantities are evaluated at the wave front.

The Jacobian J is given by (6). Since $\mathbf{F} = \mathbf{F}^E\mathbf{F}^P$, we also have $J = J^E J^P$, where the elastic Jacobian J^E and a plastic Jacobian J^P are given by

$$J^E \equiv \det \mathbf{F}^E = \det \mathbf{V}^E = \lambda_1^E\lambda_2^E\lambda_3^E\,, \quad J^P \equiv \det \mathbf{F}^P = \det \mathbf{U}^P = \lambda_1^P\lambda_2^P\lambda_3^P\,. \tag{21}$$

The principal values λ_i^P of $\mathbf{U}^P$ are the principal plastic stretches. The principal distortional elastic stretches $\overline{\lambda_i^E}$ and the principal distortional plastic stretches $\overline{\lambda_i^P}$ are defined as follows (i, j, k distinct):

$$\overline{\lambda_i^E} \equiv \frac{\lambda_i^E}{(J^E)^{1/3}} = \left(\frac{\lambda_i^E}{\lambda_j^E}\frac{\lambda_i^E}{\lambda_k^E}\right)^{1/3}, \qquad \overline{\lambda_i^P} \equiv \frac{\lambda_i^P}{(J^P)^{1/3}} = \left(\frac{\lambda_i^P}{\lambda_j^P}\frac{\lambda_i^P}{\lambda_k^P}\right)^{1/3}. \tag{22}$$

Then $\overline{\lambda_i^E}$ and $\overline{\lambda_i^P}$ are independent of J^E and J^P, respectively, and thus are measures of elastic and plastic distortion only. They satisfy the constraints

$$\overline{\lambda_1^E}\,\overline{\lambda_2^E}\,\overline{\lambda_3^E} = 1\,, \qquad \overline{\lambda_1^P}\,\overline{\lambda_2^P}\,\overline{\lambda_3^P} = 1\,. \tag{23}$$

If plastic incompressibility is assumed, then $J^P \equiv 1$, and hence $\overline{\lambda_i^P} = \lambda_i^P$.

We wish to apply these results to the case of a uniaxial strain history ahead of the transverse wave. Isotropy of $\hat{\mathcal{T}}$ implies that $\sigma_2 = \sigma_3$ if $\lambda_2^E = \lambda_3^E$, but this latter condition does not follow from the constituitive assumptions made up to this point. We now make the following assumptions (which are reasonable for isotropic materials): for a uniaxial strain history from the undeformed state, $\mathbf{C}^P$ (equivalently, $\mathbf{U}^P$) is coaxial with $\mathbf{F}$, and $\lambda_2^P = \lambda_3^P$. Then[5]

$$\lambda_1 = \lambda_1^E \lambda_1^P , \qquad \lambda_3^E = \lambda_2^E = 1/\lambda_2^P = 1/\lambda_3^P . \tag{24}$$

As noted above, $(24)_2$ implies $\sigma_2 = \sigma_3$. The relations (22), (24), and (8) imply

$$\overline{\lambda_1^E} = \left(\frac{\lambda_1^E}{\lambda_2^E}\right)^{2/3}, \qquad \overline{\lambda_1^P} = \left(\frac{\lambda_1^P}{\lambda_2^P}\right)^{2/3}, \qquad \left(\frac{\lambda_2^E}{\lambda_1^E}\right)^2 = \frac{1}{\left(\overline{\lambda_1^E}\right)^3} = \tilde{\rho}^2 \left(\overline{\lambda_1^P}\right)^3. \tag{25}$$

On substituting $(25)_{3,4}$ into (20) and using (4), we obtain the following exact formulas for the speed U_T of a transverse acceleration wave propagating in the longitudinal direction into uniaxially strained material:[6]

$$\rho\, U_T^2 = \frac{2\tau}{\dfrac{1}{\left(\overline{\lambda_1^E}\right)^3} - 1} = \frac{2\tau}{\tilde{\rho}^2 \left(\overline{\lambda_1^P}\right)^3 - 1}, \tag{26}$$

where all quantities are evaluated at the wave front. We may solve (26) for τ, and then (4) and (5) for p and σ_2. The results are given by (10)–(12) with $\tilde{\rho}^2$ replaced by $\tilde{\rho}^2 \left(\overline{\lambda_1^P}\right)^3$. To actually calculate τ, p, and σ_2 from these formulas we need to know $\overline{\lambda_1^P}$ in addition to U_T, $\tilde{\rho}$, and σ_1. On the other hand, if σ_2 is known (from lateral stress gauge data or additional constitutive assumptions), then (26) could be used to calculate $\overline{\lambda_1^P}$, assuming U_T, $\tilde{\rho}$, and σ_1 are known.

[5]To see this, note that for uniaxial strain, $\mathbf{F} = \mathbf{V}$ is symmetric positive-definite, and hence so is $\mathbf{F}(\mathbf{U}^P)^{-1}$, since $\mathbf{F}$ and $\mathbf{U}^P$ are coaxial. $\mathbf{V}^E\mathbf{R}^\sharp$ is the left polar decomposition of $\mathbf{F}(\mathbf{U}^P)^{-1}$ (cf. $(15)_3$), so the uniqueness of the polar decompostion implies $\mathbf{R}^\sharp$ is the identity in this case, since $\mathbf{F}(\mathbf{U}^P)^{-1}$ and $\mathbf{V}^E$ are symmetric positive-definite. Thus $\mathbf{F} = \mathbf{V}^E\mathbf{U}^P$ with all three tensors coaxial (this may also be inferred from $(15)_2$ and (14)). Then $\lambda_i = \lambda_i^E \lambda_i^P$, which, together with $\lambda_2 = \lambda_3 = 1$ and $\lambda_2^P = \lambda_3^P$, implies (24).

[6]The Lagrangian wave speed is U_T/λ_1. For compressive uniaxial strain we expect that $\lambda_1^E < \lambda_2^E$, which, by $(25)_1$, is equivalent to $\overline{\lambda_1^E} < 1$. Then the denominator in (26) is positive, and $\tau > 0$ (equivalently $\sigma_1 > \sigma_2$) is necessary for the existence of a transverse wave. Note that (26) reduces to (9) when there is no plastic deformation, i.e., when $\lambda_i^P = \overline{\lambda_i^P} = 1$.

4 Discussion

The results in §2 and §3 are actually valid under more general assumptions than previously described. Suppose, for example, that

$$\mathbf{T} = \mathcal{T}(\mathbf{B}; \varepsilon_1, \ldots, \varepsilon_N), \tag{27}$$

where $\varepsilon_1, \ldots, \varepsilon_N$ are internal state variables that evolve according to

$$\dot{\varepsilon}_k = f_k(\mathbf{T}, \mathbf{B}; \varepsilon_1, \ldots, \varepsilon_N) \quad \text{if} \quad g_k(\mathbf{T}, \mathbf{B}; \varepsilon_1, \ldots, \varepsilon_N) > 0\,. \tag{28}$$

Here $\mathcal{T}$, f_k, and g_k ($k = 1, \ldots, N$) are assumed to be isotropic functions of their tensor arguments for fixed values of $\varepsilon_1, \ldots, \varepsilon_N$. Thus the response is isotropic elastic for fixed values of the internal variables, but the elastic response may change with changes in the internal variables, so that the overall response is inelastic. We assume that the ε_k are continuous across an acceleration wave. Then $(28)_1$ implies that the rates $\dot{\varepsilon}_k$ are also continuous across the wave:

$$[\![\varepsilon_k]\!] = 0\,, \qquad [\![\dot{\varepsilon}_k]\!] = 0\,. \tag{29}$$

The acceleration wave analysis proceeds just as in the elastic case, and Ericksen's formula (7) continues to hold; cf. Bowen and Wang [10]. Hence the formulas (9)–(12) for transverse waves propagating into uniaxially strained material remain valid. Of course, the values of the stresses in these formulas will now depend on the current values of the internal variables, but the relations between τ, p, σ_2, σ_1, and U_T remain unchanged.

The viscoplasticity model may be generalized by replacing (16) with

$$\mathbf{T} = \mathcal{T}(\mathbf{B}^E; \varepsilon_1, \ldots, \varepsilon_N)\,. \tag{30}$$

Here the evolution equations for the ε_k might also involve the plastic stretch $\mathbf{U}^P$ (as well as any internal variables characterizing the plastic state of the material). Conversely, the evolution equation for $\mathbf{U}^P$ (and any plastic internal variables) would generally involve the ε_k. We will not consider specific forms for these equations. However, if we assume that the jump conditions (29) and (18) are satisfied across an acceleration wave, then the jump conditions (19) continue to hold, and the generalization (20) of Ericksen's formula remains valid. Likewise, (26) continues to hold if we retain the assumption that for uniaxial strain histories, $\mathbf{U}^P$ and $\mathbf{F}$ are coaxial and $\lambda_2^P = \lambda_3^P$.

We may also include dependence on the temperature θ in any of the models discussed here. Assume the material conducts heat by Fourier's law, where the thermal conductivity may depend on θ, ρ, and the ε_k. Assuming $[\![\theta]\!] = 0$ across an acceleration wave, it can be shown that $[\![\dot{\theta}]\!] = 0$ also; cf. [8] and [10]. Then all of our previous results continue to hold.

We conclude by emphasizing that in the derivation of our results, no small strain assumptions have been made, even for the elastic strains. For the classes of materials considered here, this actually simplifies the analysis, resulting in simple, exact universal relations for the transverse wave speed.

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13. ABSTRACT *(Maximum 200 words)*

For a transverse acceleration wave propagating along a principal axis of strain in a nonlinear isotropic elastic solid, a simple formula due to Ericksen relates the wave speed to the stress and strain state at the wave front. We derive the appropriate generalization of this result for finite deformation viscoplasticity models based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts. The inclusion of scalar internal state variables (e.g., to model damage) is also considered. The results may be used to obtain information on the stress state ahead of the wave if the strain state and wave speed are known. We discuss applications to the analysis of oblique plate impact tests, where the transverse wave propagates into uniaxially strained material.

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