

**NAVAL POSTGRADUATE SCHOOL**  
**Monterey, California**



**THESIS**

**NONLINEAR ANALYSIS OF COUPLED  
ROLL/SWAY/YAW STABILITY  
CHARACTERISTICS OF SUBMERSIBLE  
VEHICLES**

by

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March, 1997

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19971112 087

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# REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503.

1. AGENCY USE ONLY <i>(Leave blank)</i>	2. REPORT DATE March 1997	3. REPORT TYPE AND DATES COVERED Master's Thesis	
4. TITLE AND SUBTITLE <b>NONLINEAR ANALYSIS OF COUPLED ROLL/SWAY/YAW STABILITY CHARACTERISTICS OF SUBMERSIBLE VEHICLES</b>		5. FUNDING NUMBERS	
6. AUTHOR(S) Sotirios E. Tsamilis		8. PERFORMING ORGANIZATION REPORT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Naval Postgraduate School Monterey CA 93943-5000			
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSORING/MONITORING AGENCY REPORT NUMBER	
11. SUPPLEMENTARY NOTES The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.			
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.		12b. DISTRIBUTION CODE	
13. ABSTRACT <i>(maximum 200 words)</i> The problem of coupled roll, sway, and yaw stability analysis of submersible vehicles is analyzed, with particular emphasis on nonlinear studies. Previous results had indicated that a primary loss of stability is through the development of limit cycles. This loss of stability is due to the coupling of roll into sway and yaw and cannot be predicted by considering the uncoupled dynamics. In this study, it is shown that the mechanism of loss of stability is through bifurcations to periodic solutions. These are characterized as either subcritical or supercritical, depending on the sign of a certain nonlinear coefficient. Implications of these results to vehicle performance and operations are discussed.			
14. SUBJECT TERMS SHIP STEERING BIFURCATIONS LIMIT CYCLES			15. NUMBER OF PAGES 84
			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)  
Prescribed by ANSI Std. Z39-18 298-102



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**NONLINEAR ANALYSIS OF COUPLED ROLL/SWAY/YAW  
STABILITY CHARACTERISTICS OF SUBMERSIBLE VEHICLES**

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Submitted in partial fulfillment  
of the requirements for the degree of

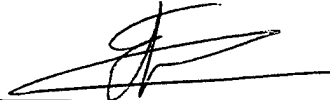
**MASTER OF SCIENCE IN MECHANICAL ENGINEERING**

from the

**NAVAL POSTGRADUATE SCHOOL**

**March 1997**

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## ABSTRACT

The problem of coupled roll, sway, and yaw stability analysis of submersible vehicles is analyzed, with particular emphasis on nonlinear studies. Previous results had indicated that a primary loss of stability is through the development of limit cycles. This loss of stability is due to the coupling of roll into sway and yaw and cannot be predicted by considering the uncoupled dynamics. In this study, it is shown that the mechanism of loss of stability is through bifurcations to periodic solutions. These are characterized as either subcritical or supercritical, depending on the sign of a certain nonlinear coefficient. Implications of these results to vehicle performance and operations are discussed.



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## I. INTRODUCTION

### A. PROBLEM STATEMENT

The dynamic response of a submersible vehicle operating at the extremes of its operational envelope is becoming increasingly important in order to enhance vehicle operations. Traditionally, dynamic stability of motion is studied using eigenvalue analysis where the equations of motion are linearized around nominal straight line level flight paths [Arentzen & Mandel (1960)], [Clayton & Bishop (1982)], [Feldman (1987)]. Directional stability in the horizontal plane is normally studied assuming that coupling between sway/yaw and roll does not exist. Relaxing this approximation can lead to an oscillatory loss of directional stability [Cunningham (1993)] which cannot be predicted by uncoupled sway/yaw motions only. This oscillatory loss of stability can generate limit cycles in the system, as was confirmed numerically in previous studies [Cunningham (1993)]. In order to gain a better understanding of the mechanism of this type of loss of stability and the stability properties of the resulting limit cycles, it is necessary to perform a systematic nonlinear analysis which is precisely the scope of this work.

## B. OBJECTIVES AND OUTLINE

In this work we examine the problem of stability of motion with controls fixed in the horizontal plane, with particular emphasis on the mechanism of loss of stability of straight line motion. Coupling between sway/yaw and roll is taken into consideration. This has its origins in both inertial and hydrodynamic coupling. We concentrate on an oscillatory loss of stability case, where one pair of complex conjugate of eigenvalues of the linearized system matrix crosses the imaginary axis. This loss of stability occurs in the form of generic bifurcations to periodic solutions [Guckenheimer & Holmes (1983)]. Taylor expansions and center manifold approximations are employed in order to isolate the main nonlinear terms that influence system response after the initial loss of stability [Hassard & Wan (1978)]. Integral averaging is performed in order to combine the nonlinear terms into a design stability coefficient [Chow & Mallet-Paret (1977)]. Special attention is paid to the study of the quadratic drag terms as they constitute some of the main nonlinear terms of the equations of motion. The difficulty associated with the nonsmoothness of the absolute value nonlinearities is dealt with by employing the concept of generalized gradient [Clarke (1983)], a technique which was utilized in [Papadimitriou (1994)]. This has the advantage of keeping the linear terms constant, unlike the linear/cubic approximation typically used in ship roll motion studies [Dalzell (1978)], where the linear damping coefficient is a function of the assumed amplitude of motion.

Vehicle modeling in this work follows standard notation [Gertler & Hagen

(1976)], [Smith *et al* (1978)], and numerical results are presented for a variant of the Swimmer Delivery Model used in [Cunningham (1993)] for which a set of hydrodynamic coefficients and geometric properties is available. Although the main results and conclusions of this work are derived for a submerged vehicle, similar techniques can be applied to surface ships as well.



## II. EQUATIONS OF MOTION

### A. COORDINATE SYSTEM

In our analysis we are going to use a moving coordinate system  $(x, y, z)$ , attached on the vehicle. The origin of this system coincides with the center of buoyancy, B. The  $x$ -axis is attached to the longitudinal plane of symmetry for the vehicle, the  $y$ -axis is positive starboard, and the  $z$ -axis is positive downwards. All main symbols used in the development of the equations of motion in this work are summarized in Table 1.

### B. GENERAL FORM OF THE EQUATIONS OF MOTION

The equations of motion for a submersible vehicle in the horizontal plane are written as follows:

Sway equation:

$$\begin{aligned}
 & m[\dot{v} + Ur - wp + x_G(pq + \dot{r}) - y_G(p^2 + r^2) + z_G(qr - \dot{p})] = \\
 & Y_{\dot{p}}\dot{p} + Y_{\dot{r}}\dot{r} + Y_{pq}pq + Y_{qr}qr + Y_vUv + Y_{vw}vw + \\
 & Y_{\delta_r}U^2\delta_r + Y_{\dot{v}}\dot{v} + Y_pUp + Y_rUr + Y_{vq}vq + Y_{wp}wp + Y_{wr}wr + \\
 & (W - B) \cos \theta \sin \phi - \\
 & \int_{x_{\text{tail}}}^{x_{\text{nose}}} [C_{D_y}h(x)(v + xr)^2 + C_{D_z}b(x)(w - xq)^2] \frac{(v + xr)}{U_{cf}(x)} dx . \quad (1)
 \end{aligned}$$

Yaw equation:

$$\begin{aligned}
& I_{zz}\dot{r} + (I_{yy} - I_{xx})pq - I_{xy}(p^2 - q^2) - I_{yz}(pr + q) + I_{xz}(qr - \dot{p}) + \\
& m[x_G(\dot{v} + Ur - wp) - y_G(\dot{U} - vr + wq)] = \\
& N_{\dot{p}}\dot{p} + N_{\dot{r}}\dot{r} + N_{pq}pq + N_{qr}qr + N_vUv + N_{vw}vw + \\
& N_{\delta_r}U^2\delta_r + N_{\dot{v}}\dot{v} + N_pUp + N_rUr + N_{vq}vq + N_{wp}wp + N_{wr}wr + \\
& (x_GW - x_BB) \cos \theta \sin \phi + (y_GW - y_BB) \sin \theta + U^2N_{\text{prop}} - \\
& \int_{x_{\text{tail}}}^{x_{\text{nose}}} \left[ C_{D_y}h(x)(v + xr)^2 + C_{D_z}b(x)(w - xq)^2 \right] \frac{(v + xr)}{U_{cf}(x)} x dx . \quad (2)
\end{aligned}$$

Roll equation:

$$\begin{aligned}
& I_{xx}\dot{p} + (I_{zz} - I_{yy})qr + I_{xy}(pr - \dot{q}) - I_{yz}(q^2 - r^2) - I_{xz}(pq + \dot{r}) + \\
& m[y_G(\dot{w} - Uq + vp) - z_G(\dot{v} + Ur - wp)] = \\
& K_{\dot{p}}\dot{p} + K_{\dot{r}}\dot{r} + K_{pq}pq + K_{qr}qr + K_vUv + K_{vw}vw + \\
& K_{\text{prop}}U^2 + K_{\dot{v}}\dot{v} + K_pUp + K_rUr + K_{vq}vq + K_{wp}wp + K_{wr}wr + \\
& (y_GW - y_BB) \cos \theta \cos \phi - (z_GW - z_BB) \cos \theta \sin \phi . \quad (3)
\end{aligned}$$

The rotational velocity equation around the  $x$ -axis is simply,

$$\dot{\phi} = p , \quad (4)$$

while  $U_{cf}$  denotes the cross-flow velocity,

$$U_{cf}(x) = \sqrt{(v + xr)^2 + (w - xq)^2} .$$

$b(x)$	local beam of the hull
$C_D$	quadratic drag coefficient
$\delta_r$	rudder deflection
$h(x)$	local height of hull
$(I_{xx}, I_{yy}, I_{zz})$	vehicle mass moments of inertia about body axes
$(I_{xy}, I_{yz}, I_{zx})$	cross products of inertia
$(K, M, N)$	moment components along the three axes
$m$	vehicle mass
$(p, q, r)$	rotational velocity components along the body axes
$(\phi, \theta, \psi)$	Euler angles
$U$	constant vehicle speed along the $x$ -axis
$(u, v, w)$	translational velocities about $(x, y, z)$ axes
$(x, y, z)$	distances along the three body axes
$(X, Y, Z)$	force components along the body axes
$(x_G, y_G, z_G)$	coordinates of the center of gravity
$(x_B, y_B, z_B)$	coordinates of the center of buoyancy
$x_{\text{nose}}$	fore coordinate of vehicle body
$x_{\text{tail}}$	aft coordinate of vehicle body

Table 1: Nomenclature

### C. SIMPLIFICATIONS

Before we proceed with the analysis, we must simplify the above equations of motion in order to reflect the fact that we are analyzing motion in the horizontal plane only. The simplifications that we employ are the following:

- Velocity,  $w$ , and acceleration,  $\dot{w}$ , in the  $z$ -direction are zero.
- Acceleration in the longitudinal direction,  $\dot{u}$ , is zero.
- Rotational velocity,  $q$ , and acceleration,  $\dot{q}$ , in the  $y$ -direction are zero.
- Pitch angle,  $\theta$ , is zero.

## D. SIMPLIFIED EQUATIONS OF MOTION

After the application of the above simplifying assumptions, the equations of motion become:

Sway equation:

$$\begin{aligned}
 & (m - Y_{\dot{v}})\dot{v} + (mx_G - Y_{\dot{r}})\dot{r} - (mz_G + Y_{\dot{p}})\dot{p} = \\
 & Y_v U v + (Y_r U - mU)r + Y_p U p + y_G p^2 + y_G r^2 + (W - B) \sin \phi + Y_{\delta_r} U^2 \delta_r - \\
 & \int_{x_{\text{tail}}}^{x_{\text{nose}}} C_{D_y} h(x) (v + xr)^2 \frac{(v + xr)}{U_{cf}(x)} dx . \tag{5}
 \end{aligned}$$

Yaw equation:

$$\begin{aligned}
 & (mx_G - N_{\dot{v}})\dot{v} + (I_{zz} - N_{\dot{r}})\dot{r} - (I_{xz} + N_{\dot{p}})\dot{p} = \\
 & N_v U v + (N_r U - mx_G U)r + N_p U p + I_{xy} p^2 + I_{yz} p r + y_G v r + \\
 & (x_G W - x_B B) \sin \phi + N_{\delta_r} U^2 \delta_r + U^2 N_{\text{prop}} - \\
 & \int_{x_{\text{tail}}}^{x_{\text{nose}}} C_{D_y} h(x) (v + xr)^2 \frac{(v + xr)}{U_{cf}(x)} x dx . \tag{6}
 \end{aligned}$$

Roll equation:

$$\begin{aligned}
 & (-mz_G - K_{\dot{v}})\dot{v} + (I_{xz} - K_{\dot{r}})\dot{r} + (I_{xx} - K_{\dot{p}})\dot{p} = \\
 & K_v U v + (K_r U + mz_G U)r + K_p U p - I_{xy} p r - I_{yz} r^2 - my_G v p + \\
 & (y_G W - y_B B) \cos \phi - (z_G W - z_B B) \sin \phi + U^2 K_{\text{prop}} . \tag{7}
 \end{aligned}$$

Roll rate:

$$\dot{\phi} = p . \tag{8}$$

### III. LINEAR ANALYSIS

#### A. LINEARIZATION

The simplified equations of motion can be written in matrix form as follows:

$$\mathbf{A}\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + \mathbf{g}(\mathbf{x}), \quad (9)$$

where the state vector  $\mathbf{x}$  is defined as,

$$\mathbf{x} = \begin{bmatrix} v \\ r \\ p \\ \phi \end{bmatrix},$$

and the state matrices are,

$$\mathbf{A} = \begin{bmatrix} m - Y_{\dot{v}} & mx_G - Y_{\dot{r}} & -mz_G - Y_{\dot{p}} & 0 \\ mx_G - N_{\dot{v}} & I_{zz} - N_{\dot{r}} & -I_{xz} - N_{\dot{p}} & 0 \\ -mz_G - K_{\dot{v}} & I_{xz} - K_{\dot{r}} & I_{xx} - K_{\dot{p}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\mathbf{B} = \begin{bmatrix} Y_v U & Y_r U - mU & Y_p U & 0 \\ N_v U & -mx_G U + N_r U & N_p U & 0 \\ K_v U & mz_G U + K_r U & K_p U & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The term  $\mathbf{g}(\mathbf{x})$  contains all the nonlinear terms,

$$g_1 = y_G p^2 + y_G r^2 + (W - B) \sin \phi + Y_{\delta_r} U^2 \delta_r - C_{D_y} \int_{x_{\text{tail}}}^{x_{\text{nose}}} h(x)(v + xr)|v + xr| dx, \quad (10)$$

$$g_2 = I_{xy} p^2 + I_{yz} pr + y_G vr + (x_G W - x_B B) \sin \phi + N_{\delta_r} U^2 \delta_r - C_{D_y} \int_{x_{\text{tail}}}^{x_{\text{nose}}} h(x)(v + xr)|v + xr|x dx, \quad (11)$$

$$g_3 = -I_{xy} pr - I_{yz} r^2 - m y_G v p + U^2 K_{\text{prop}} +$$

$$(y_G W - y_B B) \cos \phi - (z_G W - z_B B) \sin \phi , \quad (12)$$

$$g_4 = 0 . \quad (13)$$

We want to linearize the nonlinear terms about a nominal point

$$\mathbf{x}_0 = [v_0, r_0, p_0, \phi_0]^T = 0 .$$

After applying Taylor series expansion of the non-linear terms about the nominal point  $\mathbf{x}_0$  and keeping only the linear components, the linearized equations of motion are written in matrix form as:

$$\mathbf{A}' \dot{\mathbf{x}} = \mathbf{B}' \mathbf{x} , \quad (14)$$

where

$$\mathbf{A}' = \mathbf{A} ,$$

and

$$\mathbf{B}' = \begin{bmatrix} Y_v U & Y_r U - mU & Y_p U & W - B \\ N_v U & -m x_G U + N_r U & N_p U & x_G W - x_B B \\ K_v U & m z_G U + K_r U & K_p U & -z_G W + z_B B \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Equation (14) is our linearized system. Eigenvalue analysis for this system is performed in the next section in order to assess the dynamic stability of the vehicle.

## B. LOSS OF STABILITY

Stability of the linearized system depends on the location of the four eigenvalues of the system, in other words the roots of the characteristic equation:

$$\det(\mathbf{B}' - \lambda \mathbf{A}') = 0 , \quad (15)$$

or, in polynomial form,

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0. \quad (16)$$

The coefficients of the polynomial equation (16) are given by:

$$\begin{aligned} A &= (I_{xx} - K_{\dot{p}})[(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})] \\ &+ (mz_G + K_{\dot{v}})[(-I_{zz} + N_{\dot{r}})(mz_G + Y_{\dot{p}}) + (N_{\dot{p}} + I_{xz})(mx_G - Y_{\dot{r}})] \\ &- (K_{\dot{r}})[(-mx_G + N_{\dot{v}})(mz_G + Y_{\dot{p}}) + (N_{\dot{p}} + I_{xz})(m - Y_{\dot{v}})], \end{aligned} \quad (17)$$

$$\begin{aligned} B &= K_v U [(-I_{zz} + N_{\dot{r}})(mz_G + Y_{\dot{p}}) + (N_{\dot{p}} + I_{xz})(mx_G - Y_{\dot{r}})] \\ &- (mz_G + K_{\dot{v}})[(I_{zz} - N_{\dot{r}})(Y_p U) - (UN_r - Umx_G)(mz_G + Y_{\dot{p}})] \\ &+ (N_{\dot{p}} + I_{xz})(UY_r - Um) - (N_p U)(mx_G - Y_{\dot{r}})] \\ &- (I_{xx} - K_{\dot{p}})[(m - Y_{\dot{v}})(UN_r - Umx_G) + (Y_v U)(I_{zz} - N_{\dot{r}})] \\ &- (mx_G - Y_{\dot{r}})(N_v U) - (mx_G - N_{\dot{v}})(UY_r - Um)] \\ &- K_p U [(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}})] \\ &+ (K_{\dot{r}} - I_{xz}) [(mx_G - N_{\dot{v}})(Y_p U) - (N_v U)(mz_G + Y_{\dot{p}})] \\ &+ (N_{\dot{p}} + I_{xz})(Y_v U) - (N_p U)(m - Y_{\dot{v}})] \\ &- (mz_G U + K_r U) [(-mx_G + N_{\dot{v}})(mz_G + Y_{\dot{p}})] \\ &+ (N_{\dot{p}} + I_{xz})(m - Y_{\dot{v}})] \end{aligned} \quad (18)$$

$$\begin{aligned} C &= (I_{xx} - K_{\dot{p}})[(Y_v U)(UN_r - mUx_G) - (UY_r - Um)(N_v U)] \\ &+ K_p U [(m - Y_{\dot{v}})(UN_r - mUx_G) + (Y_v U)(I_{zz} - N_{\dot{r}})] \\ &- (mx_G - Y_{\dot{r}})(N_v U) - (mx_G - N_{\dot{v}})(UY_r - Um)] \\ &+ (mz_G U + K_r U) [(mx_G - N_{\dot{v}})(Y_p U) - (N_v U)(mz_G + Y_{\dot{p}})] \\ &+ (N_{\dot{p}} + I_{xz})(Y_v U) - (N_p U)(m - Y_{\dot{v}})] \end{aligned}$$

$$\begin{aligned}
& + (z_G W - z_B B)[(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (m x_G - Y_{\dot{r}})(m x_G - N_{\dot{v}})] \\
& - (m z_G + K_{\dot{v}})[(x_B B - x_G W)(m x_G - Y_{\dot{r}}) - (U N_r - U m x_G)(Y_p U)] \\
& + (U Y_r - U m)(N_p U) \\
& - K_v U[(I_{zz} - N_{\dot{r}})(Y_p U) - (U N_r - U m x_G)(m z_G + Y_{\dot{p}})] \\
& + (N_{\dot{p}} + I_{xz})(U Y_r - U m) - (N_p U)(m x_G - Y_{\dot{r}}) \\
& + (K_{\dot{r}} - I_{xz})[(x_B B - x_G W)(m - Y_{\dot{v}}) \\
& - (N_v U)(Y_p U) + (N_p U)(Y_v U)] , \tag{19}
\end{aligned}$$

$$\begin{aligned}
D & = (m z_G U + K_r U)[(x_B B - x_G W)(m - Y_{\dot{v}}) \\
& - (N_v U)(Y_p U) + (N_p U)(Y_v U)] \\
& - (K_{\dot{r}} - I_{xz})(x_B B - x_G W)(Y_v U) \\
& + (m z_G + K_{\dot{v}})(x_B B - x_G W)(U Y_r - U m) \\
& - K_v U[(x_B B - x_G W)(m x_G - Y_{\dot{r}}) \\
& - (U N_r - m U x_G)(Y_p U) + (U Y_r - U m)(N_p U)] \\
& - K_p U[(Y_v U)(U N_r - m U x_G) - (U Y_r - U m)(N_v U)] \\
& - (z_G W - z_B B)[(m - Y_{\dot{v}})(U N_r - m U x_G) + (Y_v U)(I_{zz} - N_{\dot{r}}) \\
& - (m x_G - Y_{\dot{r}})(N_v U) - (m x_G - N_{\dot{v}})(U Y_r - U m)] , \tag{20}
\end{aligned}$$

$$\begin{aligned}
E & = (z_G W - z_B B)[(Y_v U)(U N_r - m U x_G) - (U Y_r - U m)(N_v U)] \\
& + (K_v U)(x_B B - x_G W)(U Y_r - U m) \\
& - (m z_G U + K_r U)(x_B B - x_G W)(Y_v U) \tag{21}
\end{aligned}$$

We can examine the stability of the system by utilizing Routh's criterion. Application of this criterion to the characteristic equation (16), reveals that

the following two conditions must be satisfied in order to ensure that all roots of (16) have negative real parts:

$$BCD - AD^2 - EB^2 > 0, \quad (22)$$

$$E > 0. \quad (23)$$

If  $E$  is less than zero, one real root of (16) becomes positive and the system will become unstable in a divergent manner (Guckenheimer and Holmes, 1983). This is the case of a directionally unstable ship which is well known in the literature (Clayton and Bishop, 1982). If, however, condition (22) is violated, the system will exhibit an oscillatory motion due to the presence of complex conjugate roots with positive real parts. This form of instability is caused by the coupling of roll into sway and yaw and is further analyzed in this work.

In order to compute the limiting case of loss of stability, we consider equation (24),

$$BCD - AD^2 - EB^2 = 0. \quad (24)$$

The result of this equation will produce the limiting value of  $z_G$  as a function of  $x_G$  for loss of stability. A curve of this functional form,

$$z_G = f(x_G),$$

will be our locus of loss of stability. After some algebra, we can express the coefficients of equation (24) in the following form:

$$A = A_1 z_G^2 + A_2 z_G + A_3, \quad (25)$$

where

$$\begin{aligned}
A_1 &= -m^2(I_{zz} - N_{\dot{r}}) \\
A_2 &= -mY_{\dot{p}}(I_{zz} - N_{\dot{r}}) - mK_{\dot{v}}(I_{zz} - N_{\dot{r}}) + m(N_{\dot{p}} + I_{xz})(mx_G - Y_{\dot{r}}) \\
&\quad + m(K_{\dot{r}} - I_{xz})(mx_G - N_{\dot{v}}) \\
A_3 &= (I_{xx} - K_{\dot{p}})(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (I_{xx} - K_{\dot{p}})(mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}}) \\
&\quad - K_{\dot{v}}Y_{\dot{p}}(I_{zz} - N_{\dot{r}}) + K_{\dot{v}}(N_{\dot{p}} + I_{xz})(mx_G - Y_{\dot{r}}) \\
&\quad + Y_{\dot{p}}(K_{\dot{r}} - I_{xz})(mx_G - N_{\dot{v}}) - (K_{\dot{r}} - I_{xz})(N_{\dot{p}} + I_{xz})(m - Y_{\dot{v}}) \\
B &= B_1z_G^2 + B_2z_G + B_3, \tag{26}
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= m^2(UN_r - Umx_G) + m^2U(mx_G - N_{\dot{v}}) \\
B_2 &= -m(K_vU)(I_{zz} - N_{\dot{r}}) - m(I_{zz} - N_{\dot{r}})(Y_pU) \\
&\quad + mY_{\dot{p}}(UN_r - Umx_G) + mK_{\dot{v}}(UN_r - Umx_G) \\
&\quad - m(N_{\dot{p}} + I_{xz})(UY_r - Um) + m(N_pU)(mx_G - Y_{\dot{r}}) \\
&\quad - m(K_{\dot{r}} - I_{xz})(N_vU) + mUY_{\dot{p}}(mx_G - N_{\dot{v}}) \\
&\quad - mU(N_{\dot{p}} + I_{xz})(m - Y_{\dot{v}}) + mUK_r(mx_G - N_{\dot{v}}) \\
B_3 &= -Y_{\dot{p}}(K_vU)(I_{zz} - N_{\dot{r}}) + (K_vU)(N_{\dot{p}} + I_{xz})(mx_G - Y_{\dot{r}}) \\
&\quad - K_{\dot{v}}(I_{zz} - N_{\dot{r}})(Y_pU) + K_{\dot{v}}Y_{\dot{p}}(UN_r - Umx_G) \\
&\quad - K_{\dot{v}}(N_{\dot{p}} + I_{xz})(UY_r - Um) + K_{\dot{v}}(N_pU)(mx_G - Y_{\dot{r}}) \\
&\quad - (I_{xx} - K_{\dot{p}})(m - Y_{\dot{v}})(UN_r - Umx_G) - (I_{xx} - K_{\dot{p}})(Y_vU)(I_{zz} - N_{\dot{r}}) \\
&\quad + (I_{xx} - K_{\dot{p}})(mx_G - Y_{\dot{r}})(N_vU) + (I_{xx} - K_{\dot{p}})(mx_G - N_{\dot{v}})(UY_r - Um)
\end{aligned}$$

$$\begin{aligned}
& - (K_p U)(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) + (K_p U)(mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}}) \\
& + (K_{\dot{r}} - I_{xz})(mx_G - N_{\dot{v}})(Y_p U) - Y_{\dot{p}}(K_{\dot{r}} - I_{xz})(N_v U) \\
& + (N_{\dot{p}} + I_{xz})(Y_v U)(K_{\dot{r}} - I_{xz}) - (K_{\dot{r}} - I_{xz})(N_p U)(m - Y_{\dot{v}}) \\
& + UK_r Y_{\dot{p}}(mx_G - N_{\dot{v}})
\end{aligned}$$

$$C = C_1 z_G^2 + C_2 z_G + C_3, \quad (27)$$

where

$$\begin{aligned}
C_1 &= -m^2 U(N_v U) \\
C_2 &= mU(mx_G - N_{\dot{v}})(Y_p U) - mUY_{\dot{p}}(N_v U) - mUK_r(N_v U) \\
& + mU(N_{\dot{p}} + I_{xz})(Y_v U) - mU(N_p U)(m - Y_{\dot{v}}) \\
& + W(m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - W(mx_G - Y_{\dot{r}})(mx_G - N_{\dot{v}}) \\
& - m(X_B B - x_G W)(mx_G - Y_{\dot{r}}) + m(UN_r - Umx_G)(Y_p U) \\
& - m(UY_r - Um)(N_p U) + m(K_v U)(UN_r - Umx_G) \\
C_3 &= (I_{xx} - K_{\dot{p}})(Y_v U)(UN_r - Umx_G) - (I_{xx} - K_{\dot{p}})(UY_r - Um)(N_v U) \\
& + (K_p U)(m - Y_{\dot{v}})(UN_r - Umx_G) + (K_p U)(Y_v U)(I_{zz} - N_{\dot{r}}) \\
& + (K_p U)(mx_G - Y_{\dot{r}})(N_v U) - (K_p U)(mx_G - N_{\dot{v}})(UY_r - Um) \\
& + UK_r(mx_G - N_{\dot{v}})(Y_p U) - UK_r Y_{\dot{p}}(N_v U) \\
& + UK_r(N_{\dot{p}} + I_{xz})(Y_v U) - UK_r(N_p U)(m - Y_{\dot{v}}) \\
& - K_{\dot{v}}(X_B B - x_G W)(mx_G - Y_{\dot{r}}) + K_{\dot{v}}(UN_r - Umx_G)(Y_p U) \\
& - K_{\dot{v}}(UY_r - Um)(N_p U) - (K_v U)(I_{zz} - N_{\dot{r}})(Y_p U) \\
& + Y_{\dot{p}}(K_v U)(UN_r - Umx_G) - (K_v U)(N_{\dot{p}} + I_{xz})(UY_r - Um) \\
& + (K_v U)(N_p U)(mx_G - Y_{\dot{r}}) + (K_{\dot{r}} - I_{xz})(X_B B - x_G W)(m - Y_{\dot{v}})
\end{aligned}$$

$$- (K_{\dot{r}} - I_{xz})(N_v U)(Y_p U) + (K_{\dot{r}} - I_{xz})(N_p U)(Y_v U)$$

$$D = D_1 z_G + D_2, \quad (28)$$

where

$$\begin{aligned} D_1 &= mU(x_{BB} - x_G W)(m - Y_{\dot{v}}) - mU(N_v U)(Y_p U) \\ &+ mU(N_p U)(Y_v U) + m(x_{BB} - x_G W)(UY_r - Um) \\ &- W(m - Y_{\dot{v}})(UN_r - Um x_G) - W(Y_v U)(I_{zz} - N_{\dot{r}}) \\ &+ W(mx_G - Y_{\dot{r}})(N_v U) + W(mx_G - N_{\dot{v}})(UY_r - Um) \\ D_2 &= UK_r(x_{BB} - x_G W)(m - Y_{\dot{v}}) - UK_r(N_v U)(Y_p U) \\ &+ UK_r(N_p U)(Y_v U) - (K_{\dot{r}} - I_{xz})(x_{BB} - x_G W)(Y_v U) \\ &+ K_{\dot{v}}(x_{BB} - x_G W)(UY_r - Um) - (K_v U)(x_{BB} - x_G W)(mx_G - Y_{\dot{r}}) \\ &+ (K_v U)(UN_r - Um x_G)(Y_p U) - (K_v U)(UY_r - Um)(N_p U) \\ &- (K_p U)(Y_v U)(UN_r - Um x_G) + (K_p U)(UK_r - Um)(N_v U) \end{aligned}$$

and

$$E = E_1 z_G + E_2, \quad (29)$$

where

$$\begin{aligned} E_1 &= W(Y_v U)(UN_r - Um x_G) - W(UY_r - Um)(N_v U) \\ &- mU(x_{BB} - x_G W)(Y_v U) \\ E_2 &= (K_v U)(x_{BB} - x_G W)(UY_r - Um) - UK_r(x_{BB} - x_G W)(Y_v U) \end{aligned}$$

If we apply the stability criterion (24) utilizing expressions (25) through (29), we get a fifth order polynomial equation in the metacentric height  $z_G$  of

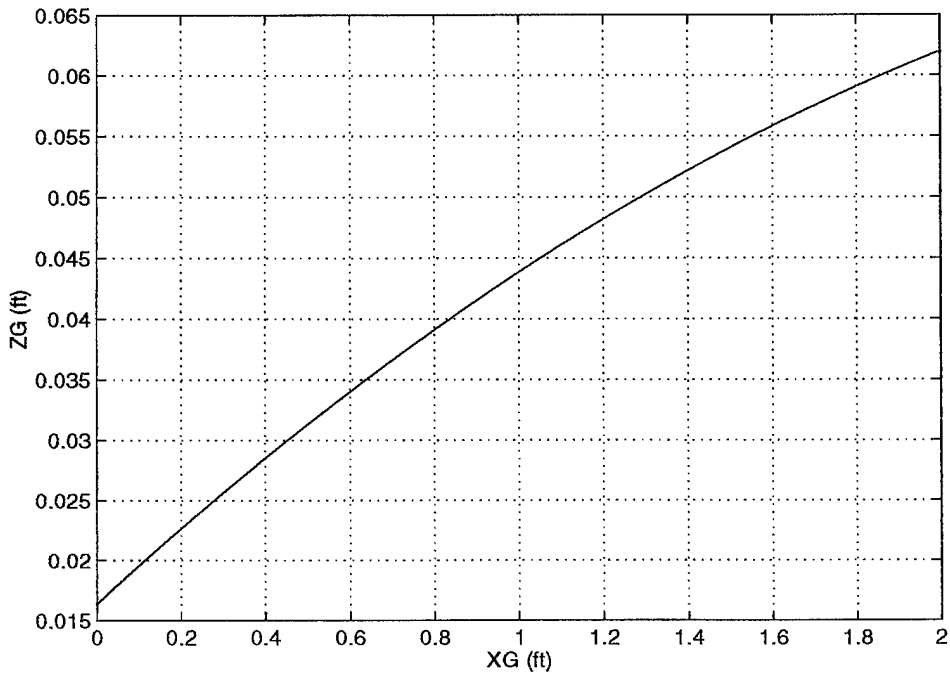


Figure 1: Critical value of  $z_G$  versus  $x_G$  for  $U = 5$  ft/sec

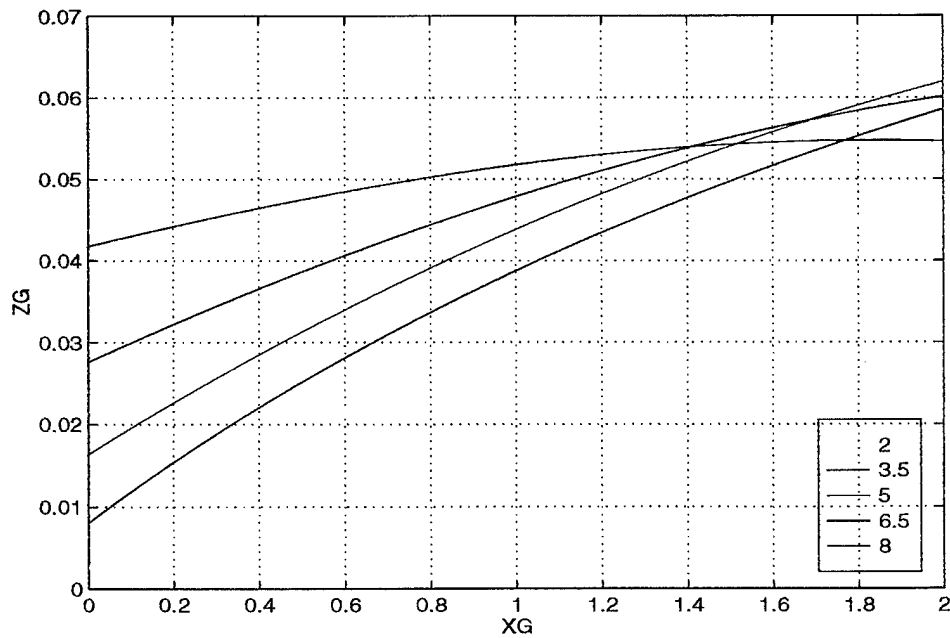


Figure 2: Critical value of  $z_G$  versus  $x_G$  for different values of  $U$  (ft/sec)

the following form,

$$F_5 z_G^5 + F_4 z_G^4 + F_3 z_G^3 + F_2 z_G^2 + F_1 z_G + F_0 = 0, \quad (30)$$

where,

$$F_0 = B_3 C_3 D_2 - A_3 D_2^2 - E_2 B_3^2,$$

$$F_1 = B_3 C_3 D_1 + (B_3 C_2 + B_2 C_3) D_2 - 2E_2 B_2 B_3 - E_1 B_3^2 -$$

$$2D_2 D_1 A_3 - D_2^2 A_2,$$

$$F_2 = -E_2 (B_2^2 + 2B_1 B_3) - 2E_1 B_2 B_3 + (B_3 C_2 + B_2 C_3) D_1 +$$

$$(B_3 C_1 + B_2 C_2 + B_1 C_3) D_2 - D_1^2 A_3 - 2D_2 D_1 A_2 - D_2^2 A_1,$$

$$F_3 = -D_1^2 A_2 - 2D_2 D_1 A_1 - 2E_2 B_1 B_2 - E_1 (B_2^2 + 2B_1 B_3) +$$

$$D_1 (B_3 C_1 + B_2 C_2 + B_1 C_3) + D_2 (B_2 C_1 + B_1 C_2),$$

$$F_4 = D_1 (B_2 C_1 + B_1 C_2) + B_1 C_1 D_2 - D_1^2 A_1 - E_2 B_1^2 - 2E_1 B_1 B_2,$$

$$F_5 = -E_1 B_1^2 + B_1 C_1 D_1.$$

Using the corresponding Matlab program shown in Appendix A, we can solve equation (30) using a typical constant forward velocity  $U = 5$  ft/sec. The value of  $x_G$  is given in the range from 0 to 2 ft. In this way we get five solutions in  $z_G$  for a given value of  $x_G$ . Investigating these solutions, we can see that only one satisfies the second criterion (23). This solution corresponds to the locus of loss of stability. For values of  $z_G$  greater than the critical value, the system is stable, and for values less than its critical value, the system is unstable, see Figure 1. The calculations are repeated for a range of forward speeds  $U$  from 2 to 8 ft/sec. The results are shown parametrically in Figure 2. We observe

an upwards movement of the curves as the speed is increased. This means that the system experiences a tendency to become less stable at higher speeds which, in turn, calls for higher metacentric heights to ensure stability.



## IV. NONLINEAR ANALYSIS

### A. INTRODUCTION

From the linearized analysis on the loss of stability that was done in the previous chapter, we can see that as a specific parameter of the system, such as  $(x_G, z_G)$ , is varied, it is possible to pass from a region of stability to a region of instability. This case of loss of stability is associated with one pair of complex conjugate eigenvalues of the system crossing the imaginary axis. This loss of stability is usually accompanied by self sustained oscillations, and it is called Hopf Bifurcation.

There are two cases of Hopf Bifurcations, supercritical and subcritical. In the supercritical case, limit cycles are created just after the loss of stability. A limit cycle is a constant amplitude oscillatory motion of our system. This amplitude is usually larger as we move away from the bifurcation point. When the limit cycles have small amplitudes the situation is not very critical for our vehicle and is not much different from stable conditions. In the subcritical case we may have convergence to limit cycles, even before our system loses its stability. Furthermore, the limit cycle amplitudes are considerably higher.

The analysis that will follow will be performed in order to verify the existence of the limit cycles and to find out which case of Hopf Bifurcations we have in our model. We will also examine the stability of the resulting limit cycles. This analysis is necessary because we can predict the behavior of a

vehicle when some of its parameters change and we have operation in the proximity of a bifurcation point. The results of the non linear analysis will be verified by a numerical simulation of the system's motion, both above and below the bifurcation point.

## B. THIRD ORDER EXPANSIONS

From the linearization procedure we see that our system is written in the form of matrix equation (14), where we have ignored the non linear terms. If we take into account the non linear terms up to third order, equation (14) becomes:

$$\mathbf{A}'\dot{\mathbf{x}} = \mathbf{B}'\mathbf{x} + \mathbf{g}(\mathbf{x}) , \quad (31)$$

where,

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{bmatrix} ,$$

Keeping terms up to third order the vector of non linear terms can be written:

$$g(x) = g^{(2)}(x) + g^{(3)}(x) + c(x) , \quad (32)$$

where  $g^{(2)}(x)$  contains the second order non linear terms,  $g^{(3)}(x)$  contains the third order non linear terms, and  $c(x)$  contains the constant terms. The cross flow integrals can be written as follows:

$$\begin{aligned} I_v &= C_{D_y} \int_{x_{\text{tail}}}^{x_{\text{nose}}} h(x)(v + xr)|v + xr| dx \\ I_r &= C_{D_y} \int_{x_{\text{tail}}}^{x_{\text{nose}}} h(x)(v + xr)|v + xr|x dx \end{aligned}$$

(33)

The second order non linear terms are:

$$\mathbf{g}^{(2)}(\mathbf{x}) = \begin{bmatrix} g_1^{(2)}(x) \\ g_2^{(2)}(x) \\ g_3^{(2)}(x) \\ g_4^{(2)}(x) \end{bmatrix},$$

where

$$g_1^{(2)} = y_G p^2 + y_G r^2 - I_v^{(2)}$$

$$g_2^{(2)} = I_{xy} p^2 + I_{yz} p r + y_G v r - I_r^{(2)}$$

$$g_3^{(2)} = -I_{xy} p r - I_{yz} r^2 - m y_G v r - (y_G W - y_B B) \phi^2$$

$$g_4^{(2)} = 0.$$

The third order non linear terms are:

$$\mathbf{g}^{(3)}(\mathbf{x}) = \begin{bmatrix} g_1^{(3)}(x) \\ g_2^{(3)}(x) \\ g_3^{(3)}(x) \\ g_4^{(3)}(x) \end{bmatrix},$$

where,

$$g_1^{(3)} = -I_v^{(3)} - \frac{1}{6}(W - B)\phi^3$$

$$g_2^{(3)} = -I_r^{(3)} - \frac{1}{6}(x_G W - x_B B)\phi^3$$

$$g_3^{(3)} = \frac{1}{6}(z_G W - z_B B)\phi^3$$

$$g_4^{(3)} = 0.$$

The constant terms are:

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} c_1(x) \\ c_2(x) \\ c_3(x) \\ c_4(x) \end{bmatrix},$$

where,

$$\begin{aligned}
c_1(x) &= Y_{\delta_r} U^2 \delta_r \\
c_2(x) &= N_{\delta_r} U^2 \delta_r + U^2 N_{prop} \\
c_3(x) &= U^2 K_{prop} + y_G W - y_B B \\
c_4(x) &= 0.
\end{aligned}$$

In order to get the second and third order non linear terms of the cross flow integrals, we must expand in Taylor series a function of the form:

$$f(\xi) = \xi|\xi| \quad (34)$$

about a nominal point  $\xi_0$ :

$$\xi|\xi| = \xi_0|\xi_0| + 2|\xi_0|(\xi - \xi_0) + \text{sign}(\xi_0)(\xi - \xi_0)^2 + f^{(3)}(\xi). \quad (35)$$

The sign function in equation (33) can be approximated by:

$$\text{sign}(\xi_0) = \lim_{\gamma \rightarrow 0} \tanh\left(\frac{\xi_0}{\gamma}\right), \quad (36)$$

where the approximation gets better as  $\gamma$  gets smaller.

If we choose  $\xi_0 = 0$  as our nominal point, equation (35) becomes:

$$\xi|\xi| = \frac{1}{6\gamma} \xi^3 \quad (37)$$

In our case  $\xi$  is  $(v + xr)$  so we have:

$$(v + xr)|v + xr| = \frac{1}{6\gamma} (v + xr)^3 \quad (38)$$

or

$$(v + xr)|v + xr| = \frac{1}{6\gamma} (v^3 + x^3 r^3 + 3v^3 xr + 3x^2 r^2 v) \quad (39)$$

Using equation (39) the cross flow integrals become:

$$I_v = \frac{C_{Dy}}{6\gamma}(E_0v^3 + 3E_1v^2r + 3E_2vr^2 + E_3r^3) \quad (40)$$

$$I_r = \frac{C_{Dy}}{6\gamma}(E_1v^3 + 3E_2v^2r + 3E_3vr^2 + E_4r^3) \quad (41)$$

where,

$$E_i = \int_{x_{\text{tail}}}^{x_{\text{nose}}} x^i h(x) dx, \quad i = 0, 1, 2, 3, 4 \quad (42)$$

By using this approximation we see that the expansion of the cross flow integrals give only third order terms, so we have:

$$I_v^{(2)} = I_r^{(2)} = 0 \quad (43)$$

Now we want to write our matrix equation in state space form, so we have to find the inverse of the system matrix:

$$(\mathbf{A}')^{-1} = \begin{bmatrix} \frac{a_{11}}{D} & \frac{a_{12}}{D} & \frac{a_{13}}{D} & 0 \\ \frac{a_{21}}{D} & \frac{a_{22}}{D} & \frac{a_{23}}{D} & 0 \\ \frac{a_{31}}{D} & \frac{a_{32}}{D} & \frac{a_{33}}{D} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where,

$$\begin{aligned} a_{11} &= (I_{zz} - N_{\dot{r}})(I_{xx} - K_{\dot{p}}) - (I_{xz} - K_{\dot{r}})(-I_{xz} - N_{\dot{p}}) \\ a_{12} &= (Y_{\dot{r}} - mx_G)(I_{xx} - K_{\dot{p}}) + (I_{xz} - K_{\dot{r}})(-mz_G - Y_{\dot{p}}) \\ a_{13} &= (mx_G - Y_{\dot{r}})(-I_{xz} - N_{\dot{p}}) - (I_{zz} - N_{\dot{r}})(-mz_G - Y_{\dot{p}}) \\ a_{21} &= -(mx_G - N_{\dot{v}})(I_{xx} - K_{\dot{p}}) + (-mz_G - K_{\dot{v}})(-I_{xz} - N_{\dot{p}}) \\ a_{22} &= (m - Y_{\dot{v}})(I_{xx} - K_{\dot{p}}) - (-mz_G - K_{\dot{v}})(-mz_G - Y_{\dot{p}}) \\ a_{23} &= -(m - Y_{\dot{v}})(-I_{xz} - N_{\dot{p}}) + (mx_G - N_{\dot{v}})(-mz_G - Y_{\dot{p}}) \\ a_{31} &= (mx_G - N_{\dot{v}})(I_{xz} - K_{\dot{r}}) - (-mz_G - K_{\dot{v}})(I_{zz} - N_{\dot{r}}) \end{aligned}$$

$$a_{32} = -(m - Y_{\dot{v}})(I_{xz} - K_{\dot{r}}) + (-mz_G - K_{\dot{v}})(mx_G - Y_{\dot{r}})$$

$$a_{33} = (m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}}) - (mx_G - N_{\dot{v}})(mx_G - Y_{\dot{r}})$$

and

$$\begin{aligned} D &= (m - Y_{\dot{v}})(I_{zz} - N_{\dot{r}})(I_{xx} - K_{\dot{p}}) \\ &\quad - (m - Y_{\dot{v}})(I_{xz} - K_{\dot{r}})(-I_{xz} - N_{\dot{p}}) \\ &\quad - (mx_G - N_{\dot{v}})(mx_G - Y_{\dot{r}})(I_{xx} - K_{\dot{p}}) \\ &\quad + (mx_G - N_{\dot{v}})(I_{xz} - K_{\dot{r}})(-mz_G - Y_{\dot{p}}) \\ &\quad + (-mz_G - K_{\dot{v}})(mx_G - Y_{\dot{r}})(-I_{xz} - N_{\dot{p}}) \\ &\quad - (-mz_G - K_{\dot{v}})(I_{zz} - N_{\dot{r}})(-mz_G - Y_{\dot{p}}) \end{aligned}$$

If we multiply equation (31) by  $(A')^{-1}$  from the left side we get:

$$(A')^{-1}A'\dot{x} = (A')^{-1}B'x + (A')^{-1}g(x) \quad (44)$$

or

$$\dot{x} = Fx + G(x) \quad (45)$$

where,

$$F = (A')^{-1}B', \quad (46)$$

$$G(x) = (A')^{-1}g(x) \quad (47)$$

Then matrix  $F$  is a  $4 \times 4$  matrix defined as follows:

$$F = \begin{bmatrix} \frac{F_{11}}{D} & \frac{F_{12}}{D} & \frac{F_{13}}{D} & \frac{F_{14}}{D} \\ \frac{F_{21}}{D} & \frac{F_{22}}{D} & \frac{F_{23}}{D} & \frac{F_{24}}{D} \\ \frac{F_{31}}{D} & \frac{F_{32}}{D} & \frac{F_{33}}{D} & \frac{F_{34}}{D} \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where,

$$\begin{aligned}
F_{11} &= a_{11}(Y_v U) + a_{12}(N_v U) + a_{13}(K_v U) \\
F_{12} &= a_{11}(Y_r - m)U + a_{12}(N_r - m x_G)U + a_{13}(K_r + m z_G)U \\
F_{13} &= a_{11}(Y_p U) + a_{12}(N_p U) + a_{13}(K_p U) \\
F_{14} &= a_{11}(W - B) + a_{12}(x_G W - x_B B) + a_{13}(-z_G W + z_B B) \\
F_{21} &= a_{21}(Y_v U) + a_{22}(N_v U) + a_{23}(K_v U) \\
F_{22} &= a_{21}(Y_r - m)U + a_{22}(N_r - m x_G)U + a_{23}(K_r + m z_G)U \\
F_{23} &= a_{21}(Y_p U) + a_{22}(N_p U) + a_{23}(K_p U) \\
F_{24} &= a_{21}(W - B) + a_{22}(x_G W - x_B B) + a_{23}(-z_G W + z_B B) \\
F_{31} &= a_{31}(Y_v U) + a_{32}(N_v U) + a_{33}(K_v U) \\
F_{32} &= a_{31}(Y_r - m)U + a_{32}(N_r - m x_G)U + a_{33}(K_r + m z_G)U \\
F_{33} &= a_{31}(Y_p U) + a_{32}(N_p U) + a_{33}(K_p U) \\
F_{34} &= a_{31}(W - B) + a_{32}(x_G W - x_B B) + a_{33}(-z_G W + z_B B)
\end{aligned}$$

From equations (45),(47) we see that:

$$G(x) = (A')^{-1}g(x) = \begin{bmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \end{bmatrix},$$

where,

$$\begin{aligned}
G_1(x) &= \frac{a_{11}}{D}g_1(x) + \frac{a_{12}}{D}g_2(x) + \frac{a_{13}}{D}g_3(x) \\
G_2(x) &= \frac{a_{21}}{D}g_1(x) + \frac{a_{22}}{D}g_2(x) + \frac{a_{23}}{D}g_3(x) \\
G_3(x) &= \frac{a_{31}}{D}g_1(x) + \frac{a_{32}}{D}g_2(x) + \frac{a_{33}}{D}g_3(x) \\
G_4(x) &= 0
\end{aligned}$$

Equation (45) can also be written in the following way:

$$\dot{x} = Fx + G^{(2)}(x) + G^{(3)}(x) + k \quad (48)$$

where,

$$G^{(2)}(x) = \begin{bmatrix} G_1^{(2)}(x) \\ G_2^{(2)}(x) \\ G_3^{(2)}(x) \\ G_4^{(2)}(x) \end{bmatrix},$$

$$G^{(3)}(x) = \begin{bmatrix} G_1^{(3)}(x) \\ G_2^{(3)}(x) \\ G_3^{(3)}(x) \\ G_4^{(3)}(x) \end{bmatrix},$$

and

$$k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix},$$

Each element of the above non linear terms vectors can be written as follows:

$$G_1^{(2)}(x) = \frac{a_{11}}{D}g_1^{(2)}(x) + \frac{a_{12}}{D}g_2^{(2)}(x) + \frac{a_{13}}{D}g_3^{(2)}(x)$$

$$G_2^{(2)}(x) = \frac{a_{21}}{D}g_1^{(2)}(x) + \frac{a_{22}}{D}g_2^{(2)}(x) + \frac{a_{23}}{D}g_3^{(2)}(x)$$

$$G_3^{(2)}(x) = \frac{a_{31}}{D}g_1^{(2)}(x) + \frac{a_{32}}{D}g_2^{(2)}(x) + \frac{a_{33}}{D}g_3^{(2)}(x)$$

$$G_4^{(2)}(x) = 0$$

and

$$G_1^{(3)}(x) = \frac{a_{11}}{D}g_1^{(3)}(x) + \frac{a_{12}}{D}g_2^{(3)}(x) + \frac{a_{13}}{D}g_3^{(3)}(x)$$

$$G_2^{(3)}(x) = \frac{a_{21}}{D}g_1^{(3)}(x) + \frac{a_{22}}{D}g_2^{(3)}(x) + \frac{a_{23}}{D}g_3^{(3)}(x)$$

$$G_3^{(3)}(x) = \frac{a_{31}}{D}g_1^{(3)}(x) + \frac{a_{32}}{D}g_2^{(3)}(x) + \frac{a_{33}}{D}g_3^{(3)}(x)$$

$$G_4^{(3)}(x) = 0$$

Finally, the constant terms are:

$$\begin{aligned}
 k_1 &= \frac{a_{11}}{D}c_1 + \frac{a_{12}}{D}c_2 + \frac{a_{13}}{D}c_3 \\
 k_2 &= \frac{a_{21}}{D}c_1 + \frac{a_{22}}{D}c_2 + \frac{a_{23}}{D}c_3 \\
 k_3 &= \frac{a_{31}}{D}c_1 + \frac{a_{32}}{D}c_2 + \frac{a_{33}}{D}c_3 \\
 k_4 &= 0.
 \end{aligned}$$

### C. COORDINATE TRANSFORMATIONS

From equations (45) and (48) it is obvious that the stability of our system depends on the eigenvalues of matrix  $F$ . Since we want to investigate the behavior of our system around the Hopf Bifurcation point, it is useful to bring our system in its normal coordinate form. This can be done by applying a transformation of the coordinate system, using as transformation matrix  $T$ , the modal matrix of eigenvectors of  $F$ , evaluated at a critical point. This critical point will be a pair of  $z_G$  and  $x_G$  values, that belong to the critical line of loss of stability. The applied transformation will be as follows:

$$x = Tz \quad (49)$$

or,

$$z = T^{-1}x \quad (50)$$

Then equation (48) can be written as follows:

$$T\dot{z} = FTz + G^{(2)}(Tz) + G^{(3)}(Tz) + k \quad (51)$$

or,

$$\dot{z} = T^{-1}FTz + T^{-1}G^{(2)}(Tz) + T^{-1}G^{(3)}(Tz) + T^{-1}k \quad (52)$$

In this new coordinate system, the system's matrix is  $T^{-1}FT$ , and at the Hopf Bifurcation point can be written as follows:

$$\mathbf{T}^{-1}\mathbf{F}\mathbf{T} = \begin{bmatrix} 0 & -\omega_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix},$$

Where  $\omega_0$  is the imaginary part of the critical pair of eigenvalues, and the remaining eigenvalues  $P_1$ ,  $P_2$  are negative. For values close to the Hopf Bifurcation line, we can write the system matrix as follows:

$$\mathbf{T}^{-1}\mathbf{F}\mathbf{T} = \begin{bmatrix} a'\epsilon & -(\omega_0 + \omega'\epsilon) & 0 & 0 \\ (\omega_0 + \omega'\epsilon) & a'\epsilon & 0 & 0 \\ 0 & 0 & (P_1 + P_1'\epsilon) & 0 \\ 0 & 0 & 0 & (P_2 + P_2'\epsilon) \end{bmatrix},$$

where:  $\epsilon =$  Difference from the critical point ( $z_G - z_{G_c}$ ).

$a' =$  Derivative of the real part of the critical eigenvalue with respect to  $\epsilon$ .

$\omega' =$  Derivative of the imaginary part of the critical eigenvalue with respect to  $\epsilon$ .

$P_1' =$  Derivative of  $P_1$  with respect to  $\epsilon$ .

$P_2' =$  Derivative of  $P_2$  with respect to  $\epsilon$ .

#### D. CENTER MANIFOLD EXPANSIONS

By using the coordinate transformation described in the previous chapter,

and from equation (50), we see that we have a new variables vector, which is:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

The first two coordinates  $z_1$ ,  $z_2$ , are the critical coordinates and correspond to a pair of complex conjugate eigenvalues. The remaining two coordinates  $z_3$ ,  $z_4$ , are the stable coordinates and they always correspond to eigenvalues that are negative. The center manifold theory predicts that the relationship between the critical coordinates  $z_1$ ,  $z_2$ , and the stable coordinates  $z_3$ ,  $z_4$ , is at least of quadratic order. After this assumption, the two stable coordinates can be written as follows:

$$z_3 = h_{11}z_1^2 + h_{12}z_1z_2 + h_{22}z_2^2 \quad (53)$$

$$z_4 = s_{11}z_1^2 + s_{12}z_1z_2 + s_{22}z_2^2 \quad (54)$$

The coefficients  $h_{ij}$ ,  $s_{ij}$ , of equations (53), (54), need to be determined. By differentiating equations (53), (54), we get the following:

$$\dot{z}_3 = 2h_{11}z_1\dot{z}_1 + h_{12}(z_1\dot{z}_2 + \dot{z}_1z_2) + 2h_{22}z_2\dot{z}_2 \quad (55)$$

$$\dot{z}_4 = 2s_{11}z_1\dot{z}_1 + s_{12}(z_1\dot{z}_2 + \dot{z}_1z_2) + 2s_{22}z_2\dot{z}_2 \quad (56)$$

From equation (52) and if we ignore the higher order terms, we take:

$$\dot{z}_1 = -\omega_0 z_2 \quad (57)$$

$$\dot{z}_2 = \omega_0 z_1 \quad (58)$$

If we substitute equations (57), (58), into equations (55), (56), we get the

following:

$$\dot{z}_3 = h_{12}\omega_0 z_1^2 + 2(h_{22} - h_{11})\omega_0 z_1 z_2 - h_{12}\omega_0 z_2^2 \quad (59)$$

$$\dot{z}_4 = s_{12}\omega_0 z_1^2 + 2(s_{22} - s_{11})\omega_0 z_1 z_2 - s_{12}\omega_0 z_2^2 \quad (60)$$

The third and the fourth equations of matrix equation (52) can be written in the following way:

$$\dot{z}_3 = P_1 z_3 + [T^{-1}G^{(2)}(Tz)]_{(3,3)} + T^{-1}k_{(3,3)} \quad (61)$$

$$\dot{z}_4 = P_2 z_4 + [T^{-1}G^{(2)}(Tz)]_{(4,4)} + T^{-1}k_{(4,4)} \quad (62)$$

In equations (61), (62), we kept terms up to second order.

The transformation matrix  $T$  and it's inverse  $T^{-1}$  are 4x4 matrices, and their elements can be denoted by:

$$T = [m_{ij}], \quad T^{-1} = [n_{ij}], \quad i, j = 1, 2, 3, 4 \quad (63)$$

From equation (52) we have:

$$T^{-1}G^{(2)}(Tz) = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

where,

$$d_1 = n_{11}G_1^{(2)}(Tz) + n_{12}G_2^{(2)}(Tz) + n_{13}G_3^{(2)}(Tz) \quad (64)$$

$$d_2 = n_{21}G_1^{(2)}(Tz) + n_{22}G_2^{(2)}(Tz) + n_{23}G_3^{(2)}(Tz) \quad (65)$$

$$d_3 = n_{31}G_1^{(2)}(Tz) + n_{32}G_2^{(2)}(Tz) + n_{33}G_3^{(2)}(Tz) \quad (66)$$

$$d_4 = n_{41}G_1^{(2)}(Tz) + n_{42}G_2^{(2)}(Tz) + n_{43}G_3^{(2)}(Tz) \quad (67)$$

Also from coordinate transformation, the relationship between the old and new coordinates is as follows:

$$v = m_{11}z_1 + m_{12}z_2 + m_{13}z_3 + m_{14}z_4 \quad (68)$$

$$r = m_{21}z_1 + m_{22}z_2 + m_{23}z_3 + m_{24}z_4 \quad (69)$$

$$p = m_{31}z_1 + m_{32}z_2 + m_{33}z_3 + m_{34}z_4 \quad (70)$$

$$\phi = m_{41}z_1 + m_{42}z_2 + m_{43}z_3 + m_{44}z_4 \quad (71)$$

Finally if we substitute equations (68), (69), (70), (71), and the expressions for  $G_1, G_2, G_3, G_4$  into equations (64), (65), (66), (67), we get the final expressions for the coefficients  $d_i$ :

$$\begin{aligned} d_1 = & n_{11}(l_{15}z_1^2 + l_{16}z_1z_2 + l_{17}z_2^2) \\ & + n_{12}(l_{25}z_1^2 + l_{26}z_1z_2 + l_{27}z_2^2) \\ & + n_{13}(l_{35}z_1^2 + l_{36}z_1z_2 + l_{37}z_2^2) \end{aligned} \quad (72)$$

$$\begin{aligned} d_2 = & n_{21}(l_{15}z_1^2 + l_{16}z_1z_2 + l_{17}z_2^2) \\ & + n_{22}(l_{25}z_1^2 + l_{26}z_1z_2 + l_{27}z_2^2) \\ & + n_{23}(l_{35}z_1^2 + l_{36}z_1z_2 + l_{37}z_2^2) \end{aligned} \quad (73)$$

$$\begin{aligned} d_3 = & n_{31}(l_{15}z_1^2 + l_{16}z_1z_2 + l_{17}z_2^2) \\ & + n_{32}(l_{25}z_1^2 + l_{26}z_1z_2 + l_{27}z_2^2) \\ & + n_{33}(l_{35}z_1^2 + l_{36}z_1z_2 + l_{37}z_2^2) \end{aligned} \quad (74)$$

$$\begin{aligned} d_4 = & n_{41}(l_{15}z_1^2 + l_{16}z_1z_2 + l_{17}z_2^2) \\ & + n_{42}(l_{25}z_1^2 + l_{26}z_1z_2 + l_{27}z_2^2) \\ & + n_{43}(l_{35}z_1^2 + l_{36}z_1z_2 + l_{37}z_2^2) \end{aligned} \quad (75)$$

Coefficients  $l_{ij}, i = 1, 2, 3, j = 5, 6, 7$  in equations (72), (73), (74), (75), are as follows:

$$\begin{aligned}
l_{1,5} &= \frac{a_{11}}{D} y_G (m_{31}^2 + m_{21}^2) \\
&+ \frac{a_{12}}{D} (I_{xy} m_{31}^2 + I_{yz} m_{31} m_{21} + y_G m_{21} m_{11}) \\
&- \frac{a_{13}}{D} (I_{xy} m_{31} m_{21} + I_{yz} m_{21}^2 + m y_G m_{11} m_{31} \\
&+ y_G W m_{41}^2 - y_B B m_{41}^2) \tag{76}
\end{aligned}$$

$$\begin{aligned}
l_{1,6} &= \frac{a_{11}}{D} y_G (2m_{31} m_{32} + 2m_{21} + m_{22}) \\
&+ \frac{a_{12}}{D} [2I_{xy} m_{31} m_{32} + I_{yz} (m_{31} m_{22} + m_{32} m_{21}) \\
&+ y_G (m_{21} m_{12} + m_{22} m_{11})] \\
&- \frac{a_{13}}{D} [I_{xy} (m_{31} m_{22} + m_{32} m_{21}) + 2I_{yz} m_{21} m_{22} \\
&+ m y_G (m_{11} m_{32} + m_{12} m_{31}) + 2(y_G W - y_B B) m_{41} m_{42}] \tag{77}
\end{aligned}$$

$$\begin{aligned}
l_{1,7} &= \frac{a_{11}}{D} y_G (m_{32}^2 + m_{22}^2) \\
&+ \frac{a_{12}}{D} (I_{xy} m_{32}^2 + I_{yz} m_{32} m_{22} + y_G m_{22} m_{12}) \\
&- \frac{a_{13}}{D} [I_{xy} m_{32} m_{22} + I_{yz} m_{22}^2 + m y_G m_{12} m_{32} + (y_G W - y_B B) m_{42}^2] \tag{78}
\end{aligned}$$

$$\begin{aligned}
l_{2,5} &= \frac{a_{21}}{D} y_G (m_{31}^2 + m_{21}^2) \\
&+ \frac{a_{22}}{D} (I_{xy} m_{31}^2 + I_{yz} m_{31} m_{21} + y_G m_{21} m_{11}) \\
&- \frac{a_{23}}{D} (I_{xy} m_{31} m_{21} + I_{yz} m_{21}^2 + m y_G m_{11} m_{31} \\
&+ y_G W m_{41}^2 - y_B B m_{41}^2) \tag{79}
\end{aligned}$$

$$\begin{aligned}
l_{2,6} &= \frac{a_{21}}{D} y_G (2m_{31} m_{32} + 2m_{21} + m_{22}) \\
&+ \frac{a_{22}}{D} [2I_{xy} m_{31} m_{32} + I_{yz} (m_{31} m_{22} + m_{32} m_{21}) \\
&+ y_G (m_{21} m_{12} + m_{22} m_{11})]
\end{aligned}$$

$$\begin{aligned}
& - \frac{a_{23}}{D} [I_{xy}(m_{31}m_{22} + m_{32}m_{21}) + 2I_{yz}m_{21}m_{22}] \\
& + my_G(m_{11}m_{32} + m_{12}m_{31}) + 2(y_GW - y_BB)m_{41}m_{42}] \quad (80)
\end{aligned}$$

$$\begin{aligned}
l_{2,7} &= \frac{a_{21}}{D} y_G(m_{32}^2 + m_{22}^2) \\
& + \frac{a_{22}}{D} (I_{xy}m_{32}^2 + I_{yz}m_{32}m_{22} + y_Gm_{22}m_{12}) \\
& - \frac{a_{23}}{D} [I_{xy}m_{32}m_{22} + I_{yz}m_{22}^2 + my_Gm_{12}m_{32} + (y_GW - y_BB)m_{42}^2] \quad (81)
\end{aligned}$$

$$\begin{aligned}
l_{3,5} &= \frac{a_{31}}{D} y_G(m_{31}^2 + m_{21}^2) \\
& + \frac{a_{32}}{D} (I_{xy}m_{31}^2 + I_{yz}m_{31}m_{21} + y_Gm_{21}m_{11}) \\
& - \frac{a_{33}}{D} (I_{xy}m_{31}m_{21} + I_{yz}m_{21}^2 + my_Gm_{11}m_{31} \\
& + y_GWm_{41}^2 - y_BBm_{41}^2) \quad (82)
\end{aligned}$$

$$\begin{aligned}
l_{3,6} &= \frac{a_{31}}{D} y_G(2m_{31}m_{32} + 2m_{21} + m_{22}) \\
& + \frac{a_{32}}{D} [2I_{xy}m_{31}m_{32} + I_{yz}(m_{31}m_{22} + m_{32}m_{21}) \\
& + y_G(m_{21}m_{12} + m_{22}m_{11})] \\
& - \frac{a_{33}}{D} [I_{xy}(m_{31}m_{22} + m_{32}m_{21}) + 2I_{yz}m_{21}m_{22} \\
& + my_G(m_{11}m_{32} + m_{12}m_{31}) + 2(y_GW - y_BB)m_{41}m_{42}] \quad (83)
\end{aligned}$$

$$\begin{aligned}
l_{3,7} &= \frac{a_{31}}{D} y_G(m_{32}^2 + m_{22}^2) \\
& + \frac{a_{32}}{D} (I_{xy}m_{32}^2 + I_{yz}m_{32}m_{22} + y_Gm_{22}m_{12}) \\
& - \frac{a_{33}}{D} [I_{xy}m_{32}m_{22} + I_{yz}m_{22}^2 + my_Gm_{12}m_{32} + (y_GW - y_BB)m_{42}^2] \quad (84)
\end{aligned}$$

Then equations (61), (62), can be written as follows:

$$\dot{z}_3 = P_1 z_3 + d_3 + e_3 \quad (85)$$

$$\dot{z}_4 = P_2 z_4 + d_4 + e_4 \quad (86)$$

where  $e_3, e_4$  come from the constant terms as:

$$T^{-1}k = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

and

$$e_1 = k_1n_{11} + k_2n_{12} + k_3n_{13}$$

$$e_2 = k_1n_{21} + k_2n_{22} + k_3n_{23}$$

$$e_3 = k_1n_{31} + k_2n_{32} + k_3n_{33}$$

$$e_4 = k_1n_{41} + k_2n_{42} + k_3n_{43}$$

Using equations (53), (54), we can write equations (85), (86), as follows:

$$\dot{z}_3 = P_1(h_{11}z_1^2 + h_{12}z_1z_2 + h_{22}z_2^2) + d_3 + e_3 \quad (87)$$

$$\dot{z}_4 = P_2(s_{11}z_1^2 + s_{12}z_1z_2 + s_{22}z_2^2) + d_4 + e_4 \quad (88)$$

And using equations (72), (73), (74), (75), they become:

$$\begin{aligned} \dot{z}_3 &= (P_1h_{11} + n_{31}l_{15} + n_{32}l_{25} + n_{33}l_{35})z_1^2 \\ &+ (P_1h_{12} + n_{31}l_{16} + n_{32}l_{26} + n_{33}l_{36})z_1z_2 \\ &+ (P_1h_{22} + n_{31}l_{17} + n_{32}l_{27} + n_{33}l_{37})z_2^2 + e_3 \end{aligned} \quad (89)$$

$$\begin{aligned} \dot{z}_4 &= (P_2s_{11} + n_{31}l_{15} + n_{32}l_{25} + n_{33}l_{35})z_1^2 \\ &+ (P_2s_{12} + n_{31}l_{16} + n_{32}l_{26} + n_{33}l_{36})z_1z_2 \\ &+ (P_2s_{22} + n_{31}l_{17} + n_{32}l_{27} + n_{33}l_{37})z_2^2 + e_4 \end{aligned} \quad (90)$$

Comparing the coefficients of equations (89), (90), with the coefficients of equations (59), (60), we get:

$$-P_1h_{11} + \omega_0h_{12} = n_{31}l_{15} + n_{32}l_{25} + n_{33}l_{35}$$

$$-2\omega_0 h_{11} - P_1 h_{12} + 2\omega_0 h_{22} = n_{31} l_{16} + n_{32} l_{26} + n_{33} l_{36}$$

$$-\omega_0 h_{12} - P_1 h_{22} = n_{31} l_{17} + n_{32} l_{27} + n_{33} l_{37}$$

Solution of the above 3x3 linear system of equations, gives us coefficients  $h_{11}$ ,  $h_{12}$ ,  $h_{22}$ .

Also in the same way:

$$-P_2 s_{11} + \omega_0 s_{12} = n_{41} l_{15} + n_{42} l_{25} + n_{43} l_{35}$$

$$-2\omega_0 s_{11} - P_2 s_{12} + 2\omega_0 s_{22} = n_{41} l_{16} + n_{42} l_{26} + n_{43} l_{36}$$

$$-\omega_0 s_{12} - P_2 s_{22} = n_{41} l_{17} + n_{42} l_{27} + n_{43} l_{37}$$

Solution of the above 3x3 linear system of equations, gives us coefficients  $s_{11}$ ,  $s_{12}$ ,  $s_{22}$ .

## E. AVERAGING

In this part of our analysis we are going to take into account the third order terms of equation (52):

$$T^{-1}G^{(3)}(Tz) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Where,

$$f_1 = n_{11}G_1^{(3)}(Tz) + n_{12}G_2^{(3)}(Tz) + n_{13}G_3^{(3)}(Tz) \quad (91)$$

$$f_2 = n_{21}G_1^{(3)}(Tz) + n_{22}G_2^{(3)}(Tz) + n_{23}G_3^{(3)}(Tz) \quad (92)$$

$$f_3 = n_{31}G_1^{(3)}(Tz) + n_{32}G_2^{(3)}(Tz) + n_{33}G_3^{(3)}(Tz) \quad (93)$$

$$f_4 = n_{41}G_1^{(3)}(Tz) + n_{42}G_2^{(3)}(Tz) + n_{43}G_3^{(3)}(Tz) \quad (94)$$

If we substitute equations (68), (69), (70), (71) and the expressions for  $G_1$ ,  $G_2$ ,  $G_3$  into equations (91) through (94), we get:

$$\begin{aligned} f_1 &= n_{11}(l_{11}z_1^3 + l_{12}z_1^2z_2 + l_{13}z_1z_2^2 + l_{14}z_2^3) \\ &+ n_{12}(l_{21}z_1^3 + l_{22}z_1^2z_2 + l_{23}z_1z_2^2 + l_{24}z_2^3) \\ &+ n_{13}(l_{31}z_1^3 + l_{32}z_1^2z_2 + l_{33}z_1z_2^2 + l_{34}z_2^3) \end{aligned} \quad (95)$$

$$\begin{aligned} f_2 &= n_{21}(l_{11}z_1^3 + l_{12}z_1^2z_2 + l_{13}z_1z_2^2 + l_{14}z_2^3) \\ &+ n_{22}(l_{21}z_1^3 + l_{22}z_1^2z_2 + l_{23}z_1z_2^2 + l_{24}z_2^3) \\ &+ n_{23}(l_{31}z_1^3 + l_{32}z_1^2z_2 + l_{33}z_1z_2^2 + l_{34}z_2^3) \end{aligned} \quad (96)$$

$$\begin{aligned} f_3 &= n_{31}(l_{11}z_1^3 + l_{12}z_1^2z_2 + l_{13}z_1z_2^2 + l_{14}z_2^3) \\ &+ n_{32}(l_{21}z_1^3 + l_{22}z_1^2z_2 + l_{23}z_1z_2^2 + l_{24}z_2^3) \\ &+ n_{33}(l_{31}z_1^3 + l_{32}z_1^2z_2 + l_{33}z_1z_2^2 + l_{34}z_2^3) \end{aligned} \quad (97)$$

$$\begin{aligned} f_4 &= n_{41}(l_{11}z_1^3 + l_{12}z_1^2z_2 + l_{13}z_1z_2^2 + l_{14}z_2^3) \\ &+ n_{42}(l_{21}z_1^3 + l_{22}z_1^2z_2 + l_{23}z_1z_2^2 + l_{24}z_2^3) \\ &+ n_{43}(l_{31}z_1^3 + l_{32}z_1^2z_2 + l_{33}z_1z_2^2 + l_{34}z_2^3) \end{aligned} \quad (98)$$

From equation (52) and from the system matrix, the derivatives of the first two modified coordinates,  $\dot{z}_1$  and  $\dot{z}_2$  can be written as follows:

$$\dot{z}_1 = (a'\epsilon)z_1 - (\omega_0 + \omega'\epsilon)z_2 + FF_1(z_1, z_2) \quad (99)$$

$$\dot{z}_2 = (\omega_0 + \omega'\epsilon)z_1 + a'\epsilon z_2 + FF_2(z_1, z_2) \quad (100)$$

where,

$$FF_1(z_1, z_2) = d_1 + f_1 + e_1 \quad (101)$$

$$FF_2(z_1, z_2) = d_2 + f_2 + e_2 \quad (102)$$

If we combine equations (101) and (102) with equations (72) through (75), and (95) through (98), we get:

$$\begin{aligned} FF_1(z_1, z_2) &= r_{11}z_1^3 + r_{12}z_1^2z_2 + r_{13}z_1z_2^2 + r_{14}z_2^3 \\ &+ p_{11}z_1^2 + p_{12}z_1z_2 + p_{13}z_2^2 + e_1 \end{aligned} \quad (103)$$

$$\begin{aligned} FF_2(z_1, z_2) &= r_{21}z_1^3 + r_{22}z_1^2z_2 + r_{23}z_1z_2^2 + r_{24}z_2^3 \\ &+ p_{21}z_1^2 + p_{22}z_1z_2 + p_{23}z_2^2 + e_1 \end{aligned} \quad (104)$$

where coefficients  $r_{ij}$  and  $p_{ij}$  are:

$$r_{11} = n_{11}l_{11} + n_{12}l_{21} + n_{13}l_{31}$$

$$r_{12} = n_{11}l_{12} + n_{12}l_{22} + n_{13}l_{32}$$

$$r_{13} = n_{11}l_{13} + n_{12}l_{23} + n_{13}l_{33}$$

$$r_{14} = n_{11}l_{14} + n_{12}l_{24} + n_{13}l_{34}$$

$$r_{21} = n_{21}l_{11} + n_{22}l_{21} + n_{23}l_{31}$$

$$r_{22} = n_{21}l_{12} + n_{22}l_{22} + n_{23}l_{32}$$

$$r_{23} = n_{21}l_{13} + n_{22}l_{23} + n_{23}l_{33}$$

$$r_{24} = n_{21}l_{14} + n_{22}l_{24} + n_{23}l_{34}$$

or generally,

$$r_{ij} = n_{i1}l_{1j} + n_{i2}l_{2j} + n_{i3}l_{3j} \quad i = 1, 2 \quad j = 1, 2, 3, 4 \quad (105)$$

also,

$$p_{11} = n_{11}l_{15} + n_{12}l_{25} + n_{13}l_{35}$$

$$p_{12} = n_{11}l_{16} + n_{12}l_{26} + n_{13}l_{36}$$

$$p_{13} = n_{11}l_{17} + n_{12}l_{27} + n_{13}l_{37}$$

$$p_{21} = n_{21}l_{15} + n_{22}l_{25} + n_{23}l_{35}$$

$$p_{22} = n_{21}l_{16} + n_{22}l_{26} + n_{23}l_{36}$$

$$p_{23} = n_{21}l_{17} + n_{22}l_{27} + n_{23}l_{37}$$

or generally,

$$p_{ij} = n_{i1}l_{1k} + n_{i2}l_{2k} + n_{i3}l_{3k} \quad i = 1, 2 \quad j = 1, 2, 3 \quad k = j + 4 \quad (106)$$

The coefficients  $l_{ij}$   $i = 1, 2, 3$   $j = 1, 2, 3, 4$  are as follows:

$$\begin{aligned} l_{11} = & -\frac{a_{11}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_0 m_{11}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{11} m_{21}^2 + E_3 m_{21}^3) \right. \\ & + \frac{1}{6} (W - B) m_{41}^3 \\ & - \frac{a_{12}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_1 m_{11}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{11} m_{21}^2 + E_4 m_{21}^3) \right. \\ & + \frac{1}{6} (x_G W - x_B B) m_{41}^3 \left. \right] + \frac{a_{13}}{6D} (z_G W - z_B B) m_{41}^3 \end{aligned} \quad (107)$$

$$\begin{aligned} l_{12} = & -\frac{a_{11}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11}^2 m_{12} + 3E_1 (m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21}) \right. \\ & + 3E_2 (m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_3 m_{21}^2 m_{22}) + \frac{1}{6} (W - B) 3m_{41}^2 m_{42} \left. \right] \\ & - \frac{a_{12}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11}^2 m_{12} + 3E_2 ((m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21}) \right. \\ & + 3E_3 ((m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_4 m_{21}^2 m_{22}) \\ & + \frac{1}{6} (x_G W - x_B B) 3m_{41}^2 m_{42} \left. \right] + \frac{a_{13}}{6D} (z_G W - z_B B) 3m_{41}^2 m_{42} \end{aligned} \quad (108)$$

$$\begin{aligned} l_{13} = & -\frac{a_{11}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11} m_{12}^2 + 3E_1 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22}) \right. \\ & + 3E_2 (m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_3 m_{21} m_{22}^2 \\ & + \frac{1}{6} (W - B) 3m_{41} m_{42}^2 \end{aligned}$$

$$\begin{aligned}
& - \frac{a_{12}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11} m_{12}^2 + 3E_2 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22})) \right. \\
& + 3E_3 (m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_4 m_{21} m_{22}^2) \\
& + \frac{1}{6} (x_G W - x_B B) 3m_{41} m_{42}^2] + \frac{a_{13}}{6D} (z_G W - z_B B) 3m_{41} m_{42}^2 \quad (109)
\end{aligned}$$

$$\begin{aligned}
l_{14} & = - \frac{a_{11}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_0 m_{12}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{12} m_{22}^2 + E_3 m_{22}^3) \right. \\
& + \frac{1}{6} (W - B) m_{42}^3] \\
& - \frac{a_{12}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_1 m_{12}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{12} m_{22}^2 + E_4 m_{22}^3) \right. \\
& + \frac{1}{6} (x_G W - x_B B) m_{42}^3] + \frac{a_{13}}{6D} (z_G W - z_B B) m_{42}^3 \quad (110)
\end{aligned}$$

$$\begin{aligned}
l_{21} & = - \frac{a_{21}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_0 m_{11}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{11} m_{21}^2 + E_3 m_{21}^3) \right. \\
& + \frac{1}{6} (W - B) m_{41}^3] \\
& - \frac{a_{22}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_1 m_{11}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{11} m_{21}^2 + E_4 m_{21}^3) \right. \\
& + \frac{1}{6} (x_G W - x_B B) m_{41}^3] + \frac{a_{23}}{6D} (z_G W - z_B B) m_{41}^3 \quad (111)
\end{aligned}$$

$$\begin{aligned}
l_{22} & = - \frac{a_{21}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11}^2 m_{12} + 3E_1 (m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21})) \right. \\
& + 3E_2 (m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_3 m_{21}^2 m_{22}) + \frac{1}{6} (W - B) 3m_{41}^2 m_{42}] \\
& - \frac{a_{22}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11}^2 m_{12} + 3E_2 ((m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21})) \right. \\
& + 3E_3 ((m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_4 m_{21}^2 m_{22}) \\
& + \frac{1}{6} (x_G W - x_B B) 3m_{41}^2 m_{42}] + \frac{a_{23}}{6D} (z_G W - z_B B) 3m_{41}^2 m_{42} \quad (112)
\end{aligned}$$

$$\begin{aligned}
l_{23} & = - \frac{a_{21}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11} m_{12}^2 + 3E_1 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22})) \right. \\
& + 3E_2 (m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_3 m_{21} m_{22}^2) \\
& + \frac{1}{6} (W - B) 3m_{41} m_{42}^2] \\
& - \frac{a_{22}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11} m_{12}^2 + 3E_2 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22})) \right.
\end{aligned}$$

$$\begin{aligned}
& + 3E_3(m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_4 m_{21} m_{22}^2) \\
& + \frac{1}{6}(x_G W - x_B B) 3m_{41} m_{42}^2] + \frac{a_{23}}{6D}(z_G W - z_B B) 3m_{41} m_{42}^2 \quad (113)
\end{aligned}$$

$$\begin{aligned}
l_{24} & = -\frac{a_{21}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_0 m_{12}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{12} m_{22}^2 + E_3 m_{22}^3) \right. \\
& + \frac{1}{6}(W - B) m_{42}^3] \\
& - \frac{a_{22}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_1 m_{12}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{12} m_{22}^2 + E_4 m_{22}^3) \right. \\
& + \frac{1}{6}(x_G W - x_B B) m_{42}^3] + \frac{a_{23}}{6D} (z_G W - z_B B) m_{42}^3 \quad (114)
\end{aligned}$$

$$\begin{aligned}
l_{31} & = -\frac{a_{31}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_0 m_{11}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{11} m_{21}^2 + E_3 m_{21}^3) \right. \\
& + \frac{1}{6}(W - B) m_{41}^3] \\
& - \frac{a_{32}}{D} \left[ \frac{C_{Dy}}{6\gamma} (E_1 m_{11}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{11} m_{21}^2 + E_4 m_{21}^3) \right. \\
& + \frac{1}{6}(x_G W - x_B B) m_{41}^3] + \frac{a_{33}}{6D} (z_G W - z_B B) m_{41}^3 \quad (115)
\end{aligned}$$

$$\begin{aligned}
l_{32} & = -\frac{a_{31}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11}^2 m_{12} + 3E_1 (m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21}) \right. \\
& + 3E_2 (m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_3 m_{21}^2 m_{22}) + \frac{1}{6}(W - B) 3m_{41}^2 m_{42}] \\
& - \frac{a_{32}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11}^2 m_{12} + 3E_2 ((m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21}) \right. \\
& + 3E_3 ((m_{12} m_{21}^2 + 2m_{21} m_{22} m_{11}) + 3E_4 m_{21}^2 m_{22}) \\
& + \frac{1}{6}(x_G W - x_B B) 3m_{41}^2 m_{42}] + \frac{a_{33}}{6D} (z_G W - z_B B) 3m_{41}^2 m_{42} \quad (116)
\end{aligned}$$

$$\begin{aligned}
l_{33} & = -\frac{a_{31}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_0 m_{11} m_{12}^2 + 3E_1 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22}) \right. \\
& + 3E_2 (m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_3 m_{21} m_{22}^2) \\
& + \frac{1}{6}(W - B) 3m_{41} m_{42}^2] \\
& - \frac{a_{32}}{D} \left[ \frac{C_{Dy}}{6\gamma} (3E_1 m_{11} m_{12}^2 + 3E_2 (m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22}) \right. \\
& + 3E_3 (m_{22}^2 m_{11} + 2m_{21} m_{22} m_{12}) + 3E_4 m_{21} m_{22}^2)
\end{aligned}$$

$$+ \frac{1}{6}(x_G W - x_B B)3m_{41}m_{42}^2] + \frac{a_{33}}{6D}(z_G W - z_B B)3m_{41}m_{42}^2 \quad (117)$$

$$\begin{aligned} l_{34} = & -\frac{a_{31}}{D}\left[\frac{C_{D_y}}{6\gamma}(E_0 m_{12}^3 + 3E_1 m_{11}^2 m_{21} + 3E_2 m_{12} m_{22}^2 + E_3 m_{22}^3)\right. \\ & + \frac{1}{6}(W - B)m_{42}^3] \\ & - \frac{a_{32}}{D}\left[\frac{C_{D_y}}{6\gamma}(E_1 m_{12}^3 + 3E_2 m_{11}^2 m_{21} + 3E_3 m_{12} m_{22}^2 + E_4 m_{22}^3)\right. \\ & + \left.\frac{1}{6}(x_G W - x_B B)m_{42}^3\right] + \frac{a_{33}}{6D}(z_G W - z_B B)m_{42}^3 \end{aligned} \quad (118)$$

The next step is to introduce polar coordinates in the form:

$$z_1 = R \cos \theta \quad (119)$$

$$z_2 = R \sin \theta \quad (120)$$

We use polar coordinates, because it is easier in this way to investigate the existence of limit cycles.

Substituting equations (119), (120), into equations (99), (100), we get:

$$\begin{aligned} \dot{R} \cos \theta - R(\sin \theta)\dot{\theta} &= (\omega_0 + \omega'\epsilon)R \cos \theta + a'\epsilon R \sin \theta + \\ & P_1(\theta)R^3 + Q_1(\theta)R^2 + e_1 \end{aligned} \quad (121)$$

$$\begin{aligned} \dot{R} \sin \theta + R(\cos \theta)\dot{\theta} &= (\omega_0 + \omega'\epsilon)R \cos \theta + a'\epsilon R \sin \theta + \\ & P_2(\theta)R^3 + Q_2(\theta)R^2 + e_2 \end{aligned} \quad (122)$$

where,

$$P_1(\theta) = r_{11} \cos^3 \theta + r_{12} \cos^2 \theta \sin \theta + r_{13} \cos \theta \sin^2 \theta + r_{14} \sin^3 \theta \quad (123)$$

$$Q_1(\theta) = p_{11} \cos^2 \theta + p_{12} \cos \theta \sin \theta + p_{13} \sin^2 \theta \quad (124)$$

$$P_2(\theta) = r_{21} \cos^3 \theta + r_{22} \cos^2 \theta \sin \theta + r_{23} \cos \theta \sin^2 \theta + r_{24} \sin^3 \theta \quad (125)$$

$$Q_2(\theta) = p_{21} \cos^2 \theta + p_{22} \cos \theta \sin \theta + p_{23} \sin^2 \theta \quad (126)$$

If we multiply equation (121) by  $\cos \theta$  and equation (122) by  $\sin \theta$ , and add the two resulting equations, we get:

$$\dot{R} = a'\epsilon R + P(\theta)R^3 + Q(\theta)R^2 + (e_1 \cos \theta + e_2 \sin \theta) \quad (127)$$

where,

$$P(\theta) = P_1(\theta) \cos \theta + P_2(\theta) \sin \theta \quad (128)$$

$$Q(\theta) = Q_1(\theta) \cos \theta + Q_2(\theta) \sin \theta \quad (129)$$

Equation (127) contains one variable that varies slowly in time ( $R$ ) and a fast variable ( $\theta$ ).

If we average this equation over one complete cycle in  $\theta$ , from 0 to  $2\pi$ , equation (127) becomes:

$$\dot{R} = a'\epsilon R + KR^3 + LR^2 + M \quad (130)$$

where,

$$\begin{aligned} K &= \frac{1}{2\pi} \int_0^{2\pi} P(\theta) d\theta \\ &= \frac{1}{8}(3r_{11} + r_{13} + r_{22} + 3r_{24}) \end{aligned} \quad (131)$$

$$L = \frac{1}{2\pi} \int_0^{2\pi} Q(\theta) d\theta \quad (132)$$

$$\begin{aligned} M &= \frac{1}{2\pi} \int_0^{2\pi} e_1 \cos \theta d\theta \\ &+ \frac{1}{2\pi} \int_0^{2\pi} e_2 \sin \theta d\theta \\ &= \frac{e_1}{2\pi} [\sin \theta]_0^{2\pi} - \frac{e_2}{2\pi} [\cos \theta]_0^{2\pi} \\ &= 0 \end{aligned} \quad (133)$$

Finally equation (135) becomes:

$$\dot{R} = a'\epsilon R + KR^3 \quad (134)$$

## F. LIMIT CYCLE ANALYSIS

At steady state  $\dot{R} = 0$ , and equation (134) becomes:

$$0 = R(a'\epsilon + KR^2) \quad (135)$$

Equation (135) has two solutions. The first solution is  $R = 0$ . This is the trivial solution and it does not give us much information. The second solution is:

$$R = \sqrt{\frac{-a'\epsilon}{K}} \quad (136)$$

This solution gives us a limit cycle of constant amplitude  $R$  in the  $z_1, z_2$  cartesian coordinate system. This limit cycle exists if the quantity inside the square root is positive, or

$$\frac{-a'\epsilon}{K} > 0 \quad (137)$$

Condition (137) is necessary for the amplitude of the limit cycle,  $R$ , to be a real number.

In our case  $a'$  is always **negative**, because for constant  $x_G$ , as we decrease  $\epsilon$  (Figure 1), the real part of the critical pair of eigenvalues increases, due to further loss of stability. In other words we can say that:

$$a' < 0 \quad (138)$$

From conditions (137), (138) we see that the existence of the limit cycles depends on the value of parameter  $K$ . We can see that:

- If  $K < 0$ , periodic solutions exist for  $\epsilon < 0$  or  $z_G - z_{G_c} < 0$  or  $z_G < z_{G_c}$ , and they are stable.
- If  $K > 0$ , periodic solutions exist for  $\epsilon > 0$  or  $z_G - z_{G_c} > 0$  or  $z_G > z_{G_c}$ , and they are unstable.

The characteristic root of equation (134) in the vicinity of (136) is:

$$\beta = -2a'\epsilon \quad (139)$$

The sign of this characteristic root, assigns the stability of the periodic solutions.

## G. RESULTS AND DISCUSSION

Typical results in terms of the nonlinear stability coefficient  $K$  are presented in Figures 3 through 6. The stability coefficient  $K$  is shown in its normalized form as  $K \cdot \gamma$  (Papadimitriou, 1994). Figure 3 shows  $K$  versus the LCG/LCB separation distance  $x_G$  (in ft) for a given vehicle speed and for different values of the drag coefficient. It can be seen that  $K$  is everywhere negative, which means that all bifurcations to periodic solutions are supercritical. Higher values of the drag coefficient result in stronger supercritical bifurcations which means that the corresponding limit cycle amplitudes will be smaller.

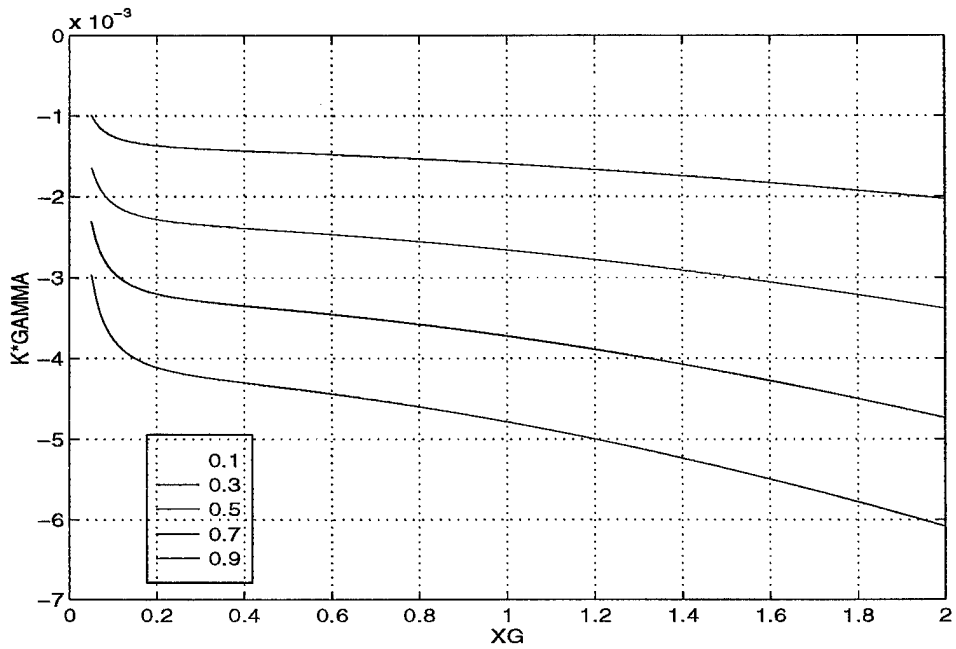


Figure 3:  $K \cdot \gamma$  versus  $x_G$  for  $U = 5$  ft/sec and different values of  $C_{Dy}$

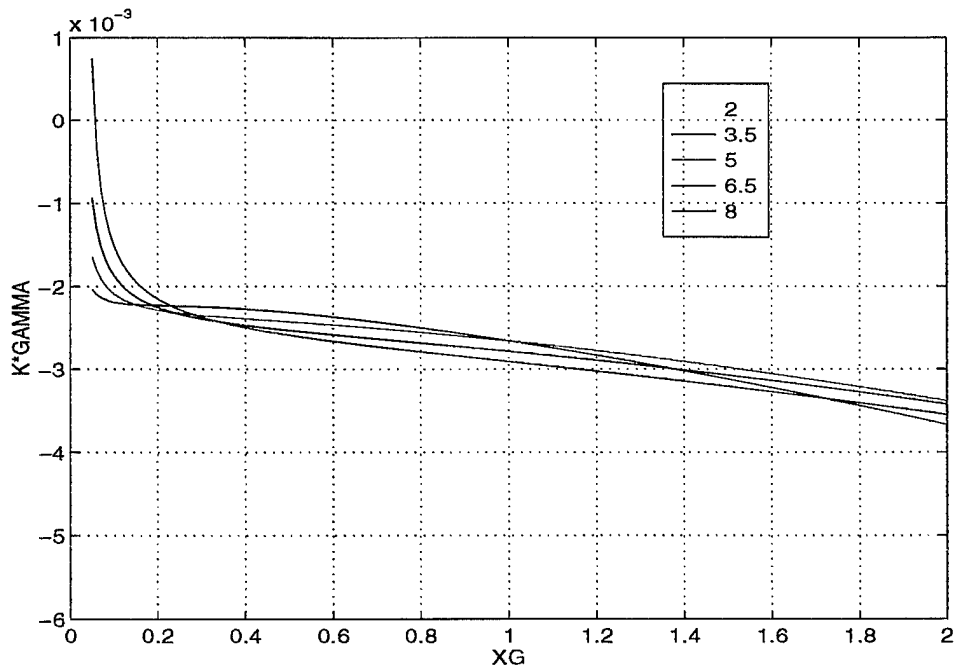


Figure 4:  $K \cdot \gamma$  versus  $x_G$  for  $C_{Dy} = 0.5$  and different values of  $U$  (ft/sec)

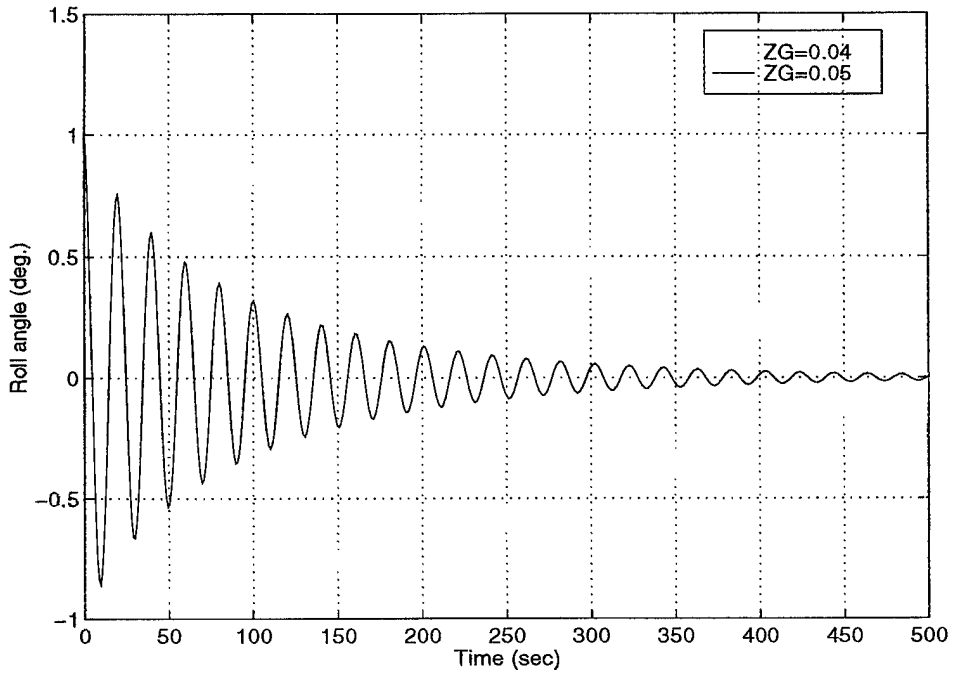


Figure 5: Simulation results ( $\phi, t$ ) for  $C_{Dy} = 0.5$ ,  $U = 5$  ft/sec, and  $x_G = 1$  ft

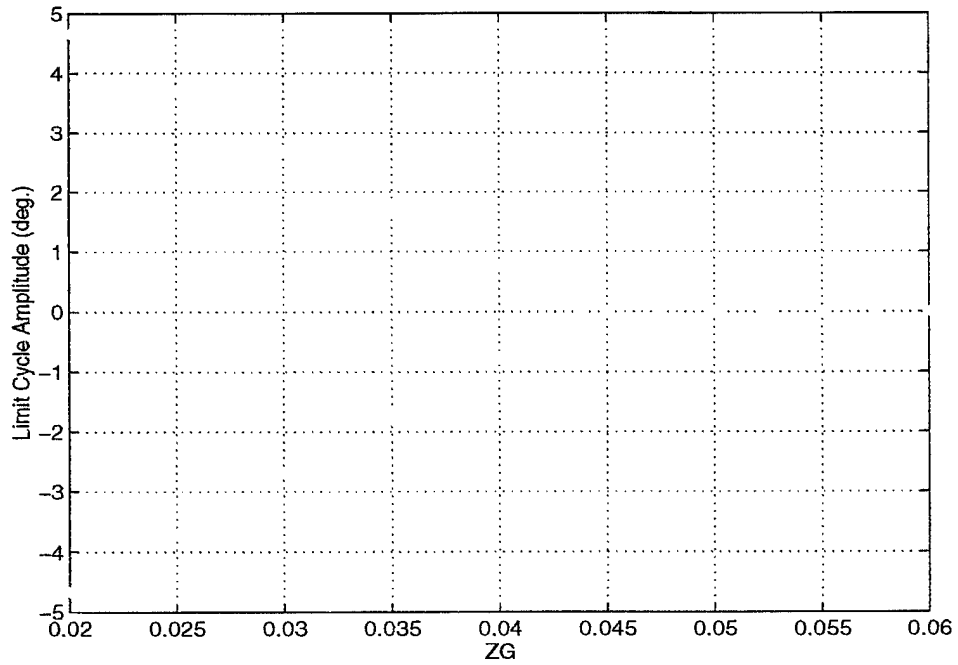


Figure 6: Limit cycle amplitudes (in  $\phi$ ) versus  $z_G$  for  $U = 5$  ft/sec and  $x_G = 1$  ft

Figure 4 shows  $K$  versus  $x_G$  for a given drag coefficient and for different forward speeds. It can be seen that the bifurcations are supercritical with the possible exception of large speeds and small  $x_G$ . This means that it is possible for a properly trimmed vehicle at relatively high speeds to experience an oscillatory behavior even before stability is lost. This demonstrates a destabilizing effect which could not have been predicted by linear techniques.

Figures 5 and 6 shows numerical simulation results which confirm the theoretical predictions. Both figures correspond to a supercritical bifurcation case and they show a continuous increase in limit cycle amplitudes as the bifurcation point is crossed.



## V. CONCLUSIONS AND RECOMMENDATIONS

This thesis presented a comprehensive nonlinear study of straight line stability of motion of submersibles in coupled sway/yaw/roll motions under open loop conditions. Primary loss of stability was shown to occur in the form of Hopf bifurcations to periodic solutions. This loss of stability is characteristic of the coupling of roll into sway and yaw and cannot be predicted by considering the uncoupled motions. The critical point where instability occurs was computed in terms of vehicle metacentric height, longitudinal separation of the centers of buoyancy and gravity, and the forward speed. Analysis of the periodic solutions that resulted from the Hopf bifurcations was accomplished through Taylor expansions, up to third order, of the equations of motion. A consistent approximation, utilizing the generalized gradient, was used to study the non-analytic quadratic cross flow integral drag terms. The results indicated that loss of stability occurs always in the form of supercritical Hopf bifurcations with stable limit cycles. It was shown that this is mainly due to the stabilizing effect of the drag forces at high angles of attack. Subcritical bifurcations, however, with considerably higher limit cycle amplitudes may develop for sufficiently high forward speeds and small LCG/LCB separations. Simulation studies of these subcritical bifurcations along with the effects of vertical plane coupling constitute recommendations for further research.



## APPENDIX

The following is a list and description of the computer programs used in this thesis. The programs are written in FORTRAN or MATLAB. Complete print-outs of the programs follow after the list.

- **STABILITY.M**

MATLAB program for performing linear stability analysis.

- **SIM.M**

MATLAB program and functions for numerical simulation.

- **HOPF.FOR**

FORTRAN program for calculating the nonlinear stability analysis coefficient  $K$ . It requires data from STABILITY.M and standard numerical linear algebra subroutines.

```

% STABILITY.M
%
% LOSS OF STABILITY
% *****
a=1
W=12000;
IXX=1760; IYY=9450;
IZZ=10700; IXZ=0; IXY=0;
IYZ=0; L=17.425; RHO=1.94;
G=32.2; U=6.5; M=W/G; B=W;

ND1=0.5*RHO*L^2;

% DEFINE HYDRODYNAMIC COEFFICIENTS

YPDOT=1.270e-04*ND1*L^2;
YVDOT=-5.550e-02*ND1*L;
YRDOT=1.240e-03*ND1*L^2;
YP=3.055e-03*ND1*L;
YV=-9.310e-02*ND1;
YR=-5.940e-02*ND1*L;

NPDOT=-3.370e-05*ND1*L^3;
NVDOT=1.240e-03*ND1*L^2;
NRDOT=-3.400e-03*ND1*L^3;
NP=-8.405e-04*ND1*L^2;
NV=-1.484e-02*ND1*L;
NR=-1.640e-02*ND1*L^2;

KPDOT=-1.01e-03*ND1*L^3;
KVDOT=1.27e-04*ND1*L^2;
KRDOT=-3.37e-05*ND1*L^3;
KP=-1.10e-02*ND1*L^2;
KV=3.055e-03*ND1*L;
KR=-8.41e-04*ND1*L^2;

flag=0;
for XG=0:0.01:2,

flag=flag+1;

```

```

xg(flag)=XG;
a=IXX-KPDOT; b=KP*U; e=KV*U;
f=KRDOT; i=YP*U; j=M-YVDOT; k=YV*U;
l=XG*M-YRDOT; m=U*(YR-M); o=NPDOT;
p=NP*U; q=-XG*W; r=XG*M-NVDOT;
w=U*(NR-XG*M); x=NV*U; u=IZZ-NRDOT;

a1=-u*M^2;
a2=-u*M*YPDOT-u*M*KVDOT+M*o*l+f*r*M;
a3=a*j*u-a*l*r-u*KVDOT*YPDOT+KVDOT*o*l+f*r*YPDOT-f*o*j;

b1=(w*(M^2))+(r*U*(M^2));
b2=-M*e*u-M*u*i+w*M*YPDOT+w*KVDOT*M-o*m*M+p*l*M-f*x*M+...
r*M*U*YPDOT-o*j*M*U+r*KR*U*M;
b3=-e*u*YPDOT+e*o*l-u*i*KVDOT+w*KVDOT*YPDOT-KVDOT*o*m+KVDOT*p*l-...
a*j*w-a*k*u+a*l*x+a*r*m-b*j*u+b*l*r+f*r*i-f*x*YPDOT+...
o*k*f-f*p*j+r*KR*U*YPDOT;

c1=-x*(M^2)*U;
c2=r*i*M*U-x*M*U*YPDOT-x*KR*U*M+o*k*M*U-p*j*M*U+j*u*W-l*r*W-...
q*l*M+w*i*M-m*p*M+e*w*M;
c3=a*k*w-a*m*x+b*j*w+b*k*u-b*l*x-b*r*m+r*i*KR*U-x*KR*U*YPDOT+...
o*k*KR*U-p*j*KR*U-q*l*KVDOT+w*i*KVDOT-m*p*KVDOT-e*u*i+e*w*YPDOT-...
e*o*m+e*p*l+f*q*j-f*x*i+f*p*k;

d2=q*j*M*U-x*i*M*U+p*k*M*U+q*m*M-j*w*W-k*u*W+l*x*W+r*m*W;
d3=q*j*KR*U-x*i*KR*U+p*k*KR*U-f*q*k+q*m*KVDOT-e*q*l+e*w*i-...
e*m*p-b*k*w+b*m*x;

e2=k*w*W-m*x*W-q*k*M*U;
e3=e*q*m-q*k*KR*U;

f5=(-e2*(b1^2))+b1*c1*d2;
f4=((b2*c1+b1*c2)*d2)+b1*c1*d3-a1*(d2^2)-e3*(b1^2)-2*e2*b1*b2;
f3=-a2*(d2^2)-2*d3*d2*a1-2*e3*b1*b2-e2*((b2^2)+2*b1*b3)+...
d2*(b3*c1+b2*c2+b1*c3)+d3*(b2*c1+b1*c2);
f2=-e3*((b2^2)+2*b1*b3)-2*e2*b2*b3+d2*(b3*c2+b2*c3)+...
d3*(b3*c1+b2*c2+b1*c3)-a3*(d2^2)-2*d3*d2*a2-a1*(d3^2);
f1=b3*c3*d2+d3*(b3*c2+b2*c3)-2*e3*b2*b3-e2*(b3^2)-...
2*d3*d2*a3-a2*(d3^2);
f0=b3*c3*d3-a3*(d3^2)-e3*(b3^2);

```

```
coef=[f5 f4 f3 f2 f1 f0];
ZG=roots(coef);

tot(flag)=ZG(5,1);
end

plot(xg,tot),grid;
title('ZG versus XG plot in the point of loss of stability')
xlabel('XG in ft');
ylabel('ZG in ft');
```

```
% NON LINEAR SIMULATION PROGRAM
```

```
t0=0;  
tfinal=500;  
q=(1/180)*pi;  
y0=[0 0 0 q];  
[t,y]=ode45('vdpo2',t0,tfinal,y0);
```

```
figure(1),plot(t(:,1),57.29578*y(:,4)),grid  
title('plot of fi with time');  
ylabel('fi'),xlabel('time')
```

% NON LINEAR SIMULATION PROGRAM

function yprime=vdpo2(t,y)

W=12000;  
IXX=1760; IYY=9450;  
IZZ=10700; IXZ=0; IXY=0;  
IYZ=0; L=17.425; RHO=1.94; ZB=0;  
G=32.2; U=5; M=W/G; B=W; XB=0;  
ZG=0.05; CD=0.5; YG=0;  
XG=1; YB=0;

ND1=0.5\*RHO\*L^2;

% DEFINE HYDRODYNAMIC COEFFICIENTS

YPDOT=1.270e-04\*ND1\*L^2;  
YVDOT=-5.550e-02\*ND1\*L;  
YRDOT=1.240e-03\*ND1\*L^2;  
YP=3.055e-03\*ND1\*L;  
YV=-9.310e-02\*ND1;  
YR=-5.940e-02\*ND1\*L;

NPDOT=-3.370e-05\*ND1\*L^3;  
NVDOT=1.240e-03\*ND1\*L^2;  
NRDOT=-3.400e-03\*ND1\*L^3;  
NP=-8.405e-04\*ND1\*L^2;  
NV=-1.484e-02\*ND1\*L;  
NR=-1.640e-02\*ND1\*L^2;

KPDOT=-1.01e-03\*ND1\*L^3;  
KVDOT=1.27e-04\*ND1\*L^2;  
KRDOT=-3.37e-05\*ND1\*L^3;  
KP=-1.10e-02\*ND1\*L^2;  
KV=3.055e-03\*ND1\*L;  
KR=-8.41e-04\*ND1\*L^2;

D=((M-YVDOT)\*(IZZ-NRDOT)\*(IXX-KPDOT))...  
-((M-YVDOT)\*(IXZ-KRDOT)\*(-IXZ-NPDOT))...  
-((M\*XG-NVDOT)\*(M\*XG-YRDOT)\*(IXX-KPDOT))...  
+((M\*XG-NVDOT)\*(IXZ-KRDOT)\*(-M\*ZG-YPDOT))...  
+((-M\*ZG-KVDOT)\*(M\*XG-YRDOT)\*(-IXZ-NPDOT))...

```

-((-M*ZG-KVDOT)*(IZZ-NRDOT)*(-M*ZG-YPDOT));

A11=((IZZ-NRDOT)*(IXX-KPDOT))-((IXZ-KRDOT)*(-IXZ-NPDOT));
A12=(-M*XG+YRDOT)*(IXX-KPDOT)+((IXZ-KRDOT)*(-M*ZG-YPDOT));
A13=(M*XG-YRDOT)*(-IXZ-NPDOT)-((IZZ-NRDOT)*(-M*ZG-YPDOT));
A21=(-M*XG+NVDOT)*(IXX-KPDOT)+((-M*ZG-KVDOT)*(-IXZ-NPDOT));
A22=(M-YVDOT)*(IXX-KPDOT)-((-M*ZG-KVDOT)*(-M*ZG-YPDOT));
A23=(-M+YVDOT)*(-IXZ-NPDOT)+((M*XG-NVDOT)*(-M*ZG-YPDOT));
A31=(M*XG-NVDOT)*(IXZ-KRDOT)-((-M*ZG-KVDOT)*(IZZ-NRDOT));
A32=(-M+YVDOT)*(IXZ-KRDOT)+((-M*ZG-KVDOT)*(M*XG-YRDOT));
A33=(M-YVDOT)*(IZZ-NRDOT)-((M*XG-NVDOT)*(M*XG-YRDOT));
%=====
%           EVALUATE TRANSFORMATION MATRIX OF EIGENVECTORS
%
F(1,1)=(A11*YV*U+A12*NV*U+A13*KV*U)/D;
F(1,2)=(A11*(YR*U-M*U)+A12*(-M*XG*U+NR*U)+A13*(M*ZG*U+KR*U))/D;
F(1,3)=(A11*YP*U+A12*NP*U+A13*KP*U)/D;
F(1,4)=0;
F(2,1)=(A21*YV*U+A22*NV*U+A23*KV*U)/D;
F(2,2)=(A21*(YR*U-M*U)+A22*(-M*XG*U+NR*U)+A23*(M*ZG*U+KR*U))/D;
F(2,3)=(A21*YP*U+A22*NP*U+A23*KP*U)/D;
F(2,4)=0;
F(3,1)=(A31*YV*U+A32*NV*U+A33*KV*U)/D;
F(3,2)=(A31*(YR*U-M*U)+A32*(-M*XG*U+NR*U)+A33*(M*ZG*U+KR*U))/D;
F(3,3)=(A31*YP*U+A32*NP*U+A33*KP*U)/D;
F(3,4)=0;
F(4,1)=0;
F(4,2)=0;
F(4,3)=1;
F(4,4)=0;
%=====
% CALCULATION OF THE INTEGRATION TERMS Ei
%=====

% DEFINE THE LENGTH X AND HEIGHT H TERMS FOR THE INTEGRATION
% ALL IN FEET.
XL(1)=-105.9/12;
XL(2)=-104.3/12;
XL(3)=-99.3/12;
XL(4)=-94.3/12;
XL(5)=-87.3/12;
XL(6)=-76.8/12;

```

```
XL(7)=-66.3/12;  
XL(8)=-55.8/12;  
XL(9)=72.7/12;  
XL(10)=79.2/12;  
XL(11)=83.2/12;  
XL(12)=87.2/12;  
XL(13)=91.2/12;  
XL(14)=95.2/12;  
XL(15)=99.2/12;  
XL(16)=101.2/12;  
XL(17)=102.1/12;  
XL(18)=103.2/12;
```

```
HT(1)=0;  
HT(2)=2.28/12;  
HT(3)=8.24/12;  
HT(4)=13.96/12;  
HT(5)=19.76/12;  
HT(6)=25.1/12;  
HT(7)=29.36/12;  
HT(8)=31.85/12;  
HT(9)=31.85/12;  
HT(10)=30.00/12;  
HT(11)=27.84/12;  
HT(12)=25.12/12;  
HT(13)=21.44/12;  
HT(14)=17.12/12;  
HT(15)=12.0/12;  
HT(16)=9.12/12;  
HT(17)=6.72/12;  
HT(18)=0;
```

```
for i=1:18,  
VEC1(i)=HT(i)*(y(1)+XL(i)*y(2))*(abs(y(1)+XL(i)*y(2)));  
VEC2(i)=HT(i)*(y(1)+XL(i)*y(2))*(abs(y(1)+XL(i)*y(2)))*XL(i);  
end
```

```
OUT=0;  
for j=1:17,  
OUT1=0.5*(VEC1(j)+VEC1(j+1))*(XL(j+1)-XL(j));  
OUT=OUT+OUT1;  
end
```

```

IV=CD*OUT;

OUT=0;
for j=1:17,
OUT2=0.5*(VEC2(j)+VEC2(j+1))*(XL(j+1)-XL(j));
OUT=OUT+OUT2;
end
IR=CD*OUT;
%=====

yprime=[F(1,1)*y(1)+F(1,2)*y(2)+F(1,3)*y(3)+F(1,4)*y(4)+...
(A11/D)*(YG*(y(3)^2+y(2)^2)-IV+(W-B)*sin(y(4)))+...
(A12/D)*(IXY*y(3)^2+IYZ*y(3)*y(2)+YG*y(1)*y(2)-IR+(XG*W-XB*B)*sin(y(4)))+...
(A13/D)*(-IXY*y(3)*y(2)-IYZ*y(2)^2-M*YG*y(1)*y(3)+(YG*W-YB*B)*cos(y(4))...
-(ZG*W-ZB*B)*sin(y(4)));...
F(2,1)*y(1)+F(2,2)*y(2)+F(2,3)*y(3)+F(2,4)*y(4)+...
(A21/D)*(YG*(y(3)^2+y(2)^2)-IV+(W-B)*sin(y(4)))+...
(A22/D)*(IXY*y(3)^2+IYZ*y(3)*y(2)+YG*y(1)*y(2)-IR+(XG*W-XB*B)*sin(y(4)))+...
(A23/D)*(-IXY*y(3)*y(2)-IYZ*y(2)^2-M*YG*y(1)*y(3)+(YG*W-YB*B)*cos(y(4))...
-(ZG*W-ZB*B)*sin(y(4)));...
F(3,1)*y(1)+F(3,2)*y(2)+F(3,3)*y(3)+F(3,4)*y(4)+...
(A31/D)*(YG*(y(3)^2+y(2)^2)-IV+(W-B)*sin(y(4)))+...
(A32/D)*(IXY*y(3)^2+IYZ*y(3)*y(2)+YG*y(1)*y(2)-IR+(XG*W-XB*B)*sin(y(4)))+...
(A33/D)*(-IXY*y(3)*y(2)-IYZ*y(2)^2-M*YG*y(1)*y(3)+(YG*W-YB*B)*cos(y(4))...
-(ZG*W-ZB*B)*sin(y(4)));...
1*y(3)];

```

```

C   HOPF.FOR
C
C   HOPF BIFURCATIONS PROGRAM
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL*8 L,IYY,M,YPDOT,YVDOT,ND1,YRDOT
      REAL*8 YP,YV,YR,NPDOT,NVDOT,NRDOT,NP,NV,NR
      REAL*8 GAMA,U,KVDOT,KRDOT,KPDOT,D
      REAL*8 E0,E1,E2,E3,E4,XG,ZG,KR,KP,KV,XG1,ZG1
C
      REAL*8 M11,M12,M13,M14,M21,M22,M23
      REAL*8 M24,M31,M32,M33,M34,M41,M42,M43,M44
      REAL*8 N11,N12,N13,N14,N21,N22,N23,N24
      REAL*8 N31,N32,N33,N34,N41,N42,N43,N44
      REAL*8 L11,L12,L13,L14,L21,L22,L23,L24,L31
      REAL*8 L32,L33,L34,L15,L16,L17,L25,L26,L27,L35
      REAL*8 L36,L37,L1A,L2A,L3A,L4A,L5A,L6A,L7A,L8A,L9A
      REAL*8 L10A,L11A,L12A,L1,L2,L3,L4,L5,L6,L7
      REAL*8 L8,L9,R11,R12,R13,R14,R21,R22,R23,R24
      REAL*8 P11,P12,P13,P21,P22,P23
C
      DIMENSION F(4,4),T(4,4),TINV(4,4),FV1(4),IV1(4),YYY(4,4)
      DIMENSION WR(4),WI(4),TSAVE(4,4),TLUD(4,4),IVLUD(4),SVLUD(4)
      DIMENSION ASAVE(4,4),A2(4,4),XL(18),HT(18),ZGG(197),FF(4,4)
      DIMENSION VECO(18),VEC1(18),VEC2(18),VEC3(18),VEC4(18),XGG(197)
C
      INTEGER I,J,K
C
      OPEN (20,FILE='HOPF.RES',STATUS='OLD')
      OPEN (21,FILE='DATA.DAT',STATUS='OLD')
      OPEN (23,FILE='HOPF1.RES',STATUS='OLD')
      OPEN (25,FILE='HOPF2.RES',STATUS='OLD')
C
      WEIGHT=12000.0
      IXX  =1760.0
      IYY  =9450.0
      IZZ  =10700.0
      IXZ  =0.0
      IXY  =0.0
      IYZ  =0.0
      L    =17.425
      RHO  =1.94

```

```

WRITE (*,*) ' ENTER CD'
READ (*,*) CD
G      =32.2
XB     =0.0
ZB     =0.0
YG     =0.0
YB     =0.0
YDELTA=0.0
DELTA  =0.0
NDELTA=0.0
NPROP  =0.0
M= WEIGHT/G
B= WEIGHT
W=WEIGHT
U=8.0
ND1=0.5*RHO*L**2
C
C  DEFINE HYDRODYNAMIC COEFFICIENTS
C
YPDOOT=1.270E-04*ND1*L**2
YVDOOT=-5.550E-02*ND1*L
YRDOOT=1.240E-03*ND1*L**2
YP=3.055E-03*ND1*L
YV=-9.310E-02*ND1
YR=-5.940E-02*ND1*L
C
NPDOOT=-3.370E-05*ND1*L**3
NVDOOT=1.240E-03*ND1*L**2
NRDOOT=-3.400E-03*ND1*L**3
NP=-8.405E-04*ND1*L**2
NV=-1.484E-02*ND1*L
NR=-1.640E-02*ND1*L**2
C
KPDOOT=-1.01E-03*ND1*L**3
KVDOOT=1.27E-04*ND1*L**2
KRDOOT=-3.37E-05*ND1*L**3
KP=-1.10E-02*ND1*L**2
KV=3.055E-03*ND1*L
KR=-8.41E-04*ND1*L**2
C
C  DEFINE THE LENGTH X AND HEIGHT H TERMS FOR
C  THE INTEGRATION, ALL IN FEET.

```

C

XL( 1)=-105.9/12.0  
XL( 2)=-104.3/12.0  
XL( 3)=-99.3/12.0  
XL( 4)=-94.3/12.0  
XL( 5)=-87.3/12.0  
XL( 6)=-76.8/12.0  
XL( 7)=-66.3/12.0  
XL( 8)=-55.8/12.0  
XL( 9)=72.7/12.0  
XL(10)=79.2/12.0  
XL(11)=83.2/12.0  
XL(12)=87.2/12.0  
XL(13)=91.2/12.0  
XL(14)=95.2/12.0  
XL(15)=99.2/12.0  
XL(16)=101.2/12.0  
XL(17)=102.1/12.0  
XL(18)=103.2/12.0

C

HT( 1)= 0.000  
HT( 2)= 2.28/12.0  
HT( 3)= 8.24/12.0  
HT( 4)= 13.96/12.0  
HT( 5)= 19.76/12.0  
HT( 6)= 25.1/12.0  
HT( 7)= 29.36/12.0  
HT( 8)= 31.85/12.0  
HT( 9)= 31.85/12.0  
HT(10)= 30.00/12.0  
HT(11)= 27.84/12.0  
HT(12)= 25.12/12.0  
HT(13)= 21.44/12.0  
HT(14)= 17.12/12.0  
HT(15)= 12.0/12.0  
HT(16)= 9.12/12.0  
HT(17)= 6.72/12.0  
HT(18)= 0.00

C

DO 104 K = 1,18  
VECO(K)=HT(K)  
VEC1(K)=XL(K)\*HT(K)

```

      VEC2(K)=XL(K)*XL(K)*HT(K)
      VEC3(K)=XL(K)*XL(K)*XL(K)*HT(K)
      VEC4(K)=XL(K)*XL(K)*XL(K)*XL(K)*HT(K)
104 CONTINUE
      CALL TRAP(18,VECO,XL,E0)
      CALL TRAP(18,VEC1,XL,E1)
      CALL TRAP(18,VEC2,XL,E2)
      CALL TRAP(18,VEC3,XL,E3)
      CALL TRAP(18,VEC4,XL,E4)
C
      WRITE (*,*) ' ENTER GAMA '
      READ (*,*) GAMA
C=====
C      READ THE CRITICAL VALUES FOR XG AND ZG FROM FILE DATA.DAT
C
      XGG(1)=0.0
      ZGG(1)=0.016358083
      DO 1 I=2,197
      READ (21,*)XG,ZG
      XGG(I)=XG
      ZGG(I)=ZG
C=====
C      DETERMINE [F] COEFFICIENTS
C
      D=((M-YVDOT)*(IZZ-NRDOT)*(IXX-KPDOT))
& -((M-YVDOT)*(IXZ-KRDOT)*(-IXZ-NPDOT))
& -((M*XG-NVDOT)*(M*XG-YRDOT)*(IXX-KPDOT))
& +((M*XG-NVDOT)*(IXZ-KRDOT)*(-M*ZG-YPDOT))
& +((-M*ZG-KVDOT)*(M*XG-YRDOT)*(-IXZ-NPDOT))
& -((-M*ZG-KVDOT)*(IZZ-NRDOT)*(-M*ZG-YPDOT))
C
      A11=((IZZ-NRDOT)*(IXX-KPDOT))-((IXZ-KRDOT)*(-IXZ-NPDOT))
      A12=(-M*XG+YRDOT)*(IXX-KPDOT)+((IXZ-KRDOT)*(-M*ZG-YPDOT))
      A13=((M*XG-YRDOT)*(-IXZ-NPDOT))-((IZZ-NRDOT)*(-M*ZG-YPDOT))
      A21=(-M*XG+NVDOT)*(IXX-KPDOT)+((-M*ZG-KVDOT)*(-IXZ-NPDOT))
      A22=((M-YVDOT)*(IXX-KPDOT))-((-M*ZG-KVDOT)*(-M*ZG-YPDOT))
      A23=(-M+YVDOT)*(-IXZ-NPDOT)+((M*XG-NVDOT)*(-M*ZG-YPDOT))
      A31=((M*XG-NVDOT)*(IXZ-KRDOT))-((-M*ZG-KVDOT)*(IZZ-NRDOT))
      A32=(-M+YVDOT)*(IXZ-KRDOT)+((-M*ZG-KVDOT)*(M*XG-YRDOT))
      A33=((M-YVDOT)*(IZZ-NRDOT))-((M*XG-NVDOT)*(M*XG-YRDOT))
C
      F(1,1)=(A11*YV*U+A12*NV*U+A13*KV*U)/D

```

```

F(1,2)=(A11*(YR*U-M*U)+A12*(-M*XG*U+NR*U)+A13*(M*ZG*U+KR*U))/D
F(1,3)=(A11*YP*U+A12*NP*U+A13*KP*U)/D
F(1,4)=(A11*(W-B)+A12*(XG*W-XB*B)+A13*(-ZG*W+ZB*B))/D
F(2,1)=(A21*YV*U+A22*NV*U+A23*KV*U)/D
F(2,2)=(A21*(YR*U-M*U)+A22*(-M*XG*U+NR*U)+A23*(M*ZG*U+KR*U))/D
F(2,3)=(A21*YP*U+A22*NP*U+A23*KP*U)/D
F(2,4)=(A21*(W-B)+A22*(XG*W-XB*B)+A23*(-ZG*W+ZB*B))/D
F(3,1)=(A31*YV*U+A32*NV*U+A33*KV*U)/D
F(3,2)=(A31*(YR*U-M*U)+A32*(-M*XG*U+NR*U)+A33*(M*ZG*U+KR*U))/D
F(3,3)=(A31*YP*U+A32*NP*U+A33*KP*U)/D
F(3,4)=(A31*(W-B)+A32*(XG*W-XB*B)+A33*(-ZG*W+ZB*B))/D
F(4,1)=0.0
F(4,2)=0.0
F(4,3)=1.0
F(4,4)=0.0

```

C  
C  
C

EVALUATE TRANSFORMATION MATRIX OF EIGENVECTORS

```

DO 11 K=1,4
  DO 12 J=1,4
    ASAVE(K,J)=F(K,J)
12  CONTINUE
11  CONTINUE
    CALL RG(4,4,F,WR,WI,1,YYY,IV1,FV1,IERR)
    WRITE(23,1007)WR(1),WR(2),WR(3),WR(4)
    CALL DSOMEG(IEV,WR,WI,OMEGA,CHECK)
    OMEGAO=OMEGA
    DO 5 J=1,4
      T(J,1)= YYY(J,IEV)
      T(J,2)=-YYY(J,IEV+1)
5  CONTINUE
    IF (IEV.EQ.1.0) GO TO 13
    IF (IEV.EQ.2.0) GO TO 18
    IF (IEV.EQ.3.0) GO TO 14
    STOP 3004
14  DO 6 J=1,4
      T(J,3)=YYY(J,1)
      T(J,4)=YYY(J,2)
6  CONTINUE
    GO TO 17
18  DO 19 J=1,4
      T(J,3)=YYY(J,1)

```

```

        T(J,4)=YYY(J,4)
19  CONTINUE
    GO TO 17
13  DO 16 J=1,4
        T(J,3)=YYY(J,3)
        T(J,4)=YYY(J,4)
16  CONTINUE
17  CONTINUE
C
C  NORMALIZATION OF THE CRITICAL EIGENVECTOR
C
    CALL NORMAL(T)
C
C  INVERT TRANSFORMATION MATRIX
C
    DO 2 K=1,4
        DO 3 J=1,4
            TINV(K,J)=0.0
            TSAVE(K,J)=T(K,J)
3    CONTINUE
2    CONTINUE
    CALL DLU(4,4,TSAVE,4,TLUD,IVLUD)
    DO 4 J=1,4
        IF (IVLUD(J).EQ.0) STOP 3003
4    CONTINUE
    CALL DILU(4,4,TLUD,IVLUD,SVLUD)
    DO 8 K=1,4
        DO 9 J=1,4
            TINV(K,J)=TLUD(K,J)
9    CONTINUE
8    CONTINUE
C
C  CHECK Inv(T)*A*T
C
    CALL MULT(TINV,ASAVE,T,A2)
C
    P1=A2(3,3)
    P2=A2(4,4)
    PEIG1=P1
    PEIG2=P2
    WRITE(25,1008)A2(1,1),A2(2,2),P1,P2
C

```

C            DEFINITION OF Nij  
C

N11=TINV(1,1)  
N12=TINV(1,2)  
N13=TINV(1,3)  
N14=TINV(1,4)  
N21=TINV(2,1)  
N22=TINV(2,2)  
N23=TINV(2,3)  
N24=TINV(2,4)  
N31=TINV(3,1)  
N32=TINV(3,2)  
N33=TINV(3,3)  
N34=TINV(3,4)  
N41=TINV(4,1)  
N42=TINV(4,2)  
N43=TINV(4,3)  
N44=TINV(4,4)

C  
C            DEFINITION OF Mij  
C

M11=T(1,1)  
M12=T(1,2)  
M13=T(1,3)  
M14=T(1,4)  
M21=T(2,1)  
M22=T(2,2)  
M23=T(2,3)  
M24=T(2,4)  
M31=T(3,1)  
M32=T(3,2)  
M33=T(3,3)  
M34=T(3,4)  
M41=T(4,1)  
M42=T(4,2)  
M43=T(4,3)  
M44=T(4,4)

C  
C            DEFINITION OF Lij  
C

L1=YG\*((M31\*\*2)+(M21\*\*2))  
L2=IXY\*(M31\*\*2)+IYZ\*M31\*M21+YG\*M21\*M11

$L3=IXY*M31*M21+IYZ*(M21**2)+M*YG*M11*M31+YG*W*(M41**2)$   
 &  $-YB*B*(M41**2)$   
 $L4=YG*(2.0*M31*M32+2.0*M21*M22)$   
 $L5=2.0*IXY*M31*M32+IYZ*(M31*M22+M32*M21)+YG*(M21*M12+M22*M11)$   
 $L6=IXY*(M31*M22+M32*M21)+2.0*IYZ*M21*M22+M*YG*(M11*M32+M12*M31)$   
 &  $+2.0*(YG*W-YB*B)*M41*M42$   
 $L7=YG*((M32**2)+(M22**2))$   
 $L8=IXY*(M32**2)+IYZ*M32*M22+YG*M22*M12$   
 $L9=IXY*M32*M22+IYZ*(M22**2)+M*YG*M12*M32+(YG*W-YB*B)*(M42**2)$

C  
C

$L15=(A11/D)*L1+(A12/D)*L2-(A13/D)*L3$   
 $L16=(A11/D)*L4+(A12/D)*L5-(A13/D)*L6$   
 $L17=(A11/D)*L7+(A12/D)*L8-(A13/D)*L9$   
 $L25=(A21/D)*L1+(A22/D)*L2-(A23/D)*L3$   
 $L26=(A21/D)*L4+(A22/D)*L5-(A23/D)*L6$   
 $L27=(A21/D)*L7+(A22/D)*L8-(A23/D)*L9$   
 $L35=(A31/D)*L1+(A32/D)*L2-(A33/D)*L3$   
 $L36=(A31/D)*L4+(A32/D)*L5-(A33/D)*L6$   
 $L37=(A31/D)*L7+(A32/D)*L8-(A33/D)*L9$

C

$C=CD/(6.0*GAMA)$   
 $L1A=C*(E0*(M11**3)+3.0*E1*(M11**2)*M21+3.0*E2*M11*(M21**2)$   
 &  $+E3*(M21**3))+(1.0/6.0)*(W-B)*(M41**3)$   
 $L2A=C*(E1*(M11**3)+3.0*E2*(M11**2)*M21+3.0*E3*M11*(M21**2)$   
 &  $+E4*(M21**3))+(1.0/6.0)*(XG*W-XB*B)*(M41**3)$   
 $L3A=(1.0/6.0)*(ZG*W-ZB*B)*(M41**3)$   
 $L4A=C*(3.0*E0*(M11**2)*M12+3.0*E1*((M11**2)*M22+2.0*M11*M12*M21)$   
 &  $+3.0*E2*(M12*(M21**2)+2.0*M21*M22*M11)+3.0*E3*(M21**2)*M22)$   
 &  $+(1.0/6.0)*(W-B)*3.0*(M41**2)*M42$   
 $L5A=C*(3.0*E1*(M11**2)*M12+3.0*E2*((M11**2)*M22+2.0*M11*M12*M21)$   
 &  $+3.0*E3*(M12*(M21**2)+2.0*M21*M22*M11)+3.0*E4*(M21**2)*M22)$   
 &  $+(1.0/6.0)*(XG*W-XB*B)*3.0*(M41**2)*M42$   
 $L6A=(1.0/6.0)*(ZG*W-ZB*B)*3.0*(M41**2)*M42$   
 $L7A=C*(3.0*E0*M11*(M12**2)+3.0*E1*((M12**2)*M21+2.0*M11*M12*M22)$   
 &  $+3.0*E2*((M22**2)*M11+2.0*M21*M22*M12)+3.0*E3*M21*(M22**2))$   
 &  $+(1.0/6.0)*(W-B)*3.0*M41*(M42**2)$   
 $L8A=C*(3.0*E1*M11*(M12**2)+3.0*E2*((M12**2)*M21+2.0*M11*M12*M22)$   
 &  $+3.0*E3*((M22**2)*M11+2.0*M21*M22*M12)+3.0*E4*M21*(M22**2))$   
 &  $+(1.0/6.0)*(XG*W-XB*B)*3.0*M41*(M42**2)$   
 $L9A=(1.0/6.0)*(ZG*W-ZB*B)*3.0*M41*(M42**2)$   
 $L10A=C*(E0*(M12**3)+3.0*E1*(M11**2)*M21+3.0*E2*M12*(M22**2)$

```

&      +E3*(M22**3))+(1.0/6.0)*(W-B)*(M42**3)
L11A=C*(E1*(M12**3)+3.0*E2*(M11**2)*M21+3.0*E3*M12*(M22**2)
&      +E4*(M22**3))+(1.0/6.0)*(XG*W-XB*B)*(M42**3)
L12A=(1.0/6.0)*(ZG*W-ZB*B)*(M42**3)
C
L11=(-A11/D)*L1A+(-A12/D)*L2A+(A13/D)*L3A
L12=(-A11/D)*L4A+(-A12/D)*L5A+(A13/D)*L6A
L13=(-A11/D)*L7A+(-A12/D)*L8A+(A13/D)*L9A
L14=(-A11/D)*L10A+(-A12/D)*L11A+(A13/D)*L12A
C
L21=(-A21/D)*L1A+(-A22/D)*L2A+(A23/D)*L3A
L22=(-A21/D)*L4A+(-A22/D)*L5A+(A23/D)*L6A
L23=(-A21/D)*L7A+(-A22/D)*L8A+(A23/D)*L9A
L24=(-A21/D)*L10A+(-A22/D)*L11A+(A23/D)*L12A
C
L31=(-A31/D)*L1A+(-A32/D)*L2A+(A33/D)*L3A
L32=(-A31/D)*L4A+(-A32/D)*L5A+(A33/D)*L6A
L33=(-A31/D)*L7A+(-A32/D)*L8A+(A33/D)*L9A
L34=(-A31/D)*L10A+(-A32/D)*L11A+(A33/D)*L12A
C
R11=(N11*L11)+(N12*L21)+(N13*L31)
R12=(N11*L12)+(N12*L22)+(N13*L32)
R13=(N11*L13)+(N12*L23)+(N13*L33)
R14=(N11*L14)+(N12*L24)+(N13*L34)
R21=(N21*L11)+(N22*L21)+(N23*L31)
R22=(N21*L12)+(N22*L22)+(N23*L32)
R23=(N21*L13)+(N22*L23)+(N23*L33)
R24=(N21*L14)+(N22*L24)+(N23*L34)
C
P11=(N11*L15)+(N12*L25)+(N13*L35)
P12=(N11*L16)+(N12*L26)+(N13*L36)
P13=(N11*L17)+(N12*L27)+(N13*L37)
P21=(N21*L15)+(N22*L25)+(N23*L35)
P22=(N21*L16)+(N22*L26)+(N23*L36)
P23=(N21*L17)+(N22*L27)+(N23*L37)
C
C      EVALUATE DALPHA AND DOMEGA
C
ZGR  =ZGG(I)
ZGL  =ZGG(I-1)
ZG1  =ZGR
XG1  =XGG(I)

```

```

C
    CALL FMATRIX(ZG1,XG1,FF)
C
    CALL RG(4,4,FF,WR,WI,0,YYY,IV1,FV1,IERR)
    CALL DSTABL(DEOS,WR,WI,FREQ)
    ALPHR=DEOS
    OMEGR=FREQ
C
    ZG1  =ZGL
    XG1  =XGG(I)
C
    CALL FMATRIX(ZG1,XG1,FF)
C
    CALL RG(4,4,FF,WR,WI,0,YYY,IV1,FV1,IERR)
    CALL DSTABL(DEOS,WR,WI,FREQ)
    ALPHL=DEOS
    OMEGL=FREQ
C
    DALPHA=(ALPHR-ALPHL)/(ZGR-ZGL)
    DOMEGA=(OMEGR-OMEGL)/(ZGR-ZGL)
C
    EVALUATION OF HOPF BIFURCATION COEFFICIENTS
C
    COEF1=(1.0/8.0)*(3.0*R11+R13+R22+3.0*R24)
    COEF2=(1.0/8.0)*(3.0*R11+R23-R12-3.0*R14)
    WRITE (20,2001)XG,ZG,COEF1,DALPHA,OMEGAO,PEIG1,PEIG2
1 CONTINUE
C
    STOP
2001 FORMAT (7E14.5)
1007 FORMAT (4E14.5)
1008 FORMAT (4E14.5)
    END

```



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