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**13. ABSTRACT** *(Maximum 200 words)*

In this Final Report, we will describe the work we have performed in robust control theory and nonlinear control, and the utilization of techniques in image processing and computer vision for problems in visual tracking.

In our research, we have studied problems in $\mu$–synthesis and analysis for a number of various perturbation classes, the extension of $H^\infty$ control to a broad class of nonlinear systems, optimization of distributed parameter systems, a new active contour ("snakes") paradigm for visual tracking and shape modeling, and nonlinear geometric invariant diffusion equations for image enhancement and smoothing. We now have a general solution of the standard problem for general classes of nonlinear systems using our nonlinear interpolation procedure. We have emphasized the power of control techniques for some of the key problems in computer vision and image processing, and the importance of the utilization of visual information in a feedback loop. The interface between control and computer vision is one of the important directions in our research program which has emerged from this AFOSR contract.

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# Final Report for the AFOSR Contract AFOSR-AF/F49620-94-1-00S8DEF entitled "Analysis and Design of Multivariable Feedback Systems"

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# 1 Introduction

In our just completed AFOSR contract, we have carried out an extensive research program for the study of robust system control using methodologies from interpolation theory, dilation theory, and functional analysis. We have also become interested in image processing and computer vision, and their application to visual tracking problems.

In this Final Report, we will summarize some of our key results. A number of the research directions described here are being continued in our new three year AFOSR contract as well. In particular, we will discuss our work in nonlinear $H^\infty$ theory, distributed parameter control, robust stability under various classes of perturbations, and visual tracking.

Under the support of AFOSR-AF/F49620-94-1-00S8DEF, we have continued to develop our operator–theoretic methods for nonlinear robust control which we had already successfully applied to the linear case. One of the pillars of our theory is a novel causal dilation result which combines the classical dilation theory of Sarason-Sz.Nagy-Foias, and the nested algebra setting of Arveson. This we believe also has important implications to time-varying systems which is an ongoing topic of research for us. In this Final Report, we will sketch a solution for the full standard nonlinear $H^\infty$ control problem in this framework as well.

As part of our AFOSR sponsored research, we have been investigating various issues in distributed parameter control. Recently a monograph [71] which extensively describes our *skew Toeplitz* approach to distributed $H^\infty$ design and analysis has been published. This is based primarily on frequency domain ideas. We have been more recently trying to meld time-domain and frequency-domain methods for robust distributed parameter control into a new synthesis to get the advantages of both points of view.

We have also done extensive work on problems in $\mu$-synthesis and analysis. We have developed a general lifting procedure, by which we can interpret the $D$-scaled upper bound for the structured singular value as a singular value on a certain extended space. We have shown in which cases lifting is unnecessary, i.e., the upper bound gives a non-conservative measure of robustness. This is very important since the upper bound for $\mu$ is log-convex in the scalings and so can be computed, while $\mu$ cannot. Our results work directly for systems, not just finite matrices. This allows us to study broad classes of structured perturbations using this tool. We have also been continuing our work to build a rigorous $\mu$–synthesis procedure based on structured interpolation.

A major direction in our research is the design of novel techniques for using visual information in control systems, and in particular the problem of *visual tracking*. Even though tracking in the presence of a disturbance is a classical control issue, because of the highly uncertain nature of the disturbance, this type of control/vision problem is very difficult and challenging. We believe that this effort will lead to enhanced man-machine interfaces for interactions with computers and more complicated systems such as remote controlled weapons and vehicles. In particular, we have been employing our control/vision methodology for the airborne laser (ABL) program at Phillips Laboratory in our continuing collaboration with Don Washburn and Sal Cusumano. Versions of some of our software are already running there for use in the project.

Our interest in visual tracking has naturally led us to consider certain problems in computer vision and image processing. Thus, we have been conducting research into advanced algorithms in image processing and computer vision for a variety of uses: image smoothing and enhancement, image segmentation, morphology, denoising algorithms, edge detection, shape recognition, stereo disparity, optical flow, deformable contours ("snakes") for tracking.

Our techniques have already been applied to a variety of imagery (military and medical), and have been used to define a novel affine invariant scale-space. Our ideas are based on certain types of geometric invariant flows rooted in the mathematical theory of curve and surface evolution. There are now available powerful numerical algorithms based on Hamilton-Jacobi type equations and the associated theory of viscosity solutions for the computer implementation of this methodology.

It is important to note that visual tracking differs from standard tracking problems in that the feedback signal is measured using imaging sensors. In particular, it has to be extracted via computer vision and image processing algorithms and interpreted by a reasoning algorithm before being used in the control loop. Furthermore, the response speed is a critical aspect. As we have indicated above, we have been developing robust control algorithms for some years now, valid for general classes of distributed parameter and nonlinear systems based on interpolation and operator theoretic methods. In our continuing research program in this area, we have been explicitly combining our robust control techniques and the new approach to image processing which we have just sketched, in order to develop "state of the art" visual tracking algorithms.

In summary, in this Final Report, we will describe a variety of problems concerning the control of systems in the presence of uncertainty which relate to a number of practical and theoretical issues which we have investigated and which will provide the impetus for our AFOSR sponsored research for the next several years.

# 2 Nonlinear Robust Control

We have been pursuing for some time now the problem of finding suitable extensions of $H^\infty$ to the nonlinear framework. Indeed, some of the first papers in this connection are [10, 11]. This research has been continued in AFOSR-AF/F49620-94-1-00S8DEF.

It turns out that this research direction has brought out a number of intriguing questions regarding the causality of input/output operators. This has led us to some new results in which we have been able to place an explicit causality constraint in the commutant lifting framework for the first time [66, 68, 81].

More precisely, in one key approach that we have been considering for the extension of generalized interpolation theory (the mathematical basis of $H^\infty$) to nonlinear input/output operators [67], we find that we must apply the linear commutant lifting theorem to an $H^2$-space defined on the $n$-disc $D^n$. The difficulty is that when one applies the standard theory to $D^n$ ($n \geq 2$), even though time-invariance is preserved (that is, commutation with the appropriate shift), causality is generally lost. Thus to successfully find a nonlinear generalization of linear $H^\infty$ suitable for control applications, we need to include the causality constraint explicitly in the formulation of the interpolation problem. We will now sketch some of the relevant results from [81, 67, 68] which allows us to accomplish this.

## 2.1 Causal Operators

A heuristic definition of "causality" for a given input/output operator is that the past output is independent of the future inputs. Formally, let $S$ denote an isometry on a Hilbert space $\mathcal{G}$, and let $T$ denote a contraction on a Hilbert space $\mathcal{H}$. Let $P_{j0}$, $j \geq 1$ denote a sequence of orthogonal projections in $\mathcal{G}$ satisfying the following conditions:

$$\begin{aligned} P_{10} &\leq P_{20} \leq \ldots & (1)\\ P_{j0} &\leq I - S^j S^{*j}, \quad j = 1, 2, \ldots & (2)\\ P_{j+1,0} S(I - P_{j0}) &= 0, \quad j = 1, 2, \ldots. & (3) \end{aligned}$$

For $U : \mathcal{K} \to \mathcal{K}$ a minimal isometric dilation of the given contraction $T$, let $B : \mathcal{G} \to \mathcal{H}$ intertwine $S$ with $U$, that is,

$$UB = BS. \tag{4}$$

From this it is easy to see that

$$(I - U^j U^{*j})B = (I - U^j U^{*j})B(I - S^j S^{*j}), \quad j = 1, 2, \ldots. \tag{5}$$

We can now define "causality":

**Definition.** An operator $B$ satisfying (4) is called $(P_{10}, P_{20}, \ldots)$-*causal* (and if the sequence $\{P_{j0}\}_{j=1}^{\infty}$ is fixed, *causal*) if

$$(I - U^j U^{*j})B = (I - U^j U^{*j})BP_{j0}, \quad j \geq 1, \tag{6}$$

or equivalently,

$$(I - P_{j0})B^* = (I - P_{j0})B^* U^j U^{*j}, \quad j \geq 1. \tag{7}$$

Note that $B$ is always $(I - SS^*, I - S^2 S^{*2}, \ldots)$-causal. We now fix the sequence $P_{10}, P_{20}, \ldots$ relative to which causality will be taken in the sequel. Let $A : \mathcal{G} \to \mathcal{H}$ be an operator intertwining $S$ and $T$, that is, $AS = TA$. Then an *intertwining lifting (or dilation) of* $A$ is an operator $B : \mathcal{G} \to \mathcal{K}$ such that $BS = UB$, and $PB = A$ where $P : \mathcal{K} \to \mathcal{H}$ denotes orthogonal projection.

Define

$$\nu_\infty(A) := \inf\{\|B\| : B \text{ is a causal intertwining dilation of } A\},$$

and

$$\mu(A) := \min\{M \geq 0 : \|A\| \leq M, \ \|(I - P_{j0})A^* h\| \leq M\|T^{*j} h\|, \ h \in \mathcal{H}, \ j \geq 1\}.$$

We can then show that $\mu(A) \leq \nu_\infty(A)$.

We will also need a functional which lies between $\mu(A)$ and $\nu_\infty(A)$. To this aim, we call a sequence of operators $\Gamma_j : \mathcal{G}_j := (I - P_{0j})\mathcal{G} \to \mathcal{H}$, $j = 0, 2, \ldots$ a *resolution of* $A$ if

$$\begin{aligned} \Gamma_0 &= A,\\ \Gamma_j | \mathcal{G}_{j+1} &= T\Gamma_{j+1} \quad \forall j \geq 0,\\ \Gamma_j &= \Gamma_{j+1} S | \mathcal{G}_j \quad \forall j \geq 0. \end{aligned}$$

(Note that we take $P_{00} := 0$, so that $\mathcal{G}_0 = \mathcal{G}$.) We now define

$$\bar{\mu}(A) :=$$

$$\min\{M \geq 0 : \|A\| \leq M \text{ and there exists a resolution of } A, \Gamma_j, \text{ with } \|\Gamma_j\| \leq M, \ \forall j \geq 0\}.$$

We can now state the following results from [81]:

**Theorem 1 (Causal Commutant Lifting Theorem)** *Notation as above. Then*

$$\nu_\infty(A) = \bar{\mu}(A).$$

We also have the following variant of Theorem 1:

**Corollary 1** *If* $\ker T = \{0\}$, *then* $\mu(A) = \nu_\infty(A)$.

Using these ideas, for the compressed shift (the case which arises in control), we can give an explicit method for designing nonlinear causal compensators which extends linear $H^\infty$ theory to nonlinear systems [66, 67]. This is done by using a rather straightforward procedure by which the construction of a causal dilation may be reduced to a classical interpolation problem.

We will now briefly indicate how the above theory looks for analytic mappings on Hilbert space. following [81]. Accordingly, consider an analytic map $\phi$ with $\mathcal{G} = \mathcal{H} = H^2$ (the standard Hardy space on the disc $D$). Clearly,

$$H^2 \otimes \cdots \otimes H^2 = (H^2)^{\otimes n} \cong H^2(D^n)$$

where we map $1 \otimes \cdots \otimes z \otimes \cdots \otimes 1$ ($z$ in the $i$-th place) to $z_i$, $i = 1, \cdots, n$. In the usual way, we say that $\phi$ *shift-invariant* (or *time-invariant*) if

$$\phi_n S^{\otimes n} = S\phi_n \quad \forall n \geq 1,$$

where $S : H^2 \to H^2$ denotes the canonical unilateral right shift. (Equivalently, this means that $S\phi = \phi \circ S$ on some open ball about the origin in which $\phi$ is defined.)

Next set

$$P_{(j)}^{(n)} := P_{(j)} \otimes \cdots \otimes P_{(j)} \ \ (n \text{ times}), \quad j \geq 1, \ \ n \geq 1,$$

where

$$P_{(j)} := I - S^j S^{*j}.$$

Then we say that $\phi$ is *causal* if

$$P_{(j)}\phi_n = P_{(j)}\phi_n P_{(j)}^{(n)}, \quad j \geq 1, \ \ n \geq 1.$$

For $\phi : H^2 \to H^2$ linear and time-invariant (i.e., intertwines with the shift), it is easy to see that $\phi$ is causal. In the nonlinear setting however, time-invariance may not imply causality. See [81] for a concrete example.

Now it is easy to show that the conditions (1), (2), and (3) given above are verified for the $P_{(j)}^{(n)}$ and so the causal commutant lifting theorem applies. We will next see how this may be used to develop an *iterative design procedure* for the construction of nonlinear robust controllers [67, 81, 79, 68].

## 2.2 Iterated Design of Nonlinear Controllers

In this section, we will indicate how the causal lifting methodology described above leads to a natural extension of $H^\infty$ theory for nonlinear plants. For simplicity, we assume that our systems are SISO systems. Define an *admissible operator* to be an analytic input/output operator $\phi : H^2 \to H^2$ which is causal, time-invariant, and $\phi(0) = 0$. Denote the set of admissible operators by $\mathcal{C}_a$. We take the plant $P$ and the weight $W$ to be admissible, and we assume that $W$ admits an admissible inverse.

Consider the sensitivity minimization (one block) problem of finding

$$\mu_\delta := \inf_C \sup_{\|v\| \leq \delta} \|[(I + P \circ C)^{-1} \circ W]v\|, \tag{8}$$

where we take the infimum over all stabilizing controllers. (In what follows, we let $\| \ \|$ denote the 2-norm $\| \ \|_2$ on $H^2$ as well as the associated operator norm. The context will make the meaning clear.) Hence we are considering a worst case disturbance attenuation problem where the energy of the signals $v$ is required to be bounded by some pre-specified level $\delta$. (In the linear case since everything scales, we can always without loss of generality take $\delta = 1$. For nonlinear systems, we must pre-specify the energy bound.) Using standard transformations, one gets that (8) is equivalent to the problem of finding

$$\mu_\delta = \inf_{q \in \mathcal{C}_a} \sup_{\|v\| \leq \delta} \|(W - P \circ q)v\|. \tag{9}$$

Our iterated lifting procedure gives an approach for approximating a solution to this type of problem. Indeed, we express

$$\begin{aligned} W &= W_1 + W_2 + \cdots, \\ P &= P_1 + P_2 + \cdots, \\ q &= q_1 + q_2 + \cdots, \end{aligned}$$

where $W_j, P_j, q_j$ are homogeneous polynomials of degree $j$. Note that

$$\mu_\delta = \delta \inf_{q_1 \in H^\infty} \|W_1 - P_1 q_1\| + O(\delta^2), \tag{10}$$

where the latter norm is the operator norm (i.e., $H^\infty$ norm). From the classical commutant lifting theorem we can find an optimal (linear, causal, time-invariant) $q_{1,opt} \in H^\infty$ such that

$$\mu_\delta = \delta \|W_1 - P_1 q_{1,opt}\| + O(\delta^2). \tag{11}$$

Now the iterative causal commutant lifting procedure gives a way of finding higher order corrections to this linearization. We illustrate the method via a second order correction. So, having fixed the linear part $q_{1,opt}$ of $q$ in (9), we observe that

$$W(v) - P(q(v)) - (W_1 - P_1 q_{1,opt})(v) =$$

$$W_2(v) - P_2(q_{1,opt}(v)) - P_1 q_2(v) + \text{higher order terms.}$$

Regarding $W_2, P_2, q_2$ as linear operators on $H^2 \otimes H^2 \cong H^2(D^2, \mathbf{C})$, we get that

$$\sup_{\|v\| \leq \delta} \|(W - P \circ q)(v) - (W_1 - P_1 q_{1,opt})v\| \leq \delta^2 \|\hat{W}_2 - P_1 q_2\| + O(\delta^3),$$

where the "weight" $\hat{W}_2$ is given by

$$\hat{W}_2 := W_2 - P_2(q_{1,opt} \otimes q_{1,opt}).$$

Using the control version of the causal commutant lifting theorem (see [67] and Section 2.3), we can derive an optimal admissible $q_{2,opt}$, and so on. This is our iterated lifting approach to nonlinear $H^\infty$.

Consequently, instead of just designing a linear compensator for a linearization of the given nonlinear system, this technique allows us to explicitly take into account the higher order terms of the nonlinear plant, and so to increase the ball of operation for the nonlinear controller. Further, if the linear part of the plant is rational, then we have shown that the iterative procedure may be reduced to a series of *finite dimensional matrix computations.* (See [67, 78] for details.)

## 2.3 General Formulation of Standard Problem

The methods we have described above have been extended to the full nonlinear standard problem. We will indicate in this section, a formulation of the causal commutant lifting theorem which is easily implementable for a nonlinear version of the standard control problem. This is based on [69].

Accordingly, for the standard problem, we have the following general mathematical framework. Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ be Hilbert spaces equipped with the unilateral shifts $S_{\mathcal{E}_1}, S_{\mathcal{E}_2}, S_{\mathcal{F}_1}, S_{\mathcal{F}_2}$, respectively. Let $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$ be a co-isometry intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{F}_1}$ (i.e., $\Theta_1 S_{\mathcal{E}_1} = S_{\mathcal{F}_1}\Theta_1$), and let $\Theta_2 : \mathcal{F}_2 \to \mathcal{E}_2$ be an isometry intertwining $S_{\mathcal{E}_2}$ with $S_{\mathcal{F}_2}$. We let $U_{\mathcal{E}_1}$, be the minimal unitary dilation of $S_{\mathcal{E}_1}$ on $\mathcal{K}_{\mathcal{E}_1}$, and similarly for $U_{\mathcal{E}_2}$ on $\mathcal{K}_{\mathcal{E}_2}$, $U_{\mathcal{F}_1}$ on $\mathcal{K}_{\mathcal{F}_1}$, and $U_{\mathcal{F}_2}$ on $\mathcal{K}_{\mathcal{F}_2}$.

Now let

$$P_{\mathcal{E}_2}^{(n)} := (I - S_{\mathcal{E}_2}^n S_{\mathcal{E}_2}^{*n}), \;\; P_{\mathcal{F}_2}^{(n)} := (I - S_{\mathcal{F}_2}^n S_{\mathcal{F}_2}^{*n}), \quad n \geq 0.$$

We let the sequence $P_{\mathcal{E}_2}^{(n)}$ define the causal structure on $\mathcal{E}_2$, and similarly the causal structure of $\mathcal{F}_2$ is defined by the sequence $P_{\mathcal{F}_2}^{(n)}$. Moreover, the causal structure on $\mathcal{E}_1$ is defined by a general sequence of operators $P_1^{(n)}, \;\; n \geq 0$ satisfying the causal structure conditions (1), 2, 3) given above, and similarly the causal structure on $\mathcal{F}_1$ is defined by a sequence of operators $P_2^{(n)}, \;\; n \geq 0$ satisfying these conditions as well. We assume that the input/output operators $\Theta_1, \;\Theta_2$, are causal with respect to the above structures. We let $W : \mathcal{E}_1 \to \mathcal{E}_2$ denote a causal operator intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{E}_2}$. Thus causality for $W$ means that $P_{\mathcal{E}_2}^{(n)} W P_1^{(n)} = P_{\mathcal{E}_2}^{(n)} W, \;\; \forall n \geq 0$. Finally, $Q : \mathcal{F}_1 \to \mathcal{F}_2$ will denote a causal operator intertwining $S_{\mathcal{F}_1}$ with $S_{\mathcal{F}_2}$.

Next let

$$\mathcal{E}_1^{(n)} := (I - P_1^{(n)})\mathcal{E}_1, \;\; \forall n \geq 0,$$

and

$$W_n := S_{\mathcal{E}_2}^{*n} W | \mathcal{E}_1^{(n)}.$$

Set

$$\mathcal{E}_1^{(c)} := \overline{\mathcal{E}_1^{(co)}},$$

where

$$\mathcal{E}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_1}^{*j} \mathcal{E}_1^{(j)} \subset \mathcal{K}_{\mathcal{E}_1}, \;\; S_{\mathcal{E}_1}^{(c)} := U_{\mathcal{E}_1} | \mathcal{E}_1^{(c)}.$$

Finally, we define $W_c : \mathcal{E}_1^{(co)} \to \mathcal{E}_2$, by

$$W_c g := W_n g_n,$$

for $g = U_{\mathcal{E}_1}^{*n} g_n, \;\; g_n \in \mathcal{E}_1^{(n)}, \;\; n \geq 0$. Now we can make a similar construction on the spaces $\mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$. In particular, for a causal $Q : \mathcal{F}_1 \to \mathcal{F}_2$, such that $QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q$, we can define $Q_c : \mathcal{F}_1^{(co)} \to \mathcal{F}_2$, where

$$\mathcal{E}_2^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_2}^{*j} \mathcal{E}_2^{(j)}.$$

Clearly both $W_c$ and $Q_c$ extend by continuity to the closure $\mathcal{E}_1^{(c)}$, respectively $\mathcal{F}_1^{(c)} = \overline{\mathcal{F}_1^{(co)}}$. Further,

$$\|W_c\| = \|W\|, \;\; W_c | \mathcal{E}_1 = W, \;\; W_c S_{\mathcal{E}_1}^{(c)} = S_{\mathcal{E}_2} W_c,$$

and

$$\|W - \Theta_2 Q \Theta_1\| = \|(W - \Theta_2 Q \Theta_1)_c\|. \tag{12}$$

We put

$$\mu(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This is a general formulation of the *classical standard control problem.* We also set

$$\mu_c(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : Q \text{ causal}, \;\; QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This is the general formulation of the *causal standard control problem.*

Let $\hat{\Theta}_1 : \mathcal{K}_{\mathcal{E}_1} \to \mathcal{K}_{\mathcal{F}_1}$ denote the extension of the co-isometry $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$, that is uniquely defined by

$$\hat{\Theta}_1 U_{\mathcal{E}_1}^{*n} e_1 = U_{\mathcal{F}_1}^{*n} \Theta_1 e_1, \;\; \forall e_1 \in \mathcal{E}_1.$$

Note that $\hat{\Theta}_1$ is also isometric and $\hat{\Theta}_1 U_{\mathcal{E}_1} = U_{\mathcal{F}_1} \hat{\Theta}_1$. We now have following theorem [69]:

**Theorem 2 (Control Causal Commutant Lifting Theorem)** *Notation as above.*

1. $\mu_c(W, \Theta_1, \Theta_2) = \mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2)$.

2. $Q_{opt}$ *is a causal optimal solution, i.e.,*

$$\mu_c(W, \Theta_1, \Theta_2) = \|W - \Theta_1 Q_{opt} \Theta_2\|$$

*if and only if* $Q_{opt,c}$ *is such that*

$$\mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2) = \|W_c - \Theta_2 Q_{opt,c} \hat{\Theta}_1 | \mathcal{E}_1^{(c)}\|.$$

We summarize how the classical standard problem can be solved using the commutant lifting theorem. Define

$$\begin{aligned}\mathcal{H}_1 &:= \mathcal{E}_1^{(c)} \ominus (\hat{\Theta}_1|\mathcal{E}_1^{(c)})^* \mathcal{E}_1^{(c)},\\ \mathcal{H}_2 &:= \mathcal{E}_2 \ominus \Theta_2 \mathcal{F}_2.\end{aligned}$$

Let $P : \mathcal{E}_2 \to \mathcal{H}_2$ denote orthogonal projection. Then we define the operator

$$\Lambda = \Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2) : \mathcal{H}_1 \to \mathcal{H}_2, \tag{13}$$

by

$$\Lambda h := PW_c h, \quad h \in \mathcal{H}_1. \tag{14}$$

Then using the commutant lifting theorem [193], one may show that

$$\|\Lambda\| = \mu(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2).$$

From the control causal commutant lifting theorem, we conclude the following:

**Corollary 2** *Notation as above. Then*

$$\mu_c(W, \Theta_1, \Theta_2) = \|\Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2)\|.$$

The key point to note is that Theorem 2 and Corollary 2 reduce a causal optimization problem to one involving classical interpolation. This we will exploit for the nonlinear standard problem.

## 2.4 Control Formulation of Nonlinear Standard Problem

We have just described the mathematical framework into which the nonlinear standard problem may be put. We will now give some details about this physical control problem itself.

Let $H_k$ denote the standard Hardy space of $\mathbf{C}^k$-valued functions on the unit disc. As in Section 2.2, we say an analytic input/output operator $\phi : H_k \to H_m$ is *admissible* if it is causal, time-invariant, majorizable, and $\phi(0) = 0$. The space of admissible operators is denoted by $\mathcal{C}_a$. For simplicity, $\mathcal{C}_a$ will denote the set of admissible operators for any $k$ and $m$. We now can define the control problem. Referring to Figure 1, $G$ represents the generalized plant which we assume is modelled by an admissible operator, and $K$ the compensator. Let $\mathcal{F}(G, K)$ denote the input/output operator from $w$ to $z$. Then we want to "minimize" $\mathcal{F}(G, K)$ over all inputs of bounded energy (of fixed given bound) in the sense which will be given below.

For admissible $G$, we can write

$$\mathcal{F}(G, K) = W - P \circ Q \circ R,$$

for admissible operators $W, P, R$ which depend only on the generalized plant $G$. We follow here the convention that for given $\phi \in \mathcal{C}_a$, $\phi_n$ will denote the bounded linear map on the space $(H_k^2)^{\otimes n} \cong H^2(D^n, \mathbf{C}^K)$ (with $K = k^n$) associated to the $n$-linear part of $\phi$ which we

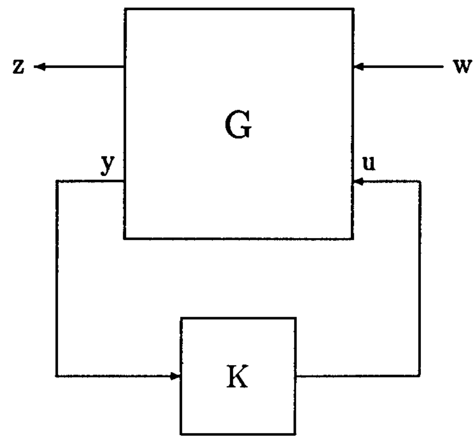


Figure 1: Standard Feedback Configuration

also denote by $\phi_n$. We can define the notions of "optimality" and "amelioration" just as in [67, 79]. We will now indicate an iterative procedure as in Section 2.2 which leads to an optimal design in the standard nonlinear setting derived by solving a sequence of *linear* $H^\infty$ problems.

For the admissible operators $W, P, R$, we suppose that $P_1$ (the linear part of $P$) is an isometry, and that $R_1$ is a co-isometry. Using the above notation, we take $\mathcal{E}_1 := H^2(D^n, \mathbf{C}^{k_1})$, $\mathcal{E}_2 := H^2(D, \mathbf{C}^{k_2})$, $\mathcal{F}_1 := H^2(D^n, \mathbf{C}^{k_3})$, and $\mathcal{F}_2 := H^2(D, \mathbf{C}^{k_4})$. Then one may show

$$(P \circ Q \circ R)_n = \sum_{1 \leq k \leq n} \sum_{i_1 + \cdots + i_k = n} P_k(Q_{i_1}(R^{\otimes i_1}), \ldots, Q_{i_k}(R^{\otimes i_k})).$$

Note that we may in term write that

$$Q_j(R^{\otimes j}) = \sum_{k_1, \ldots, k_j} Q_j(R_{k_1} \otimes \ldots \otimes R_{k_j}).$$

Thus we see that

$$(W - P \circ Q \circ R)_n = \hat{W}_n - P_1 Q_n(R_1^{\otimes n}),$$

where

$$\hat{W}_n = W_n + A(Q_1, \ldots, Q_{n-1}),$$

and $A(Q_1, \ldots, Q_{n-1})$ is an explicitly computable function of $Q_1, \ldots, Q_{n-1}$.

Here then is an iterative procedure to approximate a nonlinear causal compensator. From the classical commutant lifting theorem, we may choose $Q_1$ causal such that

$$\|W_1 - P_1 Q_1 R_1\| = \|\Lambda(W_1, R_1, P_1)\|.$$

Having chosen $Q_1$, we can choose a causal $Q_2$ such that

$$\begin{aligned}&\|W_2 - P_1Q_1R_2 - P_2(Q_1R_1 \otimes Q_1R_1) - P_1Q_2(R_1 \otimes R_1)\| \\ &= \|\Lambda((W_2 - P_1Q_1R_2 - P_2(Q_1R_1 \otimes Q_1R_1)_c, \widehat{R_1 \otimes R_1}|\mathcal{E}_1^{(c)}, P_1)\|.\end{aligned}$$

Inductively, given causal $Q_1, \ldots, Q_{n-1}$, we may choose $Q_n$ causal such that

$$\|\hat{W}_n - P_1Q_n(R_1^{\otimes n})\| = \|\Lambda((\hat{W}_n)_c, \widehat{R_1^{\otimes n}}|\mathcal{E}_1^{(c)}, P_1)\|. \tag{15}$$

As before in each step of the procedure, the new "weight" $\hat{W}_n$ is determined by $W_n, P_1, R_1^{\otimes n}$, and the optimal causal parameters chosen. Thus, we have a series of causal lifting problems each of which may be reduced to a classical dilation problem

As in [79], we may prove that optimal compensating parameters always exist. For the spaces $\mathcal{E}_1 := H^2(D^n, \mathbf{C}^{k_1})$, $\mathcal{E}_2 := H^2(D, \mathbf{C}^{k_2})$, $\mathcal{F}_1 := H^2(D^n, \mathbf{C}^{k_3})$, and $\mathcal{F}_2 := H^2(D, \mathbf{C}^{k_4})$, we are now working on expressing Theorem 2 as a "reduction theorem" (via the Fourier representation), using similar techniques as in [67]. In the next several months, we will be developing the above framework into an implementable design procedure. Accordingly, we are having a student, program part of the procedure symbolically in Mathematica. We will use our skew Toeplitz methods to realize the linear steps in the iterative construction of the nonlinear compensator. We then plan to apply this to some specific nonlinear design examples. For some results in this regard, see [66, 67].

# 3 Robust Control of Distributed Parameter Systems

Under AFOSR-AF/F49620-94-1-00S8DEF, we have continued our research into the control of distributed parameter systems using the frequency based *skew Toeplitz theory* which we have developed for $H^\infty$ optimization. In 1996, our monograph [71] appeared which explains in great detail this methodology in a completely self-contained manner together with a number of illustrative design examples. We will now briefly review some relevant facts on which this approach is based as well as the newer methods which were given in [71].

Skew Toeplitz theory gives a practical way of computing the optimal $H^\infty$–performance and the corresponding optimal and suboptimal controllers for distributed parameter systems. Indeed, via this methodology the standard $H^\infty$–optimization problem can be reduced to a finite dimensional matrix problem even in the infinite dimensional case. Matlab code has been written to implement the whole scheme.

## 3.1 Mixed Sensitivity Optimization

To illustrate our methods, we will sketch how several two block $H^\infty$-minimization problems reduce to the computation of the norm of a certain skew Toeplitz operator, and indicate how this norm may be computed. We begin with some notation. The Hardy spaces $H^2$ and $H^\infty$ are defined on the unit disc in the standard way. We denote

$$RH^\infty := \{\text{rational functions in } H^\infty\}.$$

We consider the standard feedback configuration with the plant

$$P = \frac{G_n}{G_d},$$

where $G_n \in H^\infty$, $G_d \in RH^\infty$. We assume that (i) $G_n = m_n G_{no}$, where $m_n \in H^\infty$ is inner (arbitrary) and $G_{no} \in H^\infty$ is outer, and (ii) $G_n$, $G_d$ have no common zeros in the closed unit disc. We also write $G_d = m_d G_{do}$ where $m_d \in RH^\infty$ is inner and $G_{do} \in RH^\infty$ is outer. Under these assumptions there exist $X \in RH^\infty$ and $Y \in H^\infty$ such that

$$XG_n + YG_d = 1. \tag{16}$$

The set of all controllers which stabilize the plant can now be written in the form

$$C = \frac{X + QG_d}{Y - QG_n}$$

for some $Q \in H^\infty$. Now let $S := (1 + PC)^{-1}$ and note that

$$S = 1 - XG_n - QG_nG_d. \tag{17}$$

In [138, 139], using the commutant lifting theorem, we showed that the computation of

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2(S-1) \end{bmatrix} \right\|$$

where $W_1, W_2 \in RH^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in RH^\infty$ may be reduced to computing the norm of the operator

$$\mathbf{A} := \begin{bmatrix} \mathbf{P}_{H(m_v)} \left( W_0(\mathbf{S}) - \hat{W}_0(\mathbf{S}) m(\mathbf{S}) \right) \\ G_0(\mathbf{S}) \end{bmatrix}, \tag{18}$$

where $\mathbf{S} : H^2 \to H^2$ denotes the unilateral shift, $H(m_v) := H^2 \ominus m_v H^2$ and $\mathbf{P}_{H(m_v)}$ the orthogonal projection onto $H(m_v)$, for $m, m_v$ inner functions associated to the plant and weighting filters, and where $W_0, \hat{W}_0, G_0$ are rational $H^\infty$ functions computed from the plant and weighting filters. This reduction is true for plants with arbitrary outer parts.

In [138, 139], we developed an approach to computing the singular values and vectors of operators of the form (18). We remark here that it is easy to compute the essential norm of the operator $\mathbf{A}$, which will be denoted by $\|\mathbf{A}\|_e$. We can now state the following result:

**Theorem 3** *Let $n$ denote the maximum of the McMillan degrees of the weighting filters $W_1$ and $W_2$, and let $\ell$ denote the number of unstable poles of the plant $P$. Then the singular vectors and values of $\mathbf{A}$ which are $> \|A\|_e$ may be derived from an explicitly computable system of $3n + 2\ell$ linear equations (the "singular system").*

In [139], the singular system of equations is explicitly written down. Again the number of equations only depends on the McMillan degrees of the weighting filters, and the number of right half plane poles of the plant. The computation of the *maximal* singular value and the associated singular vectors of $\mathbf{A}$, then allows us to find the optimal performance $\mu$ of our original control problem and the corresponding optimal compensator. (Similar results have been worked out in Flamm-Yang [61]. The mixed sensitivity optimization problem for stable distributed plants was first solved in Zames-Mitter [194].)

Several benchmark examples have been worked out using this methodology including problems in flight control [53], and the control of flexible beams [115].

### 3.2 Young Operator

In most typical frequency domain techniques, the standard problem needs to be put into a certain *four block* form (this was illustrated above in the two block case; see also [85]). This reduction involves a number of factorizations, some of which are quite difficult and time-consuming to perform. Indeed, a major advantage of state space methods [52] is that these factorizations may be avoided. On the other hand, one of the key disadvantages of these state space methods is that their practical applicability to distributed systems seems to be very difficult. (On an infinite dimensional state space one gets infinite dimensional, i.e., operator-valued Riccati equations. See [48].)

In our monograph [71], these issues have been treated by using the so-called *Young operator* [193]. We would like to sketch this approach very briefly now.

Recall that via the Youla parametrization, the standard problem may be formulated as finding

$$\inf_{Q\in H^\infty} \|T_1 - T_2QT_3\|_\infty,$$

where $T_1, T_2, T_3, Q$ are matrix-valued $H^\infty$ functions of compatible sizes. More precisely, let $\mathcal{E}_i, \mathcal{F}_i$ denote finite dimensional complex Hilbert spaces for $i = 1, 2$. Then we take $T_1 \in H^\infty(\mathcal{E}_1, \mathcal{E}_2)$, $T_2 \in H^\infty(\mathcal{F}_2, \mathcal{E}_2)$, $T_3 \in H^\infty(\mathcal{E}_1, \mathcal{F}_1)$, and the parameter $Q \in H^\infty(\mathcal{F}_1, \mathcal{F}_2)$. (In general, for two complex separable Hilbert spaces $\mathcal{E}, \mathcal{F}$ we define $H^\infty(\mathcal{E}, \mathcal{F})$ to be the space of all uniformly bounded analytic functions in the open unit disc, whose values are operators from $\mathcal{E}$ to $\mathcal{F}$.)

Let

$$\begin{aligned}
\mathcal{H}_1 &:= T_3^{-1}H^2(\mathcal{F}_1)\\
&:= \{f \in L^2(\mathcal{E}_1) : T_3 f \in H^2(\mathcal{F}_1)\},\\
\mathcal{H}_2 &:= L^2(\mathcal{E}_2) \ominus (T_2H^2(\mathcal{F}_2))^-.
\end{aligned}$$

Define the operator (see [193], [85], [63]) $\Lambda : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$\Lambda f := P_{\mathcal{H}_2} T_1 f,$$

where $P_{\mathcal{H}_2}$ denotes orthogonal projection on $\mathcal{H}_2$. Then using the commutant lifting theorem, one may show that

$$\inf_{Q\in H^\infty} \|T_1 - T_2QT_3\|_\infty = \|\Lambda\|.$$

In [71], we have parametrized the optimal compensators directly from the Young operator along the same lines that we followed for the four block problem ([72, 73, 140]), completely avoiding the reduction of the standard problem to its four block form.

## 4 Structured Perturbations

Under AFOSR support, we have been carrying out research in the analysis and synthesis of robust feedback control in the presence of structured uncertainty models using the structured singular value methodology pioneered by Doyle and Safonov [50, 150]. During AFOSR-AF/F49620-94-1-00S8DEF, we have formulated a novel *ampliation* or *lifting* approach for the study of robustness with respect to a number of key perturbation classes. Details about our work on the structured singular value may be found in [23, 21, 22, 29, 24].

### 4.0.1 Lifting of Structured Perturbations

In this section, we will discuss some of the key aspects of the lifting method we developed to analyze the structured singular value. In order to do this, we will first need to make some preliminary definitions.

Let $A$ be a linear operator on a Hilbert space $\mathcal{E}$, and let $\Delta$ be an algebra of operators on $\mathcal{E}$. The *structured singular value* of $A$ (relative to $\Delta$) is defined by

$$\mu_\Delta(A) := 1/\inf\{\|X\| : X \in \Delta,\ -1 \in \sigma(AX)\}\,.$$

(This quantity was defined in [50, 150] under a more restrictive context.) In robust system analysis, $\mu_\Delta(A)$ gives a measure of robust stability with respect to certain perturbation measures. Unfortunately, $\mu_\Delta(A)$ is very difficult to calculate, and so in applications an upper bound is employed. This upper bound is given by

$$\widehat{\mu}_\Delta(A) := \inf\{\|XAX^{-1}\| : X \in \Delta', X \text{ invertible}\},$$

where $\Delta'$ is the commutant of the algebra $\Delta$.

The basic result underlying the lifting approach is that $\widehat{\mu}_\Delta(A)$ can be shown to be equal to the structured singular value of an operator on a bigger Hilbert space. (In [27] this was done for finite dimensional Hilbert spaces, and then in [24] this was extended to the infinite dimensional case. For another proof of this type of lifting result in finite dimensions, see [56].) The problem with this work is that the size of the ampliation necessary to get $\widehat{\mu}_\Delta(A)$ equal to a structured singular value, is equal to the dimension of the underlying Hilbert space. Hence in the infinite dimensional case we needed an infinite lifting. (Note that in this context, we will be using the terms "ampliation" and "lifting" interchangeably.) A new result that came out of the work in [27] was that for the first time the upper bound $\widehat{\mu}$ was rigorously shown to be *continuous.*

Now in the recent paper [23], we have shown that in fact, one can always get by with a finite lifting. For the block diagonal algebras of interest in robust control, the lifting only depends on the number of blocks of the given perturbation structure.

We will denote by $\mathcal{L}(\mathcal{E})$ the algebra of all bounded linear operators on the (complex, separable) Hilbert space $\mathcal{E}$. Fix an operator $A \in \mathcal{L}(\mathcal{E})$ and a subalgebra $\Delta \subset \mathcal{L}(\mathcal{E})$. Observe that $\Delta \subset \Delta''$ and $\Delta''' = (\Delta'')' = \Delta'$ so that we have the inequalities

$$\mu_\Delta(A) \leq \mu_{\Delta''}(A),\quad \widehat{\mu}_\Delta(A) = \widehat{\mu}_{\Delta''}(A)\,.$$

In our study we will need further singular values which we now define. For $n \in \{1, 2, \ldots, \infty\}$ we denote by $\mathcal{E}^{(n)}$ the orthogonal sum of $n$ copies of $\mathcal{E}$, and by $T^{(n)}$ the orthogonal sum of $n$ copies of $T \in \mathcal{L}(\mathcal{E})$. Operators on $\mathcal{E}^{(n)}$ can be represented as $n \times n$ matrices of operators in $\mathcal{L}(\mathcal{E})$, and $T^{(n)}$ is represented by a diagonal matrix, with diagonal entries equal to $T$.

Denote by $\Delta_n$ the algebra of all operators on $\mathcal{E}^{(n)}$ whose matrix entries belong to $\Delta$, and observe that $(\Delta_n)'' = (\Delta'')_n$, and $(\Delta_n)' = (\Delta')^{(n)} = \{T^{(n)} : T \in \Delta'\}$. Therefore we will denote these algebras by $\Delta_n''$ and $\Delta_n'$, respectively.

We can now formulate our lifting result from [23], relating $\mu_\Delta(A)$ and $\widehat{\mu}_\Delta(A)$. The proof of the theorem makes use of some of our operator-theoretic work on the relative numerical range, and the continuity of the spectrum on closed similarity orbits in [21, 22].

**Theorem 4** *Assume that $\Delta'$ is a $*$-algebra of finite dimension $n$. Then*

$$\widehat{\mu}_\Delta(A) = \mu_{\Delta''_n}(A^{(n)})$$

*for every $A \in \mathcal{L}(\mathcal{E})$.*

In the cases of interest in control, $\Delta'' = \Delta$, and so one has from Theorem 4 that $\mu_{\Delta_n}(A) = \hat{\mu}_\Delta(A)$.

### 4.0.2 Nonconservative Measures of Robustness

We will discuss some of the conditions from [23] when $\mu = \hat{\mu}$ without any need for lifting. In such cases, $\hat{\mu}$ gives a nonconservative measure of robustness relative to the given perturbation structure. For constant matrices, the most famous result of thsi kind is due to Doyle [50], who showed that no lifting is necessary for perturbation structures with three or fewer blocks.

First of all, call *critical* any $A_0 \in \overline{\mathcal{O}_{\Delta'}(A)}$ satisfying

$$\limsup_{\epsilon \downarrow 0} \|(I - \epsilon X) A_0 (I - \epsilon X)^{-1}\| \geq \|A_0\|, \ \forall X \in \Delta'.$$

Then we have

**Lemma 1** *If $A_0$ is a critical operator in $\overline{\mathcal{O}_{\Delta'}(A)}$, then it enjoys the following property $(\mathcal{O})$:*

$$0 \in W_Q(\|A_0\|^2 X - A_0^* X A_0), \ X \in \Delta',$$

*where $Q = \|A_0\|^2 I - A_0^* A_0$.*

The next lemma is the key step in adapting the proof of Theorem 4 in order to show that $\mu_\Delta(A) = \widehat{\mu}_\Delta(A)$ in several interesting cases.

**Lemma 2** *Let $A_0$ be an operator on $\mathcal{E}$ which satisfies the essential version of property $(\mathcal{O})$, property $(\mathcal{O}^0)$, namely*

$$0 \in W_Q^0(\|A_0\|^2 X - A_0^* X A_0), \ X \in \Delta',$$

*where $Q = \|A_0\|^2 I - A_0^* A_0$. Then there exists a sequence $\{h_k\}_{k=1}^{\infty} \subset \mathcal{E}$, $\|h_k\| = 1$, $k = 1, 2, \ldots$, such that*

$$Qh_k \to 0 \text{ strongly and } \langle (\|A_0\|^2 X - A_0^* X A_0) h_k, h_k \rangle \to 0,$$

*for all $X \in \Delta'$.*

We will also need the following result:

**Theorem 5** *If there exists a critical operator $A_0$ satisfying property $\mathcal{O}^0$ in the closed $\Delta'$-orbit of $A$, then*

$$\mu_{\Delta''}(A) = \hat{\mu}_\Delta(A).$$

**Remark.** Under the hypotheses of Theorem 5, when $\Delta'' = \Delta$ (which happens in all cases of interest in control), we have that

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

Let $L(\Delta' A \Delta')$ denote the linear space generated by

$$\Delta' A \Delta' = \{XAY : X, Y \in \Delta'\}.$$

Obviously $L(\Delta' A \Delta')$ is finite dimensional, and therefore closed. Hence $\overline{\mathcal{O}_{\Delta'}(A)} \subset L(\Delta' A \Delta')$.

**Corollary 3** *If for every $B \in L(\Delta' A \Delta')$, $B \neq 0$, the norm of $B$ is not attained (that is, there is no $h \in \mathcal{H}$ such that $\|Bh\| = \|B\|\|h\| \neq 0$), then*

$$\mu_{\Delta''}(A) = \hat{\mu}_\Delta(A).$$

We can now use our lifting methodology in order to derive an elegant result of Magretski [120], and Shamma [162] on the structured singular value of a Toeplitz operator, i.e., a linear time invariant system. See also [56, 102, 103] for related work in this area.

Let $\mathcal{E} = H^2(\mathbf{C}^n)$ and let $A$ denote the multiplication (analytic Toeplitz) operator on $\mathcal{E}$ defined by

$$(Ah)(z) = A(z)h(z), \ \ |z| < 1, \ h \in \mathcal{E},$$

where

$$A(z) = [a_{jk}]_{j,k=1}^n, \ \ |z| < 1,$$

has $H^\infty$ entries. Let $\Delta'$ be any $*$-subalgebra of $\mathcal{L}(\mathbf{C}^n)$, the elements of which are regarded as multiplication operators on $\mathcal{E}$. Note that in this case, $\Delta'' = \Delta$ is the algebra generated by operators of the form

$$(Bh)(z) = B(z)h(z), \ \ |z| < 1, \ h \in \mathcal{E}$$

with $B(z)X = XB(z)$, $|z| < 1$, $X \in \Delta'$ as well as of the form

$$B \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} Yh_1 \\ Yh_2 \\ \vdots \\ Yh_n \end{bmatrix},$$

with $Y \in \mathcal{L}(H^2(\mathbf{C}))$ arbitrary. We can now state:

**Lemma 3** *Let $A_0$ be an analytic Toeplitz operator. Then if $A_0$ has property $(\mathcal{O})$, it also has property $(\mathcal{O}^0)$.*

**Corollary 4 ([120, 162])** *For $A$ and $\Delta'$ as above, we have that*

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

### 4.1 General Input-Output Operators

It is of interest to put some the general results stated above into a system-theoretic framework. Let $\ell_+^2$ be the space of square summable one-sided sequences in $\mathbf{C}$, let $\mathcal{C}$ denote the set of all bounded linear operators on $\ell_+^2$. Further, let $A : \ell_+^2(\mathbf{C}^n) \to \ell_+^2(\mathbf{C}^n)$ be an arbitrary bounded linear operator. Thus $A$ defines a (possibly) time-varying system. (Here $\ell_+^2(\mathbf{C}^n)$ the space of of square summable sequences in $\mathbf{C}^n$, i.e., the space of finite energy vector-valued signals with $n$ components.) Then we want to interpret $\widehat{\mu}_\Delta(A)$ as a structured singular value on an extended space with an enhanced perturbation structure. Note $\mathcal{E}$ in this case is the Hilbert space $\ell_+^2(\mathbf{C}^n)$.

Define the algebra of perturbations

$$\Delta := \{ \begin{bmatrix} \delta_1 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \delta_n \end{bmatrix} : \delta_i \in \mathcal{C}, i = 1, \dots, n \}.$$

Then the commutant of $\Delta$ is the finite dimensional $C^*$-algebra,

$$\Delta' := \{ \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} : d_i \in \mathbf{C}, i = 1, \dots, n \}.$$

Note that a constant $d \in \mathbf{C}$ defines an operator on $\ell_+^2$ via multiplication.

We now have the following interpretation of $\widehat{\mu}_\Delta(A)$. We lift $A$ to $A^{(n)} : \mathcal{E}^{(n)} \to \mathcal{E}^{(n)}$. Then

$$(\Delta_n)'' \cong \{ \begin{bmatrix} \tilde{\Delta}_1 & 0 & 0 & 0 \\ 0 & \tilde{\Delta}_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\Delta}_n \end{bmatrix} : \tilde{\Delta}_j \in \Delta \}.$$

$(\Delta_n)''$ is a space of time-varying perturbations, and we have from Theorem 4 that

$$\widehat{\mu}_\Delta(A) = \mu_{\Delta_n''}(A^{(n)}).$$

This is true for *arbitrary time-varying systems A*. In case $A$ is Toeplitz, i.e., the system is time-invariant, then as we have seen,

$$\widehat{\mu}_\Delta(A) = \mu_\Delta(A).$$

## 5 Visual Tracking

As a key application of the robust control techniques we have been developing over the past few years, we have been considering the problem of visual tracking. This has led us to consider relevant problems in active vision and image processing which we have been treating using certain invariant geometric flows which we will now very briefly describe. It is important to note that these equations themselves are very much motivated by ideas in optimal control; see [110] and the references therein.

One of the key techniques in active vision and tracking is that of *deformable contours* or *snakes*. These are autonomous processes which ultilize image coherence in order to track features of interest over time. After some preliminary remarks about curve evolution, we will discuss our snake model from [104].

## 5.1 Planar Curve Evolution

The theory of planar curve evolution has been considered in a variety of fields such as differential geometry [87, 90, 130, 131, 152], theory of parabolic equations [6], numerical analysis [135, 158], computer vision [58, 59, 107, 108, 105, 144, 148, 153, 109, 180], viscosity solutions [88, 55, 168], phase transitions [93], and image processing [3, 148, 149, 156].

Formally, let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ denote a family of closed embedded curves, where $t$ parametrizes the family, and $p$ parametrizes each curve. We assume that this family evolves according to the following equation:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{cases} \tag{19}$$

where $\vec{\mathcal{N}}$ is the inward Euclidean unit normal, $\vec{\mathcal{T}}$ is the unit tangent, and $\alpha$ and $\beta$ are the tangent and normal components of the evolution velocity $\vec{\nu}$, respectively. One can show that $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}} \quad \text{and} \quad \hat{\mathcal{C}}_t = \bar{\beta} \mathcal{N},$$

respectively. (Here $\mathrm{Img}[\cdot]$ denotes the image of the given parametrized curve in $\mathbf{R}^2$.) Thus the tangential component affects only the parametrization, and not $\mathrm{Img}[\cdot]$ (which is independent of the parametrization by definition). Therefore, assuming that the normal component $\beta$ of $\vec{\nu}$ (the curve evolution velocity) in (19) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \vec{\mathcal{N}}, \tag{20}$$

where $\beta = \vec{\nu} \cdot \vec{\mathcal{N}}$, i.e., the projection of the velocity vector on the normal direction.

The evolution (20) was studied by different researchers for different functions $\beta$. One of the most important of such flows is derived when a planar curve deforms in the direction of the Euclidean normal, with speed equal to the Euclidean curvature, i.e., when $\beta = \kappa$, for $\kappa$ is the Euclidean curvature:

$$\frac{\partial C}{\partial t} = \kappa \vec{\mathcal{N}}. \tag{21}$$

The flow given by (21) is called the *Euclidean shortening flow*, since the curve perimeter shrinks as fast as possible using only local information [90]. Gage and Hamilton [87] proved that a simple and smooth convex curve evolving according to (21), converges to a round point. Grayson [90] proved that an embedded planar curve converges to a simple convex one when evolving according to (21), and so any embedded curve in the plane converges to a round point via (21). For other results related to the Euclidean shortening flow, see [5, 6, 87, 90, 91, 109, 184].

Next note that if $v$ denotes the Euclidean arc length, then [169]

$$\kappa\vec{\mathcal{N}} = \frac{\partial^2 \mathcal{C}}{\partial v^2}.$$

Therefore, equation (21) can be written as

$$\mathcal{C}_t = \mathcal{C}_{vv}. \tag{22}$$

Equation (22) is not linear, since $v$ is a function of time (the arc length gives a time dependent parametrization). Equation (22) is also called the *geometric heat equation.*

Recently, we introduced a new curve evolution equation, the *affine geometric heat flow* [152, 153]:

$$\left\{ \begin{array}{l} \dfrac{\partial \mathcal{C}(p,t)}{\partial t} = \dfrac{\partial^2 \mathcal{C}(p,t)}{\partial s^2}, \\ \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{array} \right. \tag{23}$$

where

$$s(p) = \int_0^p [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3} dp, \tag{24}$$

is the *affine arc length* ($[\mathcal{C}_s, \mathcal{C}_{ss}] \equiv 1$), i.e., the simplest affine invariant parametrization [34]. $\mathcal{C}_{ss}$ is called the *affine normal* [92]. In contrast with the Euclidean version, the affine arc length is based on area, and not on length (note that $[\mathcal{C}_p, \mathcal{C}_{pp}]$ is the oriented area between $\mathcal{C}_p$ and $\mathcal{C}_{pp}$). This is clear since length is not affine invariant, whereas area is the simplest geometric affine invariant. This evolution is the affine analogue of equation (21), and admits affine invariant solutions, i.e., if a family $\mathcal{C}(p,t)$ of curves is a solution of (23), the family obtained from it via unimodular affine mappings, is a solution as well. We have shown that any simple and smooth convex curve evolving according to (23), converges to an ellipse [152]. Since the affine normal $\mathcal{C}_{ss}$ exists just for non-inflection points, we formulated the natural extension of the flow (25) for non-convex initial curves in [153, 155]:

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = \left\{ \begin{array}{ll} 0, & p \text{ an inflection point,} \\ \mathcal{C}_{ss}(p,t), & \text{otherwise,} \end{array} \right. \tag{25}$$

together with the initial condition $\mathcal{C}(p,0) = \mathcal{C}_0(p)$. The flow (25) defines a geometric, affine invariant, multiscale representation of planar shapes. Indeed, in [153], we proved that this flow satisfies all the required properties of (morphological) scale-space such as causality and order preservation. For this flow, we proved (see also [7]) that the curve first becomes convex, as in the Euclidean case, and after that it converges into an ellipse according to the results of [152]. An equivalent model was independently derived in [4] from a an elegant axiomatic point of view. See [153] for a number of explicit examples of planar shape smoothing.

We should also add that in [155], we give a general method for writing down invariant flows with respect to any Lie group action on $\mathbb{R}^2$. The idea is to consider the evolution given by $\mathcal{C}_t = \mathcal{C}_{rr}$ where $r$ is the group invariant arc length. This was formalized, together with uniqueness results, in [130], and extended to hypersurfaces in [131]. Results for the projective group were recently reported in [57, 58].

In great generality, the following result gives the form of an invariant flow for a hypersurface in an arbitrary number of space dimensions $p$:

**Theorem 6** *Let $G$ be a transformation group, and let $L\,dx = Ldx^1 \wedge \ldots \wedge dx^p$ be a $G$-invariant Lagrangian with nonzero variational (Euler-Lagrange) derivative $E(L)$. Then every $G$–invariant evolution equation has the form*

$$u_t = \frac{L}{E(L)} I, \tag{26}$$

*where $I$ is a arbitrary differential invariant of $G$.*

The Euclidean group is a special case of a *volume-preserving* transformation group $G$. This means that it leaves the $(p+1)$–form $dx \wedge du = dx^1 \wedge \cdots \wedge dx^p \wedge du$ invariant.

**Proposition 1** *Suppose $G$ is a connected transformation group, and $L\,dx$ a $G$–invariant $p$-form such that $E(L) \neq 0$. Then $E(L)$ is a differential invariant if and only if $G$ is volume–preserving.*

**Corollary 5** *Let $G$ be a connected volume preserving transformation group. Then, up to constant multiple, the $G$–invariant flow of lowest order has the form*

$$u_t = L, \tag{27}$$

*where $\omega = L\,dx^1 \wedge \ldots \wedge dx^p$ is the invariant $p$-form of minimal order such that $E(L) \neq 0$.*

These results allow one to write down the simplest invariant flows in any dimension with respect to any transformation group.

## 5.2 Euclidean image processing

In this section, we review a number of algorithms for image processing which are related to the Euclidean shortening flow (21). The algorithms were developed in continuous spaces, and tested on digital computers by very accurate and stable numerical implementations. These numerical implementations were developed by the various authors for their specific algorithm. Only the basic concepts of the algorithms are given here. For more details, see the appropriate references given below.

In general, $\Phi_0 : \mathbb{R}^2 \to \mathbb{R}^+$ represents a gray-level image, where $\Phi_0(x, y)$ is the gray-level value. The algorithms that we describe are based on the formulation of partial differential equations, with $\Phi_0$ as initial condition. The solution $\Phi(x, y, t)$ of the differential equation gives the processed image.

Osher and Rudin [134] formulated a method for image enhancement based on shock filters. In this case, the image $\Phi(x, y, t)$ evolves according to

$$\Phi_t = - \parallel \nabla\Phi \parallel F(\mathcal{L}(\Phi)), \tag{28}$$

where the function $F(u)$ satisfies certain technical conditions (given explicitly in [134]), and $\mathcal{L}$ is a second order (generally) nonlinear elliptic operator. An image evolving according to (28) develops shocks where $\mathcal{L} = 0$. One of the goals of this method is to get as close as possible to the inverse heat equation [134]. The algorithm was tested on images artificially degraded by the classical diffusion equation, and very good "inverse" diffusions were obtained.

Rudin *et al.* [149] presented an algorithm for noise removal, based on the minimization of the total first variation of $\Phi$, i.e.,

$$\int_{Image} \| \nabla\Phi \| \, dxdy. \tag{29}$$

The minimization is performed under certain constraints and boundary conditions (zero flow on the boundary). The constraints they employed are zero mean value and given variance $\sigma^2$ of the noise, but other constraints clearly can be considered as well. More precisely, if the noise is additive, the constraints are given by

$$\int_{Image} \Phi \, dxdy = \int_{Image} \Phi_0 \, dxdy, \qquad \int_{Image} (\Phi - \Phi_0)^2 \, dxdy = 2\sigma^2. \tag{30}$$

Note that $\kappa$, the Euclidean curvature of the level-sets, is exactly the Euler-Lagrange derivative of this total variation. Then, for the minimization of (29) with the constraints given by (30), the following gradient-descent flow is obtained:

$$\Phi_t = \kappa - \lambda(\Phi - \Phi_0), \tag{31}$$

and the solution to the variational problem is given when $\Phi$ achieves steady state. The level-sets curvature $\kappa$ may be computed via standard formulas for curves defined by implicit functions. The quantity $\sigma$ is used in the computation of $\lambda$. The authors computed $\lambda$ from the steady state solution ($\Phi_t = 0$). Rudin and co-workers have in general shown these total variational methods to be a powerful tool for a number of image processing problems.

Alvarez *et al.* [3] described an algorithm for image selective smoothing and edge detection. In this case, the image evolves according to

$$\Phi_t = \phi(\| G * \nabla\Phi \|) \, \| \nabla\Phi \| \, \mathrm{div}\left(\frac{\nabla\Phi}{\| \nabla\Phi \|}\right), \tag{32}$$

where $G$ is a smoothing kernel (for example, a Gaussian), and $\phi(w)$ is a nonincreasing function which tends to zero as $w \to \infty$. Note that

$$\| \nabla\Phi \| \, \mathrm{div}\left(\frac{\nabla\Phi}{\| \nabla\Phi \|}\right)$$

is equal to $\Phi_{\xi\xi}$, where $\xi$ is the direction normal to $\nabla\Phi$. Thus it diffuses $\Phi$ in the direction orthogonal to the gradient $\nabla\Phi$, and does not diffuse in the direction of $\nabla\Phi$. This means that the image is being smoothed on both sides of the edge, with minimal smoothing at the edge itself. Note that the evolution

$$\Phi_t = \| \nabla\Phi \| \, \mathrm{div}\left(\frac{\nabla\Phi}{\| \nabla\Phi \|}\right) = \kappa \, \| \nabla\Phi \| \tag{33}$$

is such that the level-sets of $\Phi$ move according to the Euclidean shortening flow given by equation (21) [3, 135]. Finally, the term

$$\phi(\| G * \nabla\Phi \|)$$

is used for the enhancement of the edges. If $\| \nabla\Phi \|$ is "small", then the diffusion is strong. If $\| \nabla\Phi \|$ is "large" at a certain point $(x, y)$, this point is considered as an edge point, and the diffusion is weak.

In summary, equation (32) gives anisotropic or edge preserving (and enhancement) diffusion, extending the ideas proposed by Perona and Malik [145]. The equation looks like the level-sets of $\Phi$ are moving according to (21), with the velocity value "altered" by the function $\phi(w)$.

## 5.3 Affine image processing

As we just saw, there is a close relationship between the curve evolution flow (21), and recently developed image enhancement and smoothing algorithms (see equation (33)). In this section, we consider the use of the affine shortening flow (25) for a similar purpose.

It is well-known in the theory of curve evolution, that if the velocity $\mathcal{V} = \mathcal{C}_t$ of the evolution is a geometric function of the curve, then the geometric behavior of the curve is affected only by the normal component of this velocity, i.e., by $< \mathcal{V}, \mathcal{N} >$ . Since the tangential velocity component only affects the parametrization of the evolving curve, instead of looking at (25), we can consider a Euclidean-type formulation of it. In [152], we proved that the normal component of $\mathcal{C}_{ss}$ is equal to $\kappa^{1/3}\mathcal{N}$. Since $\kappa = 0$ at inflection points, and inflection points are affine invariant, we obtain that the evolution given by

$$\mathcal{C}_t = \kappa^{1/3}\mathcal{N}, \tag{34}$$

is geometrically equivalent to the affine shortening flow (25). Then the trace (or image) of the solution to (34) is affine invariant.

It is important to note that the affine invariant property of (34) was also pointed out by Alvarez *et al.* [4], based on a completely different approach. They proved that this flow is unique under certain conditions (uniqueness is obtained also from the results in [130]). Moreover, they give an extensive characterization of PDE based multiscale analysis, and remarked that the flows (21) and (34) are well-defined also for non-smooth curves, using the theory of viscosity solutions [47]. This is also true for the corresponding image flows, where the level-sets deform according to the geometric heat flows [88, 55] (see below). The existence of the Euclidean and affine geometric heat flows for Lipschitz functions is obtained from the results in [6, 7] as well. These results on extensions of the flows to non-smooth data are fundamental for all image processing applications, since images are non-smooth. The results prove that the flows are mathematically correct (well-defined and stable).

We proceed now to show how the affine curve flow (34) can be extended to process images. The technique of embedding a curve as the zero level set in the graph of a surface, and looking at the evolution of the level-sets was developed by Osher and Sethian [135], and is frequently used for the digital implementation of curve evolution flows. (See also the recent book [160] for the numerous uses and a complete set of references about this important method.) Let us consider now what occurs when the level-sets of $\Phi$ evolve according to (34). It is easy to show that the corresponding evolution equation for $\Phi$ is given by

$$\Phi_t = \kappa^{1/3} \| \nabla\Phi \| = (\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy})^{1/3}. \tag{35}$$

This equation was used in [153] for the implementation of the novel affine invariant scale-space for planar curves. It was also used in [4, 156] for image denoising. Note again that, based on the theory of viscosity solutions, equations (33) and (35) can be analyzed even if the level-sets (or the image itself), are non-smooth; see [4, 88, 47, 55]. This flow is well-posed and stable. The maximum principle holds, meaning that the flow is smoothing the image.

If we compare (33) with (35), we observe that *the denominator is eliminated.* This not only makes the evolution (34) affine invariant [4, 153], it also makes the numerical implementation more stable [135]. The $1/3$ power is the *unique one* which eliminates this denominator. This is of course an important advantage of the affine flow over the Euclidean one in image processing. Moreover, for high curvatures, $\kappa^{1/3}$ is smaller than $\kappa$, which further prevents sharp regions from moving. Finally, since the symmetry group (the affine group) of (35) is much larger than that of equation (33) (the Euclidean heat flow), more structure is preserved up to a higher degree of smoothing. This phenomenon has been observed, for example, in Niessen *et al.* [128] in which elliptical structures of MRI images of the brain were preserved up to a very high degree of smoothing using equation (35). One can combine this smoother with an affine invariant edge map as in [132] to perform affine invariant edge preserving anisotropic diffusion as in the Euclidean case.

## 5.4 Geometric Gradient Active Contours

In the past few years, a number of approaches have been proposed for the problem of *snakes* or *active contours.* The underlying principle in these works is based upon the utilization of deformable contours which conform to various object shapes and motions. Snakes have been used for edge and curve detection, segmentation, shape modelling, and especially for *visual tracking.* The recent book by Blake and Yuille [33] contains an excellent collection of papers on the theory and practice of deformable contours together with a large list of references.

In [104], we have proposed a novel deformable contour model which was motivated by the elegant approach in of Caselles *et al.* [41] and Malladi *et al.* [122]. (A similar approach was independently formulated in [42, 161].) In these works, a level set curve evolution method is presented to solve the problem. Our idea is simply to note that both these approaches are based on Euclidean curve shortening evolution which in turn defines the gradient direction in which the Euclidean perimeter is shrinking as fast as possible. Our snake model is then based on the geometric intuition of multiplying the Euclidean arc length by a function tailored to the features of interest to which we want to flow, and then writing down the resulting *gradient evolution equations.* Mathematically, this amounts to defining a new Riemannian metric in the plane tailored to the given image, and then computing the corresponding gradient flow. This leads to some intriguing new snake models which efficiently attract the given active contour to the features of interest (which basically lie at the bottom of a *potential well*). The method also allows us to naturally write down 3-D active surface models as well.

Let us briefly review some of the details from [104]. First of all, Caselles *et al.* [41] and Malladi *et al.* [122] propose a snake model based on the level set formulation of the Euclidean curve shortening equation. More precisely, their model is

$$\frac{\partial \Psi}{\partial t} = \phi(x,y)\|\nabla\Psi\|(\mathrm{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nu). \tag{36}$$

Here the function $\phi(x, y)$ depends on the given image and is used as a "stopping term." For example, the term $\phi(x, y)$ may chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. One may take [41, 122]

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^2}, \tag{37}$$

where $I$ is the (grey-scale) image and $G_\sigma$ is a Gaussian (smoothing filter) filter. The function $\Psi(x, y, t)$ evolves in (36) according to the associated level set flow for planar curve evolution in the normal direction with speed a function of curvature which was introduced in the work of Osher-Sethian [135, 159].

It is important to note that the Euclidean curve shortening part of this evolution, namely

$$\frac{\partial \Psi}{\partial t} = \|\nabla \Psi\| \mathrm{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) \tag{38}$$

is derived as a gradient flow for shrinking the perimeter as quickly as possible. As is explained in [41, 122], the constant *inflation term* $\nu$ is added in (36) in order to keep the evolution moving in the proper direction. Note that we are taking $\Psi$ to be negative in the interior and positive in the exterior of the zero level set.

We would like to modify the model (36) in a manner suggested by Euclidean curve shortening. Namely, we will change the ordinary Euclidean arc length function along a curve $C = (x(p), y(p))^T$ with parameter $p$ given by

$$ds = (x_p^2 + y_p^2)^{1/2} dp,$$

to

$$ds_\phi = \phi (x_p^2 + y_p^2)^{1/2} dp,$$

where $\phi(x, y)$ is a positive differentiable function. Then we want to compute the corresponding gradient flow for shortening length relative to the new metric $ds_\phi$.

Accordingly set

$$L_\phi(t) := \int_0^1 \|\frac{\partial C}{\partial p}\| \phi dp.$$

Let

$$\vec{\mathcal{T}} := \frac{\partial C}{\partial p} / \|\frac{\partial C}{\partial p}\| ,$$

denote the unit tangent. Then taking the first variation of the modified length function $L_\phi$, and using integration by parts (see [104]), we get that

$$L'_\phi(t) = -\int_0^{L_\phi(t)} \langle \frac{\partial C}{\partial t}, \phi\kappa\vec{\mathcal{N}} - (\nabla\phi \cdot \vec{\mathcal{N}})\vec{\mathcal{N}} \rangle ds$$

which means that the direction in which the $L_\phi$ perimeter is shrinking as fast as possible is given by

$$\frac{\partial C}{\partial t} = (\phi\kappa - (\nabla\phi \cdot \vec{\mathcal{N}}))\vec{\mathcal{N}}. \tag{39}$$

This is precisely the gradient flow corresponding to the miminization of the length functional $L_\phi$. The level set version of this is

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \mathrm{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) + \nabla\phi \cdot \nabla\Psi. \tag{40}$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the potential well described by the gradient flow (40). As in [41, 122], we may also add a constant inflation term, and so derive a modified model of (36) given by

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| (\operatorname{div}(\frac{\nabla \Psi}{\|\nabla \Psi\|}) + \nu) + \nabla \phi \cdot \nabla \Psi. \tag{41}$$

Notice that for $\phi$ as in (37), $\nabla \phi$ will look like a doublet near an edge. Of course, one may choose other candidates for $\phi$ in order to pick out other features.

We have implemented this snake model based on the level set type algorithms in [135, 159] and [122]. We are also studying an affine invariant snake model for tracking based on our work in [132]. (The evolution itself works using a level set model of $\kappa^{1/3}\vec{\mathcal{N}}$ as discussed in the previous section.)

All of our methods are extendable to 3D pictures. Indeed, we have developed affine invariant volumetric smoothers in [131]. We also have 3D active contour evolvers for image segmentation, shape modelling, and edge detection based on both snakes (inward deformations) and bubbles (outward deformations) based on our work in [104, 192].

Finally, we should note that we have developed algortihms for optical flow and stereo disparity under AFOSR-AF/F49620-94-1-00S8DEF (see [112, 113]) which we will be exploiting for visual tracking in our new AFOSR sponsored research contract.

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# 7 Papers of Allen Tannenbaum and Collaborators under AFOSR-AF/F49620-94-1-00S8DEF

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2. "Causality in commutant lifting theory" (with C. Foias), *Journal of Functonal Analysis* **118** (1993), 407–441.

3. "On combined $H^{\infty}$–$H^{2}$ suboptimal interpolants" (with C. Foias and A. Frazho), *Linear Algebra and Its Applications* **203-204** (1994), pp. 443–469.

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6. "Nonlinear $H^{\infty}$ optimization: a causal power series approach," *SIAM J. Control and Optimization* **33** (1995), pp. 185–207.

7. "Shapes, shocks, and deformations, I: the components of shape and the reaction-diffusion space" (with B. Kimia and S. Zucker), *International Journal of Computer Vision* **15** (1995), 189-224.

8. "Affine invariant scale-space (with G. Sapiro), *International Journal of Computer Vision* **11** (1993), 25-44.

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13. "Stability margin optimization via interpolation and conformal mappings" (with J. Cockburn and Y. Sidar), *IEEE Trans. Aut. Control* **40** (1995), pp. 1066–1070.

14. "On a lifting theorem for the structured singular value" (with H. Bercovici, C. Foias, and P. Khargonekar), *Journal of Math. Analysis and Applications* **187** (1994), pp. 617–627.

15. "Classification and uniqueness of invariant geometric flows" (with P. Olver and G. Sapiro), *Comptes Rendus Acad. Sci. (Paris)* **319** (1994), pp. 339–344.

16. "Multivariable stability margin optimization: a spectral tangential interpolation approach" (with Juan Cockburn), *Int. J. Control* **63** (1996), pp. 557-590.

17. "The structured singular value for linear input/output systems," (with H. Bercovici and C. Foias), *SIAM J. Control and Optimization* **34** (1996), pp. 1392–1404.

18. "Invariant geometric evolutions of surfaces and volumetric smoothing" (with P. Olver and G. Sapiro), *SIAM J. Applied Math.* **57** (1997), pp. 176-194.

19. "Optical flow: a curve evolution approach" (with A. Kumar and G. Balas), *IEEE Trans. Image Processing* **5** (1996), pp. 598–611.

20. "Conformal curvature flows: from phase transitions to active contours" (with S. Kichenesamy, A. Kumar, P. Olver, and A. Yezzi), *Archive for Rational Mechanics and Analysis* **134** (1996), pp. 275–301.

21. "The equivalence among the solutions of the $H^\infty$ optimal sensitivity computation problem" (with K. Hirata and Y. Yamamoto), *Trans. of the Society of Instrument and Control Engineers* **31** (1995), pp. 1954–1961.

22. "Three snippets of curve evolution theory in computer vision," *Mathematical and Computer Modelling Journal* **24** (1996), pp. 103–119.

23. "On skew Toeplitz operators, II" (with H. Bercovici and C. Foias), to appear in *Operator Theory: Advances and Applications.*

24. "Behavioral analysis of anisotropic diffusion in image processing" (with Y. You, M. Kaveh, W. Xu), *IEEE Trans. Image Processing* **5** (1996), pp. 1539–1553.

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26. "New solution to the two block $H^\infty$ problem for infinite dimensional stable plants" (with K. Hirata, Y. Yamamoto, T. Katayama), *Trans. of the Society of Instrument and Control Engineers* **32** (1996), pp. 1416–1424.

27. "On the nonlinear standard $H^\infty$ problem" (with C. Foias and C. Gu), to appear in *JMAA.*

28. "Affine invariant edge maps and active contours" (with P. Olver and G. Sapiro), to appear in *CVIU.*

29. "Geometric active contours for segmentation of medical imagery," (with S. Kichenesamy, A. Kumar, P. Olver, and A. Yezzi), *IEEE Trans. Medical Imaging* **16** (1997), pp. 199–209.

30. "Differential and numerically invariant signature curves applied to object recognition" (with E. Calabi, P. Olver, C. Shakiban), to appear in *International Journal of Computer Vision.*

31. "Some remarks on Hamiltonians and the infinite-dimensional one block $H^\infty$ problem" (with K. Hirata and Y. Yamamoto), *Systems and Control Letters* **29** (1996), pp. 111-117.

32. "Area and length minimizing flows for segmentation" (with Y. Lauziere, K. Siddiqi, and S. Zucker), to appear in *IEEE Trans. Image Processing.*

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34. "Shapes, shocks, and wiggles" (with K. Siddiqi, B. Kimia, and S. Zucker), to appear in *Journal of Imaging and Vision Computation.*

35. "Curve evolution models for real-time identification with application to plasma etching" (with J. Berg and A. Yezzi), to appear in *IEEE Trans. Aut. Control.*

36. "Skew Toeplitz solution to the $H^\infty$ problem for infinite dimensional unstable plants" (with K. Hirata, Y. Yamamoto, and T. Katayama), to appear in *Trans. of the Society of Instrument and Control Engineers.*

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37. *Robust Control of Distributed Parameter Systems* (with Ciprian Foias and Hitay Özbay), *Lecture Notes in Control and Information Sciences* **209**, Springer-Verlag, New York, 1995.

38. *Feedback Control, Uncertainty, and Complexity*, edited by Bruce Francis and Allen Tannenbaum, *Lecture Notes in Control and Information Sciences* **202**, Springer-Verlag, New York, 1995.

### Papers Submitted for Publication

39. "On a nonlinear causal commutant lifting theorem" (with C. Foias and C. Gu), submitted for publication to *Integral Equations and Operator Theory.*

40. "The shape triangle: parts, protrusions, and bends" (with B. Kimia, K. Siddiqi, and S. Zucker), submitted for publication in *Vision Research.*

41. "On the affine invariant heat equation for nonconvex curves" (with S. Angenent and G. Sapiro), submitted for publication to *Journal of the American Math. Society.*

42. "On a state space solution to the singular value problem of Toeplitz operators and the computation of the gap" (with K. Hirata and Y. Yamamoto), submitted for publication to *Systems and Control Letters.*

### Book Chapters

43. "From curve detection to shape description" (with A. Dobbins, L. Iverson, B. Kimia, and S. Zucker), in *Computer Vision: Systems, Theory, and Applications*, edited by A. Basu and X. Li, World Scientific, Singapore, 1993, pages 25-39.

44. "Generalized interpolation theory and its application to robust control design," *Digital and Numeric Techniques and Their Applications in Control Systems,* edited by C. T. Leondes, Academic Press, 1993, pages 163-217.

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### Students of A. Tannenbaum Supported by AFOSR-AF/F49620-94-1-00S8DEF

1. Juan Cockburn (Ph. D.)
2. Arun Kumar (Ph. D.)
3. Anthony Yezzi (Ph. D.)

### Awards of A. Tannenbaum During AFOSR-AF/F49620-94-1-00SDEF

1. Keynote Speaker at American Mathematical Society Annual Meeting (1994).

2. Plenary Speaker at American Mathematical Society Meeting (1997).

3. George Taylor Research Award (University of Minnesota).

4. Plenary Speaker for AFOSR Workshop on Optimal Design and Control (1997).

5. SICE Best Paper Award for "New solution to the two block $H^\infty$ problem for infinite dimensional stable plants" (with K. Hirata, Y. Yamamoto, T. Katayama), *Trans. of the Society of Instrument and Control Engineers* **32** (1996), pp. 1416–1424.