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FUNCTIONAL EQUATIONS OF THERMAL RADIATION IN THE PRESENCE OF AN ABSORBING AND DIFFUSING MEDIUM

[This is a translation of an article written by Yu. A. Surinov in Dok AK Nauk SSSR (Reports of the Acad Sci USSR), 1952, Vol. LXXXIV, No. 6, pages 1159-1162.]

This contribution is devoted to the development of certain basic integral equations of macroscopic (phenomenological) kinetics of radiation for a system of gray bodies separated by an absorbing and diffusing medium, and containing sources (outflows) of heat. Such equations form a most important means for the analytical examination of the problems of radiation exchange in such radiating systems. The development of the equations is extremely simple and follows from the so-called classification of the forms of hemispherical and volumetric radiation*, to which also belongs the expression for the intensity of the radiation B, (M, s) obtained in solving the equation of the transfer of radiant energy. The latter takes the following form (3-6) for the non-stationary field of radiation:

$$\frac{\partial B_\nu(M, s, \tau)}{\partial s} + \frac{1}{c}\frac{\partial B_\nu(M, s, \tau)}{\partial \tau} = -k_\nu(M) B_\nu(M, s, \tau) + \frac{1}{4\pi}\eta_{\text{эф.}\nu}(M, s, \tau), \quad (1)$$

Where $k_\nu = \alpha_\nu + \beta_\nu$ is the extinction coefficient of the medium and is equal to the sum of the coefficient of volume absorption (α_ν) and diffusion (β_ν) of the medium; $\eta_{\text{эф.}\nu}$ is the density of the effective volumetric radiation and is equal to the sum of the densities of the actual $\eta_{\text{соб.}\nu}$ and of the scattered $\eta_{\text{рас.}\nu}$ radiation:

$$\eta_{\text{эф.}\nu}(M, s, \tau) =$$

$$= \eta_{c.\nu}(M, \tau) + \beta_\nu(M) \int_{(4\pi)} B_\nu(M, s, \tau)\gamma_\nu(M, s, s')\, d\omega \,^{**}. \quad (2)$$

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*The development of the integral equations of thermal radiation, as applied to a system of gray bodies separated by a diathermic medium, can be found in the author's works (1, 2).

**Where $\eta_{c.\nu} = \alpha_\nu \eta_{0.\nu}$ Kirchhoff's law; $\eta_{0.\nu} = \eta(T, \nu)$ Plank's function.

The values of k_ν and $\eta_{k,\nu}$ are the functions of the point M at time τ and of the frequency ν of the radiations. The intensity B_ν also depends of the direction S.

Classification of the Forms of Radiation (1)

1. The spherical vector of radiation

$$E_{4\pi,\nu}(M) = \int\limits_{(4\pi)} B_\nu(M, s)\,\mathbf{r}_1 d\omega = \begin{cases} \mathbf{E}^{(+)}_{2\pi,\nu}(M, n) + \mathbf{E}^{(-)}_{2\pi,\nu}(M, n), & M \in \mathfrak{V}_V; \\ \mathbf{n}_1 E_{\text{эф},\nu}(M) + \mathbf{E}_{2\pi,\nu}(M), & M \in \mathfrak{V}_F. \end{cases} \quad (3)$$

2. The spatial density of the incident radiation

$$\eta_{\text{пад},\nu}(M) = c\,U_\nu(M) = \int\limits_{(4\pi)} B_\nu(M, s)\,d\omega, \quad M \in \mathfrak{V}_V. \quad (4)$$

The function $\eta_{\text{incident}\,\nu}$ suffers a discontinuity at the boundary F, as the potential of a double layer. Therefore, we obtain (7, 8):

$$\eta_{\text{пад},\nu}(M_i) = \eta_{\text{пад},\nu}(M) + 2E_{\text{эф},\nu}(M) = \int\limits_{(2\pi)} B_\nu(M, s)\,d\omega + 2E_{\text{эф},\nu}(M). \quad (5)$$

Where M is the boundary point $(M \in \mathfrak{V}_F)$ and M_i is an infinitely close inner point.

3. The density of the volumetric resultant absorption

$$\eta_{\text{рез},\nu} = a_\nu \eta_{\text{пад},\nu} - \eta_{\text{с},\nu} = k_\nu \eta_{\text{пад},\nu} - \eta_{\text{эф},\nu}{}^{*}, \quad M \in \mathfrak{V}_V. \quad (6)$$

4. The intensity of the radiation for a stationary field of radiation

$$B_\nu(M, s) = B_{\text{эф},\nu}(N, s)\,e^{-h_\nu} + \frac{1}{4\pi}\int\limits_0^r \eta_{\text{эф},\nu}(P)\,e^{-\Delta h_\nu}ds, \quad M \in \mathfrak{V}_{V+F}, \quad N \in \mathfrak{V}_F{}^{**}, \quad (7)$$

$$\text{где } h_\nu = -\int\limits_0^r k_\nu(P)\,ds; \quad \Delta h_\nu = -\int\limits_{\rho}^r k_\nu(P)\,ds; \quad B_{\text{эф},\nu}(N, s) = B_{\text{эф},\nu}(N) =$$

$= E_{\text{эф},\nu}(N)/\pi$ — Lampert's law.

*Where $\eta_{\text{эф},\nu}$ is taken for the spherical scattering $\gamma_\nu(M, s, s') \equiv 1$.

**The expression (7) presents the solution of the equation for the stationary problem (i.e., when $\partial B_\nu/\partial\tau \equiv 0$), which will be the only one to be considered below. All the expressions given above are also valid for the nonstationary state.

On the basis of the above expressions, determining and relating the various forms of radiation, it is simple to obtain the integral equations for all possible ways of posing the problem and for all forms of a hemispherical and volumetric radiation. We shall make the following preliminary statement: Integrating (scalarly) all members of the equation of transfer (1) over the solid angle $\omega = 4\pi$ and over the spectrum we obtain:

$$\operatorname{div} \mathbf{E}_{4\pi} + \frac{\partial U}{\partial \tau} + \eta_{pe3} = \operatorname{div} \vec{\Omega} + \frac{\partial (U + \rho i)}{\partial \tau} - W = 0^{***}. \qquad (8)$$

Differential equation (8) is formed for a moving medium and takes into account the nonstationary system, the presence of the sources (outflows) of heat W, as well as the inductive transfer of heat. The total field vector will be in this case equal to $\vec{\Omega} = \mathbf{E}_{4\pi} + \mathbf{q} + \rho i \mathbf{W}$ ****
where q = γ grad T is the vector of heat conductivity,

***Integrating the terms of equation (1) vectorially over the solid angle $\omega = 4\pi$ and over the spectrum we obtain a vectorial equation of the form (6, 7)

$$\operatorname{div} \Pi (M, \tau) + \frac{1}{c} \frac{\partial \mathbf{E}_{4\pi} (M, \tau)}{\partial \tau} = - \int_0^\infty k_\nu (M) \, \mathbf{E}_{4\pi, \nu} (M, \tau) \, d\nu, \qquad (8a)$$

Where $\Pi (M, \tau)$ is the "affinor" /?/ of radiation, characterized by the components $\Pi_{ik} = \int B(M, s, \tau) \cos \theta_M \cos \theta_N \, d\omega$ $(i, k = x, y, z)$. We note that (as is evident from (8) even in the case of a stationary field of radiation, when $\partial \mathbf{E}_{4\pi} / \partial \tau \equiv 0$ the vector $\mathbf{E}_{4\pi, \nu}$ cannot be presented either as a potential or a quasipotential vector. This is because div II is reducible to a gradient from the corresponding potential function (grad U) only under conditions approximating the thermodynamic equilibrium.

****Here $\mathbf{E}_{4\pi}$ /∞ is the integral spherical vector of radiation $\mathbf{E}_{4\pi} = \int_0 \mathbf{E}_{4\pi, \nu} \, d\nu$, where ν is the frequency of the radiation. Similarly, all other spectral and integral characteristics of the field of radiation (2) are related. Differential equation (8) is an Umov type (9).

$\rho i w$ the vector of the convectional heat transfer, $i = c_p T$ the heat content, ρ the density of the medium [3].

Integral equations of radiation. The configuration and the dimensions of the system are assumed to be given for all examined conditions of the problem. The same pertains to the fields of the optical constants both for the volume of the medium $(\alpha, \beta$ and for the boundary F of the system (R).

The first condition of the problem, consisting in the determination of the field densities of various forms of radiation over the given field of temperatures for the volume V, and upon the boundary F of the system, brings about the following system of two integral equations for the functions

$$E_{\text{пад},\nu}(M) - \int_{(F)} R(N)\, E_{\text{пад},\nu}(N)\, K(M, N)\, dF_N -$$

$$- \int_{(V)} \eta_{\text{эф},\nu}(P)\, L(M, P)\, dV_P = \int_{(F)} E_{c,\nu}(N)\, K(M, N)\, dF_N, \qquad (9)$$

$$\eta_{\text{эф},\nu}(M, s) - \beta_\nu(M) \int_{(V)} \eta_{\text{эф},\nu}(P, s)\, L_1(M, P)\, dV_P -$$

$$- \beta_\nu(M) \int_{(F)} R(N)\, E_{\text{пад},\nu}(N)\, K_1(M, N)\, dF_N =$$

$$= \eta_{c\nu}(M) + \beta_\nu(M) \int_{(F)} E_{c,\nu}(N)\, K_1(M, N)\, dF_N, \qquad (10)$$

where $K(M, N) = e^{-h_\nu} \dfrac{\cos\theta_M \cos\theta_N}{\pi r^2_{MN}}$; $\quad L(M, P) = e^{-\Delta h_\nu} \dfrac{\cos\theta_M}{4\pi r^2_{MP}}$;

$$K_1(M, N) = e^{-h_\nu} \gamma_\nu(M, s, s') \frac{\cos\theta_N}{\pi r^2_{MN}}; \quad L_1(M, P) = e^{-\Delta h_\nu} \frac{\gamma_\nu(M, s, s')}{4\pi r^2_{MP}}. \qquad (11)$$

On determining (upon the basis of equations (9) and (10) $E_{\text{pad}\,\nu}$ and $\eta_{\text{эф},\nu}$, we may find, elementarily, all other forms of radiation [1, 2] on the basis of the classification of the forms of radiation.

The second condition of the problem differs from the first by the fact that, instead of T being given over the volume of the medium, η_{res} is given. In this case, equation (10) in the system (8, 10) has to be replaced (for $\gamma_\nu \equiv 1$) by

$$\eta_{эф,\nu}(M)-k_\nu(M)\int_{(V)}\eta_{эф,\nu}(P)L_1(M,P)\,dV_P-k_\nu(M)\int_{(F)}E_{эф,\nu}(N)K_1(M,N)\,dF_N=-\eta_{рез,\nu}(M). \quad (12)$$

On integrating this equation over the spectrum we obtain, in the general case of a selective spectrum of the medium, a nonlinear equation*. From (12) for the special case of a uniform problem and $\eta_{res} = 0$, we can obtain the nonlinear integral equation of Hopfe Kuznetsov [4, 5].

The third condition of the problem differs from the first by E_{res} being given as a boundary condition instead of T. Here we obtain a system of equations: consisting of equation (10) and one of the forms

$$E_{эф,\nu}(M)-\int_{(F)}E_{эф,\nu}(N)K(M,N)\,dF_N-\int_{(V)}\eta_{эф,\nu}(P)L(M,P)\,dV_P=-E_{рез,\nu}(M). \quad (13)$$

The fourth condition of the problem consists in the determination of the temperature field and amplitude fields of various forms of radiation over the volume V and along the boundary F of the system, providing that η_{res} is given over the volume and E_{res} for the boundary. This case will be described by the system of equation (12) and (13). The substantial difference between this system and that of (9) and (10) is that on passing to the special cases of strictly absorbing media $(\beta_\nu \equiv 0)$ and of black boundary bodies $(R \equiv 0)$ They are not reduced quadratically**. All the foregoing systems of nonhomoge-

*When we substitute into this equation the value of η_{res} arising from (8), it changes into a nonlinear integral differential equation for the temperature range, and this maintains its nonlinearity even in the case of a gray medium for which the integral equation of radiation is linear.

**We note that the system (12) (13) is reducible to one integral equation with one unknown function ($E_{эф,\nu}$ or $\eta_{эф,\nu}$), but only in the two simplest special cases (of opposite characters), namely (a) when the system of gray bodies is separated by a diathermic medium, and (b) when the medium is infinite and no boundary bodies are present. In the latter case we have:

$$\eta_{эф,\nu}(M)-k_\nu(M)\int_{(V)}\eta_{эф,\nu}(P)\frac{e^{-\Delta h_\nu}}{(r-r_p)^2}\,dV=-\eta_{рез,\nu}(M).$$

nous integral equations have solutions* (except the system (12) (13), because the corresponding system of homogenous equations have only trivial solutions (zero solutions). This follows from the equations of closed systems given below, their deductions being realizable on the basis of the second law of thermodynamics. According to the latter we have for the radiating system, in case of a thermodynamic equilibrium: $\mathbf{E}_{4\pi,\nu} = E_{\text{рез},\nu} = \eta_{\text{рез},\nu} = -\operatorname{div}\mathbf{E}_{4\pi} = 0$; $\eta_{\text{эф},\nu} = k_\nu \eta_{\text{пад},\nu} = k_\nu \eta_{0,\nu} = 4k_\nu E_{0,\nu} = \text{const}$; $E_{\text{пад},\nu} = E_{\text{эф},\nu} = E_{0,\nu} = \pi B_{\text{эф},\nu} = \pi B_{0,\nu} = \text{const}$.
Taking these relationships into consideration we obtain on the basis of (7)

$$e^{-h_\nu} + \int_0^r k_\nu(P)\, e^{-\Delta h_\nu} ds = 1^{**}. \qquad (14)$$

A similar equation (for an absorbing and refracting medium) has been examined first by Hilbert [12] who built, on its basis, a proof of Kirchhoff's law for the volumetric rediation of the medium.

*The homogencus equations corresponding to the equations (12) (13) and their special cases, have single fundamental functions from which Kirchhoff's law follows in both its forms: $\eta_{c,\nu}/\alpha_\nu = \eta_{0,\nu} = \text{const}$ and $E_{c,\nu}/A = E_{0,\nu} = \text{const}$.

**And also

$$\int_{(4\pi)} e^{-h_\nu} d\varphi_n(M,s) + \int_{(4\pi)} \int_0^r k_\nu(P)\, e^{-\Delta h_\nu} ds\, d\varphi_n(M,s) = 0. \qquad (14б)$$

Equations (14) are for the corresponding generalizations of the Hilbertians [12].

It is evident that in the case of an infinite medium and in absence of boundary bodies the equations (14) and (14b) acquire the form $\int_0^r k_\nu(P) e^{-\Delta h_\nu} ds = 1$; $\int_V k_\nu(P) L(M,P) dV_P = 4\pi$,

because $h_\nu = \int_0^\infty k_\nu(P)\, ds = \infty$. Similar equations were developed by D. Hilbert [12] in connection with his proof of Kirchhoff's law.

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