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# Nonlinear Robust Control Theory and Applications

John C. Doyle,\* Principal Investigator

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## Introduction

Carolyn Beck (now an Assistant Professor in the Electrical Engineering Department at University of Pittsburgh<sup>1</sup>) was supported by this AASERT fellowship while she completed her PhD in Electrical Engineering at Caltech. Her research and thesis were on model reduction of uncertain systems, which will be summarized in this report.

Model based control methods are commonly used in the design of large, complex systems. Specifically, a mathematical model of the system is constructed, utilizing, for example, first principles analysis and experimental data, which is then used for subsequent control system design and analysis. For the purposes of feedback control highly accurate models are desired. However, such accuracy often requires that complicated high-order models be used, which in turn lead to more difficult control design problems from both an engineering and a computational perspective. The emphasis of this research is on the development of methods for reducing the size and complexity of the model while retaining the essential features of the system description. The main goal of these methods is to find a simplified system model which describes the physical system accurately enough so that controllers designed based on this simplified model perform well when implemented on the real system. Directly related to the topic of model re-

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duction are the realization theory concepts of minimality and its converse reducibility, which are also addressed in detail in this thesis.

A fundamental limitation in achieving desired system performance via any control design process is the inherent uncertainty in modelling the dynamics of the system under consideration. This uncertainty arises during the modelling process, which requires making a number of assumptions, estimations and simplifications; for example, uncertainty is often attributed to unmodelled dynamics such as nonlinearities and disturbances, and to incomplete knowledge of exact values for many of the system parameters. The effects of model uncertainty in feedback control may be substantial, particularly for high performance systems, since many control strategies attempt to utilize all system information present in the model in order to optimize system performance. The main approach taken to account for model uncertainty, is to design controllers that perform well on a set of models, rather than on a single model. The model set is defined using a nominal model which is considered to be perturbed by a prescribed uncertainty set; that is, the model itself explicitly includes an uncertainty description. By appropriately defining and structuring the uncertainty set, a model set is constructed which covers a range of possible system behavior, without allowing for too many unlikely or impossible models. These models and the systems they represent are referred to as *uncertain systems*.

There has been much research activity on model reduction methods in recent years, however, previous reduction methods have addressed only reduction of the state dimension of the model (that is, the nominal model) and fail to address the issue of reducing the uncertainty description. In notable contrast to such methods, this research presents a systematic model reduction method to reduce both the state dimension and the uncertainty description, providing a greater reduction in the overall size and complexity of the model. Furthermore, related realization theory for uncertain systems, including an explicit method to determine the existence of, and compute, minimal order equivalent realizations for uncertain system models is addressed. Both the model reduction methods and the realization theory developed in this research are applicable to multi-dimensional system realizations, and include the standard one-dimensional (1D) results as the simplest case.

The development of earlier theory relevant to this research proceeded along two somewhat separate paths: one related to the *robustness* framework originally proposed by Zames in 1966 [1], and the other to the state-space realization theory developed mainly in the '60s by Gilbert [2], Zadeh and Desoer [3], Kalman [4], Rosenbrock [5] and others.

In [1], Zames introduced the small gain theorem, which provides an exact

robust stability test for systems perturbed by *unstructured* dynamic uncertainty. This test is said to be robust in that it holds when the nominal model is subjected to all allowable values of the uncertainty. These exact results for unstructured uncertainty give sufficient conditions for robust stability of systems with respect to *structured* uncertainty. However, for structured uncertainty, these results are often conservative. As a result, the notion of rearranging the uncertainty into block diagonal form and using structured scaling matrices to reduce conservativeness in the tests was suggested in the early '80s by Doyle [6], and Safonov [7]. We consider the framework developed by Doyle and coworkers for modelling systems with structured uncertainty, that of dynamic perturbations to a nominal system which enter in a linear fractional manner; see [8], [9], [10], [11] and the references therein for further details.

More recently, synthesis methods have been developed which provide systematic techniques to construct controllers for systems subject to structured uncertainty, and for which robust stability and performance are guaranteed (see for example [12], [11], [13]). These controllers have at least the same state dimensions and uncertainty set complexity as the original system model. Moreover, the synthesis of these controllers and the subsequent system analysis often rely on complicated computational solutions which become increasingly difficult to implement as the model size and complexity grows. Thus, the need for reducing both the nominal model and uncertainty description has become apparent.

A number of methods for reducing the state dimension of models were proposed in the '80s; examples include the balanced truncation model reduction method and its additive  $H_\infty$  norm error bound, and the optimal Hankel norm model reduction method and its Hankel norm error bound. These are state-space methods, and rely to a large extent on the notion of finding *balanced realizations* for systems. The use of balanced realizations was first proposed by Moore [14] as a means of better analyzing realizations for reducibility based on the comparative controllability and observability of the system states. This was intended as a more computable alternative to the problem of finding minimal state-space realizations, originally put forth by Kalman [4] and Gilbert [2]. Thus, from its inception, the notion of balanced model reduction has been intertwined with the notions of minimality, controllability and observability, and solutions to state-space Lyapunov equations. Specifically, when the controllability and observability Gramians, the solutions to the Lyapunov equations, are equivalent and diagonal the associated state-space model is said to be balanced. The states corresponding to the small-valued elements of the balanced Gramian are both

weakly controllable and weakly observable and can be truncated with relatively little resulting error. The guaranteed a priori error bounds for the balanced model reduction method were found independently by Enns [15] and Glover [16]; the corresponding bounds for discrete-time systems were presented by Hinrichsen and Pritchard [17].

This work builds on the balanced truncation method for 1D systems, generalizing these techniques and related realization theory to the linear fractional transformation (LFT) setting. The LFT models and results discussed herein are applicable to uncertain systems, multi-dimensional systems, or formal power series. The main results include a necessary and sufficient condition for the exact reducibility of LFT systems, leading naturally to a notion of minimality for these systems. Furthermore, systematic model reduction methods with guaranteed a priori upper error bounds are given for uncertain and multi-dimensional systems models.

## Summary of Main Results

We begin the summary with a brief review of the LFT modelling framework commonly used to represent uncertain systems, followed by a short discussion of existing model reduction and realization theory results for uncertain systems. For complete details on the following material, see [18].

We consider the LFT paradigm shown in Figure 1, where  $\Delta$  represents uncertainty, or a dynamic element, and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a realization of the input-output mapping

$$\Delta \star M = D + C\Delta(I - A\Delta)^{-1}B;$$

we assume throughout that the inverse is well-defined. If we let  $\Delta$  represent repeated copies of the integral or shift operator (e.g.,  $1/s$ ) then we recover the transfer function  $(1/s) \star M = D + C(sI - A)^{-1}B$  and a standard state-space realization with state  $x$ , input  $u$  and output  $y$ . By simply allowing the  $\Delta$  block to represent more general system operators, LFT systems provide a convenient framework for adding uncertainty in which essentially all of the major state space results can then be generalized (see [19] and the references therein).

We assume  $\Delta$  lies in a prescribed set,

$$\Delta = \{ \text{diag} [\delta_1 I_{n_1}, \dots, \delta_p I_{n_p}] : \delta_i \in \mathcal{L}(l_2) \}. \quad (1)$$

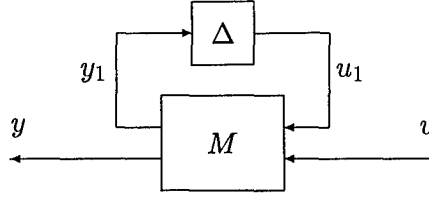


Figure 1: Uncertain System

We often consider  $\Delta$  which lie in a unity norm-bounded subset of  $\Delta$ , denoted by  $\mathbf{B}_\Delta$ . Note that  $\delta_i \in \mathcal{L}(l_2)$  allows time-varying operators on  $l_2$ , which are not commutative. Furthermore, it is typically assumed that  $M$  is an LTI system, but it is equivalent and simpler to assume that  $M$  is a constant and include the shift or integral operator as one of the  $\delta_i$ . This then gives us a state-space realization for uncertain systems which is analogous to standard or 1-D realizations.

### Reduction Results

In [20] and [18] it is shown that a general version of similarity transformations, system Lyapunov equations, and controllability and observability Gramians in balanced truncation model reduction and in terms of quantifying system minimality hold for uncertain systems modelled using the LFT framework. Namely, given a realization  $(\Delta, M)$  and any  $\epsilon \geq 0$ , a lower order realization  $(\Delta_r, M_r)$  exists such that  $\sup_{\Delta \in \mathbf{B}_\Delta} \|(\Delta \star M) - (\Delta_r \star M_r)\|_{l_2 \rightarrow l_2} \leq \epsilon$  if and only if there exist block diagonal structured solutions,  $X \geq 0$  and  $Y \geq 0$ , to the system Lyapunov inequalities:

$$\begin{aligned} AYA^* - Y + BB^* &\leq 0 \\ A^*XA - X + C^*C &\leq 0, \end{aligned} \tag{2}$$

where  $\lambda_{\min}(XY) = \epsilon^2$  with multiplicity corresponding to the difference in the dimensions of the full and reduced realizations. Existing LMI solvers may thus be used to find feasible solutions to (2).

### Realization Theory for Uncertain Systems

In the case where  $\epsilon = 0$ , we obtain a minimality result which is completely analogous to standard realization theory. At the same time, we can find a Kalman-like decomposition structure for the uncertain system realization matrices. That is, via the proof of the minimality condition, it is clear

that the existence of a singular structured Gramian implies that an equivalent realization can be found which has an uncontrollable and unobservable decomposition. Not surprisingly, we can also construct controllability and observability matrices on which rank tests may be performed giving us an equivalent minimality result. For example, given an uncertain system realization  $(\Delta, M)$ , where  $\Delta$  is structured as in (1), then the **controllability matrix** is defined by

$$\Gamma = \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{1p}B_p & A_{11}^2B_1 & \cdots & A_{11}A_{1p}B_p & A_{12}A_{21}B_1 & \cdots \\ B_2 & A_{21}B_1 & \cdots & A_{2p}B_p & A_{21}A_{11}B_1 & \cdots & A_{21}A_{1p}B_p & A_{22}A_{21}B_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_p & A_{p1}B_1 & \cdots & A_{pp}B_p & A_{p1}A_{11}B_1 & \cdots & A_{p1}A_{1p}B_p & A_{p2}A_{21}B_1 & \cdots \end{bmatrix}$$

We denote the block rows by  $\Gamma_i = [B_i \ A_{i1}B_1 \ \cdots]$ . We can then show that there exists a singular, block structured matrix  $Y \geq 0$  satisfying  $AY A^* - Y + BB^* \leq 0$  if and only if  $\text{rank}(\Gamma_i) < n_i$  for some  $i = 1, \dots, p$ . These results may also be found in [18], [21], [22] and [23].

### Formal Power Series and LFTs

Connections between the notion of minimality for LFT realizations, as discussed above, and the notion of minimal representations for formal power series (FPS), developed mainly in the '70s in the context of nonlinear system realization theory, are readily found. If we consider LFT realizations where the only structure we assume for the uncertainty set is the spatial structure of repeated scalar blocks, then the resulting LFT systems may also be viewed as a representation of rational functions in multiple noncommuting indeterminates, that is, as a particular realization of a FPS. The form of the FPS representations and the definition of minimality used differ from those used for the LFT representations we consider; we show in [18] and [24] that given a minimal FPS representation or a minimal LFT realization, the other (minimal) form can be directly computed.

### Computational Solutions and Applications

Computational methods are presented for reducing uncertain system models with the guaranteed upper error bounds mentioned above. Ideally, we would like to find minimum rank, structured Gramians  $Y \geq 0$  and  $X \geq 0$  to the above LMIs, i.e., solutions  $Y$  and  $X$  for which the product  $YX$  has the smallest-valued minimum eigenvalue with the highest multiplicity. Although feasible solutions,  $X$  and  $Y$ , are easily computed using convex programming

methods (or any of the recent LMI solvers [25, 26]), the optimization problem itself is a *reduced rank LMI problem* and as such does not yield a convex optimization problem, thus we cannot directly apply LMI algorithms to obtain solutions. However, we have constructed a straightforward heuristic algorithm using existing LMI techniques to obtain solutions for the model reduction and minimality problem, which have given quite good results not only in numerical tests, but in applications as well. A detailed description of the algorithm and applications to a power plant are given in [18] and [27].

### Recent and Ongoing Research

Although we can compute guaranteed upper error bounds using these methods, we cannot simultaneously compute lower bounds. For standard 1-D continuous systems, both upper and lower error bounds for balanced truncation model reduction can be computed using the singular values of the associated Hankel operator. In the case of uncertain and discrete time systems, the actual system Gramians are not used, but instead non-unique solutions to the LMIs in (2) are found. Thus we cannot strictly relate the solutions  $Y$  and  $X$  for the LMIs to a system Hankel operator and Hankel singular values. However, we may construct Hankel matrices for the uncertain systems we consider using the realization matrix  $M$ .

The relevance of Hankel matrices for uncertain systems has recently been considered, mainly in the context of minimality, in addition to the associated Hankel operators and the use of such in computing lower bounds on system norms and for reduction. We define Hankel matrices for uncertain systems in a manner similar to those defined for formal power series [28]; structured Hankel matrices have also been considered. We use the so-called controllability and observability matrices for uncertain systems defined in [18] for the construction of these Hankel matrices for uncertain systems. The singular values of the Hankel operators we construct provide reasonable lower bounds for the system norm, but appear to be conservative for model reduction lower bounds. Preliminary results may be found in <http://www.cds.caltech.edu/cds/reports/report-cgi/reports.cgi>.

From the reduction and realization theory developed for uncertain input-output models up to this point, computing reduced models with error bounds, and determining minimality for kernel representations of behavioral uncertain systems is currently under investigation. We consider the behavioral framework originally proposed by Willems [29]. In order to incorporate uncertainty into our models, we adopt the output nulling or kernel representation defined by Weiland [30] to describe 1-D behavioral systems. In this



framework, both minimality and the evaluation of model reduction error bounds become more complex. For example, in the input-output framework, we have necessary and sufficient LMI conditions for minimality and model reduction bounds. In the behavioral framework, the LMI conditions are only sufficient. In fact, even in the 1-D behavioral case (i.e., no uncertainty) there exist only sufficient conditions. Additionally, if we consider a kernel representation of a behavior, then minimality also involves the issues of output injection and detectability, and if we consider model reduction of a kernel representation, then the error bounds should be interpreted in a gap-like metric. Furthermore, to apply the model reduction techniques previously described, stability and *contractiveness* of the uncertain behavioral representations,  $(\Delta, M)$ , are desired. Stable  $M$  generalizes the use of stable coprime factor representations for input-output systems and as such norms can be used to define generalizations of normalized coprime factors.

In <http://www.cds.caltech.edu/cds/reports/report-cgi/reports.cgi> we address these issues, first introduced in [31], that are associated with normalization and minimality for uncertain behavioral system representations in more detail. Algorithms and associated upper error bounds for model reduction of behavioral uncertain systems are also discussed.

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