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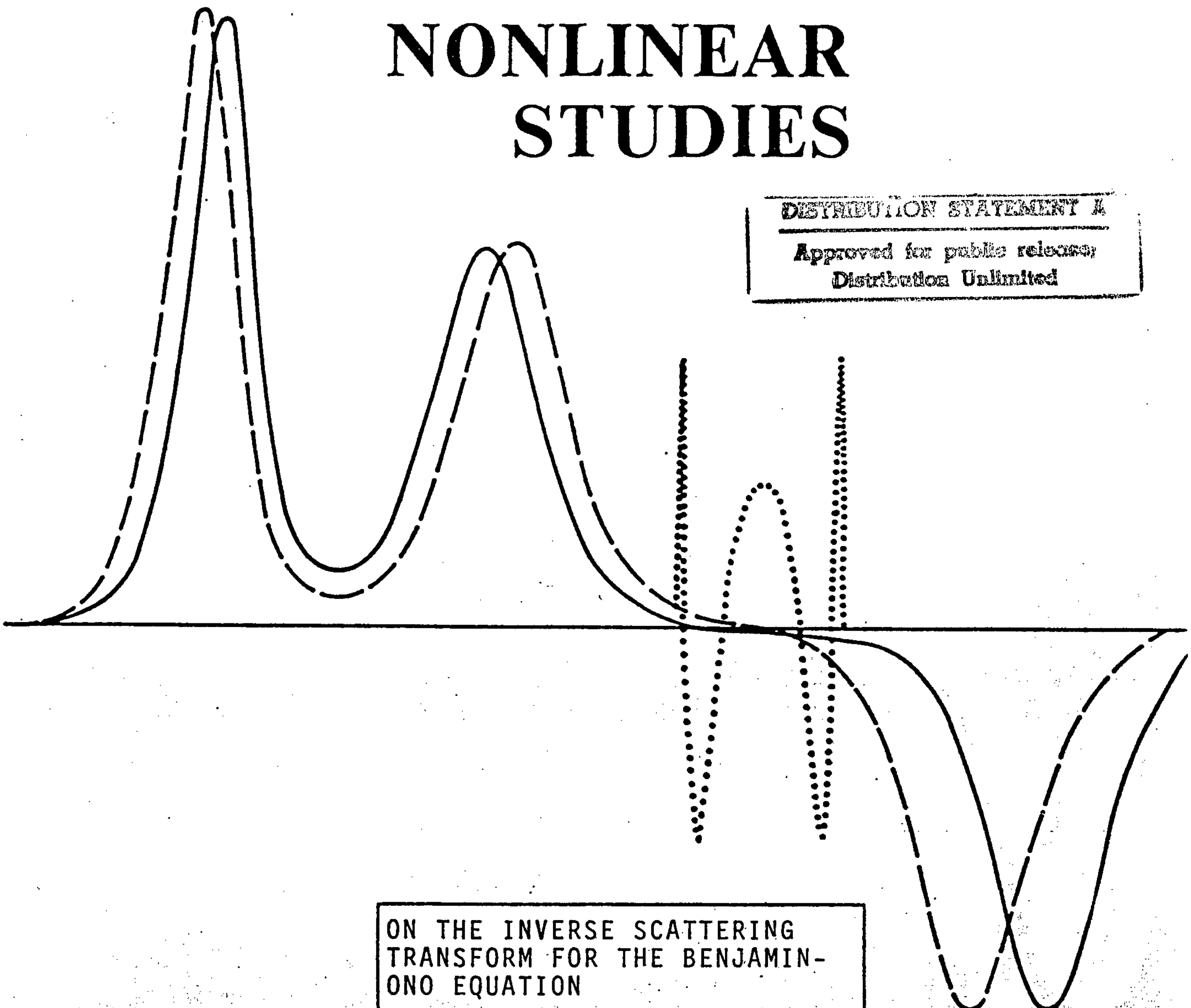

ON THE INVERSE SCATTERING TRANSFORM FOR THE BENJAMIN-ONO EQUATION

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ON THE INVERSE SCATTERING TRANSFORM FOR THE BENJAMIN-ONO EQUATION

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ABSTRACT

We extend the IST for the Benjamin-Ono (BO) equation, given in Ref.[1], in two important ways. First, we restrict the IST to purely real potentials, in which case, the scattering data and the inverse scattering equations simplify. Second, we also extend the analysis of the asymptotics of the Jost functions and the scattering data to include the nongeneric classes of potentials, which include, but may not be limited to, all N-soliton solutions. In the process, we also study the adjoint equation of the eigenvalue problem for the BO equation, from which, for real potentials, we find a very simple relation between the functions $\beta(\lambda)$ and $f(\lambda)$, introduced in Ref.[1]. Furthermore, we show that the reflection coefficient also defines a phase shift, which can be interpreted as the phase shift between the left Jost function and the right Jost function. This phase shift leads to an analogy of Levinson's Theorem, as well as a condition on the number on possible bound states that can be contained in the initial data. We also study the structure of the scattering data and the Jost functions for pure soliton solutions, and obtain remarkably simple solutions for these Jost functions. Since they are examples of nongeneric potentials, they demonstrate the asymptotics for nongeneric potentials. We then carefully detail the asymptotics in the limit $\lambda \to 0^+$ for generic and nongeneric potentials. Lastly, we show how to obtain the infinity of conserved quantities from one of the Jost functions of the BO equation, and also how to obtain these conserved quantities in terms of the various moments of the scattering data.

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I INTRODUCTION

The Benjamin–Ono (BO) equation [2]

$$u_t + 2uu_x + Hu_{xx} = 0\,, \quad Hv(x) \equiv \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(\xi)}{\xi - x} d\xi\,, \tag{1}$$

where P indicates the principal value of the integral, is one of the important integrable equations which can be linearized and solved by an inverse scattering transform (IST). It has important physical limits of internal gravity waves in a stratified fluid [2] as well as long waves in a stratified shear flow [3, 4]. Thus this equation is very important for the understanding of the evolution of stratified flows. It has soliton solutions [5, 6, 7] and a Lax Pair [8, 9]. Interactions of BO solitons have been studied by Y. Matsuno [10], as well as the asymptotics of the linearized BO equation, which describe the decay of the continuous spectra. A perturbation theory for the 1-soliton solutions of the BO equation has been obtained directly by Chen and Kaup [11], without reference to, or the use of the IST. Stability of the periodic solutions of the BO equation has been studied in one and two dimensions [12]. Physical applications of soliton solutions of the BO equation have been studied in various contexts by Matsuno. He has developed a multisoliton perturbation theory of BO solitons [13], and has applied it to two cases. These cases were weak dissipation and/or higher order dispersions, and he determined the shifts in the soliton amplitudes and the phases due to these effects. Effects of uneven bottoms have also been studied by Matsuno [14] wherein he has demonstrated that the topography could capture, repel, or even generate a stationary state.

Fokas and Ablowitz [1] presented the IST solution for the BO equation. This solution was novel and significant in several ways. First, it was the solution of the direct and inverse scattering problems of an integro-differential equation, which was a nonlocal Riemann-Hilbert problem in space. As such, the direct problem was uniquely different, requiring Fredholm theory to detail the properties of the continuous spectra, the discrete spectra, and the normalization constants. An important feature of this problem is that the generic potentials of this problem will in general only vanish algebraic, preventing one from using compact support, which was a key simplifying feature, useful in the analysis of the standard Schrödinger and Zakharov-Shabat eigenvalue problems [15]. Furthermore, they demonstrated that this IST could be formulated without having to detail the full properties of the scattering data of the problem. Also, they solved these problems for a general complex generic potential. The direct and inverse scattering problems for a related scattering problem (originating from a different, but equivalent Lax pair), has been studied by Coiffman and Wickerhauser [16]. There they have described many of the features of this scattering problem, particularly in the limit of very small initial data.

Here we extend the results of Ref. [1] in several ways. First, we detail the symmetries of the scattering data when the potential is purely real. We also treat the case of nongeneric potentials, which includes all the N-soliton solutions. As we shall see, the nongeneric case has distinctly different asymptotics from the generic case treated in Ref. [1]. When the potential is real, one can also relate the two reflection coefficients of Ref. [1], by a complex conjugation. As a consequence of the latter result, we then find that the inverse scattering equations for the BO equation can be simplified. Then by including the asymptotics of the Jost functions for nongeneric potentials, we furthermore find that the inverse scattering equations of Ref. [1] can be extended to include this case as well. Also, we will show how to define a phase shift for this scattering problem, the change of which, between the extreme limits of the spectral parameter, can be related to the number of bound states. This phase shift is related to the spectral integral in the Anderson-Taflin conservation law [16, 17], which then shows that the Anderson-Taflin conservation law is the analog of Levinson's Theorem [18] for the BO equation.

A key feature of these new results is the treatment of the adjoint problem of the eigenvalue problem for the BO equation, by which we are able to define inner products of the Jost functions. Such a study is important because it allows one to obtain important data about the scattering coefficients and the orthogonality of the Jost functions. Such a treatment will undoubtedly become a key for a later development of a general perturbation theory for the BO equation.

In the next section, we review the results from Ref. [1]. They are reproduced here only for completeness, and we will only list those relations that we will need to refer back to and use. We will use their notation for the definitions of the Jost functions and the various scattering coefficients. In later sections, we will introduce new quantities as needed. In Section III, we study the adjoint problem of the BO eigenvalue problem and define their inner products. From these relations, it naturally follows that the two reflection coefficients of Ref. [1], $f(\lambda)$ and $\beta(\lambda)$, are related by a complex conjugation when the potential, $u(x)$ is purely real. Note that essentially all the analysis in Ref. [1] was done for $u(x)$ complex in general, in which case these two reflection coefficients would be independent. Thus it is not surprising that if one requires $u(x)$ to be purely real, simplifications will occur. In Section IV, we briefly study the Jost functions for pure N-soliton solutions. These N-soliton solutions are in the class of nongeneric potentials and have features distinctly different from the generic potentials studied in Ref. [1]. It also is an interesting pedagogical exercise to solve the BO eigenvalue problem for these pure N-soliton solutions. It is only for such a potential that closed form solutions for the Jost functions are known. We shall see that although these Jost functions exist for almost all values of the spectral parameter, λ, they in general will have logarithmic singularities in the upper-half complex x-plane. These singularities in x then vanish *only* when λ is either the bound state eigenvalue ($\lambda_j < 0$) or in the continuous spectra (λ real and > 0). In Section V, we extend the results of Ref. [1, 19] for the asymptotics of the Jost functions, in the limit of $\lambda \to 0^+$, to include the general case of nongeneric, as well as generic potentials. In Section VI, we simplify the inverse scattering equations of Ref. [1] to the case of a purely real potential, and also extend them to include the possibility of nongeneric potentials. We also specify the restricted scattering data for real potentials and briefly mention the time dependence of this scattering data for the BO equation, which is given in Ref. [1]. (For simplicity and clarity, we shall otherwise ignore the time dependence in all the Jost functions and the scattering data.) In Section VII, we obtain the infinity of the conserved quantities for the BO equation. These are obtained from an asymptotic expansion of the Jost function $\overline{N}(x, \lambda)$ for large λ. We also give these conserved quantities in terms of the scattering data. The first of these conserved quantities relates the energy in the BO field to the energy in the scattering data. The energy of the soliton portion and the energy density of the radiation are each positive definite and additive. From the conservation of the total area of the BO field [16, 17], we find an analogy of Levinson's Theorem for the BO equation. Here we can relate the total change in a phase shift and the total area of the BO field to the total number of bound states.

II BACKGROUND

The following is material taken from Ref. [1], and is reproduced here for completeness, since we shall be using this notation and will need to refer to these equations. The eigenvalue problem for the BO equation may be taken to be

$$v_x - i\lambda v - i[uv]^+ = 0\,, \tag{2}$$

where v is the eigenfunction, analytic in the upper half x-plane, λ is the eigenvalue (spectral parameter), and the brackets with the superscript "+" indicates that one is to take that part of the argument which is analytic in the upper half complex x-plane. We will indicate the same for the lower half x-plane with a "-" superscript. Thus, as in Ref. [1], we take

$$[v(x)]^{\pm} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{v(\xi)}{\xi - x \mp i\epsilon} d\xi\,, \tag{3}$$

for $\epsilon \to 0^+$. There are two Jost functions of (2), only one of which is linearly dependent. First, there is the solution $N(x, \lambda)$ defined on the right, and there is the solution $\overline{M}(x, \lambda)$ defined on the left.

$$N(x \to +\infty, \lambda) \to e^{i\lambda x}\,, \quad \overline{M}(x \to -\infty, \lambda) \to e^{i\lambda x}\,. \tag{4}$$

These solutions exist for all positive values of λ. They can be given as solutions of integral equations [1], such as

$$N(x, \lambda)e^{-i\lambda x} = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi u(\xi) N(\xi, \lambda) \int_{x}^{\infty} \frac{e^{-i\lambda z} dz}{\xi - z - i\epsilon}\,. \tag{5}$$

There also can be a discrete set of bound state solutions, $\Phi_j(x)$, of (2), for $\lambda = \lambda_j < 0\,, \quad (j = 1, 2, \dots, J)$ where J is the total number of such bound states, possibly infinite. Due to (2), each of these eigenstates is an analytic function of x in the upper half complex x-plane. The bound state eigenfunctions can be normalized as

$$\Phi_j(x \to +\infty) \to \frac{1}{x}\,. \tag{6}$$

Upon taking the limit of $x \to \infty$ of (2), from (3) and (6), it follows that

$$\lambda_j = -\frac{i}{2\pi}\int_{-\infty}^{\infty} \Phi_j(\xi)u(\xi)d\xi\,, \tag{7}$$

and one finds in a similar manner that (6) is also true in the limit of $x \to -\infty$. These functions satisfy the homogeneous form of (5), which is

$$\Phi_j(x)e^{-i\lambda_j x} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi u(\xi)\Phi_j(\xi)\int_x^{\infty} \frac{e^{-i\lambda_j z}dz}{\xi - z - i\epsilon}\,. \tag{8}$$

In addition to these eigenfunctions, one can also defined two auxiliary functions, $\overline{N}(x,\lambda)$ and $M(x,\lambda)$, each of which are a particular solution of the *inhomogeneous* equation

$$V_x - i\lambda V - i[uV]^+ = -i\lambda\,, \tag{9}$$

and are defined by the boundary conditions

$$\overline{N}(x \to +\infty, \lambda) \to 1\,, \quad M(x \to -\infty, \lambda) \to 1\,. \tag{10}$$

For real positive λ, they can be obtained from N and $\overline{M}$ by differentiating with respect to λ, provided $f(\lambda)$ and $g(\lambda)$ are nonzero. In particular

$$f(\lambda)\overline{N}(x,\lambda) = e^{i\lambda x}\partial_\lambda[N(x,\lambda)e^{-i\lambda x}]\,, \tag{11}$$
$$g(\lambda)M(x,\lambda) = e^{i\lambda x}\partial_\lambda[\overline{M}(x,\lambda)e^{-i\lambda x}]\,, \tag{12}$$

where

$$f(\lambda) = \frac{-1}{2\pi\lambda}\int_{-\infty}^{\infty} u(\xi)N(\xi,\lambda)d\xi\,, \tag{13}$$
$$g(\lambda) = \frac{-1}{2\pi\lambda}\int_{-\infty}^{\infty} u(\xi)\overline{M}(\xi,\lambda)d\xi\,. \tag{14}$$

They also satisfy the integral equations,

$$\overline{N}(x,\lambda)e^{-i\lambda x} = e^{-i\lambda x} - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi u(\xi)\overline{N}(\xi,\lambda)\int_x^{\infty}\frac{e^{-i\lambda z}dz}{\xi - z - i\epsilon}\,. \tag{15}$$

These functions also have analytical properties with respect to λ. $M(x,\lambda)$ is analytic in the upper half complex λ-plane while $\overline{N}(x,\lambda)$ is analytic in the lower half complex λ-plane. By Fredholm theory, Fokas and Ablowitz then were able to show that, in the appropriate complex λ half-plane, as λ approaches any of the bound state eigenvalues, each of these functions would approach the limit:

$$\overline{N}(x, \lambda \to \lambda_j) \to \frac{-i}{\lambda - \lambda_j}\Phi_j(x) + (x + \gamma_j)\Phi_j(x) + \dots \,, \tag{16}$$

$$M(x, \lambda \to \lambda_j) \to \frac{-i}{\lambda - \lambda_j}\Phi_j(x) + (x + \gamma_j)\Phi_j(x) + \dots \,. \tag{17}$$

The complex quantity γ_j is a necessary piece of the scattering data. It has been shown that for the case of a pure N-soliton solution [20] and also indirectly from the limit of the ILW equation [19], its imaginary part is positive and is given by $-1/(2\lambda_j)$. Later, in Section III, we shall show that one can obtain also easily obtain this same result directly from the inverse scattering equations of the BO equation.

On the real, positive λ–axis, $M(x, \lambda)$ is related to $\overline{N}(x, \lambda)$ and $N(x, \lambda)$ by the scattering equation

$$M(x, \lambda) = \overline{N}(x, \lambda) + \beta(\lambda) N(x, \lambda) \,, \tag{18}$$

where $\beta(\lambda)$ is a reflection coefficient, which can be given by

$$\beta(\lambda) = i \int_{-\infty}^{\infty} u(\xi) M(\xi, \lambda) e^{-i\lambda\xi} d\xi \,. \tag{19}$$

From the above scattering equation and the analytical properties of $\overline{N}$ and M, Fokas and Ablowitz obtained a key relation for the solution of the inverse scattering problem. This relation relates $\overline{N}(x, \lambda)$, for λ in the lower half complex λ-plane, to the eigenfunctions $N(x, \lambda)$ on the real λ-axis and the bound states $\Phi_j(x)$. This is

$$\overline{N}(x, \lambda) = 1 + \frac{1}{2\pi i} \int_0^{\infty} \frac{\beta(\lambda') N(x, \lambda') d\lambda'}{\lambda' - \lambda + i\epsilon} - i \sum_j \frac{1}{\lambda - \lambda_j} \Phi_j(x) \,. \tag{20}$$

Requiring (20) to satisfy (9), one obtains the means for reconstructing the potential $u(x)$ from the scattering data and the Jost functions. This is

$$[u]^+(x) = \frac{1}{2\pi i} \int_0^{\infty} \beta(\lambda') N(x, \lambda') d\lambda' + i \sum_j \Phi_j(x) \,, \tag{21}$$

where $u(x) = [u]^+(x) +$ complex conjugate.

The first of the inverse scattering equations is a set of linear dispersion relations that relate the bound states, $\Phi_j(x)$, and the continuous eigenstates, $N(x, \lambda)$. This relation follows from (16) and (20), upon taking the limit of $\lambda \to \lambda_k$ in (20). Fokas and Ablowitz obtained

$$(x + \gamma_k)\Phi_k(x) + i \sum_{j \neq k} \frac{1}{\lambda_k - \lambda_j} \Phi_j(x) - \frac{1}{2\pi i} \int_0^{\infty} \frac{\beta(\lambda') N(x, \lambda') d\lambda'}{\lambda' - \lambda_k} = 1 \,, \tag{22}$$

which is Eq. (27) of Ref. [1].

In addition to this set of linear dispersion relations, one also needs a linear dispersion relation for the continuous spectra, that will allow one to also determine $N(x,\lambda)$. This will follow upon integrating (11) with respect to λ, and then using (20) to eliminate $\overline{N}(x,\lambda)$. This will give an integral equation involving integrals with respect to λ, but containing only $N(x,\lambda)$ and $\Phi_j(x)$, which is Eqs. (26) and (28) in Ref. [1] and Eqs. (30) - (32) in Ref. [19]. The solution of these, in conjunction with (22), provide the solution of the inverse scattering problem. We shall not give these forms here since they were obtained for complex, generic potentials and we wish to simplify them to the specific case of real potentials, and to extend them so that they may also include even the case of nongeneric potentials.

III THE ADJOINT PROBLEM

From now on, we shall restrict our analysis to the case when the potential, $u(x)$, is strictly real. This will provide us with several simplifications and will relate the two Fokas and Ablowitz reflection coefficients. Consider the adjoint problem of (2). It is

$$v_x^A + i\lambda v^A - iu[v^A]^- = 0\,. \tag{23}$$

Its solution is rather simple. First, take the part of (23) which is analytic in the lower half complex x-plane. Then one sees that one simply has the complex conjugate of (2). Second, take the part of (23) which is analytic in the upper half complex x-plane. This is a simple, first order, inhomogeneous ODE. Combining these two solutions together, one has that a solution of the adjoint problem, $N^A(x,\lambda)$, is

$$N^A(x,\lambda) = \overline{M}^*(x,\lambda) - i\int_{-\infty}^{x} [u\overline{M}^*]^+(\xi,\lambda)e^{-i\lambda(x-\xi)}d\xi\,, \tag{24}$$

where $\overline{M}$ is defined by (4). The last term in (24) is the part which is analytic in the upper half complex x-plane.

We define the adjoints of the bound state eigenfunctions, $N_j^A(x)$, similarly.

$$N_j^A(x) = \Phi_j^*(x) - i\int_{-\infty}^{x} [u\Phi_j^*]^+(\xi)e^{-i\lambda_j(x-\xi)}d\xi\,. \tag{25}$$

Next, we obtain some relations between our Jost functions and scattering coefficients. Note that N and $\overline{M}$ are not independent solutions. Thus they must be proportional [19]. For $\lambda > 0$,

$$\overline{M}(x,\lambda) = \Gamma(\lambda)N(x,\lambda)\,, \tag{26}$$

where $\Gamma(\lambda)$ has two equivalent forms:

$$\begin{aligned}\Gamma(\lambda) &= 1+i\int_{-\infty}^{\infty} u(\xi)\overline{M}(\xi,\lambda)e^{-i\lambda\xi}d\xi\,, \\ &= \left[1-i\int_{-\infty}^{\infty} u(\xi)N(\xi,\lambda)e^{-i\lambda\xi}d\xi\right]^{-1}.\end{aligned} \tag{27}$$

Thus by (13) and (14),

$$g(\lambda)=\Gamma(\lambda)f(\lambda)\,. \tag{28}$$

Also, from (11), (12), (26) and (28), it follows that for real, positive λ,

$$\beta(\lambda)=\frac{1}{g(\lambda)}\partial_\lambda\Gamma(\lambda)\,, \tag{29}$$

which relates Γ, β and g.

We will need the asymptotics of these various Jost functions. These all follow from the definitions and (5), (15), and (23). For $x\to+\infty$ and $\lambda>0$,

$$\begin{aligned}N(x,\lambda) &\to e^{i\lambda x}\,, && (30)\\ \overline{N}(x,\lambda) &\to 1\,, && (31)\\ \overline{M}(x,\lambda) &\to e^{i\lambda x}\Gamma(\lambda)\,, && (32)\\ M(x,\lambda) &\to 1+\beta(\lambda)e^{i\lambda x}\,, && (33)\\ N^A(x,\lambda) &\to e^{-i\lambda x}\Gamma^*(\lambda)\,, && (34)\end{aligned}$$

while for $x\to-\infty$ and $\lambda>0$,

$$\begin{aligned}N(x,\lambda) &\to \Gamma^{-1}(\lambda)e^{i\lambda x}\,, && (35)\\ \overline{N}(x,\lambda) &\to 1-\frac{\beta(\lambda)}{\Gamma(\lambda)}e^{i\lambda x}\,, && (36)\\ \overline{M}(x,\lambda) &\to e^{i\lambda x}\,, && (37)\\ M(x,\lambda) &\to 1\,, && (38)\\ N^A(x,\lambda) &\to e^{-i\lambda x}\,. && (39)\end{aligned}$$

For the bound states, for $x\to+\infty$ and $\lambda=\lambda_j<0$,

$$\begin{aligned}\Phi_j(x) &\to \frac{1}{x}\,, && (40)\\ N_j^A(x) &\to -ie^{-i\lambda_j x}\int_{-\infty}^{\infty} u(\xi)\Phi_j^*(\xi)e^{i\lambda_j\xi}d\xi\,, && (41)\end{aligned}$$

while for $x\to-\infty$ and $\lambda=\lambda_j<0$,

$$\begin{aligned}\Phi_j(x) &\to \frac{1}{x}\,, && (42)\\ N_j^A(x) &\to 0\,. && (43)\end{aligned}$$

Now, from the Wronskian relation, we can relate the asymptotics on the left to those on the right, and also define inner products. It follows that, for any solution, $v(x,\lambda)$, of (2) and for any solution, $v^A(x,\lambda')$, of the adjoint problem, (23),

$$v^A(x,\lambda')v(x,\lambda)\Big|_{x=-\infty}^{\infty} = i(\lambda-\lambda')\int_{-\infty}^{\infty} v^A(x,\lambda')v(x,\lambda)dx\,. \tag{44}$$

Thus for $\lambda'=\lambda$, $v^A=N^A$, $v=N$ and from (30), (32), (35) and (37), it follows that

$$\Gamma^*(\lambda)\Gamma(\lambda)=1\,. \tag{45}$$

Whence we may define a real phase shift, $\theta(\lambda)$, by

$$\Gamma(\lambda)=e^{-i\theta(\lambda)}\,. \tag{46}$$

Let us now define a norm.

$$\langle v^A|v\rangle \equiv \int_{-\infty}^{\infty} v^A(x)v(x)dx\,. \tag{47}$$

Then from the Wronskian relation, (44), it follows that

$$\langle N^A(\lambda')|N(\lambda)\rangle = \frac{2\pi}{\Gamma(\lambda)}\delta(\lambda-\lambda')\,, \tag{48}$$

where $\delta(\lambda-\lambda')$ is the Dirac delta function. Also

$$\langle N_j^A|N(\lambda)\rangle = 0 = \langle N^A(\lambda')|\Phi_j\rangle\,, \tag{49}$$

$$\langle N_j^A|\Phi_k\rangle = 0\,,\quad \text{if}\, j\neq k\,, \tag{50}$$

for λ and λ' real and positive.

One can also use the Wronskian relation with the function $\overline{N}-1$. From (9) and (23), one obtains the general relation

$$\begin{aligned} N^A(x,\lambda')[\overline{N}(x,\lambda)-1]\Big|_{x=-\infty}^{\infty} &= i(\lambda-\lambda')\int_{-\infty}^{\infty} N^A(\xi,\lambda')[\overline{N}(\xi,\lambda)-1]d\xi \\ &\quad -i\int_{-\infty}^{\infty} u(\xi)[N^A(\lambda')]^-(\xi)d\xi\,. \end{aligned} \tag{51}$$

We consider various cases for (51). For $\lambda'=\lambda_j<0$, from (20), (31), (36), (41), (43), (49), (50) and (51), and then using (7) and (25) in obtaining the second line, there results

$$\begin{aligned} \langle N_j^A|\Phi_j\rangle &= i\int_{-\infty}^{\infty} u(\xi)[N_j^A]^-(\xi)d\xi\,, \\ &= -2\pi\lambda_j\,. \end{aligned} \tag{52}$$

This gives us our nonzero inner products of the bound states. Next, for $\lambda' > 0$, from (20), (31), (34), (36), (39), (48), and (51), and again, then using (13), (24), (26) and (45) in obtaining the second line, there results the key relation, necessary for relating $f(\lambda)$ and $\beta(\lambda)$.

$$\begin{aligned} \beta(\lambda') &= -i\Gamma(\lambda')\int_{-\infty}^{\infty} u(\xi)[N^A(\lambda')]^-(\xi)d\xi\,, \\ &= -2\pi i\lambda' f^*(\lambda')\,, \end{aligned} \tag{53}$$

from which we have

$$f(\lambda) = \frac{\beta^*(\lambda)}{2\pi i\lambda}\,, \tag{54}$$

which is exactly the relation that one would find from the linear limit. When we combine (29) with (28) and (53), we also find

$$\partial_\lambda \theta(\lambda) = \frac{\beta^*\beta(\lambda)}{2\pi\lambda}\,. \tag{55}$$

Thus the phase shift will always be a monotonic function of λ.

We can now demonstrate that the imaginary part of γ_j is related to the eigenvalue λ_j, as mentioned under Equation (17). As mentioned before, this already has been shown to follow from the BO limit of the ILW equation [19]. Here, we simply show how to obtain this result purely from the BO eigenvalue problem. Take the inner product of (22) with the state $N_j^A(x)$, and then take the imaginary part. One obtains

$$\mathrm{Im}\int_{-\infty}^{\infty} N_j^A(x)[x\Phi_j(x) - 1]dx = 2\pi\lambda_j \mathrm{Im}\gamma_j\,. \tag{56}$$

Now use (25). The second part of $N_j^A(x)$ is the part of $N_j^A(x)$ which is analytic in the upper half complex x-plane, as is also $[x\Phi_j(x) - 1]$, with the latter vanishing like $1/x$ for large $|x|$. Thus this part gives a zero contribution to the integral. The first part of $N_j^A(x)$ combined with the $x\Phi_j(x)$ term gives a real quantity, so neither does it contribute. What one is left with is simply

$$-\mathrm{Im}\int_{-\infty}^{\infty} \Phi_j^*(x)dx = 2\pi\lambda_j \mathrm{Im}\gamma_j\,. \tag{57}$$

Evaluating the final integral along an infinite semicircle in the lower half complex x-plane, gives the final result

$$\mathrm{Im}\gamma_j = \frac{-1}{2\lambda_j}\,. \tag{58}$$

IV JOST FUNCTIONS FOR A PURE SOLITON SOLUTION

Although N-soliton solutions of the BO equation have been studied often, so far, it seems that there has been no study of the Jost functions for a single soliton. These are very simple, very instructive solutions and their structure demonstrates some of the features of this scattering problem. Thus we give them here and point out some of their more important and unique features.

When the potential is a pure 1-soliton solution, the reflection coefficient $\beta(\lambda)$ must vanish. Then one has $f(\lambda) = 0$ by (54). We can immediately obtain the bound states from (22) and then $\overline{N}(\lambda)$ from (20). Let the bound state eigenvalue be $\lambda_1 = -\frac{1}{2}a$ where $a > 0$. From (22), it follows that

$$\Phi_1(x) = \frac{1}{x + \gamma_1}, \tag{59}$$

where from (58), we take

$$\gamma_1 = -x_0 + i/a, \tag{60}$$

where x_0 will be the center of the soliton and a will be the amplitude. Note that since $a > 0$, we have $\Phi_1(x)$ analytic in the upper half complex x-plane. The potential can now be recovered by (21), which gives

$$\begin{aligned} [u]^+(x) &= \frac{i}{x + \gamma_1}, \\ u(x) &= \frac{2/a}{(x + \gamma_1)(x + \gamma_1^*)}. \end{aligned} \tag{61}$$

Although we are able to recover the N-soliton potential, at the moment, we have no mechanism for the direct construction of the Jost function, $N(x, \lambda)$. Clearly, we can obtain $\overline{N}(x, \lambda)$ from (20) since $\beta(\lambda) = 0$. However, since $f(\lambda)$ is zero, we can only obtain from (11), the seemingly trivial result

$$N(x, \lambda) = n(x)e^{i\lambda x}, \tag{62}$$

where $n(x)$ is independent of λ. This result will hold for any of the N-soliton solutions. (As we shall see later, one can determine $n(x)$, but it will be necessary to first establish the asymptotics of the nongeneric case in the limit of $\lambda \to 0^+$.)

Otherwise, the only means that we have at the moment for reconstruct of $N(x, \lambda)$ is by means of the direct scattering problem, (2) and (4). We illustrate how to do this in general for the one-soliton potential, (61). The general multisoliton case can be treated in the same way. Noting that N is to be analytic in the upper half complex x-plane, the function $[uN]^+$ can be evaluated using the residue theorem, which gives

$$[uN]^+ = u(x)N(x,\lambda) + i\frac{N(x_0+i/a,\lambda)}{x-x_0-i/a}\,. \tag{63}$$

Note that the last term has a pole in the upper half complex x-plane. It now is relatively easily to integrate (2) under the boundary condition (4). The result can be expressed in the form

$$N(x,\lambda) = e^{i\lambda x}\frac{x-x_0-\frac{i}{a}}{x-x_0+\frac{i}{a}}\Big[1+N(x_0+\frac{i}{a},\lambda)\Big\{(1+\frac{2\lambda}{a})e^{\lambda/a}E_1(z)+\frac{2i}{a}\frac{e^{-i\lambda x}}{x-x_0-\frac{i}{a}}\Big\}\Big]\,, \tag{64}$$

where $z = i\lambda(x-x_0-i/a)$ and $E_1(z)$ is the exponential integral, defined by

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt\,, \quad |\arg(z)| < \pi\,. \tag{65}$$

The expressions (63) and (64) still contain the unknown factor $N(x_0+i/a,\lambda)$. However, in view of the fact that $E_1(z)$ has a logarithmic singularity at $z=0$ ($E_1(z) \sim -\gamma - \ln z + O(z)$ as $z \to 0$, where γ is Euler's constant), which is in the *upper* half complex x-plane, one can see that in general, the result (64), is *not* analytic in the upper half plane, unless the products of the coefficients of this term, $N(x_0+i/a,\lambda)(1+2\lambda/a)$ happen to vanish.

If we choose $N(x_0+i/a,\lambda)=0$, then (64) takes on the particularly simple form

$$N(x,\lambda) = e^{i\lambda x}\frac{x-x_0-i/a}{x-x_0+i/a}\,, \tag{66}$$

which is exactly of the form predicted by (62). Furthermore, in the limit of $x \to -\infty$, from (4), (26), and (66), it follows that $\Gamma(\lambda)=1$ for the soliton solution. Note that this is in accordance with (45), (46) and (55), since θ must be a constant for $\beta(\lambda)=0$. It also can be confirmed directly from (27), (61) and (66).

The solution (64) is the continuous spectrum, $\lambda > 0$. For the bound states, we must have $\lambda < 0$. This solutions will be the same as (64), except that the first term, 1, in the first bracket of (64) will be absent. Again, the term $E_1(z)$ can be made to vanish, but now, by taking $(1+\frac{2\lambda}{a})=0$. However, this is just a restatement of (58), in that the imaginary part of γ_1, $(1/a)$, is related to the bound state eigenvalue by $\lambda_1 = -\frac{1}{2}a$. With this, the solution reduces to the bound state solution given by (59).

Another important point of these nongeneric solutions is that from (61) and (66), one obtains

$$\int_{-\infty}^{\infty} u(x)N(x,\lambda)e^{-i\lambda x}dx = 0\,. \tag{67}$$

As we shall shortly see, this has implications for the asymptotics of the Jost functions for nongeneric potentials in the limit of $\lambda \to 0^+$. It shows that the nongeneric potentials can have this integral as being zero, in contrast to generic potentials, for which this integral is always nonzero [1, 19].

V ASYMPTOTICS OF THE JOST FUNCTIONS

In order to extend the inverse scattering equations of Ref. [1] to include the nongeneric cases, it will be necessary to address the asymptotics of these Jost functions, $N(x,\lambda)$ and $\overline{N}(x,\lambda)$, and the reflection coefficient, $\beta(\lambda)$, in the limit of $\lambda \to 0^+$. Let us start with the Green's function form of the solutions [1], (5) and (15). One can reduce these to:

$$\begin{aligned} N(x,\lambda)e^{-i\lambda x} - 1 + i\int_x^\infty u(\xi)N(\xi,\lambda)e^{-i\lambda\xi}d\xi & \\ -\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N(\xi,\lambda)e^{-i\lambda\xi}E_1[i\lambda(x-\xi-i\epsilon)]d\xi &= 0\,, \end{aligned} \tag{68}$$

$$\begin{aligned} \overline{N}(x,\lambda)e^{-i\lambda x} - e^{-i\lambda x} + i\int_x^\infty u(\xi)\overline{N}(\xi,\lambda)e^{-i\lambda\xi}d\xi & \\ -\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)\overline{N}(\xi,\lambda)e^{-i\lambda\xi}E_1[i\lambda(x-\xi-i\epsilon)]d\xi &= 0\,, \end{aligned} \tag{69}$$

where $E_1(z)$ is the exponential integral, defined by (65). Now define

$$\mathcal{E}_1(z,\lambda) = E_1[i\lambda(z-i\epsilon)] + \gamma + \ln(i\lambda)\,, \tag{70}$$

where γ is Euler's constant. In the limit of $\lambda \to 0^+$, we have

$$\mathcal{E}_1(z,\lambda\to 0^+) \to -\ln z + O(z\lambda)\,. \tag{71}$$

The asymptotics given in [1] and [19] handle the generic case quite well. The asymptotics for the nongeneric case can also be included in a like manner, provided one allows for the vanishing of an integral, namely (67), in the limit of $\lambda \to 0^+$. To do this, first note that without any approximations, (68) can very nicely be put into the form

$$\begin{aligned} N_0(x,\lambda)e^{-i\lambda x} - 1 + i\int_x^\infty u(\xi)N_0(\xi,\lambda)e^{-i\lambda\xi}d\xi & \\ -\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N_0(\xi,\lambda)e^{-i\lambda\xi}d\xi\,\mathcal{E}_1[i\lambda(x-\xi-i\epsilon)] &= 0\,, \end{aligned} \tag{72}$$

where

$$N_0(x,\lambda) = N(x,\lambda)/\mathcal{N}(\lambda)\,, \tag{73}$$

$$\begin{aligned} \mathcal{N}(\lambda) &= 1 - [\gamma + \ln(i\lambda)]\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N(\xi,\lambda)e^{-i\lambda\xi}d\xi\,, \\ &= \frac{1}{1 + [\gamma+\ln(i\lambda)]\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N_0(\xi,\lambda)e^{-i\lambda\xi}d\xi}\,, \end{aligned} \tag{74}$$

with $\mathcal{N}(\lambda)$ being a constant, independent of x. As is clear from (72), the function $N_0(x,\lambda)$ satisfies a *non*homogeneous integral equation and therefore is uniquely determined. Thus it cannot be normalized. Furthermore, for $\lambda \to 0^+$, due to (70), the solution for $N_0(x,\lambda)$ can be expanded in an asymptotic power series about $\lambda = 0$, and when so expanded, will have no $\ln\lambda$ terms present. Thus

$$N_0(x,\lambda) \to N_{00}(x) + O(\lambda)\,, \tag{75}$$

where $N_{00}(x)$ is the solution of (72) for $\lambda = 0$ and satisfies

$$N_{00}(x) - 1 + i\int_x^\infty u(\xi)N_{00}(\xi)d\xi + \frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N_{00}(\xi)\ln(x-\xi-i\epsilon)d\xi = 0\,. \tag{76}$$

Clearly, $\overline{N}(x,\lambda)$ will have a similar structure and can be put into a similar form.

$$\begin{aligned}\overline{N}_0(x,\lambda)e^{-i\lambda x} - 1 + \frac{1-e^{-i\lambda x}}{\overline{\mathcal{N}}(\lambda)} + i\int_x^\infty u(\xi)\overline{N}_0(\xi,\lambda)e^{-i\lambda\xi}d\xi \\ -\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)N_0(\xi,\lambda)e^{-i\lambda\xi}d\xi\mathcal{E}_1[i\lambda(x-\xi-i\epsilon)] &= 0\,,\end{aligned} \tag{77}$$

where

$$\overline{N}_0(x,\lambda) = \overline{N}(x,\lambda)/\overline{\mathcal{N}}(\lambda)\,, \tag{78}$$

$$\overline{\mathcal{N}}(\lambda) = \frac{1}{1+[\gamma+\ln(i\lambda)]\frac{1}{2\pi}\int_{-\infty}^\infty u(\xi)\overline{N}_0(\xi,\lambda)e^{-i\lambda\xi}d\xi}\,. \tag{79}$$

However, these asymptotics will be slightly different. Due to the presence of $\overline{\mathcal{N}}(\lambda)$ in (77), we have instead

$$\overline{N}_0(x,\lambda) \to N_{00}(x) + O(\lambda) + O\Big(\frac{\lambda}{\overline{\mathcal{N}}(\lambda)}\Big)\,, \tag{80}$$

where the function $N_{00}(x)$ is the same function as in (75), which is given by (76).

The terms $\mathcal{N}(\lambda)$ and $\overline{\mathcal{N}}(\lambda)$ contain the asymptotic forms of the λ dependence of the Jost functions for all possible cases, generic and nongeneric. We already know that there are solutions for which the integral

$$I_{00} \equiv \int_{-\infty}^\infty u(\xi)N_{00}(\xi)d\xi\,, \tag{81}$$

is exactly zero, (67), which are the N-soliton solutions. It is not known if there are other examples, but it is possible that initial value problems where $u(x)$ has a vanishing small amplitude with a vanishing area may also have this integral zero.

Before continuing, we briefly comment concerning the asymptotic forms of these Jost functions. It should be clear that the above forms will be useful only as long as $|\lambda x| << 1$, since it is only then that we can find the expansion (71) useful. However all equations given here, from (68) to (80), are exact, except for the three asymptotic forms; (71), (75) and (80). Now, take a wider view of these asymptotics. For any given, sufficiently small value of $\lambda > 0$, there will always be 4 different regions, each containing different asymptotics for the Jost functions. First, there is the region where $u(x)$ is essentially nonzero. Second, (and considering only the regions to the right of the potential, there are also similar regions to the left) there is the region where $u(x)$ has become effectively zero, but one still has $x\lambda << 1$. Here, from (76), one has $N_{00}(x) \to 1 - (\ln x)\frac{1}{2\pi}I_{00}$, which, when the integral is nonzero, contains the $\ln x$ dependence found in [1] and [19] for the generic case. Continuing, for any given $\lambda > 0$, there will exist values of x for which $\lambda x = O(1)$, with the potential also being effectively zero. This is the third region, where we no longer can use the asymptotics provided by (70), but must return to the original form given by (68). This is a transition region, where the nature of the solution is changing from an interior solution to an exterior solution. The asymptotics here are now determined by $E_1(x\lambda)$ instead. Lastly, there is the fourth region, where for any given $\lambda > 0$, x has become so large that $x\lambda >> 1$. Here there is no remaining $\ln x$ dependence in the Jost functions and instead the Jost functions are approaching the appropriate values at $x = \infty$.

Now, let us turn to the asymptotics of the scattering data. As in Ref. [1], we can obtain this information from the spatial integrals over a product of the potential with a Jost function. To evaluate the asymptotics of such a integral, we only need to consider the asymptotics of the Jost function in the first region, since we will be assuming that $u(x)$ is sufficiently localized to ensure that the resulting limit of the integral is valid. The integrals that we will consider are (13), (19), and (27). We then use (18) to eliminate $M(x,\lambda)$ and (54) to eliminate $f(\lambda)$. This gives us the exact relations

$$\beta^*(\lambda) = -i\int_{-\infty}^{\infty} u(\xi)N(\xi,\lambda)d\xi\,, \tag{82}$$

$$\beta(\lambda) = i\Gamma(\lambda)\int_{-\infty}^{\infty} u(\xi)\overline{N}(\xi,\lambda)e^{-i\lambda\xi}d\xi\,, \tag{83}$$

$$\frac{1}{\Gamma(\lambda)} = 1 - i\int_{-\infty}^{\infty} u(\xi)N(\xi,\lambda)e^{-i\lambda\xi}d\xi\,. \tag{84}$$

(Due to the reality of $u(x)$, we are guaranteed by (54) that (82) will be the complex conjugate of (83). One finds that either relation may be used to obtain the asymptotics of $\beta(\lambda)$.) Now consider the Jost functions and the above scattering quantities in the limit of $\lambda \to 0^+$. We have two possibilities. Either the integral I_{00} is zero or it is nonzero. If it is nonzero, then we have the generic case where

$$\mathcal{N}(\lambda) \to \frac{2\pi}{I_{00}\ln\lambda} + O(\frac{1}{\ln^2\lambda})\,, \tag{85}$$

$$\overline{\mathcal{N}}(\lambda) \to \frac{2\pi}{I_{00}\ln\lambda} + O(\frac{1}{\ln^2\lambda})\,, \tag{86}$$

$$N(x,\lambda) \to \frac{2\pi N_{00}(x)}{I_{00}\ln\lambda} + O(\frac{1}{\ln^2\lambda})\,, \tag{87}$$

$$\overline{N}(x,\lambda) \rightarrow \frac{2\pi N_{00}(x)}{I_{00}\ln\lambda} + O(\frac{1}{\ln^2\lambda}), \tag{88}$$

$$\beta(\lambda) \rightarrow i\frac{2\pi}{\ln\lambda} + O(\frac{1}{\ln^2\lambda}), \tag{89}$$

$$\Gamma(\lambda) \rightarrow 1 + i\frac{2\pi}{\ln\lambda} + O(\frac{1}{\ln^2\lambda}). \tag{90}$$

If the integral I_{00} is zero, we then have the nongeneric case, where instead

$$\mathcal{N}(\lambda) \rightarrow 1 + O(\lambda\ln\lambda), \tag{91}$$

$$\overline{\mathcal{N}}(\lambda) \rightarrow 1 + O(\lambda\ln\lambda), \tag{92}$$

$$N(x,\lambda) \rightarrow N_{00}(x) + O(\lambda\ln\lambda), \tag{93}$$

$$\overline{N}(x,\lambda) \rightarrow N_{00}(x) + O(\lambda\ln\lambda), \tag{94}$$

$$\beta(\lambda) \rightarrow O(\lambda\ln\lambda), \tag{95}$$

$$\Gamma(\lambda) \rightarrow 1 + O(\lambda\ln\lambda). \tag{96}$$

From (55), (90) and (96), we now have the full solution for $\theta(\lambda)$. It is

$$\theta(\lambda) = \frac{1}{2\pi}\int_0^\lambda \frac{d\lambda}{\lambda}\beta^*\beta(\lambda). \tag{97}$$

Given initial scattering data for the BO equation, it is a simple matter to determine whether one has the case I_{00} zero (the nongeneric case) or nonzero (the generic case). Simply look at the asymptotics of $\beta(\lambda \rightarrow 0^+)$ and determine how fast the reflection coefficient is vanishing. Note that we do not assume that the nongeneric case *only* includes the N-soliton solutions, which requires $\beta(\lambda) = 0$. Although the nongeneric case does contain the N-soliton solutions, we have not been able to exclude the possibility of the existence of nongeneric cases where $\beta(\lambda) \neq 0$.

VI SIMPLIFICATION OF, AND EXTENSION OF THE IST FOR THE BO EQUATION

With the above, we may now simplify the last linear dispersion relation (Eq.(26) of Ref. [1]) to the case of a real potential, and also extend it so as to include nongeneric, as well as generic potentials. We shall proceed as in Ref. [1] except for one key difference. We shall first subtract off a collection of terms that, upon integration, would give singular integrals.

Consider (11) and (20). If one attempts to integrate the resulting expression directly, one obtains a singular result in the limit of $\lambda \rightarrow 0^+$. In Ref. [1], this was handled by subtracting off the singular part of the reflection coefficient, $f(\lambda)$, into the factor $f_s(\lambda)$, and then integrating the terms in such a fashion, so that the singular integrals would cancell. Here, we shall use a slightly different procedure for handling these singular terms. We shall simply subtract off the limit of a Jost function in the limit of $\lambda \to 0^+$. This Jost function vanishes in this limit for generic potentials, but for nongeneric potentials, it is a key component of the solution. To see this, take the limit of $\lambda \to 0^+$ in (20). There results

$$\overline{N}(x,0) = 1 + \frac{1}{2\pi i}\int_0^\infty \frac{\beta(\lambda')N(x,\lambda')d\lambda'}{\lambda'} + i\sum_j \frac{1}{\lambda_j}\Phi_j(x)\,. \tag{98}$$

As we have seen in the previous section, the function $\overline{N}(x,0)$ is either 0 or $N_{00}(x)$, depending on whether I_{00} is nonzero (generic) or zero (nongeneric). Note that in either case, the integral in (98) is not divergent as $\lambda \to 0^+$. (Note also that in the generic case, (98) is then an *identity* and therefore a *constraint* on the Jost functions and scattering data, particularly the parts that give rise to the singular integrals. Thus this proceedure could have been used equally well in Ref. [1], which would have given them a simplify version of their Eq. (26).)

Let us now use (98) to subtract off the parts of (20) which would give singular integrals. Simply subtracting gives

$$\overline{N}(x,\lambda) = \overline{N}(x,0) + \frac{\lambda}{2\pi i}\int_0^\infty \frac{\beta(\lambda')N(x,\lambda')d\lambda'}{\lambda'(\lambda'-\lambda+i\epsilon)} - i\lambda\sum_{j=1}^{J}\frac{1}{\lambda_j(\lambda-\lambda_j)}\Phi_j(x)\,. \tag{99}$$

Note that the last two terms are now proportional to λ.

Using (11) to eliminate $\overline{N}(x,\lambda)$ from (99), then as in Ref. [1], we may integrate the result from $\lambda \to 0^+$ to λ, and noting that $N(x,0) = \overline{N}(x,0)$, we obtain

$$\begin{aligned} N(x,\lambda)e^{-i\lambda x} \;=\;& \overline{N}(x,0) - iw(x,\lambda,0)\overline{N}(x,0) - \sum_{j=1}^{J}\frac{1}{\lambda_j}\Phi_j(x)w(x,\lambda,\lambda_j) \\ &+\frac{1}{2\pi}\int_0^\infty \frac{d\lambda'}{\lambda'}\beta(\lambda')N(x,\lambda')w(x,\lambda,\lambda')\,, \end{aligned} \tag{100}$$

where

$$w(x,\lambda,\lambda') = \frac{1}{2\pi}\int_0^\lambda \frac{\beta^*(\ell)e^{-i\ell x}d\ell}{\ell-\lambda'-i\epsilon}\,, \tag{101}$$

for λ' real (and either positive or negative). This is the simplified and extended form of Eqs. (26) and (28) of Ref. [1]. As one may readily verify from (87)–(95), all the integrals in (100) and (101) are well defined and nonsingular at the lower limit, except for the term $w(x,\lambda,0)$ in the generic case ($I_{00} \neq 0$). However in this case, since $\overline{N}(x,0)$ is zero anyway, the quantity $w(x,\lambda,0)$ is never required.

In the nongeneric case, ($I_{00} = 0$), we close our set of equations by including (98) in the set of linear dispersion relations. Our final set of inverse scattering equations, valid only for real potentials, but extended to include nongeneric potentials, are therefore Eqs. (22), (98), and (100), with recovery of the potential being achieved by (21). Note that once these solutions are obtained, then one may construct $\overline{N}(x,\lambda)$ from either (20) or (99).

For N-soliton solutions, since $\beta(\lambda)=0$, all terms in (100) vanish, except for the first term, $\overline{N}(x,0)$, which can now be given by (98). In this case, (100) reduces to (62) with its value being given by (98).

This particular structure of a set of inverse scattering equations deserves some comments. As already noted, in the generic case ($I_{00}\neq 0$), (98) gives a *constraint* on the scattering data, since the left-hand side is zero. This can be understood as follows. Due to (87), (88), and (89), the reflection coefficient and the Jost functions have been predetermined at one point, $\lambda\to 0^+$. Thus one degree of freedom is missing from the continuous spectrum, which is taken up by the nonzero value of I_{00}. On the other hand, in the nongeneric case, ($I_{00}=0$), then the reflection coefficient and the Jost functions are not predetermined in this limit (only the order is known, not the coefficient), and (98) and (100) then are independent equations, with no degrees of freedom missing.

The time dependence of the scattering data for the BO equation was first given in Ref. [1]. We include it here for completeness, and it is simply

$$\frac{d\lambda_j(t)}{dt} = 0\,, \quad j=1,2,\ldots,J\,, \tag{102}$$

$$\frac{d\gamma_j(t)}{dt} = 2\lambda_j(t)\,, \quad j=1,2,\ldots,J\,, \tag{103}$$

$$\frac{d\beta(\lambda,t)}{dt} = i\lambda^2\beta(\lambda,t)\,. \tag{104}$$

VII AN INFINITY OF CONSERVATION LAWS

Although it has long been known that the BO equation has an infinity of conserved quantities [8, 9, 21], no general scheme has ever been given for their generation from the scattering problem [22, 23, 24], nor has anyone ever given them in terms of the scattering data. This we shall do here.

Start with the eigenvalue equation for $\overline{N}(x,\lambda)$, (9), and its time evolution

$$\overline{N}_x - i\lambda\overline{N} - i[u\overline{N}]^+ = -i\lambda\,, \tag{105}$$

$$\overline{N}_t - 2\lambda\overline{N}_x - i\overline{N}_{xx} - 2[u]_x^+\overline{N} = 0\,. \tag{106}$$

Then from (1), (105) and (106), one has that

$$\int_{-\infty}^{\infty}[u(x)\overline{N}(x,\lambda)]_t dx = 0\,. \tag{107}$$

Thus the quantity $\int_{-\infty}^{\infty} u(x)\overline{N}(x,\lambda)dx$ is conserved. This conserved quantity can be evaluated in terms of the scattering data. It follows from (20) that

$$\int_{-\infty}^{\infty} u(x)[\overline{N}(x,\lambda)-1]dx = \frac{1}{2\pi}\int_0^{\infty} \frac{\beta^*\beta(\lambda',0)}{\lambda'-\lambda+i\epsilon}d\lambda' + 2\pi\sum_{j=1}^{J}\frac{\lambda_j}{\lambda-\lambda_j}, \qquad (108)$$

where (7), (13), (54) and (104) have been used.

Now, expand $\overline{N}(x,\lambda)$ in inverse powers of λ, as

$$\overline{N}(x,\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n\overline{N}_{n+1}(x)}{\lambda^n}, \quad (\overline{N}_1(x)\equiv 1). \qquad (109)$$

Substitute this into (105). One can now determine $\overline{N}_{n+1}(x)$ successively from:

$$\overline{N}_1 = 1, \quad \overline{N}_{n+1} = i\overline{N}_{n,x} + [u\overline{N}_n]^+, \ (n\geq 1). \qquad (110)$$

Expanding (108) also in an asymptotic series for λ large, then substituting (109) into the result and comparing the coefficients of λ^{-n} on both sides, we obtain

$$\begin{aligned} I_n &\equiv \int_{-\infty}^{\infty} u(x)\overline{N}_n(x)\,dx, \\ &= \frac{1}{2\pi}(-1)^n\int_0^{\infty}\lambda'^{n-2}\beta^*(\lambda')\beta(\lambda')d\lambda' + 2\pi\sum_{j=1}^{J}(-\lambda_j)^{n-1}, \quad (n\geq 2). \quad (111) \end{aligned}$$

which are the infinity of conserved quantities. Clearly, one can generate $\overline{N}_n(x)$ to any order, and therefore any conserved quantity. Furthermore, note that each conserved quantity is simply the $(n-2)$th moment of the radiation density, $\beta^*(\lambda)\beta(\lambda)$, plus a corresponding term for the soliton contribution. We note that the first conserved quantity, I_2, is

$$\int_{-\infty}^{\infty}\frac{1}{2}u^2dx = 2\pi\sum_{j=1}^{J}(-\lambda_j) + \frac{1}{2\pi}\int_0^{\infty} d\lambda\beta^*(\lambda)\beta(\lambda), \qquad (112)$$

which relates the energy in $u(x)$ to the energy in the scattering data.

While the quantity $\int_{-\infty}^{\infty} u(x)\,dx$ is also conserved, its expansion cannot be obtained from the above. Instead, it has to be obtained by a separate consideration, which we shall now detail. It follows directly upon multiplying (98) by $u(x)$ and integrating from $-\infty$ to ∞, by which we obtain

$$2\pi J = \frac{1}{2\pi}\int_0^{\infty}\frac{d\lambda}{\lambda}\beta^*(\lambda)\beta(\lambda) + \int_{-\infty}^{\infty} u(x)dx, \qquad (113)$$

which relates the number of bound states, J, and the total "quanta" of radiation to the total area under $u(x)$. This is the Anderson-Taflin [17] conservation law and its convergence has been discussed in Ref. [16]. It is the analog of Levinson's Theorem [18] for the BO equation.

An important feature of this conservation law follows from (97). Expressed it as

$$2\pi J = \theta(\infty) + \int_{-\infty}^{\infty} u(x)dx\,, \tag{114}$$

shows that the total number of bound states depends only on the value of the total phase shift, $\theta(\infty)$, and the total area under the profile, both of which are time independent. Also, we see that for a given area, there is no limit on the number of solitons that could be generated, unless the total phase shift was somehow limited. But on the other hand, from (112), we see that for a given total area of a profile and a given total energy, there is a limit on the total number of possible solitons that are larger than any given size.

The form of (114) is very suggestive. Since both J and the total phase shift are positive definite, we see that the total area under the profile is expressible as a positive quantity, $2\pi J$, minus another positive quantity, $\theta(\infty)$. Thus, in some fashion, it seems that we can expect the positive regions of $u(x)$ to be the sources of the solitons while the negative regions would be the sources of the radiation. In fact, in the small dispersion limit of the BO equation, one can derive exactly this condition [25, 26, 27], whereby the strictly positive parts of the profile area determine the number of solitons.

VIII CONCLUSIONS

We have restricted the results of Ref. [1] to real potentials, obtaining a reduction in the number of reflection coefficients required. We have also extended the same to include nongeneric potentials and have generalized their inverse scattering equations to include this case.

When the potential is real, we have shown that $f(\lambda)$ is directly proportional to $\beta^*(\lambda)$. We have also generalized the asymptotics of the Jost functions and the scattering data to the nongeneric case. With only one minor change, we have modified the set of inverse scattering equations introduced in Ref. [1], by the introduction of one new function, $\overline{N}(x,0)$, which vanishes for generic potentials, but is nonzero, and is a key function, for nongeneric potentials.

We note that although N-soliton solutions, for which $\beta(\lambda)$ vanishes identically, are one set of examples of nongeneric potentials, we do not know if there are other sets. This class of potentials is distinguished from generic potentials by the vanishing of a single integral, I_{00}, complex in general, and thus with only two degrees of freedom. Considering this, it then seems quite unlikely that the vanishing of this single integral, with only its two degrees of freedom, would be sufficient to guarantee that $\beta(\lambda)$, would have to vanish identically, for all λ. Considering such, then one expects that there must exist other examples of nongeneric potentials besides N-soliton solutions. We have already noted that one such example could be a zero area pulse with a vanishing small amplitude.

We have also presented the necessary recurrence relations for generating the infinity of conserved quantities from the Jost function, $\overline{N}(x,\lambda)$, and also have given these quantities in terms of the scattering data. We have shown that the Anderson-Taflin [17] conservation law is the analogy of Levinson's Theorem for the BO equation and that it relates the total area of the BO field to a total phase shift and the total number of bound states.

We have also obtained the inner products of the Jost functions with their adjoints. These functions can be expected to be a closed set for any function analytic in the upper half x-plane, and a proof of such could be expected to follow from a similar proof for the ILW equation in Ref. [28]. If we adjoin to these functions their complex conjugates, then the total set would be expected to be complete for any L_2 function. Also, from the inner products, assuming closure, one finds a very suggestive structure of the closure relation. It may be possible to interpret this in some manner, perhaps equivalent to bound state eigenvalues being given by the zeros of $\Gamma(\lambda)$, somehow continued into the negative real λ-axis. This was a key feature of the Zakharov-Shabat IST [15]. This also seems to be implied from the BO limit of the ILW equation [19].

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