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Detection Performance of Signals in Dependent Noise From a Gaussian Mixture Uncertainty Class

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Detection performance in terms of probabilities of detection and false alarm (P_D and P_F) is derived for a desired signal embedded in dependent (correlated) multivariate noise from a Gaussian mixture uncertainty class. This uncertainty class is defined using upper and lower bounding functions on the univariate Gaussian mixing distribution function. For a given detector operating point, expressions for P_D and P_F are derived. In addition, nontrivial lower and upper bounds of performance are derived which are functions of the bounding functions of the uncertainty class, the detection threshold, and the desired signal strength (only for P_D).

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CONTENTS

DETECTION PERFORMANCE OF SIGNALS IN DEPENDENT NOISE FROM A GAUSSIAN MIXTURE UNCERTAINTY CLASS

I. INTRODUCTION

The discrete-time detection of signals in non-Gaussian, dependent (correlated) noise is a difficult problem of fundamental interest. Aside from the non-Gaussian aspect, the problem is complicated by the correlation in the noise data. To attack the detection problem requires specification of a multivariate non-Gaussian distribution that accounts for both the correlation function and the non-Gaussian marginal distributions. It is desirable to find a multivariate distribution of sufficient generality to model the desired noise characteristics (i.e. correlation and non-Gaussian marginals), yet of sufficient tractability to yield a solution to the detection problem. A general class of distributions that allows one to accomplish this is the Gaussian mixture distributions (GMD).

The GMD has found utility in modeling radar clutter [1-6,18]. A Rayleigh mixture distribution (RMD), which is closely associated with the GMD, is often used to model the amplitude characteristics of radar clutter [1-6]. Such common amplitude clutter models as Weibull, K, and contaminated Rayleigh can be represented as RMD's [1,6]. These models were formulated to account for the target-like spikes that were observed in experimental clutter data [2,3,7-12]. For multivariate clutter modeling, the RMD has a natural extension to modeling clutter as a multivariate GMD where often the only knowledge one has of the inputs is the univariate amplitude distribution and the correlation matrix [1,6,18].

Hence much of the information necessary to define a detection scheme for a desired signal embedded in correlated clutter modeled as a GMD is known. For our purposes, this detection structure takes the form of the Generalized Likelihood Ratio Test (GLRT) [13] whereby the unknown complex amplitude of the desired signal is estimated.

Quite often the univariate mixing distribution of the RMD (or GMD) is unknown but can be lower and upper bounded (e.g., using histograms constructed from measured input data). Hence, if one designs a detector for a given GMD which is within this class of bounded distributions, the detection performance will vary from the expected nominal performance when the inputs are not this same GMD but still a member of this class.

A minimax detector approach (see. e.g., [14,15]) for finding a good detector for a given input uncertainty class is as follows. Find a distribution, $\tilde{P}_{op}$, that is a member of the given input uncertainty class on which we base our detection scheme (this is sometimes called the detection operating point) which has the following saddle point property; $R(\tilde{P}_{op}, P_{op}) \leq R(\tilde{P}_{op}, \tilde{P}_{op}) \leq R(P_i, \tilde{P}_{op})$. For these inequalities $R(\cdot,\cdot)$ is some measure of the detector's performance where the first argument indicates the input distribution and the second argument indicates the detector operating point distribution. The larger R is the better we judge the detector's performance. P_{op} and P_i are any distributions from the input uncertainty class.

We do not use the minimax detector approach in that we choose any distribution from within the uncertainty class as a detector operating point. Thereafter we quantify the performance of the detector when the input distribution is any from the uncertainty class. The performance will degrade when the input distribution is not matched with the detector's operating point distribution. Specifically, in this paper, we derive the GLRT and performance measures (probabilities of detection and false alarm) associated with a given GMD (the designed operating point) for inputs from a given GMD uncertainty class of bounded distributions. In addition, nontrivial upper and lower bounds of performance are derived

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which are functions of the bounding functions of the uncertainty class, the detection threshold, and for the detection probability, the desired signal strength.

II. PRELIMINARIES

A. Gaussian Mixture Detectors

Consider the following binary hypotheses

$$\begin{aligned} H_0 &: \quad \mathbf{z} = \mathbf{x} \\ H_1 &: \quad \mathbf{z} = a\mathbf{s}_0 + \mathbf{x} \end{aligned} \tag{1}$$

where $\mathbf{z}$, $\mathbf{x}$, and $\mathbf{s}_0$ are N-length vectors belonging to the N-dimensional complex space, C^N, and $\mathbf{z} = \{z_n;\ n = 1, 2, \ldots, N\}$ is a sequence of observations or samples. The H_0 hypothesis corresponds to the case when no desired signal is present. The H_1 hypothesis corresponds to the case where the desired signal vector, $a\mathbf{s}_0$, is corrupted by an additive zero-mean noise vector, $\mathbf{x}$. Here a is the unknown complex amplitude of a known signal vector, $\mathbf{s}_0$, and $\mathbf{s}_0$ is normalized so that $\mathbf{s}_0^H\, \mathbf{s}_0 = 1$ where H denotes the conjugate transpose operation.

If the noise is modelled as a GMD then $\mathbf{z}$ has the following N-variate probability density function (pdf) under H_i, $(i = 0,1)$:

$$f_i(\mathbf{z}) = \frac{1}{\pi^N \, det\, \Phi} \int_{\Omega_\tau} \frac{1}{\tau^N} \exp\left[-\frac{q_i}{\tau}\right] dP_\tau \tag{2}$$

where

$$q_i = \begin{cases} \mathbf{z}^H\, \Phi^{-1}\, \mathbf{z} & , i = 0, \\ (\mathbf{z} - a\mathbf{s}_0)^H\, \Phi^{-1}\, (\mathbf{z} - a\mathbf{s}_0) & , i = 1, \end{cases} \tag{3}$$

Φ = normalized covariance matrix of $\mathbf{x}$ (assumed nonsingular),
τ = variance of any element of $\mathbf{x}$,
P_τ = Gaussian mixing distribution,
Ω_τ = support of τ, $\Omega_\tau \subseteq [0, \infty)$,

and det denotes the matrix determinant. The normalized covariance matrix is defined as follows. Set $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T$ where T denote the transpose operation. Then

$$\Phi = \frac{E\{\mathbf{x}\mathbf{x}^H\}}{E\{\tau\}}$$

where E denotes the expected value. For the case where the N inputs are equi-spaced time samples from a wide-sense stationary process, Φ is a normalized correlation matrix with the Hermitian-Toeplitz form where each diagonal element is equal to one.

If the complex amplitude, a, were known then the Neyman-Pearson likelihood ratio (LR) statistic is given by

$$\lambda = \frac{f_1(\mathbf{z})}{f_0(\mathbf{z})}. \tag{4}$$

Define the function F_N as

$$F_N(\beta) = \int_{\Omega_\tau} \frac{1}{\tau^N} \exp\left[-\frac{\beta}{\tau}\right] dP_\tau. \tag{5}$$

Then it is seen that the LR detector is given by

$$\lambda = \frac{F_N((\mathbf{z} - a\mathbf{s}_0)^{\mathrm{H}}\ \Phi^{-1}\ (\mathbf{z} - a\mathbf{s}_0))}{F_N(\mathbf{z}^{\mathrm{H}}\ \Phi^{-1}\ \mathbf{z})}. \tag{6}$$

For many detection problems, the complex signal amplitude of the desired signal vector is unknown to the detector. In general if one has unknown parameters in the LR detector, a scheme known as the Generalized Likelihood Ratio Test (GLRT) is commonly used whereby the unknown parameters are found using their maximum likelihood (ML) estimates under each hypothesis [13]. These estimates are then inserted into the respective pdf's (under H_0 and H_1) and the LR is formed. We now derive the GLRT for detection of a signal with unknown complex amplitude in additive Gaussian mixture noise.

The GLRT for our problem is given by

$$\lambda = \frac{F_N((\mathbf{z} - \hat{a}\mathbf{s}_0)^{\mathrm{H}}\ \Phi^{-1}\ (\mathbf{z} - \hat{a}\mathbf{s}_0))}{F_N(\mathbf{z}^{\mathrm{H}}\ \Phi^{-1}\ \mathbf{z})} \tag{7}$$

where

$$\hat{a} = \arg\max_a F_N((\mathbf{z} - a\mathbf{s}_0)^{\mathrm{H}}\ \Phi^{-1}\ (\mathbf{z} - a\mathbf{s}_0)). \tag{8}$$

The maximum of F_N with respect to a can be found by setting $dF_N/da = 0$ and solving for a. Now

$$\frac{dF_N}{da} = \frac{d\beta}{da}\frac{dF_N}{d\beta} = 0 \tag{9}$$

where

$$\beta = (\mathbf{z} - a\mathbf{s}_0)^{\mathrm{H}}\ \Phi^{-1}\ (\mathbf{z} - a\mathbf{s}_0). \tag{10}$$

Using (5), we can show

$$\frac{dF_N}{d\beta} = - \int_{\Omega_\tau} \frac{1}{\tau^{N+1}} \exp\left[-\frac{\beta}{\tau}\right] dP_\tau . \tag{11}$$

From (11) it is seen that $dF_N/d\beta < 0$. Thus all solutions for a in Eq. (9) must satisfy $d\beta/da = 0$. It is straightforward to show that this equation has only one solution which is given by

$$\hat{a} = \frac{\mathbf{s}_0^H \, \Phi^{-1}\mathbf{z}}{\mathbf{s}_0^H \, \Phi^{-1}\mathbf{s}_0}. \tag{12}$$

Note that (12) is the ML estimate of the complex signal amplitude when the contaminating noise is purely Gaussian. Inserting (12) into (7) results in the GLRT for Gaussian mixtures:

$$\lambda = \frac{F_N(\mathbf{z}^H \, \Phi^{-1} \, \mathbf{z} \, (1 - |\rho|^2))}{F_N(\mathbf{z}^H \, \Phi^{-1} \, \mathbf{z})} \tag{13}$$

where $|\cdot|$ denotes magnitude,

$$|\rho|^2 = \frac{|\mathbf{s}_0^H \, \Phi^{-1} \, \mathbf{z}|^2}{(\mathbf{z}^H \, \Phi^{-1} \, \mathbf{z}) \, (\mathbf{s}_0^H \, \Phi^{-1} \, \mathbf{s}_0)}, \tag{14}$$

and $0 \leq |\rho| \leq 1$.

B. Problem Statement

Let $\mathcal{P}$ be the set of all univariate distribution functions that have support $\Omega \subseteq [0,\infty]$, are right-continuous at $\tau = 0$, and equal zero at $\tau = 0$. Let $\underline{G}$ and $\bar{G}$ be non-negative, right-continuous, bounded monotonically increasing functions (not necessarily distribution functions) on $[0,\infty]$ which are zero at $\tau = 0$. Define the uncertainty class $\mathcal{P}_\tau$ as

$$\mathcal{P}_\tau = \{P_\tau \,|\, \underline{G} \leq P_\tau \leq \bar{G}, \; P_\tau \in \mathcal{P}\}. \tag{15}$$

Often one implements a detection scheme based on a representative element of $\mathcal{P}_\tau$, say $P_\tau^{(1)}$; i.e. the GLRT given in the previous subsection is implemented with $P_\tau = P_\tau^{(1)}$. $P_\tau^{(1)}$ will be called the detector operating point. If the actual multivariate inputs had an underlying distribution $P_\tau^{(2)} \in \mathcal{P}_\tau$, we are interested in formulating the detection and false alarm probability performance which we denote by $P_D(P_\tau^{(1)}, P_\tau^{(2)}, T)$ and $P_F(P_\tau^{(1)}, P_\tau^{(2)}, T)$, respectively. In addition, we would like to find nontrivial upper and lower bounds on these performance measures such that

$$L_D(\underline{G}, \bar{G}, T) \leq P_D(P_\tau^{(1)}, P_\tau^{(2)}, T) \leq U_D(\underline{G}, \bar{G}, T) \tag{16}$$

and

$$L_F(\underline{G},\bar{G},T) \leq P_F(P_T^{(1)},P_T^{(2)},T) \leq U_F(\underline{G},\bar{G},T) \tag{17}$$

where the upper and lower bounds are functions of just $\underline{G}$, $\bar{G}$, and the detection threshold, T. The detection measures, P_D, L_D, and U_D, are also a function of the signal-to-noise ratio. In the following sections, expressions for P_D, P_F, L_D, U_D, L_F, and U_F are derived.

III. P_D AND P_F DERIVATION

A. Equivalent GLRT

In this subsection we present a simplified equivalent GLRT which will allow us in the next section to find explicit results for the GMD detector's performance. To this end, it is known [16] that if Φ is non-singular, then we can matrix transform the input vector, $\mathbf{z}$, by a non-singular $N \times N$ matrix A, which has the properties that the input noise vector is whitened, each input element has noise power normalized to one, and

$$A\mathbf{s}_0 = ((\mathbf{s}_0^{\mathrm{H}} \Phi^{-1} \mathbf{s}_0)^{1/2}, 0,0,\ldots,0)^{\mathrm{T}} \equiv \tilde{\mathbf{s}}_0. \tag{18}$$

We note that all of the desired signal has been placed into the first element by the matrix transform. Thus if $\mathbf{u} = A\mathbf{z}$ where $\mathbf{u} = (u_1, u_2, \ldots, u_N)^{\mathrm{T}}$, then under H_0

$$E\{\mathbf{u}\mathbf{u}^{\mathrm{H}}\} = I_N \tag{19}$$

where I_N is the $N \times N$ identity matrix. Transforming the input by a non-singular matrix transform, A, does not change the performance measures of the GLRT that were discussed in the preceding section.

If we use the vector, $\mathbf{u}$, as our starting point in determining the GLRT, it can be shown that the N-variate pdf of $\mathbf{u}$ under the two hypothesis is given by (2) and (3) by setting $\Phi = I_N$. Furthermore, the GLRT is given by (13) and (14) with $\Phi = I_N$ as

$$\tilde{\lambda} = \frac{F_N(\mathbf{u}^{\mathrm{H}}\,\mathbf{u}\,(1 - |\tilde{\rho}|^2)}{F_N\,(\mathbf{u}^{\mathrm{H}}\,\mathbf{u})} \tag{20}$$

where F_N is defined by (5) and

$$|\tilde{\rho}|^2 = \frac{|\tilde{\mathbf{s}}_0^{\mathrm{H}}\,\mathbf{u}|^2}{(\mathbf{u}^{\mathrm{H}}\,\mathbf{u})\,(\tilde{\mathbf{s}}_0^{\mathrm{H}}\,\tilde{\mathbf{s}}_0)} = \frac{|u_1|^2}{\sum_{n=1}^{N} |u_n|^2}. \tag{21}$$

Substituting (21) into (20) results in

$$\tilde{\lambda} = \frac{F_N\left[\sum_{n=2}^{N} |u_n|^2\right]}{F_N\left[\sum_{n=1}^{N} |u_n|^2\right]}. \tag{22}$$

Define the N-variate function, Λ, such that $\tilde{\lambda} = \Lambda(\mathbf{u})$. A functional block diagram of the equivalent GLRT is shown in Fig. 1. In this figure, T is the detector threshold.

B. Derivation

In this section, we derive closed-form solutions for the probabilities of detection, $P_D\,(P_\tau^{(1)}, P_\tau^{(2)}, T)$, and false alarm, $P_F\,(P_\tau^{(1)}, P_\tau^{(2)}, T)$, associated with the GLRT discussed in the previous section. After the A matrix transformation, the desired signal is contained only in the first element of $\mathbf{u}$ under H_1, and is equal to $a(s_0^H\, \Phi^{-1}\, s_0)^{1/2}$. We set $a = a_0\, e^{j\phi}$ where ϕ is the phase and a_0 the magnitude of a. Note that at this point in processing, the first element of $\mathbf{u}$ is the matched filter output and that the signal-to-noise (S/N) associated with the matched filter denoted by $(S/N)_{opt}$ can be expressed as

$$(S/N)_{opt} = a_0^2\, s_0^H\, \Phi^{-1}\, s_0. \tag{23}$$

Set $\tilde{a}_0 = a_0(s_0^H\, \Phi^{-1}\, s_0)^{1/2}$. We now analyze the statistically equivalent GLRT presented in the previous subsection. Set

$$w = |u_1|^2 \tag{24}$$

$$v = \sum_{n=2}^{N} |u_n|^2. \tag{25}$$

We note that the random variables (r.v.'s) $|u_1|^2, |u_2|^2, \ldots, |u_N|^2$ when conditioned on τ are statistically independent. Thus w and v when conditioned on τ are statistically independent. It is straightforward to show that v when conditioned on τ under H_0 or H_1 has a $2(N-1)$ Chi-square pdf [17] with $\sigma^2 = .5$ and is given by

$$p_{v|\tau}(v|\tau) = \frac{1}{\tau^{N-1}(N-2)!}\, v^{N-2}\, e^{-v/\tau} \tag{26}$$

where $p_{\cdot|\tau}\,(\cdot|\tau)$ denotes that the pdf is conditioned on τ.

Under H_0 and conditioned on τ, u_1 is a complex circular Gaussian r.v. with variance equal to τ and under H_1, $|u_1|$ has a non-central Rayleigh distribution. In fact, it is straightforward to show

$$p_{w|\tau}(w|\tau) = \begin{cases} \dfrac{1}{\tau}\, e^{-w/\tau} & , w \geq 0, \text{ under } H_0 \\ \dfrac{1}{\tau}\, e^{-(w + \tilde{a}_0^2)}\, I_0\left[\dfrac{2\tilde{a}_0\sqrt{w}}{\tau}\right] & , w \geq 0, \text{ under } H_1 \end{cases} \tag{27}$$

where I_0 is the modified 0'th order Bessel function of the first kind.

The decision rule in terms of w and v is

$$\tilde{\lambda} = \frac{F_N^{(1)}(v)}{F_N^{(1)}(w + v)} \underset{H_0}{\overset{H_1}{\gtrless}} T. \tag{28}$$

where the superscript on the F_N indicates that F_N is evaluated using (5) with $P_\tau = P_\tau^{(1)}$. It was shown in the previous section that F_N is strictly monotonically decreasing. As a result, since $T \geq 1$, and because F_N is a one-to-one function, an equivalent decision rule for (28) is

$$w \underset{H_0}{\overset{H_1}{\gtrless}} F_N^{(1)-1}\left[\frac{F_N^{(1)}(v)}{T}\right] - v \equiv T_0(v,T) \tag{29}$$

where F_N^{-1} is the inverse function of F_N. Define $P(D|v,\tau)$ to be the probability of detection when conditioned on v and τ. Thus

$$\begin{aligned} P(D|v,\tau) &= \int_{T_0(v,T)}^{\infty} \frac{1}{\tau}\, e^{-(w + \tilde{a}_0^2)/\tau}\, I_0\left(\frac{2\tilde{a}_0\sqrt{w}}{\tau}\right) dw \\ &= Q\left(\tilde{a}_0\sqrt{\frac{2}{\tau}}, \sqrt{\frac{2T_0(v,T)}{\tau}}\right) \end{aligned} \tag{30}$$

where $Q(\cdot, \cdot)$ is Marcum's Q function [13]. Finally,

$$P_D\,(P_\tau^{(1)}, P_\tau^{(2)}, T) = \int_{\Omega_\tau} \int_0^\infty P(D|v,\tau)\, p_{v|\tau}(v|\tau)\, dv dP_\tau^{(2)}. \tag{31}$$

The false alarm probability can be found by setting $\tilde{a}_0 = 0$ in (30). It will be found that

$$P_F\,(P_\tau^{(1)}, P_\tau^{(2)}, T) = \int_{\Omega_\tau} \int_0^\infty p_{v|\tau}(v|\tau) \exp -\left[\frac{F_N^{(1)-1}\,(F_N^{(1)}(v)/T) - v}{\tau}\right] dv dP_\tau^{(2)}. \tag{32}$$

IV. PERFORMANCE BOUNDS

A. Useful Lemmas

In this section, we prove three lemmas that will be used to derive upper and lower bounds of performance for $P_D(P_\tau^{(1)}, P_\tau^{(2)}, T)$ and $P_F(P_\tau^{(1)}, P_\tau^{(2)}, T)$.

Lemma 1: Let $f(\tau)$ be a differentiable function on $[0,\infty]$ with $f(0) < \infty$ and $f(\infty) = 0$. Define

$$G_U(\tau) = \begin{cases} \bar{G}(\tau) & \text{if} \quad -f'(\tau) \geq 0 \\ \underline{G}(\tau) & \text{otherwise} \end{cases} \tag{33}$$

$$G_L(\tau) = \begin{cases} \underline{G}(\tau) & \text{if} \quad -f'(\tau) \geq 0 \\ \bar{G}(\tau) & \text{otherwise.} \end{cases} \tag{34}$$

where $'$ denotes differentiation with respect to τ. If $P_\tau \in \mathcal{P}_\tau$, then

$$\int_0^\infty f(\tau)\, dG_L \leq \int_{\Omega_\tau} f(\tau)\, dP_\tau \leq \int_0^\infty f(\tau)\, dG_U . \tag{35}$$

Proof: Integrating by parts, we can show

$$\int_{\Omega_\tau} f(\tau)\, dP_\tau = \int_0^\infty f(\tau)\, dP_\tau = -\int_0^\infty f'(\tau) P_\tau(\tau)\, d\tau. \tag{36}$$

Using (33) and (34) and the fact that $\underline{G} \leq P_\tau \leq \bar{G}$, it follows that

$$-\int_0^\infty f'(\tau)\, G_L(\tau)\, d\tau \leq -\int_0^\infty f'(\tau)\, P_\tau(\tau)\, d\tau \leq -\int_0^\infty f'(\tau)\, G_U(\tau)\, d\tau . \tag{37}$$

Integrating each of the above three integrals by parts results in (35). □

Lemma 2: Define

$$\bar{F}(\beta) = \int_0^{\beta/N} \frac{1}{\alpha^N} e^{-\beta/\alpha}\, d\underline{G} + \int_{\beta/N}^\infty \frac{1}{\alpha^N} e^{-\beta/\alpha}\, d\bar{G} \tag{38}$$

$$\underline{F}(\beta) = \int_0^{\beta/N} \frac{1}{\alpha^N} e^{-\beta/\alpha}\, d\bar{G} + \int_{\beta/N}^\infty \frac{1}{\alpha^N} e^{-\beta/\alpha}\, d\underline{G} \tag{39}$$

Then

$$\underline{F}(\beta) \le F_N(\beta) \le \bar{F}(\beta). \tag{40}$$

Proof: Integrating by parts, we can show

$$F_N(\beta) = \int_0^\infty \left[- \frac{d}{d\alpha} \left(\frac{e^{-\beta/\alpha}}{\alpha^N} \right) \right] P_\tau(\tau)\, d\tau . \tag{41}$$

Set

$$\bar{V}(\alpha) = \begin{cases} \bar{G} & \text{if} \quad -(e^{-\beta/\alpha}/\alpha^N)' \ge 0 \ \ ('\text{denotes}\, d/d\alpha) \\ \underline{G} & \text{otherwise} \end{cases} \tag{42}$$

and

$$\underline{V}(\alpha) = \begin{cases} \underline{G} & \text{if} \quad -(e^{-\beta/\alpha}/\alpha^N)' \ge 0 \\ \bar{G} & \text{otherwise.} \end{cases} \tag{43}$$

It is straightforward to show that $-(e^{-\beta/\alpha}/\alpha^N)' \ge 0$ *iff* $\alpha \ge \beta/N$. Also from (42) and (43), it follows that

$$\int_0^\infty \left[- \frac{d}{d\alpha} \left(\frac{e^{-\beta/\alpha}}{\alpha^N} \right) \right] \underline{V}(\alpha)\, d\alpha \le \int_0^\infty \left[- \frac{d}{d\alpha} \left(\frac{e^{-\beta/\alpha}}{\alpha^N} \right) \right] P_\tau(\tau)\, d\tau \le \int_0^\infty \left[- \frac{d}{d\alpha} \left(\frac{e^{-\beta/\alpha}}{\alpha^N} \right) \right] \bar{V}(\alpha)\, d\alpha \tag{44}$$

Integrating each of the above integrals by parts and using the definitions given by (38) and (39), Eq. (40) results. □

Using the definitions of F_N, $\bar{F}$, and $\underline{F}$ given by (5), (38), and (39), respectively, it can be shown that F_N, $\bar{F}$, and $\underline{F}$ are strictly monotonically decreasing functions of β on $[0, \infty]$. Let Ω_{F_N}, $\Omega_{\bar{F}}$, and $\Omega_{\underline{F}}$ be the ranges of F_N, $\bar{F}$, and $\underline{F}$ respectively. Each of these ranges includes 0 as one end point ($\beta = \infty$). The other end point of the ranges may or may not be infinite. Now F_N^{-1}, $\bar{F}^{-1}$, and $\underline{F}^{-1}$ are strictly monotonically decreasing functions. We define on $[0, \infty]$ the following functions:

$$\bar{F}_0^{-1}(\lambda) = \begin{cases} \bar{F}^{-1}(\lambda) & \text{if} \quad \lambda \in \Omega_{\bar{F}} \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

$$\underline{F}_0^{-1}(\lambda) = \begin{cases} \underline{F}^{-1}(\lambda) & \text{if} \quad \lambda \in \Omega_{\underline{F}} \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

and

$$F_{N,0}^{-1}(\lambda) = \begin{cases} F_N^{-1}(\lambda) & \text{if} \quad \lambda \in \Omega_{F_N} \\ 0 & \text{otherwise}. \end{cases} \tag{47}$$

Lemma 3: For $T \geq 1$

$$\underline{F}_0^{-1}\,(\bar{F}(v)/T) \leq F_N^{-1}\,(F_N(v)/T) \leq \bar{F}_0^{-1}\,(\underline{F}(v)/T). \tag{48}$$

Proof: For $T \geq 1$, $F_N\,(v)/T$ is always in the range of F_N. From Lemma 2, $\underline{F} \leq F_N \leq \bar{F}$. Using this and the fact that $\underline{F}$, $\bar{F}$, and F_N are strictly monotonically decreasing, it can be shown that

$\underline{F}_0^{-1} \leq F_{N,0}^{-1} \leq \bar{F}_0^{-1}$

Thus

$$F_{N,0}^{-1}(F_N(v)/T) \geq \underline{F}_0^{-1}\,(F_N(v)/T) \geq \underline{F}_0^{-1}\,(\bar{F}(v)/T) \tag{49}$$

and

$$F_{N,0}^{-1}\,(F_N(v)/T) \leq \bar{F}_0^{-1}\,(F_N(v)/T) \leq \bar{F}_0^{-1}\,(\underline{F}(v)/T). \tag{50}$$

The last inequality in (49) follows because $\underline{F}_0^{-1}$ is monotonically decreasing and $F_N(v)/T \leq \bar{F}(v)/T$. A similar argument establishes the last inequality of (50). Since, for $T \geq 1$, $F_{N,0}^{-1}\,(F_N(v)/T) = F_N^{-1}(F_N(v)/T)$, and the lemma follows.□

B. Derivation

In this subsection, we derive nontrivial lower and upper bounds of performance for $P_D(P_\tau^{(1)}, P_\tau^{(2)}, T)$ and $P_F(P_\tau^{(1)}, P_\tau^{(2)}, T)$.

Theorem 1:

$$L_D(\underline{G}, \bar{G}, T) \leq P_D(P_\tau^{(1)}, P_\tau^{(2)}, T) \leq U_D(\underline{G}, \bar{G}, T) \tag{51}$$

where

$$L_D(\underline{G}, \bar{G}, T) = \int_0^\infty \underline{H}_D(\tau, T)\, dG_{LD} \tag{52}$$

$$U_D(\underline{G}, \bar{G}, T) = \int_0^\infty \bar{H}_D(\tau, T)\, dG_{UD} \tag{53}$$

$$\underline{H}_D(\tau, T) = \int_0^\infty J_D\, [\bar{F}_0^{-1}\, (\underline{F}(v)/T) - v,\ \tau]\, p_{v|\tau}(v|\tau)\, dv \tag{54}$$

$$\bar{H}_D(\tau, T) = \int_0^\infty J_D\, [\underline{F}_0^{-1}\, (\bar{F}(v)/T) - v,\ \tau]\, p_{v|\tau}(v|\tau)\, dv \tag{55}$$

$$G_{LD}(\tau) = \begin{cases} \underline{G}(\tau) & \text{if} \quad -\,\partial \underline{H}_D/\partial\tau \geq 0 \\ \bar{G}(\tau) & \text{otherwise} \end{cases} \tag{56}$$

$$G_{UD}(\tau) = \begin{cases} \bar{G}(\tau) & \text{if} \quad -\,\partial \bar{H}_D/\partial\tau \geq 0 \\ \underline{G}(\tau) & \text{otherwise} \end{cases} \tag{57}$$

and

$$J_D(t, \tau) = \int_t^\infty \frac{1}{\tau}\, e^{-(w + \tilde{a}_0^2)/\tau}\, I_0(2\tilde{a}_0\sqrt{w}\,/\tau)dw = Q\left(\tilde{a}_0\sqrt{\frac{2}{\tau}}, \sqrt{\frac{2t}{\tau}}\right). \tag{58}$$

Proof: From the results of Section III, it can be demonstrated that

$$P_D(P_\tau^{(1)}, P_\tau^{(2)}, T) = \int_{\Omega_\tau} H_D^{(1)}(\tau, T)\, dP_\tau^{(2)} \tag{59}$$

where

$$H_D^{(1)}(\tau, T) = \int_0^\infty \int_{T_0(v,T)}^\infty \frac{1}{\tau}\, e^{-(w + \tilde{a}_0^2)/\tau}\, I_0(2\tilde{a}_0\sqrt{w}/\tau)\, p_{v|\tau}(v|\tau)\, dw dv, \tag{60}$$

$p_{v|\tau}$ is given by (26), and $T_0(v,T)$ is given by (29). Furthermore,

$$H_D^{(1)}(\tau, T) = \int_0^\infty J_D\, [(F_N^{-1}(F_N(v)/T) - v,\ \tau]\, p_{v|\tau}(v|\tau)\, dv. \tag{61}$$

Using Lemma 3, since J_D is monotonically decreasing with respect to its first argument, we can show

$$J_D[\bar{F}_0^{-1}(\underline{F}(v)/T) - v, \tau] \leq J_D\,[F_N^{-1}(F_N(v)/T) - v, \tau] \\ \leq J_D(\underline{F}_0^{-1}\,(\bar{F}(v)/T) - v, \tau]. \tag{62}$$

Since $p_{v|\tau}(v|\tau) \geq 0$, it follows that

$$\underline{H}_D(\tau,T) \leq H_D^{(1)}(\tau,T) \leq \bar{H}_D(\tau,T) \tag{63}$$

and

$$\int_0^\infty \underline{H}_D(\tau,T)\; dP_\tau^{(2)} \leq \int_{\Omega_\tau} H_D^{(1)}(\tau,T)\; dP_\tau^{(2)} \leq \int_0^\infty \bar{H}_D(\tau,T)\; dP_\tau^{(2)}. \tag{64}$$

It is straightforward to show that $\underline{H}_D$ and $\bar{H}_D < \infty$ for $\tau \in [0, \infty]$, $\underline{H}_D(\infty,T) = \bar{H}_D(\infty,T) = 0$, and $\underline{H}_D$ and $\bar{H}_D$ are differentiable on $[0, \infty]$. Thus invoking the appropriate inequalities of Lemma 1:

$$\int_0^\infty \underline{H}_D(\tau,T)\; dG_{LD} \leq \int_0^\infty \underline{H}_D(\tau,T)\; dP_\tau^{(2)} \tag{65}$$

and

$$\int_0^\infty \bar{H}_D(\tau,T)\; dP_\tau^{(2)} \leq \int_0^\infty \bar{H}_D(\tau,T)\; dG_{UD} \tag{66}$$

where G_{LD} and G_{UD} are defined by (56) and (57), respectively. The results of (64)-(66) prove the theorem. □

In similar fashion we can prove

Theorem 2:

$$L_F(\underline{G}, \bar{G},T) \leq P_F(P_\tau^{(1)}, P_\tau^{(2)},T) \leq U_F(\underline{G}, \bar{G},T) \tag{67}$$

where

$$L_F(\underline{G}, \bar{G},T) = \int_0^\infty \underline{H}_F(\tau,T)\; dG_{LF} \tag{68}$$

$$U_F(\underline{G}, \bar{G},T) = \int_0^\infty \bar{H}_F(\tau,T)\; dG_{UF} \tag{69}$$

$$\underline{H}_F(\tau,T) = \int_0^\infty \exp - \left[\frac{\bar{F}_0^{-1}(\underline{F}(v)/T) - v}{\tau} \right] p_{v|\tau}(v|\tau)\ dv \tag{70}$$

$$\bar{H}_F(\tau,T) = \int_0^\infty \exp - \left[\frac{\underline{F}_0^{(1)}(\bar{F}(v)/T) - v}{\tau} \right] p_{v|\tau}(v|\tau)\ dv \tag{71}$$

and

$$G_{LF}(\tau) = \begin{cases} \underline{G}(\tau) & \text{if} \quad -\partial \underline{H}_F/\partial\tau \geq 0 \\ \bar{G}(\tau) & \text{otherwise} \end{cases} \tag{72}$$

$$G_{UF}(\tau) = \begin{cases} \bar{G}(\tau) & \text{if} \quad -\partial \bar{H}_F/\partial\tau \geq 0 \\ \underline{G}(\tau) & \text{otherwise.} \end{cases} \tag{73}$$

C. Performance Spread

For a given detector operating point and threshold, it is desirable to quantify the performance measure spread over the input uncertainty class. Thus we define

$$S_D(P_D^{(1)},T) = \max_{P_\tau^{(2)},\, P_\tau^{(3)} \in \mathcal{P}_\tau} |\ P_D(P_\tau^{(1)},\ P_\tau^{(2)},T) - P_D(P_\tau^{(1)},\ P_\tau^{(3)},T)| \tag{74}$$

and

$$S_F(P_F^{(1)},T) = \max_{P_\tau^{(2)},\, P_\tau^{(3)} \in \mathcal{P}_\tau} |P_F(P_\tau^{(1)},\ P_\tau^{(2)},T) - P_F(P_\tau^{(1)},\ P_\tau^{(3)},T)|. \tag{75}$$

We desire to upper bound the maximum performance measure spread. To this end, define

$$\bar{S}_D(T) = \max_{P_\tau^{(1)} \in \mathcal{P}_\tau} S_D(P_\tau^{(1)},T) \tag{76}$$

$$\bar{S}_F(T) = \max_{P_\tau^{(1)} \in \mathcal{P}_\tau} S_F(P_\tau^{(1)},T). \tag{77}$$

Using Theorems 1 and 2 it is trivial to show that

$$\bar{S}_D(T) \leq U_D(\underline{G},\ \bar{G},T) - L_D(\underline{G},\ \bar{G},T) \tag{78}$$

$$\bar{S}_F(T) \leq U_F(\underline{G}, \bar{G}\ T) - L_F(\underline{G}, \bar{G}, T). \tag{79}$$

The following theorem provides a looser set of upper bounds on $\bar{S}_D$ and $\bar{S}_F$.

Theorem 3: Let $\underline{G}$ and $\bar{G}$ admit density functions $\underline{g}$ and $\bar{g}$, almost everywhere with respect to measure, τ (a.e. [τ]) respectively where $\underline{g} = d\underline{G}/d\tau$, $\bar{g} = d\bar{G}/d\tau$. Then

$$\bar{S}_D(T) \leq \int_0^\infty \bar{H}_D(\tau,T)\ |\bar{g} - \underline{g}|\, d\tau \tag{80}$$

$$\bar{S}_F(T) \leq \int_0^\infty \bar{H}_F(\tau,T)\ |\bar{g} - \underline{g}|\, d\tau. \tag{81}$$

Proof: The proofs of (80) and (81) are similar so that we only present the proof of (80). From (51)-(53), it follows that

$$S_D(P_\tau^{(1)},T) \leq \int_0^\infty H_D^{(1)}(\tau,T)\ d(G_{UD} - G_{LD}). \tag{82}$$

If $\underline{G}$ and $\bar{G}$ admit densities, then G_{LD} and G_{UD} defined by (56) and (57) admit densities a.e. [τ]. Set $g_{LD} = dG_{LD}/d\tau$, $g_{UD} = dG_{UD}/d\tau$. In addition, $|g_{UD} - g_{LD}| = |\bar{g} - \underline{g}|$. Using (82)

$$S_D(P_\tau^{(1)},T) \leq \int_0^\infty H_D^{(1)}(\tau,T)\ (g_{UD} - g_{LD})\ d\tau$$

$$\leq \int_0^\infty \bar{H}_D(\tau,T)\ |g_{UD} - g_{LD}|\ d\tau = \int_0^\infty \bar{H}_D(\tau,T)\ |\bar{g} - \underline{g}|\ d\tau. \tag{83}$$

Since the expression on the right above is independent of $P_\tau^{(1)}$, (80) follows.□

The bounds given above might be utilized as follows. First, find the threshold T, such that a given desired probability of false alarm is equal to $U_F(\underline{G}, \bar{G}, T)$. Hence we can guarantee the false alarm performance over the uncertainty class for any detector operating point that is chosen from the uncertainty class. Next, find $U_D(\underline{G}, \bar{G}, T)$ and $L_D(\underline{G}, \bar{G}, T)$ and ascertain 1) is L_D sufficient to guarantee a given level of detection performance and 2) what is the potential spread (Use (78) or (80)) of detection performance?

V. SUMMARY

Detection performance in terms of probabilities of detection and false alarm (P_D and P_F) was derived for a desired signal embedded in correlated multivariate noise from a Gaussian mixture uncertainty class. This uncertainty class is defined using upper and lower bounding functions on the univariate Gaussian mixing distribution function. For a given detector operating point, expressions for P_D and P_F were derived. In addition, nontrivial lower and upper bounds of performance were derived which are functions of the bounding functions of the uncertainty class, the detection threshold, and the matched filter signal-to-noise power ratio, $(S/N)_{opt}$ (only for P_D).

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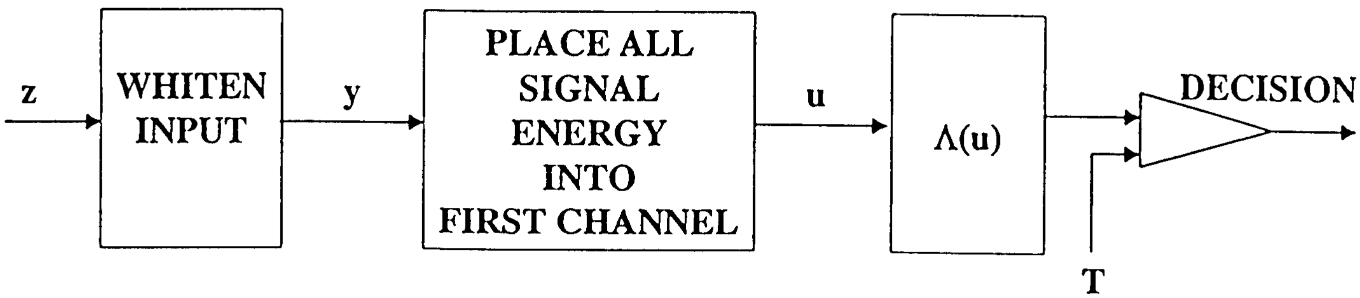

Fig. 1 — Equivalent detector