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THE DIGITAL TOPOLOGY OF SETS OF CONVEX VOXELS

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Abstract

Classical digital geometry deals with sets of cubical voxels (or square pixels) that can share faces, edges, or vertices; but basic parts of digital geometry can be generalized to sets S of convex voxels (or pixels) that can have arbitrary intersections. In particular, it can be shown that if each voxel P of S has only finitely many neightbors (voxels of S that intersect P), and if any nonempty intersection of neighbors of P intersects P, then the neighborhood $N(P)$ of every voxel P is simply connected, and if the topology of $N(P)$ does not change when P is deleted (i.e., P is a "simple" voxel), then deletion of P does not change the topology of S.

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1 Introduction

Classical digital geometry deals with sets of cubical voxels (or square pixels) that can share faces, edges, or vertices. Some authors [1] have studied digital geometry on other regular grids (in 2D: hexagonal or triangular), and other authors have generalized digital geometry to various types of abstract discrete spaces.

In this paper we show that basic parts of digital geometry can be generalized to sets S of convex voxels (or pixels) that can have arbitrary intersections. In particular, we show that if each voxel P of S has only finitely many neightbors (voxels of S that intersect P), and if any nonempty intersection of neighbors of P intersects P, then the neighborhood $N(P)$ of every voxel is simply connected, and if the topology of $N(P)$ does not change when P is deleted (i.e., if P is a "simple" voxel), then deletion of P does not change the topology of S. [In early work on digital convexity, Sklansky [2] considered tessellations of the plane into convex tiles, but he did not allow the tiles to overlap or to have gaps between them.]

The approach used in this paper originated in an earlier paper by the authors [3] which studied sets of (not necessarily regular) tetrahedra whose pairwise intersections have empty interiors. In that paper it was shown that the neighborhood of any tetrahedron (the union of the tetrahedra that intersect it) is simply connected if the tetrahedra satisfy a property called strong normality: For all $T, T_1, \ldots, T_n (n \geq 1)$, if each T_i intersects T and $I = T_1 \cap \cdots \cap T_n \neq \emptyset$, then I intersects T. In [4] the authors showed that this result is also true for sets of convex polygons or polyhedra, and that the converse is also true: simple connectedness of the neighborhood implies strong normality. It was suggested to the authors by an anonymous referee of [2] that these results might actually be true in a very general setting, involving sets of arbitrary simply-connected sets whose pairwise intersections are simply connected. This suggestion is in fact too general; in Section 5 we will show by example that it is false if the sets are not convex. On the other hand, as we will show in Sections 2 and 3, the results are true for sets of convex sets. (Note that an intersection of convex sets is convex, and hence simply connected.) In Section 4 we will show that when the strong normality property holds, it is easy to identify a "simple" voxel (= a voxel whose deletion does not change the topology of its neighborhood), and the deletion of a simple voxel does not change the topology of the set S of voxels.

2 Strongly normal sets of tiles

Let $\mathcal{P}$ be a set of closed, bounded convex sets in R^3; the elements of $\mathcal{P}$ will be called *tiles*,* and the union of all the elements of $\mathcal{P}$ will be denoted by $\mathcal{U}(\mathcal{P})$. $\mathcal{P}$ will be called *normal* (or "locally finite") if, for any $P \in \mathcal{P}$, the number of tiles that intersect P is finite. $\mathcal{P}$ will be called *strongly normal* (SN) if for all $P, P_1, P_2, \cdots, P_n (n \geq 1) \in \mathcal{P}$, if each P_i intersects P and $I = P_1 \cap P_2 \cap \cdots \cap P_n$ is nonempty, then I intersects P. It is not difficult to see that both normality and strong normality are hereditary: If they hold for $\mathcal{P}$, they hold for every $\mathcal{P}' \subseteq \mathcal{P}$.

*The "tiles" correspond to voxels (or in R^2 (see Section 5), to pixels); note that tiles can have arbitrary intersections. The definitions in this and the next paragraph all generalize immediately to any R^m; but the theorems in this and the next section are proved only for R^3 (and R^2).

The *neighborhood* of P in $\mathcal{P}$, denoted by $N_{\mathcal{P}}(P)$, is the union of all $Q \in \mathcal{P}$ that intersect P (including P itself). The interior of P, denoted by $interior(P)$, is the largest open set contained in P; the *border* of P is the set $P - interior(P)$. From now on we will assume that $\mathcal{P}$ is normal. In this and the next section we will show that a normal set of tiles $\mathcal{P}$ is SN iff, for every $\mathcal{P}' \subseteq \mathcal{P}$ and every $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected; thus SN is equivalent to hereditary "local simple connectedness".

A plane π will be called a *supporting plane* (or *plane of support*) of P if $\pi \cap P$ is nonempty, and P is contained in one of the closed halfspaces bounded by π. (Note that if P is contained in a plane, it is contained in both halfspaces bounded by that plane.) Such a halfspace is called a *supporting halfspace* of P. It is well known that a closed, bounded convex set is the intersection of all its supporting halfspaces.

Theorem 1 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ cannot have a cavity (i.e., the complement of $N_{\mathcal{P}'}(P)$ is connected.)*

Proof: Suppose $N_{\mathcal{P}'}(P)$ had a cavity K; since K is a component of the complement of the closed set $N_{\mathcal{P}'}(P)$, K is an open set. Also, since the neighbors Q of P (including P itself) surround K, every point on the border of K (the set $\bar{K} - K$, where $\bar{K}$ is the closure of K) must be in one of the Q's.

Since P is the intersection of all the supporting halfspaces of P, K cannot be contained in all of these halfspaces. Let π be a supporting plane of P such that P is contained in (at least) one of the halfspaces bounded by π and K intersects (at least) the other halfspace. Translate π parallel to itself, away from P, until it no longer intersects K. (Since K is surrounded by P and its neighbors, which are bounded, K must be bounded.) Let π' be the position of π just when this happens; thus π' contains at least one point p such that any neighborhood of p intersects K. Thus p is on the border of K, so that it lies in some set of neighbors of P.

Suppose first that p is only in one neighbor Q of P. Then there must exist a neighborhood $n(p)$ of p in π' (an open disk) that meets no other Q'. Thus any point p' on or near $n(p)$, in the halfspace H' bounded by π' that does not contain P, is in Q; and any point near $n(p)$ in the opposite halfspace H is in K. If there were a point q' of Q anywhere in H, some line segment $p'q'$ would thus intersect K, contradicting the convexity of Q. Hence Q lies entirely in H'; but this is impossible since Q is a neighbor of P.

In general, let p be on the borders of the neighbors Q_i of P. Since K is open, and p is on its border, the Q_i's cannot fill all of H in the vicinity of p. Hence there exists a nondegenerate solid angular sector s, emanating from p into H, that is contained in K in some neighborhood of p. Since s and Q_i are convex and their interiors are disjoint, there exists a supporting plane π_i of Q_i through p such that Q_i and s are in different halfspaces of π_i. Let H_i be the halfspace bounded by π_i that contains Q_i and let H_i' be the halfspace bounded by π_i that contains s. Since there are only finitely many Q_i's, the intersection of all the H_i''s forms a polyhedral angular sector t, containing s, that emanates from p into H and whose interior is contained in K and does not intersect any of the Q_i's. Let t' be the polyhedral angular sector constructed by continuing the faces of t through p into H'. Obviouly, t' is the intersection of the H_i's; thus t' contains the intersection I of the Q_i's.

Since $interior(t)$ is contained in K, t is contained in H; hence t' (and thus I) is contained in H'. Hence I cannot intersect P, which is contained in H; this contradicts SN. □

Theorem 2 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ cannot have a tunnel.*

Proof: Suppose $N_{\mathcal{P}'}(P)$ had a tunnel; then there exists a closed curve in $N_{\mathcal{P}'}(P)$ that cannot be reduced to a point. Any curve in $N_{\mathcal{P}'}(P)$ can be decomposed into nondegenerate (closed) arcs such that the interior of each arc is contained in one of the tiles of $N_{\mathcal{P}'}(P)$. Let C be such a curve that has a decomposition into as few such arcs as possible, say $C_1, \ldots, C_m$. If $m = 2$, C is contained in the union of two tiles of $N_{\mathcal{P}'}(P)$, and the intersection of these tiles is nonempty (it contains the common endpoints of the arcs); but since the tiles are convex, the union of two intersecting tiles is evidently simply connected, so C can be deformed to a point, contradiction. For each i, let Q_i be a tile that contains C_i; by the minimality of m, successive Q_i's must be distinct. Let C leave Q_i and enter Q_{i+1} (modulo m) at p_i, which is a point of $Q_i \cap Q_{i+1}$. Since Q_i is convex, the arc C_i from p_{i-1} to p_i can be deformed into the line segment $p_{i-1}p_i$, which lies in Q_i. Suppose Q_{i-1}, Q_i, Q_{i+1} had a common point p. Then we could continuously deform C by moving p_{i-1} in $Q_{i-1} \cap Q_i$ and p_i in $Q_i \cap Q_{i+1}$ until they both coincide with p; this reduces $p_{i-1}p_i$ to the single point p, so that C_i is now a degenerate arc, contradicting the minimality of m. Hence any three successive Q's must be disjoint. Since $\mathcal{P}'$ is SN, $Q_{i-1} \cap Q_i$ and $Q_i \cap Q_{i+1}$ must both intersect P; hence we can continuously deform C by moving p_{i-1} in $Q_{i-1} \cap Q_i$ and p_i in $Q_i \cap Q_{i+1}$ until they both reach P. The line segment $p_{i-1}p_i$ then lies in P, so we can replace Q_i by P. As just shown, $Q_i = P, Q_{i+1}$, and Q_{i+2} must be disjoint; but this implies that $Q_{i+1} \cap Q_{i+2}$ must be disjoint from P, contradicting SN. □

Theorems 1 and 2 immediately imply

Theorem 3 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ is simply connected..* □

3 The converse

In this section we prove that the converse of Theorem 3 is also true: if, for any normal $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected, then $\mathcal{P}$ is SN.

Lemma 1 *Let P be a tile in a normal set of tiles $\mathcal{P}$. If $Q_1, Q_2, \cdots, Q_n$ is a minimal set of neighbors of P in $\mathcal{P}$ that violates SN, then n is either 2 or 3.*

Proof. Evidently a single neighbor cannot violate SN, so we need only show that for $n > 3$, $Q_1, Q_2, \cdots, Q_n$ cannot be a minimal set of neighbors of P that violates SN. Suppose $Q_1, Q_2, \cdots, Q_n$, where $n > 3$, were such a minimal set of neighbors. Then the intersection of the Q_i's would be nonempty and disjoint from P. Let p be a point in the intersection of the Q_i's. Since $Q_1, Q_2, \cdots, Q_n$ is minimal, for every $1 \leq i \leq n$, $P \cap (\bigcap_{j \neq i} Q_j)$ must be nonempty. Let p_i be a point in $P \cap (\bigcap_{j \neq i} Q_j)$. This gives us n points $p_1, p_2, \cdots, p_n$ such that every Q_i

contains all of the p_j's except p_i, and hence contains the convex hull H_i of all the p_j's except p_i; and P contains the convex hull H of all n p_i's.

Let π be a plane that does not contain p and that intersects every line pp_i, say at q_i; and let X_i be the convex hull of all the q_j's except q_i. We shall now show that $\bigcap_i X_i$ cannot be nonempty. Evidently, each X_i is the projection of H_i (through p) on π. Hence if there were a point x in every X_i, the line px would intersect every H_i. Let h_i be the first point at which px meets H_i. All the h_i's must be on the same side of p, since otherwise p would be in the convex set H, which is contained in P, contradicting the fact that p is in the intersection of the Q's, which is disjoint from P. Let h be the h_i that is closest to p. Each Q_i is a convex set that contains p and all the p_j's except p_i; hence it contains p and H_i, hence contains the line segment ph_i. Thus every Q_i contains h. But h is in $H \subseteq P$; thus $P \cap Q_1 \cap \cdots \cap Q_n$ is nonempty, contradicting the assumption that the Q's violate SN.

If $n > 3$, we can choose four q's that form a (possibly degenerate) quadrilateral $q_1q_2q_3q_4$ in π such that each X_i contains at least one of the four (possibly degenerate) triangles $q_1q_2q_3$, $q_2q_3q_4$, $q_1q_2q_4$, and $q_1q_3q_4$. But the intersection of the four triangles is the intersection of the diagonals of the quadrilateral, so is always nonempty; hence $\bigcap_i X_i$ is nonempty, contradiction. □

If $k = 2$, $P \cap Q_1$ and $P \cap Q_2$ must be disjoint. If $k = 3$, $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$ must be nonempty and disjoint.

Theorem 4 *Let $\mathcal{P}$ be such that, for any normal $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ is simply connected; then $\mathcal{P}$ is SN.*

Proof: Suppose $\mathcal{P}$ is not SN. By the Lemma, a minimal set of Q's that violate SN has either two or three elements.

Suppose first that it has two elements Q_1, Q_2; let $\mathcal{P}' = \{P, Q_1, Q_2\}$. Let C be a closed curve in $N_{\mathcal{P}'}(P) = P \cup Q_1 \cup Q_2$ that passes through each of the intersections $P \cap Q_1$, $P \cap Q_2$ and $Q_1 \cap Q_2$. Suppose we could deform C so that it leaves any of the three tiles, say Q_1. Before C leaves Q_1 it has an arc from a point of $P \cap Q_1$ to a point of $Q_1 \cap Q_2$, passing through Q_1. Hence just after C leaves Q_1 it must have points arbitrarily close to $P \cap Q_1$ and $Q_1 \cap Q_2$. Since $P \cap Q_1$ is disjoint from Q_2, the end of the arc that was previously in $P \cap Q_1$ cannot be in Q_2; hence it must be in P. Similarly, since $Q_1 \cap Q_2$ is disjoint from P, the end that was previously close to $Q_1 \cap Q_2$ cannot be in P; hence it must be in Q_2. Since the arc no longer lies in Q_1, to get from the endpoint in P to the endpoint in Q_2 it must pass through $P \cap Q_2$. Just after the arc leaves Q_1, it must be arbitrarily close to Q_1; hence it cannot pass through $P \cap Q_2$, which is disjoint from Q_1. Thus the curve cannot leave Q_1, and similarly it cannot leave Q_2 or P, so it cannot be reduced to a point,which proves that $N_{\mathcal{P}'}(P) = P \cup Q_1 \cup Q_2$ is not simply connected.

Next, suppose that a minimal set of Q's that violates SN has three elements Q_1, Q_2, Q_3. Let $\mathcal{P}' = \{P, Q_1, Q_2, Q_3\}$; since the Q's violate SN, their intersection must be nonempty; and $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$ must also be nonempty. Let p, p_3, p_1 and p_2 be points in $Q_1 \cap Q_2 \cap Q_3$, $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$, respectively, such that the volume of the tetrahedron T defined by the four points p, p_3, p_1 and p_2 is minimum. Then *interior*(T) cannot intersect any of $Q_1 \cap Q_2 \cap Q_3$, $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, or $P \cap Q_3 \cap Q_1$. Note that p, p_1, p_2 are all in Q_3; p, p_2, p_3 are all in Q_1; p, p_3, p_1 are all in Q_2; and p_1, p_2, p_3 are all in P. Thus each of P, Q_1, Q_2, Q_3 contains a face of T. On the other hand, since the Q's violate SN, p is not in P, p_1 is not in Q_1, p_2 is not in Q_2, and p_3 is not in Q_3, so that none of P, Q_1, Q_2, Q_3 contains T. This also implies that T is nondegenerate. [Indeed, if its vertices were coplanar (or collinear), the intersection of the triangles (possibly degenerate) would contain the (nonempty) intersection of the diagonals of the quadrilateral (possibly degenerate) $pp_1p_2p_3$; but this implies that $P \cap Q_1 \cap Q_2 \cap Q_3$ is nonempty, contradiction.] Now the interior of T is surrounded by the faces of T; hence it is surrounded by $P \cup Q_1 \cup Q_2 \cup Q_3 (= N_{\mathcal{P}'}(P))$. If we can show that the interior of T is not contained in $P \cup Q_1 \cup Q_2 \cap Q_3$, it will follow that $N_{\mathcal{P}'}(P)$ has a cavity, and hence is not simply connected.

As we have just seen, none of the tiles P, Q_1, Q_2 Q_3 can contain (the interior of) T. Thus if the intersection of no two of them intersects the interior, we are done. Let $Q_1 \cap Q_2$ intersect $interior(T)$; since $Q_1 \cap Q_2$ cannot contain the entire interior of T (otherwise, each of Q_1 and Q_2 would contain it), $interior(T)$ must intersect the border of $Q_1 \cap Q_2$. Let x be a point in $border(Q_1 \cap Q_2) \cap interior(T)$ such that the area of the triangle xp_1p_2 is minimum. Since P, Q_1, Q_2 and Q_3 are all closed and no three of them intersect $interior(T)$, there must exist some neighborhood $n(x)$ of x in the triangle xp_1p_2 such that neither P nor Q_3 intersects $n(x)$. Therefore one of the following three cases must occur: (1) $n(x)$ is entirely contained in Q_1; (2) $n(x)$ is entirely contained in Q_2; (3) $n(x)$ intersects $Q_1 \cap Q_2$. If (3) occurs, let y be a point in $n(x)$ and in $Q_1 \cap Q_2$. Obviously the area of triangle yp_1p_2 is less than that of xp_1p_2; but this contradicts the assumption that xp_1p_2 has minimum area. If (1) occurs, the interior of xp_2 must intersect $Q_1 \cap Q_2$, which leads to the same contradiction as (3); and if (2) occurs, we reach the same contradiction in the same way. Therefore the union of P, Q_1, Q_2 and Q_3 fails to occupy the entire interior of T. This proves that $N_{\mathcal{P}'}(P)$ contains a cavity, so that it is not simply connected. □

4 Simple tiles

A tile P is called *simple* in $\mathcal{P}$ if deleting P from $\mathcal{P}$ does not change the topology of $N_{\mathcal{P}}(P)$. Define $N^*_{\mathcal{P}}(P)$, the *excluded neighborhood* of P in $\mathcal{P}$, as the union of all $Q \in \mathcal{P}$, excluding P itself, that intersect P; thus $N_{\mathcal{P}}(P) = N^*_{\mathcal{P}}(P) \cup P$, and P is simple in $\mathcal{P}$ iff $N_{\mathcal{P}}(P)$ and $N^*_{\mathcal{P}}(P)$ are topologically equivalent. Define $N^s_{\mathcal{P}}(P)$, the *shared subset* of P in $\mathcal{P}$, as the set $N^*_{\mathcal{P}}(P) \cap P$.

Theorem 5 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$, $P \in \mathcal{P}'$ is simple in $\mathcal{P}'$ iff $N^s_{\mathcal{P}'}(P)$ is simply connected.*

Proof: By Theorem 3, $N_{\mathcal{P}'}(P)$ is simply connected; and it contains the simply connected subset P. Hence [5] there exists a topology-preserving transformation τ (a deformation retract) that takes $N_{\mathcal{P}'}(P)$ into P. (Note that this eliminates all the tiles of $N_{\mathcal{P}'}(P)$ except P.) Since τ removes $N_{\mathcal{P}'}(P) - P$ from $N_{\mathcal{P}'}(P)$, it takes the subset $N^*_{\mathcal{P}'}(P)$ of $N_{\mathcal{P}'}(P)$ into $N^*_{\mathcal{P}'}(P) - (N_{\mathcal{P}'}(P) - P) = N^*_{\mathcal{P}'}(P) \cap P = N^s_{\mathcal{P}'}(P)$. $N^*_{\mathcal{P}'}(P)$ and $N^s_{\mathcal{P}'}(P)$ must thus be topologically equivalent; hence P is simple iff $N^*_{\mathcal{P}'}(P)$ is topologically equivalent to $N_{\mathcal{P}'}(P)$ (i.e., simply connected) iff $N^s_{\mathcal{P}'}(P)$ is simply connected. □

Theorem 6 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$, if $P \in \mathcal{P}'$ is simple in $\mathcal{P}'$, the deletion of P from $\mathcal{P}'$ does not change the topology of $\mathcal{U}(\mathcal{P}')$.*

Proof: By Theorem 5, $N^s_{\mathcal{P}'}(P)$ is simply connected, and it is contained in the simply connected set P. Hence [5] there exists a topology-preserving transformation σ (a deformation retract) that takes P into $N^s_{\mathcal{P}'}(P)$, which is a subset of $N^*_{\mathcal{P}'}(P)$. Thus σ deletes P from $\mathcal{U}(P')$, and it is topology preserving. (Note that P may be contained in $N^*_{\mathcal{P}'}(P)$ (hence equal to $N^s_{\mathcal{P}'}(P)$), in which case σ is the identity mapping, and "deletion" of P actually leaves $\mathcal{U}(P')$ unchanged.) □

When $\mathcal{P}$ is SN (so that $N_{\mathcal{P}}(P)$ is simply connected), the local topological changes when P is deleted depend on the numbers of components, tunnels and cavities in $N^*_{\mathcal{P}}(P)$. In [6] we gave efficient methods of identifying simple tiles, and measuring the local topological changes when a non-simple tile is deleted, in the case where the tiles are polyhedral (or polygonal, in 2D). Unfortunately, this does not seem to be possible for general convex tiles.

If $\mathcal{P}$ is not SN, the topology of $\mathcal{U}(\mathcal{P})$ may change when a tile P is deleted from $\mathcal{P}$ even if P is simple. An example is shown in Figure 1, where P is simple (both $N_{\mathcal{P}}(P)$ and $N^*_{\mathcal{P}}(P)$ have one component, one tunnel, and no cavities; note that tile R is not in $N_{\mathcal{P}}(P)$), but the topology of $\mathcal{U}(\mathcal{P})$ changes when P is deleted (prior to deleting P, $\mathcal{U}(\mathcal{P})$ has one component, one tunnel, and no cavities; after deleting P, $\mathcal{U}(\mathcal{P})$ has one component but no tunnels or cavities).

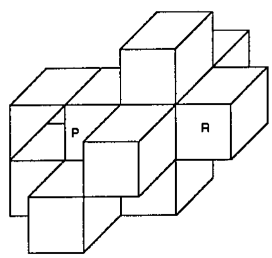

Figure 1: If $\mathcal{P}$ is not SN, the topology of $\mathcal{U}(\mathcal{P})$ may change even when a simple tile P is deleted. (Note that in this example, one of the tiles is a triangular prism, not a cube.)

5 Concluding remarks

The theorems in this paper also hold in R^2; in this case a tile is the intersection of all its supporting halfplanes (= closed halfplanes bounded by supporting lines, where a supporting line of P is a line l such that $l \cap P \neq \emptyset$, and P is contained in one of the closed halfplanes bounded by l), and the proofs of the theorems must be reworded appropriately. Note that in R^2, Theorems 1 and 2 reduce to the same statement: If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ and any $P \in \mathcal{P}'$, $N_{\mathcal{P}'}(P)$ cannot have a hole.

Convexity of the P's seems to be essential to obtaining the main results in this paper; it is not sufficient for the P's and their pairwise intersections to be simply connected. For example, let P be a closed cube and let Q_1, Q_2 be halves of a hollow closed hemisphere (divided by a vertical plane) sitting on top of P. Thus P, Q_1, Q_2 and their pairwise intersections are all nonempty and simply connected. Also, Q_1, Q_2 and $Q_1 \cap Q_2$ all intersect P, so that $\{P, Q_1, Q_2\}$ is SN. But $P \cup Q_1 \cup Q_2$ has a cavity, so that Theorem 1 does not hold.

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13. ABSTRACT *(Maximum 200 words)*

Classical digital geometry deals with sets of cubical voxels (or square pixels) that can share faces, edges, or vertices; but basic parts of digital geometry can be generalized to sets S of convex voxels (or pixels) that can have arbitrary intersections. In particular, it can be shown that if each voxel P of S has only finitely many neighbors (voxels of S that intersect P), and if any nonempty intersection of neighbors of P intersects P, then the neighborhood $N(P)$ of every voxel P is simply connected, and if the topology of $N(P)$ does not change when P is deleted (i.e., P is a "simple" voxel), then deletion of P does not change the topology of S.

14. SUBJECT TERMS

Digital geometry, Digital topology, Voxels, Pixels, Convex sets, Simple voxels, Simple pixels

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