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METHODS TO DETERMINE WIDEBAND SYSTEM RESPONSE UTILIZING ONLY NARROWBAND INFORMATION

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13. ABSTRACT *(Maximum 200 words)*

In this report a novel interpolation/extrapolation is developed, based on the Cauchy method, which interpolates/extrapolates system responses in a narrow frequency band to system responses over a broad frequency band. The given information can be either theoretical data points or measured experimental data over the narrowband. For theoretical data interpolation/ extrapolation, the sampled values (magnitude and phase) of the function and, optimally, a few of its derivatives have been used to reconstruct the function. For measured data, only measured values of the parameter are used to create the broadband information as the derivative information is too noisy. This method assumes that the parameter to be interpolated/extra-polated as a function of frequency is a ratio of two polynomials. The problem then is to determine the required order of the polynomials and the polynomial coefficients. The required order of the polynomials is estimated by applying Singular Value Decomposition (SVD) to the system matrix to determine the singular values and vectors. The number of nonzero singular values and their ratio of largest to smallest value are used to estimate the required polynomial order and the applicability of the method to the given data set. Polynomial coefficients then are determined using the method of Total Least Squares.

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1.0 Introduction

In a host of applications in engineering, it is necessary to obtain information about a system over a broad range. In most cases it is not possible to evaluate the parameter of interest in closed form. However, either theoretical or experimental data is available in a narrow band. Generation of the data over the broadband is not possible or may be extremely time consuming. The principle of analytic continuation is utilized to extrapolate/interpolate the data over a wide band. The method of Cauchy [1] has been chosen to carry out the analytic continuation.

The Cauchy method deals with approximating a function by a ratio of two polynomials. Given the value of the function and its derivatives at a few points, the order of the polynomials and their coefficients are evaluated. Once the coefficients of the two polynomials are known, they can be used to generate the parameter over the entire band of interest.

In this chapter, Cauchy's method is utilized to generate broadband currents on a body from which its Radar Cross Section (RCS) is calculated. This is done from narrowband calculations of the currents. Particularly in the Method of Moments [2] generation of the response at each frequency point is very time consuming. However, the current and its derivatives with respect to frequency can be calculated at a few points using the Method of Moments. Then Cauchy's method can be used to extrapolate/interpolate the current over a broad frequency range from which the RCS can be calculated.

Another example is the characterization of optical systems. The time required to evaluate a response over a broad range of the size parameter would be prohibitive. The Cauchy method can be used to generate the response of interest over a broadband from the value of the function at some discrete points.

The Cauchy method has wide application. A third example is in the area of filter analysis. In the laboratory it is not always possible to make accurate broadband measurements. This problem is especially severe in the case of measuring the transfer function of a filter in the stop band. The signal to noise ratio may be too low to be confident about the measurements of filter characteristics. Here the Cauchy method could be used to generate broadband information from measured narrowband data.

Yet another area of application for the Cauchy method is that of device characterization. A very useful tool in computer aided design would be an online description of the characteristics of many devices. But, since each device could be used under different operating conditions, each with its own frequency characteristic, the memory required to describe all devices would be prohibitive. Here the Cauchy method could be used to generate broadband information while storing the measured data at only a few points.

The choice of polynomial orders is restricted by the information we have. While it is true that the more information we have the higher we can choose the orders, this is not always desirable. In filter analysis especially, the choice of the order of the polynomials is very important.

In this work the Cauchy technique is used to solve the above problems. In each of the cases mentioned above the Cauchy technique would save a significant amount of program execution time or computer memory while still producing accurate results over broadband frequencies. The method is tested and numerical results are presented along with two examples of the method as a time-saving device.

2.0 The Cauchy Method

Consider a system function $H(s)$. The objective is to approximate $H(s)$ by a ratio of two polynomials $A(s)$ and $B(s)$ so that $H(s)$ can be represented by fewer variables.

Consider

$$H(s) \simeq \frac{A(s)}{B(s)} = \frac{\sum_{k=0}^{P} a_k s^k}{\sum_{k=0}^{Q} b_k s^k} \tag{2.1}$$

Here the given information could be the value of $H(s)$ and its N_j derivatives at some frequency points s_j, $j = 1, \ldots J$. If $H^n(s_j)$ represents the n^{th} derivative of $H(s)$ at point $s = s_j$, the Cauchy problem is:

Given $H^{(n)}(s_j)$ for $n = 0, \ldots N_j$, $j = 1, \ldots J$, find $P, Q, \{a_k, k = 0, \ldots P\}, and \{b_k, k = 0, \ldots Q\}$.

The solution for $\{a_k\}$ and $\{b_k\}$ is unique if the total number of samples is greater than or equal to the total number of unknown coefficients $P + Q + 2$ [1], i.e.

$$N \equiv \sum_{j=1}^{J} (N_j + 1) \geq P + Q + 2.$$

By enforcing the equality in equation (2.1) one obtains

$$A(s) = H(s)B(s) \tag{2.2}$$

Differentiating the above equation n times, and evaluating the expressions at point s_j, results in the binomial expansion,

$$A^{(n)}(s_j) = \sum_{i=0}^{n} {}^nC_i H^{(n-i)}(s_j) B^{(i)}(s_j) \tag{2.3}$$

where,

$${}^nC_i = \frac{n!}{(n-i)!i!},$$

n! represents the factorial of n.

Using the polynomial expansions for $A(s)$ and $B(s)$, equation (2.3) can be rewritten as

$$\sum_{k=0}^{P} \mathbf{A}_{(j,n),k} a_k = \sum_{k=0}^{Q} \mathbf{B}_{(j,n),k} b_k \tag{2.4}$$

where

$$\mathbf{A}_{(j,n),k} = \frac{k!}{(k-n)!} s_j^{(k-n)} u(k-n) \tag{2.5}$$

$$\mathbf{B}_{(j,n),k} = \sum_{i=0}^{n} {}^nC_i \frac{k!}{(k-i)!} H^{(n-i)}(s_j) s^{(k-i)} u(k-i) \tag{2.6}$$

$n = 0, 1, \ldots N_j$, $j = 1, ..J$, where $u(k) = 0$ for $k < 0$ and $= 1$ otherwise.

Define,

$$\mathbf{A} = \begin{bmatrix} \\ A_{(j,n),0}, A_{(j,n),1}, \ldots A_{(j,n),P} \\ \\ \end{bmatrix} \tag{2.7}$$

$$\mathbf{B} = \begin{bmatrix} \\ B_{(j,n),0}, B_{(j,n),1}, \ldots B_{(j,n),Q} \\ \\ \end{bmatrix} \tag{2.8}$$

$$[a] = \left[\; a_0, a_1, a_2, \ldots a_P \; \right]^T \tag{2.9}$$

$$[b] = \left[\; b_0, b_1, b_2, \ldots b_Q \; \right]^T \tag{2.10}$$

The order of matrix $\mathbf{A}$ is $N \times (P+1)$ and that of $\mathbf{B}$ is $N \times (Q+1)$.
Then, equation (2.4) becomes

$$[\mathbf{A}]a = [\mathbf{B}]b \tag{2.11}$$

or

$$\left[\ \mathbf{A}| - \mathbf{B}\ \right] \left[\begin{array}{c} a \\ b \end{array} \right] = 0 \tag{2.12}$$

For ease of notation, define $[\mathbf{C}] \equiv [\mathbf{A}| - \mathbf{B}]$. $\mathbf{C}$ is of order $N \times (P + Q + 2)$. A Singular Value Decomposition (SVD) of the matrix $\mathbf{C}$ will give us a gauge of the required values of P and Q [3]. A SVD results in the equation

$$[\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^H \left[\begin{array}{c} a \\ b \end{array} \right] = 0 \tag{2.13}$$

where the superscript H indicates the matrix Hermitian i.e. the conjugate transpose of the matrix. The matrices $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices and $\mathbf{\Sigma}$ is a diagonal matrix with the singular values of $\mathbf{C}$ in descending order as its entries. The columns of $\mathbf{U}$ are the left eigenvectors of $\mathbf{C}$ or the eigenvectors of CC^H. The columns of $\mathbf{V}$ are the right eigenvectors of $\mathbf{C}$ or the eigenvectors of C^HC. The singular values are the square roots of the eigenvalues of the matrix C^HC. Therefore, the singular values of any matrix are real and positive. The number of nonzero singular values is the rank of the matrix in equation (2.12) and so gives us an idea of the information in this system of simultaneous equations. If R is the number of nonzero singular values, the dimension of the right null space of C is $P+Q+2-R$. Our solution vector belongs to this null space. Hence to make this solution unique, we need to make the dimension of this null space 1 so that only one vector defines this space. Hence P and Q must satisfy the relation

$$R + 1 = P + Q + 2 \tag{2.14}$$

Hence, the solution algorithm must include a method to estimate R. This is done by starting out with the choices of P and Q that are higher than can be expected for the system at hand. Then we get an estimate for R from the number of non-zero singular values of the matrix $\mathbf{C}$. Now, using equation (2.14) we get better estimates for P and Q. Letting P and Q stand for these new estimates of the polynomial orders, we can recalculate the matrices $\mathbf{A}$ and $\mathbf{B}$. Therefore, we come back to the relation

$$[\mathbf{C}] \left[\begin{array}{c} a \\ b \end{array} \right] \equiv \left[\ \mathbf{A}| - \mathbf{B}\ \right] \left[\begin{array}{c} a \\ b \end{array} \right] = 0 \tag{2.15}$$

Many methods to solve equation (2.15) are well documented [3]. For reasons indicated below, we choose the method of Total Least Squares (TLS)

The usual approach to solve equation (2.15) is that of Least Squares (LS). In this approach, the equation is rewritten as:

$$[\mathbf{C}]^H[\mathbf{C}] \left[\begin{array}{c} a \\ b \end{array} \right] = 0 \tag{2.16}$$

The solution for $\left[\begin{array}{c} a \\ b \end{array} \right]$ is taken as the eigenvector corresponding to the zero eigenvalue of the resulting matrix. However, as we have seen, it is important to limit the rank of the null space of the matrix [C] to one. But, this approach has an extra step of a matrix multiplication. Also, since the eigenvalues are not sorted, it is additional work to find the number of non-zero eigenvalues.

A better approach would be the Total Least Squares(TLS) [4]. Since the elements of the matrix **C** are affected by measurement errors and noise, the optimal solution of this equation must take into account the effect of the noise in the matrix elements. The LS approach fails to do so. As shown in [4], the Total Least Squares is the optimal approach when the matrix is affected by noise.

[C] is a rectangular matrix with more rows than columns. Another SVD of this matrix brings us back to the equation

$$[\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^H \begin{bmatrix} a \\ b \end{bmatrix} = 0 \tag{2.17}$$

By the theory of the TLS, the solution is proportional to the last column of the matrix **V**. Since any constants of proportionality cancel out while dividing the two polynomials, we can choose

$$\begin{bmatrix} a \\ b \end{bmatrix} = [\mathbf{V}]_{P+Q+2} \tag{2.18}$$

3.0 Applications of the Cauchy Method

3.1 The Method of Moments

The Method of Moments yields remarkably accurate solutions to integral equations arising in electromagnetic scattering and radiation problems. It approximates the interactions of complicated bodies with a set of smaller, easily solvable interactions. The currents are approximated by a linear combination of some known basis functions. The problem then of extrapolating the current density, as a function of frequency, reduces to finding the coefficients in the linear combination. This approach allows the problem to be written as a matrix equation with the unknown coefficients as the solution to the equation. The Method of Moments finds it greatest advantage in the widespread use of the computer. But, its major limitation is that a large matrix equation has to be solved at every frequency point of interest. If a large system is to be studied, the program execution time may be as long as days.

The Cauchy method can partially solve this problem. The Method of Moments program could generate information over a limited band from which the Cauchy method could generate broadband information.

3.1.1 Interfacing with the Method of Moments

The Cauchy method can easily be incorporated as part of a Method of Moments program. The Method of Moments converts a linear operator equation into a matrix equation of the form

$$[V] = [Z][I] \tag{3.1}$$

Here, $[I]$ is the vector of coefficients in the representation of the current as a linear combination of basis functions. $[V]$ is the known excitation to the system, while $[Z]$ is the matrix that describes the interaction of the currents and the excitation.

Differentiating the above equation with respect to frequency results in a binomial expansion.

$$\begin{aligned} [V]' &= [Z]'[I] + [Z][I]' \\ \Rightarrow [I]' &= [Z]^{-1}\left[[V]' - [Z]'[I]\right] \end{aligned} \tag{3.2}$$

In general,

$$\begin{aligned} [V]^{(n)} &= \sum_{i=1}^{n} {}^{n}C_i [Z]^{(n-i)}[I]^{(i)} \\ \Rightarrow [I]^{(n)} &= [Z]^{-1}\left[[V]^{(n)} - \sum_{i=1}^{n-1} {}^{n}C_i [Z]^{(n-i)}[I]^{(i)}\right] \end{aligned} \tag{3.3}$$

In the above equations, $[V]^{(n)}$ is the vector with each element of $[V]$ differentiated with respect to frequency n times. Similarly, $[Z]^{(n)}$ is the matrix generated by differentiating each element of the matrix $[Z]$ with respect to frequency n times.

Hence, using a Method of Moments program, we can generate all the information needed to apply the Cauchy method. The use of derivative information saves execution time because no new matrix inversions are required to generate the additional information. Hence, evaluation of a derivative at a frequency point required $O(N^2)$ operations as opposed to solving for the currents at a frequency point which takes $O(N^3)$ operations. Each element of the solution current ($[I]$) vector is treated as our function $H(s)$. Given the current and its derivatives at some frequency points, we can use the Cauchy method to approximate the current at many more points.

3.1.2 Numerical Examples

To test the Cauchy method, Radar Cross Sections (RCS) of five different perfectly conducting three dimensional bodies were calculated over wide frequency bands. A program to evaluate the currents on an arbitrarily shaped closed or open body using the Electric Field Integral Equation and triangular patching as described in [5] was used. The triangular patching approximates the geometry of the surface of the body with a set of adjacent triangles. The program then uses these currents to evaluate the RCS of the body. It was modified to also calculate the first four derivatives of the currents with respect to frequency. This information was used as input to a Cauchy subroutine. The original Method of Moments program was used to calculate the RCS without the Cauchy method. The two RCS plots were compared to show the accuracy of the Cauchy method.

The bodies chosen were a sphere, a square plate, a disk, a concave, and a convex hemisphere. In all cases the currents and their first four derivatives were evaluated at five frequency points. Hence, the total information allows a maximum of $5 \times (4+1) = 25$ coefficients combined in the two polynomials of equation (2.1). In the application of Cauchy method to the Method of Moments, it was found that no singular values of the original matrix $[A| - B]$ are zero. This is to be expected, since, the current, as a function of frequency, is not a ratio of two polynomials. Hence, the higher the polynomial orders we choose, the more accurate the approximation would be. Therefore, in this application, the step of estimating $R, P,$and Q, in equation (2.14) is bypassed. Given the 25 samples, the numerator polynomial was of order 11 while the denominator was a polynomial of order 12. Physically we know that for the polynomial approximation to be stable, the numerator polynomial must be of lower order than the denominator polynomial.

The motivation to apply the Cauchy method to the Method of Moments is to save program execution time. To get an idea of how much time can be saved, the program was timed for two of the above bodies and compared to the original Method of Moments program. The two bodies chosen were the sphere and the plate.

In the first example a sphere of radius $0.3m$ was analyzed. The sphere was triangularized using 182 nodes and 540 edges. Because the sphere is a closed object, this results in 540 unknowns in the expansion of the current in terms of the basis functions. The currents on the sphere and its first four derivatives with respect to frequency were evaluated at five frequency points. The points chosen were in the range $\lambda = 0.30m$ and $\lambda = 0.84m$ at a spacing of 0.135m. Using this information and the Cauchy method the current on the sphere was calculated for 51 points in the same frequency range. Using these currents the RCS of the sphere was calculated at the 51 frequency points. The time taken for this calculation is compared to the time taken by the original Method of Moments program to evaluate the RCS at five frequency points in the same range.

Method of Moments (5 points)	:47mins. 56secs.
Cauchy Method (51 points)	:57mins. 57secs.

To generate the same information at 51 points the Method of Moments program would take approximately 8hrs.8mins.

In Figure 3.1 we see the results of applying the Cauchy method to the evaluation of the RCS of a sphere. Here the RCS is plotted over a decade bandwidth. This bandwidth was broken up into 3 ranges:

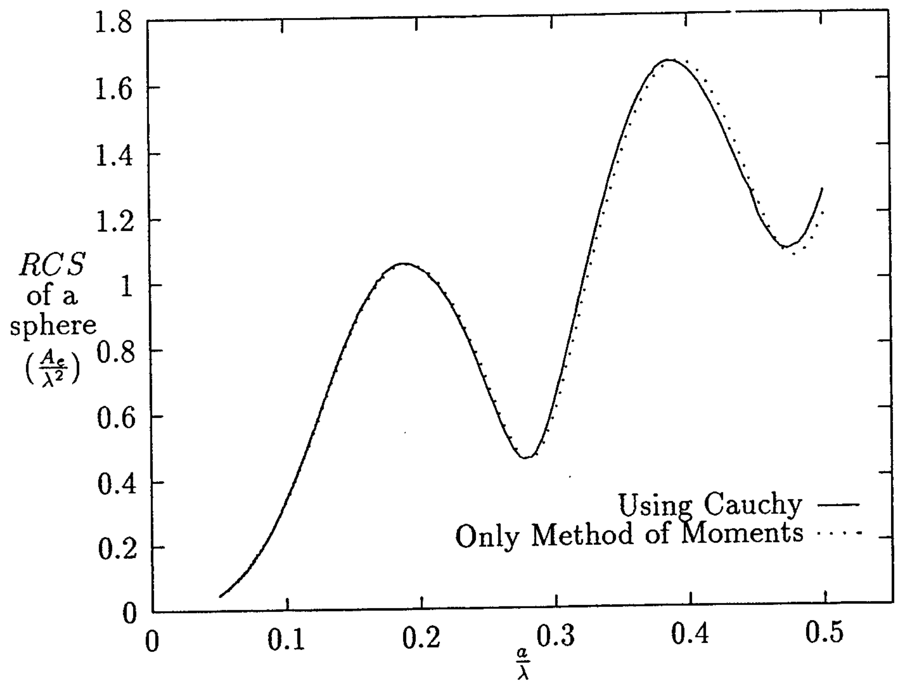


Figure 3.1: Radar Cross Section of a sphere

$$0.6m \leq \lambda \leq 1.0m$$
$$1.0m \leq \lambda \leq 1.8m$$
$$1.8m \leq \lambda \leq 6.0m$$

In each of the three ranges the current and its first four derivatives were evaluated at five equally spaced points using the Method of Moments program. Using this information the polynomials in equation (2.1) were formed. This rational polynomial was used to evaluate the current at 51 points in each range. Also, the original Method of Moments program was used to calculate the currents at a few points in the decade bandwidth. The currents were used to calculate the RCS of the sphere in this bandwidth. As can be seen from the figure, the agreement between the results from the use of the Cauchy program and the original Method of Moments program is excellent.

As a second example, a square plate of side $0.3m$ was analyzed. The plate was triangularized using 169 triangle nodes and 456 edges. In the Method of Moments formulation the nodes on the boundary of an open object are not unknowns. Hence, the number of unknowns in this case was only 408. The procedure followed is similar to the analysis of the sphere. Here the five frequency points chosen were in the range $\lambda = 0.15m$ and $\lambda = 0.30m$ at intervals of $0.0375m$. Using this information and the Cauchy method the currents on the plate were evaluated at 201 frequency points. The time taken for this calculation is compared to the time taken by the original Method of Moments program to evaluate the RCS at five frequency points in the same range.

Method of Moments (5 points) :21mins. 50secs.
Cauchy Method (201 points) :27mins. 47secs.

To generate the same information at 201 points the Method of Moments program would take approximately 14hrs.38mins.

All programs were executed on an IBM RS6000 platform running AIX.

Figure 3.2 shows the application of this technique to the evaluation of the RCS of a plate. Again, to evaluate the RCS over a decade bandwidth, three intervals were chosen and the two polynomials of equation (2.1) formed in each interval. The numerator polynomial had order 11 while the denominator polynomial had order 12. The intervals chosen were:

$$0.1m \leq \lambda \leq 0.15m$$
$$0.15m \leq \lambda \leq 0.3m$$
$$0.3m \leq \lambda \leq 1.0m$$

The rational polynomial was used to evaluate the currents at 201 points in each range. The original Method of Moments program was used to evaluate the RCS of the plate

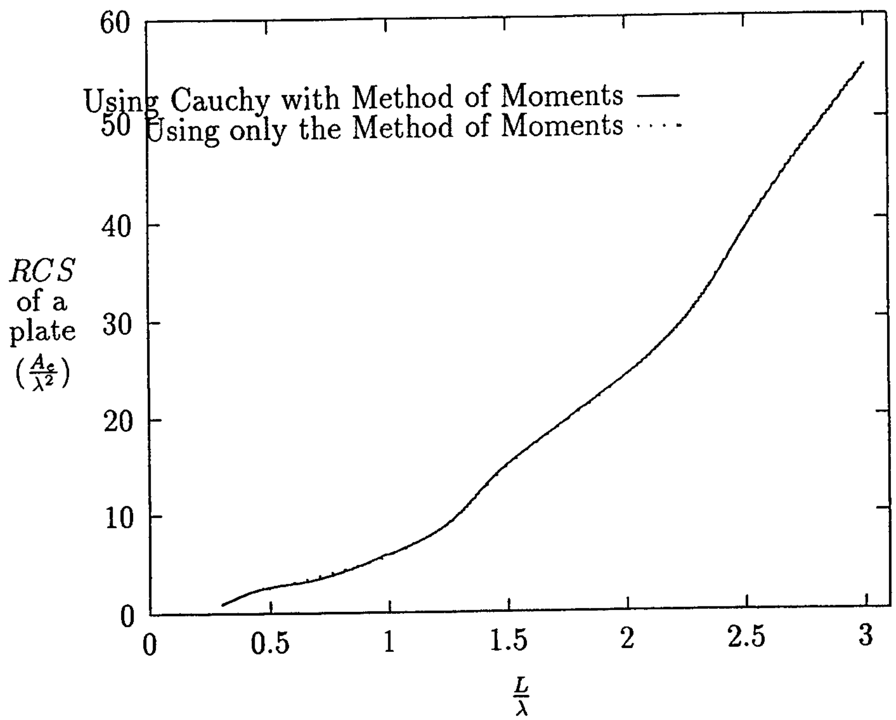


Figure 3.2: Radar Cross Section of a square plate

in this decade bandwidth. As can be seen from the figure, the agreement between the two results is excellent.

The third example is a disk of radius $0.3m$. The disk was triangularized using 142 nodes and 460 edges. Of these only 440 were interior nodes. Figure 3.3 shows the RCS of the disk over a decade bandwidth. Here too, the decade bandwidth was broken up into three intervals and polynomials of order 11 and 12 formed in each interval. The rational polynomial was used to evaluate the currents and then the RCS of the disk at 51 frequency points in each range. The intervals chosen were:

$$
\begin{aligned}
0.6m \leq \lambda \leq 2.4m \\
2.4m \leq \lambda \leq 4.2m \\
4.2m \leq \lambda \leq 6.0m
\end{aligned}
$$

Figures 3.4(a) and 3.4(b) show the results of the final example of the Cauchy technique applied to the Method of Moments. The RCS of a convex and a concave hemisphere was calculated over a decade bandwidth. Figure 3.4(a) shows the RCS of a convex hemisphere while Figure 3.4(b) shows the RCS of a concave hemisphere. The radius of both hemispheres was $0.3m$. The convex hemisphere had 257 nodes and 736 edges. This resulted in a problem with 704 unknowns. The concave hemisphere had 316 nodes and 910 edges. Of these 875 were interior nodes. The decade bandwidth was broken into the following ranges:

$$
\begin{aligned}
0.6m \leq \lambda \leq 1.0m \\
1.0m \leq \lambda \leq 2.6m \\
2.6m \leq \lambda \leq 6.0m
\end{aligned}
$$

As in the case of the disk, for the hemispheres too, the two polynomials of equation (2.1) were formed in each of the three ranges. In both cases the Method of Moments program evaluated the currents and its first four derivatives with respect to frequency at five points in each range. This information was used by the Cauchy subroutine to approximate the currents at 51 points in each range from which the RCS of the hemispheres were calculated at 51 points in each range. Also, the original Method of Moments program was used to calculate the RCS over the decade bandwidth. As can be seen from Figure 3.4, the agreement in each case is excellent.

3.2 Optical Computations

The calculation of either the scattering efficiency or the intensity is highly computationally intensive. If these parameters are desired over a broad range *and* at finely spaced points of the size parameter, the time required for the calculations could be prohibitive. The Cauchy method would solve this problem by needing the calculations to be done at a much coarser spacing and interpolate the parameter of interest.

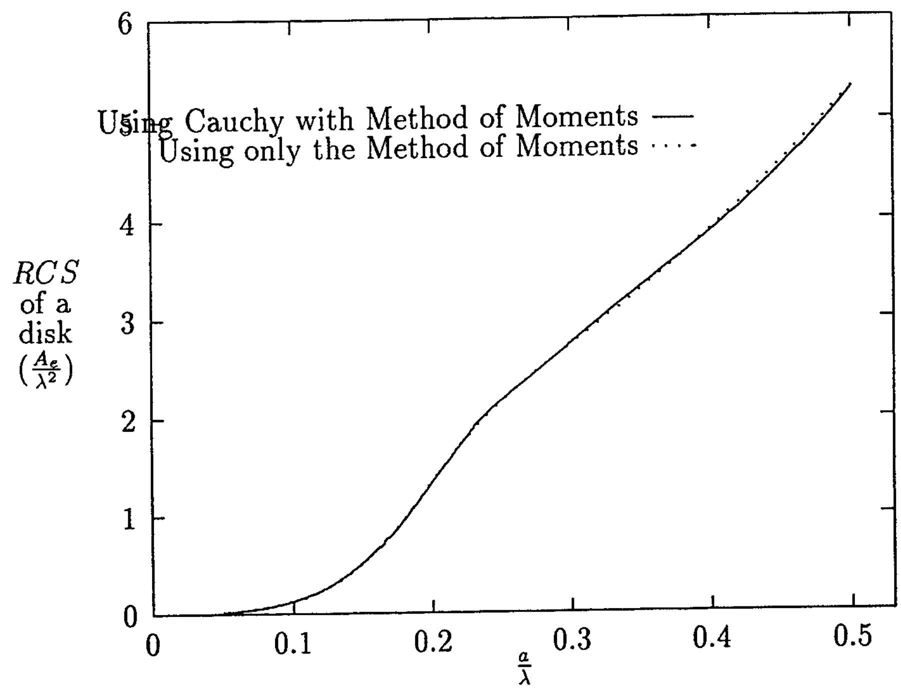
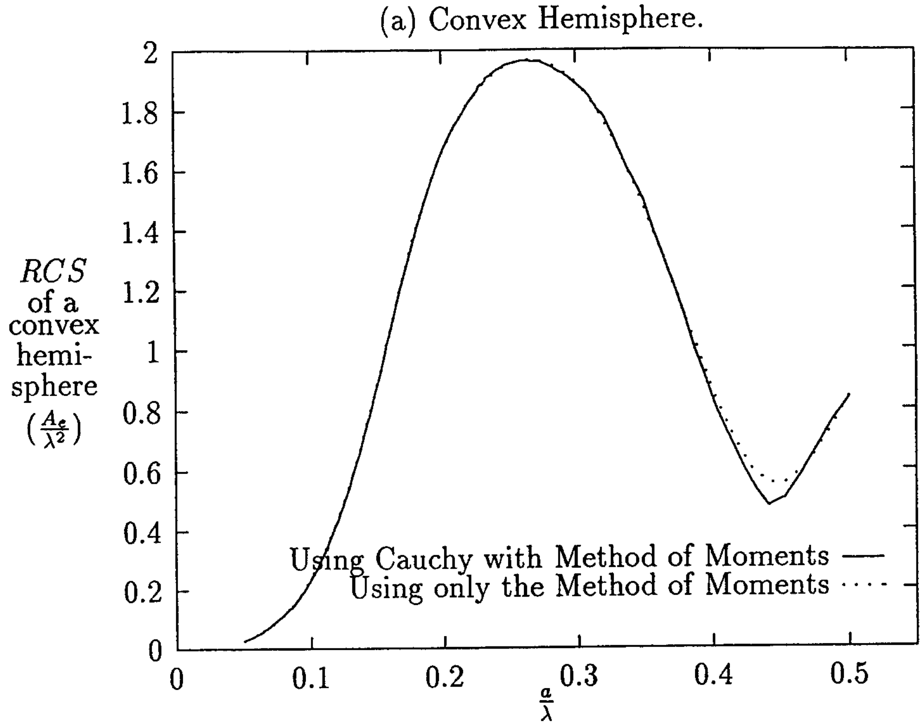


Figure 3.3: Radar Cross Section of a Disk



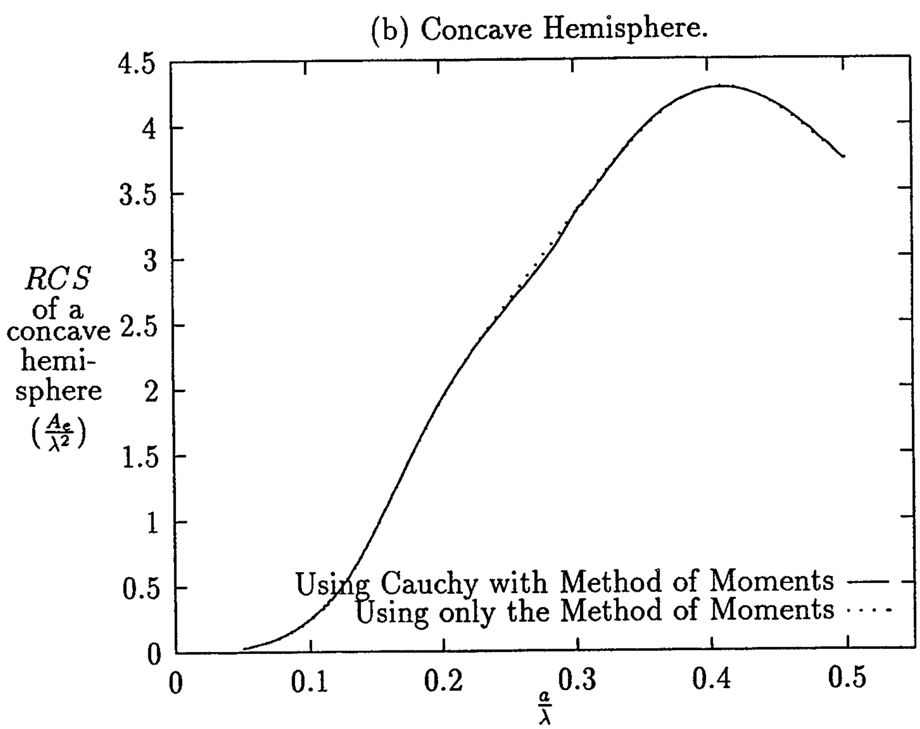


Figure 3.4: Radar Cross Section of a Hemisphere

This application was tested on the scattering efficiency of a sphere as a function of size parameter[6]. The sphere had an index of refraction of 2.0. The original data calculated the scattering efficiency at a spacing 0.002 in the size parameter. The range of the size parameter was from 7.0 to 8.0. Hence, the original data had 501 points. The Cauchy program needed a spacing of 0.01 in the size parameter to accurately calculate the scattering efficiency of the sphere at the original 501 points. This cuts down the program execution time by a factor of 5. The input to the Cauchy program is shown in Figure 3.5(a).

Because all computer calculations suffer from round-off error, most of the singular values returned from the SVD subroutine are not exactly zero. The choice of the threshold was such that a singular value was considered zero if it was 18 orders of magnitude lower than the largest singular value. This choice of the cutoff was chosen by the limited precision of the computer arithmatic i.e. the limited dynamic range of the computer. Using this threshold, only 72 singular values could be considered not equal to zero. Hence, the estimate for R is 72. Using this and equation (2.14), the choice for the polynomial orders was reduced to 35 for the numerator and 36 for the denominator. Using these polynomials the scattering efficiency was calculated at the original 501 points.

Figure 3.5(b) shows the results of the application of the Cauchy method to optical computations. The dotted line represents the original data while the unbroken line the interpolated data. As can be seen, the two plots are nearly visually indistinguishable. Also, even though the input data to the program did not have of the peaks of the Scattering Parameter, the Cauchy program was able to reproduce them.

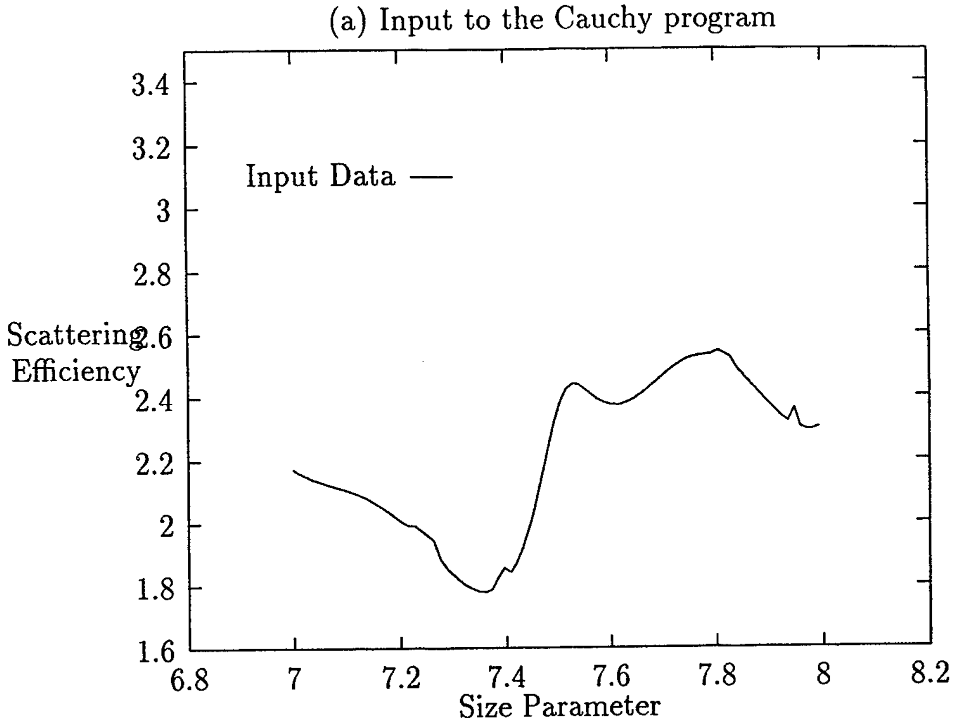
3.3 Filter Analysis

The Cauchy method can also be used in analysis of filters over broad frequency ranges. A filter response is a ratio of two polynomials and hence lends itself easily to the use of a Cauchy program. This has practical application to the problem of generating the stop band response given the pass band response or the reverse i.e. generating the pass band response given some data from the stop band.

A filter transfer function (S_{21}) was measured using a network analyzer at frequency points in and out of the filter passband. The filter had its 3dB points at 4.98GHz and 6.61GHz. Hence, the filter had a passband of 1.63GHz with a center frequency of 5.80GHz. The filter response was measured at 415 equally spaced points in the frequency range 4.31GHz to 7.42GHz.

In the first application, the response over the entire band of measurement was recovered using mostly passband information. 51 equally spaced points, in the frequency range 4.79GHz to 6.96GHz, were chosen as input to a Cauchy program. Because we are dealing with measured data, we do not have any information about the derivative of the transfer function with respect to frequency.

The threshold was chosen such that a singular value was considered zero if it was 14 orders of magnitude lower than the largest singular value. After the program



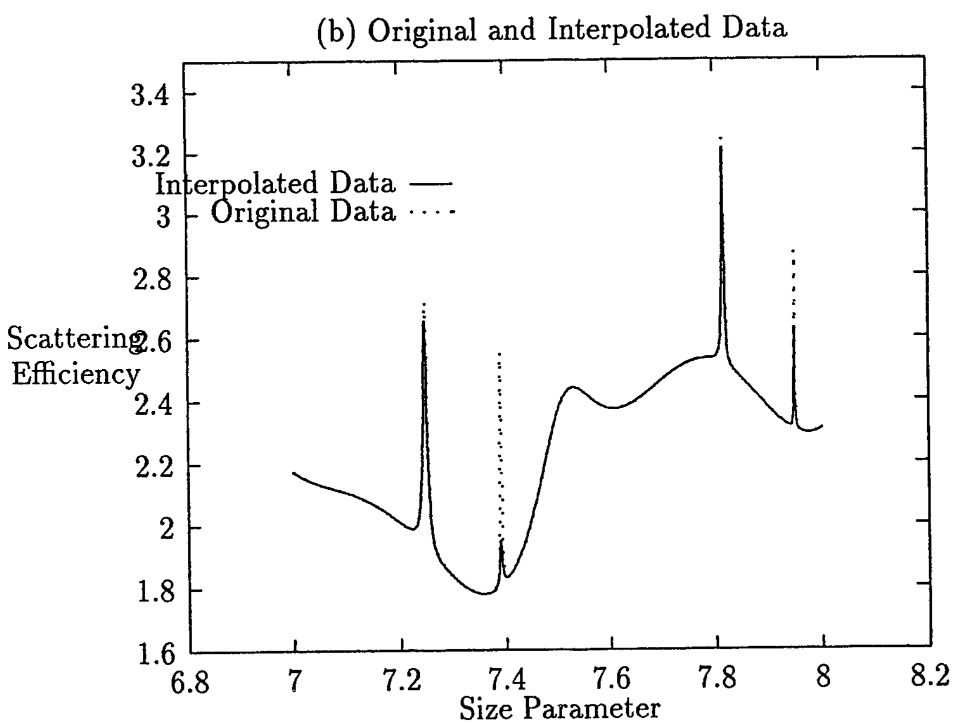


Figure 3.5: Scattering efficiency as a function of size parameter

checked for the number of non-zero singular values, the estimate for R was 16. The order of the numerator polynomial was chosen to be 7 while that of the denominator polynomial was chosen to be 8.

In Figures 3.6(a) and 3.6(b) the results from the Cauchy program are described. Figure 3.6(a) shows the magnitude response while Figure 3.6(b) shows the phase response of the filter. As is often the case with filters, the magnitude response was considered more important. Hence, the phase response was allowed to show a poor agreement so as to maximize the agreement of the magnitude response. If a 10% error in the magnitude were acceptable, the extrapolation is valid for 0.39GHz. This is 6.7% of the center frequency and 23.9% of the bandwidth. For frequencies beyond the passband, the extrapolation is accurate within 10% up to 7.42GHz, the frequency till which data was available. Hence, we have generated accurate data over a bandwidth of 3.32GHz starting with data over a bandwidth of 2.16GHz.

In the second application, data from the stop band and a little from the passband was used to interpolate into the passband. Here too the choice of threshold is very important. Using the same threshold as in the first application, the estimate of R remained the same. Hence, in this case too, the numerator polynomial had order 7 while the denominator had order 8. In this case, 23 equally spaced points from 4.31GHz up to 5.35GHz and 28 equally spaced points from 6.20GHz to 7.42GHz were used to interpolate into the passband. This represents on interpolation of 0.85GHz, which is 14.6% of the center frequency or 52.7% of the bandwidth. Figures 3.7(a) and 3.7(b) show the results of this application. Figure 3.7(a) is the reconstructed magnitude response and Figure 3.7(b) is the reconstructed phase response. Again, since more attention is paid to the magnitude response, the phase response shows poorer agreement with the true response.

In both figures the dotted line represents the measured data and the continuous line the results of the Cauchy program.

3.4 Device Characterization

An application of the Cauchy method is in creating a database of many devices working in varying operating conditions. The Cauchy program would require the value of a parameter at a few frequency points and use this information to evaluate the parameter over a wide frequency band. Over many devices, and their operating conditions, this would yield significant savings in memory requirements.

To test this application the Y-parameters of a UM PHEMT were measured over the range of 1.0-40.0GHz. Just five of these points were used as input to the Cauchy program. The points chosen were at the frequency points 1.0GHz, 10.0GHz, 20.0GHz, 30.0GHz, and 40.0GHz. This resulted in a numerator polynomial of order 1 and denominator polynomial of order 2. Here again, the step of estimating R, P, and Q is bypassed. Figure 3.8(a) shows the magnitude ($|Y_{11}|$) reconstructed over this broad frequency range. Figure 3.8(b) shows the phase ($\angle Y_{11}$) over the same range. As can be seen the agreement with the measured values and the interpolated values is excellent.

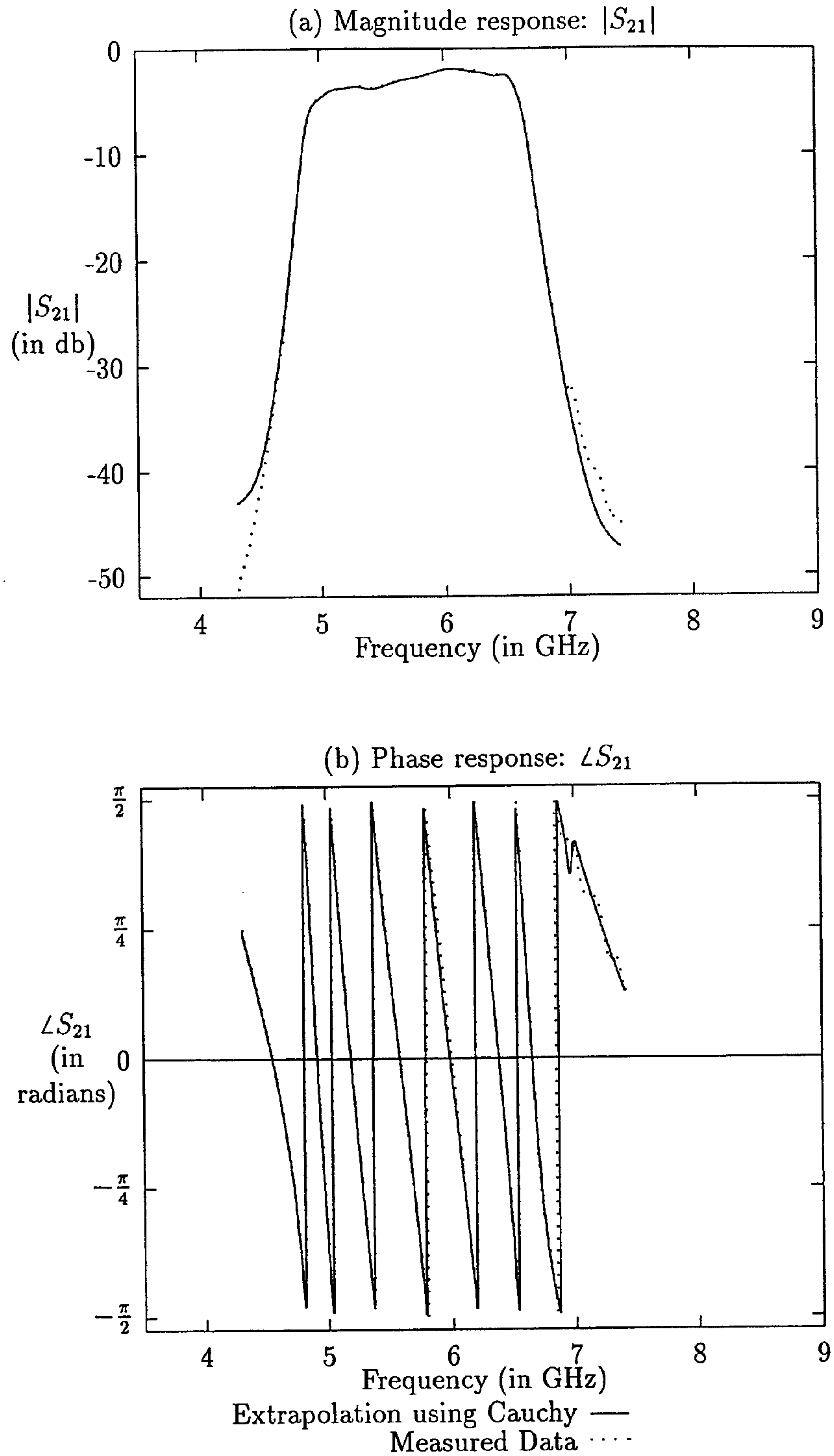
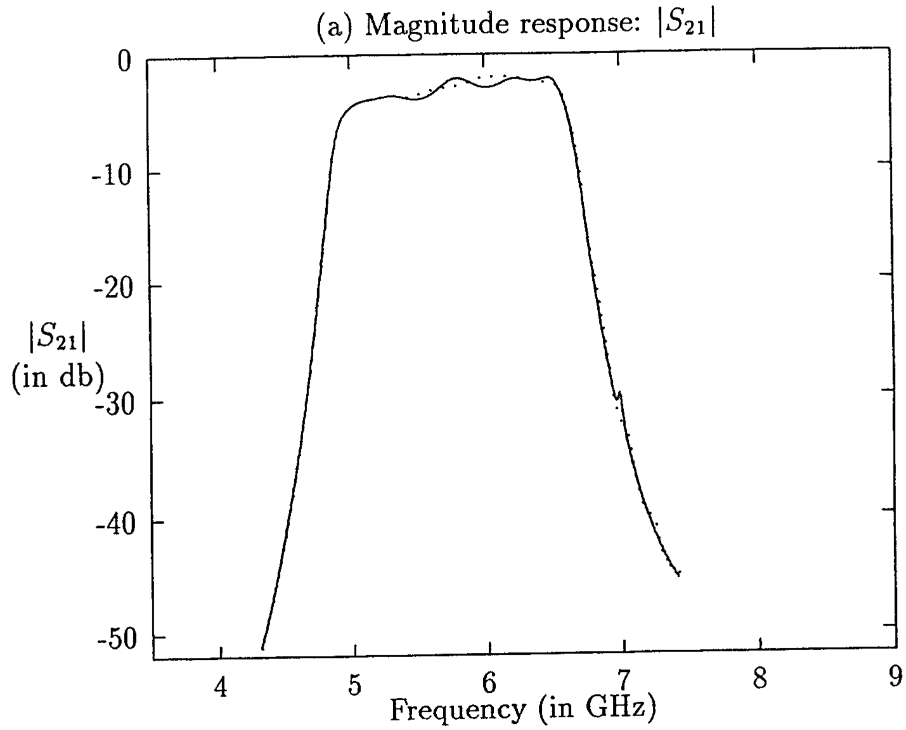


Figure 3.6: Generation of stop band response using pass band data.



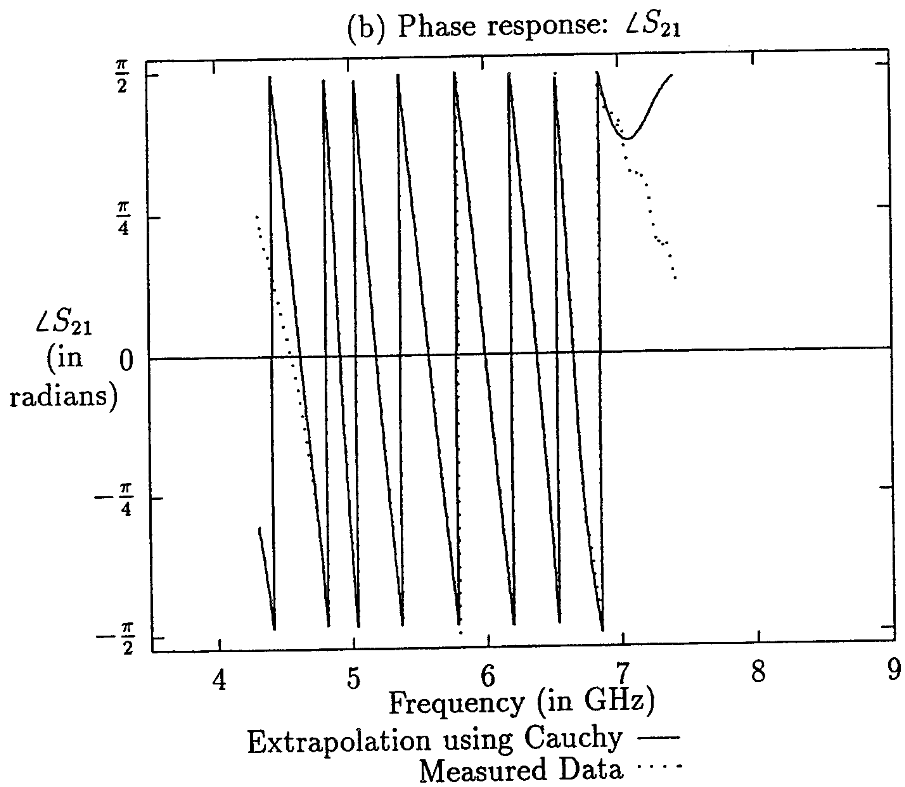


Figure 3.7: Generation of pass band response using stop band data.

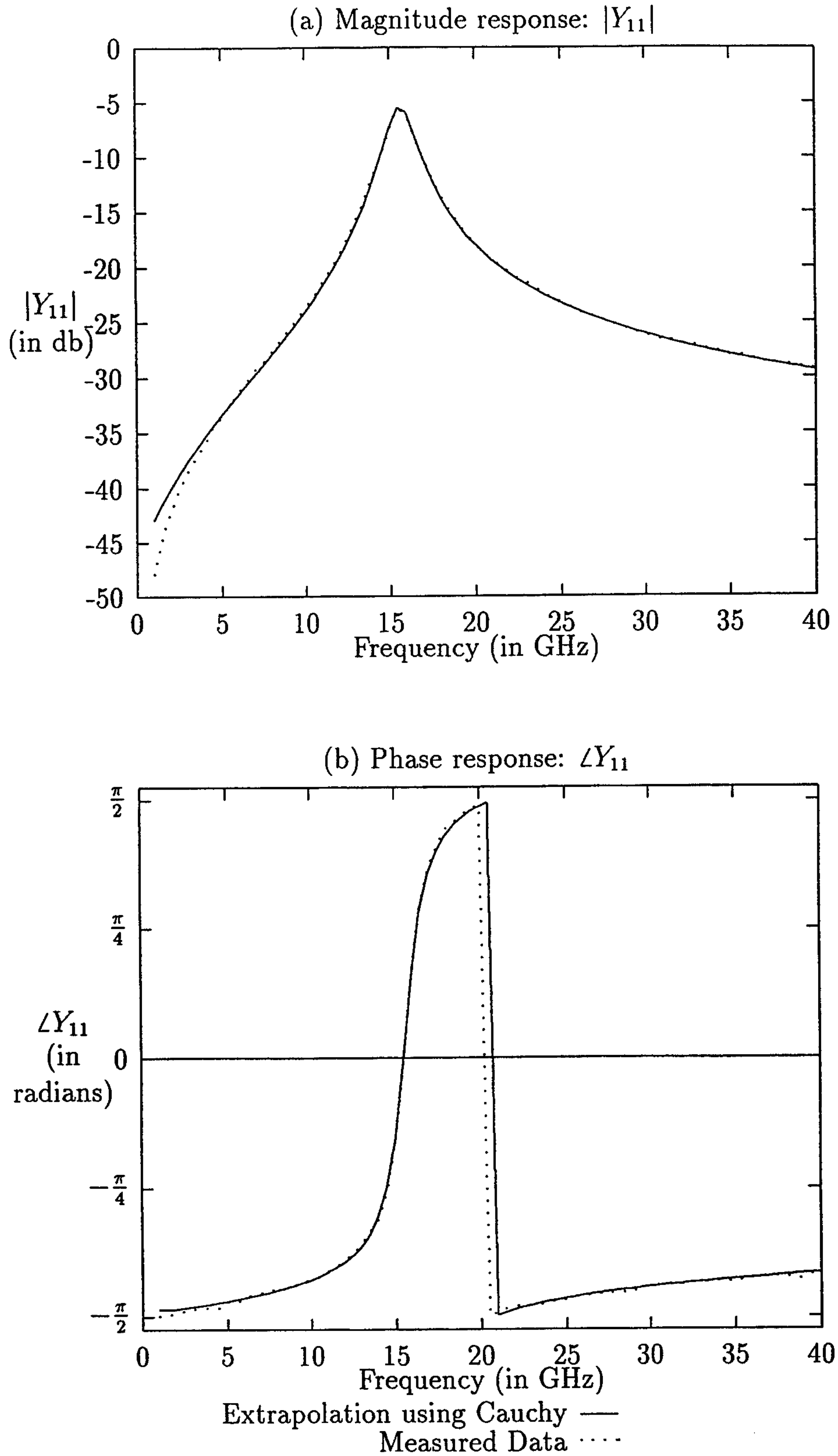


Figure 3.8: The Cauchy method applied to Device Characterization.

4.0 Effect of Noise in the Data on the Cauchy Method

The Cauchy method [7] has been shown to provide accurate broadband information from narrowband data. The method deals with approximating a function by a ratio of two polynomials. Given the value of the function and its derivatives at a few points, the order of the polynomials and their coefficients are evaluated. Once the coefficients of the two polynomials are known, they can be used to generate the parameter over the entire band of interest. The Cauchy method was shown to be applicable to the cases where the input data was measured values of the function and not its theoretical values.

However, no measuring instrument is perfect. Each measurement has, added to the signal, an unwanted noise component. Reference [7] does not discuss how this noise affects the results from the Cauchy method. The presence of noise in the data limits the effectiveness of the method. In this chapter we try to quantify the limitations of the Cauchy method when the input data is subject to contamination by noise.

Throughout this work, we assume that the noise is additive, stationary, zero mean, and Gaussian. This assumption is approximately valid for most measuring instruments. Using this assumption, the probability density function (PDF) of the parameter, as a function of frequency, is evaluated. This is compared to the PDF approximated by a computer numerical simulation.

To make the problem tractable, certain simplifying assumptions are necessary. This includes assuming that the coefficients of the polynomials in the Cauchy method are independent random variables. As the theory will show, this assumption is not strictly true. However, the error introduced in the PDF due to this assumption is minimal.

Another assumption made is that the noise affects each measurement independently. The noise is also assumed to affect each measurement, on average, equally. This means, that the average power in the noise in each measurement is assumed constant over repeated measurements. This assumption too is approximately valid for most measuring systems.

Using these assumptions we derive the theoretical PDF of the estimate of the parameter as a function of frequency. The theoretical PDF was verified by numerical simulations.

4.1 The Effect of Noise on the Solution Vector

As shown in Chapter 2.0, the solution vector belongs to the invariant subspace that is spanned by the right singular vector $[\mathbf{V}]_{P+Q+2}$. This singular vector is associated with the smallest singular value. However, because of the noise in the data, the entries of matrix $\mathbf{C}$ are perturbed from their true values. Hence, the solution vector is also perturbed. We need to quantify the perturbation of this subspace.

Notation In this work, a perturbed parameter or matrix will be represented by a tilde ($\sim$) above the corresponding unperturbed parameter or matrix.

4.1.1 Perturbation of Invariant Subspaces

Let, $\mathcal{R}$ denote the set of real numbers, $\mathcal{R}^n$, the set of real vectors of length n, and $\mathcal{R}^{n\times p}$ the set of real matrices of order $n \times p$.

Consider an arbitrary matrix $A \in \mathcal{R}^{N\times P}$ with $P \leq N$. Let $\tilde{A} = A + E$, where E is the perturbation to the matrix A, and

$$[U^T][A][V] = \begin{array}{cc} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ P-1 \\ N-P \end{array} \\ \begin{array}{cc} 1 \quad & \quad P-1 \end{array} & \end{array} \tag{4.1}$$

Here the figures below the matrix indicate the number of columns in each submatrix while the figures to the side of the matrix indicate the number columns in each submatrix. Also,

$$\begin{aligned} \mathbf{U} &= (u_1 \,|\, U_2 \,|\, U_3) \\ \mathbf{V} &= (v_1 \,|\, V_2) \end{aligned}$$

Here, $u_1 \in \mathcal{R}^N$, $U_2 \in \mathcal{R}^{N\times(P-1)}$, $U_3 \in \mathcal{R}^{N\times(N-P)}$, $v_1 \in \mathcal{R}^P$, and $V_2 \in \mathcal{R}^{P\times(P-1)}$. σ_1 is the singular value corresponding to the left singular vector u_1 and the right singular vector v_1. This singular value can be the one of interest and not just the largest singular value. In the Cauchy method the singular value of interest is the smallest or the zero singular value. Σ_2 is the diagonal matrix with the rest of the singular values of C as its entries. These singular values can be ordered arbitrarily as long as the columns of U and V are permuted appropriately so as to maintain the equality of equation (4.1).

If,

$$[U]^T[E][V] = \begin{pmatrix} \gamma_{11} & g_{12}^T \\ g_{21} & G_{22} \\ g_{31} & G_{32} \end{pmatrix} \tag{4.2}$$

where $\gamma_{11} \in \mathcal{R}$, $g_{12}, g_{21}, g_{31} \in \mathcal{R}^{P-1}$, $G_{22} \in \mathcal{R}^{P-1\times P-1}$, and $G_{32} \in \mathcal{R}^{N-P-1\times P-1}$, and if σ_1 is not repeated as a singular value, then [8]

$$\tilde{v}_1 = v_1 + V_2\left(\sigma_1^2 I - \Sigma_2^2\right)^{-1} h + O\left(\|E\|_2^2\right) \tag{4.3}$$

where,

$$h = \sigma_1 g_{12} + \Sigma_2 g_{21}$$

4.1.2 Perturbation of the Solution in the Cauchy Method

In all measurements the true value of the measured parameter (here H_i) is perturbed by an additive noise component. Hence,

$$\tilde{H}_i = H_i + e_i$$

where, $\tilde{H}_i$ is the value of H_i after it has been perturbed by noise e_i.

In the following discussion we are assuming:
1) The noise is only in the measurement of the parameter $[H(s)]$, not in the measurement of the frequency (s).

2)$\mathbf{C}x = 0$ has a solution which is unique to within a constant.

$\Rightarrow$ a) $x = v_1$, with $\sigma_1 = 0$
b) $\sigma_1 = 0$ is a simple singular value. This assumption is valid because in the solution procedure we made sure that the rank of the null space of $\mathbf{C}$ is one.
3) $\tilde{H}(s_i) = H(s_i) + e_i$, $\{e_i\}_{i=1}^{N}$ are zero mean, Gaussian, uncorrelated, and have equal variances σ^2.

Using the above notation for a perturbed matrix and equation (2.15), we get

$$\left[\,\tilde{\mathbf{C}}\,\right]\begin{bmatrix}\tilde{a}\\ \tilde{b}\end{bmatrix} \simeq 0 \tag{4.4}$$

where,

$$[\tilde{\mathbf{C}}] = \begin{bmatrix} 1 & s_1 \ldots s_1^P & -\tilde{H}_1 & -\tilde{H}_1 s_1 \ldots - \tilde{H}_1 s_1^Q \\ 1 & s_2 \ldots s_2^P & -\tilde{H}_2 & -\tilde{H}_2 s_2 \ldots - \tilde{H}_2 s_2^Q \\ \vdots & \vdots & \vdots & \vdots \\ 1 & s_N \ldots s_N^P & -\tilde{H}_N & -\tilde{H}_N s_N \ldots - \tilde{H}_N s_N^Q \end{bmatrix} \tag{4.5}$$

where,
P= estimate of the order of the numerator,
Q= estimate of the order of the denominator,
N= number of sample points.

$$\Rightarrow [\tilde{\mathbf{C}}] = [\mathbf{C}] + [\mathbf{E}] \tag{4.6}$$

where, $\mathbf{E}$ is the additive error to the matrix $\mathbf{C}$ due to noise in the data. Hence,

$$[E] = \ [\mathbf{0}\,|\,\mathbf{E}_1] \tag{4.7}$$

where, $[\mathbf{0}]$ is a zero matrix of order $N \times P+1$ and

$$[E_1] = -\begin{bmatrix} e_1 & e_1 s_1 & e_1 s_1^2 \ldots e_1 s_1^Q \\ e_2 & e_2 s_2 & e_2 s_2^2 \ldots e_2 s_2^Q \\ \vdots & \vdots & \vdots \\ e_N & e_N s_N & e_N s_N^2 \ldots e_N s_N^Q \end{bmatrix}_{N \times Q+1} \tag{4.8}$$

$$\Rightarrow E_1 = -\underbrace{\begin{bmatrix} e_1 & 0 & 0 & \dots & 0 \\ 0 & e_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_N \end{bmatrix}}_{N\times N}\underbrace{\begin{bmatrix} 1 & s_1 & s_1^2 & \dots & s_1^Q \\ 1 & s_2 & s_2^2 & \dots & s_2^Q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & s_N & s_N^2 & \dots & s_N^Q \end{bmatrix}}_{N\times Q+1} \tag{4.9}$$

$$\Rightarrow [U]^T[E][V] = [U]^T[0|E_1] \begin{array}{cc} \left[\begin{array}{c|c} v_1' & V_2' \\ v_1'' & V_2'' \end{array}\right] & \begin{array}{c} P+1 \\ Q+1 \end{array} \\ \begin{array}{cc} 1 & P+Q+2 \end{array} & \end{array} \tag{4.10}$$

$$= \left[U^T E_1 v_1'' \mid U^T E_1 V_2''\right] \tag{4.11}$$

Using the notation of section 3.1,

$$[U]^T[E][V] = \left[U^T E_1 v_1'' | U^T E_1 V_2''\right] = \begin{pmatrix} \gamma_{11} & g_{12}^T \\ g_{21} & G_{22} \\ g_{31} & G_{32} \end{pmatrix} \tag{4.12}$$

Since v_1 is the solution of the unperturbed Cauchy equation

$$\left[\, \mathrm{C} \,\right] \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

and v_1'' is the vector of the last $Q+1$ entries of v_1, v_1'' is the vector of denominator coefficients. Also, the singular value of interest (σ_1) is zero. Hence, in the notation of section 3.1,

$$\begin{aligned} h &= \sigma_1 g_{12} + \Sigma_2 g_{21} \\ &= \Sigma_2 g_{21} \end{aligned} \tag{4.13}$$

$$\tag{4.14}$$

also, using equation (4.3)

$$\tilde{v}_1 = v_1 + V_2 \Sigma_2^{-1} g_{21} + O\left(\|E\|_2^2\right) \tag{4.15}$$

Hence, g_{12} is of no consequence.

From equation (4.12),

$$[U]^T[E_1][v_1''] = \begin{pmatrix} \gamma_{11} \\ g_{21} \\ g_{31} \end{pmatrix} \tag{4.16}$$

Using equation (4.9), and the fact that v_1'' is the vector of denominator coefficients,

$$E_1 v_1'' = -\begin{bmatrix} e_1 & 0 & 0 & \dots & 0 \\ 0 & e_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_N \end{bmatrix} \begin{bmatrix} de(s_1) \\ de(s_2) \\ \vdots \\ de(s_N) \end{bmatrix} \tag{4.17}$$

where, $de(s_i) = \sum_{k=0}^{Q} b_k s_i^k$, is the value of the unperturbed denominator polynomial evaluated at s_i. For convenience we define a new vector $\underline{\tilde{e}}$ as

$$\Rightarrow \underline{\tilde{e}} \equiv E_1 v_1^{''} = - \begin{bmatrix} e_1 de(s_1) \\ e_2 de(s_2) \\ \vdots \\ e_N de(s_N) \end{bmatrix} \tag{4.18}$$

Using this equation, the fact that $U = [u_1|U_2|U_3]$, and equation (4.16),

$$g_{21} = U_2^T \underline{\tilde{e}} \tag{4.19}$$

Therefore, using equation (4.15),

$$\tilde{v}_1 = v_1 + V_2 \Sigma_2^{-1} U_2^T \underline{\tilde{e}} + O\left(\|E\|_2^2\right) \tag{4.20}$$

Since the elements of $\underline{\tilde{e}}$ are Gaussian random variables, $\tilde{v}_1$ is, to the *first order of approximation* a Gaussian random vector.

Now, using equation (4.1) and the fact that $\sigma_1 = 0$,

$$C = U_2 \Sigma_2 V_2^T$$
$$\Rightarrow V_2 \Sigma_2^{-1} U_2^T = C^+ \tag{4.21}$$

where C^+ is the psuedo-inverse of C.

Therefore, to the first order of approximation,

$$\tilde{v}_1 = v_1 + C^+ \underline{\tilde{e}} \tag{4.22}$$

Using the fact that C^+ is unperturbed and the noise is zero mean, the expectation value of the solution vector (v_1) is given by

$$\begin{aligned} \mathbf{E}(v_1) &= v_1 + C^+ \mathbf{E}(\underline{\tilde{e}}) && (4.23) \\ &= v_1 && (4.24) \end{aligned}$$

Here $\mathbf{E}$ is the expectation operator and not the error matrix. Therefore, to the *first order of approximation*, the estimator is *unbiased.*

The covariance matrix of v_1 is given by,

$$cov(v_1) = \mathbf{E}[(\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T]$$

Using equation (4.22), we have

$$\begin{aligned} \mathbf{E}[(\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T] &= \mathbf{E}[C^+ \underline{\tilde{e}}\, \underline{\tilde{e}}^T C^{+^T}] \\ &= C^+ \mathbf{E}[\underline{\tilde{e}}\, \underline{\tilde{e}}^T] C^{+^T} && (4.25) \end{aligned}$$

Now,

$$\underset{\sim}{\tilde{e}}\,\underset{\sim}{\tilde{e}}^T = \begin{bmatrix} e_1 de(s_1) \\ e_2 de(s_2) \\ \vdots \\ e_N de(s_N) \end{bmatrix} [e_1 de(s_1) e_2 de(s_2) \ldots e_N de(s_N)] \tag{4.26}$$

Therefore, the ij-th entry of this matrix is given by,

$$[\underset{\sim}{\tilde{e}}\,\underset{\sim}{\tilde{e}}^T]_{ij} = e_i e_j de(s_i) de(s_j) \tag{4.27}$$

Since e_i and e_j are assumed to be zero mean, independent, and identically distributed with variance σ^2,

$$\mathbf{E}[e_i e_j] = \sigma^2 \delta_{ij} \tag{4.28}$$

where,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathbf{E}[\underset{\sim}{\tilde{e}}\,\underset{\sim}{\tilde{e}}^T] = \sigma^2 \begin{bmatrix} de^2(s_1) & 0 & 0 & \ldots & 0 \\ 0 & de^2(s_2) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & de^2(s_N) \end{bmatrix} \tag{4.29}$$

$$\Rightarrow \mathbf{E}[(\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T] = \sigma^2 C^+ \begin{bmatrix} de^2(s_1) & 0 & 0 & \ldots & 0 \\ 0 & de^2(s_2) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & de^2(s_N) \end{bmatrix} C^{+^T} \tag{4.30}$$

Letting $C^+_{ij} = c_{ij}$, the autocovariance of the i-th entry of $\tilde{v}_1$ is given by

$$\mathbf{E}[C^+ \underset{\sim}{\tilde{e}}\,\underset{\sim}{\tilde{e}}^T C^+]_{ii} = \sigma^2 \sum_{j=1}^{N} c_{ij}^2 de^2(s_j) \tag{4.31}$$

This is the variance of the i-th entry in the vector of coefficients. Hence, if $i \leq P+1$, we are dealing with a numerator coefficient, else we are dealing with a denominator coefficient.

Since, we have solved a matrix equation in which the elements of the matrix are Gaussian random variables, each element of the solution vector is a Gaussian random variable. Also, the numerator and denominators are linear combinations of the coefficients. Hence, the numerator and denominator are Gaussian random variables *as functions of frequency.* Hence, to completely characterize the numerator and denominator random variables, we only need their expectation values and variances.

To make this problem of the ratio of two Gaussians solvable, we have to assume that any two coefficients are independent of each other. Hence, the cross-covariance matrix of v_1 is assumed to be diagonal.

Now,

$$\tilde{A}(s) = \sum_{k=0}^{P} \tilde{a}_k s^k \tag{4.32}$$

$$\tilde{B}(s) = \sum_{k=0}^{Q} \tilde{b}_k s^k \tag{4.33}$$

Therefore,

$$\mathbf{E}[\tilde{A}(s)] = \sum_{k=0}^{P} \mathbf{E}[\tilde{a}_k] s^k \tag{4.34}$$

and

$$\mathbf{E}[\tilde{B}(s)] = \sum_{k=0}^{Q} \mathbf{E}[\tilde{b}_k] s^k \tag{4.35}$$

However, since to the first order of approximation the coefficients are unbiased,

$$\mathbf{E}[\tilde{A}(s)] = \sum_{k=0}^{P} a_k s^k \tag{4.36}$$

and

$$\mathbf{E}[\tilde{B}(s)] = \sum_{k=0}^{Q} b_k s^k \tag{4.37}$$

Therefore, the estimators for the numerator and denominator as a function of frequency are unbiased. However, as we will see, since the ratio of two variables is not a linear function, this does not mean the final estimator is unbiased.

To calculate the variances of the numerator and denominator as a function of frequency,

$$var[A(s)] = var\left[\sum_{k=0}^{P} \tilde{a}_i s^i\right] \tag{4.38}$$

Using the assumption that each coefficient is independent of the others,

$$var[A(s)] = \sum_{k=0}^{P} var(\tilde{a}_i) s^{2i} \tag{4.39}$$

Therefore, from equation (4.31),

$$\Rightarrow var[A(s)] = \sigma^2 \sum_{i=1}^{P+1} s^{2i} \sum_{j=1}^{N} c_{ij}^2 de^2(s_j) \tag{4.40}$$

Similarly,

$$var[B(s)] = \sigma^2 \sum_{i=P+2}^{P+Q+2} s^{2i} \sum_{j=1}^{N} c_{ij}^2 de^2(s_j) \tag{4.41}$$

Let $\bar{N} \equiv \mathbf{E}[A(s)]$, $\bar{D} \equiv \mathbf{E}[B(s)]$, $a^2 \equiv var[A(s)]$, and $b^2 \equiv var[B(s)]$. Therefore, the problem has reduced to: Given the means and variances of two independent

Gaussian random variables, what is the PDF of their ratio? This problem has been solved in reference [9].

In the notation of [9], if N and D are independent Gaussian random variables with means $\bar{N}$ and $\bar{D}$ respectively, and variances a^2 and b^2 respectively, and if

$$R = \frac{N}{D}$$

then, the probability density function of R is given by

$$f_R(r) = \sqrt{\frac{(ab)^3}{\pi}} \frac{1}{b^2 r^2 + a^2} e^{-\left(\frac{\bar{N}^2}{2a^2} + \frac{\bar{D}^2}{2b^2}\right)} \left[Z erf(Z) \exp(Z^2) + \frac{1}{\sqrt{\pi}} \right] \tag{4.42}$$

where,

$$Z = \frac{1}{\sqrt{2b^2a^2}} \left(\frac{b^2 \bar{N} r + a^2 \bar{D}}{\sqrt{b^2 r^2 + a^2}} \right)$$

and the error function is defined as

$$erf(Z) = \frac{2}{\sqrt{\pi}} \int_0^Z e^{-t^2} dt$$

Hence, we have the theoretical PDF of the ratio of two random variables. However, this density function is an approximation of the true density function. To obtain the true density function we would need to take into account the cross-correlation between the coefficients. This leads to a problem that is highly difficult to solve.

4.2 Numerical Examples

To test the above theory, the Cauchy method was tested with a simple example. As an example the function was chosen to be the testing function was

$$H(s) = \frac{\sum_{k=0}^{4} k s^k}{\sum_{k=0}^{5} (k+1) s^k} \tag{4.43}$$

This ratio of two polynomials was evaluated at a 31 points in the range s=2.0 and s=4.0. Two tests were performed on this data.

In the first test, Gaussian noise was added to the data directly. A numerical Gaussian random number generator was used. The power in the noise was chosen such that the signal to noise ratio (SNR) was 30dB. This perturbed data was used as input to the Cauchy program. The resulting polynomials were used to evaluate the parameter at s=3.0. This was considered to be one sample of the random variable *at s=3.0.* 1001 such samples were taken. A PDF estimator was used to estimate the PDF *at s=3.0.* Figure 4.1 shows the PDF found using this method. This is the plot marked 'Adding noise to the data'.

In the second test, the original unperturbed data between s=2.0 and s=4.0 were used as input to the Cauchy program. The unperturbed numerator and denominator coefficients were evaluated. The means of the numerator and denominator were evaluated using equations (4.36) and (4.37) respectively. Also, the variances of the numerator and denominator were evaluated using equations (4.40) and (4.41) respectively. Using these values of means and variances, a Gaussian random variable, with the numerator mean and variance, was divided with another Gaussian random variable with the denominator mean and variance. This was repeated 1001 times. The Gaussian random numbers were generated using the same random number generator as in the first test.

The 1001 samples got from this test were used as input to the same PDF estimator. The result from this estimation of the PDF is shown in Figure 4.1. This plot is labeled 'Numerical Simulation'.

Finally, these two PDFs are compared with the theoretical PDF in equation (4.42). The choices of $\bar{N}$, $\bar{D}$, a^2, and b^2 are obtained from the theoretical means and variances used in the second test.

At s=3.0, using the above function

Actual Value :0.2126
Mean (adding noise to the data):0.2124
Mean (dividing two Gaussians with the theoretical means and variances):0.1804

Figure 4.2 shows the same three PDFs for a signal to noise ratio of 40dB. Here the agreement is better than in the earlier case. This is to be expected since the assumptions come closer to be being satisfied as the noise reduces.

The work in this chapter is also published in [10]

5.0 Cauchy Method with Magnitude Only Data

In Chapter 2.0 the theory of the Cauchy method is developed assuming the knowledge of both the real and imaginary parts of the measured or numerically generated data i.e. the knowledge of the phase response is assumed to be known. However, in many important applicatons, it is difficult to accurately measure the phase response. A classic example of an important situtation where this difficulty arises is the measuring the RCS of a target in the X-band. The magnitude response can be measured accurately, however calibration for the phase is difficult, if not impossible.

In this section we propose possible approaches to extending the Cauchy method to the case where only the magnitude of the data is known. Two entirely different approaches are possible. In the first approach, the Cauchy method can be used to extrapolate or interpolate the magnitude only data. The extrapolated magnitude

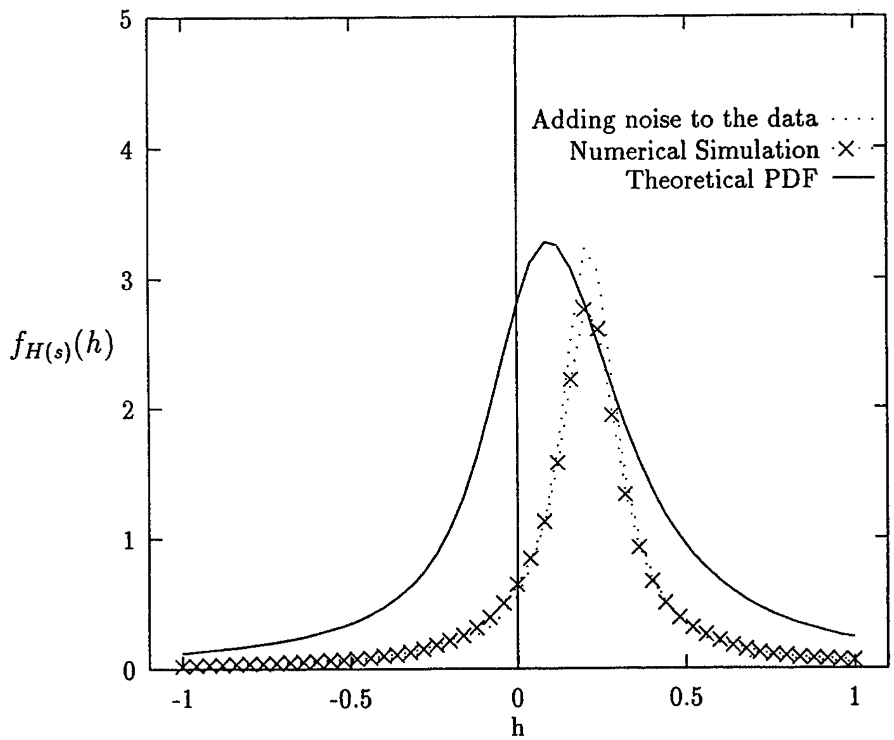


Figure 4.1: Comparison of theoretical PDF and numerically simulated PDFs. SNR =30dB

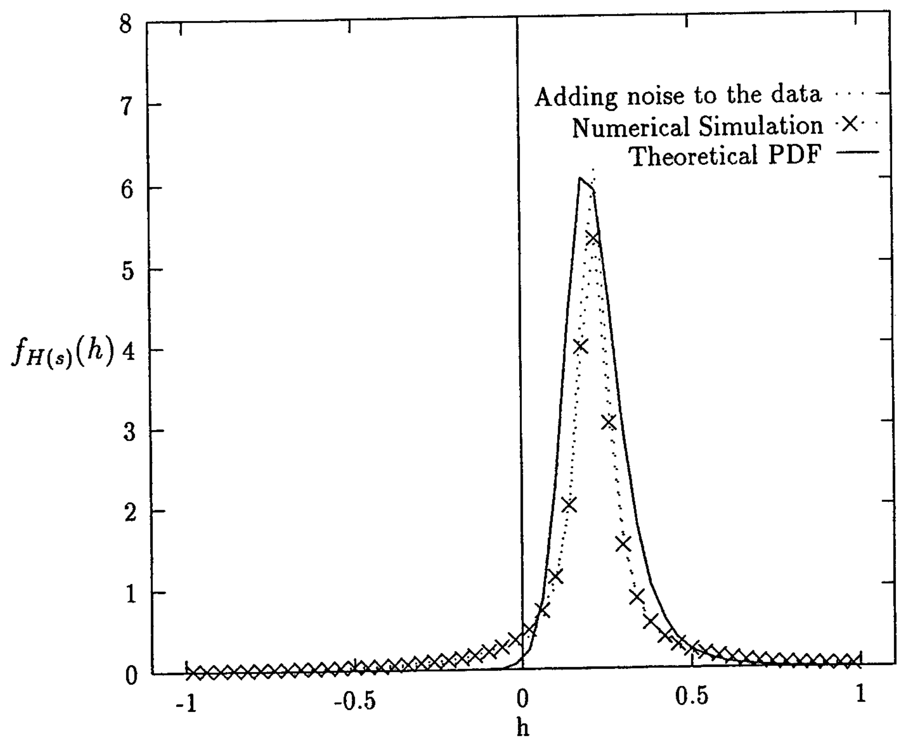


Figure 4.2: Comparison of theoretical PDF and numerically simulated PDFs. SNR =40dB

only data can then be used as input to a program to retrieve the phase from the magnitude.

The second possible approach is the reverse of the above process. The magnitude only data can be used as input to a program that retrieves the phase from the given magnitude. This complex data can then be used by the Cauchy method described in Chapter 2.0 to extrapolate with respect to the variable of interest.

In case of the second approach the theory of the Cauchy method does not require any changes. The theory of retrieving the phase from the magntiude is presented later in this chapter. In the case of the first approach, the theory of the Cauchy method must be modified to account for the fact that the given data has only magnitude information.

5.1 Extrapolation with magnitude only data

To apply the Cauchy method to magnitude only data, we begin by recognizing that the magnitude response of a linear, time invariant, system is an even function of frequency i.e., the magnitude response at f_0 is the same as the response at $-f_0$. Hence, when modeling the magnitude response using a ratio of polynomials, the odd powers of frequency in equation (2.1) do not contribute to the extrapolation. The magnitude response $|H(s)|$ as a function of frequency can, therefore, be represented as

$$|H(s)| \simeq \frac{A(s)}{B(s)} = \frac{\sum_{k=0}^{P/2} a_k s^{2k}}{\sum_{k=0}^{Q/2} b_k s^{2k}} \tag{5.1}$$

Since the magnitude response is real, the coefficients $a(k), k = 0, \ldots, P/2$ and $b(k), k = 0, \ldots, Q/2$ that define the two polynomials are real. Hence all the Hermitian operations in equations (2.13), (2.16) and (2.17) must be replaced with the transpose operator. Other than these modifications, the Cauchy method, as described in Chapter 2.0 is completely applicable.

5.2 Phase retrieval from magnitude only data

Reconstruction of the phase from magnitude only data is an important problem. For minimum phase systems, the reconstruction of phase from magnitude only data is relatively straightforward as the phase response is given by the Hilbert transform of the log of the magnitude response [11], [12], [13]. Given the magnitude response $|H(s)|$, the phase response $arg[H(s)]$ is obtained by

$$arg[H(s)] = \frac{1}{\pi} P \left[\int_{-\infty}^{\infty} \frac{\ln |H(x)|}{x - s} dx \right] \tag{5.2}$$

where P indicates the prinicpal value integral as the integrand has a singularity and is not integrable in the usual sense.

This relationship between the magnitude and the phase does not hold if a system is not minimum phase. A system is minimum phase if all zeros of the transfer function are in the left half plane ($x = \sigma + \jmath\omega, \sigma < 0$). Unfortunately, most electromagnetic systems are non-minimum phase and hence equation (5.2) does not apply. However, there is a more general result, based on causality, that can be applied to the phase retrieval problem.

It must be noted that the phase retrieval problem is inherently non-unique. A pure linear phase can be added to the reconstructed phase without affecting the magnitude spectrum. This arises from the fact that a linear phase is equivalent to a pure time delay in the time domain. the principle of causality implies that the time response has to be zero for negative times i.e., $f(t) = 0, t < 0$. All physical systems must be causal. A given magnitude response can represent a causal system only if it satisfies the Paley-Wiener criterion

$$\int_{-\infty}^{\infty} \frac{\ln(|H(s)|)}{1+s^2} ds < \infty \tag{5.3}$$

i.e. the integral must be bounded. Note that this is possible only if $|H(s)|$ is non-zero for all frequencies. Otherwise, if $|H(s)|$ is zero over any finite support, the term $\ln|H(s)|$ would be infinity. If $H(s) = R(s) + jX(s)$, $R(s)$ and $X(s)$ are related by the Hilbert transform [14] i.e.

$$R(s) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(w)}{s-w} dw \tag{5.4}$$

$$X(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(w)}{s-w} dw \tag{5.5}$$

Also,

$$h(t) = \frac{2}{\pi}\int_{0}^{\infty} R(s)\cos(st) ds \tag{5.6}$$

$$h(t) = -\frac{2}{\pi}\int_{0}^{\infty} X(s)\sin(st) ds \tag{5.7}$$

and

$$\int_{0}^{\infty} |x(t)|^2 dt = \frac{1}{\pi}\int_{-\infty}^{\infty} |R(s)|^2 = \frac{1}{\pi}\int_{-\infty}^{\infty} |X(s)|^2 \tag{5.8}$$

In our case, the magnitude is given over a limited frequency range and at discrete frequency points. Hence, the integrals in the above equations must be replaced by summations. Assuming that real part $R(s)$ is even with period 2π from $-\pi$ to π, we can write the Fourier series,

$$R(s) = a_0 + \sum_{n=1}^{\infty} a_n \cos(ns + \phi_n) \qquad 0 < s < \pi \tag{5.9}$$

and by the discrete version of equation (5.5

$$X(s) = -\sum_{n=1}^{\infty} a_n \sin(ns + \phi_n) \qquad 0 < s < \pi \tag{5.10}$$

i.e. the real and imaginary parts of the frequency response are defined by the same Fourier coefficients.

Now,

$$\begin{aligned} |X(s)|^2 &= R^2(s) + X^2(s) \\ &\simeq \left[a_0 + \sum_{n=1}^{N} a_n \cos(ns + \phi_n)\right]^2 + \left[\sum_{n=1}^{N} a_n \sin(ns + \phi_n)\right]^2 \end{aligned} \tag{5.11}$$

where, N is chosen as a cut off to limit the number of unknowns to a finite number based on the information available. We can now use an optimization routine to minimize the error,

$$E(s) = \left\{ |X(s)|^2 - \left[a_0 + \sum_{n=1}^{N} a_n \cos(ns + \phi_n)\right]^2 - \left[\sum_{n=1}^{N} a_n \sin(ns + \phi_n)\right]^2 \right\} \tag{5.12}$$

and the phase can be retrieved from

$$\phi(s) = \frac{-\left[\sum_{n=1}^{N} a_n \sin(ns + \phi_n)\right]}{\left[a_0 + \sum_{n=1}^{N} a_n \cos(ns + \phi_n)\right]} \tag{5.13}$$

6.0 Conclusions

This work has presented a technique for the determination of wideband response of a system given only narrowband information. Based on the Cauchy method starts with assuming that the parameter of interest, as a function of frequency, can be approximated by a simple rational polynomial function. This assumption is valid for many physical systems. The method evaluates the order of the polynomials and the coefficients that define them. Using this form the parameter is evaluated at many frequency points. It is shown that the technique has applications to many practical problems. In our research the technique is applied to the Method of Moments, optical systems, filter analysis, and device characterization. In all applications the Cauchy method has shown to save time and memory.

This work has also begun to explore the possibility of applying the Cauchy method to magnitude only data. Two possible approaches are conisdered, both of which draw on the ability to recover the phase from the magnitude only data. A phase retrieval algorithm has been presented.

It must be pointed out that the Cauchy method is completely general and can be used to extrapolate or interpolate with respect to any variable other than frequency. However, in many applications in electromagnetics, frequency is the variable of interest.

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