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13. ABSTRACT (Maximum 200 words): We have explored the problem of extending the linear  $H^\infty$  theory to nonlinear systems, robust distributed parameter control, and the robust analysis and synthesis of controllers for systems in the presence of various classes of structured perturbations. We have employed operator theory, partial differential equations, optimization theory, and invariant theory to study the problem of utilizing visual information in a feedback loop and the closely related application of visual tracking. We have considered a combination of methods from robust control and computer vision for this purpose. Our approach to computer vision and image processing is based on certain novel curvature dependent evolution equations that may be employed for image enhancement, denoising, active contours, edge detection, morphology, shape recognition, shape-from-shading, stereo disparity, and optical flow. We have combined this with our longstanding work in robust control, and applying these these ideas to certain benchmark problems in visual tracking.

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"Nonlinear and Structured Interpolation for Robust Control"

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## 1 Introduction

In the research program supported by DAAH04-94-G-0054, we have carried out an extensive study of robust system control using methodologies from interpolation theory, dilation theory, and functional analysis. Some of these methodologies have been applied to certain benchmark problems involving visual tracking.

A key part of our effort has centered around a novel *iterative commutant lifting theorem* which gives an explicit design procedure for nonlinear systems and captures the  $H^\infty$ -control problem in the nonlinear framework [61], [62], [63]. In this area, we have defined a notion of *rationality* for nonlinear systems, and we have proven that the iterative commutant lifting procedure produces rational controllers (in this nonlinear sense) if we start from rational data. The procedure has already been applied to certain systems with noninvertible nonlinearities (in collaboration with colleagues at Honeywell in Minneapolis). This framework has also led to new directions in introducing notions of *causality* [64] in commutant lifting theory, and moreover has led to the possible formulation of a global nonlinear commutant lifting theorem

as a saddle-point result. We are particularly interested in the treatment (and understanding the control limitations) of "hard" nonlinearities such as dead-zone, backlash, and saturation.

We have been working as well in distributed parameter control as part of the research sponsored by ARO Grant DAAH04-94-G-0054. Recently a monograph [54] which extensively describes our *skew Toeplitz* approach to distributed  $H^\infty$  design and analysis has been published. We have also explored problems in  $\mu$ -synthesis and analysis; see Section 2. We have developed a general lifting procedure, by which we can interpret the  $D$ -scaled upper bound for the structured singular value as a structured singular value on a certain extended space. We have shown in which cases lifting is unnecessary, i.e., the upper bound gives a non-conservative measure of robustness. This is very important since the upper bound for  $\mu$  is log-convex in the scalings and so can be computed, while  $\mu$  cannot. Our results work directly for systems, not just finite matrices. This allows us to study broad classes of structured perturbations using this tool. We have also been continuing our work for a rigorous  $\mu$ -synthesis procedure based on structured interpolation.

We have applied our methods to the key area of visual tracking which may be employed for a number of problems in robotics, manufacturing, as well as automatic target recognition. Even though tracking in the presence of a disturbance is a classical control issue, because of the highly uncertain nature of the disturbance, this type of problem is very difficult and challenging. Visual tracking differs from standard tracking problems in that the feedback signal is measured using imaging sensors. In particular, it has to be extracted via computer vision and image processing algorithms and interpreted by a reasoning algorithm before being used in the control loop. Furthermore, the response speed is a critical aspect. As alluded to above, we have developed robust control algorithms valid for general classes of distributed parameter and nonlinear systems based on interpolation and operator theoretic methods. In our Army Research Office sponsored research project, we have been explicitly combining our robust control techniques with our novel approach to image processing in order to develop "state of the art" visual tracking algorithms.

Indeed, because of our interest in controlled active vision, we have been conducting research into advanced algorithms in image processing and computer vision for a variety of uses: image smoothing and enhancement, image segmentation, morphology, denoising algorithms, shape recognition, edge detection, optical flow, shape-from-shading, and deformable contours ("snakes"). Our techniques have already been applied to medical imaging, and have been used to define a novel affine invariant scale-space. Our ideas are motivated by certain types of geometric invariant flows rooted in the mathematical theory of curve and surface evolution. There are now available powerful numerical algorithms based on Hamilton-Jacobi type equations and the associated theory of viscosity solutions for the computer implementation of this methodology. In addition to systems and control, our work in vision can also have had an impact on ATR. This is based on the nonlinear filters we have developed for image enhancement that are natural corollaries of our work. These filters can smooth while preserving such key features as edges and ridges.

A complete list of publications which acknowledges support of DAAH04-94-G-0054 is included in Section 7. We now sketch the work done on this contract.

## 2 Structured Singular Values and $\mu$ -Synthesis

During the past funding period, we have investigated a number of the key properties of the *structured singular value* which was introduced into robust control by John Doyle and Michael Safonov ([39], [116]) to handle problems involving structured perturbations which includes both  $H^\infty$  and the multivariable gain margin as special cases. In particular, we have introduced a new *lifting* or *ampliation* method for the study of robustness under a variety of structured perturbation models. We will now briefly elucidate some of the key results in this area. Details about this methodology can be found in our papers [20, 18, 19, 26, 21].

### 2.1 Preliminary Results on Structured Singular Values

In this section, we will introduce some notation and elementary results about the structured singular value which we will need to explain our ampliation method.

Let  $A$  be a linear operator on a Hilbert space  $\mathcal{E}$ , and let  $\Delta$  be an algebra of operators on  $\mathcal{E}$ . The *structured singular value* of  $A$  (relative to  $\Delta$ ) is the number

$$\mu_\Delta(A) = 1/\inf\{\|X\| : X \in \Delta, -1 \in \sigma(AX)\}.$$

(This quantity was defined in [39, 116] under a more restrictive context.) Recall that in system analysis, the structured singular value gives a measure of robust stability with respect to certain perturbation measures. Unfortunately,  $\mu_\Delta(A)$  is very difficult to calculate, and in practice an upper bound for it is used. This upper bound is defined by

$$\hat{\mu}_\Delta(A) := \inf\{\|XAX^{-1}\| : X \in \Delta', X \text{ invertible}\},$$

where  $\Delta'$  is the commutant of the algebra  $\Delta$ .

In [24, 21], we formulated a lifting technique for the study of the structured singular value. The basic idea is that  $\hat{\mu}_\Delta(A)$  can be shown to be equal to the structured singular value of an operator on a bigger Hilbert space. (In [24] this was done for finite dimensional Hilbert spaces, and then in [21] this was extended to the infinite dimensional case.) The problem with these results is that the size of the ampliation necessary to get  $\hat{\mu}_\Delta(A)$  equal to a structured singular value, was equal to the dimension of the underlying Hilbert space. Hence in the infinite dimensional case we needed an infinite ampliation. We should note, that one important result that came out of the work in [24] was that for the first time the upper bound  $\hat{\mu}$  was shown to be *continuous*. Since this is the quantity actually used in  $\mu$  analysis and synthesis, this result is certainly of importance.

In our most recent work [20], we have shown that in fact, one can always get by with a finite ampliation. (Note that in this context, we will be using the terms “ampliation” and “lifting” interchangeably.) For the block diagonal algebras of interest in robust control, the ampliation only depends on the number of blocks of the given perturbation structure. We will moreover, give a new criterion below showing when  $\hat{\mu}_\Delta(A) = \mu_\Delta(A)$ , that is, when no lifting is necessary and so  $\hat{\mu}_\Delta(A)$  gives a *nonconservative measure of robustness*. This is then used to derive an elegant result of Megretski and Shamma [93, 125] on Toeplitz operators. See also [44, 79, 80] for related work in this area.

We will denote by  $\mathcal{L}(\mathcal{E})$  the algebra of all bounded linear operators on the (complex, separable) Hilbert space  $\mathcal{E}$ . Fix an operator  $A \in \mathcal{L}(\mathcal{E})$  and a subalgebra  $\Delta \subset \mathcal{L}(\mathcal{E})$ . Observe that  $\Delta \subset \Delta''$  and  $\Delta''' = (\Delta'')' = \Delta'$  so that we have the inequalities

$$\mu_\Delta(A) \leq \mu_{\Delta''}(A), \quad \hat{\mu}_\Delta(A) = \hat{\mu}_{\Delta''}(A).$$

In our study we will need further singular values which we now define. For  $n \in \{1, 2, \dots, \infty\}$  we denote by  $\mathcal{E}^{(n)}$  the orthogonal sum of  $n$  copies of  $\mathcal{E}$ , and by  $T^{(n)}$  the orthogonal sum of  $n$  copies of  $T \in \mathcal{L}(\mathcal{E})$ . Operators on  $\mathcal{E}^{(n)}$  can be represented as  $n \times n$  matrices of operators in  $\mathcal{L}(\mathcal{E})$ , and  $T^{(n)}$  is represented by a diagonal matrix, with diagonal entries equal to  $T$ .

Denote by  $\Delta_n$  the algebra of all operators on  $\mathcal{E}^{(n)}$  whose matrix entries belong to  $\Delta$ , and observe that  $(\Delta_n)'' = (\Delta'')_n$ , and  $(\Delta_n)' = (\Delta')^{(n)} = \{T^{(n)} : T \in \Delta'\}$ . Therefore we will denote these algebras by  $\Delta_n''$  and  $\Delta_n'$ , respectively.

## 2.2 Ampliations of Perturbations

We can now formulate our lifting result from [20], relating  $\mu_\Delta(A)$  and  $\hat{\mu}_\Delta(A)$ . For finite dimensional  $\mathcal{E}$ , a lifting result of this type was first proven in [24]. The result was then generalized to the infinite dimensional case in [21]. (For another proof of this type of lifting result in finite dimensions, see [44].) In these previous works, the lifting or ampliation of the operator  $A$  and perturbation structure  $\Delta$  depends on the dimension of  $\mathcal{E}$ . Thus if  $\mathcal{E}$  is infinite dimensional, we get an infinite lifting. In Theorem 1 stated below, we only have to lift up to the dimension of  $\Delta'$  which in the cases of interest in the control applications of this theory only depends on the number of blocks of the given perturbation structure. The proof of the theorem makes use of some work that we did on the *relative numerical range* and the continuity of the spectrum on closed similarity orbits in [18, 19] which we believe also has independent interest.

**Theorem 1** *Assume that  $\Delta'$  is a  $*$ -algebra of finite dimension  $n$ . Then*

$$\hat{\mu}_\Delta(A) = \mu_{\Delta_n''}(A^{(n)})$$

for every  $A \in \mathcal{L}(\mathcal{E})$ .

**Remark.** In the cases of interest in control,

$$\Delta'' = \Delta,$$

and so one has from Theorem 1 that

$$\mu_{\Delta_n}(A) = \hat{\mu}_\Delta(A).$$

## 2.3 Conditions for $\mu = \hat{\mu}$

We will discuss some the conditions from [20] when  $\mu = \hat{\mu}$  without any need for lifting or ampliation. In such cases,  $\hat{\mu}$  gives a nonconservative measure of robustness relative to the given perturbation structure. In the finite dimensional case, there have been some results of this kind, the most famous of which is that of Doyle [39], who showed that no lifting is necessary for perturbation structures with three or fewer blocks.

First of all, call *critical* any  $A_0 \in \overline{\mathcal{O}_{\Delta'}(A)}$  satisfying

$$\limsup_{\epsilon \downarrow 0} \|(I - \epsilon X)A_0(I - \epsilon X)^{-1}\| \geq \|A_0\|, \quad \forall X \in \Delta'.$$

Then we have

**Lemma 1** *If  $A_0$  is a critical operator in  $\overline{\mathcal{O}_{\Delta'}(A)}$ , then it enjoys the following property ( $\mathcal{O}$ ):*

$$0 \in W_Q(\|A_0\|^2 X - A_0^* X A_0), X \in \Delta',$$

where  $Q = \|A_0\|^2 I - A_0^* A_0$ .

The next lemma is the key step in adapting the proof of Theorem 1 in order to show that

$$\mu_{\Delta}(A) = \widehat{\mu}_{\Delta}(A)$$

in several interesting cases.

**Lemma 2** *Let  $A_0$  be an operator on  $\mathcal{E}$  which satisfies the essential version of property ( $\mathcal{O}$ ), property ( $\mathcal{O}^0$ ), namely*

$$0 \in W_Q^0(\|A_0\|^2 X - A_0^* X A_0), X \in \Delta',$$

where  $Q = \|A_0\|^2 I - A_0^* A_0$ . Then there exists a sequence  $\{h_k\}_{k=1}^{\infty} \subset \mathcal{E}$ ,  $\|h_k\| = 1$ ,  $k = 1, 2, \dots$ , such that

$$Qh_k \rightarrow 0 \text{ strongly and } \langle (\|A_0\|^2 X - A_0^* X A_0)h_k, h_k \rangle \rightarrow 0,$$

for all  $X \in \Delta'$ .

We can now state another key result:

**Theorem 2** *If there exists a critical operator  $A_0$  satisfying property  $\mathcal{O}^0$  in the closed  $\Delta'$ -orbit of  $A$ , then*

$$\mu_{\Delta''}(A) = \widehat{\mu}_{\Delta}(A).$$

**Remark.** Under the hypotheses of Theorem 2, when  $\Delta'' = \Delta$  (which happens in all cases of interest in control), we have that

$$\mu_{\Delta}(A) = \widehat{\mu}_{\Delta}(A).$$

Let  $L(\Delta' A \Delta')$  denote the linear space generated by

$$\Delta' A \Delta' = \{XAY : X, Y \in \Delta'\}.$$

Obviously  $L(\Delta' A \Delta')$  is finite dimensional, and therefore closed. Hence  $\overline{\mathcal{O}_{\Delta'}(A)} \subset L(\Delta' A \Delta')$ .

**Corollary 1** *If for every  $B \in L(\Delta' A \Delta')$ ,  $B \neq 0$ , the norm of  $B$  is not attained (that is, there is no  $h \in \mathcal{H}$  such that  $\|Bh\| = \|B\|\|h\| \neq 0$ ), then*

$$\mu_{\Delta''}(A) = \widehat{\mu}_{\Delta}(A).$$

## 2.4 Toeplitz Operators

In this section, we want to use our lifting methodology in order to derive a beautiful result of Magretski [93], and Shamma [125] on the structured singular value of a Toeplitz operator, i.e., a linear time invariant system.

Accordingly, set  $\mathcal{E} = H^2(\mathbb{C}^n)$  and let  $A$  denote the multiplication (analytic Toeplitz) operator on  $\mathcal{E}$  defined by

$$(Ah)(z) = A(z)h(z), \quad |z| < 1, \quad h \in \mathcal{E},$$

where

$$A(z) = [a_{jk}]_{j,k=1}^n, \quad |z| < 1,$$

has  $H^\infty$  entries. Let  $\Delta'$  be any  $*$ -subalgebra of  $\mathcal{L}(\mathbb{C}^n)$ , the elements of which are regarded as multiplication operators on  $\mathcal{E}$ . Note that in this case,  $\Delta'' = \Delta$  is the algebra generated by operators of the form

$$(Bh)(z) = B(z)h(z), \quad |z| < 1, \quad h \in \mathcal{E}$$

with  $B(z)X = XB(z)$ ,  $|z| < 1$ ,  $X \in \Delta'$  as well as of the form

$$B \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} Yh_1 \\ Yh_2 \\ \vdots \\ Yh_n \end{bmatrix},$$

with  $Y \in \mathcal{L}(H^2(\mathbb{C}))$  arbitrary. We can now state:

**Lemma 3** *Let  $A_0$  be an analytic Toeplitz operator. Then if  $A_0$  has property  $(\mathcal{O})$ , it also has property  $(\mathcal{O}^0)$ .*

**Corollary 2** ([93, 125]) *For  $A$  and  $\Delta'$  as above, we have that*

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

## 3 Causality and Robust Nonlinear Control

In our work in nonlinear robust control, and especially the extension of  $H^\infty$  to the nonlinear framework, we have discovered a number of intriguing problems which do not occur in the linear case. Indeed, besides the theoretical and practical questions involved in finding an implementable design methodology, it is interesting to note that certain associated problems of causality have arisen in working in this area, which we would like to briefly sketch. In fact, as a result of this effort we have been able to put an explicit causality constraint in commutant lifting theory for the first time [49, 51, 64, 17].

There have been several attempts to extend dilation theoretic techniques to nonlinear input/output operators, especially those which admit a Volterra series expansion (see [50] and the references therein). In several of these approaches, one is reduced to applying the classical (linear) commutant lifting theorem to an  $H^2$ -space defined on some  $D^n$  (where  $D$  denotes the unit disc). Now when one applies the classical result to  $D^n$  ( $n \geq 2$ ), even though

time-invariance is preserved (that is, commutation with the appropriate shift), causality may be lost. Indeed, for analytic functions on the disc  $D$ , time-invariance (that is, commutation with the shift) implies causality. For analytic functions on the  $n$ -disc ( $n > 1$ ), this is not necessarily the case. For dynamical system control design and for any physical application, this is of course a major drawback. Hence for a dilation result in  $H^2(D^n)$  we need to include the causality constraint explicitly in the set-up of the dilation problem. We will discuss a way of doing this now based on [64, 50, 51], and some of the beautiful associated problems that arise in operator theory and systems and control as a result of this method.

### 3.1 Causality in Nonlinear Systems

Recall that causality basically means that for a given input/output system the past output is independent of the future inputs. This may be given precise mathematical formulation in terms of a family of projections which we shall now do. We follow treatment of [64].

Let  $S$  denote an isometry on a Hilbert space  $\mathcal{G}$ , and let  $T$  denote a contraction on a Hilbert space  $\mathcal{H}$ . Let  $P_{j0}$ ,  $j \geq 1$  denote a sequence of orthogonal projections in  $\mathcal{G}$  satisfying the following conditions:

$$P_{10} \leq P_{20} \leq \dots \quad (1)$$

$$P_{j0} \leq I - S^j S^{*j} \quad j = 1, 2, \dots \quad (2)$$

$$P_{j+1,0} S(I - P_{j0}) = 0 \quad j = 1, 2, \dots \quad (3)$$

Next  $U : \mathcal{K} \rightarrow \mathcal{K}$  will denote a minimal isometric dilation of the given contraction  $T$  defined on  $\mathcal{H}$ . Let  $B : \mathcal{G} \rightarrow \mathcal{H}$  intertwine  $S$  with  $U$ , that is,

$$UB = BS. \quad (4)$$

Note that this latter condition implies  $U^j B = BS^j$  ( $j \geq 1$ ), hence  $U^j U^{*j} BS^j S^{*j} = BS^j B^{*j}$ , and so

$$(I - U^j U^{*j})B = (I - U^j U^{*j})B(I - S^j S^{*j}) \quad j = 1, 2, \dots \quad (5)$$

We now make the following key definition:

**Definition.** An operator  $B$  satisfying (4) is called  $(P_{10}, P_{20}, \dots)$ -causal (and if the sequence  $\{P_{j0}\}_{j=1}^{\infty}$  is fixed, *causal*) if

$$(I - U^j U^{*j})B = (I - U^j U^{*j})BP_{j0} \quad j \geq 1, \quad (6)$$

or equivalently,

$$(I - P_{j0})B^* = (I - P_{j0})B^*U^j U^{*j} \quad j \geq 1. \quad (7)$$

Note that  $B$  is always  $(I - SS^*, I - S^2 S^{*2}, \dots)$ -causal. In what follows the sequence  $P_{10}, P_{20}, \dots$  will be fixed and causality will always be defined relative to this sequence.

Next we let  $A : \mathcal{G} \rightarrow \mathcal{H}$  be an operator intertwining  $S$  and  $T$ , that is,  $AS = TA$ . Then an *intertwining lifting (or dilation)* of  $A$  is an operator  $B : \mathcal{G} \rightarrow \mathcal{K}$  such that  $BS = UB$ , and  $PB = A$  where  $P : \mathcal{K} \rightarrow \mathcal{H}$  denotes orthogonal projection.

We define

$$\nu_\infty(A) := \inf\{\|B\| : B \text{ is a causal intertwining dilation of } A\},$$

and

$$\mu(A) := \min\{M \geq 0 : \|A\| \leq M, \|(I - P_{j0})A^*h\| \leq M\|T^{*j}h\|, h \in \mathcal{H} \ j \geq 1\}.$$

We can then show that

$$\mu(A) \leq \nu_\infty(A). \tag{8}$$

We will also need a functional which lies between  $\mu(A)$  and  $\nu_\infty(A)$ . To this aim, we call a sequence of operators  $\Gamma_j : \mathcal{G}_j := (I - P_{0j})\mathcal{G} \rightarrow \mathcal{H}$ ,  $j = 0, 2, \dots$  a *resolution of A* if

$$\begin{aligned} \Gamma_0 &= A \\ \Gamma_j|_{\mathcal{G}_{j+1}} &= T\Gamma_{j+1} \quad \forall j \geq 0, \\ \Gamma_j &= \Gamma_{j+1}S|_{\mathcal{G}_j} \quad \forall j \geq 0. \end{aligned}$$

(Note that we take  $P_{00} := 0$ , so that  $\mathcal{G}_0 = \mathcal{G}$ .) We now define

$$\bar{\mu}(A) :=$$

$$\min\{M \geq 0 : \|A\| \leq M \text{ and there exists a resolution of } A, \Gamma_j, \text{ with } \|\Gamma_j\| \leq M, \forall j \geq 0\}.$$

Then, we have the following results from [64]:

**Theorem 3 (Causal Commutant Lifting Theorem)** *Notation as above. Then*

$$\nu_\infty(A) = \bar{\mu}(A).$$

We also have the following variant of the Causal Commutant Lifting Theorem:

**Corollary 3** *If  $\ker T = \{0\}$ , then  $\mu(A) = \nu_\infty(A)$ .*

In the case of the compressed shift, which is of course of greatest interest in control, based on these results we have developed a completely constructive method for designing nonlinear causal controllers which generalize the  $H^\infty$  procedure. Indeed, in this case one can write down *explicitly* the causal dilations (see [49]). In fact, in [49, 50] via a *Reduction Theorem*, we show how the construction of a causal dilation may be reduced to a *classical interpolation problem*.

### 3.2 Causal Analytic Mappings

We will now specialize the discussion to analytic mappings on Hilbert space. We use the standard definitions from [64].

So, we consider an analytic map  $\phi$  with  $\mathcal{G} = \mathcal{H} = H^2$  (the standard Hardy space on the disc  $D$ ). Note that

$$H^2 \otimes \cdots \otimes H^2 = (H^2)^{\otimes n} \cong H^2(D^n)$$

where we map  $1 \otimes \cdots \otimes z \otimes \cdots \otimes 1$  ( $z$  in the  $i$ -th place) to  $z_i$ ,  $i = 1, \dots, n$ . In the usual way, we say that  $\phi$  *shift-invariant* (or *time-invariant*) if

$$\phi_n S^{\otimes n} = S \phi_n \quad \forall n \geq 1,$$

where  $S : H^2 \rightarrow H^2$  denotes the canonical unilateral right shift. (Equivalently, this means that  $S\phi = \phi \circ S$  on some open ball about the origin in which  $\phi$  is defined.)

Now set

$$P_{(j)}^{(n)} := P_{(j)} \otimes \cdots \otimes P_{(j)} \quad (n \text{ times}), \quad j \geq 1, \quad n \geq 1,$$

where

$$P_{(j)} := I - S^j S^{*j}.$$

Then we say that  $\phi$  is *causal* if

$$P_{(j)} \phi_n = P_{(j)} \phi_n P_{(j)}^{(n)}, \quad j \geq 1, \quad n \geq 1.$$

For  $\phi : H^2 \rightarrow H^2$  linear and time-invariant (i.e., intertwines with the shift), it is easy to see that  $\phi$  is causal. In the nonlinear setting however, time-invariance may not imply causality. As a concrete example, let  $\phi_o : (H^2)^{\otimes 2} \rightarrow H^2$  be a linear operator such that  $S^{\otimes 2} \phi_o = \phi_o S$ , defined by

$$(\phi_o(f \otimes g))(z) := \sum_{k=0}^{\infty} (f_{k+1}g_k + f_k g_{k+1}) z^k,$$

where

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad g(z) = \sum_{k=0}^{\infty} g_k z^k.$$

Now set

$$\phi(f) := \phi_o(f \otimes f), \quad f \in H^2.$$

Then  $\phi$  is an analytic time-invariant map. (In fact  $\phi$  is a homogeneous polynomial of degree 2.) But  $\phi$  is not causal. Indeed,

$$\begin{aligned} (P_{(1)} \phi(f))(z) &= 2f_1 f_0 + f_0^2, \quad z \in D \\ (P_{(1)} \phi(P_{(1)} f))(z) &= f_0^2, \quad z \in D. \end{aligned}$$

Thus  $P_{(1)} \phi(f) \neq P_{(1)} \phi(P_{(1)} f)$ , for example for  $f(z) := 1 + z$  for  $z \in D$ . It is for this reason, that a causality constraint must be explicitly included for nonlinear  $H^\infty$  design.

Now it is easy to show that the conditions (1), (2), and (3) given above are verified for the  $P_{(j)}^{(n)}$  and so the causal commutant lifting theorem applies. In the next section, we will see how one can base an *iterative design procedure* on this result [50, 64, 62, 51].

### 3.3 Iterated Nonlinear Optimization

We will now consider, how the above ideas, lead to an optimization procedure for nonlinear systems which generalizes  $H^\infty$  in a completely natural manner. For convenience, we will only treat SISO systems here. Let us call an analytic input/output operator  $\phi : H^2 \rightarrow H^2$  *admissible* if it is causal, time-invariant, majorizable, and  $\phi(0) = 0$ . Denote the set of admissible operators by  $\mathcal{C}_a$ . In what follows below, we assume  $P, W \in \mathcal{C}_a$ , and that  $W$  admits an admissible inverse.

We consider the (one block) problem of finding

$$\mu_\delta := \inf_C \sup_{\|v\| \leq \delta} \|[(I + P \circ C)^{-1} \circ W]v\|, \quad (9)$$

where we take the infimum over all stabilizing controllers. (In what follows, we let  $\|\cdot\|$  denote the 2-norm  $\|\cdot\|_2$  on  $H^2$  as well as the associated operator norm. The context will make the meaning clear.) Thus we are looking at a worst case disturbance attenuation problem where the energy of the signals  $v$  is required to be bounded by some pre-specified level  $\delta$ . (In the linear case of course since everything scales, we can always without loss of generality take  $\delta = 1$ . For nonlinear systems, we must specify the energy bound *a priori*.) Then one sees that (9) is equivalent to the problem of finding the problem of finding

$$\mu_\delta = \inf_{q \in \mathcal{C}_a} \sup_{\|v\| \leq \delta} \|(W - P \circ q)v\|. \quad (10)$$

The iterated causal commutant lifting procedure gives an approach for approximating a solution to such a problem. Briefly, the idea is that we write

$$\begin{aligned} W &= W_1 + W_2 + \dots, \\ P &= P_1 + P_2 + \dots, \\ q &= q_1 + q_2 + \dots, \end{aligned}$$

where  $W_j, P_j, q_j$  are homogeneous polynomials of degree  $j$ . Notice that

$$\mu_\delta = \delta \inf_{q_1 \in H^\infty} \|W_1 - P_1 q_1\| + O(\delta^2), \quad (11)$$

where the latter norm is the operator norm (i.e.,  $H^\infty$  norm). From the classical commutant lifting theorem we can find an optimal (linear, causal, time-invariant)  $q_{1,opt} \in H^\infty$  such that

$$\mu_\delta = \delta \|W_1 - P_1 q_{1,opt}\| + O(\delta^2). \quad (12)$$

Now the iterative procedure gives a way of finding higher order corrections to this linearization. Let us illustrate this now with the second order correction. Indeed, having fixed the linear part  $q_{1,opt}$  of  $q$  in (10), we note that

$$\begin{aligned} W(v) - P(q(v)) - (W_1 - P_1 q_{1,opt})(v) = \\ W_2(v) - P_2(q_{1,opt}(v)) - P_1 q_2(v) + \text{higher order terms.} \end{aligned}$$

Regarding  $W_2, P_2, q_2$  as linear operators on  $H^2 \otimes H^2 \cong H^2(D^2, \mathbb{C})$  as above, we see that

$$\sup_{\|v\| \leq \delta} \|(W - P \circ q)(v) - (W_1 - P_1 q_{1,opt})v\| \leq \delta^2 \|\hat{W}_2 - P_1 q_2\| + O(\delta^3),$$

where the “weight”  $\hat{W}_2$  is given by

$$\hat{W}_2 := W_2 - P_2(q_{1,opt} \otimes q_{1,opt}).$$

Using the control version of the causal commutant lifting theorem (see [50] and Section 4.1 below), we can now construct an optimal admissible  $q_{2,opt}$ , and so on. This is our *iterated optimization procedure*.

In short, instead of simply designing a linear compensator for a linearization of the given nonlinear system, this methodology allows one to explicitly take into account the higher order terms of the nonlinear plant, and therefore increase the ball of operation for the nonlinear controller. Moreover, if the linear part of the plant is rational, our iterative procedure may be reduced to a series of finite dimensional matrix computations. (See [50, 61] for discussions of rationality in the nonlinear framework.)

In our continuing research, we will be developing the above framework into an implementable design procedure. Accordingly, we are having a doctoral student, program part of the procedure symbolically in Mathematica. Notice that since the nonlinear optimization problem has now been reduced to a series of linear  $H^\infty$  problems, we can use any of the standard methods in order to iteratively design the nonlinear compensator. We will then be looking to apply this to some specific nonlinear design examples. For some preliminary work in this direction, see [49, 50].

### 3.4 Game Theory and Nonlinear Optimization

We have just considered an iterative commutant lifting approach to nonlinear system design. The iterative commutant lifting technique is basically a local analytic method for nonlinear system synthesis. We have also been exploring a very different approach applicable to certain systems with saturations (and “hard” noninvertible nonlinearities) based on a game-theoretic interpretation of the classical commutant lifting theorem [65]. This motivates us to formulate a nonlinear commutant lifting result in such a saddle-point, game-theoretic framework.

A related approach to nonlinear design has already been employed by a number of researchers; see [10, 11, 12, 13, 76, 77, 78, 135, 14] and the references therein. As is well known, game theoretic ideas have already been extensively applied in linear  $H^\infty$  theory.

In our research, instead of considering general nonlinear systems we have limited ourselves to the concrete (but certainly interesting case) of linear systems with input saturations. Such systems occur, of course, all the time in “nature.” We should add that a similar approach should be valid for many of the hard, memoryless, noninvertible nonlinearities which appear in control.

Mathematically, the case of the saturation is very interesting since it represents a non-invertible nonlinearity, and in a certain sense is a nonlinear analogue of an inner (non-minimum phase) element. In fact, it seems that the saturation “acts” like an inner function whose spectrum is spread throughout the unit circle. Thus the problem of sensitivity minimization for such elements (that is, a form of weighted inversion) is particularly difficult, and will be an important topic to be considered in our research program. Again, we will discuss some preliminary results along these lines based on [65].

In order to motivate our game-theoretic approach to nonlinear  $H^\infty$ , we will first give a “saddle-point” interpretation of the classical Sarason theorem [129] in a special case. We let  $w, m \in H^\infty$  with  $m$  inner. Set  $H(m) := H^2 \ominus mH^2$ , we let  $P_{H(m)} : H^2 \rightarrow H(m)$  denote

orthogonal projection, and  $S(m)$  denote the compressed shift. We let  $\|\cdot\|$  denote the 2-norm  $\|\cdot\|_2$  on  $H^2$  as well as the associated induced operator norm.

In [65], we prove that

$$\inf_{q \in H^\infty} \sup_{\|f\| \leq 1} \|(w - mq)f\| = \sup_{\|f\| \leq 1} \inf_{q \in H^\infty} \|(w - mq)f\| = \sup_{\|f\| \leq 1} \inf_{g \in H^2} \|w - mg\|.$$

Now it is easy to show there always an optimal  $q_o$ ; see e.g., [65]. We now assume that

$$\|w(S(m))\|_{ess} < \|w(S(m))\|,$$

where  $\|\cdot\|_{ess}$  denotes the essential norm. Then there exists  $f_o \in H^2$ ,  $\|f_o\| = 1$  (a maximal vector), such that

$$\begin{aligned} \|(w - mq_o)(S(m))f_o\| &= \|w(S(m))f_o\| \\ &= \|w(S(m))\| \\ &= \|(w - mq_o)(S(m))\|. \end{aligned}$$

Now

$$\begin{aligned} P_{H(m)}(w - mq_o)f_o &= (w - mq_o)(S(m))f_o = w(S(m))f_o, \\ (w - mq_o)f_o &= w(S(m))f_o. \end{aligned}$$

So

$$\|(w - mq_o)f\| \leq \|(w - mq_o)(S(m))f_o\| = \|(w - mq_o)f_o\|$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ . Moreover,

$$\|w(S(m))f_o\| = \|(w - mq_o)(S(m))f_o\| \leq \|(w - mq_o)f_o\|.$$

Hence, we get that

$$\|(w - mq_o)f\| \leq \|(w - mq_o)f_o\| \leq \|(w - mq)f_o\| \tag{13}$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ , and for all  $q \in H^\infty$ . It is a nonlinear analogue of the saddle-point condition (13) that we want to analyze for saturated systems. Indeed, assuming the saddle-point condition (13), in [65] we derive all of the standard consequences of the Sarason theorem. Thus it is precisely the existence of a saddle-point which we would like to explore in the nonlinear setting.

By virtue of interpretation of the commuting lifting theorem as asserting the existence of a saddle-point, we would like to derive a global approach to sensitivity minimization for input saturated systems. Thus for  $\sigma_\theta$ , a saturation of magnitude  $\theta < 1$  (see [65] for all the precise definitions), and  $m \in H^\infty$  inner, we want to know when there exist  $f_o \in H^2$ ,  $\|f_o\| \leq 1$ ,  $q_o$  continuous, causal, time-invariant, such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \leq \|(w - m\sigma_\theta \circ q_o)f_o\| \leq \|(w - m\sigma_\theta \circ q)f_o\|$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ ,  $q$  continuous, causal, time-invariant. Such a  $q_o$  (when it exists) will correspond to the optimal compensator, and

$$\mu := \|(w - m\sigma_\theta \circ q_o)f_o\|$$

will be the optimal performance in the weighted sensitivity minimization problem. But this is equivalent to finding  $g_o = q_o(f_o) \in H^2$  such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \leq \|w(f_o) - m\sigma_\theta(g_o)\| \leq \|(w - m\sigma_\theta \circ q)f_o\|. \quad (14)$$

The approach we want to explore now in our present research is to follow an analogous line of reasoning which we employed in our analysis of the saddle-point condition in the linear case. We believe that at the end of the line, we will discover a novel nonlinear commutant lifting theorem valid on a *convex space* which can be used to develop a global robust design procedure for nonlinear plants with hard nonlinearities.

## 4 Curve Evolution in Controlled Active Vision

In the past few years, we have become very interested in visual tracking and the general area of the use of visual information in a feedback loop. This is a central area in which the robust control methods developed over the past fifteen years could have a major impact. In order to work on visual tracking, we have had to learn some of the key techniques from image processing and computer vision, which has led in turn to a new research direction. Indeed, we have been using geometric invariant flows for various problems in active vision. These flows themselves are very much motivated by ideas in *optimal control*; see [87]. We will now discuss some of the key ideas in curve evolution. These will be applied to certain problems in controlled active vision as described below.

### 4.1 Introduction to Curve Evolution

A geometric set or shape can be defined by its boundary. In the case of bounded planar shapes for example, this boundary consists of closed planar curves. We will only deal with closed planar curves, keeping in mind that these curves are boundaries of planar shapes.

A curve may be regarded as a trajectory of a point moving in the plane. Formally, we define a curve  $\mathcal{C}(\cdot)$  as the map  $\mathcal{C}(p) : S^1 \rightarrow \mathbf{R}^2$  (where  $S^1$  denotes the unit circle).  $\mathcal{C}$  can be written using Cartesian coordinates, i.e.,  $\mathcal{C}(p) = [x(p), y(p)]^T$ , where  $x(\cdot)$  and  $y(\cdot)$  are maps from  $S^1$  to  $\mathbf{R}$ . We assume that all of our mappings are sufficiently smooth, so that all the relevant derivatives may be defined. We also assume that our curves are have no self-intersections, i.e., are embedded.

We now consider plane curves deforming in time. Let  $\mathcal{C}(p, t) : S^1 \times [0, \tau] \rightarrow \mathbf{R}^2$  denote a family of closed embedded curves, where  $t$  parametrizes the family, and  $p$  parametrizes each curve. Assume that this family evolves according to the following equation:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{T} + \beta \vec{N} \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p) \end{cases} \quad (15)$$

where  $\vec{N}$  is the inward Euclidean unit normal,  $\vec{T}$  is the unit tangent, and  $\alpha$  and  $\beta$  are the tangent and normal components of the evolution velocity  $\vec{v}$ , respectively. In fact, it is easy to show that  $\text{Img}[\mathcal{C}(p, t)] = \text{Img}[\hat{\mathcal{C}}(w, t)]$ , where  $\mathcal{C}(p, t)$  and  $\hat{\mathcal{C}}(w, t)$  are the solutions of

$$\mathcal{C}_t = \alpha \vec{T} + \beta \vec{N} \quad \text{and} \quad \hat{\mathcal{C}}_t = \bar{\beta} \vec{N},$$

respectively. (Here  $\text{Img}[\cdot]$  denotes the image of the given parametrized curve in  $\mathbf{R}^2$ .) Thus the tangential component affects only the parametrization, and not  $\text{Img}[\cdot]$  (which is independent

of the parametrization by definition). Therefore, assuming that the normal component  $\beta$  of  $\vec{v}$  (the curve evolution velocity) in (15) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial C}{\partial t} = \beta \vec{\mathcal{N}}, \quad (16)$$

where  $\beta = \vec{v} \cdot \vec{\mathcal{N}}$ , i.e., the projection of the velocity vector on the normal direction.

The evolution (16) was studied by different researchers for different functions  $\beta$ . This type of flow was introduced into the theory of shape in [82, 83, 84, 85]. One of the most studied evolution equations is obtained for  $\beta = \kappa$ , where  $\kappa$  is the Euclidean curvature [73, 128]:

$$\frac{\partial C}{\partial t} = \kappa \vec{\mathcal{N}}. \quad (17)$$

Equation (17) has its origins in physical phenomena [3, 72]. It is called the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (17); see [72]. Gage and Hamilton [70] proved that a planar embedded convex curve converges to a round point when evolving according to (17). (A round point is a point that, when the curve is normalized in order to enclose an area equal to  $\pi$ , it is equal to the unit disk.) Grayson [71] proved that a planar embedded non-convex curve converges to a convex one, and from there to a round point from Gage and Hamilton result. Note that in spite of the local character of the evolution, global properties are obtained, which is a very interesting feature of this evolution. For other results related to the Euclidean shortening flow, see [3, 4, 70, 71, 72, 85, 137].

Next note that if  $s$  denotes the Euclidean arc-length, then [128]

$$\kappa \vec{\mathcal{N}} = \frac{\partial^2 C}{\partial s^2}.$$

Therefore, equation (17) can be written as

$$C_t = C_{ss}. \quad (18)$$

Equation (18) is not linear, since  $s$  is a function of time (the arc-length gives a time dependent parametrization). This equation is also called the *geometric heat equation*.

Another interesting example is obtained when one sets  $\beta = 1$  in equation (16):

$$\frac{\partial C}{\partial t} = \vec{\mathcal{N}}. \quad (19)$$

This equation simulates, under certain conditions, the grassfire flow [30, 82, 83, 85, 123]. (More precisely, the unique weak solution of (19) which satisfies the *entropy* condition [123] gives the grassfire flow.) This grassfire flow is also the morphological scale-space created by a disk. Moreover, one can prove that with different selections of  $\beta$ , other morphological scale-spaces are obtained [87].

In [82, 83, 85], we have studied the following equation in order to develop a hierarchy of shape,

$$\frac{\partial C}{\partial t} = (1 + \epsilon \kappa) \vec{\mathcal{N}}. \quad (20)$$

If  $\epsilon \rightarrow 0$  in (20), the grassfire flow is obtained, and this introduces singularities (*shocks*) in the evolving curve. (The shocks define the well-known skeleton.) On the other hand, if  $\epsilon \rightarrow \infty$ , equation (20) reduces to the classical Euclidean curve shortening flow, which smoothes the curve [71, 124]. The combination of these two opposite features gives very interesting properties [82].

When a curve evolves according to (20), the evolution of the curve slope satisfies a reaction-diffusion equation [126]. The reaction term, which tends to create singularities, competes with the diffusion term which tends to smooth the curve. For each different value of  $\epsilon$ , a scale-space is obtained by looking at the solution of (20), and considering the time  $t$  as the scale parameter. We have called the set of all the scale-spaces obtained for all values of  $\epsilon$ , the *reaction-diffusion scale-space* [83].

In particular, we see that the much studied Euclidean shortening flow (equation (17)) defines an Euclidean invariant scale-space (the equation admits Euclidean invariant solutions). In contrast with other scale-spaces, like the one obtained from the classical linear heat equation, this one is a full geometric scale-space. The progressive smoothing given by  $\kappa$  (or  $\mathcal{C}_{ss}$ ), is geometrically intrinsic to the curve.

We now discuss the affine analogue of the Euclidean shortening flow. (The affine group  $SA_2$  is the group generated by unimodular transformations and translations of  $\mathbf{R}^2$ . Under certain natural conditions, it provides a good approximation to the full group of perspective projective transformations.) Then in [99, 118], we show that the simplest non-trivial affine invariant flow in the plane is given by

$$\mathcal{C}_t = \kappa^{1/3} \vec{\mathcal{N}}. \quad (21)$$

The question now is what happens when a non-convex curve evolves according to (21). The following result answers this question [5]:

**Theorem 4** *Let  $\mathcal{C}(\cdot, 0) : S^1 \rightarrow \mathbf{R}^2$  be a smooth embedded curve in the plane. Then there exists a family  $\mathcal{C} : S^1 \times [0, T) \rightarrow \mathbf{R}^2$  satisfying*

$$\mathcal{C}_t = \kappa^{1/3} \vec{\mathcal{N}},$$

*such that  $\mathcal{C}(\cdot, t)$  is smooth for all  $t < T$ , and moreover there is a  $t_0 < T$  such that for all  $t > t_0$ ,  $\mathcal{C}(\cdot, t)$  is smooth and convex.*

Theorem 4 means that just as in the Euclidean case, a non-convex curve first becomes convex when evolving according to (21). After this, the curve converges to an ellipse from our results in [118]. Because of this, and other related properties (see [119]), we can conclude that equation (21) is the affine analogue of (17) for smooth embedded curves, and thus is called the *affine shortening flow*. (It is also the affine invariant formulation of the geometric heat equation.) In the next section, we will use it to construct an *affine invariant scale-space* for planar shapes.

## 4.2 Affine invariant scale-space

In the previous section, we indicated that equation (21) is the affine analogue of (17). Of course, in addition to the Euclidean invariant property of the Euclidean curve shortening flow,

this affine flow admits also unimodular affine invariant solutions. Due to this analogy, and certain key properties [119], we can use the affine shortening flow for defining a new *affine invariant scale-space*. More precisely, the representation of shape obtained from the evolving solution of equation (21), defines the affine invariant scale-space. The scale parameter is given by the time  $t$

Note the curve shrinks when evolving according to (21). This curve can be normalized in order to keep constant area. In fact, in [120], we have evolutions that preserve either area or length. The affine scale-space can therefore be re-defined by means of these normalized dilated curves. Since the process of normalization modifies  $\kappa$ , and other geometric properties just by constant multiplication [118], all the key scale-space features for the family of curves  $\mathcal{C}$  satisfying (21), also hold for the normalized scale-space.

When defining scale-spaces, a number of desired properties must be verified [75, 88, 110, 138]. One of the most important properties is the *causality criterion*, which requires that no new features should be introduced in the curve (or image) in passing from fine to coarse scales in the scale-space. In general, “features” are related to the salient characteristics of the signal, which are important for the image description, and are easily identified.

Hummel in [75] showed that the causality criterion may be expressed in terms of the Maximum Principle [112]. The Maximum Principle states that under certain conditions, a given function satisfying a parabolic partial differential equation, attains its maximum (minimum) on the boundary [112]. (The Maximum Principle is very frequently used in the theory of curve evolution for proving important properties.) Hummel showed that under certain conditions, when the Maximum Principle holds for a given parabolic operator, which defines the scale-space, zero-crossings are never created at a non-zero scale. (This result was used in [110] for demonstrating the causality principle.) In our work [5, 119], we show that the affine shortening flow satisfies a Maximum Principle, and so because of this and other related properties, the affine invariant scale-space is an ideal vehicle for the multi-scale study of shape.

### 4.3 Visual Tracking

Much of our recent research in image processing and computer vision has been motivated by problems in controlled active vision, especially visual tracking. We have already described some of the relevant work in control above, and so we would like to consider now some of the key tools we plan to employ from our work in computer vision and image processing. These include active contours, optical flow and stereo disparity, and certain results from invariant theory for invariant object recognition, as well as the curve and surface evolution methodology sketched above in Section 4. These methods have played an integral part in our study of the utilization of visual information in a feedback loop.

### 4.4 Geometric Active Contours

In this section, we will describe a new paradigm for *snakes* or *active contours* based on principles from geometric optimization theory. Active contours may be regarded as autonomous processes which employ image coherence in order to track various features of interest over time. Such deformable contours have the ability to conform to various object shapes and motions. Snakes have been utilized for segmentation, edge detection, shape modeling, and visual tracking. Active contours have also been widely applied for various applications in medical

imaging. For example, snakes have been employed for the segmentation of myocardial heart boundaries as a prerequisite from which such vital information such as ejection-fraction ratio, heart output, and ventricular volume ratio can be computed.

In the classical theory of snakes, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain constraints set by the user. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters. Moreover, natural criteria for the splitting and merging of contours (or for the treatment of multiple contours) are not readily available in this framework.

In [81], we have proposed a novel deformable contour model to successfully solve such problems, and which will become one of our key techniques for tracking. Our method is based on the Euclidean curve shortening evolution (see Section 4.1) which defines the gradient direction in which a given curve is shrinking as fast as possible relative to Euclidean arc-length, and on the theory of conformal metrics. Namely, we multiply the Euclidean arc-length by a function tailored to the features of interest which we want to extract, and then we compute the corresponding gradient evolution equations. The features which we want to capture therefore lie at the bottom of a potential well to which the initial contour will flow. Further, our model may be easily extended to extract 3D contours based on motion by mean curvature [81, 139].

Let us briefly review some of the details from [81]. First of all, in [35] and [95], a snake model based on the level set formulation of the Euclidean curve shortening equation is proposed. More precisely, the model is

$$\frac{\partial \Psi}{\partial t} = \phi(x, y) \|\nabla \Psi\| \left( \operatorname{div} \left( \frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nu \right). \quad (22)$$

Here the function  $\phi(x, y)$  depends on the given image and is used as a “stopping term.” For example, the term  $\phi(x, y)$  may be chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. One may take [35, 95]

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^2}, \quad (23)$$

where  $I$  is the (grey-scale) image and  $G_\sigma$  is a Gaussian (smoothing filter) filter. The function  $\Psi(x, y, t)$  evolves in (22) according to the associated level set flow for planar curve evolution in the normal direction with speed a function of curvature which was introduced in [102, 123, 124].

It is important to note that the Euclidean curve shortening part of this evolution, namely

$$\frac{\partial \Psi}{\partial t} = \|\nabla \Psi\| \operatorname{div} \left( \frac{\nabla \Psi}{\|\nabla \Psi\|} \right) \quad (24)$$

is derived as a gradient flow for shrinking the perimeter as quickly as possible. As is explained in [35], the constant *inflation term*  $\nu$  is added in (22) in order to keep the evolution moving in the proper direction. Note that we are taking  $\Psi$  to be negative in the interior and positive in the exterior of the zero level set.

We would like to modify the model (22) in a manner suggested by Euclidean curve shortening. Namely, we will change the ordinary Euclidean arc-length function along a curve  $C = (x(p), y(p))^T$  with parameter  $p$  given by

$$ds = (x_p^2 + y_p^2)^{1/2} dp,$$

to

$$ds_\phi = (x_p^2 + y_p^2)^{1/2} \phi dp,$$

where  $\phi(x, y)$  is a positive differentiable function. Then we want to compute the corresponding gradient flow for shortening length relative to the new metric  $ds_\phi$ .

Accordingly set

$$L_\phi(t) := \int_0^1 \left\| \frac{\partial C}{\partial p} \right\| \phi dp.$$

Let

$$\vec{\tau} := \frac{\partial C}{\partial p} / \left\| \frac{\partial C}{\partial p} \right\|,$$

denote the unit tangent. Then taking the first variation of the modified length function  $L_\phi$ , and using integration by parts (see [81]), we get that

$$L'_\phi(t) = - \int_0^{L_\phi(t)} \left\langle \frac{\partial C}{\partial t}, \phi \kappa \vec{N} - (\nabla \phi \cdot \vec{N}) \vec{N} \right\rangle ds$$

which means that the direction in which the  $L_\phi$  perimeter is shrinking as fast as possible is given by

$$\frac{\partial C}{\partial t} = (\phi \kappa - (\nabla \phi \cdot \vec{N})) \vec{N}. \quad (25)$$

This is precisely the gradient flow corresponding to the minimization of the length functional  $L_\phi$ . The level set version of this is

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \operatorname{div} \left( \frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nabla \phi \cdot \nabla \Psi. \quad (26)$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the potential well described by the gradient flow (26). As in [35, 95], we may also add a constant inflation term, and so derive a modified model of (22) given by

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \left( \operatorname{div} \left( \frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nu \right) + \nabla \phi \cdot \nabla \Psi. \quad (27)$$

Notice that for  $\phi$  as in (23),  $\nabla \phi$  will look like a doublet near an edge. Of course, one may choose other candidates for  $\phi$  in order to pick out other features.

We now have very fast implementations of these snake algorithms based on level set methods [102, 123]. Clearly, the ability of our snakes to change topology, and quickly capture the desired features will make them an indispensable tool for our visual tracking algorithms.

We are also studying an affine invariant snake model for tracking based on our work in [101]. (The evolution itself works using a level set model of  $\kappa^{1/3} \vec{N}$  as discussed in Section 4.1.)

Finally, our methods are extendable to 3D images. Indeed, we have developed affine invariant volumetric smoothers in [100]. We also have 3D active contour evolvers for image segmentation, shape modeling, and edge detection based on both snakes (inward deformations) and bubbles (outward deformations) in our work [81, 139]. We also intend to use our affine smoothers in movies as a preprocessing tool for motion estimation.

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## 6 Students of Allen Tannenbaum Supported by DAAH04-94-G-0054

1. Juan Cockburn (Ph. D.)
2. Arun Kumar (Ph. D.)

## 7 Papers of Allen Tannenbaum and Collaborators under DAAH04-94-G-0054

### Refereed Journal Papers

1. "On combined  $H^\infty$ - $H^2$  suboptimal interpolants" (with C. Foias and A. Frazho), *Linear Algebra and Its Applications* **203-204** (1994), pp. 443-469.
2. "On affine plane curve evolution" (with G. Sapiro), *Journal of Functional Analysis* **119** (1994), pp. 79-120.
3. "Nonlinear  $H^\infty$  optimization: a causal power series approach," *SIAM J. Control and Optimization* **33** (1995), pp. 185-207.
4. "Shapes, shocks, and deformations, I: the components of shape and the reaction-diffusion space" (with B. Kimia and S. Zucker), *International Journal of Computer Vision* **15** (1995), 189-224.
5. "Affine invariant scale-space (with G. Sapiro), *International Journal of Computer Vision* **11** (1993), 25-44.
6. "A relative Toeplitz-Hausdorff theorem" (with H. Bercovici and C. Foias), *Operator Theory: Advances and Applications* **71** (1994), pp. 29-34.
7. "Continuity of the spectrum on closed similarity orbits" (with H. Bercovici and C. Foias), *Integral Equations and Operator Theory* **18** (1994), 242-246.
8. "On area and length preserving geometric invariant curve evolutions" (with G. Sapiro), *IEEE Trans. on Pattern Analysis and Machine Intelligence* **17** (1995), pp. 67-72.
9. "Stability margin optimization via interpolation and conformal mappings" (with J. Cockburn and Y. Sidar), *IEEE Trans. Aut. Control* **40** (1995), pp. 1066-1070.
10. "On a lifting theorem for the structured singular value" (with H. Bercovici, C. Foias, and P. Khargonekar), *Journal of Math. Analysis and Applications* **187** (1994), pp. 617-627.
11. "Classification and uniqueness of invariant geometric flows" (with P. Olver and G. Sapiro), *Comptes Rendus Acad. Sci. (Paris)* **319** (1994), pp. 339-344.
12. "Multivariable stability margin optimization: a spectral tangential interpolation approach" (with Juan Cockburn), *Int. J. Control* **63** (1996), pp. 557-590.
13. "The structured singular value for linear input/output systems," (with H. Bercovici and C. Foias), *SIAM J. Control and Optimization* **34** (1996), pp. 1392-1404.
14. "Invariant geometric evolutions of surfaces and volumetric smoothing" (with P. Olver and G. Sapiro), *SIAM J. Applied Math.* **57** (1997), pp. 176-194.
15. "Optical flow: a curve evolution approach" (with A. Kumar and G. Balas), *IEEE Trans. Image Processing* **5** (1996), pp. 598-611.

16. "Conformal curvature flows: from phase transitions to active contours" (with S. Kichenesamy, A. Kumar, P. Olver, and A. Yezzi), *Archive for Rational Mechanics and Analysis* **134** (1996), pp. 275–301.
17. "The equivalence among the solutions of the  $H^\infty$  optimal sensitivity computation problem" (with K. Hirata and Y. Yamamoto), *Trans. of the Society of Instrument and Control Engineers* **31** (1995), pp. 1954–1961.
18. "Three snippets of curve evolution theory in computer vision," *Mathematical and Computer Modelling Journal* **24** (1996), pp. 103–119.
19. "On skew Toeplitz operators, II" (with H. Bercovici and C. Foias), to appear in *Operator Theory: Advances and Applications*.
20. "Behavioral analysis of anisotropic diffusion in image processing" (with Y. You, M. Kaveh, W. Xu), *IEEE Trans. Image Processing* **5** (1996), pp. 1539–1553.
21. "Affine geometry, curve flows and invariant numerical approximations" (with E. Calabi and P. Olver), *Advances in Mathematics* **124** (1996), pp. 154–196
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