NPS-OR-99-001

NAVAL POSTGRADUATE SCHOOL
Monterey, California

In Search of Military Unit Formations: G-String Models

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January 1999

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Prepared for: Institute for Joint Warfare Analysis
Naval Postgraduate School
Monterey, California 93943

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REPORT DOCUMENTATION PAGE

Form approved

OMB No 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	January 1999	Technical

4. TITLE AND SUBTITLE
In Search of Military Unit Formations: G-String Models

5. FUNDING
IJWA

6. AUTHOR(S)
Donald P. Gaver and Patricia A. Jacobs

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Postgraduate School
Monterey, CA 93943

8. PERFORMING ORGANIZATION REPORT NUMBER
NPS-OR-99-001

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Institute for Joint Warfare Analysis
Naval Postgraduate School
Monterey, CA 93943

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)
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Calculations are in terms of (i) a convenient parametric model for the spatial arrangement and concentration of an entire unit, the effective size of target items within that unit and their disposition, the sensor's field of view and glance rate, and the speed and course of the sensor-equipped platform's travels across the region containing elements of the large force. Alternatively, calculations are made when (ii) single target items are distributed in any manner in the region, (iii) a sensor-bearing platform wanders in accordance with a random walk; (iv) combinations of the above.

14. SUBJECT TERMS
moving sensor platform, diffuse targets, search and detection

15. NUMBER OF PAGES
24

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	UL

NSN 7540-01-280-5800

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std 239-18

In Search of Military Unit Formations: G-String Models

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0. Overview and Outline

This report provides a way of quickly calculating a measure of what a sensor-equipped platform (e.g. JSTARS or a RPV/UAV) "sees" when it flies across a region containing a large (e.g. brigade-sized) unit. Such observations form the basis for inferring the size and type of the unit, or units.

Calculations are in terms of (i) a convenient parametric model for the spatial arrangement and concentration of an entire unit, the effective size of target items within that unit and their disposition, the sensor's field of view and glance rate, and the speed and course of the sensor-equipped platform's travels across the region containing elements of the large force. Alternatively, calculations are made when (ii) single target items are distributed in any manner in the region, (iii) a sensor-bearing platform wanders in accordance with a random walk; (iv) combinations of the above.

1. Diffuse-Concentrated Target Unit Models

Suppose an opponent force, $\mathcal{T}$, e.g. a military unit such as an Army brigade, or a Naval battlegroup, is located in spatial formation centered and oriented at a position (around a center) in theater-space. $\mathcal{T}$ is the subject of search by a platform, $\mathcal{S}$, that moves across it. Detection and classification of $\mathcal{T}$ by $\mathcal{S}$ depends upon the numbers of individual items (optionally tanks, or tank companies, etc.) encountered during a surveillance passage by the sensor platform. The elements of the force $\mathcal{T}$ are cohesive groups such as tank companies that are spatially separated within their own rough boundaries. These are themselves concentrated-but-diffuse in space, within a flexible and changing unit boundary. There is a need for a simple, accurate-enough depiction of such a structure when attempting to model sensor interaction with a large unit such as a brigade.

1.1 Crisp "Cookie-Cutter" Unit Templates

Such a generic setup can be modeled by characterizing $\mathcal{T}$ as a geometrical shape such as a large *outline rectangle* (cookie-cutter template) within which items (tanks or ships, or suitable groups of these) are arranged, perhaps in rectangular or circular subformations. The path of a searcher can then be treated as having given sweep width, that crosses part of the outline rectangle, "seeing" some of the individual items that lie within the path. Such a model can be simulated, but with some difficulty because of the many special cases potentially involved; see Gaver and Jacobs (1998); very specific geometric/trigonometric problems occur. It is also "too crisp", in that sharp-boundary depiction of formation shape is unrealistic, besides often leading to the mentioned analytical/computational difficulties. We seek, and provide, *analytical formulas* (the equivalent of multi-dimensional look-up tables often found in campaign models) that represent such situations efficiently, and even somewhat more realistically than does a crisp cookie cutter. The availability of such formulas shortcuts detailed event-by-event simulations, thus allowing models of theater-level operations or campaigns to be run expeditiously.

Surprisingly, a great diversity of individual sensor behaviors can be explicitly modeled in entirely or nearly closed form, leading to great computational economy.

1.2 The Gaussian Diffuse-Concentrated Target Unit

As an alternative to the above, we first consider a target formation representation that is *concentrated, but smoothly and diffusely*: its elements (individual items such as tanks, or tank components) occur with high density near its center, that density falling off *relatively gradually*, and not necessarily symmetrically, with distance from the center. The target density is thus a smooth bump, not a sharply-defined block. The bump (unit outline) can be long and thin in a chosen direction, or (nearly) symmetrical, at the choice of the analyst. We first consider a "fixed bump", but the results can be generalized to a "moving bump", and one with varying item types. Several separated bumps can be represented, if so desired.

An extremely convenient parametric example of such is a *bivariate Gaussian*: measure distance from formation center as origin, and let the density of (Red) units at (x, y) be

$$r(x,y;\boldsymbol{\tau})=\frac{1}{\sqrt{2\pi}\,\tau_x}e^{-\frac{1}{2}(x/\tau_x)^2}\cdot\frac{1}{\sqrt{2\pi}\,\tau_y}e^{-\frac{1}{2}(y/\tau_y)^2} \tag{1.1}$$

where the orientation is arbitrarily chosen to be along x and y axes (this is readily changed). It is of course clear that equal density contours are ellipses, and that the ellipse

$$(x/\tau_x)^2+(y/\tau_y)^2=+2\ln 2\equiv 1.386\ldots \tag{1.2}$$

defines a natural two-dimensional "median density": for (x, y) closer to $(0, 0)$ than the boundary of (1.2) — inside that ellipse — the density exceeds ½ of the peak density $\left(\frac{1}{(2\pi)\tau_x\tau_y}\right)$, while outside the ellipse the density is less than ½ of that peak, falling rapidly to zero, but not symmetrically. The *shape* of the bump is otherwise not adjustable. However, there is nothing to prevent a meta-unit from being a collection of Gaussian bumps that even move cooperatively (and suffer attrition, or enjoy reinforcement).

2. Modeling Sightings by the Sensor

Suppose that at time t a sensor (platform such as JSTARS or a UAV) is located/centered at coordinates $(a_x + b_x t,\ a_y + b_y t)$. Its glance rate at location (u, v) is *assumed* proportional *to a Gaussian*:

$$g(u,v;t,\theta) = \theta \cdot \left(\frac{e^{-\frac{1}{2}(u-a_x-b_x t)^2/\omega_x^2}}{\sqrt{2\pi}\,\omega_x} \right) \left(\frac{e^{-\frac{1}{2}(v-a_y-b_y t)^2/\omega_y^2}}{\sqrt{2\pi}\,\omega_y} \right), \tag{2.1}$$

where θ is the overall glance rate per unit time. Thus the sensor is responsive to actual units within a region $O(\omega_x, \omega_y)$ of its location at any time, roughly its field of view.

The probability that a glance at point (u, v) *hits*, or potentially reveals the presence of a component or item that is present and centered at (x, y) is conveniently modeled as Gauss-like:

$$\begin{aligned} d(x,y;u,v) &= p\,e^{-\frac{1}{2}\left[(x-u)^2/\delta_x^2 + (y-v)^2/\delta_y^2\right]} \\ &= p\,e^{-\frac{1}{2}\left[(x-u)^2/\delta_x^2\right]} \cdot e^{-\frac{1}{2}\left[(y-v)^2/\delta_y^2\right]} \end{aligned} \tag{2.2}$$

where $0 \le p \le 1$. Think of (δ_x, δ_y) as defining the vulnerability of the unit component at (x, y) (e.g. tank company) to being seen by a glance "at" (u, v). Given (x, y), the probability that a single glance from a sensor located at $(a_x + b_x t,\ a_y + b_y t)$ is effective, i.e. records a sighting, is seen to be proportional to a convolution of Gaussian densities (i.e. a two-fold Gauss or *G-string*):

$$\bar{d}\left(x, y; a_x + b_x t, a_y + b_y t\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(x, y; u, v) g(u, v; t, 1)\, du\, dv$$

$$= p \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-u)^2/\delta_x^2} \frac{e^{-\frac{1}{2}(u-a_x-b_x t)^2/\omega_x^2}}{\sqrt{2\pi}\omega_x} du \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-v)^2/\delta_y^2} \frac{e^{-\frac{1}{2}(v-a_y-b_y t)^2/\omega_y^2}}{\sqrt{2\pi}\omega_y} dv \qquad (2.3)$$

$$= p\sqrt{2\pi}\,\delta_x \cdot \frac{e^{-\frac{1}{2}(x-a_x-b_x t)^2/\left(\omega_x^2+\delta_x^2\right)}}{\sqrt{2\pi}\left(\omega_x^2+\delta_x^2\right)^{\frac{1}{2}}} \cdot \sqrt{2\pi}\,\delta_y \cdot \frac{e^{-\frac{1}{2}(y-a_y-b_y t)^2/\left(\omega_y^2+\delta_y^2\right)}}{\sqrt{2\pi}\left(\omega_y^2+\delta_y^2\right)^{\frac{1}{2}}}$$

$$= p \frac{\delta_x}{\left(\omega_x^2+\delta_x^2\right)^{\frac{1}{2}}} \frac{\delta_y}{\left(\omega_y^2+\delta_y^2\right)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-a_x-b_x t)^2/\left(\omega_x^2+\delta_x^2\right)} e^{-\frac{1}{2}(y-a_y-b_y t)^2/\left(\omega_y^2+\delta_y^2\right)}.$$

This expression is evidently a *probability*, with value dependent upon sensor field-of-view parameter (ω_x, ω_y; conveniently $\omega = \omega_x = \omega_y$), and target item's size/dimension-visibility, as quantified by (δ_x, δ_y). The latter depends on an item's physical dimension and orientation (δ_x and δ_y are not necessarily equal), but also implicitly on terrain and background contrast and clutter, at least in an average sense; for instance, $\delta_y = 2\delta_x$ might characterize a tank company lined up parallel to the y (e.g. North-South) axis. Strictly speaking, the present model only represents a target item that is selectively non-symmetric, e.g. extended in parallel with the direction of the axes of symmetry of the entire unit, but the actual "shape" of the item is here left implicit.

Use the assumed bivariate Gaussian representation of item density, (1.1), to calculate the probability that a sensor glance at t reveals a target item:

$$r(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{d}\left(x, y; a_x + b_x t, a_y + b_y t\right) r(x, y; \boldsymbol{\tau})\, dx\, dy. \qquad (2.4)$$

where $\boldsymbol{\tau} = (\tau_x, \tau_y)$.

By direct analogy with the calculation leading to (2.3),

$$r(t) = p\frac{\delta_x}{\left(\tau_x^2+\omega_x^2+\delta_x^2\right)^{\frac{1}{2}}}\frac{\delta_y}{\left(\tau_y^2+\omega_y^2+\delta_y^2\right)^{\frac{1}{2}}}$$
$$\times e^{-\frac{1}{2}(a_x+b_xt)^2/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)}e^{-\frac{1}{2}(a_y+b_yt)^2/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)} \tag{2.5}$$

Expected Sightings Over a Linear Sensor Path

If θ is the glance rate from the (moving) sensor and R is the total number of unit items, *then the expected number of sightings/detections over a sensor path* $(a_x + b_xt, a_y + b_yt;\ t_l \le t \le t_u)$ is given by the integral of $\theta r(t)R$ with respect to t over whatever time interval the sensor maintains activity, e.g. $t_l \le t \le t_u$. Recognizably, $\theta r(t)R$ is the rate of sightings of items (however defined) at (x, y) by a sensor at $(a_x + b_xt,\ a_y + b_yt)$; it is entirely possible that some, even many, of these are *re-sightings*, i.e. that the events are not all novel to the sensor as it moves and glances repeatedly. Whatever *is* seen must depend on the speed with which the sensor platform travels; that speed is implicit in (b_x, b_y).

In Appendix I it is shown that the calculation of the required integral of $r(t)$ is conveniently based on new parameters, μ and σ^2, that are simple functions of the initial ones:

$$\int_{t_l}^{t_u} e^{-\frac{1}{2}(a_x+b_xt)^2/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)} \times e^{-\frac{1}{2}(a_y+b_yt)^2/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)}dt$$
$$\equiv \int_{t_l}^{t_u} e^{-\frac{1}{2}(t-\mu)^2/\sigma^2}e^{-\frac{1}{2}K^2}dt = \left(\sqrt{2\pi}\right)\sigma e^{-\frac{1}{2}K^2}\left(\Phi\left(\frac{t_u-\mu}{\sigma}\right)-\Phi\left(\frac{t_l-\mu}{\sigma}\right)\right) \tag{2.6}$$

where μ and σ^2 are expressed in terms of the basic a_x, b_x, a_y, b_y, etc. parameters.

Assemble the parts into one formula (set $\omega_x = \omega_y = \omega$):

$$\tilde{r}(t_l, t_u; \bullet) = R\theta p \sigma \sqrt{2\pi} \frac{\delta_x \delta_y}{\left[\left(\tau_x^2 + \omega^2 + \delta_x^2\right)\left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right]^{\frac{1}{2}}} e^{-K^2/2}\left(\Phi\left(\frac{t_u - \mu}{\sigma}\right) - \Phi\left(\frac{t_l - \mu}{\sigma}\right)\right)$$

with

$$\frac{1}{\sigma^2} = \frac{b_x^2}{\left(\tau_x^2 + \omega^2 + \delta_x^2\right)} + \frac{b_y^2}{\left(\tau_y^2 + \omega^2 + \delta_y^2\right)}$$

$$\mu = -\frac{\left[a_x b_x / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + a_y b_y / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right]}{\left[b_x^2 / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + b_y^2 / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right]} \tag{2.7}$$

$$K^2 = \left[\left(a_x + b_x \mu\right)^2 / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + \left(a_y + b_y \mu\right)^2 / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right]$$

It is more "physical" to represent the above in terms of the platform speed, s, and direction:

$$b_x = s \cos \varphi, \qquad b_y = s \sin \varphi$$

where φ is the angle the sensor course makes with the x-axis. Then

$$\sigma^2 = 1 \Big/ s^2 \left[\frac{\cos^2 \varphi}{\left(\tau_x^2 + \omega^2 + \delta_x^2\right)} + \frac{\sin^2 \varphi}{\left(\tau_y^2 + \omega^2 + \delta_y^2\right)}\right]$$

and

$$\mu = -\frac{1}{s}\left[\frac{\left(a_x \cos \varphi\right) / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + \left(a_y \sin \varphi / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right)}{\cos^2 \varphi / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + \sin^2 \varphi / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)}\right], \tag{2.8}$$

$$K^2 = \left[\left(a_x + c_x / s\right)^2 / \left(\tau_x^2 + \omega^2 + \delta_x^2\right) + \left(a_y + c_y / s\right)^2 / \left(\tau_y^2 + \omega^2 + \delta_y^2\right)\right]$$

where c_x, c_y do not depend on s.

All calculations are extremely straightforward once course and speed of the sensor is specified ((a_x, b_x, a_y, b_y) as well as the times. A look-up table or quickly calculated approximation for the $\Phi(z)$ cumulative unit Gaussian is needed. Just setting the bracketed Φ's term equal to unity ($t_u = \infty$, $t_l = -\infty$) gives a crude upper bound on the number of sightings by the sensor as it flies over the target unit. This might be adequate to characterize what a satellite sees.

Usefulness. There is a connection between the total number of sightings over a fixed sensor course and the identity of any unit that happens to be encountered (*and* that unit's size, and disposition, e.g. parameters R and (τ_x, τ_y)). If the platform is a satellite with preset course (a_x, a_y, b_x, b_y) the sighting of an (expected) number greater than a threshold can trigger a UAV hunt. This can be modeled in the present level of detail, *or* just its effect modeled as a probability.

2.1 Uncertainty of the UAV Course

In operational practice a cued platform such as a UAV will not necessarily follow the correct course, e.g. over the formation center. To account for this possible variability effect one may *randomize* (once and for all) on (b_x, b_y), by averaging out (taking the conditional expectation) over an appropriate (posterior) distribution. Of course this does not account for realistic *within-flight cueing*: when this occurs the UAV sequentially picks up indications of where to go next; such a realism is not yet modeled.

It is still possible to carry out a calculation leading towards (2.5) and later formulas explicitly, up to a point. The practical import is that another stage of realism can be handled analytically rather than by carrying out Monte Carlo sampling (provided course uncertainty can be represented adequately by a Gaussian). Here the components of the course error are taken to be independent, again with

$$f_{b_x}\left(u;\beta_x,\gamma_x\right)=\frac{e^{-\frac{1}{2}(u-\beta_x)^2/\gamma_x^2}}{\sqrt{2\pi}\,\gamma_x} \tag{2.9,a}$$

and

$$f_{b_y}\left(v;\beta_y,\gamma_y\right)=\frac{e^{-\frac{1}{2}(v-\beta_y)^2/\gamma_y^2}}{\sqrt{2\pi}\,\gamma_y} \tag{2.9,b}$$

Thus, another G-string component appears.

Appeal to (2.5) with $\omega_x = \omega_y = \omega$ and remove conditions on b_x and b_y (independently for convenience, but this is not essential): the rate of sightings/detections at time t is

$$\begin{aligned}\bar{r}\left(t;\beta_x,\sigma_x^2,\beta_y,\sigma_y^2\right)=R\theta&\frac{\delta_x}{\left(\tau_x^2+\omega_x^2+\delta_x^2\right)^{\frac{1}{2}}}\cdot\frac{\delta_y}{\left(\tau_y^2+\omega_y^2+\delta_y^2\right)^{\frac{1}{2}}}\\ &\times\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(a_x+ut)^2/\left(\tau_x^2+\omega^2+\delta_x^2\right)}}{\sqrt{2\pi}\left(\tau_x^2+\omega^2+\delta_x^2\right)^{\frac{1}{2}}}\frac{e^{-\frac{1}{2}(u-\beta_x)^2/\gamma_x^2}}{\sqrt{2\pi}\,\gamma_x}du\\ &\times\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(a_y+vt)^2/\left(\tau_y^2+\omega^2+\delta_y^2\right)}}{\sqrt{2\pi}\left(\tau_y^2+\omega^2+\delta_y^2\right)^{\frac{1}{2}}}\frac{e^{-\frac{1}{2}(v-\beta_y)^2/\gamma_y^2}}{\sqrt{2\pi}\,\gamma_y}dv.\end{aligned} \tag{2.10}$$

Interpretation of the integrals as Gaussian convolutions or thinking in terms of adding independent Gaussian random variables gives this formula

$$\bar{r}\left(t;\beta_x,\sigma_x^2,\beta_y,\sigma_y^2\right)=R\theta\frac{e^{-\frac{1}{2}(a_x+\beta_x t)^2/\left(\tau_x^2+\omega^2+\delta_x^2+\gamma_x^2t^2\right)}}{\sqrt{2\pi}\sqrt{\tau_x^2+\omega^2+\delta_x^2+\gamma_x^2t^2}}\cdot\frac{e^{-\frac{1}{2}(a_y+\beta_y t)^2/\left(\tau_y^2+\omega^2+\delta_y^2+\gamma_y^2t^2\right)}}{\sqrt{2\pi}\sqrt{\tau_y^2+\omega^2+\delta_y^2+\gamma_y^2t^2}} \tag{2.11}$$

To continue the analytical program that leads to the expected number of contacts over a time period, as in (2.5), we must integrate (2.11) with respect to t. This is not an *easy* closed form integral, but simple numerical integration should be adequate, or Monte Carlo sampling of glance times according to an exponential distribution is possible. This would represent an enhancement of Metron's ISP idea. Whatever time either approach takes is likely to be insignificant as compared to a detailed event-by-event simulation.

2.2 Random Walking Sensor

Consider a sensor that "random walks" around the region of interest (this idea has been mentioned by R. Blacksten). We analyze first a simple random walk in terms of Brownian motions with drift, but it is also possible to allow the sensor to execute *Brownian-bridge* motion, whereby the sensor returns to a particular location after a given time S, the on-station time. Having finished, it leaves the region. It is possible that such sensor maneuvering will turn up more items than a straight-line course, unless the latter is very well directed.

A random-walking sensor's course can be represented as $(a_x + b_x t + \sigma_x W_x(t),\ a_y + b_y t + \sigma_y W_y(t),\ t_l \le t \le t_u)$, where $(W_x(t),\ W_y(t),\ t_l \le t \le t_u)$ is a pair of independently evolving Brownian motions or Wiener processes; each has mean 0 and variance t and further has independent Gaussian increments. Thus at time t the $r(t)$-value is, conditional on the past history of the sensor, obtained by replacing $a_x + b_x t$ by $a_x + b_x t + \sigma_x W_x(t)$, and correspondingly $a_y + b_y t$ by $a_y + b_y t + \sigma_y W_y(t)$ in (2.5). Now remove the condition on the past:

$$\bar{r}(t) = \theta R p \frac{\delta_x}{\left(\tau_x^2 + \omega_x^2 + \delta_x^2\right)^{\frac{1}{2}}} \cdot \frac{\delta_y}{\left(\tau_y^2 + \omega_y^2 + \delta_y^2\right)^{\frac{1}{2}}}$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(a_x + b_x t + \sigma_x \sqrt{t} w\right)^2 / \left(\tau_x^2 + \omega_x^2 + \delta_x^2\right)} \cdot \frac{e^{-\frac{1}{2}w^2}\, dw}{\sqrt{2\pi}} \tag{2.12}$$

$$\times \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(a_y + b_y t + \sigma_y \sqrt{t} z\right)^2 / \left(\tau_x^2 + \omega_x^2 + \delta_x^2\right)} \frac{e^{-\frac{1}{2}z^2}\, dz}{\sqrt{2\pi}}.$$

Straightforward manipulation shows that the above integrals are again Gaussian convolutions that are explicitly of Gaussian form: the rate of sightings/detections at time t is

$$\bar{r}(t) = \theta R p \frac{\delta_x}{\left(\tau_x^2 + \omega_x^2 + \delta_x^2 + \sigma_x^2 t\right)^{\frac{1}{2}}} \cdot \frac{\delta_y}{\left(\tau_y^2 + \omega_y^2 + \delta_y^2 + \sigma_y^2 t\right)^{\frac{1}{2}}}$$
$$\times e^{-\frac{1}{2}(a_x + b_x t)^2 / \left(\tau_x^2 + \omega_x^2 + \delta_x^2 + \sigma_x^2 t\right)}$$
$$\times e^{-\frac{1}{2}(a_y + b_y t)^2 / \left(\tau_y^2 + \omega_y^2 + \delta_y^2 + \sigma_y^2 t\right)}. \tag{2.13}$$

Numerical integration over $t_l \leq t \leq t_u$ again provides the mean number of detections over the path, and exponentially distributed times between glance sampling provides a probabilistic/stochastic version.

Note: the formulas given are approximations that must be calibrated to properly reflect the influence of sensor speed, s.

3. G-String Models Explained and Numerically Illustrated

3.1 An Example

Consider a very special simple scenario: a single sensor-bearing platform on a (cued) course travels over a stylized brigade size unit; here composed of 9 company-size "items", each of specific size, and all scattered or assembled over a plausibly-sized spatial region. The sensor package scans at about 1 scan/second; each scan directs a *glance* (a visual bullet) at some point in space; if the glance encounters/is near enough to an item a *hit* is generated with a probability governed by *how close* the glance was to the target size. With a given probability, then, the hit is converted to detection; (the above probability *could be* the probability (= fraction of time) some items are moving and not hidden by terrain).

3.2 Example Mini-Scenario 1: A Brigade-Sized Unit Investigated by a Cued UAV

(a) Represent the Brigade as a collection of 9 companies (here called "items") that are clustered within a region of approximately 3-km radius around a center.

(a-1) Model: The locations of the companies are independently and bivariate-Gaussian distributed with mean (0,0) (center) and x-coordinate and y-coordinate variances $\tau_x^2(\mathrm{km}^2)$ and $\tau_y^2(\mathrm{km}^2) = 3^2(\mathrm{km})^2$. These items could be arranged in standard formations ("two up and one back") if the analyst wishes to commit to precise locations).

(b) Each company (item) has an effective detectability or size: depending on the particular terrain, and on whatever other conditions prevail (to be specified). For the present example, the item size is on the order of $\delta_x^2 = \delta_y^2 = 0.25\mathrm{km}^2$; this is easily adjusted.

(b-1) Model: The effect of the item size or visibility is represented by a probability of sensor glance hit that is proportional to a bivariate-Gaussian variable with x-coordinate and y-coordinate variances $\delta_x^2(\mathrm{km}^2)$ and $\delta_y^2(\mathrm{km}^2)$. The probability of detection thus falls off gradually with the distance of a sensor glance location from the center of the item.

The above specific figures were abstracted from the CGSC Student Text 100-7 Soviet-Style Threat Tactical Handbook, U.S. Army Command and General Staff College, Fort Leavenworth, Kansas, 1 March 1992. We are grateful for advice from LTC C. Shaw.

(c) A sensor on the platform is characterized by a glance *rate* θ and a sensor footprint or region of glance concentration or field of view. Repeated glances fall over that footprint.

(c-1) Model: The sensor footprint is proportional to a bivariate-Gaussian variable with x-coordinate and y-coordinate variances $\omega_x^2(\mathrm{km}^2)$ and $\omega_y^2(\mathrm{km}^2)$. We assume $\omega_x = \omega_y = \omega$ so the sensor's attention is spread uniformly, angularly, around the sensor (platform) instantaneous location, but does not extend much beyond 2 to $3 \times \omega$ from the current sensor location. If a glance location is close enough to the target item's center it registers a hit, and a potential detection. We introduce another random effect, detection probability p, to reflect that every hit may not be a detection. Assume here that the human glance *rate* $\theta = 1$ per sec. The basic notion of the model is that, at rate θ, the particular sensor system instantaneously focuses at locations in space picked randomly and independently in a manner governed by a bivariate-Gaussian ("normal") distribution. Depending upon how close such a glance location is to a target item, a hit or potential detection is registered, as governed by target item size, as in (b).

(d) The platform travels linearly at speed, s, here 100 miles per hour, which is 161km per hour; the course is directly over the center of the brigade area. Thus, the platform will spend 1.5/161 = 0.003 hours~34 seconds to fly 3×0.5km. It will spend 36/161 = 0.22 hours~805 sec to fly 3×6km. Thus, the sensor will have approximately 805 glances along its pass over the area occupied by the brigade and approximately 34 glances over an area occupied by a company-sized unit. Note: these specific numbers are only for illustration.

3.3 Sensor Course Specifics (Illustration)

Assume the sensor is perfectly cued and travels along the y-axis starting at $a_x = 0$, $a_y = -18$. It moves at a speed $s = 0.045$km/second for 805 seconds; thus, $b_x = 0$ and $b_y = 0.045$ (km/sec): The positions of the 9 items (here companies) have been drawn from a bivariate-Gaussian/normal distribution centered about (0,0) with standard deviations $\tau_x =$ 3km and $\tau_y =$3km. The size of a unit is characterized by $\delta_x =0.25\text{km} = \delta_y$. The rate of glances is $\theta = 1$ per second. Given the position of a glance of the sensor and the position of a unit, the probability the item (company) is detected is given by (2.2) with $p = 0.7$. The footprint of the sensor is $\omega_x = \omega_y = 0.5$km.

3.4 Detailed Simulation with Randomly Located Glances

The results of the model with the above parameters were compared to those of a simulation. Each replication of the simulation generates glance instants from a Poisson process with rate $\theta = 1$ per second. The position of the sensor is computed at each glance time for the Poisson process.

Note: Connection with ISP and classical search

This is the equivalent of glancing at exponentially distributed intervals. If target items were uniformly distributed in space along the sensor path then this would be closely related to the Implicit Search and Prosecution (ISP) model suggested by Metron (Metron (1997)): times to *detection* would be exponential. The G.-S. model discussed *accounts for clustering of target items as they might occur in space in "standard" military formations.* For G.-S. the times to detection will not generally be exponentially distributed but account for "simple" modifications that depend on the general (or specific) locations of the target items.

In the simulation, a "1" is assigned to a unit if a hit occurs, and "0" otherwise.

In each replication of the sensor traversal of the region, the positions of the 9 units are generated according to a bivariate-Gaussian distribution with standard deviations $\tau_x = \tau_y =$ 3km. This Gaussian sampling does not have to be done if another item configuration generator (cookie cutter) is preferred; explicit integration again seems feasible.

For 500 replications of the simulation using the parameters given above, the mean total number of hits for a flight through the areas is 7.08 with a standard deviation of the mean of 0.22. The mean number of hits given by the model (2.7) is 7.19, so on the average the model does well and is extremely inexpensive and quick from a computational viewpoint.

Figures 1-4 present displays of the possible items' locations within the unit (centered at (0,0)) and the results of a number of sensor passes (replications) over the unit area with the items' locations fixed. In general the analytical model (2.7) is useful against the individual item configurations.

Usefulness: Consider a decision rule that states that a unit is present if the mean number of detections exceeds a threshold (to be determined). The model results quickly show whether a sensor has encountered a unit of specified size (and type). A more refined approach could be Bayesian, using the non-stationary Poisson number of detections implied by the exponential sampling model as likelihood, to be combined with an appropriate prior, based on knowledge of the number of items R in the unit of item sizes (measured by (δ_x, δ_y)) and spatial spread of the unit (τ_x, τ_y).

FLIGHT OF SENSOR PLATFORM

∇=POSITION OF UNITS

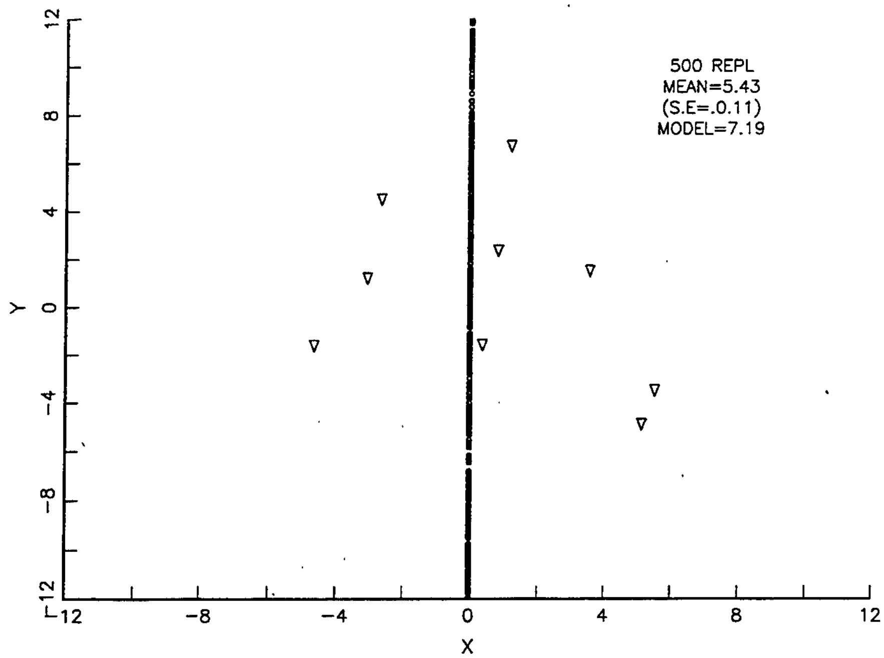

Figure 1

FLIGHT OF SENSOR PLATFORM

∇=POSITION OF UNITS

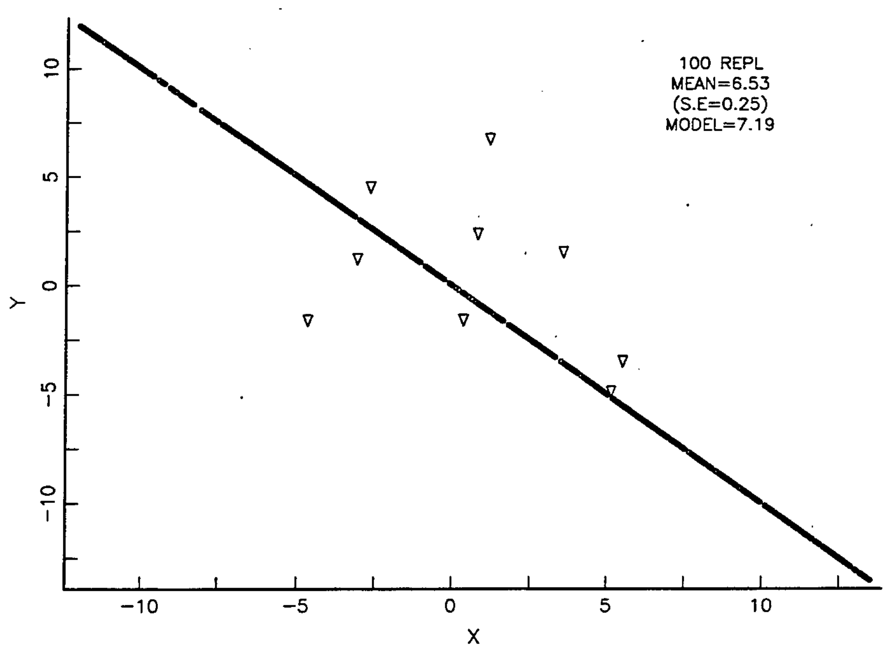

Figure 2

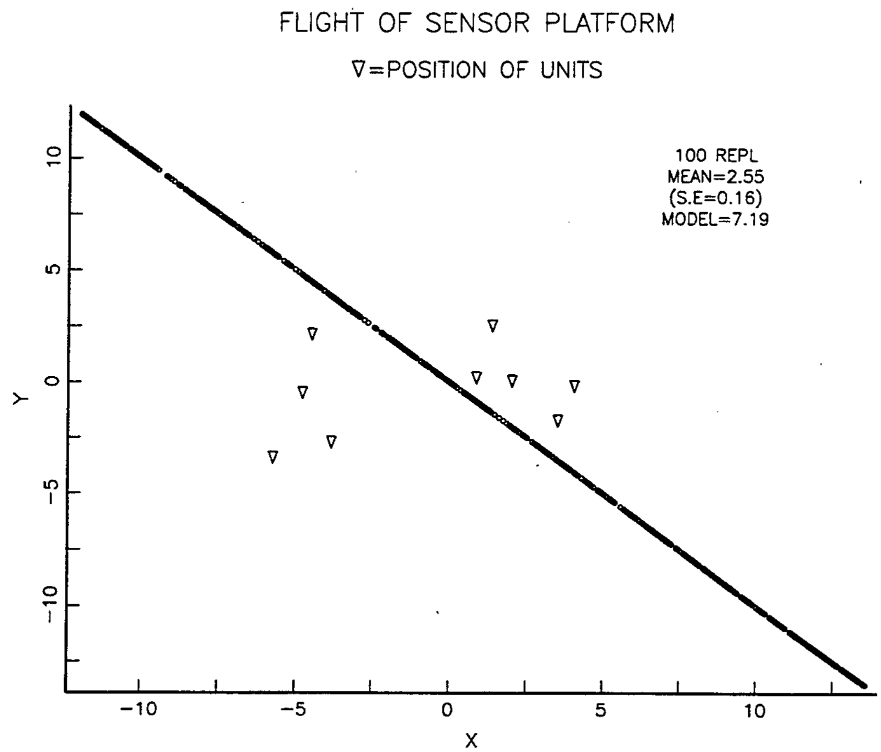

Figure 3

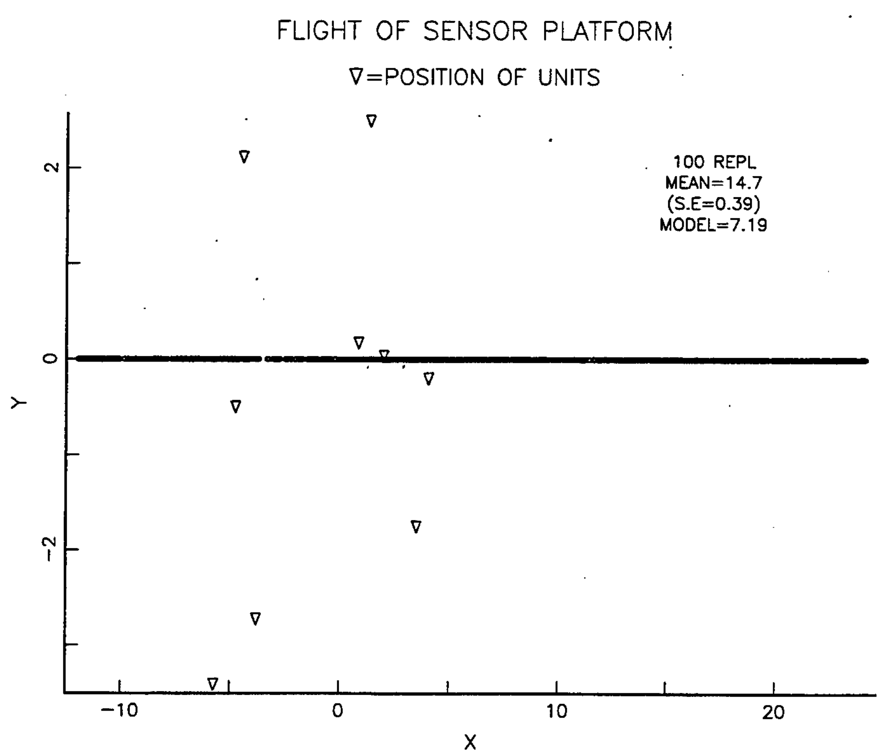

Figure 4

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APPENDIX I

Note that we can write

$$-\frac{1}{2}(a_x+b_xt)^2\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)-\frac{1}{2}\left(a_y+b_yt\right)^2\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)$$
$$\equiv-\frac{1}{2}(t-\mu)^2\Big/\sigma^2+\frac{1}{2}K^2. \tag{AI.1}$$

Now differentiate both sides of the above:

$$\frac{d}{dt}(\bullet)=-(a_x+b_xt)b_x\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)-\left(a_y+b_yt\right)b_y\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)$$
$$\equiv-(t-\mu)1\big/\sigma^2. \tag{AI.2}$$

Identify coefficients of powers of t (first, 0^{th}):

Coeff t: $$\frac{1}{\sigma^2}=b_x^2\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)+b_y^2\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right) \tag{AI.3}$$

Coeff 1: $$\frac{\mu}{\sigma^2}=-\left[a_xb_x\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)+a_yb_y\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)\right] \tag{AI.4}$$

$$\therefore\quad \mu=-\frac{\left[a_xb_x\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)+a_yb_y\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)\right]}{\left[b_x^2\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)+b_y^2\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right)\right]}. \tag{AI.5}$$

Also, if we put $t=\mu$, the

$$K^2=-(a_x+b_x\mu)^2\Big/\left(\tau_x^2+\omega_x^2+\delta_x^2\right)-\left(a_y+b_y\mu\right)^2\Big/\left(\tau_y^2+\omega_y^2+\delta_y^2\right). \tag{AI.6}$$

APPENDIX II

Glances at a Single Item

Suppose a target is in place at $(\boldsymbol{x}, \boldsymbol{y})$. Let it be the object of search by a platform-sensor that is on course $(a_x + b_x t,\ a_y + b_y t;\ t_l \le t \le t_u)$.

(a) Use the detection model to calculate the probability that a glance at time t detects/sees the target, using the exponential probability model (2.2); it is seen that this is given by (2.3) (take $\omega_x = \omega_y = \omega$, $\delta_x = \delta_y = \delta$):

$$\bar{d}(\boldsymbol{x}, \boldsymbol{y}; a_x + b_x t, a_y + b_y t) = p\frac{\delta^2}{\omega^2 + \delta^2} e^{-\frac{1}{2}\left[(x-a_x-b_xt)^2+(y-a_y-b_yt)^2\right]/\omega^2+\delta^2} \qquad \text{(AII.1)}$$

Note: A *simple* formula does *not* (yet) result if we put a circular cookie cutter at $(\boldsymbol{x}, \boldsymbol{y})$; see Eckler, Ross and Buss (1972).

(b) Formulas already given, (2.7), give the total expected number of glances at $(\boldsymbol{x}, \boldsymbol{y})$ over $t_l \le t \le t_u$: simply do not integrate out (x, y), but do integrate on t, which can be seen to require only

(b-1) remove/set to zero τ_x^2, τ_y^2;

(b-2) replace a_x by $a_x - \boldsymbol{x}$, a_y by $a_y - \boldsymbol{y}$.

Thus the assumption of a Gaussian-bump complex target has now been replaced by a single target at $(\boldsymbol{x}, \boldsymbol{y})$ with size δ. We calculate the expected number of glances "at" that target as the platform-sensor combination proceeds along the specified path.

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