

**A PRELIMINARY STUDY OF SENSITIVITY ANALYSIS AND ITS  
APPLICATIONS TO STRUCTURAL CONTROL PROBLEMS**

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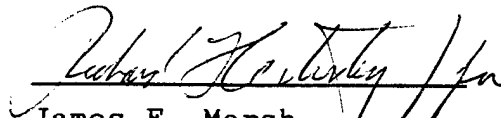
Foreword

The technical effort reported herein was performed as a part of Task 4.2-35 of Air Force Contract No. F33615-87-C-3228 under which Anamet Laboratories, Inc. operates the Aerospace Structures Information and Analysis Center (ASIAC) for the Flight Dynamics Laboratory at Wright-Patterson Air Force Base in Ohio. The report was prepared by Professor Yan Young, Department of Ocean Engineering, Florida Atlantic University, Boca Raton, Florida for Anamet Laboratories, Inc. under Purchase Order No. 7810.

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## ABSTRACT

This report presents a preliminary study of the sensitivity analysis for dynamic systems with emphasis on its applications to structural control. Definitions are first given for different sensitivity functions in the time and the frequency domains. Since most physical quantities of dynamic systems cannot be expressed in analytical forms, we introduce an indirect approach to determine their sensitivity derivatives from the sensitivity equations derived from governing equations. A direct application of the sensitivity analysis can be found in the integrated control and optimization in which design variables and control variables are treated equally as the system parameters active in optimization. An extensive review and evaluation of the existing techniques in this area are given to identify a feasible algorithm for future improvements. Finally, a new control algorithm, called optimization based instant control, is proposed for those systems subjected to general deterministic or random excitations. Unlike the conventional algorithm, the optimal control is designed and implemented according to instant information of the excitations. The important feature of this approach is that the original optimal control becomes a problem of static parameter optimization. The formulation layout makes it possible to apply the newly developed compound scaling algorithm [20] in optimal structural control.

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## 1. INTRODUCTION

Sensitivity, as the term suggested, is of concern with the effects of the variation of the parameters to the system response and characteristics. This concept was first introduced by mathematicians to evaluate the behavior of the solutions for differential equations affected by their coefficients [17]. Early application of this technique was mainly in the assessment of the discrepancy between the actual system and its theoretical model. The situation has drastically changed during last three decades. Growing interest of the sensitivity analysis has widely expanded to many research areas. In particular, many sophisticated and optimization theories have emerged recently from this important technique.

The optimal design is the approach of searching a set of system parameters that correspond to minimum values of certain objective functions. The parameter search is guided by the sensitivity derivatives of those functions, which determine the steepest gradient directions. How to determine the sensitivity derivatives, hence, is essential for the optimization. For the case of static loading, theory has been well developed and research efforts have been concentrated on the development of more efficient numerical techniques [18,19,1]. For the case of dynamic loading, complication arises in the parameter optimization. This is due to the fact that, in general, a structural response cannot be expressed in a closed form and many physical variables are time

dependent. It is conceivable that the determination of the sensitivity derivatives for such a case, if possible, may lead to a very cumbersome process. Thus, the possibility of calculating the sensitivity derivatives and the efficiency of numerical computations become two important issues for the researchers in the area.

Application of the sensitivity analysis in the control was first performed by Bode [2], who established different sensitivity measures in the frequency domain. These sensitivity measures became very important in the classical control theory. However, the classical control theory has limited applications for structural system since most of the structures have multiple inputs and multiple outputs. Structural control, thus, must be pursued using modern control theory or optimal control theory. The sensitivity analysis again is playing an increasingly important role in the modern control theory [5].

Recently, Hale, et al. [7] and Haftka, et al. [6] discussed the possibility of improving control performance by choosing proper system parameters. They suggested that the control and parameter optimization be performed simultaneously in order to achieve the best control design. Many research works along this line have shown promise results for the case where the structure is experience zero or white noise excitations [8,9,13,14]. Very few have shown their successes for cases otherwise. One of the objectives of this preliminary research is to evaluate existing techniques in the integrated control and optimization, and identify

the most feasible approach to be adopted in the future.

It is known that the optimal control theory is established on the basis of optimization, in which the sensitivity derivatives play very important roles. However, the conventional optimization scheme is not necessary if the external excitations are zeros or white noises. It can be proved that the optimal design for such cases are corresponding to the solution of the Riccati equation, a nonlinear ordinary differential equation with closed-form coefficients [11]. In reality, however, most external excitations applied to structures are not white-noise; thus, the conventional optimization scheme must be adopted. In this report, we propose a new approach, called optimization based instant control, which is valid for structures subjected to general excitations. The main feature of this approach is that one only needs to solve a static parameter optimization problem since the optimal control problem can be converted into this form.

## 2. SENSITIVITY ANALYSIS

### 2.1 Definitions

Consider a real or complex function  $F(\alpha)$  with parameter  $\alpha$ . The sensitivity function at the nominal value  $\alpha = \alpha^0$  is defined as

$$S = \frac{\partial F(\alpha)}{\partial \alpha} \Big|_{\alpha = \alpha^0} \quad (2-1)$$

which indicates the rate of change of the function  $F$  at the nominal value  $\alpha^0$ . The reason of using partial derivative is that, in general,  $F$  also may be function of other parameters, or variables. This definition only refers to the sensitivity with respect to  $\alpha$ . Equation (2-1) is also called an absolute sensitivity function.

Now we extend the definition for the sensitivity to the case of  $r$  parameters, indicated by the vector  $\alpha = [\alpha_1, \dots, \alpha_r]^T$ . The sensitivity of  $F(\alpha)$  is characterized by a sensitivity vector  $S$  with the  $j$ -th component

$$S_j = \frac{\partial F(\alpha)}{\partial \alpha_j} \Big|_{\alpha = \alpha^0} \quad j = 1, \dots, r \quad (2-2)$$

Apparently, equation (2-2) is identical to equation (2-1) except for  $\alpha$  being a vector.

In a general case where the system response or other characteristics are described by a vector, say  $F(\alpha) = \{F_1(\alpha), \dots, F_n(\alpha)\}^T$ , we define an  $n \times r$  sensitivity matrix  $S$ . of which

$$S_{kj} = \frac{\partial F_k(\alpha)}{\partial \alpha_j} \Big|_{\alpha=\alpha^0} \quad j = 1, \dots, r; \quad k = 1, \dots, n \quad (2-3)$$

Equation (2-3) represents the sensitivity of the k-th components of  $\mathbf{F}$  with respect to the j-th component of parameter vector  $\alpha$ .

In the classic control theory, it is useful to employ the relative sensitivity function in the frequency domain [2]. The relative sensitivity function is defined as

$$Q = \frac{\partial \ln F(\alpha)}{\partial \ln \alpha} \Big|_{\alpha=\alpha^0} \quad (2-4)$$

The relativity of this sensitivity function may not be obvious, unless we rewrite equation (2-4) as

$$Q = \frac{\partial F(\alpha)/F(\alpha)}{\partial \alpha/\alpha} \Big|_{\alpha=\alpha^0} \quad (2-5)$$

which represents the ratio of the percentage change of function  $F$  to the percentage change of parameter  $\alpha$ . The relative sensitivity function is related to the absolute sensitivity function by

$$Q = S \frac{F(\alpha_0)}{\alpha_0} \quad (2-6)$$

If  $\mathbf{F}$  and  $\alpha$  are both vectors, the relative sensitivity is defined by a rectangular matrix  $\mathbf{Q}$  with elements

$$Q_{kj} = S_{kj} \frac{F_k(\alpha)}{\alpha_j} \Big|_{\alpha=\alpha_0}; \quad j=1, \dots, r; \quad k=1, \dots, n \quad (2-7)$$

## 2.2 General Rules

It has been seen that a sensitivity functions are the partial derivatives with respect to a parameter vector. The general rules thus are basically the rules for the partial derivatives. For the sake of the completeness, we list these rules in the following:

a. summation rule

$$S_{\alpha}^{F_1+F_2} = S_{\alpha}^{F_1} + S_{\alpha}^{F_2} \quad (2-8)$$

b. product rule

$$S_{\alpha}^{F_1 F_2} = S_{\alpha}^{F_1} F_2(\alpha_0) + F_1(\alpha_0) S_{\alpha}^{F_2} \quad (2-9)$$

c. chain rule

$$S_{\alpha}^{F_1(F_2)} = S_{F_2}^{F_1} S_{\alpha}^{F_2} \quad (2-10)$$

d. quotient rule

$$S_{\alpha}^{F_1/F_2} = \frac{S_{\alpha}^{F_1}}{F_2(\alpha_0)} - \frac{F_1(\alpha_0) S_{\alpha}^{F_2}}{F_2^2(\alpha_0)} \quad (2-11)$$

where  $\alpha^0$  is the nominal value. In the above equations, the super-index indicates the function for which the sensitivity is calculated, and the sub-index indicates the parameter(s) with respect to which the sensitivity function is measured. For the relative sensitivity function, we also have

e. product rule

$$Q_{\alpha}^{F_1 F_2} = Q_{\alpha}^{F_1} + Q_{\alpha}^{F_2} \quad (2-12)$$

f. quotient rule

$$Q_{\alpha}^{F_1/F_2} = Q_{\alpha}^{F_1} - Q_{\alpha}^{F_2} \quad (2-13)$$

g. chain rule

$$Q_{\alpha}^{F_1(F_2)} = Q_{F_2}^{F_1} Q_{\alpha}^{F_2} \quad (2-14)$$

### 2.3 Output Sensitivity

It is important to note that the sensitivity functions can be determined directly from their definitions provided in the last section. This is not the case for dynamic systems in which the objective functions, such as state variables, are involved in differential equations. The objective function is implicitly given. Therefore, we must look for other alternatives and determine the sensitivity functions indirectly.

Consider a system governed by the differential equation

$$F(y^{(n)}, \dots, y^{(1)}, y, u, \alpha, t) = 0; \quad (2-15)$$

$$y^{(j)}(0) = y^{(j)0}, \quad j=1, \dots, n$$

in which  $F$  is a nonlinear function,  $y(t)$  an output function and  $u(t)$  an input function. The numbers within the parentheses indicate the order of the derivatives respect to time  $t$ . Our goal is to calculate the sensitivity functions  $S_{\alpha_j}^y$ ,  $j=0, 1, \dots, n$ , at the nominal values  $\alpha_j^0$ ,  $j=0, 1, \dots, n$ .

Taking derivatives with respect to  $\alpha_j$  on both sides of

differential equation (2-15) results in

$$\frac{\partial F}{\partial y^{(n)}} \frac{\partial y^{(n)}}{\partial \alpha_j} + \dots + \frac{\partial F}{\partial y^{(1)}} \frac{\partial y^{(1)}}{\partial \alpha_j} + F \frac{\partial y}{\partial \alpha_j} + \frac{\partial F}{\partial \alpha_j} = 0; \quad (2-16)$$

$$\frac{\partial y^{(k)}}{\partial \alpha_j} \Big|_{t=0} = 0; \quad k=1, \dots, n; j=1, \dots, r$$

Use is made of the fact that input  $u$  and initial condition are independent of parameter vector  $\alpha$ . Recognizing that derivatives with respect to  $t$  and  $\alpha_j$  are exchangeable, we evaluate equation (2-16) at the nominal value  $\alpha^0$ , and obtain

$$\frac{\partial F}{\partial \alpha_j} \Big|_{\alpha^0} S_j^{(n)} + \dots + \frac{\partial F}{\partial \alpha_j} \Big|_{\alpha^0} S_j^{(1)} + \frac{\partial F}{\partial \alpha_j} \Big|_{\alpha^0} S_j = - \frac{\partial F}{\partial \alpha_j} \Big|_{\alpha^0}; \quad (2-17)$$

$$S_j^{(k)}(0) = 0; \quad k=1, \dots, n; j=1, \dots, r$$

in which

$$S_j = \frac{\partial y(t, \alpha)}{\partial \alpha_j} \Big|_{\alpha = \alpha^0} \quad (2-18)$$

It is interesting to see that the sensitivity equations represent a time variant linear system in spite of the original governing equations being nonlinear.

For a special case where the system is governed by a linear time-invariant differential equation

$$b_n(\alpha)y^{(n)} + \dots + b_1(\alpha)y^{(1)} + b_0(\alpha)y = u(t); \quad (2-19)$$

$$y^{(k)}(0) = y_0^{(k)}; \quad k=1, \dots, n$$

The corresponding sensitivity equation is

$$b_n S_j^{(n)} + \dots + b_1 S_j^{(1)} + b_0 S_j = - \frac{\partial b_n}{\partial \alpha_j} \Big|_{\alpha^0} y^{(n)}(\alpha^0) - \dots - \frac{\partial b_1}{\partial \alpha_j} \Big|_{\alpha^0} y^{(1)}(\alpha^0) - \frac{\partial b_0}{\partial \alpha_j} \Big|_{\alpha^0} y(\alpha^0);$$

$$S_j^{(k)}(0) = 0 \quad k=1, \dots, n; \quad j=1, \dots, r$$

where  $y(\alpha^0, t)$  is the nominal solution of equation (2-19),

$$y(\alpha^0, t) = \int_0^t h(\alpha^0, t-\tau) u(\tau) d\tau \quad (2-21)$$

in which  $h(\alpha^0, t)$  is an impulse response function. Equation (2-20) is identical to the governing equation (2-19) except for the input term on the right hand side. Solution for equation (2-20) can be written as a convolution integral; namely,

$$S_j = \int_0^t h(\alpha^0, \tau-t) \left[ \frac{\partial b_n}{\partial \alpha_j} \Big|_{\alpha^0} y^{(n)}(\tau, \alpha^0) + \dots + \frac{\partial b_1}{\partial \alpha_j} \Big|_{\alpha^0} y^{(1)}(\alpha^0, \tau) + \frac{\partial b_0}{\partial \alpha_j} \Big|_{\alpha^0} y(\alpha^0, \tau) \right] d\tau \quad (2-22)$$

$$j=1, \dots, r$$

#### 2.4 Trajectory Sensitivity Function

The output sensitivity functions defined above are for the systems with a single degree of freedom. They can be determined by solving the high-order ordinary linear differential equation. In

theory the same approach can be used for a system with multiple degrees of freedom. This will be, however, a enormous task in practice. Nevertheless, the governing equation can be conveniently written in terms of the state variables. The similar approach then can be adopted in the determination of the sensitivity functions.

Consider a system govern by the differential equations

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t, \boldsymbol{\alpha}), \quad \mathbf{x}(t_0) = \mathbf{x}^0 \quad (2-23)$$

in which  $\mathbf{x}$  is an  $n \times 1$  state vector,  $\mathbf{F}$  an  $n \times 1$  excitation vector,  $\mathbf{u}$  an  $m \times 1$  input vector, and  $\boldsymbol{\alpha}$  an  $r \times 1$  parameter vector. Taking the partial derivatives on both sides of equation (2-23) with respect to  $\alpha_j$  yields

$$\frac{\partial \mathbf{x}}{\partial \alpha_j} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \alpha_j} + \frac{\partial \mathbf{F}}{\partial \alpha_j}, \quad \frac{\partial \mathbf{x}}{\partial \alpha_j} \Big|_{t=0} = \mathbf{0} \quad (2-24)$$

$$j = 1, \dots, r$$

where  $\partial \mathbf{F} / \partial \mathbf{x}$  is an  $n \times n$  Jacobian matrix,

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (2-25)$$

Again, interchanging the derivatives with respect to the time  $t$  and parameter  $\alpha_j$ , and evaluating the above equation at the nominal value  $\boldsymbol{\alpha}^0$ , we arrive at

$$\dot{\xi}_j = \frac{\partial F}{\partial x} \Big|_{\alpha^0} \xi_j + \frac{\partial F}{\partial \alpha_j} \Big|_{\alpha^0}; \quad \xi_j(0) = 0 \quad (2-26)$$

$$j = 1, \dots, r$$

where  $\xi_j$  is called the trajectory sensitivity function with respect to the  $j$ -th component of the parameter vector  $\alpha$ , and is defined by

$$\xi_j = \frac{\partial x}{\partial \alpha_j} \Big|_{\alpha^0} \quad (2-27)$$

Equation (2-26) generally is a set of linear time variant differential equations. Since coefficients in the equation contain nominal solutions  $x(\alpha^0, t)$ , the numerical computation for this problem would be quite expensive. The situation will be very different if the system is linear, and represented by

$$\dot{x} = Ax + Bu; \quad x = x_0 \quad (2-28)$$

in which **A** and **B** are matrices containing  $\alpha$ . The corresponding sensitivity equations appear to be linear time-invariant, i.e.,

$$\dot{\xi}_j = A \xi_j + \frac{\partial A}{\partial \alpha_j} \Big|_{\alpha^0} x(\alpha_0, t); \quad \xi_j(0) = 0 \quad (2-29)$$

$$j = 1, \dots, r$$

## 2.5 Performance-Index Sensitivity

A linear quadratic regulator is an optimization problem of determining the optimal control force  $u(t)$  by minimizing the

performance index

$$J = G + \int_0^{t_f} L dt \quad (2-30)$$

where

$$G = x^T(t_f) S x(t_f) \quad (2-31)$$

and

$$L(t) = x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \quad (2-32)$$

The constraint for the performance index is given in equation (2-28). The sensitivity for the performance index is a vector defined by

$$\eta = \frac{\partial J}{\partial \alpha} \Big|_{\alpha_0} \quad (2-33)$$

Taking derivatives of equation (2-30) and evaluating it at the nominal value  $\alpha^0$  yield

$$\eta = \left[ \frac{\partial G}{\partial x(t_f)} \Big|_{\alpha_0} \right]^T \xi(\alpha^0, t_f) + \int_0^{t_f} \left[ \left( \frac{\partial L}{\partial x} \Big|_{\alpha_0} \right)^T \xi + \frac{\partial L}{\partial \alpha} \Big|_{\alpha_0} \right] dt \quad (2-34)$$

in which  $\xi$  is the trajectory sensitivity vector obtained from equation (2-26).

Methods of determine other kinds of sensitivity functions, such as eigenvalue and eigenvector sensitivity functions, can be found in References [3,4,5,15].

### 3. INTEGRATED STRUCTURAL CONTROL AND OPTIMIZATION

Structural design is of an exercise to aim the fulfillment of safety criteria and economical expenses. For a structural system under static loadings, it is a common practice to employ the optimization technique to minimize certain cost function of the system, such as the strength or stiffness, by varying system parameters in a prescribed range. This problem can be stated as a linear or nonlinear mathematical program, that minimizes the cost function  $J(\alpha)$ , where  $\alpha$  is the vector of system parameters, with constraints,  $h_0 \leq h(\alpha) \leq h_1$ . Sensitivity derivatives, the important ingredients in the optimization, can be easily determined for such a case. When a structural system is excited by dynamic loadings, however, the fulfillment of the safety criteria and economical expenses is much more difficult to achieve solely through the optimization, since either the cost function  $J$ , or the constraints may be functions of time  $t$ . Attempt has been made to change the system characteristics, such as the eigenvalues, by varying damping ratio and stiffness coefficients, such that the dynamic response can be controlled to meet some specific requirements [10,12]. This effort is also known to be passive control, a limited approach especially when control of overall performance of the structural system is desired. The reason for the ineffectiveness of this control is that it only changes the amount of energy dissipation or the resonance frequencies, but by no means alternates the effect of the input. In contrast, the

active control provides a system with external energy, capable of canceling out parts of the energy due to the external excitation, keeping its effect on the dynamic response minimum. This method has been proved to be very efficient. However, experience has shown that the structural control, in most cases, was carried out independently from the structural optimization. Possible improvement in terms of combination of the control and optimization had long been overlooked, until recent papers [6,7] that drew much attention in the structural engineer community. These papers provide a clear evidence that changes of system parameters can affect control performance significantly. They strongly suggested that control and optimization should be applied to a structural system simultaneously in order to achieve the best structural design.

It is of importance to note that the modern control theory was established on the basis of the optimization scheme. As an example, the linear quadratic optimal control is to minimize the performance index

$$J_c(\alpha, u) = \int_0^{t_f} (x^T Q x + u^T R u) dt \quad (3-1)$$

with constraints

$$\dot{x} = Ax + Bu; \quad x(0) = x^0 \quad (3-2)$$

in which  $x$  and  $u$  are vectors of state variables and active control forces, respectively, and  $Q$  and  $R$  are appropriate weighting matrices determined according to relative emphasis. Matrix  $A$  contains system parameters  $\alpha$ , which in general for a structural

system is

$$A(\alpha) = \begin{bmatrix} 0 & I \\ -M^{-1}(\alpha)K(\alpha) & -M^{-1}(\alpha)C(\alpha) \end{bmatrix} \quad (3-3)$$

where  $M$ ,  $C$  and  $K$  are the mass matrix, the damping matrix and the stiffness matrix, respectively. The dependence of  $\alpha$  in the performance index is due to the constraint which in fact is the governing equation for the structural system. For optimal control, optimization is performed on the control variables  $u$  with predetermined system parameters  $\alpha$ . In other words, if the performance index  $J_c$  and the control forces  $u$  are treated as the system response and the system parameters, respectively, the control itself is a optimization process. However, the control variables are usually not determined by the conventional optimization approach. Instead, they can be obtained indirectly by solving a set of nonlinear ordinary differential equations, called the Riccati equation [11],

$$\dot{P} + A^T P + P A - P B R^{-1} B^T P + Q = 0 \quad (3-4)$$

in which  $P$  is a matrix attributed to the optimal feed back control forces

$$u = -R^{-1} B^T P x \quad (3-5)$$

Since the Riccati equation contains the coefficients of the governing equation explicitly, the original optimization is simplified to a ordinary differential equation problem.

Many researchers suggested that the integrated structural

control and optimization be pursued by two separate paths [8,9,13,14]: the optimal control and the parameter optimization. In this two-path optimization approach, structural design variable  $\alpha$  and the control variables  $u$  are classified as two sets of parameters to be optimized. The corresponding objective functions are  $J_s(\alpha)$  and  $J_c(\alpha, u)$ , respectively. The purpose of the integrated structural control and optimization is to minimize the total objective function,  $J(\alpha, u) = J_s(\alpha) + J_c(\alpha, u)$ , with respect to  $\alpha$  and  $u$  under the conditions that

$$h_0 \leq h(\alpha) \leq h_1 \tag{3-6}$$

$$x = Ax + Bu; \quad x(0) = x_0$$

Instead, the two-path optimization is performed on each sub-objective function separately. The minimization of  $J(\alpha, u)$  with respect to  $\alpha$  only involves  $J_s(\alpha)$ , but not  $J(\alpha, u)$ . The algorithm for this path of optimization is exactly the same for the passive control. The design variables  $\alpha$  are fixed while the  $J(\alpha, u)$  is minimized with respect to  $u$ . In this manner, the total objective function is minimized to

$$J(\alpha^{**}, u^{**}) = \min_{\alpha} J_s(\alpha) + \min_u J_c(\alpha^{**}, u) \tag{3-7}$$

The second part of the minimization usually is achieved by solving the Riccati equation, which is the main advantage of this approach as far as the numerical computation is concerned. Since the coupling of the two sets of parameters is disregarded in each path

of optimization,  $\alpha^{**}$  and  $u^{**}$  are not true optimal values, and  $J(\alpha^{**}, u^{**})$  is not the true minimum value. It can be proved that

$$J(\alpha^{**}, u^{**}) \geq \min_{\alpha, u} [J_s(\alpha) + J_c(\alpha, u)] = J(\alpha^*, u^*) \quad (3-8)$$

Thus, the two-path optimization only leads to sub-optimal solutions.

Salama, et al. [16] proposed an alternative in which the coupling of the two sets of parameters are taken into account. The strategy of this optimization can be described by the following expression

$$J(\alpha^*, u^*) = \min_{\alpha} [J_s(\alpha) + \min_u J_c(\alpha, u)] = \min_{\alpha} [J_s(\alpha) + J_c(\alpha, u^*)] \quad (3-9)$$

Consider the special case where the control is activated in the period from zero to infinity.  $x(t=\infty) = 0$  then is realized. Recognizing the fact that [11]

$$\min_u J_c(\alpha, u) = \min_u \int_0^{\infty} [x^T Q x + u^T R u] dt = x_0^T P(\alpha) x_0 \quad (3-10)$$

and

$$u^* = -R^{-1} B(\alpha)^T P(\alpha) x \quad (3-11)$$

where matrix  $P(\alpha)$  is the solution for the Riccati equation (3-4).

We can rewrite equation (3-9) as

$$J(\alpha^*, u^*) = \min_{\alpha} [J_s(\alpha) + x_0^T P(\alpha) x_0] \quad (3-12)$$

The right hand of the above equation now is an implicit function of  $\alpha$ . With the imposition of constraints on  $\alpha$ , the optimal solution

can be determined by using sensitivity analysis. This algorithm requires the derivatives of matrix  $P(\alpha)$ , which must be obtained from the Riccati equation. Since the Riccati equation only can be solved numerically, calculating of the sensitivity derivatives with respect to  $\alpha$  can be cumbersome. The detailed discussion of this method can be found in [16].

The methods stated above have one thing in common: the Riccati equation being used in the optimal control part. Although solving the Riccati equation requires sophisticated techniques, this effort would be still much less than that using the conventional optimization approach. The value of these methods largely depends upon the legitimacy of the Riccati equation for specific problems. In what follows, we will carefully reexamine the conditions for which the Riccati equation is valid.

We start with the governing equation for a structural system,

$$\dot{x} = Ax + Bu + Hf; \quad x(0) = x_0 \quad (3-13)$$

where  $f(t)$  is a vector of external forces. The objective of the active control design is to minimize the performance index in equation (3-1). As stated early, equation (3-13) is of the constraints for the optimal problem. Now we combine equations (2-1) and (2-13) and form a Lagrange

$$\mathcal{L} = \int_0^t [x^T Q x + u^T R u - \lambda^T (\dot{x} - Ax - Bu - Hf)] dt \quad (3-14)$$

where  $\lambda$  is a vector of Lagrangian multiplier, which in general is time dependent.

Taking variation of equation (3-14) results in

$$\delta \mathcal{L} = \int_0^{t_f} \left\{ \left( \dot{\lambda}^T + \frac{\partial \mathcal{F}}{\partial x} \right) \delta x + \frac{\partial \mathcal{F}}{\partial u} \right\} dt + \lambda^T(0) \delta x(0) - \lambda(t_f) \delta(t_f) \quad (3-15)$$

where  $\mathcal{F}$  is the integrand in equation (3-14). The necessary condition for  $\mathcal{L}$  to be minimum is  $\delta \mathcal{L} = 0$ , which corresponds to

$$\frac{\partial \mathcal{F}}{\partial u} = 0 \quad (3-16)$$

$$\dot{\lambda} + \frac{\partial \mathcal{F}}{\partial x} = 0 \quad (3-17)$$

and

$$\lambda(t_f) = 0 \quad (3-18)$$

The second term on the right hand side of equation (3-15) vanishes due to the initial condition. Equations (3-16) and (3-17) constitute a new set differential equations of  $\lambda$ ,

$$\dot{\lambda} = -A\lambda - 2Qx; \quad \lambda(t_f) = 0 \quad (3-19)$$

For a closed-loop control,  $\lambda(t)$  is taken to be

$$\lambda(t) = P(t)x(t) \quad (3-20)$$

The corresponding optimal control force can be determined according to equation (3-16),

$$u(t) = -R^{-1}B^T\lambda(t) \quad (3-21)$$

Substituting it into equation (3-17), we obtain

$$(\dot{P} + A^T P + P A - P B R^{-1} B^T P + Q)x + P H f = 0 \quad (3-22)$$

$$P(t_f) = 0 \quad (3-23)$$

For the special case where  $f = 0$ , which usually happens for parametric excitations or structure-induced excitations, equation (3-22) reduces to the Riccati equation (3-4). It also can be proved [11] that the optimal solution for the case in which  $f$  is a white noise excitation has the same form of equation (3-21), and corresponding matrix  $P$  again can be solved from the Riccati equation (3-4). Therefore, the necessary condition for the Riccati equation to be valid for the optimal control is that the additive external excitation  $f$  is either white noise or zero. Since most excitations applied to the structural systems are not white-noise, the optimal control cannot be determined from the Riccati equation. As the consequence, theories presented early for the integrated control and optimization have limited applications for structural systems.

Dealing with a structural system subjected to a general loading, Hale and his colleagues [7] derived the necessary conditions for optimal control and optimal system parameters by using variation principle. The objective function they used is

$$J = \int_0^{t_f} [x^T Q x + u^T R u + J_s(\alpha) - \lambda^T(t) [\dot{x} - A(\alpha)x - B(\alpha)u - H(\alpha)f]] dt \quad (3-24)$$

in which  $J_s(\alpha)$  is the cost function for design variable  $\alpha$ . The

condition for the minimum  $J$  can be obtained by taking variation on equation (3-24). These conditions are

$$\dot{x} = Ax + Bu + Hf; \quad x(0) = 0 \quad (3-25)$$

$$2Qx + A^T \lambda + \dot{\lambda} = 0; \quad \lambda(t_f) = 0 \quad (3-26)$$

$$2Ru + B^T \lambda = 0 \quad (3-27)$$

$$\lambda^T (\nabla_{\alpha} Ax + \nabla_{\alpha} Bu + \nabla_{\alpha} Hf) + \nabla_{\alpha} W = 0 \quad (3-28)$$

It is of importance to note that the advantage of this approach is that it directs the solution toward to a single optimal configuration so that the true optimal design of control forces and system parameters is achieved. Bearing it in mind that matrices  $A$ ,  $B$  and  $H$  are functions of parameter  $\alpha$ , the above equations are coupled in  $\alpha$  and  $u$ . It is almost impossible to solve them simultaneously for analytical solutions. Therefore, numerical computations are necessary to search the optimal solutions. Several computational technique are available to solve equations (3-25) through (3-28) [7]. They appeared to be very time consuming. In addition, they have not been fully evaluated in terms of the rate of convergence.

#### 4. OPTIMIZATION BASED INSTANT OPTIMAL CONTROL

As mentioned before, the theory of the optimization is well developed along with some standard optimization algorithms. However, increasing applications of large scale structures in the modern time demand further developments toward to the efficiency of optimization techniques. Recent advances shows that there have been great achievements in such a mission. One of important developments in optimization algorithm attributes to a recent work by Venkayya [20], who developed a so-called compound scaling algorithm, which has been proved to be very efficient for large scale structures. The objective of current research is to extend this new method to optimal control problem, and possibly to the integrated control and optimization for active structures. In what follows, we will propose a new control algorithm in which the optimal control can be completely treated as static parameter optimization.

The conventional optimal control relying on the Riccati equation requires that the external excitations either be zero or white noise. The nature of the excitation, in reality, however, is random and usually cannot be predicted exactly. Hence, the control design must depend upon the instant information of the excitation, and should be implemented instantly. To this end, we formulate an instant performance index

$$J(t) = \mathbf{x}^T(t)Q(t)\mathbf{x}(t) + \mathbf{u}^T(t)B\mathbf{x}(t) \quad (4-1)$$

Our purpose is to minimize  $J(t)$  at every time instant with

constraints

$$\dot{x}(t) = Ax(t) + Bu(t) + Hf(t); \quad x(0) = x_0 \quad (4-2)$$

For a linear system, it can be proved that over a small time interval  $\Delta t$ , the state vector can be written as

$$x(t) = Td(t-\Delta t) + \frac{\Delta t}{2}[Bu(t) + Hf(t)] \quad (4-3)$$

where  $T$  is a transformation matrix independent of time  $t$ , and  $d(t-\Delta t)$  is a vector containing previous information of  $x(t-\Delta t)$ ,  $u(t-\Delta t)$  and  $f(t-\Delta t)$ . Equation (4-3) is equivalent to equation (4-2), and again is a constraint for the instant performance index. For the closed-loop control, the control force is directly related to the state vector  $x(t)$ , namely;

$$u(t) = P(t)x(t) \quad (4-4)$$

in which  $P(t)$  is a control gain matrix. Specific requirement can be imposed to this control gain by

$$\mathfrak{R}[P(t)] - \mathfrak{R}_0(t) = 0 \quad (4-5)$$

$\mathfrak{R}$  is a operator which may represent eigenvalues, eigenvectors, determinant or other properties of matrix  $P(t)$ . Equation (4-5) along with equation (4-4) are additional constraints for the instant performance index.

Equations (4-1) through (4-5) constitute a regular parameter optimization problem. They can be cast into a Lagrange

$$\mathcal{L} = J(t) - \sum_1^l \lambda_i(t) X_i(t) - \sum_1^m \gamma_j(t) Y_j(t) - \sum_0^n \mu_k(t) Z_k(t) \quad (4-6)$$

where  $\lambda_i(t)$ ,  $\gamma_j(t)$  and  $\mu_k(t)$  are Lagrangian multipliers corresponding to the constraints

$$X(t) = x(t) - Td(t-\Delta t) - \frac{\Delta t}{2}[Bu(t) + Hf(t)] \leq 0 \quad (4-7)$$

$$Y(t) = u(t) - P(t)x(t) \leq 0 \quad (4-8)$$

and

$$Z(t) = \mathfrak{R}[P(t)] - \mathfrak{R}_0(t) \leq 0 \quad (4-9)$$

In equation (4-6), active variables participating the optimization are  $\mathbf{x}(t)$ ,  $\mathbf{u}(t)$  and  $\mathbf{P}(t)$ , which can be cast into a  $q \times 1$  vector  $\alpha$ . The stationary conditions for the Lagrange are

$$\frac{\partial \mathcal{L}(t)}{\partial \alpha_p} = \frac{\partial J(t)}{\partial \alpha_p} - \sum_1^l \lambda_i(t) \frac{\partial X_i(t)}{\partial \alpha_p} - \sum_1^m \gamma_j(t) \frac{\partial Y_j(t)}{\partial \alpha_p} - \sum_1^n \mu_k(t) \frac{\partial Z_k(t)}{\partial \alpha_p} = 0; \quad (4-10)$$

$$p = 1, 2, \dots, q$$

Equation (43) then can be written as the Kuhn-Tucker conditions

$$\sum_1^l e_{ij} \lambda_j + \sum_1^m h_{ij} \lambda_j + \sum_1^n g_{ij} \mu_j = 1; \quad i = 1, 2, \dots, q \quad (4-11)$$

where

$$e_{ij} = \frac{\frac{\partial X_j}{\partial \alpha_j}}{\frac{\partial J}{\partial \alpha_j}} \quad (4-12)$$

$$h_{ij} = \frac{\frac{\partial Y_i}{\partial \alpha_j}}{\frac{\partial J}{\partial \alpha_j}} \quad (4-13)$$

$$g_{ij} = \frac{\frac{\partial Z_j}{\partial \alpha_j}}{\frac{\partial J}{\partial \alpha_j}} \quad (4-14)$$

are ratios of sensitivity derivatives, which can be determined using the methods in Chapter 2.

It is worthy to note that, at any time instant, equation (4-7) can be viewed as the conditions for static equilibrium. The formulations presented in equations (4-6) through (4-9) constitute a standard parameter optimization problem at any time instant  $t$ , provided that the optimal solutions  $\mathbf{x}(t-\Delta t)$  and  $\mathbf{u}(t-\Delta t)$  are available. This requires that the optimal solution  $\mathbf{x}(t)$  and  $\mathbf{u}(t)$  be obtained within a time period  $\Delta t$  in order to progressively solve the problem in a full time scale. The on-line computation must engage if better accuracy is required. Therefore, the computational efficiency is essential to the proposed scheme. Among the existing numerical techniques, the compound scaling algorithm [20] is the most feasible one for the current

optimization based instant control. This method was developed by successfully applying the scaling and resizing techniques in the optimization procedure. The remarkable feature of this approach is that, instead of searching the optimum point by point, like the conventional ones, in the design space, the search sweeps the design space and approaches the optimum in a fast rate [20].

The optimization based instant optimal control has explicit superiorities over the conventional optimal control. First, it can be applied to structural systems under general dynamic loadings. It is especially useful when the loading is random. Secondly, The performance index does not contain integrations or derivatives, and optimal control is designed for time instant  $t$ . Therefore, the problem is suitable to be treated as regular static structural optimization. Finally, the optimal control solutions satisfy the Kuhn-Tucker conditions. The compound scaling algorithm can be used; so that numerical computation is expected to be very efficient.

## 5. CONCLUDING REMARKS

In this report, we have presented several methods to indirectly determine sensitivity measures related to structural controls in the time domain. Upon reviewing the existing techniques in the area of integrated control and optimization, we conclude that approaches that relies on the Riccati equation are limited to the structures subjected to zero additive excitation or white noise random excitations. True optimal solutions should be obtained by treating design variables and control variables equally as the system parameters to be optimized. The sensitivity analysis obviously should play the key role in this approach. Future research should concentrate on the development of efficient computational schemes that can implement the sensitivity analysis in the integrated control and optimization systematically. Also, we proposed a new optimal control algorithm, capable of converting a optimal control problem to a static parameter optimization problem. This new algorithm provides us an opportunity to take the advantage of recent advances in the optimization area. This method has potential to be extended to active structures by including system parameters in the active parameters in optimization. Better performance of the active structures can be achieved due to the fact that the structural properties and control variables are optimized at every time instant. Since this report is prepared for our preliminary research, many details have not been touched thoroughly. Prospective results will be developed in the

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