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Conditional event algebra (CEA) was developed in order to represent conditional probabilities with differing antecedents by the probability evaluation of well-defined individual "conditional" events in a single larger space extending the original unconditional one. These conditional events can then be combined logically before being evaluated. A major application of CEA is to data fusion problems, especially the testing of hypotheses concerning the similarity or redundancy among inference rules through use of probabilistic distance functions which critically require probabilistic conjunctions of conditional events. Relational event algebra (REA) is a further extension of CEA, whereby functions of probabilities formally representing single event probabilities - not just divisions as in the case of CEA - are shown to represent actual "relational" events relative to appropriately determined larger probability spaces. Analogously, utilizing the logical combinations of such relational events allows for testing of hypotheses of similarity between data fusion models represented by functions of probabilities. Independent of, and prior to this work, it was proven that a major portion of fuzzy logic - a basic tool for treating natural language descriptions - can be directly related to probability theory via the use of one point random set coverage functions. In this paper, it is demonstrated that a natural extension of the one point coverage link between fuzzy logic and random set theory can be used in conjunction with CEA and REA to test for similarity of natural language descriptions.

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CONTENTS

EXTENSION OF RELATIONAL AND CONDITIONAL EVENT ALGEBRA TO RANDOM SETS WITH APPLICATIONS TO DATA FUSION

I.R. GOODMAN AND G.F. KRAMER*

Abstract. Conditional event algebra (CEA) was developed in order to represent conditional probabilities with differing antecedents by the probability evaluation of well-defined individual "conditional" events in a single larger space extending the original unconditional one. These conditional events can then be combined logically before being evaluated. A major application of CEA is to data fusion problems, especially the testing of hypotheses concerning the similarity or redundancy among inference rules through use of probabilistic distance functions which critically require probabilistic conjunctions of conditional events. Relational event algebra (REA) is a further extension of CEA, whereby functions of probabilities formally representing single event probabilities -- not just divisions as in the case of CEA -- are shown to represent actual "relational" events relative to appropriately determined larger probability spaces. Analogously, utilizing the logical combinations of such relational events allows for testing of hypotheses of similarity between data fusion models represented by functions of probabilities. Independent of, and prior to this work, it was proven that a major portion of fuzzy logic -- a basic tool for treating natural language descriptions -- can be directly related to probability theory via the use of one point random set coverage functions. In this paper, it is demonstrated that a natural extension of the one point coverage link between fuzzy logic and random set theory can be used in conjunction with CEA and REA to test for similarity of natural language descriptions.

Key words. Conditional event algebra, Conditional probability, Relational event algebra, Functions of probabilities, Random sets, One point coverage functions, Probabilistic distances, Fuzzy Logic, Data Fusion.

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1. Introduction. The work carried out here is motivated directly by the basic data fusion problem: Consider a collection of multi-source information, which, for convenience, we call "models", in the form of descriptions and/or rules of inference concerning a given situation. The sources may be expert-based or sensor system-based, utilizing the medium of natural language or probability, or a mixture of both. The main task, then, is to determine:

Goal 1. Which models can be considered similar enough to be combined or reduced in some way and which models are dissimilar so as to be considered inconsistent or contradictory and kept apart possibly until further arriving evidence resolves the issue.

Goal 2. Combine models declared as pertaining to the same situation.

Goals 1 and 2 are actually extensions of classical hypotheses testing and estimation, respectively, applied to those situations where the relevant probability distributions are either not available from standard procedures or involve, at least initially, non-probabilistic concepts such as natural language.

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1.1. Statement of the problem. In this paper, we consider only Goal 1, with future efforts to be directed toward Goal 2. Furthermore, this is restricted to pairwise testing of hypotheses for sameness, when the two models in question are given in the formal *truth-functional* forms

$$(1.1)\qquad \begin{cases} \text{Model 1}: \ P(a) = f(P(c), P(d), P(e), P(h), \ldots) \\ \qquad\qquad\text{vs.} \\ \text{Model 2}: \ P(b) = g(P(c), P(d), P(e), P(h), \ldots) \end{cases}$$

where $f, g : [0,1]^m \to [0,1]$ are known functions and contributing events $c,d,e,h,\ldots$ all belong to probability space (Ω,B,P), i.e., $c, d, e, h,\ldots \in B$, a boolean or sigma-algebra over Ω associated with probability measure P. In general, probability involves *non*-truth-functional relations, such as the joint or conjunctive probability of two events *not* being a function of the individual marginal probability evaluations of the events. In light of this fact, in most situations, the "events" a and b in the left-hand side of eq.(1.1) are only formalisms representing overall "probabilities" for Models 1 and 2, respectively. However, if events $c,d,e,h,\ldots$ are all infinite left rays of real one-dimensional space, and $f = g$ is any copula and P on the left-hand side of eq.(1.1) is replaced by the joint probability measure P_0 determined by f (see, e.g., Sklar's copula theorem [28]), then a and b actually exist in the form of corresponding infinite left rays in multidimensional real space and eq.(1.1) reduces to an actual truth-functional form.

Returning back to the general case, *suppose* legitimate events a and b could be explicitly obtained in eq.(1.1) lying in some appropriate boolean or sigma algebra B_0 extending B in the sense of containing an isomorphic imbedding of all of the contributing events $c, d, e, h,\ldots,$ independent of the choice of P, but so that for each P, the probability space (Ω_0,B_0,P_0) extends (Ω,B,P), and *suppose* $P_0(a\&b)$ could be evaluated explicitly, then one could compute any of several natural *probability distance functions* $M_{P_0}(a,b)$. Examples of such include: $D_{P_0}(a,b)$, $R_{P_0}(a,b)$, $E_{P_0}(a,b)$; where $D_{P_0}(a,b)$ is the *absolute probability distance* between a and b (actually a pseudometric over (Ω,B,P), see Kappos [21]); $R_{P_0}(a,b)$ is the *relative probability distance* between a and b; and $E_{P_0}(a,b)$ is a symmetrized log-conditional probability form between a and b. (See Section 1.4 or [13], [14].) For example, using the usual boolean notation (& for conjunction, v for disjunction, ( )′ for complement, + for symmetric difference or sum, ≤ for subevent of, etc.),

$$(1.2)\ D_{P_0}(a,b) = P_0(a+b) = P_0(a'\& b) + P_0(a \& b') = P_0(a) + P_0(b) - 2P_0(a \& b).$$

Then, by replacing the formalisms $P(a)$, $P(b)$ in eq.(1.1)), by the probabilities $P_0(a) = f(P(c),P(d),P(e),...)$ and $P_0(b) = g(P(c),P(d),P(e),...)$, together with the assumption that the evaluation $P_0(a \& b)$ is meaningful and can be explicitly determined, the full evaluation of $D_{P_0}(a,b)$ can be obtained via eq.(1.2). In turn, assuming for simplicity that the higher order probability distribution of the relative atom evaluations $P_0(a \& b)$, $P_0(a' \& b)$, $P_0(a \& b')$ are -- as P and a and b are allowed to vary -- uniformly distributed over the natural simplex of possible values, the cdf F_D of $D_{P_0}(a,b)$ under the hypothesis of being not identical can be ascertained. Thus, using F_D and the "statistic" $D_{P_0}(a,b)$ one can test the hypotheses that a and b are different or the same, up to P_0-measure zero. (See Sections 1.4, 1.5 for more details.)

Indeed, it has been shown that two relatively new mathematical tools – *conditional event algebra* (CEA) and the more general *relational event algebra* (REA) -- can be used to legitimize a and b in eq.(1.1): CEA, developed prior to REA, yields the existence and construction of such a and b when functions f and g are identical to ordinary arithmetic division for two arguments $P(c)$, $P(d)$) in Model 1 with $c \leq d$, and $P(e)$, $P(h)$ in Model 2 with $e \leq h$, i.e., for conditional probabilities [18]. REA extends this to other classes of functions of probabilities, including, weighted linear combinations, weighted exponentials, and polynomials and series, among other functions [13].

Finally, it is of interest to be able to apply the above procedure when the models in question are provided through natural language descriptions. In this case, we first convert the natural language descriptions to a corresponding fuzzy logic one. Though any choice of fuzzy logic still yields a truth-functional logic, while probability logic is non-truth functional in general, it is interesting to note that the now well-developed one point random set coverage function representation of various types of fuzzy logic [10] bridges this gap: the logic of one point coverages is truth-functional (thanks to the ability to use Sklar's copula theorem here [28]). In turn, this structure fits the format of eq.(1.1) and one can then apply CEA and/or REA, as before, to obtain full evaluations of the natural event probabilistic distance-related functions and thus test hypotheses for similarity.

1.2. Overview of effort. Preliminary aspects of this work have been published in [13], [14]. But, this paper provides for the first time a unified, cohesive approach to the problem as stated in Section 1.1 for both direct probabilistic and natural language-based formulations. Section 1.3 provides some specific examples of models which can be tested for similarity. Section 1.4 gives some additional details on the computation and tightest bounds with respect to individual event probabilities of some basic probability distance functions when the conjunctive probability is not available. Section 1.5

summarizes the distributions of these probability distance functions as test statistics and the associated tests of hypotheses. Sections 2 and 3 give summaries of CEA and REA, respectively, while Section 4 provides the background for one point random set coverage representations of fuzzy logic. Section 5 shows how CEA can be combined with one point coverage theory to yield a sound and computable extension of CEA and (unconditional) fuzzy logic to conditional fuzzy logic. Finally, Section 6 reconsiders the examples presented in Section 1.3 and sketches implementation for testing similarity hypotheses.

1.3 Some examples of models. The following three examples provide some particular illustrations of the fundamental problem.

Example 1. Models as weighted linear functions of probabilities of possibly overlapping events. Consider the estimation of the probability of enemy attack tomorrow at the shore from two different experts who take into account the contributing probabilities of good weather holding, a calm sea state, and the enemy having an adequate supply of type 1 weapons. In the simplest kind of modeling the experts may provide their respective probabilities as weighted sums of contributing probabilities of possibly non-overlapping events

$$(1.3)\qquad \begin{cases} \text{Model 1}: P(a) = (w_{11}P(c)) + (w_{12}P(d)) + (w_{13}P(e)) \\ \qquad\qquad\qquad \text{vs.} \\ \text{Model 2}: P(b) = (w_{21}P(c)) + (w_{22}P(d)) + (w_{23}P(e)) \end{cases}$$

where $0 \le w_{ij} \le 1$, $w_{i1} + w_{i2} + w_{i3} = 1$, $i = 1,2$; a = enemy attacks tomorrow, according to Expert 1; b = enemy attacks tomorrow, according to Expert 2; c = good weather will hold; d = calm sea state will hold; e = enemy has adequate supply of type 1 weapons.

Note again that c,d,e in general are not disjoint events so that the total probability theorem is not applicable here. It can be readily shown no solution exists independent of all choices of P in eq.(1.3) when a, b, c, d, e all belong literally to the *same* probability space.

Example 2. Models as conditional probabilities. Here,

$$(1.4)\qquad \begin{cases} \text{Model 1}: P(a) = P(c \mid d) \; (= P(c \,\&\, d) / P(d)) \\ \qquad\qquad\qquad \text{vs.} \\ \text{Model 2}: P(b) = P(c \mid e) \; (= P(c \,\&\, e) / P(e)) \end{cases}$$

Models 1 and 2 could represent, e.g., two inference rules "if d, then c", "if e, then c", or two posterior descriptions of parameter c via different data sources corresponding to events d,e. Lewis' Theorem ([22] -- see also comments in Section 2.1 here) directly shows that in general no possible a, b, c, d, e can exist in the same probability space, independent of the choice of probability measure.

Example 3. Models as natural language descriptions. In this case, two experts independently provide their opinions concerning the same situation of interest: namely the description of an enemy ship relative to length and visible weaponry of a certain type.

$$(1.5)\quad \begin{cases} \text{Model 1 : Ship A is very long, or has a large number of q - type weapons on deck} \\ \qquad\qquad\qquad\text{vs.} \\ \text{Model 2 : Ship A is fairly long, or if intelligence source 1 is reasonably accurate,} \\ \qquad\qquad \text{it has a medium quantity of q - type weapons on deck} \end{cases}$$

Translating the natural language form in eq.(1.5) to fuzzy logic form [4]:

$$(1.6)\quad \begin{cases} \text{Model 1}: \ t(a) = (\mathrm{f}_{\mathrm{long}}(\mathrm{lngth}(A)))^2 \ \mathrm{v}_1 \ \mathrm{f}_{\mathrm{large}}(\#(Q)) \\ \qquad\qquad\text{vs.} \\ \text{Model 2}: \ t(b) = (\mathrm{f}_{\mathrm{long}}(\mathrm{lngth}(A)))^{1.5} \ \mathrm{v}_2 \ (\mathrm{f}_{\mathrm{medium}} \mid \mathrm{f}_{\mathrm{accurate}})(\#(Q), L) \end{cases}$$

where v_1 and v_2 are appropriately chosen fuzzy logic disjunction operators over $[0,1]^2$. Also, (non-italic) $\mathrm{f}_c{:}D \to [0,1]$ denotes a fuzzy set membership function corresponding to attribute c; $\mathrm{f}_{\mathrm{long}}$:pos.reals$\to [0,1]$, $\mathrm{f}_{\mathrm{large}}$:pos.reals $\to [0,1]$, $\mathrm{f}_{\mathrm{medium}} : \{0,1,2,3,...\} \to [0,1]$, $\mathrm{f}_{\mathrm{accurate}}$: class of intelligence sources $\to [0,1]$ are appropriately determined fuzzy set membership functions *representing* the attributes "long", "large", "medium", "accurate", respectively;

$(\mathrm{f}_{\mathrm{medium}} \mid \mathrm{f}_{\mathrm{accurate}}){:}\{0,1,2,3,...\} \times$ class of intelligence sources $\to [0,1]$ is a *conditional* fuzzy set membership function (to be considered in more detail in Section 5) representing the "if-then" statement. Also, $t(a)$ = truth or possibility of the description of ship A using Model 1; $t(b)$ = truth or possibility of the description of ship A using Model 2; where A = ship A, Q = collection of q-type weapons on deck of A, L = intelligence source l; and measurement functions lngth( ) = length of ( ) in feet, #( ) = no. of ( ).

In this example the issue of determining whether one could find actual events a,b, such that they and all contributing events lie in the same probability space requires first the conversion of the fuzzy logic models in eq.(1.6) to probability form. This is seen to be possible for both the unconditional and conditional cases via the one point random set coverage representation of fuzzy sets and certain fuzzy logics. (For the unconditional case, see Section 4; for the conditional case see Section 5.) Thus, Lewis' result again is applicable, showing a negative answer to the above question.

Hence, all three examples again point up the need -- if such constructions can be accomplished -- to obtain an appropriate probability space *properly extending* the original given one where events a, b can be found, as well as the isomorphic imbedding of the contributing events (but, not the original events !) in eq.(1.1), independent of the choice of the given probability measure.

1.4. Probability distance functions. We summarize here some basic candidate probability distance functions $M_P(a,b)$ for any events a, b belonging to a probability space (Ω,B,P). The absolute distance $D_{P_o}(a,b)$ has already been defined in eq.(1.2) for a,b assumed to belong to a probability space (Ω_o,B_o,P_o) extending (Ω,B,P). For simplicity here, we consider any a, b belonging to probability space (Ω,B,P). First, the *naïve distance* $N_P(a,b)$ is given by the absolute difference of probabilities

$$(1.7) \qquad N_P(a,b) = |P(a)-P(b)| = |P(a'\& b) - P(a \& b')| .$$

Here, there is no need to determine $P(a\&b)$ and clearly N_P is a pseudometric relative to probability space (Ω,B,P). On the other hand, a chief drawback is that there can be many events a, b which have probabilities near a half and are either disjoint or close to being disjoint, yet $N_P(a,b)$ is small, while $D_P(a,b)$ for such cases remains appropriately high. However, a drawback for the latter occurs when, in addition to a,b being nearly disjoint, both events have low probabilities, in which case $D_P(a,b)$ remains small, not reflecting the distinctness between the events. A perhaps more satisfactory distance function for this situation is the relative probability distance $R_P(a,b)$ given as

$$\begin{aligned}(1.8)\ R_P(a,b) &= d_P(a,b)\,/P(a \vee b) = (P(a)+P(b)-2P(a\&b))\,/(P(a)+P(b)-P(a\&b)),\\ &= (P(a'\& b) + P(a \& b'))\,/\,(P(a'\& b) + P(a \& b') + P(a \& b))\end{aligned}$$

noting that the last example of near-disjoint small probability events yields a value of $R_P(a,b)$ close to unity, not zero as for $D_P(a,b)$. Note also in eqs.(1.7) and (1.8) the existence of both the relative atom and the marginal-conjunction forms. It can be shown (using a tedious relative atom argument) that R_P is also a pseudometric relative to probability space (Ω,B,P), just as D_P is. (Another probability "distance" function is a symmetrization of conditional probability [13].)

Various tradeoffs for the use of each of the above functions can be compiled. In addition, one can pose a number of questions concerning the characterization of these and possibly other probability distance functions ([13], Sects.1,2).

1.5. Additional properties of probability distance functions and tests of hypotheses. First, it should be remarked that eqs.(1.2) and (1.8), again point out that full computations of $D_P(a,b)$, $R_P(a,b)$, $E_P(a,b)$ (but, of course not $N_P(a,b)$) require knowledge of the two marginal probabilities $P(a)$, $P(b)$, *as well as the conjunctive probability* $P(a \& b)$. When the latter is missing, we can consider the well-known extended Fréchet-Hailperin tightest bounds [20], [3] in terms of the marginal probabilities for $P(a\&b)$ and $P(a \vee b)$:

$$\begin{aligned}(1.9) \qquad & \max(P(a) + P(b) - 1, 0) \le P(a \& b) \le \min(P(a),P(b))\\ & \le wP(a) + (1-w)P(b)\\ & \le \max(P(a),P(b)) \le P(a \vee b) \le \min(P(a) + P(b), 1),\end{aligned}$$

for any weight w, $0 \leq w \leq 1$. In turn, applying inequality (1.9) to eqs.(1.2) and (1.8) yields the corresponding tightest bounds on the computations of the probability distance functions as

$$(1.10) \qquad N_P(a,b) \leq D_P(a,b) \leq \min(P(a)+P(b), 2-P(a)-P(b)),$$

$$(1.11) \qquad 1 - \min(P(a)/P(b), P(b)/P(a)) \leq R_P(a,b) \leq \min(1, 2-P(a)-P(b)).$$

Inspection of inequalities (1.10) and (1.11) shows that considerable errors can be made in estimating the probability distance functions when $P(a\&b)$ is not obtainable. In effect, one of the roles played by CEA and REA is to address this issue through the determination of such conjunctive probabilities when a and b represent complex models as in Examples 1-3. (See Section 6.)

Eqs.(1.2), (1.7), and (1.8) also show that these probability distance functions can be expressed as functions of the relative atomic forms $P(a \,\&\, b)$, $P(a' \& b)$, $P(a \,\&\, b')$. Then, making a basic higher order probability assumption that these three quantities can be considered also with respect to different choices of P, a, and b as random variables are jointly uniformly distributed over the natural simplex of values $\{(s,t,u): 0 \leq s,t,u \leq 1 \,, s+t+u \leq 1\} \subseteq [0,1]^3$, when $a \neq b$, one can then readily derive by standard transformation of probability techniques the corresponding cdf F_M for each function $M = N, D, R, E$. Thus,

$$(1.12) \qquad F_N(t) = 1-(1-t)^3, \qquad F_D(t) = t^2(3-2t), \qquad F_R(t) = t^2,$$

for all $0 \leq t \leq 1$. To apply the above to testing hypotheses, we simply proceed in the usual way, where the null hypothesis is H_0: $a \neq b$ and the alternative is H_1: $a = b$. Here, for any observed (i.e., fully computable) probability distance function $M_P(a,b)$, we

$$(1.13) \qquad \begin{cases} \text{accept } H_0 \text{ (and reject } H_1\text{) iff } M_P(a,b) > C_\alpha \,, \\ \text{accept } H_1 \text{ (and reject } H_0\text{) iff } M_P(a,b) \leq C_\alpha \end{cases}$$

where threshold C_α is pre-determined by the significance (or type-one error) level

$$(1.14) \qquad \alpha = P(\text{reject } H_0 \mid H_0 \text{ true}) = F_{M_P}(C_\alpha).$$

Thus, for all similar tests using the same statistic outcome $M_P(a,b)$, but possibly differing significance levels α (and hence thresholds C_α), considering the fixed significant level

$$(1.15) \qquad \alpha_0 = F_m(M_P(a,b)) = P(\text{reject } H_0 \text{ using } M_P(a,b) \mid H_0 \text{ holds}),$$

$$(1.16) \qquad \begin{cases} \text{If significance level } \alpha < \alpha_0, \text{ then accept } H_0, \\ \text{If significance level } \alpha > \alpha_0, \text{ then accept } H_1. \end{cases}$$

2. Conditional event algebra. Conditional event algebra is concerned with the following problem: Given any probability space (Ω, B, P), find a space (B_0, P_0) and an associated mapping ψ: $B^2 \to B_0$ — with B_0 (and well-defined operations over it representing conjunction, disjunction, and negation in some sense) and ψ not dependent on any particular choice of P — such that $\psi(.,\Omega)$: $B \to B_0$ is an isomorphic imbedding and the following compatibility condition holds with respect to conditional probability:

$$P_0(\psi(a,b)) = P(a|b)\text{ , for all } a, b \text{ in } B, \text{ with } P(b) > 0. \tag{2.1}$$

When $(B_0, P_0; \psi)$ exists, call it a *conditional event algebra* (CEA) extending (Ω, B, P) and call each $\psi(a,b)$ a *conditional event.* For convenience, use the notation $(a|b)$ for $\psi(a,b)$. When Ω_0 exists such that (Ω_0, B_0, P_0) is a probability space extending (Ω, B, P), call $(\Omega_0, B_0, P_0; \psi)$ a *boolean* CEA extension. Finally, call any boolean CEA extension with $(\Omega_0, B_0, P_0) = (\Omega, B, P)$ a *trivializing* extension.

A basic motivation for developing CEA is as follows: Note first that a natural numerical assignment of uncertainty to an inference rule or conditional statement in the form "if b, then a" (or "a, given b", or perhaps even "a is partially caused by b") when a,b are events (denoted usually as the consequent, antecedent, respectively) belonging to a probability space (Ω, B, P) is the conditional probability $P(a|b)$, i.e., formally (noting a can be replaced by $a\&b$)

$$P(\text{if } b, \text{ then } a) = P(a|b). \tag{2.2}$$

(A number of individuals have considered this assignment as a natural one. See, e.g., Stalnaker [29], Adams [1], Rowe [26].) Then, analogous to the use of ordinary unconditional probability logic, which employs probability assignments for any well-defined logical / boolean operations on events, it is also natural to inquire if a "conditional probability logic" (or CEA), can be derived, based on sound principles, which is applicable to inference rules.

2.1. Additional general comments. For the following special cases, the problem of constructing a specific CEA extension of the original probability space can actually be avoided:

1. *All antecedents are identical, with consequents possibly varying.* In this case, the traditional development of conditional probability theory is adequate to handle computations such as the conjunctions, disjunctions and negations applied to the statements "if b, then a", "if b, then c", where, similar to the interpretation in eq.(2.2), assuming $P(b) > 0$

$$P(\text{if } b, \text{ then } a) = P(a|b) = P_b(a)\text{ ,}\quad P(\text{if } b, \text{ then } c) = P(c|b) = P_b(c)\text{ ;} \tag{2.3}$$

$$P((\text{if } b, \text{ then } a)\&(\text{if } b, \text{ then } c)) = P(a\&c\,|b) = P_b(a\&c)\text{ ,} \tag{2.4}$$

$$P((\text{if } b, \text{ then } a)\text{v}(\text{if } b, \text{ then } c)) = P(a\text{v}c\,|b) = P_b(a \text{ v } c)\text{ ,} \tag{2.5}$$

$$P(\text{not}(\text{if } b, \text{ then } a)) = P(a'|b) = P_b(a')\text{ ,} \tag{2.6}$$

where P_b is the standard notation for the conditional probability operator $P(.|b)$, a legitimate probability measure over all of B, but also restricted, without loss of generality, to the trace boolean (or sigma) algebra $b\&B = \{b\&d: d \text{ in } B\}$. A further special case of this situation is when all antecedents are identical to Ω, so that all unconditional statements or events such as a, c can be interpreted as the special conditionals "if Ω, then a", "if Ω, then c", respectively.

2. *Conditional statements are assumed statistically independent, with differing antecedents.* When it seems intuitively obvious that the conditional expressions in question should not be considered dependent on each other, one can make the reasonable assumption that, with the analogue of eq.(2.3) holding,

(2.7) $\qquad P(\text{if } b, \text{ then } a) = P(a|b)\,, \quad P(\text{if } d, \text{ then } c) = P(c|d)\,,$

the usual laws of probability are applicable and

(2.8) $\qquad P((\text{if } b, \text{ then } a)\&(\text{if } d, \text{ then } c)) = P(a|b)P(c|d)\,,$

(2.9) $\qquad P((\text{if } b, \text{ then } a)\text{v}(\text{if } d, \text{ then } c)) = P(a|b) + P(c|d) - (P(a|b)P(c|d)).$

A reasonable sufficient condition to assume independence of the conditional expressions is that $a\&b$, b are P-independent of $c\&d$, d (for each of four possible combinations of pairs).

3. *While the actual structure of a specific CEA may not be known, it is reasonable to assume that a boolean one exists, so that all laws of probability are applicable.* Rowe tacitly makes this assumption in his work ([26], Chapter 8) in applying parts of the Fréchet-Hailperin bounds -- as well as further P-independence reductions in the spirit of Comment 2 above. Thus, inequality (1.9) applied formally to conditionals "if b, then a", "if d, then c" with compatibility relation (2.7) yields the following bounds in terms of the marginal conditional probabilities $P(a|b)$, $P(c|d)$, assuming $P(b)$, $P(d) > 0$:

(2.10) $\qquad \max(P(a|b) + P(c|d) - 1, 0) \leq P((\text{if } b, \text{ then } a)\&(\text{if } d, \text{ then } c))$
$\leq \min(P(a|b), P(c|d)) \leq (w\,P(a|b)) + ((1-w)P(c|d))$
$\leq \max(P(a|b), P(c|d)) \leq P((\text{if } b, \text{ then } a)\text{v}(\text{if } d, \text{ then } c)) \leq \min(P(a|b) + P(c|d), 1).$

Again, apropos to earlier comments, inspection of eq.(2.10) shows that considerable errors can arise in not being able to determine specific probabilistic conjunctions and disjunctions via some CEA.

At first glance one may propose that there already exists a candidate within classical logic which can generate a trivializing CEA: the material conditional operator $\Rightarrow$, where, as usual, for any two events a,b belonging to probability space (Ω,B,P), $b \Rightarrow a = b'\text{v}\, a = b'\text{v}\,(a \,\&\, b)$. However, note that [7]

(2.11) $\qquad P(b \Rightarrow a) = 1 - P(b) + P(a\&b) = P(a|b) + (P(a'|b)P(b')) \geq P(a|b),$

with strict inequality holding in general, unless $P(b) = 1$ or $P(a|b) = 1$. In fact, Lewis proved the fundamental negative result: (See also the recent work [5].)

THEOREM 2.1. (D. Lewis [22]) *In general, there does not exist any trivializing CEA.* □

Nevertheless, the above result does not preclude non-trivializing boolean and other CEA's from existing. Despite many positive properties [19], the chief drawback of previously proposed CEA (all using three-valued logic approaches) is their non-boolean structure, and consequent incompatibility with many of the standard laws and extensive results of probability theory. For a history of the development of non-boolean CEA up to five years ago, again see Goodman [7].

2.2. PSCEA: a non-trivializing boolean CEA. Three groups independently derived a similar non-trivializing boolean CEA: Van Fraasen, utilizing "Stalnaker Bernoulli conditionals" [30]; McGee, motivated by utility/rational betting considerations [23]; and Goodman & Nguyen [17], utilizing an algebraic analogue with arithmetic division as an infinite series and following up a comment of Bamber [2] concerning the representation of conditional probabilities as unconditional infinite trial branching processes. In short, this CEA, which we call the product space CEA (or PSCEA, for short), is constructed as follows (see [18] for further details and proofs): Let (Ω,B,P) be a given probability space. Then, form its extension (Ω_0,B_0,P_0) and mapping $\psi = (.|..): B^2 \to B_0$ by defining (Ω_0,B_0,P_0) as that product probability space formed out of a countable infinity of copies of (Ω,B,P) (as its identical marginal). Hence, $\Omega_0 = \Omega \times \Omega \times \Omega \times\ldots$; B_0 = sigma algebra generated by $(B \times B \times B \times\ldots)$, etc. Define also, for any a, b in B, the conditional event $(a\,|b)$ as:

$$(2.12)\qquad (a\,|b) = (a\,\&b\,|b) = \bigvee_{j=0}^{+\infty} \left((b\,')^j \times (a\,\&b) \times \Omega_0\right) \qquad \text{(direct form)}$$

$$(2.13)\qquad = \bigvee_{j=0}^{k-1} \left((b\,')^j \times (a\,\&b)\times\Omega_0\right) \vee \left((b\,')^k\times(a\,|b)\right),\ k=1,2,3,\ldots \quad \text{(recursive form)},$$

where the exponential-cartesian product notation holds for any $c,d \in B$

$$(2.14)\qquad c^j \times d^k = \begin{cases} \underbrace{c \times c \times \ldots \times c}_{j \text{ factors}} \times \underbrace{d \times d \times \ldots \times d}_{k \text{ factors}}, & \text{if } j,k \text{ are positive integers} \\ \underbrace{c \times c \times \ldots \times c}_{j \text{ factors}}, & \text{if } j \text{ is a positive integer and } k = 0 \\ \underbrace{d \times d \times \ldots \times d}_{k \text{ factors}}, & \text{if } j = 0 \text{ and } k \text{ is a positive integer,} \end{cases}$$

It follows from (2.12) that the ordinary membership function $\phi(a\,|b):\Omega \to \{0,1\}$ (unlike the three-valued ones corresponding to previously proposed CEA) corresponding to $(a|b)$ is given for any $\underline{\omega} = (\omega_1, \omega_2, \omega_3,\ldots)$ in Ω_0, where for any positive integer j, (2.15) implies (2.16):

$$(2.15) \quad \phi(b)(\omega_1) = \phi(b)(\omega_2) = \ldots = \phi(b)(\omega_{j-1}) = 0 < \phi(b)(\omega_j)\ (=1),$$
$$(2.16) \quad \phi(a \mid b)(\underline{\omega}) = \phi(a \mid b)(\omega_j)\ (= \phi(a)(\omega_j)).$$

The natural isomorphic imbedding here of B into B_0 is simply:

$$(2.17) \quad a \leftrightarrow (a \mid \Omega) = a \times \Omega_0, \text{ for all } a \in B$$

and, indeed, for all $a,b \in B$ with $P(b) > 0$, $P_0((a \mid b)) = P(a \mid b)$.

2.3. Basic properties of PSCEA. A brief listing of properties of PSCEA is provided below, valid for any given probability space (Ω,B,P) and any $a,b,c,d, \in B$, all derivable from use of the basic recursive definition for k=1 in eq.(2.13) and the structure of (Ω_0,B_0,P_0) (again, see [18]):
(i) *Fixed antecedent combinations compatible with Comment 1 of Section 2.1.*

$$(2.18) \quad (a \mid b) \,\&\, (c \mid b) = (a \,\&c \mid b), \quad (a \mid b) \vee (c \mid b) = (a \vee c \mid b), \quad (a \mid b)' = (a' \mid b),$$
$$(2.19) \quad P_0((a \mid b) \,\&\, (c \mid b)) = P(a \,\&\, c \mid b), \quad P_0((a \mid b) \vee (c \mid b)) = P(a \vee c \mid b),$$
$$(2.20) \quad P_0((a \mid b)') = P(a' \mid b) = 1 - P(a \mid b) = 1 - P_0((a \mid b)).$$

(ii) *Binary logical combinations.* The following are all extendible to any number of arguments where

$$(2.21) \quad (a \mid b) \,\&\, (c \mid d) = (A \mid b \vee d), \quad (a \mid b) \vee (c \mid d) = (B \mid b \vee d) \text{ (formalisms)}$$
$$(2.22) \quad A = (a \,\&b \,\&c \,\&d) \vee ((a \,\&b \,\&d') \times (c \mid d)) \vee ((b' \&c \,\&d) \times (a \mid b)),$$
$$(2.23) \quad B = (a \,\&b) \vee (c \,\&d) \vee ((a' \&b \,\&d') \times (c \mid d)) \vee ((b' \&c' \&d) \times (a \mid b)),$$
$$(2.24) \quad P_0((a \mid b) \,\&\, (c \mid d)) = P_0(A)/P(b \vee d),$$
$$(2.25) \quad P_0((a \mid b) \vee (c \mid d)) = P_0(B)/P(b \vee d)$$
$$= P(a \mid b) + P(c \mid d) - P_0((a \mid b) \,\&\, (c \mid d)),$$
$$(2.26) \quad P_0(A) = P(a \,\&b \,\&c \,\&d) + (P(a \,\&b \,\&d')P(c \mid d)) + (P(b' \&c \,\&d)P(a \mid b)),$$
$$(2.27) \quad P_0(B) = P((a \,\&b) \vee (c \,\&d)) + (P(a' \&b \,\&d')P(c \mid d)) + (P(b' \&c' \&d)P(a \mid b)).$$

In particular, note the combinations of an unconditional and conditional

$$(2.28) \quad P_0((a \mid b) \,\&\, (c \mid \Omega)) = P(a \,\&b \,\&c) + (P(b' \&c)P(a \mid b)),$$
$$(2.29) \quad (a \mid b) \,\&\, (b \mid \Omega) = (a \,\&\, b \mid \Omega) \qquad \text{(modus ponens)}$$

whence $(a \mid b)$ and $(b \mid \Omega)$ are necessarily always P_0-independent.
(iii) *Other properties including: Higher order conditioning; Partial ordering, extending unconditional event ordering; Compatibility with probability ordering.*

2.4. Additional key properties of PSCEA. (Once more, see [18] for other results and all proofs.) In the following $a, b, c, d, a_j, b_j \in B$, $j = 1,\ldots,n$, $n = 1,2,3,\ldots$. Apropos to Comment 2 in Section 2.1 concerning the sufficiency assumption of independence of two conditional events), PSCEA satisfies:

(iv) *Sufficiency for P_o-independence.* If $a\&b$, b are (four-way) P-independent of $c\&d$, d, then $(a \mid b)$ is P_o-independent of $(c \mid d)$.

(v) *General independence property.* $(a_1 \mid b_1),\ldots,(a_n \mid b_n)$ is always P_o-independent of $(b_1 \vee \ldots \vee b_n \mid \Omega)$. (This can be extended to show $(a_1 \mid b_1),\ldots,(a_n \mid b_n)$ is always P_o-independent of any $(c \mid \Omega)$, where $c \geq b_1 \vee \ldots \vee b_n$ or $c \leq (b_1)'\&\ldots\&(b_n)'$.)

(vi) *Characterization of PSCEA among all boolean CEA:*

THEOREM 2.2 (Goodman & Nguyen [18], pp. 296-301) *Any boolean CEA which satisfies modus ponens and the general independence property must coincide with PSCEA, up to probability evaluations of all well-defined finite logical combinations (under* &, v, ( )*'*) *of conditional events.* □

(vii) *Compatibility between conditioning of measurable mappings and PSCEA conditional events.* Let $(\Omega_1, B_1, P_1) \xrightarrow{Z} (\Omega_2, B_2, P_1 \circ Z^{-1})$ indicate that $Z\colon \Omega_1 \to \Omega_2$ is a (B_1,B_2)-measurable mapping which induces probability space $(\Omega_2,B_2,P_1\circ Z^{-1})$ from probability space (Ω_1,B_1,P_1). When some P_2 is used in place of $P_1\circ Z^{-1}$, it is understood that $P_2 = P_1\circ Z^{-1}$. Also, still using the notation $(\Omega_o,B_o,P_o;(.\mid..))$ to indicate the PSCEA extension of probability space (Ω,B,P), define the *PSCEA extension of* Z to be the mapping $Z_o\colon (\Omega_1)_o \to (\Omega_2)_o$, where

(2.30) $\quad Z_o(\underline{\omega}) = (Z(\omega_1), Z(\omega_2), Z(\omega_3),\ldots)$, for all $\underline{\omega} = (\omega_1, \omega_2, \omega_3,\ldots) \in (\Omega_1)_o$.

LEMMA 2.1. (Restatement of Goodman & Nguyen [18], Sects. 3.1, 3.4) *If* $(\Omega_1, B_1, P_1) \xrightarrow{Z} (\Omega_2, B_2, P_1 \circ Z^{-1})$ *holds, then does* $((\Omega_1)_o, (B_1)_o, (P_1)_o) \xrightarrow{Z_o} ((\Omega_2)_o, (B_2)_o, (P_1 \circ Z^{-1})_o)$ *hold (where we can naturally identify* $(P_1)_o\circ Z_o^{-1}$ with $(P_1\circ Z^{-1})_o$ *) and say that* $(\;)_o$ *lifts* Z *to* Z_o. □

Next, replace Z by joint measurable mapping X,Y, in eq.(2.30), where (Ω_1,B_1,P_1) is simply (Ω,B,P), Ω_2 is replaced by $\Omega_1\times\Omega_2$, B_2 by sigma algebra generated by $(B_1 \times B_2)$, and define

(2.31) $$(X,Y)(\omega) = (X(\omega),Y(\omega)), \text{ for any } \omega \in \Omega.$$

Thus, we have the following commutative diagram:

(2.32)
$$\begin{tikzcd}
 & (\Omega_1, B_1, P\circ X^{-1}) & \\
(\Omega,B,P) \arrow[ur, "X"] \arrow[rr, "{(X,Y)}"] \arrow[dr, "Y"'] & & (\Omega_1\times\Omega_2, \mathrm{sigma}(B_1\times B_2), P\circ(X,Y)^{-1}) \arrow[ul, "\mathrm{proj}_1"'] \arrow[dl, "\mathrm{proj}_2"] \\
 & (\Omega_2, B_2, P\circ Y^{-1}) &
\end{tikzcd}$$

Finally, consider any $a \in B_1$ and $b \in B_2$ and corresponding conditional event in the form $(a \times b \mid \Omega_1 \times b)$. Then, the following holds, using Lemma 2.1 and the basic structure of PSCEA:

THEOREM 2.3. (Clarification of Goodman & Nguyen [18], Sects. 3.1, 3.4) *The commutative diagram of arbitrary joint measurable mappings in eq.(2.32) lifts to the commutative diagram*

$$
(2.33)\qquad
\begin{tikzcd}
 & & ((\Omega_1)_o, (B_1)_o, (P\circ X^{-1})_o) & \\
(\Omega_o, B_o, P_o) \arrow[urr, "X_o"] \arrow[rrr, "(X,Y)_o"] \arrow[drr, "Y_o"'] & & & ((\Omega_1 \times \Omega_2)_o, (\mathrm{sigma}(B_1 \times B_2))_o, (P\circ(X,Y)^{-1})_o) \arrow[ul, "\mathrm{proj}_1"'] \arrow[dl, "\mathrm{proj}_2"] \\
 & & ((\Omega_2)_o, (B_2)_o, (P\circ Y^{-1})_o) &
\end{tikzcd}
\quad ,
$$

where, we can naturally identify $(\Omega_1 \times \Omega_2)_o$ *with* $(\Omega_1)_o \times (\Omega_2)_o$, $(\mathrm{sigma}(B_1 \times B_2))_o$ *with* $\mathrm{sigma}((B_1)_o \times (B_2)_o)$, $(P\circ X^{-1})_o)$ *with* $P_o \circ X_o^{-1}$, $(P\circ Y^{-1})_o)$ *with* $P_o \circ Y_o^{-1}$, *and* $(P\circ(X,Y)^{-1})_o$ *with* $P_o \circ (X_o, Y_o)^{-1}$. *Moreover, the basic compatibility relations always hold between conditioning of measurable mappings in unconditional events and joint measurable mappings in conditional events, all* $a \in B_1$, $b \in B_2$*:*

$$
\begin{aligned}
(2.34)\quad & P(X \in a \mid Y \in b) \quad (= P(X^{-1}(a) \mid Y^{-1}(b))) \\
& = P_o((X,Y)_o \in (a \times b \mid \Omega_1 \times b)) \quad (= P_o((X,Y)_o^{-1}((a \times b \mid \Omega_1 \times b)))). \quad \square
\end{aligned}
$$

3. Relational event algebra. The relational event algebra (REA) problem was stated informally in Section 1.1. More rigorously, given any two functions $f,g:[0,1]^m \to [0,1]$, any probability space (Ω,B,P), find a probability space (Ω_o,B_o,P_o) extending (Ω,B,P) isomorphically (with B_o not dependent on P) and find mappings $a(f)$, $b(g)$: $B^m \to B_o$ such that the formal relations in eq.(1.1) are solvable where on the left-hand side P is replaced by P_o, a by $a(f)(c,d,e,h,\ldots)$, b by $b(g)(c, d, e, h,\ldots)$, with possibly some constraint on the c,d,e,h ,... in B and the class of probability functions P. Solving the REA problem can be succinctly put as determining the commutative diagram:

$$
(3.1)\qquad
\begin{tikzcd}
B^m \arrow[r, "{a(f)\,,\,b(g)}"] \arrow[d, "{(P(.),\, P(..),P(...),\ldots)}"'] & B_o \arrow[d, "P_o"] & \mathrm{FIND} \arrow[l] \\
{[0,1]^m} \arrow[r, "{f,g}"] & {[0,1]} &
\end{tikzcd}
$$

As mentioned before, the CEA problem is that special case of the REA problem, where f and g are each ordinary division of two probabilities with the restriction that each pair of probabilities corresponds to the first event being a subevent of the second. (See beginning of Section 2 and eq.(2.1).) Other special cases of the REA problem that have been treated include f, g being: constant-valued; weighted linear functions in multiple (common) arguments; polynomials or infinite series in one common argument; exponentials in one common argument; min and max. The first second and last case are considered here; further details for all but the last case can be found in [13].

3.1. REA problem for constant-valued functions. To begin with, it is obvious that for any given measurable space (Ω,B) other than the events $\emptyset$ and Ω, there do not exist any other constant-probability-valued events belonging to probability space (Ω,B,P), independent of all possible choices of P. However, by considering the extensions (Ω_o, B_o, P_o), in a modified sense such events can be constructed. Consider first eq.(1.1) where f and g are any constants. Let probability space (Ω,B,P), be given as before and consider any real numbers s, t in [0,1]. Next, independent of the choice of s, t, pick a fixed event, say, c in B, with $0 < P(c) < 1$ and define for any integers $1 \leq j \leq k \leq n$,

$$\mu(j,n;c) = c^{j-1} \times c' \times c^{n-j}, \tag{3.2}$$

utilizing notation similar to that in eq.(2.14). Note that all $\mu(j,n;c)$, as j varies, are mutually disjoint with the identical product probability evaluation

$$P_n(\mu(j,n;c)) = (P(c))^{n-1}P(c'), \tag{3.3}$$

where product probability space $(\Omega^n,\text{sigma}(B^n),P_n)$ has n marginal spaces, each identical to (Ω,B,P) with PSCEA extension $((\Omega^n)_o,(\text{sigma}(B^n))_o,(P_n)_o)$. Then, consider the following conditional event with evaluation due to eq.(3.3):

$$\theta(j,k,n;c) = \Big(\bigvee_{i=j+1}^{k} \mu(i,n;c) \,\Big|\, \bigvee_{i=1}^{n} \mu(i,n;c) \Big); \quad (P_n)_o(\theta(j,k,n;c)) = (k-j)/n\,. \tag{3.4}$$

In addition, it is readily shown that for any fixed n, as j, k vary freely, $1 \leq j \leq k \leq n$, the set of all finite disjoint unions of *constant-probability* events $\theta(j,k,n;c))$ is closed with respect to all well-defined logical combinations for $((\Omega^n)_o,(\text{sigma}(B^n))_o,(P_n)_o)$ and is not dependent upon the particular choice of integer n, event $c \in B$, nor probability P, except for the assumption $0 < P(c) < 1$. In order for there to exist constant-probability events which act in a universal way -- analogous to the role that the boundary constant-probability events $\emptyset$ and Ω_o play -- to accommodate all possible values of s, t simultaneously, we must let $n \to +\infty$ (or be sufficiently large). One way of accomplishing this is to first form for each value of n, a new product probability space with first factor being (Ω_o, B_o, P_o) and second factor being $((\Omega^n)_o,(\text{sigma}(B^n))_o,(P_n)_o)$ and then formally allow n to approach $+\infty$. By a slight abuse of notation, we will formally identify this space and limiting process with (Ω_o, B_o, P_o). Finally, with all of the above stated, we can choose any sequences of rationals converging to s, t, such as $(j_n/n)_{n=1,2,3,...} \to s$, $(k_n/n)_{n=1,2,3,...} \to t$ and define

(3.5) $$\theta(s,t) = \lim_{n \to +\infty} \theta(j_n,k_n,n;c)\ , \quad \theta(t) = \theta(0,t),$$

with the convention, boundary values, and evaluations

(3.6) $$\theta(s,t) = \emptyset \text{ , if } s \geq t\ , \quad \theta(0) = \emptyset\ , \quad \theta(1) = \Omega_o\ ,$$
(3.7) $$P_o(\theta(s,t)) = \max(t\text{-}s,0) \quad , \quad P_o(\theta(t)) = t \text{ , all } 0 \leq s,t \leq 1.$$

Hence, an REA solution to $f = s$ and $g = t$ constants in $[0,1]$, is simply to choose $a(f) = \theta(s)$, $b(g) = \theta(t)$. A summary of logical combination properties of such "constant-probability" events $\theta(s,t)$ and $\theta(t)$ are given below:
For all $0 \leq s_j \leq t_j \leq 1$, $0 \leq s \leq t \leq 1$, all c_i, d_j in B, i=1,...,m, j=1,...,n, $m \leq n$,

(3.8) $$\theta(s_1,t_1) \;\&\; \theta(s_2,t_2) = \theta(\max(s_1,s_2), \min(t_1,t_2)),$$
(3.9) $$\theta(t_1) \;\&\; \theta(t_2) = \theta(\min(t_1,t_2)), \qquad \theta(t_1) \vee \theta(t_2) = \theta(\max(t_1,t_2)),$$
(3.10) $$(\theta(t))' = \theta(t,1), \qquad (\theta(s))' \;\&\; \theta(t) = \theta(s,t),$$
(3.11) $$(c_1 \times ... \times c_m \times \theta(s_1,t_1)) \;\&\; (d_1 \times ... \times d_n \times \theta(s_2,t_2))$$
$$= (c_1 \& d_1) \times ... \times (c_m \& d_m) \times d_{m+1} \times ... \times d_n \times (\theta(s_1,t_1) \;\&\; \theta(s_2,t_2))\ ,$$

with all obvious corresponding probability evaluations by P_o.

3.2. REA problem for weighted linear functions. Consider the REA problem where, as before, (Ω,B,P) is a given probability space with $c_j \in B$, j=1,...,m. For all $\underline{t} = (t_1,...,t_m)$ in $[0,1]^m$, now define

(3.12) $$f(\underline{t}) = w_{11}t_1 + ... + w_{1m}t_m\ , \qquad g(\underline{t}) = w_{21}t_1 + ... + w_{2m}t_m$$
(3.13) $$0 \leq w_{ij} \leq 1\ , \; w_{i1} + ... + w_{im} = 1\ , \; i=1,2, \quad j=1,...,m.$$

The following holds for any real w_j, with disjoint $c_{\underline{q}}$ replacing non-disjoint c_j :

(3.14) $$\sum_{j=1}^{m} P(c_j) \cdot w_j = \sum_{\underline{q} \in J_m} P(c_{\underline{q}}) \cdot w_{\underline{q}}\ ; \qquad J_m = \{\emptyset,\Omega\}^m \text{-} \{(\Omega,...,\Omega)\},$$

$$\underline{q} = (q_1,...,q_m)\ , \quad w_{\underline{q}} = \sum_{\{j:\ 1 \leq j \leq m \text{ and } q_j = \emptyset\}} w_j \qquad , c_{\underline{q}} = (c_1 + q_1) \& ... \& (c_m + q_m).$$

Then, the REA solution for this case using eq.(3.14) and constant-probability events as constructed in the last section is seen to consist of the following disjoint disjunctions of cartesian products

(3.15) $$a(f)(\underline{c}) = \bigvee_{\underline{q} \in J_m} c_{\underline{q}} \times \theta(w_{1\underline{q}})\ , \qquad b(g)(\underline{c}) = \bigvee_{\underline{q} \in J_m} c_{\underline{q}} \times \theta(w_{2\underline{q}})\ ;$$

$$\underline{c} = (c_1,...,c_m)\ , \quad w_{i\underline{q}} = \sum_{\{j:1 \leq j \leq m \text{ and } q_j = \emptyset\}} w_{ij} \quad .$$

Some logical combinations of REA solutions here:

(3.16) $a(f)(\underline{c}) \,\&\, b(g)(\underline{c}) = \bigvee_{q \in J_m} c_q \times \theta(\min(w_{1q}, w_{2q}))$,

(3.17) $a(f)(\underline{c}) \vee b(g)(\underline{c}) = \bigvee_{q \in J_m} c_q \times \theta(\max(w_{1q}, w_{2q}))$,

(3.18) $(a(f)(\underline{c}))' = \bigvee_{q \in J_m} c_q \times \theta(w_{1q}, 1) \vee (c_1' \,\&\ldots\&\, c_m')$.

A typical example of the corresponding probability evaluations for (Ω_o, B_o, P_o) is

(3.19) $P_o[a(f)(\underline{c}) \,\&\, b(g)(\underline{c})] = \sum_{\underline{q} \in J_m} c_q \min(w_{1q}, w_{2q}))$.

Applications of the above results can be made to weighted coefficient polynomials and series in one variable (replacing in eq.(3.15) c_j by c^{j-1}), as well as for weighted combinations of exponentials in one or many arguments; but computational problems arise for the latter unless special cases are considered, such as independence of terms, etc. (see [15]).

3.3 REA problem for min, max. In this case, we consider in eq.(1.1) REA solutions when one or both functions f, g involve minimum or maximum operations. For simplicity, consider f by itself in the form

(3.20) $f(s,t) = \max(s,t)$, for all s, t in $[0,1]$

and seek for all events c, d belonging to probability space (Ω, B, P), a relational event $a(f)(c, d)$ belonging to probability space (Ω_o, B_o, P_o) where for all P

(3.21) $P_o(a(f)(c, d)) = \max(P(c), P(d))$.

First, it should be remarked that it can be proven that we cannot apply any techniques related to Section 3.2 where the weights are *not* dependent on P. However, a reasonable *modified* REA solution is possible based on the idea of choosing weights dependent upon P resulting in the form d, when $P(c) < P(d)$, and c, when $P(d) < P(c)$, etc. Using the REA solution in Section 3.2, we have:

(3.22) $a(f)(c,d) = (c \,\&\, d) \vee ((c \,\&\, d') \times \theta(w_{P,1})) \vee ((c' \,\&\, d) \times \theta(w_{P,2}))$,

(3.23) $w_{P,1} = \begin{cases} 1, & \text{if } P(d) < P(c), \\ 0, & \text{if } P(c) < P(d), \\ w, & \text{if } P(c) = P(d) \end{cases} \quad w_{P,2} = \begin{cases} 0, & \text{if } P(d) < P(c), \\ 1, & \text{if } P(c) < P(d), \\ 1-w, & \text{if } P(c) = P(d) \end{cases}.$

Dually, the case for $f = \min$ is also solvable by this approach.

4. One point random set coverages and coverage functions. One point random set coverages (or hittings) and their associated probabilities, called one point random set coverage functions, are the weakest way to specify random sets, analogous to the role measures of central tendency play with respect to probability distributions. In this section we show there is a class of fuzzy logics and a corresponding class of joint random sets which possess "homomorphic-like" properties with respect to probabilities of certain logical combinations of one point random set coverages, thereby identifying a part of fuzzy logic as a weakened form of probability. In turn, this motivates the proposed definition for conditional fuzzy sets in Section 5. Previous work in this area can be found in [16], [9], [10].

4.1. Preliminaries. In the following development, we assume all sets D_j finite, $j \in J$, any finite index set, and repeatedly use the measurable map notation introduced in Sect. 2.4 (vii) applied to random sets and 0-1-valued random variables. As usual, the distinctness of a measurable map is up to the probability measure it induces. For any $x \in D_j$, denote the filter class on x with respect to D_j as $F_x(D_j) = \{c: x \in c \subseteq D_j\}$ and, as before, denote the ordinary set membership (or indicator) functional as ϕ. Denote the power class of any D_j as $\mathcal{P}(D_j)$ and the double power class of D_j as $\mathcal{P}\mathcal{P}(D_j)$. If S_j is any random subset of D_j, written as $(\Omega, B, P) \xrightarrow{S_j} (\mathcal{P}(D), \mathcal{P}\mathcal{P}(D), P \circ S_j^{-1})$, use the multivariable notation $S_J = (S_j)_{j \in J}$ to indicate a collection of joint random subsets S_j of D_j, $j \in J$. Similarly, define the corresponding collection of joint 0-1 random variables $\phi(S_J) = (\phi(S_j))_{j\in J}$, noting the (x,j)-marginal random variable here is $\phi(S_j)(x)$, for any $x \in D_j$, $j \in J$, yielding the relation
$S_J \leftrightarrow \phi(S_J) = (\phi(S_j)(x))_{x \in D_j, j \in J}$. Let $\dagger$ denote the separating union:

$$\dagger(D_J) = \bigcup_{j\in J}(D_j\times\{j\}), \qquad \dagger(\mathcal{P}(D_J)) = \bigcup_{j\in J}(\mathcal{P}(D_j)\times\{j\}). \tag{4.1}$$

The following commutative diagram summarizes the above relations for all $x \in D_j$, $j \in J$:

(4.2)
$$\begin{tikzcd}
(\Omega,B,P) \arrow[rr, "S_J"] \arrow[dr, "\phi(S_J)"] \arrow[ddr, "\phi(S_j)(x)"'] & & (\dagger(\mathcal{P}(D_J)),\ \mathcal{P}(\dagger(\mathcal{P}(D_J))), P\circ S_J^{-1}) \arrow[dl, "\phi"] \\
& (\{0,1\}^{\dagger(D_J)}, \mathcal{P}(\{0,1\}^{\dagger(D_J)}), P\circ(\phi(S_J))^{-1}) \arrow[d, "\mathrm{proj}_{x,j}"] & \\
& (\{0,1\}, \mathcal{P}(\{0,1\}), P\circ(\phi(S_j)(x))^{-1}) &
\end{tikzcd}$$

For each $x \in D_j$, note the equivalences of *one point random set coverages*:

$$(4.3)\qquad \begin{cases} x \in S_j \text{ iff } S_j^{-1}(F_x(D_j)) \text{ occurs iff } \phi(S_j)(x) = 1, \\ x \notin S_j \text{ iff } S_j^{-1}(F_x(D_j)) \text{ not occurs iff } \phi(S_j)(x) = 0, \end{cases}$$

and for each S_j define its *one point coverage function* $\mathrm{f}_j : D_j \to [0,1]$, a fuzzy set membership function, by

$$(4.4)\qquad \mathrm{f}_j(x) = P(x \in S_j) = P(S_j^{-1}(F_x(D_j))) = P((\phi(S_j))(x) = 1)\,.$$

The induced probability measure $P \circ S_J^{-1}$ through $P \circ \phi(S_J)^{-1}$ is completely determined by its corresponding joint probability function g_{f_J}, or equivalently, by its corresponding joint cdf F_{f_J}, where $\mathrm{f}_J = (\mathrm{f}_j)_{j \in J} = (\mathrm{f}_j(x))_{x \in D_j, j \in J}$. Also, use the multivariable notation $D_J = (D_j)_{j \in J}$, $\mathrm{f}_J(x_J) = (\mathrm{f}_j(x_j))_{j \in J}$, $x_J = (x_j)_{j \in J}$, $c_J = (c_j)_{j \in J}$, $D_J\text{-}c_J = (D_j\text{-}c_j)_{j \in J}$, etc. A copula, written cop:$[0,1]^n \to [0,1]$, is any joint cdf of one-dimensional marginal cdf's corresponding to the uniform distribution over $[0,1]$, compatibly defined for n=1,2,... (see [3]). It will also be useful to consider the cocopula or DeMorgan transform of cop, i.e., $\mathrm{cocop}(t_J) = 1 - \mathrm{cop}(1_J - t_J)$, for all $t_J \in [0,1]^J$. By Sklar's copula theorem [28],

$$(4.5)\qquad F_{\mathrm{f}_J}(\underline{t}) = \mathrm{cop}\big((F_{\mathrm{f}_j(x)}(t_{x,j}))_{x \in D_j, j \in J}\big),\quad \underline{t} = (t_{x,j})_{x \in D_j, j \in J} \in [0,1]^{\dagger D_J}$$

$$(4.6)\qquad F_{\mathrm{f}_j(x)}(t) = \begin{cases} 0, \text{ if } t < 0, \\ 1 - \mathrm{f}_j(x), \text{ if } 0 \le t < 1, \\ 1, \text{ if } 1 \le t. \end{cases} \qquad g_{\mathrm{f}_j(x)}(t) = \begin{cases} \mathrm{f}_j(x), \text{ if } t = 1, \\ 1 - \mathrm{f}_j(x), \text{ if } t = 0, \end{cases}$$

where, in particular,

$$(4.7)\qquad F_{\mathrm{f}_j(x)}(0) = g_{\mathrm{f}_j(x)}(0) = 1 - \mathrm{f}_j(x),\ \text{ all } x \in D_j, j \in J.$$

The following is needed :

LEMMA 4.1 (A version of the Möbius inversion formula [27]) *Given any finite nonempty set J and a collection of joint 0-1-valued r.v.'s* $T_J = (T_j)_{j \in J}$, $(\Omega, B, P) \xrightarrow{T_j} (\{0,1\}, \mathcal{P}(\{0,1\}), P \circ T_j^{-1})$ *with* $P(T_j{=}1) = \lambda_j$, $P(T_j{=}0) = 1 - \lambda_j$, $j \in J$, *and given any set* $L \subseteq J$, *with the convention of letting* $P(\underset{j \in \varnothing}{\&}(T_j = 0)) = \mathrm{cop}((\lambda_j)_{j \in \varnothing}) = 1$ *and* $\mathrm{cocop}((\lambda_j)_{j \in \varnothing}) = 0$. *Then,*

$$(4.8)\qquad P\big(\underset{j \in L}{\&}(T_j = 1) \,\&\, \underset{j \in J-L}{\&}(T_j = 0)\big) = \sum_{K \subseteq L} (-1)^{\mathrm{card}(K)} P\big(\underset{j \in K \cup (J-L)}{\&}(T_j = 0)\big)$$

$$= \overline{\mathrm{cop}}(\lambda_L;\lambda_{J-L}),$$

where we define

$$(4.9)\qquad \overline{\mathrm{cop}}(\lambda_L;\lambda_{J-L}) = \sum_{K\subseteq L}(-1)^{\mathrm{card}(K)}\mathrm{cop}\big((1-\lambda_j)_{j\in K\cup(J-L)}\big)$$

$$= \delta(L{=}\emptyset) + \sum_{K\subseteq L}(-1)^{\mathrm{card}(K)+1}\mathrm{cocop}\big((\lambda_j)_{j\in K\cup(J-L)}\big),$$

$\delta(. = ..)$ denoting the Krönecker delta function, noting the special cases

$$(4.10)\qquad \overline{\mathrm{cop}}(\lambda_J) = \overline{\mathrm{cop}}(\lambda_J;\lambda_\emptyset) = \sum_{\emptyset\neq K\subseteq J}(-1)^{\mathrm{card}(K)+1}\mathrm{cocop}\big((\lambda_j)_{j\in K}\big),$$

$$(4.11)\qquad \overline{\mathrm{cop}}(\lambda_\emptyset;\lambda_J) = 1\text{-}\mathrm{cocop}\big((\lambda_j)_{j\in J}\big) = \mathrm{cop}\big((1\text{-}\lambda_j)_{j\in J}\big). \qquad \square$$

4.2. Solution class of joint random sets one point coverage equivalent to given fuzzy sets. Next, given any collection of fuzzy set membership functions f_J, $f_j{:}D_j \to [0,1]$, $j \in J$, consider $\mathcal{S}(f_J)$, the class of all collections of joint random subsets $S(f_J)$ of D_J which are *one point coverage equivalent to* f_J, i.e., each $S(f_j)$ is any random subset of D_j, which *is one point coverage equivalent to* f_j, $j \in J$, i.e.,

$$(4.12)\qquad P(x \in S(f_j)) = f_j(x), \text{ for all } x \in D_j, j \in J,$$

compatible with the converse relation in eq.(4.4). It can be shown that when $f_j{=}\phi(a_j)$, $a_j \in B$, $j \in J$, then necessarily $\mathcal{S}(f_J) = \{a_J\}$, $a_J = (a_j)_{j\in J}$.

THEOREM 4.1 (Goodman [10]) *Given any collection of fuzzy sets* f_J, *as above, then* $\mathcal{S}(f_J)$ *is bijective to* $\phi(\mathcal{S}(f_J)) = \{\phi(S(f_J)): S(f_J) \in \mathcal{S}(f_J)\}$, *which is bijective to* $\{ F_{f_J} = \mathrm{cop}((F_{f_j(x)})_{x \in D_j, j \in J})$: $\mathrm{cop}{:}[0,1]^{\dagger D_J} \to [0,1]$ is arbitrary$\}$.

Proof. This follows by noting any choice of cop always makes F_{f_J} a legitimate cdf corresponding to fixed one point coverage functions f_J. $\square$

By applying Lemma 4.1 and eq.(4.7), the explicit relation between each $S(f_J)$ and choice of cop generating them is given for any $c_J{=}(c_j)_{j\in J}$, $c_j \in B$,

$$(4.13)\qquad P(S(f_J) = c_J) = P(\underset{j\in J}{\&}(S(f_j) = c_j))$$

$$= P(\underset{x\in c_j, j\in J}{\&}(\phi(S(f_j))(x) = 1) \,\&\, \underset{x\in D_j - c_j, j\in J}{\&}(\phi(S(f_j))(x) = 0))$$

$$= \overline{\mathrm{cop}}(f_{\dagger(c_J)}(x_{\dagger(c_J)}); f_{\dagger(D_J\text{-}c_J)}(x_{\dagger(D_J\text{-}c_J)})).$$

In particular, when cop = min, one can easily show (appealing, e.g., to the unique determination of cdf's for 0-1-valued r.v.'s by all joint evaluations at 0 – see Lemma 4.1 -- and by eq.(4.5)) this is equivalent to choosing the *nested* random sets $S(f_j) = f_j^{-1}[U,1] = \{x: x \in D_j, f_j(x) \geq U\}$ as the one point coverage equivalent random sets for the *same fixed* U, a uniformly distributed random variable over [0,1]. When cop = prod (arithmetic product) the corresponding collection of joint random sets is such that $S(f_J)$ corresponds to $\phi(S(f_J))$ being a collection of independent 0-1 r.v.'s, and hence $S(f_J)$ also corresponds to the maximal entropy solution in $\mathcal{S}(f_J)$.

4.3. Homomorphic-like relations between fuzzy logic and one point coverages. Call a pair of operators $\&_1, v_1:[0,1]^n \to [0,1]$, well-defined for n=1,2,3,.., a *fuzzy logic conjunction, disjunction pair*, if both operators are pointwise nondecreasing with $\&_1 \leq v_1$, and for any $0 \leq t_j \leq 1$, $j=1,..,n$, letting $r=t_1\&_1...\&_1 t_n$, $s=t_1 v_1 ... v_1 t_n$, if for any $t_j=0$, then $r=0$ and $s= t_1 v_1 .. v_1 t_{j-1} v_1 t_{j+1} v_1 .. v_1 t_n$, and if any $t_j = 1$, then $s = 1$ and $r = t_1\&_1...\&_1 t_{j-1}\&_1 t_{j+1}\&_1...\&_1 t_n$. Note that any cop,cocop pair qualifies as a fuzzy logic conjunction, disjunction pair, as does any *t-norm, t-conorm* pair (certain associative, commutative functions, see [16]). For example, (min,max), (prod, probsum) are two fuzzy conjunction, disjunction pairs which are also both cop, cocop and t-norm, t-conorm pairs, where probsum is defined as the DeMorgan transform of prod.

THEOREM 4.2. *Let* (cop, cocop) *be arbitrary. Then (referring to eq.(4.10):*
(i) ($\overline{\text{cop}}$, cocop) *is a conjunctive disjunctive fuzzy logic operation pair which in general is non-DeMorgan.*
(ii) *For any choice of fuzzy set membership functions symbolically,* $f_J:D_J \to [0,1]^J$ *and any* $S(f_J) \in \mathcal{S}(f_J)$ *determined through* cop *(as in Theorem 4.1), and any* $x_J \in D_J$,

$$\text{(4.14)} \quad \overline{\text{cop}}(f_J(x_J)) = P(\underset{j\in J}{\&}(x_j \in S(f_j))), \quad \text{cocop}(f_J(x_J)) = P(\underset{j\in J}{\vee}(x_j \in S(f_j))).$$

(iii) *If* (cop,cocop) *is any continuous t-norm, t-conorm pair, then* $\overline{\text{cop}}$ = cop *iff* (cop,cocop) *is either* (min,max), (prod, probsum), *or any ordinal sum of* (prod,probsum).

Proof. Part (i) follows from (ii). Part (ii) left-hand side follows from Lemma 4.1 with $T_j = \phi(S(f_j))$, $L=J$. (ii) right-hand side follows from the DeMorgan expansion of $P(\underset{j\in J}{\vee}(x_j \in S(f_j))) = 1 - P(\underset{j\in J}{\&}(\phi(S(f_j))(x_j)=0))$ and the last result. (iii) follows from [10], Corollary 2.1. □

The validity for $\overline{\text{cop}}$ = cop without using the sufficiency conditions in (iii) above are apparently not known at present. Call the family of all (cop,cocop) pairs listed in Theorem 4.2 (iii), the *semi-distributive* family (see [10]), because of additional properties possessed by them. Call the much larger family of all

($\overline{\text{cop}}$,cocop) pairs the *alternating signed sum* family. $\overline{\text{cop}}$ is a more restricted function of cocop than a modular transform (i.e., when $\overline{\text{cop}}$ is evaluated at two arguments). In fact, Frank has found a nontrivial family characterizing all pairs of t-norms and t-conorms which are modular to each other, a proper subfamily of which consists of also copula, cocopula pairs which, in turn, properly contains the semi-distributive family [6]. When we choose as a fuzzy logic conjunction disjunction pair $(\&_1,v_1) = (\overline{\text{cop}},\text{cocop})$, eq.(4.14) shows there is a "homomorphic-like" relation for conjunctions and disjunctions separately between fuzzy logic and corresponding probabilities of conjunction and disjunctions of one point random set coverages. It is natural to inquire what other classes of fuzzy logic operators produce homomorphic-like relations between various well-defined *combinations* of conjunctions and disjunctions with corresponding probabilities of combinations of one point random set coverages. One such has been determined:

THEOREM 4.3 (Goodman [10]) *Suppose $(\&_1,v_1)$ is any continuous conjunction, disjunction fuzzy logic pair. Then, the following are equivalent:*
(i) *For any choice of $f_{ij}:D_{ij}\to[0,1]$, $i=1,\dots,m$, $j=1,\dots,n$, $m,n \geq 1$, there is a collection of joint one point coverage equivalent random sets $S(f_{ij})$, $i=1,\dots,m$, $j=1,\dots,n$, such that for all $x_{ij} \in D_{ij}$, the homomorphic-like relation holds*:

$$\text{(4.15)} \qquad \mathop{\&_1}_{i=1}^{m}\Big(\mathop{\mathrm{V}_1}_{j=1}^{n}(f_{ij}(x_{ij}))\Big) = P\Big(\mathop{\&}_{i=1}^{m}\Big(\mathop{\mathrm{V}}_{j=1}^{n}(x_{ij} \in S(f_{ij}))\Big)\Big)$$

(ii) *Same statement as* (i), *but with* v_1 *over* $\&_1$ *and* v *over* &.
(iii) $(\&_1,v_1)$ *is any member of the semi-distributive family.* □

By inspection, it is easily verified (using, e.g., the nested random set forms corresponding to min and the mutual independence property corresponding to prod) that the fuzzy logic operator pairs $(\&_1,v_1)$ = (min,max) and (prod,probsum) produce homomorphic-like relations between *all* well-defined finite combinations of these operators applied to fuzzy set membership values and probabilities of corresponding combinations of one point coverages. (The issue of whether ordinal sums also enjoy this property remains open.) However, it is also quickly shown for the case $D_j=D$, all $j \in J$, the pair (min,max) actually produces full conjunction-disjunction homomorphisms between arbitrary combinations of fuzzy set membership functions and corresponding combinations of one point coverage equivalent random sets. This fact was discovered in a different form many years ago by Negoita & Ralescu [24] in terms of non-random level sets, corresponding to the nested random sets discussed above. On the other hand, the pair (prod,probsum) can be ruled out of producing actual conjunction-disjunction homomorphisms by noting that the collection of all jointly independent $\phi(S(f))$ indexed by D, as $f:D \to [0,1]$ varies arbitrarily, is not even closed with respect to min, max (corresponding to set intersection and union). For example, note that for any four 0-1 r.v'sT_j, $j=1,2,3,4$, $P(\min(T_1,T_2,T_3,T_4) = 0) \neq P(\min(T_1,T_2)=0)P(\min(T_3,T_4)=0)$, generally.

4.4 Some applications of homomorphic-like representation to fuzzy logic concepts. The following concepts, in one related form or another, can be found in any comprehensive treatise on fuzzy logic [4] where $(\&_1,\mathrm{v}_1)$ is some chosen fuzzy logic conjunction, disjunction pair. However, in order to apply the homomorphic-like relations in Theorems 4.2 and 4.3, we now assume $(\&_1,\mathrm{v}_1)$ is any member of the alternating signed sum family.

(i) *Fuzzy negation.* Here, 1-f is called the fuzzy negation of $f:D \to [0,1]$. By noting the almost trivial relation, $x \in S(1\text{-f})$ iff not($x \in (S(1\text{-f}))'$ the homomorphic relations presented in Theorems 4.2 and 4.3 can be reinterpreted to include negations. However, in general it is not true that $S(\mathrm{f}) = (S(1\text{-f}))'$ even when they are generated by copulas in the semi-distributive family, compatible with the fact that fuzzy logic as a truth functional logic cannot be boolean.

(ii) *Fuzzy logic projection.* For any $\mathrm{f}:D_1{\times}D_2 \to [0,1]$, $x \in D_1$, the fuzzy logic projection of f at x is

$$(4.16)\qquad \text{fuzzy proj}_1(\mathrm{f})(x) = \underset{x\in D_2}{\mathrm{V}_1}\,(\mathrm{f}(x,y)) = P\Big(\underset{y\in D_2}{\mathrm{V}}\,(x\in S(\mathrm{f}(.,y)))\Big),$$

a probability projection of a corresponding random set.

(iii) *Fuzzy logic modifiers.* These correspond to "very", "more or less", "most", etc., where for any $\mathrm{f}:D\to [0,1]$, modifier $h:[0,1] \to [0,1]$ is applied compositionally as $h\circ\mathrm{f}:D \to [0,1]$. Then, for any $x \in D$,

$$(4.17)\qquad h(\mathrm{f}(x)) = P(x\in S(h\circ\mathrm{f})) = P(\mathrm{f}(x)\in S(h)) = P(x\in \mathrm{f}^{-1}(S(h)))$$

also with obvious probability interpretations.

(iv) *Fuzzy extension principle.* In its simplest form, let $\mathrm{f}:D\to [0,1]$, $g:D \to E$. Then the f-fuzzification of g is $g[\mathrm{f}]: E \to [0,1]$ where at any $y \in E$,

$$(4.18)\qquad g[\mathrm{f}](y) = \underset{x_J\in g^{-1}(y)}{\mathrm{V}_1}\,(\mathrm{f}(x)) \;=\; P(y\in g(S(\mathrm{f}))),$$

the one point coverage function of the g function of a random set representing f. A more complicated situation occurs when f above is defined as the fuzzy cartesian product $\times_1\mathrm{f}_J : \times D_J \to [0,1]$ of $\mathrm{f}_j:D_j \to [0,1]$, $j \in J$, given at any $x_J \in [0,1]^J$ as $\times_1\mathrm{f}_J(x_J) = \&_1(\mathrm{f}_J(x_J))$. Then, restricting $(\&_1,\mathrm{v}_1)$ to only be in the semi-distributive family, we have in place of (4.18)

$$(4.19)\qquad \begin{aligned} g[\mathrm{f}](y) &= \underset{x_J\in g^{-1}(y)}{\mathrm{V}_1}\,(\&_1(\mathrm{f}_J(x_J)))\\ &= P\Big(\underset{x_J\in g^{-1}(y)}{\mathrm{V}}\,\big(\underset{j\in J}{\&}(x_j\in S(\mathrm{f}_j))\big)\Big) = P(y\in g(S(\mathrm{f}_J))),\end{aligned}$$

a natural generalization of (4.18).

(v) *Fuzzy weighted averages.* For any $f_1{:}D_1 \to [0,1]$, $f_2{:}D_2 \to [0,1]$ and weight w, $0 \le w \le 1$, the fuzzy weighted average of f_1, f_2 at any $x \in D_1$, $y \in D_2$, is

$$(4.20) \qquad (wf_1(x)) + ((1-w)f_2(y)) = P_o(a),$$

$$(4.21) \qquad a = [(x\in S(f_1))\&(y\in S(f_2))] \text{ v } [(x\in S(f_1))\&(y\notin S(f_2)) \times \theta(w)] \\ \text{ v } [(x\notin S(f_1))\&(y\in S(f_2)) \times \theta(1-w)].$$

This is achieved by application of the REA solution to weighted linear functions of probabilities (Section 3.2); in this case the latter are the one point coverage ones.

(vi) *Fuzzy membership functions of overall populations.* This is inspired by the well-known example in approximate reasoning "Most blond Swedes are tall" where the membership functions corresponding to "blond & tall" and to "blond" are averaged over the population of Swedes and then one divides the first by the second to obtain the ratio of overall tallness to blondness, to which the fuzzy set corresponding to modifier "most" is then applied ([4], pp. 173-185). More generally, let $f_j{:}D \to [0,1]$, j=1,2, and consider two measurable mappings

$$(4.22) \qquad \begin{tikzcd} & (D, \mathcal{P}(D), P\circ X^{-1}) \\ (\Omega,B,P) \arrow[ur, "X"] \arrow[dr, "S(f)"'] & \\ & (\mathcal{P}(D), \mathcal{P}\mathcal{P}(D), P\circ S^{-1}) \end{tikzcd}$$

where X is a designated population weighting r.v. Then, the relevant fuzzy logic computations are, noting the similarity to Robbin's original application of the Fubini iterated integral theorem [25] and denoting expectation by $E_X(\ )$,

$$(4.23) \qquad E_X(f_2(X)) = \int_{x\in D} f_2(x)dP(X^{-1}(x)) = \int_{x\in D}\int_{\omega\in\Omega} \phi((S(f_2)(\omega))(x)dP(\omega)dP(X^{-1}(x))$$

$$= \int_{\omega\in\Omega}\int_{x\in D} \phi((S(f_2)(\omega))(x)\, dP(X^{-1}(x))dP(\omega) = \int_{\omega\in\Omega} P(X\in S(f_2)(\omega))\, dP(\omega)$$

$$= P(X\in S(f_2)),$$

and similarly,

$$(4.24) \qquad E_X(f_1(X)\,\&_1\, f_2(X)) = P(X\in S(f_1\&_1 f_2)) = P(X\in (S(f_1)\cap S(f_2))).$$

Hence, the overall population tendency to attribute 1, given attribute 2 is, using (4.23) and (4.24), is

$$(4.25) \qquad E_X(f_1(X)\,\&_1\, f_2(X)) \,/\, E_X(f_2(X)) = P(X\in S(f_1) \mid X\in S(f_2)),$$

showing that this proposed fuzzy logic quantity is the same as the conditional probability of one point coverage of random weighting with respect to $S(f_1)$, given a one point coverage of the random weighting with respect to $S(f_2)$.

5. Use of one point coverage functions to define conditional fuzzy logic. This work extends earlier ideas in [11]. (For a detailed history of the problem of attempting a sound definition for conditional fuzzy sets see Goodman [8].) Even for the special case of ordinary events, until the recent fuller development of PSCEA, many basic difficulties arose with the use of other CEA. It seems reasonable that whatever definitions we settle upon for conditional fuzzy sets and logic, they should generalize conditional events and logic of PSCEA. In addition, we have seen in the last sections that homomorphic-like relations can be established between aspects of fuzzy logic and a truth-functional-related part of probability theory, namely the probability evaluations of logical combinations of one point coverages for random sets corresponding to given fuzzy sets.

Thus, it is also reasonable to expect that the definition of conditional fuzzy sets and associated logic should tie-in with homomorphic-like relations with one point coverage equivalent random sets. This connection becomes more evident by considering the fundamental lifting and compatibility relations for joint measurable mappings provided in Theorem 2.3: Let $f_j : D_j \to [0,1], j=1,2$, be any two fuzzy set membership functions. In Theorem 2.3 and commutative diagrams (2.32), (2.33), let (Ω, B, P) be as before, any given probability space, but now replace X by $S(f_1)$, Y by $S(f_2)$, for any joint pair of one point coverage equivalent random sets $S(f_j) \in \mathcal{S}(f_j)$, j=1,2. Also, replace Ω_j by $\mathcal{P}(D_j)$, B_j by $\mathcal{P}\mathcal{P}(D_j)$, and in eq.(2.34), event $a \in B_1$ by filter class $F_x(D_1) \in \mathcal{P}\mathcal{P}(D_1)$ and event $b \in B_2$ by filter class $F_y(D_2) \in \mathcal{P}\mathcal{P}(D_2)$, for any choice of $x \in D_1$, $y \in D_2$. Temporarily, we assume $f_2(y) > 0$. Then, in addition to the result that diagram (2.33) lifts (2.32) via $(\ )_o$ with the corresponding joint random set and one point coverage interpretation, eq.(2.34) now becomes (recalling eq.(4.3))

$$
\begin{aligned}
(5.1) \quad & P(x \in S(f_1) \mid y \in S(f_2)) \\
& = P_o\big((S(f_1), S(f_2))_o \in (\ F_x(D_1) \times F_y(D_2) \mid \mathcal{P}(D_1) \times\ F_y(D_2))\big).
\end{aligned}
$$

Note the distinction between the expression in (5.1) and the one point coverage function of the *conditional random set* $(S(f_1) \times S(f_2) \mid D_1 \times S(f_2))$, (where as usual for any, $\omega \in \Omega$, $(S(f_1) \times S(f_2) \mid D_1 \times S(f_2))(\omega) = (S(f_1)(\omega) \times S(f_2)(\omega) \mid D_1 \times S(f_2)(\omega))$. Both $(S(f_1), S(f_2))_o$ and $(S(f_1) \times S(f_2) \mid D_1 \times S(f_2))$ are based on the same measurable spaces, but differ on the induced probability measures (see eq.(5.3). The latter in general produces complex infinite series evaluations for its one point coverage function (as originally attested to in [12]): For any $u = (x_1, y_1, x_2, y_2, \ldots)$ in $(D_1 \times D_2)_o$, using eqs.(2.12) and (4.8),

$$
(5.2) \qquad P(u \in (S(f_1) \times S(f_2) \mid D_1 \times S(f_2)))
$$

$$= \sum_{j=0}^{+\infty} P\Big((x_j \in S(\mathrm{f}_1)) \,\&\, (y_j \in S(\mathrm{f}_2)) \,\&\, \mathop{\&}_{i=1}^{j-1} (y_i \notin S(\mathrm{f}_2)) \Big)$$

$$= \sum_{j=0}^{+\infty} \overline{\mathrm{cop}}\big(\mathrm{f}_1(x_j), \mathrm{f}_2(y_j)\ ; \mathrm{f}_2(y_1), \ldots, \mathrm{f}_2(y_{j-1})\big).$$

(5.3)

$$\begin{tikzcd}[column sep=small]
 & \big(\mathcal{P}(D_1) \times \mathcal{P}(D_2)\big)_o, (\mathrm{sigma}(\mathcal{P}\mathcal{P}(D_1) \times \mathcal{P}\mathcal{P}(D_2)))_o, P_o\circ(S(\mathrm{f}_1),S(\mathrm{f}_2))_o^{-1}\big) \\
(\Omega, B, P) \arrow[ur, "(S(\mathrm{f}_1),S(\mathrm{f}_2))_o"'] \arrow[dr, "(S(\mathrm{f}_1)\times S(\mathrm{f}_2)\,|\,D_1\times S(\mathrm{f}_2))"] & \\
 & \big((\mathcal{P}(D_1) \times \mathcal{P}(D_2))_o, (\mathrm{sigma}(\mathcal{P}\mathcal{P}(D_1) \times \mathcal{P}\mathcal{P}(D_2)))_o, P\circ(S(\mathrm{f}_1)\times S(\mathrm{f}_2)\,|\,D_1\times S(\mathrm{f}_2))^{-1}\big)
\end{tikzcd}$$

On the other hand, the expression in eq.(5.1) quickly simplifies to the form

(5.4) $\quad P(x \in S(\mathrm{f}_1) \mid y \in S(\mathrm{f}_2))$

$$= P((\phi(S(\mathrm{f}_1))(x)=1) \,\&\, (\phi(S(\mathrm{f}_2))(y)=1)) \,/\, P(\phi(S(\mathrm{f}_2))(y)=1)$$

$$= \overline{\mathrm{cop}}\,(\mathrm{f}_1(x), \mathrm{f}_2(y)) \,/\, \mathrm{f}_2(y),$$

using eq.(4.14) and the assumption that $\mathrm{f}_2(y) > 0$.

Thus, we are led in this case to define the conditional fuzzy set $(\mathrm{f}_1|\mathrm{f}_2)$ as

(5.5) $\quad (\mathrm{f}_1|\mathrm{f}_2)(x,y) = P(x \in S(\mathrm{f}_1) \mid y \in S(\mathrm{f}_2)) = \overline{\mathrm{cop}}\,(\mathrm{f}_1(x), \mathrm{f}_2(y)) \,/\, \mathrm{f}_2(y)).$

The case of $\mathrm{f}_2(y) = 0$, is treated as a natural extension of the situation for the PSCEA conditional event membership function $\phi(a|b)$ in eqs.(2.15), (2.16), namely, for any $u = (x_1, y_1, x_2, y_2, \ldots)$, with x_j in D_1, y_j in D_2 :

(5.6) If $\mathrm{f}_2(y_1) = \mathrm{f}_2(y_2) = \ldots = \mathrm{f}_2(y_{j-1}) = 0 < \mathrm{f}_2(y_j)$, then, by def. $(\mathrm{f}_1|\mathrm{f}_2)(u) = (\mathrm{f}_1|\mathrm{f}_2)(x_j, y_j)$,

noting that the case for vacuous zero values in eq.(5.6) occurs for $j = 1$.

(5.7) If $\mathrm{f}_2(y_1) = \mathrm{f}_2(y_2) = \ldots = \mathrm{f}_2(y_j) = \ldots = 0$, then by def. $(\mathrm{f}_1|\mathrm{f}_2)(u) = 0$.

More concisely, eqs.(5.6), (5.7) are equivalent to

(5.8) $$(\mathrm{f}_1|\mathrm{f}_2)(u) = \sum_{j=1}^{+\infty} \prod_{i=1}^{j-1} \delta(\mathrm{f}_2(y_j)=0)\cdot(\mathrm{f}_1|\mathrm{f}_2)(x_j, y_j)\ ,$$

where $(f_1|f_2)(x_j,y_j)$ is as in eq.(5.5).

Furthermore, let us define logical operations between such conditional fuzzy membership motivated by the homomorphic-like relations discussed earlier and compatible with the definition in eqs.(5.5)-(5.8). For example, binary conjunction here between $(f_1|f_2)$ and $(f_3|f_4)$ for any arguments $u = (x_1,y_1,x_2,y_2,\ldots)$, $v = (w_1,z_1,w,z_2,\ldots)$ is defined by first determining that j and k so that $f_2(y_1) = f_2(y_2) = \ldots = f_2(y_{j-1}) = 0 < f_2(y_j)$, $f_4(z_1) = f_4(z_2) = \ldots = f_4(z_{k-1}) = 0 < f_4(z_k)$, assuming the two trivial cases of 0-value in eq.(5.7) are avoided here. Thus, $f_2(y_j)$, $f_4(z_k) > 0$. Next, choose any copula and corresponding pair $(\&_1,v_1)$ (= $(\overline{\mathrm{cop}},\mathrm{cocop})$) in the alternating sum family and consider the joint one point coverage equivalent random subsets of D_j, $S(f_j)$, j=1,2,3,4 determined through cop (as in Theorem 4.1, e.g.). Denote one point coverage events $a = (x_j \in S(f_1))$ (= $(\phi(S(f_1))=1) = (S(f_1))^{-1}(F_{x_j}(D_1))$, etc.) , $b = (y_j \in S(f_2))$, $c = (w_k \in S(f_3))$, $d = (z_k \in S(f_4))$ (all belonging to the probability space (Ω,B,P)). Finally, define conditional fuzzy logic operator $\&_1$ as

$$(5.9)\qquad ((f_1|f_2)\&_1(f_3|f_4))(u,v) = (f_1|f_2)(u)\ \&_1(f_3|f_4)(v) = P_o((a|b)\ \&\ (c|d)) = P_o(A\ )/P(b \vee d\ )) ,$$

$$(5.10)\quad P_o(A) = P\,(a\ \&b\ \&c\ \&d\ ) + (P(a\ \&b\ \&d\ ')P(c\mid d\ )) + (P(b\ '\&c\ \&d\ )P(a\mid b)),$$

using eqs.(2.24), (2.26). In turn, each of the components needed for the full evaluation of eq.(5.9) are readily obtainable using, e.g., the bottom parts of eq.(4.13) and/or eq.(4.14). Specifically, we have:

$$(5.11)\qquad P\,(a\ \&b\ \&c\ \&d\ ) = \overline{\mathrm{cop}}(f_1(x_j),f_2(y_j),f_3(w_k),f_4(z_k)),$$

$$(5.12)\qquad P(a\ \&b\ \&d\ ') = \overline{\mathrm{cop}}(f_1(x_j),f_2(y_j)\ ;\ f_4(z_k)),$$

$$(5.13)\qquad P(b\ '\&c\ \&d\ ) = \overline{\mathrm{cop}}(f_3(w_k),f_4(z_k)\ ;\ f_2(y_j)),$$

$$(5.14)\qquad P(a\mid b\ ) = (f_1|f_2)(x_j,y_j) = (\overline{\mathrm{cop}}(f_1(x_j),f_2(y_j))/f_2(y_j),$$

$$(5.15)\qquad P(c\mid d\ ) = (f_3|f_4)(w_k,z_k) = (\overline{\mathrm{cop}}(f_3(w_k),f_4(z_k))/f_4(z_k),$$

$$(5.16)\qquad P(b \vee d) = \mathrm{cocop}(f_2(y_j),\ f_4(z_k)).$$

Also, using eqs.(2.25), (2.27), we obtain similarly,

$$(5.17)\qquad ((f_1|f_2)v_1(f_3|f_4))(u,v) = (f_1|f_2)(u)\ v_1(f_3|f_4)(v) = P_o((a|b) \vee (c|d)) = P(a\mid b\ ) + P(c\mid d\ ) - P_o((a\mid b)\ \&\ (c\mid d\ )),$$

all obviously obtainable from eqs.(5.9)-(5.16). For fuzzy complementation,

$$(5.18)\qquad (f_1|f_2)\ '(u) = ((f_1|f_2)(u))\ ' = P((a|b)\ ') = 1-P(a|b) = P(a\ '|b) = 1-(f_1|f_2)(u) = 1-(f_1|f_2)(x_j,\ y_j) = (f_2(y_j)-\overline{\mathrm{cop}}(f_1(x_j),f_2(y_j)))\ /\ f_2(y_j) = \overline{\mathrm{cop}}(f_2(y_j);\ f_1(x_j))\ /\ f_2(y_j),$$

all consistent. Combinations of conjunctions and disjunctions for multiple arguments follow a similar pattern of definition. The following summarizes some of the basic properties of fuzzy conditionals sets and logic, recalling here the multivariable notation introduced earlier:

THEOREM 5.1. *Consider any collection of fuzzy set membership functions* $f_J{:}D_J \to [0,1]^J$, *any choice of cop producing joint one point coverage equivalent random subsets* $S(f_J)$ *of* D_J *with respect to probability space* (Ω,B,P), *and define fuzzy conditional membership functions and logic as in eqs.(5.5)-(5.18):*

(i) *When, any two fuzzy set membership functions reduce to ordinary set membership functions such as* $f_1 = \phi(a_1)$, $f_2 = \phi(a_2)$, $a_1, a_2 \in B$, *then*

$$(f_1|f_2) = \phi(a_1 \times a_2 \mid D_1 \times a_2)),$$

the ordinary membership function of a PSCEA conditional event in product form.

(ii) *All well-defined combinations of fuzzy logical operations over* $(f_1|f_2),(f_3|f_4),\ldots$, *when* $f_J = \phi(a_J)$, *reduce to their PSCEA counterparts.*

(iii) *When* $f_2{=}1$ *identically, we have the natural identification*

$$(f_1 \mid f_2) = f_1$$

(iv) *The following is extendible to any number of arguments*: When $f_2 = f_4 = f$,

$$(f_1|f)\&_1(f_3|f) = (f_1\&_1f_3 \mid f)\,, \qquad (f_1|f)v_1(f_3|f) = (f_1v_1f_3 \mid f)\;,$$

where for the unconditional computations $f_1\&_1f_3 = \overline{\mathrm{cop}}(f_1,f_3)$, $f_1v_1f_3 =$ $\mathrm{cocop}(f_1,f_3)$, etc.

(v) *Modus ponens holds for conditional fuzzy logic as constructed here:*

$$(f_1|f_2)\;\&_1\;f_2 = f_1\,\&_1\,f_2.$$

Proof: Straightforward, from the construction. Details are omitted here. □

Thus, a reasonable basis has been established for conditional fuzzy logic extending both PSCEA for ordinary events and unconditional fuzzy logic. The resulting conditional fuzzy logic is not significantly more difficult to compute than is PSCEA itself.

6. Three examples reconsidered. We show here briefly how the ideas developed in the previous sections can be applied to the examples of Section 1.3.

Example 1. (See eq.(1.3).) Applying the REA solution for weighted linear functions from Section 3.2, we obtain

(6.1) $P_0(a\&b) = P(c\&d\&e) + (\min(w_{11}+w_{12},\, w_{21}+w_{22}))P(c\&d\&e')$
$+ (\min(w_{11}+w_{13},\, w_{21}+w_{23}))P(c\&d'\&e) + (\min(w_{12}+w_{13},\, w_{22}+w_{23}))P(c'\&d\&e)$
$+ (\min(w_{13},w_{23}))P(c'\&d'\&e) + \min(w_{12},w_{22}))P(c'\&d\&e')$
$+ (\min(w_{11},w_{21}))P(c\&d'\&e').$

Then, choosing, e.g., the absolute probability distance function, eq.(6.1) yields

$$(6.2)\qquad D_{P_o}(a,b) = P_o(a) + P_o(b) - 2P_o(a \,\&\, b)$$
$$= |(w_{11}+w_{12})-(w_{21}+w_{22})|P(c\&d\&e') + |(w_{11}+w_{13})-(w_{21}+w_{23})|P(c\&d'\&e)$$
$$+ |(w_{12}+w_{13})-(w_{22}+w_{23})|P(c'\& d\&e) + |w_{13}-w_{23}|P(c'\&d'\&e)$$
$$+ |w_{12}-w_{22}|P(c'\&d\&e') + |w_{11}-w_{21}|P(c\&d'\&e').$$

In turn, use the above expression to test hypotheses $a \neq b$ vs. $a = b$ following the procedure in eqs.(1.13)-(1.16), where, e.g., the fixed significance level is

$$(6.3)\qquad \alpha_o = F_D(D_{P_o}(a,b)) = (D_{P_o}(a,b))^2(3 - 2D_{P_o}(a,b))$$

using eq.(6.2) for full evaluation.

Example 2. Specializing the conjunctive probability formula in eqs.(2.24), (2.26),

$$(6.4)\qquad P_o(a \,\&\, b) = P_o((c|d)\&(c|e)) = P_o(A) / P(d \vee e),$$
$$(6.5)\quad P(A) = P(c\&d\&e) + (P(c\&d\&e')P(c|e)) + (P(c\&d'\&e)P(c|d)).$$

Using, e.g., the relative distance to test the hypotheses $a \neq b$ vs. $a = b$, eq.(6.4) allows us to calculate

$$(6.6)\; R_{P_o}(a,b) = P_o(a+b)/P_o(a \vee b) = \frac{P(c \mid d) + P(c \mid e) - 2P_o(a \,\&\, b)}{P(c \mid d) + P(c \mid e) - P_o(a \,\&\, b)}.$$

Then, we can test hypotheses $a \neq b$ vs. $a = b$, by using eqs.(1.13)-(1.16), where now the fixed significance level is

$$(6.7)\qquad \alpha_o = F_R(R_{P_o}(a,b)) = (R_{P_o}(a,b))^2,$$

using eq.(6.6) for full evaluation.

Example 3. We now suppose that in the fuzzy logic interpretation in eq.(1.6) for Model 1, v_1 is max, while for Model 2, v_2 is chosen compatible with conditional fuzzy logic as outlined in Section 5. We also simplify the models further: The attribute "fairly long" and "long" are identified, thus avoiding the exponentiation of 1.5 (though this can also be treated -- but to avoid complications, we make the assumption). We also identify "large" with medium". Model 2 can be evaluated from eqs.(5.9)-(5.17), where

$$(6.8)\qquad c = (x_1 \in S(f_1)),\; d = (w_1 \in S(f_3)),\; e = (z_1 \in S(f_4)),$$
$$f_1(x_1) = f_{long}(\text{lngth}(A)),\; f_2(y_1) = 1,\; f_3(w_1) = f_{medium}(\#(Q)),$$
$$f_4(z_1) = f_{accurate}(L),\; x_1 = \text{lngth}(A),\; y_1 \text{ arbitrary},\; w_1 = \#(Q),\; z_1 = L,$$

On the other hand, we are here interested in *comparing* the two models via probability representations and use of probabilistic distance functions. In summary, all of the above simplifies to

$$(6.9)\qquad \begin{cases} \text{Model 1: } t(a) = P_o(a) = \max(P(c^2), P(d)) \\ \qquad\qquad\text{vs.} \\ \text{Model 2: } t(b) = P_o(b) = P_o((c \mid \Omega) \vee (d \mid e)) \end{cases} .$$

In turn, applying the REA approach to max in Section 3.5 and using PSCEA via eq.(5.17) with $d = \Omega$, we obtain the description events according to each expert as

$$(6.10)\qquad a = (c^2\&d) \vee ((c^2\&d')\times\theta(w_{P,1})) \vee (((c^2)'\&d)\times\theta(w_{P,2})),$$

$$(6.11)\qquad c^2 = c\times c\ ,\ (c^2)' = (c\times c')\vee c'.$$

$$(6.12)\qquad b = c \vee (d|e) = c \vee (d\&e) \vee ((c'\&e')\times(d|e))$$

Then again using PSCEA and simplifying,

$$(6.13)\quad P_o(a\&b) = P(c\&d)P(c) + P(c\&d')P(c)w_{P,1} + P(c\&d)P(c')w_{P,2} + P(c'\&d\&e)w_{P,2} + P(c'\&d\&e')P(d|e)w_{P,2}\ .$$

In turn, each expression in eq.(6.13) can be fully evaluated via the use of eq.(4.13) again:

$$(6.14)\quad P(c\&d) = \overline{\mathrm{cop}}(f_1(x_1),f_3(w_3)),\quad P(c'\&d\&e) = \overline{\mathrm{cop}}(f_3(w_1),f_4(z_1);f_1(x_1),$$
$$P(c'\&d\&e') = \overline{\mathrm{cop}}(f_3(w_1);f_1(x_1),f_4(z_1)),\quad P(c) = f_1(x_1)\ ,$$
$$P(c\&d') = \overline{\mathrm{cop}}(f_1(x_1);f_3(w_1)) = P(c) - P(c\&d)\ ,\quad P(d|e) = (f_3| f_4)(w_1,z_1)\ ,$$

etc., with all interpretations in terms of the original corresponding attributes given in eq.(6.8). Finally, eq.(6.13) can be used, together with the evaluations of the marginal probabilities $P_o(a)$, $P_o(b)$ in eq.(6.9) to obtain once more any of the probability distance measures used in Section 1.4 for testing the hypotheses of $a \neq b$ vs. $a = b$.

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