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- USSR -

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ALGORITHMIC DIFFICULTIES OF THE SYNTHESIS OF MINIMUM CONTACT CIRCUITS*

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INTRODUCTION

Each class of mathematical problems requires an enumeration of the means that are accessible for their solution. Thus, in the solution of set-theoretical problems the choice principle is admitted /1/. However, the choice principle is good and convenient for those problems, where an analysis of set-theoretical concepts is necessary. At the same time, the choice principle is absolutely inapplicable to the solution of problems in which it is not only necessary to prove the existence of a certain object, but to carry out its actual construction. It is assumed here that, being capable of performing potentially any finite number of effective elementary acts, we obtain the sought object after a finite number of steps. Means of this kind are afforded by modern mathematical logic in the form, for example, of the apparatus of normal algorithms /2/. One class of problems, for the solution of which this apparatus is essential, is the class of problems concerning the consistency /3/ of certain assumptions. Sometimes one admits as elementary acts those which, unlike the choice principle can be performed in practice, but which cannot be called effective, since nothing definite can be said concerning the object obtained. We have in mind here acts with the element of chance (tossing a coin, registering an elementary particle in a counter, etc.). Algorithms with chance elementary acts are successfully used in the solution of several computational problems, for example, in solving differential equations (the Monte Carlo method /4/). At the present time there are in cybernetics an extensive group of problems, where the existence of various objects or facts is established trivially and, within the framework of the classical definition of the algorithm, quite effectively, although the performance of the solution sometimes becomes impossible in practice, owing to its cumbersomeness. Such, for example, are problems connected with the coding of information, problems connected with the analysis and synthesis of networks /5/ etc. Here naturally, the question arises of the necessity for refining the classical definition of the algorithm. One must expect here this refinement will take into account to an even greater extent the singularities of the particular class of problem. The latter, possibly, leads to an expansion of the concept of algorithm in such a way, that individual types of algorithms can no longer be compared as regards their strength. It is premature at present to make any general forecasts of how the concept of algorithm will be refined, since we have far too little information on the specific nature of the individual classes of problems. In the present work we make an attempt to clarify the algorithmic difficulties that arise in solving cybernetics problems, which are not of trivial solutions on the basis of the classical definition of the algorithm, but this solution is not realizable in practice because of its cumbersomeness.

By way of a model object we use contact networks, which realize functions of algebraic logic. We pose for this object the question of the construction of a network, that realizes the function $f(x_1, x_2, \ldots, x_n)$ and which has a minumum number of contacts (minimal network), which we shall denote by $L(f)$. It is known that there exists a trivial algorithm for the construction of minimum contact networks. This algorithm consists of the following. Assume that it is necessary to construct a minimal contact network for the function $f(x_1; x_2; \ldots, x_n)$ of algebraic logic. Let us consider the sequence of sets

$$G_0, G_1, \ldots, G_i, \ldots;$$

where G_i consists of all the two-pole nets with i-links. Each such set has a finite number g_i of elements, and /6, 7/

$$\left(C_1 \frac{i}{\log_2^2 i}\right)^i < g_i < \left(C_2 \frac{i}{\log_2^2 i}\right)^i.$$

We shall sort out in some sequence the nets, first from the set G_0, then from G_1, etc. In each net from G_i ($i = 0, 1, \ldots$ we shall place in all possible manner on the links the symbols from the alphabet $x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n$. Obviously we shall have $(2n)^i$ methods of placing the symbols. For each individual placement we obtain a two-pole contact network $\mathfrak{A}$. Let $f_{\mathfrak{A}}(x_1, x_2, \ldots, x_n)$ be a function of algebraic logic (i.e., a function of class P_2 /8/), which describes the admittance of the network $\mathfrak{A}$, calculated either experimentally or algorithmically /5/. In the case when

$$f_{\mathfrak{A}}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n),$$

then the network $\mathfrak{A}$ gives us the necessary network, which realizes the function $f(x_1, x_2, \ldots, x_n)$. Since it is known that for each function $f(x_1, x_2, \ldots, x_n)$ from P_2 it is possible to construct a network that realizes it, with not more than $L(n)$ contacts, where

$$L(n) = \max_{f(x_1, \ldots, x_n) \in P_2} L(f)$$

(the maximum is taken over all the functions of algebraic logic, which depend on n variables), our process must lead to such a network. Furthermore, the first network $\mathfrak{A}$, constructed by this algorithm and realizing the function $f(x_1, x_2, \ldots, x_n)$ will indeed be the network with minimum number of contacts. Thus, the algorithm given here for the synthesis of networks is based on the sorting out of all the networks. Let us estimate the volume of the sorting necessary to construct with the aid of this algorithm a minimum network that realizes the function $f(x_1, x_2, \ldots, x_n)$. This volume is characterized by the number of the reviewed network and has an order of magnitude

$$\left(\frac{cL(n)}{\log_2 L(n)}\right)^{L(n)}(2n)^{L(n)}$$

If it is considered /9/, that $L(n) \sim 2^n/n$, one can already visualize the speed with which the volume of sorting increases. It is seen therefore that this algorithm has low efficiency. The large volume of sorting makes it difficult to make practical use of this algorithm. Thus, in ref. /10/, Higonnet and Grea, by sorting out all the nets with links up to 6 inclusive, obtained all the minimal networks that contain not more than 6 contact, for functions which depend on four variables. However, to make the next step, i.e., to scan nets with 7 links, is practically impossible. Shannon proposed in /11/ a scheme of a machine for the synthesis of networks, which realize functions of four variables. This machine makes it possible to find, for the majority of functions, minimal networks, but for this it is necessary to perform switchings. Consequently, to find the solution it is necessary to sort out a certain number of commutations, corresponding to the networks. Therefore the efficiency of the solution is determined by the searching time of the required commutation. This time becomes large (practically infinite even if the search is automatized electronically), if an attempt is made to construct minimal network with the aid of analogous machines for functions depending on 6 or 7 variables.

Thus, the practical use of the trivial algorithm is possible only for the first several values of n. Furthermore, the use of machine technology yields no practical advantages over human capabilities. There are grounds for assuming that for any method of constructing a minimal network for an arbitrary function from P_2, it is necessary to have some sort of a form of a sorting out of approximately the same order as the trivial algorithm. We arrive at the necessity of modifying the statement of the problem. Two volumes are possible here.

1. Network Not Minimal. In this case it is necessary for any function $f(x_1, x_2, \ldots, x_n)$ from P_2 to construct a network with a number of contacts equal to $L(f)$, but having an order not higher than $L(n)$. The

problem was first formulated in this form by Shannon /12/ and was finally solved by Lupanov /9/. The latter has shown that for any function $f(x_1, x_2, \ldots, x_n)$ it is possible to construct effectively (with a sorting that is considerably smaller than the sorting of all the networks), the number of contacts of which has an order ~~of~~ not less than $2^n/n$. It follows from the results of /9, 12/ that almost all the functions of n variables require asymptotically $2^n/n$ contacts. Consequently, neglecting the large number of functions, it is possible to construct for any of the functions from P_2 almost minimal networks. It is found here that by avoiding the sorting, it is possible to construct compact networks for a majority of functions, but these networks are in themselves complicated, so that the number of contacts grows as $2^n/n$. On the other hand, using the Lupanov algorithm (without enrolling additional information) we are not certain that we obtain compact networks (close to minimal) also for those functions, which admit of a realization of a circuit with a number of contacts having an order lower than $2^n/n$. This algorithm is more likely to yield for these functions, as a rule, a poor result. Thus, the following situation arises: the algorithm yields compact networks for functions, the minimal networks of which contain approximately $2^n/n$ contacts, i.e., for functions which are of little practical interest, and gives networks of unknown degree of compactness for functions which are of practical value.

2. Failure to consider all the functions of algebraic logic. In this case one narrows down in a sensible manner the number of functions under consideration to a certain class $Q \subset P_2$. It is then possible to expect that the construction of minimal networks, which realize all the functions of n variables from the class Q, with the aid of trivial algorithm will require substantially less sorting of the networks, than the construction of minimal networks for all the functions of algebraic logic, which depend on n variables. It will be exactly so if (see estimate of the volume of sorting on p. 76 /of source/)

$$L_Q(n) \ll L(n),$$

where

$$L_Q(n) = \max_{f(x_1, \ldots, x_n) \in Q} L(f).$$

In this case the effectiveness of the trivial algorithm (i.e., the relative applicability, determined by the maximum value of the number n, at which it is still possible in practice to construct minimal networks for all functions from Q of n variables) increases considerably. Thus, the increase in the effectiveness of the trivial algorithm is connected with the fact that the functions of n variables from the class Q admit of a substantially simpler network realization, than arbitrary functions of n variables. Consequently, the question arises of separating out the "simple" classes Q, i.e., such classes, for which $L_Q(n) \ll L(n)$. Naturally, to avoid a vicious circle, it is necessary that all the classes be defined not in terms of the properties of the network, but in terms of properties of the functions. The construction of such classes and the clarification of the possibility of the network realization of the functions from these classes has been the subject of many investigations: the realization of linear functions /13/, the realization of symmetrical functions /12, 14/, the synthesis of nonrepetitive networks /15/, the realization of functions that have a value of one on a small set of assemblies /14/, etc. Inasmuch as the determination of the "simple" class requires in the final analysis that $L_Q(n) \ll L(n)$, i.e., it is based on a comparison of certain functions with other functions of the same number of variables, having the most complicated minimal networks, the question reduces to the construction (for each n) of a function that depends on n variables and that has the most complex network realization (i.e., to the calculation of the value of $L(n)$). The problem is formulated more accurately as follows.

It is desired to construct for each n such a function $f(x_1, x_2, \ldots, x_n) \in P_2$, that $L(f) = L(n)$.

In solving this problem we encounter a principal difficulty. Namely, since we do not know beforehand the value of $L(n)$, it becomes necessary to construct for each function $f(x_1, x_2, \ldots, x_n)$ a minimal network and to calculate $L(f)$. After the values of $L(f)$ are found for all the functions that depend on n variables, we can readily obtain also one for which $L(f)$ has a maximum, i.e., has a value $L(n)$. Thus, to find the unknown function it is necessary to carry out a sorting out of all the functions that depend on n variables. Naturally, such an argument, although it does bring to mind the idea that the sorting out of all the functions remains unavoidable in the solution of this problem, it cannot serve as proof of this fact.

It is quite natural for the solution of the problem to depend on the choice of the means.

In the present paper we analyze solutions of this problem both in a class of algorithms, which admit random elements as elementary acts, and in a certain class of ordinary algorithms. It is shown in Sec 1 that in class of algorithms with random elementary acts, the problem, which represents a certain weakening of the discussed problem, admits with probability of unity a positive and very simple solution. On the other hand, as is established in Sec. 4, in the natural subclass of ordinary algorithms, i.e., in the class of the so called regular algorithms, the construction of the sequence $\{f_n\}$ of the functions $f(x_1, x_2, \ldots, x_n)$ for each $L(f_n) = L(n)$, leads to the construction of all the functions of algebraic logic, i.e., to a complete sorting out. Then we give a comparison of these two approaches to the solution of the problem. In Sec 2 we construct a family of the so called invariant classes and study their properties. In Sec 3 we clarify the possibilities of the network realization of functions from the invariant classes. In particular it is found that all the invariant classes Q_σ where σ is a parameter which can be determined in some manner $(0 \leq \sigma \leq 1)$ have a simple asymptotic expression for $L_{Q_\sigma}(n)$, namely

$$L_{Q_\sigma}(n) \sim \frac{\sigma 2^n}{n}.$$

Consequently, it becomes necessary to construct for a continual set of classes a synthesis method which gives an asymptotic value for $L_Q(n)$. We see therefore that the results of Secs. 2 and 3, are in addition to serving auxiliary purposes, are of independent interest.

Finally, we point out that this problem arose in 1954--1955. The very idea of the proof came to mind at the same time. However, the lack of an asymptotic expression for L(n) made it impossible to realize this idea. It was therefore necessary to publish in 1956 /14, 18/ a few extraneous results. The final solution occurred soon after Lupanov /9/ obtained an asymptotic expression for L(n).

1. SOLUTION OF THE PROBLEM OF THE CLASS OF ALGORITHS WITH RANDOM ELEMENTARY ACTS

The statistical approach, which we are about to discuss, uses certain peculiarities of the network realization of functions of algebraic logic, which we mentioned briefly in the introduction. We deal, primarily, with the asymptotic behavior of the quantity L(n), namely with the fact that

$$L(n) \sim \frac{2^n}{n}.$$

Secondly, we have in mind the result obtained by Shannon /12/, that for any $\varepsilon > 0$ the fraction of all the functions f of algebraic logic, which depend on n variables and for which

$$L(f) \leqslant (1-\varepsilon)\frac{2^n}{n}$$

relative to the total number of functions of algebraic logic that depend on the same n variables, tends to zero with increasing n. However, as shown by Lupanov /7/, a stronger result is indeed obtained. Namely: for any $\varepsilon > 0$ the fraction of all the functions f of algebraic logic which depend on n variables and for which

$$L(f) \leqslant \frac{2^n}{n}\left[1+(2-\varepsilon)\frac{\log_2 n}{n}\right],$$

with respect to the total number of functions of algebraic logic that depend on the same n variables, tends to zero with increasing n.

Definition. Let ε be an arbitrary fixed positive number. A function $f(x_1, x_2, \ldots, x_n)$ from P_2 is called ε-simple if

$$L(f) \leqslant (1-\varepsilon) L(n).$$

and ε-complex, if

$$L(f) > (1-\varepsilon) L(n).$$

From the preceding results it follows, incidentally, that for any $\varepsilon > 0$ the fraction of ε-simple functions from P_2, dependent on n variables, relative to the total number of functions from P_2, which depend on the same n variables, tends to zero with increasing n.

Let us now formulate our problem in the following manner: we wish to construct a set M^0 of the functions of algebraic logic in the form

$$M^0 = \{f_1(x_1),\ f_2(x_1, x_2),\ \ldots,\ f_n(x_1, x_2, \ldots, x_n),\ \ldots\},$$

for which there exists a sequence $\{n_k\}$ such that

$$\frac{L(f_{n_k})}{L(n_k)} \to 1 \quad (k \to \infty).$$

Obviously, this problem represents a certain weakening of the problem stated in the introduction. In fact, if we are able to construct for each number n a function $f_n(x_1, x_2, \ldots, x_n)$ such that

$$L(f_n) = L(n),$$

then the set

$$M^0 = \{f_1(x_1),\ f_2(x_1, x_2),\ \ldots,\ f_n(x_1, x_2, \ldots, x_n),\ \ldots\}$$

is characterized by the fact that

$$\frac{L(f_n)}{L(n)} = 1$$

for any n. Consequently, we obtain a solution of the problem just stated.

Let us consider an algorithm, which constructs a certain set of functions

$$M = \{f_1(x_1),\ f_2(x_1,\ x_2),\ \dots,\ f_n(x_1,\ x_2,\ \dots,\ x_n),\ \dots\}$$

in the form of a sequence, i.e., that in the n-th step the algorithm constructs the function $f_n(x_1, x_2, \dots, x_n)$.

The function is constructed by the writing out its tables in the following manner: on the left part of the table one places the column, in which the sets of values of the variables $x_1, x_2, \dots, x_n$ (a total of 2^n) are placed one under the other in their natural order, and on the right part, next to each set, we place successively the value 0 or 1, depending on the result of the tossing of the coin, i.e., depending on whether "heads" or "tails" are obtained.*

For example:

$x_1, \dots, x_{n-1}, x_n$	Результат бросания монеты 1)	Значение функции 2)
0 ... 0 0	Герб 3)	1
0 ... 0 1	Решетка 4)	0
0 ... 1 0	Решетка	0
0 ... 1 1	Герб	1
. . . .		
. . . .		
1 ... 1 1	Решетка	0

1) Result of tossing the coin, 2) Value of the function, 3) Heads, 4) Tails

The algorithm described admits of the tossing of the coin as one of the elementary acts. In this connec-

* It is assumed that the probability of obtaining heads or tails is the same.

tion, repeated applications of the algorithm lead, generally speaking, to different sets. Let us denote by $\mathfrak{M} = \{M\}$ the set of all possible results of the constructions. It is obvious that $\mathfrak{M}$ had the cardinality of the continuum. Let us take an arbitrary number $\varepsilon > 0$. We consider the subset $\mathfrak{M}^{\varepsilon}$ from $\mathfrak{M}$, consisting of those and only those sets M^{ε}, each of which contains the finite number ε-complex functions.

Theorem 1.1. The probability $P(\mathfrak{M}^{\varepsilon})$ that the result of construction will be a set from $\mathfrak{M}^{\varepsilon}$, is equal to zero, i.e., $P(\mathfrak{M}^{\varepsilon}) = 0$.

Proof. Since the construction of the function that depends on n variables can lead with equal probability to each function, the probability of construction of an ε-complex function is p_n, where p_n represents the fraction of the ε-complex functions which depend on n variables relative to all the functions which depend on the same variables, i.e., to 2^{2^n}. At the same time, the probability of constructing an ε-simple function, which depends on n variables, is equal to $1 - p_n$. We have seen earlier that $p_n \rightarrow 1 (n \rightarrow \infty)$. Let us consider the arbitrary set $M^{\varepsilon} \in \mathfrak{M}^{\varepsilon}$. By definition in this set, starting with a certain place (to be specific, from the $n+1$-th place), all the functions are ε-simple, and the n-th function is $\mathcal{E}$-complex. If we place under each $\mathcal{E}$-complex function the letter C, and under each $\mathcal{E}$-simple one the letter S, then the set will have the form

$$\begin{array}{l} f_1, f_2, \ldots, f_{n-1}, f_n, f_{n+1}, f_{n+2}, \ldots \\ \ldots\ldots\ldots\ldots\ldots, C, \ldots S, \ldots S, \ldots \end{array}$$

We denote by $\mathfrak{M}_n^{\mathcal{E}}$ the subset of such sets $M^{\mathcal{E}}$, that in each of these the n-th function is $\mathcal{E}$-complex, and all the succeeding ones $\mathcal{E}$-simple. Obviously

$$\mathfrak{M}^{\varepsilon} = \mathfrak{M}_0^{\varepsilon} \cup \mathfrak{M}_1^{\varepsilon} \cup \ldots$$

is the direct sum of non intersecting subsets. Let us calculate the probability $P(\mathfrak{M}_n^{\mathcal{E}})$:

$$\mathbf{P}(\mathfrak{M}_n^{\varepsilon}) = p_n \prod_{i>n} (1 - p_i) = 0, \text{ since } p_i \to 1 \ (i \to \infty).$$

From this it follows that

$$\mathbf{P}(\mathfrak{M}^{\varepsilon}) = \sum_{n=0}^{\infty} \mathbf{P}(\mathfrak{M}_n^{\varepsilon}) = 0.$$

This proves the theorem completely.

It follows directly from the theorem that the probability of constructing a set of functions, in which is contained an infinite number of $\mathcal{E}$-complex functions, i.e., the probability of constructing a set from $\mathfrak{M} \setminus \mathfrak{M}^{\mathcal{E}}$ is equal to 1.

We denote by $\mathfrak{M}^{0}$ the subset from $\mathfrak{M}$, consisting of the sets M^{0} such, that for any $\mathcal{E} > 0$ the set M^{0} contains an infinite number of $\mathcal{E}$-complex functions.

Theorem 1.2. The probability $P(\mathfrak{M}^0)$ of constructing a set M^0 from $\mathfrak{M}^0$ is equal to 1, i.e., $P(\mathfrak{M}^0) = 1$.

Proof. Let $M \bar{\in} \mathfrak{M}^0$. This means that there exists an ε such that the set M contains a finite number of ε-complex functions. It is also obvious that for any $\varepsilon' < \varepsilon$ the set M contains a finite number of ε'-complex functions. By virtue of these circumstances, we have

$$\mathfrak{M}^0 = \mathfrak{M} \setminus \bigcup_{n=1}^{\infty} \mathfrak{M}^{\varepsilon_n},$$

where

$$\varepsilon_1 > \varepsilon_2 > \dots \quad \text{and} \quad \varepsilon_n \to 0 \, (n \to \infty).$$

On the basis of the preceding theorem $P(\mathfrak{M}^{\varepsilon_n}) = 0$ for any ε_n. Therefore

$$\mathbf{P}(\mathfrak{M}^0) = \mathbf{P}(\mathfrak{M}) = 1.$$

This proves the theorem.

The last theorem shows that we can construct with probability 1 a set M^0, which for any small positive number ε, no matter how small, contains an infinite number of ε-complex functions. Consequently, we can construct with probability 1, a set M^0, for which we have for a certain sequence of numbers $n_k (n_1 < n_2 < \dots)$

$$\frac{L_{M^0}(n_k)}{L(n_k)} \to 1 \, (k \to \infty).$$

We have thus obtained, with the aid of this algorithm a solution of the problem, stated at the beginning of this section. It must be emphasized in particular that the algorithm "constructs" the sought set with probability 1, without scanning through all the functions of algebraic logic. However, in the construction an element of chance is used. By virtue of this, we are not actually sure that any individual sample of the set M, constructed by this algorithm, will actually contain for each ε an infinite number of ε-complex functions. Furthermore, since the foregoing algorithm contains chance events, it is impossible to prove by any means whatever that the set M contains for every ε an infinite number of ε-complex functions. True, there may be encountered such algorithms of random acts, for which the proof of this fact is possible, but this will be evidence that the random acts can be excluded from the algorithm.

2. INVARIANT CLASSES AND THEIR PROPERTIES.

As already noted above, the class of all the functions of algebraic logic, which depend on not more than n variables, contains almost entirely functions which have a "complex" network realization. However, in practice as a rule one does not deal with arbitrary functions and, furthermore, with the "most complex ones." Naturally, the question arises of finding the classes of functions of algebraic logic, which are not of a simpler network realization, than in the general case. Examples of synthesis of networks for individual classes have been investigated by various authors /12--20/. In this section we shall construct and investigate a sufficiently large family of classes, containing apparently all the classes that arise in practice of networks synthesis.

Let $f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ be a function of algebraic logic. The variable x_i is called nonessential or fictitious if

$$f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \equiv f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n).$$

A variable which is not fictitious is called essential.

Definition. The functions f and g are called equal, if after eliminating the fictitious variables these functions go respectively to functions f' and g' such that

$$f' \equiv g'.$$

Thus, equal functions differ, perhaps, in the presence of fictitious variables. It is most natural to assume that if the function f is specified, then all the functions equal to it are also specified.

Definition. The set Q of functions of algebraic logic is called an invariant class if:

1) For each function $f(x_1, x_2, \ldots, x_n) \in Q$, the class Q contains all the functions equal to it;

2) For each function $f(x_1, x_2, \ldots, x_n) \in Q$ the class Q contains all the functions obtained from f by renaming (without identification) of the variables;*

3) For each function $f(x_1, x_2, \ldots, x_n) \in Q$ the class Q contains the functions obtained from f by any substitution of constants in place of variables (not necessarily all the variables).

Corollary. If an invariant class Q contains the function $f(x_1, x_2, \ldots, x_n) \not\equiv$ const, then Q contains both constants 0 and 1.

That this definition is natural is quite obvious from network considerations. In fact, if a network $\mathfrak{A}$ is constructed, realizing the function $f(x_1, x_2, \ldots, x_n)$, one can obtain without difficulty the

* From now on we shall understand by "renaming" of variables the renaming of variables without identification.

networks ~~that~~ realize the functions derivable from f by employing operations 1, 2, and 3. At the same time, if a certain class of functions is realized, it is possible to assume that it is an invariant class.

Let us give examples of invariant classes:

1. The class L of all linear functions, i.e., the functions $f(x_1, x_2, \ldots, x_n)$ for which the following representation is possible

$$f(x_1, x_2, \ldots, x_n) \equiv c_0 + c_1x_1 + c_2x_2 + \ldots + c_nx_n \pmod 2.$$

2. The class S of all symmetrical functions, i.e., functions $S(x_1, x_2, \ldots, x_n)$, the values of which do not change for any rearrangement of the essential variables.

3. The class P^N of all the functions of algebraic logic, which depend essentially on not more than N variables.

4. The class M of all monotonic functions, i.e., functions $f(x_1, x_2, \ldots, x_n)$ which can be specified in the form of a formula that contains only the operations $\&$ and $\vee$.

5. The class H_0 of all the functions $f(x_1, x_2, \ldots, x_n)$ which are identically equal to zero.

In view of their obvious nature, we shall omit the proofs of the invariance of these classes.

We now proceed to clarification of the descriptive structure of invariant classes. For this purpose we introduce several concepts.

Definition. The function $g(x_1, x_2, \ldots, x_n)$ is called the generating element for the invariant class Q, if $g \overline{\in} Q$ and either $g(x_1, x_2, \ldots, x_n) \equiv$ const with $n = 0$, or for any substitution of the constants we obtain the function $g' \in Q$.

Corollary. If $g(x_1, x_2, \ldots, x_n)$ is the generating element for the invariant class Q, then all the variables $x_1, x_2, \ldots, x_n$ are essential.

Two functions g_1 and g_2 will be called equivalent, if they are obtained from each other by renaming of the variables.

Let us construct maximal systems of pairwise nonequivalent generating elements for the invariant classes, listed in the foregoing examples:

1. Class L. Since from any nonlinear function one can construct [2] by means of operations 1, 2, and 3 a nonlinear function of two variables, and any function of one variable is linear, then the generating elements for the class L will be, accurate only to equivalence, a function of the type

$$xy + Ax + By + C \pmod 2.$$

There are only six pairwise nonequivalent functions of this type, namely:

$$g_1 = xy;\ g_2 = \bar{x}y;\ g_3 = \bar{x}\bar{y};\ g_4 = x \vee y;\ g_5 = \bar{x} \vee y;\ g_6 = \bar{x} \vee \bar{y}.$$

2. Class S. Since for each of the nonsymmetrical functions $h(x_1, x_2, \ldots, x_n)$ there exist two such variables* -- for the sake of definiteness let these be x_1 and x_2 -- for which

$$h(x_1, x_2, x_3, \ldots, x_n) \not\equiv h(x_2, x_1, x_3, \ldots, x_n),$$

then it is possible by substitution of variables to obtain from it a nonsymmetrical function of two variables. We have at most two nonequivalent generating elements

$$g_1 = x\bar{y};\ g_2 = x \vee \bar{y}.$$

3. Class P^N. It is easy to see that the maximum system of pairwise nonequivalent generating elements for P^N consists of functions which depend essentially on $N + 1$ variables, and is therefore finite.

4. Class M. Since from each nonmonotonic function it is possible, by substitution of the constants, to obtain a function x, all the generating elements for class M are equivalent, $g = \bar{x}$.

5. Class H_0. As follows from the foregoing corollary, the class H_0 has a single generating element $g \equiv 1$.

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* Since any substitution can be represented as a product of transpositions.

Definition. Let $g(x_1, x_2, \ldots, x_n)$ be an arbitrary function. The set Π_g of all the functions $h(y_1, y_2, \ldots,)$ each of which can be reduced to a function $g(x_1, x_2, \ldots, x_n)$ by substitution of constants and renaming of the variables, is called a pencil, generated by the function $g(x_1, x_2, \ldots, x_n)$.

Corollary 1. If g is the generating element for the invariant class Q, then $\Pi_g \subset CQ$.

2. If g_1 and g_2 are equivalent functions, then $\Pi_{g_1} = \Pi_{g_2}$.

3. If g_1 and g_2 are nonequivalent generating elements for the invariant class Q, then $g_1 \bar{\in} \Pi_{g_2}$ and $g_2 \bar{\in} \Pi_{g_1}$, and consequently not one of the pencils Π_{g_1} and Π_{g_2} contain each other.

From this we obtain directly theorems that explain the construction of invariant sets.

Theorem 2.1. For each invariant class Q the following identity holds

$$Q = C \bigcup_i \Pi_{g_i},$$

where the sum is taken over the maximum system of pairwise nonequivalent generating elements.

Proof. Let $h(x_1, x_2, \ldots, x_n) \in Q$. Then obviously $h \in C \bigcup_i \Pi_{g_i}$. Let now $h(x_1, x_2, \ldots, x_n) \bar{\in} Q$. Let us prove that $h \bar{\in} C \bigcup_i \Pi_{g_i}$.

If $n = 0$, then h is a constant and is therefore the generating element g_i for the class Q.

If $n \gg 1$, then in any substitution the constant h goes into a function that belongs to Q, or else there exists a substitution of constants, which transforms h into a function that does not belong to the class Q and depends on less than n variables. In the former case h is equivalent to certain generating element g_i for the class Q. In the latter case, after a finite number of steps we arrive at a function that is equivalent to a certain generating element g_i for the class Q. It is shown thereby that there exists a generating element g_i for the class Q, such that $h \in \Pi_{g_i}$, consequently, $h \in C \bigcup_i \Pi_{g_i}$

This proves the theorem completely.

This theorem allows us to express the classes L, S, P^N, M, and H_0 in terms of pencils that correspond to the maximal systems constructed above, which are pairwise not equivalent to the generating elements. For example,

$$S = C(\Pi_{x\bar{y}} \cup \Pi_{x\vee\bar{y}}) \quad \text{or} \quad M = C\Pi_{\bar{x}}.$$

It is obvious that the reverse proposition also holds.

Theorem 2.2. Let $G = \{g_i\}$ be an arbitrary subset of functions of algebraic logic. Then the class $Q = C \bigcup_i \pi$ is invariant and a certain maximal system of pairwise non-equivalent generating elements for the class Q are contained in G.

Naturally, this gives further rise to the question of the cardinality of the set of all the invariant classes of functions of algebraic logic. The answer to this question is given by the following theorem.

Theorem 2.3. The cardinality of the set of the invariant classes of functions that depend on the variables $x_1, x_2, \ldots, x_n, \ldots$, is equal to $\mathfrak{c}$. [illegible]

Proof. In view of the fact that each invariant class Q can be specified in the form C $\bigcup_i \Pi_{g_i}$, then the cardinality of interest to us is not greater than the cardinality of the set of sets $\{g_i\}$, i.e., $\mathfrak{c}$. Let us show that the cardinality of the set of all invariant classes is not less than $\mathfrak{c}$. For this purpose we construct a continual family of pairwise different invariant classes. We put

$$f_i = x_1 \ldots x_{i+1} \vee \bar{x}_1 \ldots \bar{x}_{i+1} \quad (i \geqslant 1).$$

It is easy to see that $f_i \overline{\in} \Pi_{f_j}$ when $i \neq j$. Let $\alpha = \{i_1, i_2, \ldots\}$ and $\beta = \{j_1, j_2, \ldots\}$ two different subsets of natural numbers. We denote by $E_\alpha = \{f_{i_1}, f_{i_2}, \ldots\}$ and $E_\beta = \{f_{j_1}, f_{j_2}, \ldots\}$. Obviously the classes Q_0^α and Q_0^β obtained from E_α and E_β by their closures with respect to the operations 1, 2, and 3, enumerated in the definition of invariant classes, are different invariant classes. Thus, the cardinality of the set of the classes Q_0^{α} is equal to the cardinality of the subsets $\{\alpha\}$ of the natural numbers, i.e., it is equal to $\aleph$. This proves the theorem.

Now let us proceed to a study of the metric properties of invariant classes.

We denote by $P_Q(n)$ (or respectively by $P^*_Q(n)$) the number of functions of the invariant class Q, which depend on n variables $x_1, x_2, \ldots, x_n$ (which respectively depend substantially on n variables $x_1, x_2, \ldots, x_n$). For what is to come it is useful to bear in mind the trivial relation between $P_Q(n)$ and $P^*_Q(n)$:

$$P_Q(n) = C_n^0 P_Q^*(0) + C_n^1 P_Q^*(1) + \ldots + C_n^n P_Q^*(n).$$

Theorem 2.4. If the invariant class Q does not contain all the functions of algebraic logic, i.e., $Q \neq P_2$, then $\frac{P_Q(n)}{2^{2^n}} \to 0$ as $n \to \infty$.

Proof. For $Q = \Lambda$* the statement is trivial, since $P_Q(n) = 0$.

Let now $Q \neq \Lambda$. Since $Q \neq P_2$, there exists such a number m, that a certain function $g(x_1, x_2, \ldots, x_m) \bar{\in}$ Let us take $n = m + k$ and consider arbitrarily the operation** $f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}) \in Q$. For

* Here Λ is an empty set.

**Such a functions always exists, since fictitious variables are admitted.

this function we can write out its expansion in the variables $x_{m+1}, \ldots, x_{m+k}$

$$f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+k}) = \bigvee_{\alpha_{m+1} \ldots \alpha_{m+k}} x_{m+1}^{\alpha_{m+1}} \ldots x_{m+k}^{\alpha_{m+k}} f(x_1, \ldots, x_m, \alpha_{m+1}, \ldots, \alpha_{m+k})$$

(where, as always, $x^0 = x$ and $x^1 = x$). It is obvious that

$$f(x_1, \ldots, x_m, \alpha_{m+1}, \ldots, \alpha_{m+k}) \in Q$$

for any of the assemblies $(\alpha_{m+1}, \ldots, \alpha_{m+k})$. Let ρ denote the number of functions that depend on the variables $x_1, \ldots, x_m$ and which do not belong to the class Q. By the nature of the construction $\rho > 0$. Hence

$$0 < P_Q(n) < (2^{2^m} - \rho)^{2^k} \qquad (n = m + k)$$

and

$$0 < \frac{P_Q(n)}{2^{2^n}} < \left(1 - \frac{\rho}{2^{2^m}}\right)^{2^k}.$$

From the latter inequality we obtain directly the required result.

The theorem just proved shows that the classes Q, which do not contain all the functions of algebraic logic, are liquid compared with the class P_2 of all functions of algebraic logic.

Theorem 2.5. The sequence of numbers $\left\{\sqrt[2^n]{P_Q(n)}\right\}$ tends, without increasing, to a limit and $1 \leq \lim_{n\to\infty} \sqrt[2^n]{P_Q(n)}$ for any non-empty invariant class Q (i.e., $Q \neq \Lambda$).

Proof. Let $f(x_1, x_2, \ldots, x_{n+1})$ be an arbitrary function from the class Q, depending on $(n + 1)$ variables.

Let us examine the expansion

$$f(x_1, \ldots, x_n, x_{n+1}) = x_{n+1} f(x_1, \ldots, x_n, 1) \vee \bar{x}_{n+1} f(x_1, \ldots, x_n, 0).$$

Since $f(x_1, \ldots, x_n, 1)$ and $f(x_1, \ldots, x_n, 0)$ belong to the class Q, then

$$P_Q(n+1) \leqslant P_Q^2(n).$$

Hence

$$\sqrt[2^n]{P_Q(n)} \geqslant \sqrt[2^{n+1}]{P_Q(n+1)} \geqslant \ldots$$

It is clear that when $Q \neq \Lambda$, then $1 \leqslant \sqrt[2^m]{P_Q(m)} \leqslant \sqrt[2^m]{2^{2^m}} = 2$. Consequently, $\lim_{n \to \infty} \sqrt[2^n]{P_Q(n)}$ exists and is included in the segment $[1, 2]$, q.e.d.

Corollary. If the invariant class $Q \neq P_2$ then

$$\lim_{n \to \infty} \sqrt[2^n]{P_Q(n)} < 2.$$

In fact, if $Q \neq P_2$, there exists a function

$$g(x_1, x_2, \ldots, x_m) \bar{\in} Q.$$

But then

$$\lim_{n \to \infty} \sqrt[2^n]{P_Q(n)} \leqslant \sqrt[2^m]{P_Q(m)} < 2.$$

Let us now calculate the values of $\lim_{n \to \infty} \sqrt[2^n]{P_Q(n)}$ for several invariant classes.

1. Class L. $P_L(n) = 2^{n+1} \Rightarrow \lim \sqrt[2^n]{P_L(n)} = \lim \sqrt[2^n]{2^{n+1}} = 1.$

2. Class S. $P_S^*(n) = 2^{n+1} - 2$ when $n > 0$ and $P_S^*(0) = 2$. Therefore $P_S(n) = 2(3^n - 2^n) + 2$ and $\lim_{n \to \infty} \sqrt[2^n]{P_S(n)} = \lim_{n \to \infty} \sqrt[2^n]{2(3^n - 2^n) + 2} = 1.$

3. Class P^N. $P_{P^N}(n) \leqslant C_n^N 2^{2^N} \Rightarrow 1 \leqslant \lim_{n \to \infty} \sqrt[2^n]{P_{P^N}(n)} \leqslant \lim_{n \to \infty} \sqrt[2^n]{C_n^N 2^{2^N}} = 1.$

4. Class M. It is easy to show that (see, for example, /21/) $P_M(n) < n^{C_n^{[\frac{n}{2}]}} < n^{\frac{c 2^n}{\sqrt{n}}}$

Hence
$$1 \leqslant \lim_{n\to\infty} \sqrt[2^n]{P_M(n)} \leqslant \lim_{n\to\infty} \sqrt[2^n]{n^{\frac{c2^n}{\sqrt{n}}}} = \lim_{n\to\infty} n^{\frac{c}{\sqrt{n}}} = 1.$$

5. *Class* H_0. $P_{H_0}(n) = 1 \Rightarrow \lim\limits_{n\to\infty} \sqrt[2^n]{P_{H_0}(n)} = \lim\limits_{n\to\infty} \sqrt[2^n]{1} = 1.$

6. *Class* Q_0^*. $P_{Q_0^*}(n) \leqslant 2^n - n - 1 + 2\cdot 2^n < 3\cdot 2^n$

$$1 \leqslant \lim_{n\to\infty} \sqrt[2^n]{P_{Q_0^*}(n)} \leqslant \lim_{n\to\infty} \sqrt[2^n]{3\cdot 2^n} = 1.$$

Thus, $\lim\limits_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 1$ for all the invariant classes Q constructed above. From this, in particular, on the basis of theorem (2.2) we have the following theorem.

Theorem 2.6. The cardinality of the set of invariant classes Q, for which $\lim\limits_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 1$, is $\aleph$.

In connection with the analysis of the foregoing examples, the question arises of whether there exist in general invariant classes Q, for which $1 < \lim\limits_{n\to\infty} \sqrt[2^n]{P_Q(n)} < 2$. To answer this question let us consider the following example.

Example. We denote by $Q_{1/2}$ the class consisting of the functions $f(x_1, x_2, \ldots, x_n)$ such that
$$f(x_1, x_2, \ldots, x_n) = f'(x_{i_1}, \ldots, x_{i_r}) \,\&\, f''(x_1, x_2, \ldots, x_n),$$
where $f'(x_{i_1}, \ldots, x_{i_r}) = x_{i_1} + \ldots + x_{i_r} + c \pmod 2$ and $f''(x_1, x_2, \ldots,$ is an arbitrary function of algebraic logic, all the essential variables of which are contained among the variables $x_{i_1}, \ldots, x_{i_r}$.

It is obvious that $Q_{1/2}$ is an invariant class. To estimate the number $P_{Q_{1/2}}(n)$ we turn to the formula

$f = f' \& f''$, from which it is easy to see that on all assemblies $\tilde{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ such that $f'(\tilde{\alpha}) = 0$, the function assumes a value of 0, and on the remaining ~~sets~~ assemblies we have $f = f''$. Consequently, the function f, for a fixed function f', is determined completely by indicating the subset of all such assemblies $\tilde{\alpha}$, for which $f'(\tilde{\alpha}) = 1$, and $f''(\tilde{\alpha}) = 0$. Let now $f' = x_1 + x_2 + \ldots + x_n$ (mod 2) or $f' = x_1 + x_2 + \ldots + x_n + 1$ (mod 2). Since both of these functions are equal to unity exactly on 2^{n-1} sets, then we obtain in each case $2^{2^{n-1}}$ different functions f, with 0 being the only function which will be constructed in either case. Hence $P_{Q1/2}(n) \geq 2 \times 2^{2^{n-1}} - 1$. On the other hand, if $f' = x_{i_1} + \ldots + x_{i_r}$ (mod 2), we obtain exactly $2^{2^{r-1}}$ of functions f, with $2^{2^{r-1}} \leq 2^{2^{n-1}}$. If it is considered that the number of linear functions that depend on n variables, $x_1, x_2, \ldots, x_n$ is equal to 2^{n+1}, we obtain $P_{Q1/2}(n) < 2^{n+1} \times 2^{2^{n-1}}$. Thus

$$2 \cdot 2^{2^{n-1}} - 1 \leq P_{Q_{1/2}}(n) < 2^{n+1} \cdot 2^{2^{n-1}}.$$

Hence

$$\lim_{n\to\infty} \sqrt[2^n]{P_{Q_{1/2}}(n)} = \sqrt{2}.$$

Let us see now if $\lim_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 2^{\sigma}$ for $Q \neq \Lambda$. Then $0 \leq \sigma \leq 1$. We ask ourselves, can we construct for any number σ, such that $0 \leq \sigma \leq 1$, an invariant class Q such that $\lim_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 2^{\sigma}$? If there is an affirmative answer to this question, it is desirable to ascertain the cardinality of the set of all the invariant classes corresponding to one and the same value σ. The remaining portion of this section is devoted to the solution of these problems. Before we answer these questions, we shall construct a special family $\{S_\sigma\}$ of the invariant classes S_σ of symmetrical functions.

Let $S(x_1, \ldots, x_n, y_n, \ldots, y_k)$ be an arbitrary symmetrical function and $x_1, x_2, \ldots, x_n$ be all its essential variables. Let us make up of these functions a cortege of n + 1 numbers $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ where

$$\gamma_i = \begin{cases} 0, & \text{if } S(\overbrace{1, \ldots, 1}^{i}, 0, \ldots, 0, y_1, \ldots, y_k) \equiv 0; \\ 1, & \text{if } S(1, \ldots, 1, 0, \ldots, 0, y_1, \ldots, y_k) \equiv 1. \end{cases} \quad (i = 0, 1, \ldots, n)$$

It is easy to see that the cortege $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ determines fully an assembly ~~a collection~~ of working numbers /19/, and at the same time the function itself* $s(x_1, \ldots, x_n, y_1, \ldots, y_k)$. We shall therefore from now on indicate sometimes in the symbol of a symmetrical function in the form of an index that cortege, which defines the given function, namely:

$$S(x_1, \ldots, x_n, y_1, \ldots, y_k) = S_{\gamma_0\gamma_1\ldots\gamma_n}(x_1, \ldots, x_n, y_1, \ldots, y_k) = S_{\gamma(n)}(x_1, \ldots, x_n, y_1, \ldots, y_k).$$

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* Accurate to the designation of the variables.

Let S_σ be a certain invariant class of symmetrical functions $S_{\gamma(n)}(x_1, \ldots, x_n, y_1, \ldots, y_k)$. We denote by $\Gamma_\sigma = \{\gamma(n)\}$ the set of all the corteges, corresponding to the functions from the class S_σ.

We shall henceforth call such families of corteges sets of type Γ. Since the class S_σ is established uniquely by means of the set Γ_σ, a study of the class S_σ reduces to a study of the set Γ_σ. It is obvious that not any set of corteges can be a set of type Γ. In order to clarify the characteristic property of the sets of type Γ, let us give the following definition.

Definition. The cortege $\gamma'(m) = (\gamma_p, \gamma_{p+1}, \ldots, \gamma_{p+m})$ is called a segment of the cortege $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ if $0 \leqslant p \leqslant p + m \leqslant n$.

Lemma 2.1. In order for a set of corteges $\gamma(n)$ to be a set of type Γ, it is necessary and sufficient that it contain together with any cortege $\gamma(n)$ any of its segments $\gamma'(m)$.

Proof. In fact, let us consider a symmetrical function $S_{\gamma(n)}(x_1, \ldots, x_n, y_1, \ldots, y_k)$. If the constants are substituted in this function, it goes into a certain symmetrical function

$$S(x_{i_1}, \ldots, x_{i_m}, y_{j_1}, \ldots, y_{j_r}),$$

where

$$\{x_{i_1}, \ldots, x_{i_m}\} \subset \{x_1, \ldots, x_n\} \not\subset \{y_{j_1}, \ldots, y_{j_v}\} \subset \{y_1, \ldots, y_k\}.$$

Let this function be defined by the cortege $\gamma'(m)$. Let us assume that a given substitution of constants converts p essential variables from $x_1, \ldots, x_r$ into 1, then

$$\gamma'(m) = (\gamma_p, \gamma_{p+1}, \ldots, \gamma_{p+m}).$$

Consequently, $\gamma'(m)$ is a segment of the cortege $\gamma(n)$. The opposite is also true: if the cortege $\gamma'(m)$ is a segment of the cortege $\gamma(n)$ and $S_{\gamma(n)}$, $S_{\gamma'(m)}$ are any symmetrical functions defined by these corteges, then the function $S_{\gamma'(m)}$ is obtained from the function $S_{\gamma(n)}$ by employing operations 1, 2, and 3 (see definition of the invariant class). From this we readily extract also the proof of the lemma.

Thus, Γ_6 together with any cortege contains all its segments.

We shall now study the structure of the corteges $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$

For this purpose we introduce a series of numerical characteristics. Let us assign to each segment $\gamma'(m) = (\gamma_p, \gamma_{p+1}, \ldots \gamma_{p+m})$ of cortege $\gamma(n)$ a number $\gamma_p(m)$, defined by the formula

$$\gamma_p(m) = \frac{\gamma_p + \gamma_{p+1} + \cdots + \gamma_{p+m}}{m+1}$$

and called its characteristic.

Obviously, the characteristic $\gamma_p(m)$ represents the average density of distribution of the numbers 1 in the segment $\gamma'(m)$. In particular, $\nu_0(n)$ is the average density of distribution of the numbers 1 in the initial cortege $\gamma(n)$. Let furthermore

$$\lambda_{m+1} = \min_{0 \leq p \leq n-m} \nu_p(m) \ \& \ \mu_{m+1} = \max_{0 \leq p \leq n-m} \nu_p(m)$$

(here the minimum and the maximum are taken over all the segments of m + 1 numbers). We then obtain two corteges $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{n+1})$ such that

$$\lambda_{m+1} < \nu_p(m) < \mu_{m+1} \ (m = 0, 1, \ldots, n) \ \& \ \lambda_{n+1} = \mu_{n+1} = \nu_0(n).$$

We shall call these corteges respectively the λ and μ corteges.

Examples:

1) $\gamma = (0, 0, 0, 1, 1, 1)$. Then, $\lambda = (0, 0, 0, 1/4, 2/5, 3/6)$ and $\mu = (1, 1, 1, 3/4, 3/5, 3/6)$;

2) $\gamma = (0, 0, 1, 0, 1, 1)$. Then, $\lambda = (0, 0, 1/3, 1/4, 2/5, 3/6)$ and $\mu = (1, 1, 2/3, 3/4, 3/5, 3/6)$;

3) $\gamma = (0, 1, 0, 0, 1, 1)$. Then, $\lambda = (0, 0, 1/3, 1/4, 2/5, 3/6)$ and $\mu = (1, 1, 2/3, 2/4, 3/5, 3/6)$;

4) $\gamma = (0, 1, 0, 1, 0, 1)$. Then, $\lambda = (0, 1/2, 1/3, 2/4, 2/5, 3/6)$ and $\mu = (1, 1/2, 2/3, 2/4, 3/5, 3/6)$

It is easy to see that λ and μ corteges characterize the structure of the cortege, i.e., the distribution of the zeros and units in the cortege.

Let σ be an arbitrary real number such that $0 \leqslant \sigma \leqslant 1$. We denote by $\frac{\alpha_1}{1}, \frac{\alpha_2}{2}, \ldots \frac{\alpha_m}{m}$ and

$\frac{\beta_1}{1}, \frac{\beta_2}{2}, \ldots \frac{\beta_m}{m}$ its best approximation with shortage and with excess among the fractions having denominators respectively 1, 2, ..., m, ... Then

$$0 < \sigma - \frac{\alpha_m}{m} < \frac{1}{m} \text{ и } 0 \leqslant \frac{\beta_m}{m} - \sigma < \frac{1}{m} \qquad (*)$$

Let us prove one auxiliary proposition, concerning the best approximations of a given type.

Lemma 2.2. If $\frac{\beta_1}{1}$ $\frac{\beta_2}{2}, \ldots \frac{\beta_m}{m}, \ldots$ are the best approximations with excess for a real number σ, where $0 \leqslant \sigma \leqslant 1$, then

$$\beta_m \leqslant \beta_{m+1} \leqslant \beta_m + 1,$$

i.e., the numerators $\beta_1, \beta_2, \ldots, \beta_m, \ldots$ do not decrease monotonically.

Proof. From (*) we have

$$\frac{\beta_{m+1}}{m+1} \geqslant \sigma > \frac{\beta_m - 1}{m} > \frac{\beta_m - 1}{m+1}.$$

Hence $\beta_{m+1} > \beta_m - 1$ or $\beta_{m+1} \geqslant \beta_m$. On the other hand, it follows from (*) that

$$\left| \frac{\beta_{m+1}}{m+1} - \frac{\beta_m}{m} \right| < \frac{1}{m}. \qquad (**)$$

Let us put $\beta_{m+1} = \beta_m + \vartheta$. Let us show that $\vartheta \leqslant 1$. We distinguish two cases:

1) $\beta_m = m$. It then follows from $\frac{\beta_{m+1}}{m+1} \leqslant 1$ that $\beta_m + \vartheta \leqslant m + 1$ or $\vartheta \leqslant 1$;

2) $\beta_m < m$. Let us assume $\vartheta \geqslant 2$. Then

$$\frac{\beta_{m+1}}{m+1} - \frac{\beta_m}{m} = \frac{\vartheta \cdot m - \beta_m}{m(m+1)} \geqslant \frac{m - \beta_m + m}{m(m+1)} \geqslant \frac{1}{m}$$

(since $m - \beta_m > 0$, i.e., $m - \beta_m \geqslant 1$).

The last equation contradicts (**). Therefore

$\nu^c \leqslant 1$. This proves the lemma completely.

Corollary. If $\sigma < 1$ and m_0 is such that $\beta_{m_0} < m_0$, then for $m > m_0$ we have

$$\beta_m < m.$$

Let us put $\sigma = \nu_0(n)$. It is obvious that the initial cortege $\gamma(n)$ cannot have a uniform average density, equal to σ over all the segments, if $0 < \sigma < 1$. In fact, for $0 < \sigma < 1$ one can always find such $m \leqslant n$, that $\frac{\alpha_{m+1}}{m+1} < \sigma < \frac{\beta_{m+1}}{m+1}$, i.e., σ cannot be represented in the form of a fraction with a denominator equal to $m + 1$. Therefore the average density $\nu_p(m) \neq \sigma$ for any value of p. Thus, with the exception of two trivial cases, the distribution of units in the cortege $\gamma(n)$ can not be ideally uniform. To be sure, the average density $\nu_p(m)$ can approach the number σ, namely when $\nu_p(m) = \frac{\sigma_{m+1}}{m+1}$ or $\nu_p(m) = \frac{\beta_{m+1}}{m+1}$. In this connection, it is natural to define further the concept of uniform distribution, in the following manner.

Definition. A cortege $\gamma(n)$ with characteristic $\nu_0(n) = \sigma$ is called uniform if

$$\frac{\alpha_{m+1}}{m+1} \leqslant \nu_p(m) \leqslant \frac{\beta_{m+1}}{m+1}$$

for any m ($m = 0, 1, \ldots, n$) and p ($p = 0, 1, \ldots, n - m$), where $\frac{\sigma_{m+1}}{m+1}$ and $\frac{\beta_{m+1}}{m+1}$ are the best approximations with numerator $m + 1$ of the number σ on the low and on the high side.

It is easy to see that the cortege $\gamma = (0, 1, 0, 1, 0, 1)$ is uniform. Consequently, uniform corteges do exist. It is seen from this also that the concept of uniformity introduced here corresponds to our intuitive concept of uniformity. However, the difinition becomes completely meaningful if we succeed in showing that uniform corteges exists for any characteristic $\nu_0(n)$.

Lemma 2.3. For any rational number ℓ/q, where $0 \leq \ell/q \leq 1$ (the numbers ℓ and q need not necessarily be relatively prime), it is possible to construct a uniform cortege $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ with the characterist... $\nu_0(n) = \ell/q$.

Proof. Let us put n = q - one and let $\frac{\beta_1}{1}, \frac{\beta_2}{2}, \ldots \frac{\beta_{n+1}}{n+1}$ be the approximations of the number ℓ/q on the high side, the best among the fractions with denominators 1, 2, ..., n + 1 respectively. Obviously $\frac{\beta_{n+1}}{n+1} = \frac{\ell}{q}$ (since n + one = q). Let us show that the requirements of the lemma are satisfied by the cortege

$$\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n),$$

where

$$\begin{cases} \gamma_0 = \beta_1, \\ \gamma_m = \beta_{m+1} - \beta_m \ (m = 1, 2, \ldots, n). \end{cases}$$

From the preceding lemma it follows that γ_m is equal either to 0 or to 1.

Obviously, for each initial segment $(\gamma_0, \gamma_1, \dots, \gamma_m)$

$$\nu_0(m) = \frac{\gamma_0+\gamma_1+\dots+\gamma_m}{m+1} = \frac{\beta_{m+1}}{m+1} \quad (m = 0,\ 1,\ \dots,\ n).$$

Let us assume that for a certain m_0 there is a segment $(\gamma_p, \gamma_{p+1}, \dots, \gamma_{p+m_0})$ such that

$$\nu_p(m_0) > \frac{\beta_{m_0+1}}{m_0+1}, \text{ т. е. } \frac{\gamma_p+\gamma_{p+1}+\dots+\gamma_{p+m_0}}{m_0+1} > \frac{\beta_{m_0+1}}{m_0+1}.$$

Then $\gamma_{p+m_0+1} - \beta_p > \beta_{m_0+1}$, i.e., $\beta_{p+m_0+1} \geqslant \beta_{m_0+1} + 1$. From this we have, taking into consideration that $p \neq 0$

$$\frac{\beta_{p+m_0+1}}{p+m_0+1} > \frac{\beta_{p+m_0+1}}{p+m_0+1} + \frac{1}{p+m_0+1} \geqslant \min\left(\frac{\beta_p}{p}, \frac{\beta_{m_0+1}}{m_0+1}\right) + \frac{1}{p+m_0+1} > \frac{l}{q} + \frac{1}{p+m_0+1}.$$

At the same time, since $\frac{\beta_{p+m_0+1}}{p+m_0+1}$ is the best approximation of the number $1/q$ on the high side, among the fractions with denominators $p + m_0 + 1$

$$\frac{\beta_{p+m_0+1}}{p+m_0+1} < \frac{l}{q} + \frac{1}{p+m_0+1}.$$

We have arrived at a contradiction. Therefore always

$$\nu_p(m) < \frac{\beta_{m+1}}{m+1} \ \& \ \nu_0(m) = \frac{\beta_{m+1}}{m+1}.$$

Consequently,

$$\mu_{m+1} = \max_{0<p\leqslant n-m} \nu_p(m) = \frac{\beta_{m+1}}{m+1}.$$

Let us examine the numbers

$$\begin{cases} \alpha'_m = \beta_{n+1} - \beta_{n+1-m} & (m = 1,\ 2,\ \dots,\ n). \\ \alpha'_{n+1} = \beta_{n+1}. \end{cases}$$

We shall show that the fraction α'_m/m are the best approximations of the number l/q, on the low side, among the fractions with denominator equal to m, i.e., we shall show that

$$\frac{\alpha'_m}{m}=\frac{\alpha_m}{m}.$$

If m = n + 1, the statement is trivial, since

$$\frac{\alpha'_{n+1}}{n+1}=\frac{\beta_{n+1}}{n+1}=\frac{l}{q}.$$

Let now m $<$ n + 1. First we shall establish that $\dfrac{\alpha'_m}{m} \leq \dfrac{l}{q}$. Inasmuch as

$$\frac{l}{q}=\frac{\beta_{n+1}}{n+1}=\frac{\beta_{n+1}-\beta_{n+1-m}+\beta_{n+1-m}}{m+(n+1-m)}=\frac{\alpha'_m+\beta_{n+1-m}}{m+n+1-m}=\frac{l}{q}<\frac{\beta_{n+1-m}}{n+1-m}.$$

then

$$\frac{\alpha'_m}{m}<\frac{l}{q}.$$

More accurately, $\dfrac{\alpha'_m}{m}<\dfrac{l}{q}$ if $\dfrac{l}{q}<\dfrac{\beta_{n+1-m}}{n+1-m}$ and $\dfrac{\alpha'_m}{m}=\dfrac{l}{q}$ if $\dfrac{l}{q}=\dfrac{\beta_{n+1-m}}{n+1-m}$.

In order to show that $\dfrac{\alpha'_m}{m}=\dfrac{\alpha_m}{m}$, it remains to establish that

$$\frac{\alpha'_m}{m}>\frac{l}{q}-\frac{1}{m}.$$

Assume that this is not so. Then

$$\frac{\alpha'_m}{m}<\frac{l}{q}-\frac{1}{m},$$

i.e.,

$$\frac{\beta_{n+1}-\beta_{n+1-m}+1}{m}<\frac{\beta_{n+1}}{n+1}.$$

Hence

$$(n+1-m)\beta_{n+1}-(n+1)\beta_{n+1-m}+n+1<0.$$

Dividing each term by (n + 1)(n + 1 - m), we obtain

$$\frac{\beta_{n+1}}{n+1}-\frac{\beta_{n+1-m}}{n+1-m}+\frac{1}{n+1-m}<0 \text{ or } \frac{l}{q}<\frac{\beta_{n+1-m}-1}{n+1-m}.$$

The latter contradicts the fact that $\frac{\beta_{n+1-m}}{n+1-m}$ is the best approximation of the number l/q with excess amont the fractions with denominator n + 1 - m.

We have thus established that

— We shall now show that $\frac{\alpha'_m}{m} = \frac{\alpha_m}{m}$.

$$\nu_p(m) \geqslant \frac{a_{m+1}}{m+1}.$$

For the end segment of the form (γ_{n-m}, ..., γ_n), we have

$$\nu_{n-m}(m) = \frac{\gamma_{n-m} + \dots + \gamma_n}{m+1} = \frac{\beta_{n+1} - \beta_{n-m}}{m+1} = \frac{a_{m+1}}{m+1}.$$

Let us now establish that for any m and for any p($0 \leqslant p \leqslant n - m$)

$$\nu_p(m) \geqslant \frac{a_{m+1}}{m+1}.$$

If this statement is not true, then there exists an m_0 and a certain number p_0($p_0 < n - m_0$) such that

$$\nu_{p_0}(m_0) < \frac{a_{m_0+1}}{m_0+1}, \text{ i. e. } \frac{\gamma_{p_0} + \dots + \gamma_{p_0+m_0}}{m_0+1} < \frac{a_{m_0+1}}{m_0+1}.$$

Inasmuch as

$$\gamma_{p_1} + \dots + \gamma_{p_0+m_0} = \beta_{p_0+m_0+1} - \beta_{p_0} = (\beta_{n+1} - \beta_{p_0}) - (\beta_{n+1} - \beta_{p_0+m_0+1}) = \\ = a_{n+1-p_0} - a_{n-p_0-m_0},$$

then $$a_{n+1-p_0} - a_{n-p_0-m_0} < a_{m_0+1} \text{ i. e. } a_{n+1-p_0} \leqslant a_{n-p_0-m_0} + a_{m_0} - 1.$$

Hence $$\frac{a_{n+1-p_0}}{n+1-p_0} \leqslant \frac{a_{n-p_0-m_0} + a_{m_0+1}}{(n-p_0+m_0)+m_0+1} - \frac{1}{n+1-p_0} \leqslant \\ \leqslant \max\left(\frac{a_{n-p_0-m_0}}{n-p_0-m_0}, \frac{a_{m_0+1}}{m_0+1}\right) - \frac{1}{n+1-p_0} < \frac{l}{q} - \frac{1}{n+1-p_0}.$$

At the same time, since $\frac{\alpha_{n+1-p_0}}{n+1-p_0}$ is the best approximation of the number l/q on the low side among fractions with a denominator $n + 1 - p_0$,

$$\frac{\alpha_{n+1-p_0}}{n+1-p_0} > \frac{l}{q} - \frac{1}{n+1-p_0}.$$

We have arrived at a contradiction. Therefore

$$\nu_p(m) \geqslant \frac{\alpha_{m+1}}{m+1}, \text{ and since } \nu_{n-m}(m) = \frac{\alpha_{m+1}}{m+1}$$

then for any m

$$\lambda_{m+1} = \min_{0 \leqslant p \leqslant n-m} \nu_p(m) = \frac{\alpha_{m+1}}{m+1}.$$

Consequently,

$$\frac{\alpha_{m+1}}{m+1} \leqslant \nu_p(m) \leqslant \frac{\beta_{m+1}}{m+1} \quad (m = 0, 1, \ldots, n+1;\ p = 0, 1, \ldots, n-m)$$

This proves the lemma completely.

Note. We have established simultaneously that

$$\lambda_m = \frac{\alpha_m}{m} \quad \text{and} \quad \mu_m = \frac{\beta_m}{m}.$$

We have thus shown that for any rational number l/q it is possible to construct a uniform cortege with a characteristic l/q. The question of the number of such corteges, having a length $n = q - 1$, we shall leave aside. Let us give an example based on the construction of uniform corteges.

Example. It is required to construct a uniform cortege for the number 10/13 (n = 12). Let us write out the best approximations on the high side of the number 10/13, having the respective denominators 1, 2, ..., 13:

$$\frac{\beta_1}{1}=\frac{1}{1},\ \frac{\beta_2}{2}=\frac{2}{2},\ \frac{\beta_3}{3}=\frac{3}{3},\ \frac{\beta_4}{4}=\frac{4}{4},\ \frac{\beta_5}{5}=\frac{4}{5},\ \frac{\beta_6}{6}=\frac{5}{6},\ \frac{\beta_7}{7}=\frac{6}{7},$$
$$\frac{\beta_8}{8}=\frac{7}{8},\ \frac{\beta_9}{9}=\frac{7}{9},\ \frac{\beta_{10}}{10}=\frac{8}{10},\ \frac{\beta_{11}}{11}=\frac{9}{11},\ \frac{\beta_{12}}{12}=\frac{10}{12},\ \frac{\beta_{13}}{13}=\frac{10}{13}.$$

The sought cortege has the form (11110 1110 1110).

Let σ be an arbitrary real number such that $0 \leqslant \sigma \leqslant 1$ and

$$\frac{\alpha_1}{1},\ \frac{\alpha_2}{2},\ \ldots,\ \frac{\alpha_n}{n},\ \ldots;\ \frac{\beta_1}{1},\ \frac{\beta_2}{2},\ \ldots,\ \frac{\beta_n}{n},\ \ldots$$

are its best approximations on the low side and on the high side, among all the fractions with the denominators equal respectively to 1, 2, ..., n, ... Then

$$\frac{\alpha_n}{n} < \sigma \leqslant \frac{\beta_n}{n} \text{ and } \beta_n - \alpha_n \leqslant 1 \quad (n = 1, 2, \ldots).$$

The set of all the uniform corteges with characteristics $\nu_0(n) = \alpha_{n+1}/(n+1)$ and $\nu_0(n) = \beta_{n+1}/(n+1)$ $(n = 0, 1, \ldots)$ and all the segments of these corteges, by virtue of lemma 2.1 is a set of type Γ. We denote it by Γ_σ, and the corresponding invariant class of symmetrical functions will be denoted by S_σ.

Lemma 2.4. If a cortege $\gamma'(m) = (\gamma_p, \gamma_{p+1}, \ldots, \gamma_{p+m}) \in \Gamma_\sigma$ then

$$\sigma - \frac{2}{m+1} < \nu_p(m) < \sigma + \frac{2}{m+1}.$$

Proof. By definition this cortege is a segment of a certain cortege $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \Gamma_\sigma$, with $p \geqslant 0$ and $m \leqslant n$. ν is the characteristic of the cortege $\gamma(n)$ and satisfies the inequality

$$\sigma - \frac{1}{n+1} < \nu_0(n) < \sigma + \frac{1}{n+1}.$$

inasmuch as

$$\sigma - \frac{1}{n+1} < \frac{\alpha_{n+1}}{n+1} < \sigma \leqslant \frac{\beta_{n+1}}{n+1} < \sigma + \frac{1}{n+1}.$$

Let $\alpha^{n+1}_{m+1}/(m+1)$ and $\beta^{n+1}_{m+1}/(m+1)$ denote the best approximations of the numbers $\alpha_{n+1}/(n+1)$ and $\beta_{n+1}/(n+1)$ respectively on the low and on the high side among all the fractions with denominator m + 1. Then

$$\frac{\alpha^{n+1}_{m+1}}{m+1} > \frac{\alpha_{n+1}}{n+1} - \frac{1}{m+1}$$

and

$$\frac{\beta_{m+1}^{n+1}}{m+1} < \frac{\beta_{n+1}}{n+1} + \frac{1}{m+1}.$$

Consequently

$$\frac{\alpha_{m+1}^{n+1}}{m+1} > \frac{\alpha_{n+1}}{n+1} - \frac{1}{m+1} > \sigma - \frac{1}{n+1} - \frac{1}{m+1} \geqslant \sigma - \frac{2}{m+1}$$

and

$$\frac{\beta_{m+1}^{n+1}}{m+1} < \frac{\beta_{n+1}}{n+1} + \frac{1}{m+1} < \sigma + \frac{1}{n+1} + \frac{1}{m+1} \leqslant \sigma + \frac{2}{m+1}.$$

Since by virtue of the uniformity of the cortege $\gamma(n)$ we have

$$\frac{\alpha_{m+1}^{n+1}}{m+1} \leqslant \nu_p(m) \leqslant \frac{\beta_{m+1}^{n+1}}{m+1},$$

then

$$\sigma - \frac{2}{m+1} < \nu_p(m) < \sigma + \frac{2}{m+1}.$$

This proves the lemma completely.

Lemma 2.5. $\Gamma_{\sigma_1} \neq \Gamma_{\sigma_2}$ if $\sigma_1 \neq \sigma_2$.

Proof. Let $\sigma_1 \neq \sigma_2$. To be specific, we put $\sigma_1 < \sigma_2$. We choose m_0 such that

$$\sigma_1 + \frac{2}{m_0+1} < \sigma_2 - \frac{2}{m_0+1}.$$

Assume that there exists a cortege $\gamma'(m)=(\gamma_p, \gamma_{p+1}, \ldots, \gamma_{p+m})$ with $m \geqslant m_0$, which belong simultaneously to the sets Γ_{σ_1} and Γ_{σ_2}. Then, using the preceding lemma, we obtain

$$\sigma_1 - \frac{2}{m+1} < \nu_p(m) < \sigma_1 + \frac{2}{m+1}$$

and

$$\sigma_2 - \frac{2}{m+1} < \nu_p(m) < \sigma_2 + \frac{2}{m+1}.$$

Since $m > m_0$, these inequalities are incompatible with the initial equation. Consequently, the cortege $\gamma'(m)$ $(m > m_0)$ cannot belong simultaneously to the two sets Γ_{σ_1} and Γ_{σ_2}. From this we obtain the required results directly, namely

$$\Gamma_{\sigma_1} \neq \Gamma_{\sigma_2} \text{ where } \sigma_1 \neq \sigma_2.$$

At the same time we established the following stronger fact: the sets Γ_{σ_1} and Γ_{σ_2} can have as common only a finite number of corteges -- corteges with length not exceeding m_0.

Corollary. The family $\{\Gamma_\sigma\}$ has a cardinality $2^{[c]}$.

As already noted, each set Γ_σ defines simultaneously an invariant class S of symmeteric functions. We therefore obtain from the lemma just proved the following:

Theorem 2.7. $S_{\sigma_1} \neq S_{\sigma_2}$, if $\sigma_1 \neq \sigma_2$

Corollary. The family $\{S_\sigma\}$ has a cardinality 2.

Let us prove still another auxiliary statement concerning the functions of the classes S_σ. Let $\sigma < 1$. On the basis of lemma 2.4 there exists such a number $N_0 = N_0(\sigma)$, that when $m > N_0$, for any segment $\gamma'(m) \in \Gamma_\sigma$, $\nu_p(m) < 1$, i.e., the cortege $\gamma'(m)$ cannot consist of unities only.

Lemma 2.6. Let $S(x_1, \ldots, x_n, y_1, \ldots, y_k)$ be a symmetrical function from the class S_σ ($\sigma < 1$), which depends essentially on the variables $x_1, x_2, \ldots, x_n$.

Then the function S' obtained from S by substitution of constants instead of d essential variables ($d \leq n$) and instead of certain unessential variables, is a symmetrical function, whereas if $n - d > N_0$, this function either vanishes identically, or depends essentially on $n - d$ variables.

Proof. The symmetry of function of S' is obvious. Let $n - d > N_0$. If at least on of the $n - d$ variables $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-d}}$ ($1 \leq i_1 \leq i_2 \leq \ldots \leq i_{n-d} \leq n$) which remain unreplaced by constants is such that the function S' depends on it in a nonessential manner, then by virtue of the symmetry it depends in a nonessential manner on all the variables of this group. Therefore, the only functions which contain only nonessential variables, are constants, and in this case we have either the constant 0 or the constant 1.

At the same time, the functions S' correspond to the cortege $\gamma'(n-d)$ of the set Γ_σ. Since $n-d > N_0$, the characteristic $\nu_p(n-d)$ of the cortege $\gamma'(n-d)$ satisfies the inequality $\nu_p(n-d) < 1$. The latter signifies that $S' \neq 1$. Consequently, when $n-d > N_0$, either the function S' depends essentially on the variables $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-d}}$ or else is the constant 0. This proves the lemma.

Let us now construct a family of classes Q_σ ($0 \leq \sigma \leq 1$), starting out with classes S_σ. Let, as before, $N_0 = N_0(\sigma)$, where $\sigma < 1$ denotes such a number, that when $n > N_0$ we have

$$\nu_\sigma(n) < 1$$

We denote, furthermore, by P^{N_0} the invariant class consisting of all the functions which depend essentially on not more than N_0 variables, and by Q' the class of all the functions $f(x_1, \ldots, x_n, z_1, \ldots, z_m)$ of the form

$$f(x_1, \ldots, x_n, z_1, \ldots, z_m) = \\ = \& S(x_1, \ldots, x_n, y_1, \ldots, y_k) f'(x_{i_1}, \ldots, x_{i_r}, u_1, \ldots, u_q),$$

where S is an arbitrary function from the class S_σ, $x_1, \ldots, x_n$ -- are all essential variables of the functions S; f' -- arbitrary function of algebraic logic such that its essential variables $x_{i_1}, \ldots, x_{i_r}$ are contained among the variables $x_1, \ldots, x_n$, i.e.,

$$\{x_{i_1}, \ldots, x_{i_r}\} \subset \{x_1, \ldots, x_n\};$$

the set of inessential variables $\{z_1, \ldots, z_m\}$ of the function f is the joining of the sets of inessential variables $\{y_1, \ldots, y_k\}$ and $\{u_1, \ldots, u_q\}$, i.e.

$$\{z_1, \ldots, z_m\} = \{y_1, \ldots, y_k\} \cup \{u_1, \ldots, u_q\}.$$

Let us put finally

$$Q_\sigma = \begin{cases} Q'_\sigma \cup P^{N_0} & \text{when} \quad 0 \leq \sigma < 1, \\ P_2 & \text{when} \quad \sigma = 1. \end{cases}$$

Theorem 2.8. The classes Q are invariant and $Q_{\sigma_1} \neq Q_{\sigma_2}$ if

Proof. Let us show first that the class Q_σ is invariant. The statement is quite obvious if $Q_\sigma = P_2$.

Let now $f \in Q_\sigma$ and $Q_\sigma \neq P_2$. Two cases are possible:

a) $f \in P^{N_0}$. Then any function which is obtained from f by applying operations 1, 2, or 3 (see definition of invariant class) belongs to this class by virtue of the invariance of the class P^{N_0}.

b) $f \in Q'_\sigma$ and $f \bar{\in} P^{N_0}$. In this case

$$f(x_1, \ldots, x_n, z_1, \ldots, z_m) = \\ = S(x_1, \ldots, x_n, y_1, \ldots, y_k) \& f'(x_{i_1}, \ldots, x_{i_r}, u_1, \ldots, u_q),$$

with $n > N_0$. Let us verify that upon the substitution of the constants (3rd operation) we obtain a function from Q_σ. In fact, let us insert d constants instead of d essential variables and a certain number of constants instead of the nonessential variables. Then if $n-d \leq N_0$, the function obtained belongs to class P^{N_0}, but if $n-d > N_0$, then on the basis of the preceding lemma it is either a constant 0 (and therefore again belongs to class P^{N_0}), or depends essentially on all the remaining n-d variables from among the variables $x_1, \ldots, x_n$, and then it has for it the same representation as for the initial function, since it belongs to the class Q'_σ. Verification of conditions 1 and 2 (see definition of the invariant class) is obvious. We have thus established that the class Q is invariant.

Let us show now that $Q_{\sigma_1} \neq Q_{\sigma_2}$ if

$\sigma_1 \neq \sigma_2$ $(0 < \sigma_1, \sigma_2 \leq 1)$. To be specific, we put $\sigma_1 < \sigma_2$. We consider the sets Γ_{σ_1} and Γ_{σ_2}. On the basis of lemma 2.4 we have the following estimates

$$\text{for } \gamma'(n) \in \Gamma_{\sigma_1}: \qquad \sigma_1 - \frac{2}{n+1} < \nu_0'(n) < \sigma_1 + \frac{2}{n+1},$$

$$\text{for } \gamma''(n) \in \Gamma_{\sigma_2}: \qquad \sigma_2 - \frac{2}{n+1} < \nu_0''(n) < \sigma_2 + \frac{2}{n+1}.$$

Assume that n is chosen such that

$$\sigma_1 + \frac{2}{n+1} < \sigma_2 - \frac{2}{n+1} \quad \& \quad n > N_0' \quad n > N_0''$$

where $N_0' = N_0(\sigma_1)$ & $N_0'' = N_0(\sigma_2)$.

When the symmetrical function $S_{\gamma''(n)} \in Q_{\sigma_2}$ and $S_{\gamma'(n)} \overline{\in} Q_{\sigma_1}$, since in the opposite case

$$S_{\gamma''(n)}(x_1, \ldots, x_n, z_1, \ldots, z_m) = \\ = S_{\gamma'(n)}(x_1, \ldots, x_n, y_1, \ldots, y_k) \,\&\, f'(x_{i_1}, \ldots, x_{i_r}, u_1, \ldots, u_q),$$

which is impossible, since the average density $\nu_0''(n)$ of the cortege $\gamma_0''(n)$ is greater than the average density $\nu_0'(n)$ of the cortege $\gamma'(n)$. This proves the theorem completely.

Let $0 \leq \sigma \leq 1$ and $\gamma(n) = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be an arbitrary cortege from Γ_σ. Let us consider the sum

$$\mathfrak{S}(\gamma(n)) = \gamma_0 C_n^0 + \gamma_1 C_n^1 + \ldots + \gamma_n C_n^n.$$

It is easily seen that this sum characterizes the number of aggregates in which the symmetrical function

$$S_{\gamma(n)}(x_1, x_2, \ldots, x_n)$$

turns into unity.

We shall study the asymptotic behavior of the quantity $\gamma(\gamma(n))$ as $n \longrightarrow \infty$. For this purpose we use the local and integral limit theorems of Laplace /22/. Let us recall the formulation of these theorems.

Let $P_n(k) = C_n^k p^k q^{n-k}$, where $p + q = 1$ and $0 < p$, $q < 1$. Let us make the following transformation on the graph of the function $P_n(k)$: 1) we shift the graph to the left by an amount np, 2) we contract the x axis by $\sqrt{npq}$ times, and 3) we stretch the axis y by $\sqrt{npq}$ times. Obviously, under this transformation a point with abscissa k will go into a point with abscissa $x_k^{(n)} = \frac{k-np}{\sqrt{npq}}$. As to the value of the function at this point, it will be

$$\sqrt{npq}\, P_n(k) = \sqrt{npq}\, P_n(np + x_k^{(n)} \sqrt{npq}).$$

Local Limit Theorem. If $0 < p, q < 1$, then $\sqrt{npq}\, P_n(np + x_k^{(n)} \sqrt{npq}) : \frac{1}{\sqrt{2\pi}} \exp(-x_k^{(n)2}/2) \longrightarrow 1$ uniformly over $x_k^{(n)}$, as $n \longrightarrow \infty$. We denote by $P_n(a, b)$ the sum

$$\sum_{np+a\sqrt{npq} < k < np+b\sqrt{npq}} P_n(k).$$

Integral Limit Theorem. If $0 \leq p, q < 1$, $a < b$, then as $n \longrightarrow \infty$

$$P_n(a, b) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}} dz.$$

We now formulate and prove a lemma that characterizes the asymptotic behavior of the quantity $\sigma(\gamma(n))$.

Lemma. $\frac{\mathfrak{S}(\gamma(n))}{2^n} \to s \ (n \to \infty)$, if $\gamma(n) \in \Gamma_s$.

Proof. Let us estimate the quantity

$$2^{-n}\mathfrak{S}(\gamma(n)) = \gamma_0 C_n^0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^n + \gamma_1 C_n^1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{n-1} + \dots$$
$$\dots + \gamma_n C_n^n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^0.$$

If we put p = q = 1/2, we obtain

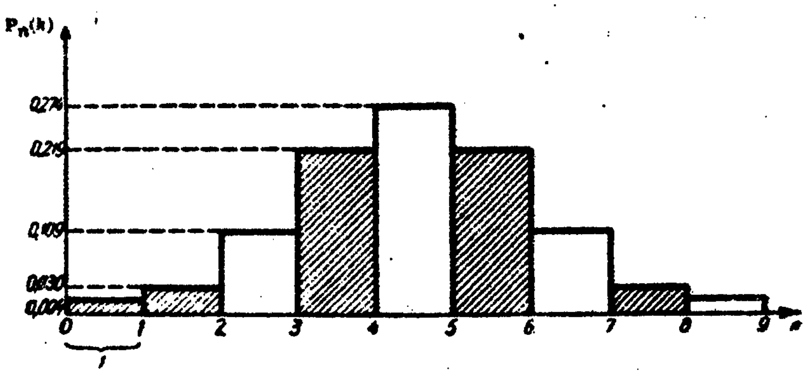
$$2^{-n}\mathfrak{S}(\gamma(n)) = \gamma_0 P_n(0) + \gamma_1 P_n(1) + \dots + \gamma_n P_n(n) = \sum_{k=0}^{n} \gamma_k P_n(k).$$

This expression can be considered as an integral sum, corresponding to the shaded portion of the step-like figure of Fig. 1 (the drawing was made for n = 8 and $\gamma(8)$ = (110101010)). Let us number in Fig. 1 the rectangles from left to right by numbers from 0 to n. The rectangle numbered k (adjacent to the segment $[k, k+1]$) is shaded if and only if $\gamma_k = 1$. In this case the quantity $\sum_{k=0}^{n} \gamma_k P_n(k)$ represents the area of the shaded figure. For what is to come we have to proceed to Fig. 2, which is obtained from Fig. 1 by the same transformations as mentioned in the formulation of the limit theorem. Since here p = q = 1/2, the shift is made by an amount $np = n/2$ and the x axis is compressed by $\sqrt{npq} = \sqrt{n}/2$ times; the y axis is stretched by $\sqrt{n}/2$ times.

As a result of these transformations, the area of the image of the stepped figure (Fig. 2) equals the area of the initial stepped figure (Fig. 1). On Fig. 2, in accordance with the given representation, there arises a natural numbering of the rectangles, namely by numbers from 0 to n from left to right. Assume we have a certain segment (piece) of the x axis. We denote by p and $p + m$ respectively the numbers of the farthest left and farthest right rectangles, the bases of which belong to the segment. We ~~identify~~ compare the given piece with the segment $\gamma'(m) = (\gamma_p, \gamma_{p+1}, \ldots \gamma_{p+n})$ of the initial cortege $\gamma(n)$.

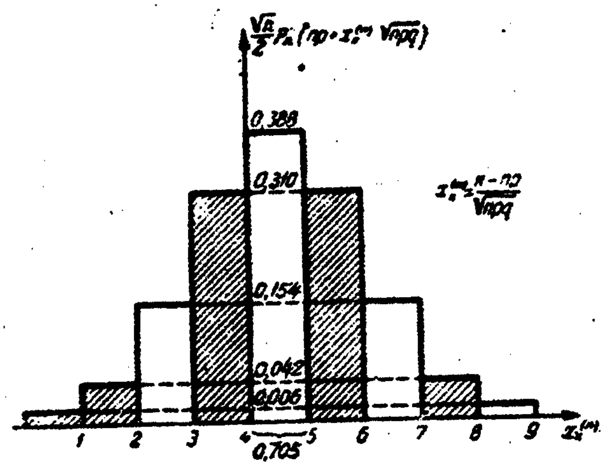
Let us now explain the idea used to prove the lemma. By virtue of the limit theorems, at sufficiently large values of n, the graph shown in Fig. 2 becomes close to the graph of the function $y = (\exp\{-x^2/2\})/\sqrt{2\pi}$ in the area corresponding to the stepped figure, determined by the segment $[a, b]$ becomes close to $\frac{1}{\sqrt{2\pi}}\int_a^b e^{-z^2/2}\,dz$. However, we are interested not in the entire area of the stepped figure, but only in the area of its shaded part, determined by the cortege $\gamma(n)$. From the fact that in the cortege $\gamma(n)$ the fraction of ones is close to σ, it does not at all follow that the fraction of the area of the crossed hatched part relative to the area of the entire stepped figure is close to σ. Nevertheless this is true, and to justify this fact we use a property of corteges from the set Γ_σ. Subdividing the x axis into sufficiently small parts, we are justified in stating that at sufficiently large n on each such a piece the fraction of the area of the shaded part relative to the area of the corresponding stepped figure is close to σ. In fact, at sufficiently large n we have the following: 1) the graph represented in Fig. 2 becomes close to the graph of the function $y = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$

and since the limit function is continuous, then on the considered piece, the heights of the rectangles change insignificantly if the piece is small, and therefore the ratio of the areas of the rectangles of interest to us is determined by the ratio of the number of shaded rectangles to the corresponding number of shaded and unshaded rectangles, i.e., in final analysis by the average density $\gamma_p(m)$ of the segment $\gamma'(m)$ of the initial cortege $\gamma(n)$, corresponding to the given piece; 2) the number m, by virtue of the contraction of the x axis, will be large and since $\gamma'(m) \in \Gamma_\sigma$, then $\gamma_p(m)$ will be close to σ. From the result obtained for sufficiently small pieces and large n we can readily obtain



Pn(k)
0,274
0,219
0,109
0,030
0,004
0
1
2
3
4
5
6
7
8
9
n
1

Fig. 1.



0,388
0,310
0,154
0,042
0,006
1
2
3
4
5
6
7
8
9
0,705

Fig. 2.

also the final result.

Let us proceed to carry out the proof itself.

1. We take $a < b$ such that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{z^2}{2}}dz+\frac{1}{\sqrt{2\pi}}\int_{b}^{\infty} e^{-\frac{z^2}{2}}dz<\frac{\varepsilon}{8}.$$

2. Let M_1 be such that when $n > M_1$

$$\left|P_n(a,b)-\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{z^2}{2}}dz\right|<\frac{\varepsilon}{8}.$$

3. Let M_2 be such that when $n > M_2$

$$\left|\frac{\sqrt{n}}{2}P_n\left(\frac{n}{2}+x_k^{(n)}\frac{\sqrt{n}}{2}\right):\frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}}-1\right|<\frac{\varepsilon}{8(b-a)}.$$

4. We take M_3 such that $M_3 \geqslant 16/\varepsilon$. Since the characteristic of the cortege $\gamma'(m)=(\gamma_p, \gamma_{p+1}, \dots \gamma_{p+n})\epsilon$ satisfies the relation

$$\sigma-\frac{2}{m+1}<\nu_p(m)<\sigma+\frac{2}{m+1}$$

we have when $m \geqslant M_3$:

$$\sigma-\frac{\varepsilon}{8}<\nu_p(m)<\sigma+\frac{\varepsilon}{8}.$$

5. Let us consider the subdivision $\Delta = \{a = a_0, a_1, \dots, a_r = b\}$ of the segment $[a, b]$. Let $\delta > 0$ be such that at maximum $|a_i - a_{i-1}| < \delta$

$$\left|\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{z^2}{2}}dz-\frac{1}{\sqrt{2\pi}}\sum_{i=1}^{r}(a_i-a_{i-1})e^{-\frac{x_i^2}{2}}\right|<\frac{\varepsilon}{8}, \text{ where } a_{i-1}\leqslant x_i\leqslant a_i.$$

6. Since the x axis was contracted by a factor $\sqrt{n}/2$, then for any subdivision Δ of the segment $[a, b]$

there exists such a number M_4, that when $n > M_4$ the number of points with abscissas $x_k^{(n)}$ in each segment $[a_{i-1}, a_i]$ is greater than M_3.

7. Let M_5 be such that $(r+1)\frac{2}{\sqrt{n}} < \frac{\varepsilon}{8}$ when $n > M_5$.

— Let us now specify ε, $1 > \varepsilon > 0$. We choose the numbers a and b in accordance with item 1. We find M_1 and M_2, as indicated in items 2--3. We take the subdivision Δ with $\max|a_i - a_{i-1}| < \delta$ where δ is chosen in accordance with ε. We choose M_3, M_4, and M_5 in accordance with items 4, 6, and 7.

We put $M = \max(M_1, M_2, M_3, M_4, M_5)$. Let $n > M$.

Consider the expression

$$\frac{\mathfrak{S}(\gamma(n))}{2^n} = \gamma_0 P_n(0) + \gamma_1 P_n(1) + \ldots + \gamma_n P_n(n) = \sum_{k=0}^{n} \gamma_k P_n(k).$$

Let us break up the sum into two parts

$$\sum_{k=0}^{n} \gamma_k P_n(k) = \sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2} < k < \frac{n}{2}+b\frac{\sqrt{n}}{2}} \gamma_k P_n(k) + \sum_{\substack{k \le \frac{n}{2}+a\frac{\sqrt{n}}{2} \\ k \ge \frac{n}{2}+b\frac{\sqrt{n}}{2}}} \gamma_k P_n(k).$$

Hence

$$0 < \frac{\mathfrak{S}(\gamma(n))}{2^n} - \sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2} < k < \frac{n}{2}+b\frac{\sqrt{n}}{2}} \gamma_k P_n(k) =$$

$$= \sum_{\substack{k \le \frac{n}{2}+a\frac{\sqrt{n}}{2} \\ k \ge \frac{n}{2}+b\frac{\sqrt{n}}{2}}} \gamma_k P_n(k) < \sum_{\substack{k \le \frac{n}{2}+a\frac{\sqrt{n}}{2} \\ k \ge \frac{n}{2}+b\frac{\sqrt{n}}{2}}} P_n(k) = 1 - P_n(a, b).$$

From items 1 -- 2 we have

$$1-P_n(a,b)<1-\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{z^2}{2}}dz+\frac{\varepsilon}{8}=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{z^2}{2}}dz+\frac{1}{\sqrt{2\pi}}\int_b^{\infty} e^{-\frac{z^2}{2}}dz+\frac{\varepsilon}{8}<\frac{\varepsilon}{4}.$$

Consequently

$$0<\frac{\mathfrak{S}(\gamma(n))}{2^n}-\sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2}<k<\frac{n}{2}+b\frac{\sqrt{n}}{2}}\gamma_k P_n(n)<\frac{\varepsilon}{4}. \qquad (1)$$

Let us transform the sum in the last inequality

$$\sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2}<k<\frac{n}{2}+b\frac{\sqrt{n}}{2}}\gamma_k P_n(k)=\sum_{a<x_k^{(n)}<b}\gamma_k\frac{2}{\sqrt{n}}\left(\frac{\sqrt{n}}{2}P_n\left(\frac{n}{2}+x_k^{(n)}\frac{\sqrt{n}}{2}\right)\right)=$$

$$=\sum_{i=1}^{r}\sum_{a_{i-1}\leq x_k^{(n)}<a_i}\gamma_k\frac{2}{\sqrt{n}}\left(\frac{\sqrt{n}}{2}P_n\left(\frac{n}{2}+x_k^{(n)}\frac{\sqrt{n}}{2}\right)\right).$$

Using item 3 we obtain

$$\frac{\sqrt{n}}{2}P_n\left(\frac{n}{2}+x_k^{(n)}\frac{\sqrt{n}}{2}\right)=\frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}}+\eta_{nk}, \quad \textit{where} \quad |\eta_{nk}|<\frac{\varepsilon}{8(b-a)}.$$

Inserting this expression into the preceding sum, we get

$$\sum_{i=1}^{r}\sum_{a_{i-1}\leq x_k^{(n)}<a_i}\gamma_k\frac{2}{\sqrt{n}}\left(\frac{\sqrt{n}}{2}P_n\left(\frac{n}{2}+x_k^{(n)}\frac{\sqrt{n}}{2}\right)\right)=$$

$$=\sum_{i=1}^{r}\sum_{a_{i-1}\leq x_k^{(n)}<a_i}\gamma_k\frac{2}{\sqrt{n}}\left(\frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}}+\eta_{nk}\right)=$$

$$=\sum_{i=1}^{r}\sum_{a_{i-1}\leq x_k^{(n)}<a_i}\gamma_k\frac{2}{\sqrt{n}}\frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}}+\sum_{i=1}^{r}\sum_{a_{i-1}\leq x_k^{(n)}<a_i}\gamma_k\frac{2\eta_{nk}}{\sqrt{n}},$$

or

$$\left| \sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2} < k < \frac{n}{2}+b\frac{\sqrt{n}}{2}} \gamma_k P_n(k) - \sum_{i=1}^{r} \sum_{a_{i-1} \leq x_k^{(n)} < a_i} \gamma_k \frac{2}{\sqrt{n}} \frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}} \right| <$$

$$< \sum_{i=1}^{r} \sum_{a_{i-1} \leq x_k^{(n)} < a_i} \frac{2}{\sqrt{n}} |\eta_{nk}| < \sum_{i=1}^{r} \sum_{a_{i-1} \leq x_k^{(n)} < a_i} \frac{2\varepsilon}{8(b-a)\sqrt{n}} = \frac{\varepsilon}{8}. \qquad (2)$$

We denote by $\bar{x}_i^{(n)}$ and $\underline{x}_i^{(n)}$ respectively the largest and the smallest values of the quantity $|x_k^{(n)}|$ on the subinterval $[a_{i-1}, a_i)$ (Fig. 3).

Taking items 4 and 6 into account, we get

$$\sum_{a_{i-1} \leq x_k^{(n)} < a_i} \gamma_k \frac{2}{\sqrt{n}} \frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}} < \frac{e^{-\frac{\underline{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}} \sum_{a_{i-1} \leq x_k^{(n)} < a_i} \gamma_k \frac{2}{\sqrt{n}} <$$

$$< \left(\sigma + \frac{\varepsilon}{8}\right)\left(a_i - a_{i-1} + \frac{2}{\sqrt{n}}\right) \frac{e^{-\frac{\underline{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}}$$

and

$$\sum_{a_{i-1} \leq x_k^{(n)} < a_i} \gamma_k \frac{2}{\sqrt{n}} \frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}} > \frac{e^{-\frac{\bar{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}} \sum_{a_{i-1} \leq x_k^{(n)} < a_i} \gamma_k \frac{2}{\sqrt{n}} >$$

$$> \left(\sigma - \frac{\varepsilon}{8}\right)\left(a_i - a_{i-1} - \frac{2}{\sqrt{n}}\right) \frac{e^{-\frac{\bar{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}},$$

where the $\pm 2/\sqrt{n}$ is added because of the interval located between a_{i-1} and the farthest from among the points $x_k^{(n)}$ to the left, and also between a_i and the farthest from the points $\bar{x}_k^{(n)}$ to the right, belonging to $[a_{i-1}, a_i)$ (see Fig. 3).

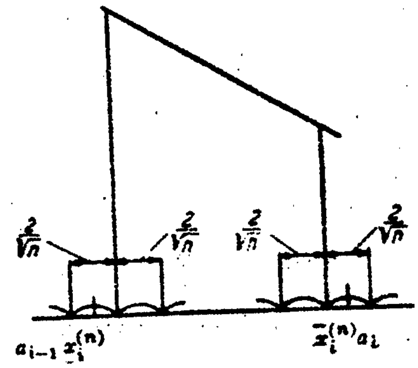


Fig. 3

From this, and also taking items 5 and 7 into account, we obtain

$$\sum_{i=1}^{r}{}' \sum_{a_{i-1}\leq x_k^{(n)}<a_i}{}' \gamma_k \frac{2}{\sqrt{n}} \frac{e^{\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}} < \left(\sigma+\frac{\varepsilon}{8}\right)\left(\sum_{i=1}^{r}{}' (a_i-a_{i-1})\frac{e^{-\frac{\bar{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}}+\frac{2}{\sqrt{n}}(r+1)\right)<$$
$$<\left(\sigma+\frac{\varepsilon}{8}\right)\left(\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{z^2}{2}}dz+\frac{2\varepsilon}{8}\right. \leq \left(\sigma+\frac{\varepsilon}{8}\right)\left(1+\frac{2\varepsilon}{8}\right)<\sigma+\frac{5\varepsilon}{8} \quad (3_1)$$

and

$$\sum_{i=1}^{r} \sum_{a_{i-1}\leq x_k^{(n)}<a_i} \gamma_k \frac{2}{\sqrt{n}} \frac{e^{-\frac{x_k^{(n)2}}{2}}}{\sqrt{2\pi}} > \left(\sigma-\frac{\varepsilon}{8}\right)\left(\sum_{i=1}^{r} (a_i-a_{i-1})\frac{e^{-\frac{\underline{x}_i^{(n)2}}{2}}}{\sqrt{2\pi}}-\right.$$
$$\left.-\frac{2}{\sqrt{n}}(r+1)\right)>\left(\sigma-\frac{\varepsilon}{8}\right)\left(\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{z^2}{2}}dz-\frac{2\varepsilon}{8}\right)\geq$$
$$\geq\left(\sigma-\frac{\varepsilon}{8}\right)\left(1-\frac{3\varepsilon}{8}\right)>\sigma-\frac{4\varepsilon}{8}. \quad (3_2$$

Combining (2), (3_1), and (3_2), we get

$$\left|\sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2}<k<\frac{n}{2}+b\frac{\sqrt{n}}{2}} \gamma_k P_n(k)-\sigma\right|<\frac{3\varepsilon}{4}.$$

From this inequality and from (1) it follows directly that when $n > M$

$$\left|\frac{\mathfrak{S}(\gamma(n))}{2^n} - c\right| < \varepsilon.$$

This proves the lemma completely.

Using approximately similar arguments as in the lemma, we can readily obtain a generalization of the integral limit theorem. For its formulation it is necessary to introduce another designation, namely: let

$$\mathfrak{S}_{a,b}(\gamma(n)) = \sum_{\frac{n}{2}+a\frac{\sqrt{n}}{2}<k<\frac{n}{2}+b\frac{\sqrt{n}}{2}} \gamma_k P_n(n).$$

Theorem 2.9. If $a < b$ and $\gamma(n) \in \Gamma_\sigma$, then as $n \rightarrow \infty$

$$\frac{\mathfrak{S}_{a,b}(\gamma(n))}{2^n} \rightarrow \frac{\sigma}{\sqrt{2\pi}} \int_a^b e^{-\frac{z^2}{2}} dz.$$

Since this theorem will be of no use to us further, we shall leave its proof aside.

Theorem 2.10. For any $\sigma (0 \leq \sigma \leq 1)$ we can construct an invariant set Q such that

$$\lim_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 2^{\sigma}.$$

Let us take for Q the class Q_σ which we have constructed, i.e., $Q = Q_\sigma$. If $\sigma = 1$, the theorem is obvious. If $\sigma < 1$, we have by definition (see p. 97 /of source/)

$$Q_\sigma = Q'_\sigma \cup P^{N_\sigma}$$

and

$$Q'_\sigma = \{S(x_1, \ldots, x_k, y_1, \ldots, y_p) \& f'(x_{i_1}, \ldots, x_{i_r}, u_1, \ldots, u_q)\},$$

with

$$S(x_1, \ldots, x_k, y_{k_1}, \ldots, y_p) = S_{\gamma(k)}(x_1, \ldots, x_k, y_1, \ldots, y_p),$$

where $\gamma(k) \in \Gamma_\sigma$ and the essential variable functions s and f' are related by $\{x_{i_1}, \ldots, x_{i_r}\} \subset \{x_1, \ldots, x_k\}$.

Let

$$f(x_1, \ldots, x_k, z_1, \ldots, z_m) =$$
$$= S(x_1, \ldots, x_k, y_1, \ldots, y_p) \& f'(x_{i_1}, \ldots, x_{i_r}, u_1, \ldots, u_q).$$

Obviously, if we put n = m + k, then

$$P_{Q'_\sigma}(n) \leqslant P_{Q_\sigma}(n) < P_{Q'_\sigma}(n) + P_{pN_0}(n).$$

Let us estimate the numbers $P_{pN_0}(n)$ and $P_{Q'_\sigma}(n)$. For the first of these we found earlier (example 3 of p. 25 /of source/) the estimate

$$P_{pN_0}(n) < C_n^{N_0} 2^{2^{N_0}}.$$

The procedure for the estimate of $P_{Q'_\sigma}(n)$ is very much the same as that for the estimate of $P_{Q_{\psi}}$ (n) (see example on p. 88 /of source/). We shall fix the function $S(x_1, \ldots, x_k, y_1, \ldots, y_p)$ and vary the function f', observing only one limitation imposed on its essential variables

$$\{x_{i_1}, \ldots, x_{i_r}\} \subset \{x_1, \ldots, x_k\}.$$

Considering the function f, where f = S & f', as a function of the variables $x_1, \ldots, x_k$, we see that always

$\bar{f} = 0$, when $S = 0$. Consequently, the function f' is determined fully by indicating the subset of all the assemblies $\tilde{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$, on which $S(\tilde{\alpha}) = 1$, and $f'(\tilde{\alpha}) = 0$. Since the number of assemblies $\tilde{\alpha}$ on which $S(\tilde{\alpha}) = 1$ is

$$\gamma_0 C_k^0 + \gamma_1 C_k^1 + \ldots + \gamma_k C_k^k = \mathfrak{S}(\gamma(k)),$$

where $\gamma(k)$ is the cortege defining the symmetrical function S, then the number of subsets of interest to us is $2^{\mathfrak{S}(\gamma(k))}$. Thus, for any function S, which depends essentially on k variables and is determined by a cortege $\gamma(k)$, we have $2^{\mathfrak{S}(\gamma(k))}$ different functions f. From this it follows, in particular, that

$$P_{Q_2'}(n) \geqslant 2^{\mathfrak{S}(\gamma(n))}.$$

Further, the choice of the function S is determined, first of all, by the separation of the subset of essential variables from among the variables $x_1, \ldots, x_n$ and, secondly, by the fixation of a certain symmetrical function of the given variables and characterized by a cortege from the set Γ_γ. For k essential variables, there are no more than

$$C_n^k 2^{k+1}.$$

such functions. Then the number of functions f, which are obtained from the symmetrical functions that depend essentially on k variables, does not exceed

$$C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))}$$

(where the maximum is taken over all the corteges from at fixed k).

We obtain

$$P_{Q_\sigma}(n) < \sum_{k=0}^{n} C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))}.$$

Thus,

$$2^{\mathfrak{S}(\gamma(n))} \leqslant P_{Q_\sigma}(n) \leqslant C_n^{N_0} 2^{2^{N_0}} + \sum_{k=0}^{n} C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))}.$$

Let $\varepsilon > 0$ be an arbitrary small number, and then according to the preceding lemma there exists such M, that when $k > M$ and $\gamma(k) \in \Gamma_\sigma$

$$2^k(\sigma-\varepsilon) < \mathfrak{S}(\gamma(k)) < 2^k(\sigma+\varepsilon).$$

We have

$$\sum_{k=0}^{n} C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))} = \sum_{k=0}^{M} C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))} + \sum_{k=M+1}^{n} C_n^k 2^{k+1} \max_{\gamma} 2^{\mathfrak{S}(\gamma(k))} \leqslant$$

$$\leqslant \max_{\substack{\gamma \\ 0 \leqslant k \leqslant M}} 2^{\mathfrak{S}(\gamma(k))} \sum_{k=0}^{M} C_n^k 2^{k+1} + 2^{(\sigma+\varepsilon)2^n} \sum_{k=0}^{n} C_n^k 2^{k+1} <$$

$$< (M+1)\, n^M 2^{M+1} \max_{\substack{\gamma \\ 0 \leqslant k \leqslant M}} 2^{\mathfrak{S}(\gamma(k))} + 2 \cdot 3^n \cdot 2^{(\sigma+\varepsilon)2^n}.$$

Thus,

$$2^{(\sigma-\varepsilon)2^n} \leqslant P_{Q_\sigma}(n) \leqslant C_n^{N_0} 2^{2^{N_0}} + (M+1) n^M 2^{M+1} \max_{\substack{\gamma \\ 0 \leqslant k \leqslant M}} 2^{\mathfrak{S}(\gamma(k))} + 2 \cdot 3^n \cdot 2^{(\sigma+\varepsilon)2^n}.$$

From this it follows that

$$2^{\sigma-\varepsilon} \leqslant \lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma}(n)} \leqslant 2^{\sigma+\varepsilon}.$$

In view of the arbitrariness of ε, we finally get

$$\lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma}(n)} = 2^\sigma.$$

This proves the theorem.

Theorem 2.11. For each $\sigma (0 \leq \sigma \leq 1)$ there exists a continuum of pairwise different invariant classes Q_σ^α with $\lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma^\alpha}(n)} = 2^\sigma$.

The correctness of this theorem $\sigma = 0$ follows from the proof of theorem 2.3 and example 6 on p. 87 /of source/.

Let R_ξ be the subset of all the real numbers from the interval (0, 1), which are separated from the number by a rational distance. We can show that the subsets $R_{\xi'}$ and $R_{\xi''}$ corresponding to the numbers ξ' and ξ'' are either the same, or else do not intersect. Thus, the interval (0, 1) is broken up into a direct sum of nonintersecting classes R_ξ of the subsets R_ξ. We denote by R the set which has in common with each subset R_ξ exatly one element (different for different classes $\tilde{R}_\xi$). R is a set which is not measurable in the sense of Lebesgue /23/ and has a continual cardinality.

The numbers $\xi \in R$ will be represented in the form of an infinite binary fraction

$$\xi = 0,\xi_1,\xi_2 \ldots ;$$

in the presence of two possible expansions, we take the expansion that contains a tail of units.

Let $\sigma \neq 0$, i.e., $0 < \sigma < 1$. We set in correspondence with each number ξ, $\xi \in R$ a subset α_ξ of natural numbers $i_1 i_2, \ldots,$ which are the numbers of only those columns, in which the expansion of the number ξ contains unity, i.e.,

$$i \begin{cases} \epsilon \alpha_\xi, & \text{if } \xi_i = 1, \\ \bar{\epsilon} \alpha_\xi, & \text{if } \xi_i = 0. \end{cases}$$

Inasmuch as every two numbers ξ' and ξ'' ($\xi' \neq \xi''$) from the set R are all at an irrational distance, their expansions

$$\xi' = 0.\xi_1'\xi_2' \ldots \text{ and } \xi'' = 0.\xi_1''\xi_2'' \ldots$$

differ on an infinite set of columns. It follows therefore that for any two different subsets $\alpha_{\xi'}$ and $\alpha_{\xi''}$ ($\xi', \xi'' \in R$) there exists a natural number i_n, as large as desired, which enters into one subset and does not enter into another, i.e.,

$$i_n \in \alpha_{\xi'} \cup \alpha_{\xi''} \text{ and } i_n \bar{\in} \alpha_{\xi'} \cap \alpha_{\xi''}.$$

The subset α_ξ, $\xi \in R$ determines and invariant class $Q_0^{\alpha_\xi}$ (see page 85 /of source/). Let $\overline{Q_0^{\alpha_\xi}}$ denote the set consisting of negations of all the functions of class $Q_0^{\alpha_\xi}$. Obviously $\overline{Q_0^{\alpha_\xi}}$ is an invariant class, with $P_{\overline{Q_0^{\alpha_\xi}}}(n) = P_{Q_0^{\alpha_\xi}}(n)$. Let furthermore

$$Q_\sigma^{\alpha_\xi} = Q_\sigma \cup \overline{Q}_0^{\alpha_\xi},$$

where Q_σ is an invariant class with $\lim_{n \to \infty} \sqrt[2^n]{P_{Q_\sigma}(n)} =$

$= 2^{\sigma}$ $(0 < \sigma < 1)$. $Q_{\sigma}^{\alpha_\xi}$ being a sum of invariant classes, is an invariant class.

Let us show that the family of classes $Q_{\sigma}^{\alpha_\xi}$, where $\xi \in R$, is continual. For this it is enough to establish that $Q_{\sigma}^{\alpha_{\xi'}} \neq Q_{\sigma}^{\alpha_{\xi''}}$ when $\xi' = \xi''$.

Since $\sigma < 1$, it is possible to choose a number σ' such that

$$\sigma < \sigma' < 1.$$

It follows from lemma 2.4 that there exists such $N_1(\sigma')$, that when $n > N_1$ the characteristic $\nu(n)$ of each cortege $\gamma(n)$ from Γ_σ satisfies the inequality

$$\nu(n) < \sigma'.$$

On the other hand, the functions

$$\bar{f}_{n-1}(x_1, \ldots, x_n) = \overline{x_1 \ldots x_n \vee \bar{x}_1 \ldots \bar{x}_n}$$

which participate in the construction of the classes $Q_0^{\alpha_\xi}$, are symmetrical, specified by the corteges $\tilde{\gamma}(n) =$ $= (1\ \underbrace{0 \ldots 1}_{n-1}\ 0)$ with characteristics $\tilde{\nu}(n) = \frac{n-2}{n-1}$.

Let N_2 be such that when $n > N_2$ we have $\frac{n-2}{n-1} > \sigma'$.

We take $N = \max(N_1, N_2)$. Then obviously we have when $n > N$

$$\bar{f}_{n-1}(x_1, \ldots, x_n) \bar{\in} Q_3.$$

At the same time, since the subsets $\alpha_{\xi'}$ and $\alpha_{\xi''}$ are different, there exists such an $i > N$ that

$$i \in \alpha_{\xi'} \cup \alpha_{\xi''} \qquad i \bar{\in} \alpha_{\xi'} \cap \alpha_{\xi''}.$$

It is obvious that the function $\bar{f}_{i-1}(x_1, \ldots, x_i)$ will be a function which belongs only to one of the classes $Q_0^{\alpha_{\xi'}}$ and $Q_0^{\alpha_{\xi''}}$. This shows that $Q_\sigma^{\alpha_{\xi'}} \neq Q_\sigma^{\alpha_{\xi''}}$ when $\xi' \neq \xi''$.

Finally,

$$P_{Q_\sigma}(n) \leqslant P_{Q_\sigma^{\alpha_\xi}}(n) \leqslant P_{Q_\sigma}(n) + P_{\bar{Q}_0^{\alpha_\xi}}(n).$$

Since

$$\lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma}(n)} = 2^\sigma \quad \text{and} \quad \lim_{n\to\infty} \sqrt[2^n]{P_{\bar{Q}_\sigma^{\alpha_\xi}}(n)} = 1,$$

then

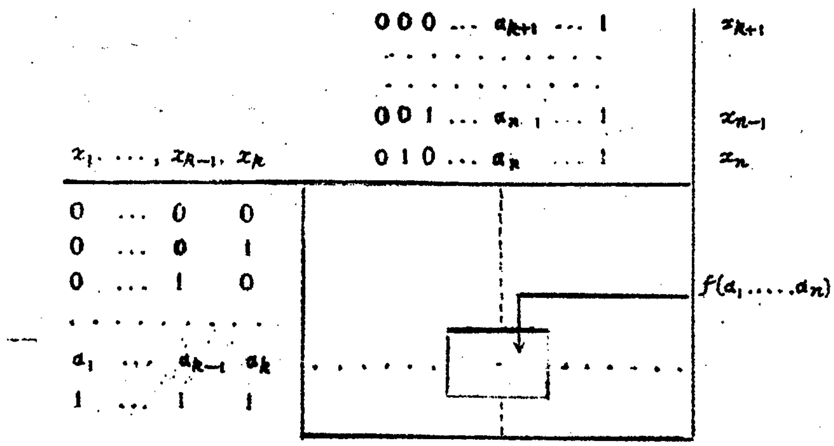
$$\lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma^{\alpha_\xi}}(n)} = 2^\sigma.$$

This proves the theorem completely.

3. REALIZATION OF THE FUNCTIONS OF CLASS Q_σ BY MEANS OF CONTACT NETWORKS

As noted in the preceding section, it is possible to assume that in realization problems we always deal with invariant classes. In this section we shall show, making use of the method of O. B. Lupanov /9/, how to construct networks that realize functions from class Q_σ. Along with these we find an asymptotic expression for the function $L_{Q_\sigma}(n)$. The latter, in addition to being directly of interest, serves as a basis for all the arguments of Sec. 4.

Let the function $f(x_1, x_2, \ldots, x_n) \in Q_\sigma$ be specified by means of a table:



in which the values $f(\alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n)$ for any $0 \leqslant \alpha_1, \alpha_2, \dots, \alpha_n \leqslant 1$ consists of the intersection of the row corresponding to the set $(\alpha_1, \dots, \alpha_k)$, and the column corresponding to the set $(\alpha_{k+1}, \dots, \alpha_n)$. The form of such a table is determined by choice of the parameter k.

We shall be interested in what follows by certain parts of the table columns, belonging to a certain aggregate of rows, i.e, the intersection of columns with a given aggregate of rows. It is easy to see that to each substitution of the constants $\alpha_{i_1}, \dots, \alpha_{i_r}, \alpha_{k+1}, \dots, \alpha_n$ in the variables $x_{i_1}, \dots, x_{i_r}, x_{k+1}, \dots, x_n$, where $1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_r \leqslant k$, corresponds to a certain intersection of the column with a system of rows, namely the column determined by the ~~set~~ assembly $(\alpha_{k+1}, \dots, \alpha_n)$ with the system of all the rows determined by sets for which $x_{i_1} = \alpha_{i_1}, \ldots, x_{i_r} = \alpha_{i_r}$. We shall say of the foregoing intersection of the column with the system of rows that it is separated by a substitution of constants. Obviously, the intersection which is separated by means of a certain substitution of constants determines a function which depends on the remaining variables and which is obtained from $f(x_1, \ldots, x_n)$ by substitution of the constants $\alpha_{i_1}, \ldots, \alpha_{i_r}, \alpha_{k+1}, \ldots, \alpha_n$ instead of the variables $x_{i_1}, \ldots, x_{i_r}, x_{k+1}, \ldots, x_n$.

Let us consider the set of assemblies $\{(\alpha_1, \ldots, \alpha_k)\}$; from the geometric point of view it represents the set of vertices of a unit k-dimensional tube. The subset of all the vertices of the tube, for which $x_{i_1} = \alpha_{i_1}, \ldots, x_{i_r} = \alpha_{i_r}$ is called an iterval of rank r. It is obvious that an interval of rank r is a $k = r$-dimensional ~~side~~ face of the initial cube.* Let us prove a lemma on the breakup of a k-dimentional cube into a direct! sum of intervals.

Lemma 3.1. If $\rho_1 2^{r_1} + \rho_2 2^{r_2} + \ldots + \rho_\ell 2^{r_\ell} = 2^k$, where $k \geqslant r_1 > \ldots > r_i \geqslant 0$, then the unit k-dimensional tube can be broken up into a direct sum of

* By "face" we understant the set of all the vertices belonging to the face.

ρ_1 r_1-dimensional faces, ρ_2 r_2-dimensional faces, etc., and finally ρ_i r_i-dimensional faces.

Proof. The k-dimensional cube can be broken up by dichotomic division into a direct sum of 2^{k-r_1} r_1-dimensional faces. Since $\rho_1 2^{r_1} \leq 2^k$ or $\rho_1 \leq 2^{k-r_1}$, we can select from them ρ_1 r_1-dimensional faces. Each of the $2^{k-r_1} - \rho_1$ remaining r_1-dimensional faces can be broken up (by dichotomic division) into a direct sum of $2^{k_1-r_2}$ r_2-dimensional faces. We obtain a total of

$$(2^{k-r_1} - \rho_1)\, 2^{r_1-r_2} = 2^{k-r_2} - \rho_1 2^{r_1-r_2}$$

r_2-dimensional faces. Since $\rho_1 2^{r_1} + \rho_2 2^{r_2} \leq 2^k$ or $\rho_2 \leq 2^{k-r_2} - \rho_1 2^{r_1-r_2}$ we can select from among $2^{k-r_2} - \rho_1 2^{r_1-r_2}$ r_2-dimensional faces a total of ρ_2 r_2-dimensional faces, etc. After selecting $\rho_{\ell-1}$ $r_{\ell-1}$-dimensional faces (ℓ-th step), we are left with

$$2^{k-r_{\ell-1}} - \rho_1 2^{r_1-r_{\ell-1}} - \ldots - \rho_{\ell-1}$$

$r_{\ell-1}$-dimensional faces. Each of these can be broken up (by dichotomic division) into $2^{r_{\ell-1}-r_\ell}$ r_ℓ-dimensional faces. We obtain a total of

$$(2^{k-r_{\ell-1}} - \rho_1 2^{r_1-r_{\ell-1}} - \ldots - \rho_{\ell-1})\, 2^{r_{\ell-1}-r_\ell}$$

r_ℓ-dimensional faces. And this is exactly equal to ρ_ℓ, and therefore we can select exactly ρ_ℓ r_ℓ-dimensional faces. We have thereby constructed the breakup of the cube in the required direct sum. This proves the lemma.

Let us assume that Q_σ is an arbitrary non-empty invariant class, such that $\lim_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma}(n)} = 2^\sigma$. Let $\varepsilon_1 > 0$ be an arbitrary small number. Then it follows from theorem 2.5 that there exists a number $N_1(\sigma_1, \varepsilon_1)$, such that when $n > N_1$

$$2^{\sigma 2^n} < P_{Q_\sigma}(n) < 2^{\sigma_1 2^n},$$

where

$$0 < \sigma_1 - \sigma < \varepsilon.$$

We can now formulate and prove the lemma that explains the construction of a table for functions from class Q_σ.

Lemma 3.2. Let ε_2 be an arbitrary positive number, $\sigma_2 = \sigma_1(1 + \varepsilon_2)$. Then, if the numbers s and k satisfy the inequality

$$\frac{2^{N_1}}{\sigma_1 \varepsilon_2} < s < 2^k,$$

the rows in the table for the function $f(x_1, x_2, \ldots, x_n)$ with parameter k (therefore $k < n$) can be broken up into two groups, in each of which there is contained exactly s rows, with the exception perhaps of one, which contains s' rows ($0 < s' \leq s$), and such that the different intersections of columns with rows of each group is not more than $2^{\sigma_2 s}$.

Proof. Let us consider the expansion of the numbers s and $s' = 2^k - (2^k/s)s$ in powers of 2

$$s = 2^{q_1} + 2^{q_2} + \ldots + 2^{q_m};$$
$$s' = 2^{q'_1} + 2^{q'_2} + \ldots + 2^{q'_{m'}}.$$

If $2^k/s$ is an integer, then $p = 2^k/s$ and $2^k = p(2^{q_1} + 2^{q_2} + \ldots + 2^{q_m})$, if $2^k/s$ is not integral, then $p = (2^k/s) + 1$ and $2^k = (p - 1)(2^{q_1} + \ldots + 2^{q_m}) + (2^{q'_1} + 2^{q'_{m'}})$. Thus, in both cases

$$2^k = \rho_1 2^{r_1} + \rho_2 2^{r_2} + \ldots + \rho_l 2^{r_l}, \quad k \geqslant r_1 > r_2 > \ldots > r_l \geqslant 0,$$

with

$$\{r_1, \ldots, r_l\} = \{q_1, \ldots, q_m\} \cup \{q'_1, \ldots, q'_{m'}\}.$$

On the basis of lemma 3.1 the unit k-dimensional cube can be broken up into the direct sum of ρ_1 r_1-dimensional intervals, ρ_2 r_2-dimensional intervals, etc., ρ_l r_l-dimensional intervals. This breakup of the cube leads to the corresponding breakdown of the rows of the table into smaller groups. It is now easy to construct, starting up with the smaller groups and the expansions of the numbers s and s', to construct the required breakdown of the rows into p groups. For this purpose each of s rows is formed by joining one small group of 2^{q_1} rows, one small group of 2^{q_2} rows, etc., finally, one small group of 2^{q_m} rows. Analogously, one constructs a group of s' rows. Let us estimate the number of different intersections of the columns with the rows of each group. For this purpose we shall transform somewhat the expressions for the numbers s and s'

$$s = 2^{q_1} + 2^{q_2} + \ldots + 2^{q_m} = 2^{q_1} + \ldots + 2^{q_t} + 2^{q_{t+1}} + \ldots + 2^{q_m} =$$
$$= 2^{q_1} + \ldots + 2^{q_t} + s_0,$$
$$s' = 2^{q'_1} + 2^{q'_2} + \ldots + 2^{q'_{m'}} = 2^{q'_1} + \ldots + 2^{q'_{t'}} + 2^{q'_{t'+1}} + \ldots + 2^{q'_{m'}} =$$
$$= 2^{q'_1} + \ldots + 2^{q'_{t'}} + s'_0,$$

where the number t and t' are chosen such to estimate

$$q_t \geqslant [\log_2(\sigma_1 \varepsilon_2 s)], \quad \textit{and} \quad q_{t+1} < [\log_2(\sigma_1 \varepsilon_2 s)],$$
$$q'_{t'} \geqslant [\log_2(\sigma_1 \varepsilon_2 s)], \quad \textit{and} \quad q'_{t'+1} < [\log_2(\sigma_1 \varepsilon_2 s)].$$

It follows therefore that $s_0 \leqslant \sigma_1 \varepsilon_2 s$ and $s'_0 \leqslant \sigma_1 \varepsilon_2 s$. In fact, if for example $s_0 > \sigma_1 \varepsilon_2 s$, then $\log_2 s_0 > \log_2(\sigma_1 \varepsilon_2 s)$ and consequently $[\log_2 s_0] \geqslant [\log_2(\sigma_1 \varepsilon_2 s)]$. The latter would denote that $q_{t+1} \geqslant [\log_2(\sigma_1 \varepsilon_2 s)]$. We have arrived at a contradiction and therefore, $s_0 \leqslant \sigma_1 \varepsilon_2 s$. We can show analogously that $s'_0 \leqslant \sigma_1 \varepsilon_2 s$. Each group of s rows (or analogously of s' rows) consists of smaller groups, and each of the smaller groups can be separated by a substitution of constants. Within each smaller group of 2^{q_i} (or respectively $2^{q'_{i'}}$) rows contains not more than $P_{Q_\sigma}(q_i)$ (or respectively $P_{Q_\sigma}(q'_{i'})$) different columns. Therefore the number of different intersections with the rows of each group of s rows and s' rows does not exceed respectively

$$P_{Q_\sigma}(q_1)\,P_{Q_\sigma}(q_2)\,\ldots\,P_{Q_\sigma}(q_t)\,2^{s_0}$$

and

$$P_{Q_\sigma}(q'_1)\,P_{Q_\sigma}(q'_2)\,\ldots\,P_{Q_\sigma}(q'_{t'})\,2^{s'_0}.$$

It is obvious that if i and i' are such that $1 \leqslant i \leqslant t$ and $1 \leqslant i' \leqslant t'$, then $q_i \geqslant [\log_2(\sigma_1 \varepsilon_2 s)]$ and $q'_{i'} \geqslant [\log_2(\sigma_1 \varepsilon_2 s)]$. Therefore, by virtue of $[\log_2(\sigma_1 \varepsilon_2 s)] \geqslant N_1$ (see formulation of the lemma) we obtain $q_i \geqslant N_1$ and $q'_{i'} \geqslant N_1$. From this we have

$$P_{Q_\sigma}(q_i) < 2^{\sigma_1 2^{q_i}} \text{ и } P_{Q_\sigma}(q'_{i'}) < 2^{\sigma_1 2^{q'_{i'}}}.$$

and thus we obtain

$$P_{Q_\sigma}(q_1)\ldots P_{Q_\sigma}(q_t)\,2^{s_0} < 2^{\sigma_1 2^{q_1}+\ldots+\sigma_1 2^{q_t}+s_0} < 2^{\sigma_1(2^{q_1}+\ldots+2^{q_t}+\varepsilon_2 s)} <$$
$$< 2^{\sigma_1 s(1+\varepsilon_2)} = 2^{\sigma s}.$$

and

$$P_{Q_\sigma}(q'_1)\ldots P_{Q_\sigma}(q'_{t'})\,2^{s'_0} < 2^{\sigma_1 2^{q'_1}+\ldots+\sigma_1 2^{q'_{t'}}+s'_0} < 2^{\sigma_1(2^{q'_1}+\ldots+2^{q'_{t'}}+\varepsilon_2 s)} <$$
$$< 2^{\sigma_1(s'+s\varepsilon_2)} < 2^{\sigma_1 s(1+\varepsilon_2)} = 2^{\sigma s}.$$

This proves the lemma completely.

Theorem 3.1. For any σ $(0 \leqslant \sigma \leqslant 1)$ and $\varepsilon > 0$ there exists such N, that when $n > N$ for any function $f(x_1, x_2, \ldots, x_n)$ from Q_σ it is possible to construct a contact network with a number of contacts not exceeding

$$\frac{(\sigma+s)2^n}{n}.$$

Proof. To construct a network that realizes the function $f(x_1, x_2, \ldots, x_n) \in Q_\sigma$, we use the method of O. B. Lupanov, somewhat refining it with the aid of the lemmas just proved. Thus, let the function f be specified in the form of a table with parameter k. We break up the rows of this table into p groups as was done in lemma 3.2. Each of the groups contains exactly s rows, with the exception perhaps of the last one, which contains s'(s' $\leq$ s) rows. Parts of the columns located between the rows of the i-th group are broken up into classes, in each of which are contained only those parts of the columns, which are identical to each other. Obviously the number of such classes is equal to the number of different parts of the columns belonging to the i-th group. Let $f_{ij}(x_1, x_2, \ldots, x_n)$ be a function coinciding with that given on the j-th class of the i-th group and equal to zero on the remaining assemblies. Then

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{i=1}^{p} \bigvee_{j} f_{ij}(x_1, x_2, \ldots, x_n),$$

with

$$f_{ij}(x_1, x_2, \ldots, x_n) = f_{ij}^{(1)}(x_1, \ldots, x_k) f_{ij}^{(2)}(x_{k+1}, \ldots, x_n),$$

where

$$f_{ij}^{(1)}(x_1, \ldots, x_k) = \bigvee x_1^{\alpha_1} \ldots x_k^{\alpha_k}$$

(the disjunction is taken over the assemblies $(\alpha_1, \ldots, \alpha_k)$, corresponding to non-zero rows of the table of functions f_{ij}) and $f^{(2)}(x_{k+1}, \ldots, x_n) = \bigvee x_{k+1}^{\alpha_{k+1}} \ldots x_n^{\alpha_n}$ (the disjunction is taken over the assemblies $(\alpha_{k+1}, \ldots, \alpha_n)$ corresponding to the columns of the j-th class of the i-th group).

We denote by

$$f_i(x_1, \ldots, x_n) = \bigvee_j f_{ij}(x_1, \ldots, x_n).$$

We assume furthermore that we have a set of vertices of a unit r-dimensional cube, where r is the power of two. Let $(\beta_1, \ldots, \beta_r)$ be an arbitrary vertex. Then the set of all the vertices, determined by the assemblies, each of which differs from the assembly $(\beta_1, \ldots, \beta_r)$ by the value of exactly one coordinate, is called a sphere. We denote by $\varphi(x_1, \ldots, x_r)$ the characteristic function of the sphere. It is known /9/ that if r is a power of 2, then the r-dimensional cube can be broken up into $2^r/r$ non-intersecting spheres.

The network $\mathfrak{A}$ which realizes the function $f(x_1, x_2, \ldots, x_n)$ is constructed by connecting in parallel the networks $\mathfrak{A}_i$, which realize the functions $f_i(x_1, x_2, \ldots, x_n)$. Let us describe the structure of the network $\mathfrak{A}_i$. For this purpose we construct

$$\frac{(\sigma+s)2^n}{n}.$$

Proof. To construct a network that realizes the function $f(x_1, x_2, \ldots, x_n) \in Q_\delta$, we use the method of O. B. Lupanov, somewhat refining it with the aid of the lemmas just proved. Thus, let the function f be specified in the form of a table with parameter k. We break up the rows of this table into p groups as was done in lemma 3.2. Each of the groups contains exactly s rows, with the exception perhaps of the last one, which contains $s'(s' \leq s)$ rows. Parts of the columns located between the rows of the i-th group are broken up into classes, in each of which are contained only those parts of the columns, which are identical to each other. Obviously the number of such classes is equal to the number of different parts of the columns belonging to the i-th group. Let $f_{ij}(x_1, x_2, \ldots, x_n)$ be a function coinciding with that given on the j-th class of the i-th group and equal to zero on the remaining assemblies. Then

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{i=1}^{p} \bigvee_{j} f_{ij}(x_1, x_2, \ldots, x_n),$$

with

$$f_{ij}(x_1, x_2, \ldots, x_n) = f_{ij}^{(1)}(x_1, \ldots, x_k) f_{ij}^{(2)}(x_{k+1}, \ldots, x_n),$$

where

$$f_i^{(1)}(x_1, \ldots, x_k) = \bigvee x_1^{\alpha_1} \ldots x_k^{\alpha_k}$$

(the disjunction is taken over the assemblies $(\alpha_1, \ldots, \alpha_k)$, corresponding to non-zero rows of the table of functions f_{ij}) and $f^{(2)}(x_{k+1}, \ldots, x_n) = \bigvee x_{k+1}^{\alpha_{k+1}} \ldots x_n^{\alpha_n}$ (the disjunction is taken over the assemblies $(\alpha_{k+1}, \ldots, \alpha_n)$ corresponding to the columns of the j-th class of the i-th group).

We denote by

$$f_i(x_1, \ldots, x_n) = \bigvee_j f_{ij}(x_1, \ldots, x_n).$$

We assume furthermore that we have a set of vertices of a unit r-dimensional cube, where r is the power of two. Let $(\beta_1, \ldots, \beta_r)$ be an arbitrary vertex. Then the set of all the vertices, determined by the assemblies, each of which differs from the assembly $(\beta_1, \ldots, \beta_r)$ by the value of exactly one coordinate, is called a sphere. We denote by $\varphi(x_1, \ldots, x_r)$ the characteristic function of the sphere. It is known /9/ that if r is a power of 2, then the r-dimensional cube can be broken up into $2^r/r$ non-intersecting spheres.

The network $\mathfrak{A}$ which realizes the function $f(x_1, x_2, \ldots, x_n)$ is constructed by connecting in parallel the networks $\mathfrak{A}_i$, which realize the functions $f_i(x_1, x_2, \ldots, x_n)$. Let us describe the structure of the network $\mathfrak{A}_i$. For this purpose we construct a system $\left[1, q\right]$ of terminal networks $M_1, M_2, \ldots, M_6$ such that each exceeding one is obtained by adding to the preceding one (Fig. 4) and $M_6 = \mathfrak{M}_i$. M_1 is a contact tree of the variables $x_{k+1}, \ldots, x_{k+r}$ is a $\left[1, 2^r\right]$-terminal network (where r is the power of two), which realizes all the conjunctions of the form $x_{k+1}^{\alpha_{k+1}} \ldots x_{k+r}^{\alpha_{k+r}}$. M_2 is obtained from M_1 by joining the outputs, corresponding to the points of the same sphere for a certain fixed breakdown of the cube $\{(\beta_{k+1}, \ldots, \beta_{k+r})\}$ into spheres. M_2 is a $\left[1, 2^r/r\right]$-terminal network, which realizes the functions $\varphi_h(x_{k+1}, \ldots, x_{k+r})$, where $h = 1, 2, \ldots, 2^r/r$. M_3 is a multi-terminal network, obtained from M_2 by connecting to each of the outputs contact trees of the variables $x_{k+r+1}, \ldots, x_n$. M_3 is thus a $\left[1, 2^r/r\right]$-terminal network, which realizes functions of the form

$$\varphi_h(x_{k+1}, \ldots, x_{k+r}) x_{k+r+1}^{\alpha_{k+r+1}} \ldots x_n^{\alpha_n}.$$

M_4 is formed from M_3 by connecting to each output corresponding to a sphere with a center at the point $(\beta^0_{k+1}, \ldots, \beta^0_{k+r})$, a $\left[1, r\right]$-terminal network of the form indicated in Fig. 5. The resultant multi-terminal network is a $1, 2^{n-k}$ -terminal network, which realizes conjunctions of the form $x_{k+1}^{\alpha_{k+1}} \ldots x_n^{\alpha_n}$. M_5 is obtained from M_4 by joining certain outputs

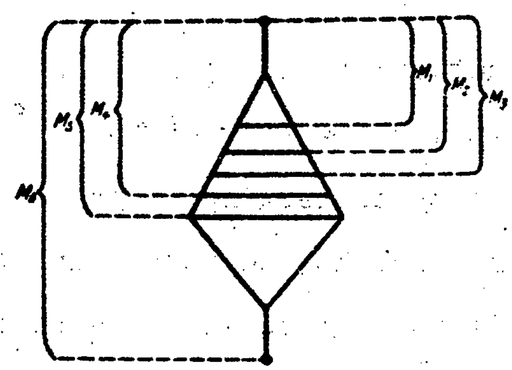


Fig. 4

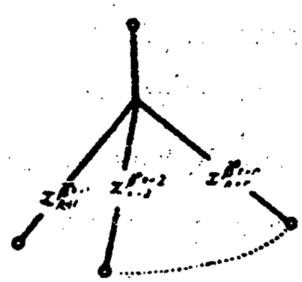


Fig. 5

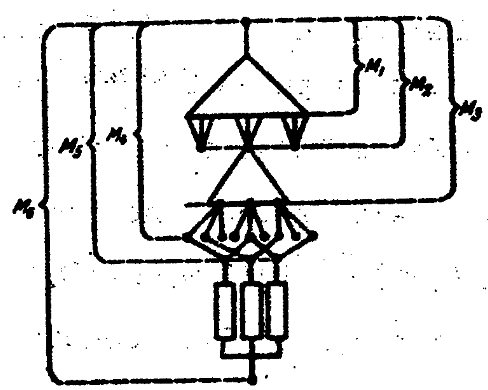


Fig. 6

corresponding to the same sphere, so that the function

$$f_{ijh}^{(2)}(x_{k+1}, \dots, x_n) = f_{ij}^{(1)}(x_{k+1}, \dots, x_n) \& \varphi_h(x_{k+1}, \dots, x_{k+r})$$

is realized on the output constructed thereby. This multi-terminal network realizes all the functions

$$f_{ijh}^{(2)} \left(i = 1, \dots, p;\ j = 1, 2, \dots \dots;\ h = 1, \dots \frac{2^r}{r}\right)$$

M_6 is constructed by connecting to the output of the multi-terminal network M_5, corresponding to the function $f_{ijh}^{(2)}$ an equivalent Π-network corresponding to the perfect disjunctive normal form of the function $f_{ij}^{(1)}(x_1, \dots, x_n)$, and by subsequent joining of all the outputs into one. The multi-terminal network M_6 (Fig. 6) is a (1, 1)-terminal network, which realizes the function $f_i(x_1, \dots, x_n)$. We omit the proof of this fact, since it makes up the content of reference /9/.

Let us put $r = 2^{[1/2 \log_2 n]}$, $k = [2 \log_2 n]$ and $s = \left[\frac{n - 2\sqrt{n}}{\sigma_2}\right]$.

We estimate the number of contacts in the network, which realizes the function f. It is obvious that

$$L_{Q_\sigma}(n) < \left(\frac{2^k}{s} + 1\right)(L(M_2) + L(M_3 \setminus M_2) + L(M_4 \setminus M_3) + L(M_6 \setminus M_5)),$$

where the symbol L(T) denotes the number of contacts of the multi-terminal network T, the factor $(2^k/s + 1)$ is the major end of the number of p networks $\mathfrak{A}_i$. We have

$$L(M_2) < 2^{r+1},\ L(M_3 \setminus M_2) < 2^{n-k-r+1} \frac{2^r}{r} = \frac{2^{n-k+1}}{r},\ L(M_4 \setminus M_3) = 2^{n-k}.$$

If in addition the parameter s satisfies the inequality

$$\frac{2^{N_1}}{\sigma_1\varepsilon_2} < s \leqslant 2^k,$$

then according to lemma 3.2 the number of classes of columns of each group does not exceed $2^{\sigma_2 s}$ and $L(M_6 \setminus M_5) \leqslant s\cdot k(2^r/r)2^{\sigma_2 s}$. Thus, when the foregoing limitations are satisfied, we have

$$L_{Q_\sigma}(n) < \left(\frac{2^k}{s}+1\right)\left(2^{r+1}+\frac{2^{n-k+1}}{r}+2^{n-k}+sk\frac{2^r}{r}2^{\sigma_2 s}\right)=$$

$$=\frac{2^n}{s}+2^{n-k}+\left(\frac{2^k}{s}+1\right)\left(2^{r+1}+\frac{2^{n-k+1}}{r}+s\cdot k\frac{2^{r+\sigma_2 s}}{r}\right).$$

Inasmuch as

$$\frac{\sqrt{n}}{2} < r \leqslant \sqrt{n},\ 2\log_2 n - 1 < k \leqslant 2\log_2 n,$$

$$\frac{n-2\sqrt{n}}{\sigma_2}-1 < s \leqslant \frac{n-2\sqrt{n}}{\sigma_2} \ \& \ \sigma < \sigma_2 < 1,$$

we obtain

$$L_{Q_\sigma}(n) < \frac{\sigma_2 2^n}{n-2\sqrt{n}-1}+\frac{2^{n+1}}{n^2}+\left(\frac{\sigma_2 n^2}{n-2\sqrt{n}-1}+1\right)\times$$

$$\times\left(2^{\sqrt{n}+1}+\frac{2^{n+2}}{n^2\sqrt{n}}+\frac{n-2\sqrt{n}}{\sigma}2\frac{2^{n-\sqrt{n}+1}}{\sqrt{n}}\log_2 n\right).$$

Hence

$$L_{Q_\sigma}(n) < \frac{\sigma_2 2^n}{n}(1+\tilde{\varepsilon}(n)),$$

where

$$\tilde{\varepsilon}(n)=\frac{2\sqrt{n}+1}{n-2\sqrt{n}-1}+\frac{2}{\sigma n}+\left(\frac{n}{n-2\sqrt{n}-1}+\frac{1}{\sigma n}\right)\times$$

$$\times\left(n^2 2^{-n+\sqrt{n}+1}+\frac{8}{\sqrt{n}}+\frac{n^2(n-2\sqrt{n})}{3\sqrt{n}}2^{-\sqrt{n}+2}\log_2 n\right)$$

Obviously, $\tilde{\varepsilon}(n) < c/\sqrt{n}$, and therefore $\tilde{\varepsilon}(n) \to 0$ as $n \to \infty$.

Let $\varepsilon > 0$ be an arbitrary small number. We choose numbers ε_1, ε_2, ε_3 such that

$$(\sigma+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) < \sigma+\varepsilon.$$

We specify a number N_2 such that when $n > N_2$ we have

$$\tilde{\varepsilon}(n) < \varepsilon_3.$$

Inasmuch as $s = \left[\frac{n-2\sqrt{n}}{\sigma_2}\right]$ and $k = [2 \log_2 n]$, there exists such a N_3, that when $n > N_3$

$$\frac{2^{N_1}}{\sigma_1\varepsilon_2} < s < 2^k.$$

If we now take $N = \max(N_1, N_2, N_3)$ we have when $n > N$

$$L_{Q_\sigma}(n) < \frac{\sigma_1 2^n}{n}(1+\tilde{\varepsilon}(n)) < \frac{2^n}{n}(\sigma+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) < \frac{(\sigma+\varepsilon)2^n}{n}.$$

This proves the theorem completely. From this theorem we readily obtain three other statements:

Theorem 3.2. For any invariant class Q_σ, for which $0 < \sigma \leq 1$, we have

$$L_{Q_\sigma}(n) \sim \frac{\sigma 2^n}{n}.$$

Proof. To estimate the lower bound of $L_{Q_\sigma}(n)$ we use a theorem by O. B. Lupanov from reference /7/, where under very weak limitations, which are satisfied under our conditions, it is proved that for any $\varepsilon > 0$ and

$n > N(\varepsilon)$ we have

$$L_Q(n) > (1-\varepsilon)\frac{\log_2 P_Q(n)}{\log_2\log_2 P_Q(n)}.$$

Since $\lim\limits_{n\to\infty} \sqrt[2^n]{P_{Q_\sigma}(n)} = 2^\sigma$, then

$$L_{Q_\sigma}(n) > (1-\varepsilon)\frac{\log_2 P_{Q_\sigma}(n)}{\log_2\log_2 P_{Q_\sigma}(n)} > (1-\varepsilon')\frac{\sigma 2^n}{n}.$$

Comparing this relation with the estimate given by theorem 3.1, we obtain finally

$$L_{Q_\sigma}(n) \sim \frac{\sigma 2^n}{n}.$$

This proves the theorem.

Thus, we have separated a continual family of classes Q_σ $(0 < \sigma \leqslant 1)$, for which there exists a sufficiently effective synthesis method. This method allows us to realize functions from class Q_σ by means of contact networks ~~with contacts~~ which contain asymptotically not more than $L_{Q_\sigma}(n)$ contacts. The latter is evidence that in a certain sense the method does admit of any essential improvement. This result disclosed the asymptotic behavior of the functions $L_{Q_\sigma}(n)$ for a continuum of classes.

An essential role is played in the establishment of this fact by the possibility of realizing a lower--bound estimate for the function $L_{Q_\sigma}(n)$ for a continual family of classes, which makes the use of the

Lupanov theorem essential.

Theorem 3.3. For any invariant class Q_σ, for which $\sigma = 0$, we have

$$\lim_{n\to\infty} \frac{L_{Q_\sigma}(n)}{L(n)} = 0.$$

Proof. Using /7, 12/ the lower-bound estimate for the quantity L(n) and the upper-bound estimate for the quantity $L_{Q_\sigma}(n)$, provided by theorem 3.2, i.e., the fact that when $n > N(\varepsilon)$

$$L(n) > (1-\varepsilon)\frac{2^n}{n} \;\&\; L_{Q_\sigma}(n) < \frac{\varepsilon 2^n}{n},$$

we obtain

$$0 < \frac{L_{Q_\sigma}(n)}{L(n)} < \frac{\varepsilon}{1-\varepsilon}.$$

This leads directly to the required result.

This theorem is contained in /18/ and shows that in the case $\sigma = 0$ the function of the class Q_σ admits of a substantially simpler network realization, than in the general case. Theorem 2.6 is evidence that ~~the~~ a situation of this kind exists for a continual family of invariant classes. In particular, the invariant classes listed in examples 1 -- 6 (p. 87 /of source/) admit of a simpler network realization. However, the application of theorem 3.3 to the majority of the indicated classes is not so interesting, since methods of a simple network realization of the functions of these classes are known without them. Worthy of attention here is only the class of monotonic functions, for which theorem 3.3 establishes the existence of a simpler network realization, than in the general case. The author expresses his confidence that for all invariant classes of practical interest $\sigma = 0$, and consequently, theorem 3.3 is valid.

Theorems 3.2 and 3.3 can be unified into a more general theorem.

Theorem 3.4. For each invariant class Q_σ $(0 \leq \sigma \leq 1)$ the following relation holds

$$\lim_{n \to \infty} \frac{L_{Q_\sigma}(n)}{L(n)} = \sigma.$$

4. Solution of the Problem in the Class of Correct Algorithms

We now turn to the question of constructing the set M^0, which contains (for any $\varepsilon > 0$) an infinite number of ε-complex functions (see Sec. 1), i.e., such that for a certain sequence $\{n_k\}$ $(n_1 < n_2 < \ldots)$ we have

$$\frac{L_{M^0}(n_k)}{L(n_k)} \to 1 \quad (k \to \infty).$$

Trivial arguments, similar to those carried out in the introduction, bring to mind the thought that to construct the function $f_n(x_1, x_2, \ldots, x_n) \in M^0$ it is necessary to review (scan) all the functions that depend on n variables. However if no limitations are imposed on the means for solving the problem, a simpler algorithm can be indicated.

Let the function F(m) be defined on a natural series and let it assume as values functions from the set M^0, namely

$$F(m) = f_m(x_{i_1}, x_{i_2}, \ldots, x_{i_n}),$$

where $f_m \in M^0$. It is obvious that F(m) is a recursive function. Let us join it, for example, to the inital recursive functions /24/. Then we can solve the problem stated, by using recursions constructed on the specified functions, in a trivial manner and quite simply: it is necessary to take the values of the function F. Incidentally, such a layout of the situation is nothing but a certain subterfuge, based on the fact that we have admitted as an elementary means a function which requires a scanning of the same order to calculate as values, as the solution of the initial problem.

Thus, if we wish to forbid the use of a trivial algorithm (complete scanning), it is necessary to impose limitations on the means of solving the problem. These limitations are aimed at breaking up the vicious cycle, at which one admits as elementary means those means, that require the same scanning (and perhaps an even greater one) as in the case of the trivial algorithm.

Thus, for convenience we assume that we have an algorithm A, which converts the natural numbers into functions of algebraic logic (say, into a table of functions) $f_m(x_{i_1(m)}, x_{i_2(m)}, \ldots, x_{i_{n(m)}}) \; m \to f_m(x_{i_1(m)}, x_{i_2(m)}, \ldots, x_{i_{n(m)}})$.

This defines the mapping F of the set of natural numbers on the subset of functions of algebraic logic, i.e.,

$$F(m) = f(x_{i_1(m)}, x_{i_2(m)}, \ldots, x_{i_{n(m)}}).$$

Definition. An algorithm A is called correct, if the image of the natural series in the mapping F, determined by the algorithm A, is an invariant class Q of functions of algebraic logic.

That this is a natural definition is dictated by the fact that usually the algorithm that constructs the function $f(x_1, x_2, \ldots, x_n)$ also constructs:

1) Any function equal to it,

2) Any function obtained from f by renaming (without identification) of the variables,

3) Any function obtained from f by any substitution of constants in place of (not necessarity all) the variables.

Thus, usually the algorithm constructs a certain invariant class Q (see definition on p. 82 /of source/).

Theorem 4.1. The closure $\widetilde{M}^0$ of the set M^0 with respect to the operations 1, 2, and 3, entering into the definition of the invariant class, as P_2, i.e., it contains all the functions of algebraic logic.

Proof. Let $\widetilde{M}^0 \equiv Q \neq P_2$. From this, on the basis of the corollary from theorem 2.5 $\lim_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 2^{\sigma}$, where $\sigma < 1$. But then by virtue of the theorem 3.4

$$\lim_{n\to\infty} \frac{L_Q(n)}{L(n)} = \sigma < 1.$$

The latter contradicts the fact that for a certain sequence $\{n_k\}$ (see definitiion of the set M^0)

$$\lim_{n\to\infty} \frac{L_{M^0}(n_k)}{L(n_k)}.$$

Consequently, the assumption that $Q \neq P_2$ is incorrect. Therefore $Q = P_2$ and this proves the theorem.

Corollary. If the set M^0 is such that for any n it contains "the most complicated" function f_n, i.e., $L(f_n) = L(n)$, then the closure of the set M^0 with respect to the operations 1, 2, and 3 contains all the functions of algebraic logic.

The latter statement was predicted by the author as a hypothesis in 1953--1954, and served as the starting point for the investigations, the culmination of which is the present paper.

From this theorem follows directly our principal proposition.

Theorem 4.2. In any correct algorithm, which constructs the set M^0, constructs all the functions of algebraic logic, i.e., in other words, this algorithm is the scanning of all the functions of algebraic logic.

In particular we also obtain the following:

Corollary. Any ~~correct~~ regular algorithm, which for any natural n constructs the "most complicated" function f_n, i.e., $L(f_n) = L(n)$, contains a complete scanning of all the functions of algebraic logic.

It is necessary to make a remark concerning the result obtained. Theorem does not state at all that in order to find an individual "most comlex" function $f_n(x_1, x_2, \ldots, x_n)$ it is necessary to scan all the functions that depend on n variables. It follows from the theorem that to construct the function f_n it is necessary to scan all the functions of algebraic logic that depend on $m(n) < n$ variables and that $m(n) \rightarrow \infty$ $(n \rightarrow \infty)$. In other words, a complete scanning is essential in order to construct the entire class M^0.

Let us now compare the two approaches to the solution of the problem concerning the construction of the set M^0. Earlier, in Sec. 1, we noted that if we admit as possible means algorithms with elementary random acts, then there exists a simple method of constructing the set M^0 with probability 1. However, in spite of this, we are unable to establish whether the result of such a construction does have the required properties or not. We have just shown that in a class of correct algorithms, the construction of the set M^0 requires a complete scanning of the functions of algebraic logic. The latter, in view of the impossibility of practical realization of a complete scanning (even at values of n that are too large) is evidence of the impossibility of solving the problem with admissible means. Under these conditions, it remains for us either to forego entirely problems of this kind, designating them as incorrect, "poorly formulated," or else to be satisfied with solutions in the class of algorithms with elementary random acts. In other words, to do what is usually done in practice -- neglect events of probability 0, i.e., assume that the given construction always gives the set M^0.

Thus, we have formulated a problem that pertains to network objects in terms that do not contain the probability concept. As to the solution, it has a probability-theory character. Consequently, in many problems the network objects come to the forefront as objects with statistical nature. The latter pertains to macroobjects, which have sufficient complexity./5/

In conclusion we note that the result obtained can be readily extended to cases of realization of functions in the class of compact-valve networks /25/, in the class of networks of the type "formulas with memory" /26/ (which include electronic networks), etc., i.e., those cases when synthesis methods have been constructed, which make it possible to establish an asymptotic value for $L(n)$. Apparently this problem lies now not in transferring the result to these types of networks, but in establishing theorems for classes of networks, in which the elements can vary over a wide range. The result acquires thereby a general-cybernetic significance.

Appendix

As shown by the analysis of Sec. 3, the complexity of a contact network $\mathfrak{A}$, which realizes the function $f(x_1, \ldots, x_n)$ and which is obtained by methods of references /9, 12, 14, 27/, depends essentially on how many functions can be obtained by substitution of constants from the function $f(x_1, \ldots, x_n)$. It is possible that the complexity of the minimal network of the function $f(x_1, \ldots, x_n)$, i.e., $L(f)$, is determined to a great extent by the number $S(f)$, which denotes the number of pairwise-unequal functions, obtained from a given function f by substitution of constants. In other words, the quantity S(f) represents a measure of information on the complexity of the minimal contact network, which can be determined from functional considerations. In this connection, the quantity S(f) is of interest. However, an estimate of the quantity S(f) in general form entails considerable difficulties, and we shall therefore estimate the quantity S(n) where

$$S(n) = \max S(f)$$

(the maximum is taken over all the functions f which depend on n variables).

Theorem.

$$S(n) \sim 3^n.$$

Proof. Since the number of possible substitutions of constants in a function of n variables is 3^n, then obviously $S(n) \leqslant 3^n$. Let us show that when $n > N(\varepsilon)$ we have the inequality $S(n) > (1 - \varepsilon)3^n$.

Fig. 7 shows on the left and on the right, in a certain sequence, all the functions f which depend on n variables $x_1, \ldots, x_n$ ($f_i \neq f_j$, if $i \neq j$). We join the function f_i from the left column to function f_j from the right column by means of a vector, which originates in f_i, if there exists at least one substitution of constants in the function f_i, which converts it into a function equal to f_j. We denote by $t(f_i)$ and

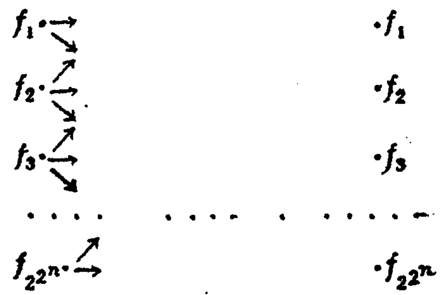


Fig. 7.

$\tau(f_j)$ the numbers of vectors which originate in f_i and enter in f_j respectively. From the construction we have

$$t(f_i) \leqslant S(f_i), \quad \sum_{i=1}^{2^{2^n}} t(f_i) = \sum_{j=1}^{2^{2^n}} \tau(f_j).$$

Let $f_j(x_1, \ldots, x_n)$ depend essentially on the variables $x_{i_1}, \ldots, x_{i_k}$ (for convenience let $i_1 = 1, \ldots, i_k = k$), i.e., $f_j(x_1, \ldots, x_n) = f_j(x_1, \ldots, x_k)$. Then each function $f_i(x_1, \ldots, x_n)$ from which we can obtain the function f_j by substitution of constants, should contain when expanded in the variables $x_{k+1}, \ldots, x_n$ a term of the form

$$x_{k+1}^{\sigma_{k+1}} \ldots x_n^{\sigma_n} f_j(x_1, \ldots, x_k).$$

Therefore

$$\tau(f_j) > 2^{2^n} - (2^{2^k} - 1)^{2^{n-k}}$$

and

$$\sum_{j=1}^{2^{2^n}} \tau(f_j) > \sum_{k=0}^{n} C_n^k P^*(k) [2^{2^n} - (2^{2^k} - 1)^{2^{n-k}}],$$

where

$$P^*(k) > 2^{2^k} - k 2^{2^{k-1}}$$

(see formula on p. 87 /of source/). Let $k > \log_2 n$, then

$$\left(1 - \frac{1}{2^{2^k}}\right)^{2^{n-k}} = e^{-\frac{2^{n-k}}{2^{2^k}} + o\left(\frac{2^{n-k}}{2^{2^k}}\right)} = 1 - \frac{2^{n-k}}{2^{2^k}} + o\left(\frac{2^{n-k}}{2^{2^k}}\right)$$

and

$$\sum_{k=0}^{n} C_n^k P^*(k) [2^{2^n} - (2^{2^k} - 1)^{2^{n-k}}] >$$

$$\geqslant 2^{2^n} \sum_{k > \log_2 n} C_n^k 2^{2^k} \left(1 - \frac{k}{2^{2^{k-1}}}\right) \left(\frac{2^{n-k}}{2^{2^k}} + o\left(\frac{2^{n-k}}{2^{2^k}}\right)\right)$$

Let furthermore, for $\log_2 n > N_1$, the inequality $k/2^{2^{k-1}} \leqslant \varepsilon/3$ be satisfied. Then

$$\sum_{j=1}^{2^{2^n}} \tau(f_j) > \left(1 - \frac{\varepsilon}{3}\right) 2^{2^n} \sum_{k > \log_2 n} C_n^k 2^{n-k} (1 + o(1)).$$

Finally, assume that when $\log_2 n > N_2$ we have

$$|o(1)| < \frac{\varepsilon}{3} \text{ and } \sum_{k \leqslant \log_2 n} C_n^k 2^{n-k} < \frac{\varepsilon}{3} 3^n.$$

We then obtain for $\log_2 n > \max(N_1, N_2)$

$$\sum_{i=1}^{2^{2^n}} S(f_i) \geqslant \sum_{j=1}^{2^{2^n}} \tau(f_j) > (1 - \varepsilon) 2^{2^n} \cdot 3^n.$$

From this it follows that

$$S(n) \cdot 2^{2^n} \geqslant \sum_{i=1}^{2^{2^n}} S(f_i) > (1 - \varepsilon) \cdot 2^{2^n} \cdot 3^n, \qquad (*)$$

or

$$S(n) > (1 - \varepsilon) \cdot 3^n.$$

This proves the theorem.

Theorem. Let $\delta > 0$ be a number as small as desired. The fraction of all the functions $f_i(x_1, \ldots, x_n)$ for which $S(f_i) < (1 - \quad)3^n$, with respect to the number of all the functions of n variables, tends to zero with increasing n.

Proof. The theorem follows from inequality (*). Let $\rho(\delta, n)$ denote the number of all the functions

$f_i(x_1, \ldots, x_n)$ of n variables, for which $S(f_i) < (1 - \delta)3^n$. We have

$$\sum_{S(f_i)\geq(1-\delta)3^n} S(f_i) + \sum_{S(f_i)<(1-\delta)3^n} S(f_i) > (1-\varepsilon)2^{2^n}\cdot 3^n.$$

From this it follows that

$$[2^{2^n} - \rho(\delta, n)]\,3^n + \rho(\delta, n)(1-\delta)3^n > (1-\varepsilon)2^{2^n}3^n,$$

or

$$\frac{\rho(\delta, n)}{2^{2^n}} < \frac{\varepsilon}{\delta}.$$

This proves the theorem completely.

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