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Abstract: In medical studies or in reliabilty analysis an investigator is often interested with the assesment of a specific risk in presence of other risk factors. In the Statistical literature it is known as the analysis of competing risks model. The competing risks model assumes that the data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out under this assumption for parametric and non parametric set up. Unfortunately in many situations, the causes of failure are not observed, even if the failure time is observed. Miyawaka (1984) obtained some of the results under the assumption that the failure time distribution is exponential. He obtained the maximum likelihood estimators and the minimum variance unbaised estimators of the unknown parameters. We provide the approximate and asymptotic properties of these estimators. Using the approximate and the asymptotic distributions we obtain confidence bounds of the parameters and also propose two different bootstrap confidence bounds. We consider the case when the failure distribution may not be exponential and use one data set to see how different methods work in real life situations.

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ANALYSIS OF INCOMPLETE DATA IN PRESENCE OF COMPETING RISKS

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1. INTRODUCTION:

In medical studies or in reliability analysis it is quite common that more than one risk factor may be present at the same time. An investigator is often interested in the assesment of a specific risk in presence of other risk factors. In the Statistical literature it is well known as the analysis of competing risks model. In analyzing the competing risks model it is assumed that the data consists of a failure time and an indicator denoting the cause of failure. Several studies have been carried out under this assumption for both parametric and nonparametric set up. For the parametric set up it is assumed that the different lifetime distributions follow some special parametric distribution, namely exponential, gamma or Weibull. Several authors for example Berkson and Elveback (1960), Cox (1959), David and Moeschberger (1978) considered this problem from the parametric point of view. In the nonparametric set up no specific lifetime distribution is assumed. Kaplan and Meier (1958), Efron (1967) and Peterson (1977) analysed the nonparametric version of this model. But in all the above cases it is assumed that the causes of failure are known if the failure times are observed. But in certain situations (Dinse; 1982 or Miyawaka; 1982) it is observed that the determination of the cause of failure may be expensive or may be very difficult to observe. Therefore it might occur that the failure time of that item/ individual is observed but the corresponding cause of failure is not observed. Miyawaka (1984) considered this model and obtained the maximum likelihood estimators (MLE's) and the uniformly minimum variance unbiased estimators (UMVUE's) of the failure rates of the different failure distribution under the assumption that they are of the exponential type. But he did not provide any distributional properties of these estimators.

In this paper we consider the same model as that of Miyawaka (1984). It is assumed that every member of a certain target population either dies of a particular cause, say cancer, or by other causes. A proportion π of the population die of cancer and the proportion $(1-\pi)$ die due to other causes. At the end of the study , we have three types of observations:

(a) Individuals who died of cancer and their lifetimes.

(b) Individuals who died of other causes and their lifetimes.

(c) Individuals who died of unknown causes their lifetimes are observed, but their causes of death are not known.

There is another dimension to this problem. The research project is financed for a fixed length of time, say, M. Some individuals could be alive at the end of the project period. For simplicity, we ignore this aspect of the problem and assume that every one in the sample can be monitored until theirs death. We assume that the lifetime distributions of the different causes of failure are exponential. It is observed that although the MLE's or the UMVUE's of the hazard rates always exist but the MLE's or the UMVUE's of the mean lifetime of the different causes may not exist always. It is not possible to obtain the UMVUE's of the mean lifetimes of the different causes. We obtain the conditional MLE's of the mean lifetime of the different causes. We obtain the exact distribution of the conditional MLE's using the conditional moment generating function approach and also obtain the asymptotic distributions of the MLE's. It is observed that the MLE of the mean lifetime usually over estimate the true parameter at least for small sample. As in complete sample, the bias can be obtained as the inverse moment of the positive binomial random variable even in the case of incomplete data. We provide small tables for the biases of the MLE's for different sample sizes and for different parameter values when 10% and 20% data are incomplete. Based on the exact and the asymptotic distributions, we propose two approximate confidence intervals of the different parameters of interest. We also use percentile bootstrap and bootstrap-t confidence intervals for the unknown parameters and compare their performances through Monte Carlo simulatons.

Since the exponential distribution has constant failure rates it might not be very practical to assume that the lifetime distribution is exponential. Two parameter Weibull distribution can be used to analyze the lifetime data because of its increasing and decreasing failure rates. We consider the model when the underlying lifetime distributions are Weibull. We obtain the maximum likelihood estimators of the different parameters and study their properties under this assumption. We also obtain the asymptotic confidence bounds and the bootstrap confidence bounds of the different parameters and compare their performances through Monte Carlo simulations. We consider one data set from Lawless (1982) and see how the different methods work in this practical situation.

The rest of the paper is organised as follows. In Section 2, we give the description of the model and give the different notations we are going to use in this paper. In Section 3, we consider the estimation of the different parameters and also obtain the exact distribution of the MLE of the mean lifetime when the lifetime distribution is exponential. Different confidence intervals of the unknown parameters are considered in Section 4. Weibull lifetime distribution is considered in Section 5. The numerical results are presented in Section 6. One data set from Lawless (1982) is being analysed in Section 7 and finally we draw conclusions from our work in section 8.

2. MODEL DESCRIPTION AND NOTATIONS:

Without loss of generality we assume that there are only two causes of failure. We use the following notations:

X_i: Lifetime of system i.

X_{ji}: Lifetime of mode j of system i, j = 1,2.

F(.): Cumulative distribution function of X_i.

$F_j(.)$: Cumulative distribution function of X_{ji}, j = 1,2.

$\bar{F}_j(.)$ = 1 - $F_j(.)$

δ_i: indiactor variable denoting the cause of failure of system i.

I[.]: indiactor function of event [.].

Gamma (α, λ): denotes the gamma random variable with density function $\frac{\lambda^\alpha}{\Gamma\alpha}x^{\alpha-1}e^{-\lambda x}$

Weibull (α, λ): denotes the Weibull random variable with density function $\alpha\lambda x^{\alpha-1}e^{-\lambda x^\alpha}$

It is assumed that (X_{1i}, X_{2i}); i = 1,2, ..., n, are n independent identically distributed (i.i.d.) random variables. X_{1i} and X_{2i} are independent for all i = 1,2, ..., n and X_i = min $\{X_{1i}, X_{2i}\}$. With out loss of generality it is assumed that the first m observations consist of failure times and also causes of failure whereas for the last (n-m) observations we only observe the failure times and not the causes of failure, i.e. the following data are being observed $(X_1, \delta_1), (X_2, \delta_2), \ldots, (X_m, \delta_m)$, $(X_{m+1}, *), \ldots, (X_n, *)$. In order to analyze the incomplete data it is assumed that the failure times are from the same population as the complete data, that is population remains unchanged irrespective of the cause of failure.

3. EXPONENTIAL FAILURE DISTRIBUTIONS, ESTIMATION:

In this section we assume that X_{ji}'s are exponential random variables with parameters λ_j for i = 1, 2, ..., n and for j = 1 and 2. The distribution function $F_j(.)$ of X_{ji} has the following form:

$$F_j(t) = (1 - e^{-\lambda_j t}), \tag{3.1}$$

for j = 1 and 2. The likelihood function of the observed data $(x_1, \delta_1), \ldots (x_m, \delta_m), (x_{m+1}, *)$, ..., $(x_n, *)$ for the general case takes the following form:

$$L = \prod_{i=1}^{m}[dF_1(x_i)\bar{F}_2(x_i)]^{I(\delta_i=1)}[dF_2(x_i)\bar{F}_1(x_i)]^{I(\delta_i=2)} \prod_{i=m+1}^{n} dF(x_i). \tag{3.2}$$

Therefore for the particular case when F_1 and F_2 are exponentials with parameters λ_1 and λ_2 respectively, the likelihood function becomes:

$$L = \lambda_1^{r_1}\lambda_2^{r_2}(\lambda_1 + \lambda_2)^{n-m}exp(-(\lambda_1 + \lambda_2)\sum_{i=1}^{n} x_i). \tag{3.3}$$

Here r_1 and r_2 denote the number of failures due to mode 1 and mode 2 respectively. Taking the logarithm of (3.3) and equating the partial derivatives to be zeros, we get the MLE's of λ_1 as

$$\hat{\lambda}_1 = \frac{nr_1}{m\sum_{i=1}^{n} x_i}, \tag{3.3}$$

and the UMVUE's of λ_1 as

$$\lambda_1^* = \frac{(n-1)r_1}{m\sum_{i=1}^{n} x_i}, \tag{3.4}$$

see Miyawaka (1984). Therefore MLE of the survival function due to cause 1 (say cancer) is

$$\hat{\bar{F}}_j(x) = e^{-\hat{\lambda}_1 x}, \tag{3.5}$$

and the MLE of the hazard rate or the instantaneous death rate due to cause 1 is given by

$$\frac{d\hat{F}_1(x)}{\hat{\bar{F}}_1(x)} = \hat{\lambda}_1.$$

The realtive risk rate, π, due to cause 1 is

$$P[X_{1i} < X_{2i}] = \int_0^\infty \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2},$$

and because of the invariance property of the MLE, the relative risk rate $\hat{\pi}$ due to cause 1 is

$$\hat{\pi} = \frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \hat{\lambda}_2} = \frac{r_1}{m}. \tag{3.6}$$

For the exponential lifetime distribution as (3.1), λ_1 represents the hazard rate and $\theta_1 = \frac{1}{\lambda_1}$ denotes the mean lifetime due to cause 1. Although the MLE's and the UMVUE's of λ_1 always exist but the same is not true for θ_1. The UMVUE of θ_1 does not exist and the MLE of θ_1 exist only when $r_1 > 0$. The conditional MLE of θ_1, say $\hat{\theta}_1$, when $r_1 > 0$ is as follows:

$$\hat{\theta}_1 = \frac{m\sum_{i=1}^{n} x_i}{nr_1}. \tag{3.7}$$

If $r_1 = 0$, $\hat{\theta}_1$ does not exist. Now we obtain the conditional distribution of the MLE of θ_1, conditioning on $r_1 > 0$. Our approach to produce the confidence bound for θ_1 is based on the distribution of $\hat{\theta}_1$ similarly as Chen and Bhattacharya (1988). Moreover, use of the MLE provides a safeguard against any obvious loss of informations and ensures asymptotic optimality of the present method. Consider the following lemmas:

Lemma 1: The conditional moment generating function (mgf) of $\hat{\theta}_1$, $\phi_{\hat{\theta}_1}(t)$, is of the following form

$$\phi_{\hat{\theta}_1}(t) = E\left[e^{t\hat{\theta}_1} | r_1 > 0\right]$$

$$
\begin{aligned}
&= (1-q^m)^{-1}\left[\sum_{i=1}^{m}\frac{m!}{i!(m-i)!}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{m-i}\left(1-\frac{tm\theta_1\theta_2}{ni(\theta_1+\theta_2)}\right)^{-n}\right] \\
&= [\sum_{i=1}^{m} p_i\left(1-\frac{tm\theta_1\theta_2}{ni(\theta_1+\theta_2)}\right)^{-n}],
\end{aligned}
$$

here

$$
q = \frac{\theta_1}{\theta_1+\theta_2} \text{ and } p_i = (1-q^m)^{-1}\frac{m!}{i!(m-i)!}\left(\frac{\theta_2}{\theta_1+\theta_2}\right)^i\left(\frac{\theta_1}{\theta_1+\theta_2}\right)^{m-i},
$$

for i = 1, 2 , ..., m.

Proof of Lemma 1:

Note that $\sum_{i=1}^{n} X_i$ is Gamma (n, $\lambda_1+\lambda_2$) random variable and r_1 is a binomial random variable with parameters m and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

$$
\begin{aligned}
\phi_{\hat{\theta}_1}(t) &= E\left[e^{t\hat{\theta}_1}|r_1>0\right] \\
&= \sum_{i=1}^{n} E\left[e^{t\hat{\theta}_1}|r_1=i\right]P(r_1=i|r_1>0).
\end{aligned}
$$

Using the facts $p_i = P(r_1 = i|r_1 > 0)$ for i = 1,2, ..., m and the moment generating function of Gamma (α, λ) is $(1-\frac{t}{\lambda})^{-\alpha}$, the result follows immediately.

Theorem 1: The conditional probabilty density function (pdf) of $\hat{\theta}_1$, say $f_{\hat{\theta}_1}$(x), or the conditional probabilty distribution function, say $F_{\hat{\theta}_1}$(x), becomes

$$
f_{\hat{\theta}_1}(x) = \sum_{i=1}^{m} p_i g_i(x), \quad F_{\hat{\theta}_1}(x) = \sum_{i=1}^{m} p_i G_i(x),
$$

where $g_i(x)$ and $G_i(x)$ denote the density function and the distribution function respectively of a gamma random variable with shape parameter n and scale parameter $\frac{ni(\theta_1+\theta_2)}{m\theta_1\theta_2}$ for i = 1, 2, ... m.

Proof of Theorem 1: Obvious from Lemma 1.

Therefore the conditional distribution of $\hat{\theta}_1$ is a mixture of m gamma random variables. It may be also noted that when m = n, we get the distribution of $\hat{\theta}_1$ in the competing risk model when the lifetime distributions of the different causes are exponential and there is no censoring (see David and Moeschberger; 1978). With the help of Theorem 1, we can obtain different conditional moments of $\hat{\theta}_1$. We give the first and the second conditional momemts of $\hat{\theta}_1$. For brevity we denote them as E($\hat{\theta}_1$) and E($\hat{\theta}_1^2$) respectively.

$$
E(\hat{\theta}_1) = \frac{m\theta_1\theta_2}{(\theta_1+\theta_2)}\sum_{i=1}^{m}\frac{p_i}{i}
$$

and

$$E(\hat{\theta}_1^2) = \frac{m^2\theta_1^2\theta_2^2 n(n+1)}{(\theta_1+\theta_2)^2 n^2} \sum_{i=1}^{m} \frac{p_i}{i^2}.$$

Since in both the cases the quantities within the summation sign denote the inverse moments of the positive binomial random variable it is not possible to give the exact expressions. Therefore it is difficult to obatin the exact bias from there. The tabulated values of the first moment of the inverse positive binomial random variable are available (see Edwin and Savage; 1954), it may be used to make some bias correction. It is not pursued here. If Z is a binomial random variable with parameters N and P, then for large N, E($\frac{1}{Z}|Z > 0) \approx \frac{1}{E(Z)} = \frac{1}{NP}$ and E($\frac{1}{Z^2}|Z > 0) \approx \frac{1}{(E(Z))^2} = \frac{1}{(NP)^2}$ see Mendenhall and Lehmann (1960). Using those approximatons, we can say that for large m and n, $E(\hat{\theta}_1) \approx \theta_1$ and $E(\hat{\theta}_1^2) \approx \theta_1^2$. It shows asymptotically $\hat{\theta}_1$ is an unbiased estimator of θ_1. As the variance of $\hat{\theta}_1$ goes to zero, $\hat{\theta}_1$ is a consistent estimator of θ_1 also.

We present two small tables which give the numerical values of the biases for different sample sizes and for different vaules of θ_2. In all these calculations we have kept $\theta_1 = 1$. Table 3.1 represents the negative value of the biases when 10% of the data are incomplete and Table 3.2 represents the negative value of the biases when 20% of the data are incomplete.

Table 3.1

Negative biases of $\hat{\theta}_1$ when 10% of the date are incomplete.

n	$\theta_2 = 1.25$	$\theta_2 = 1.50$	$\theta_2 = 1.75$	$\theta_2 = 2.00$	$\theta_2 = 2.25$	$\theta_2 = 2.50$
10	.1284	.1041	.0868	.0741	.0646	.0572
20	.0528	.0431	.0363	.0315	.0277	.0248
30	.0330	.0271	.0231	.0201	.0177	.0159
40	.0241	.0198	.0169	.0147	.0130	.0117
50	.0189	.0156	.0133	.0116	.0103	.0093

Table 3.2

Negative biases of $\hat{\theta}_1$ when 20% of the date are incomplete.

n	$\theta_2 = 1.25$	$\theta_2 = 1.50$	$\theta_2 = 1.75$	$\theta_2 = 2.00$	$\theta_2 = 2.25$	$\theta_2 = 2.50$
10	.1469	.1207	.1011	.0867	.0756	.0669
20	.0610	.0495	.0417	.0360	.0317	.0284
30	.0377	.0309	.0263	.0228	.0202	.0181
40	.0273	.0225	.0192	.0167	.0148	.0133
50	.0214	.0177	.0151	.0132	.0117	.0105

From the Table 3.1 and Table 3.2 it is clear that the bias of the MLE of $\hat{\theta}_1$ is always negative, that means $\hat{\theta}_1$ always over estimates θ_1. As the sample size increases the bias also decreases. If we have more incomplete data the bias is also more. For example, if $\theta_1 = 50$, and $\frac{\theta_2}{\theta_1} = 1.50$ and only 90 % of the data are complete for a sample of size 40, then the bias of $\hat{\theta}_1$ can be obtained from the Table 3.1 as $40 \times .0198 = .792$. Therefore, it is not a very serious bias.

4. EXPONENTIAL FAILURE DISTRIBUTIONS, CONFIDENCE INTERVALS:

In this section we propose four different confidence intervals of θ_1. The first one is based on the distribution of $\hat{\theta}_1$ under the similar kind of assumptions as that of Chen and Bhattacharya (1988). The second one is based on the asymptotic distribution of $\hat{\theta}_1$. We propose to use percentile bootstrap confidence interval and bootstrap-t confidence interval and give their implementation procedures in this section.

4.1: Approximate Confidence Bound:

First let's assume that θ_2 is known. Suppose $P_{\theta_1}[\hat{\theta}_1 \geq b]$ is a monotonically increasing function of θ_1, and let b(θ) be a function such that $\frac{\alpha}{2} = P_{\theta_1}[\hat{\theta}_1 \geq b(\theta_1)]$. Then for $\theta_1 < \theta_1'$, we have

$$\frac{\alpha}{2} = P_{\theta_1'}[\hat{\theta}_1 \geq b(\theta_1')] = P_{\theta_1}[\hat{\theta}_1 \geq b(\theta_1)] \leq P_{\theta_1'}[\hat{\theta}_1 \geq b(\theta_1)].$$

This implies that $b(\theta_1) < b(\theta_1')$, that is $b(\theta)$ is an increasing function. Thus $b^{-1}(\theta)$ exists and is an increasing function. So we get 1 - $\frac{\alpha}{2} = P_{\theta_1}[b^{-1}(\hat{\theta}_1) \leq \theta_1$], it implies $b^{-1}(\hat{\theta}_1)$ is the lower bound of the $(1-\alpha)100\%$ confidence bound of α_1. If $\hat{\theta}_{obs}$ denotes the observed value of $\hat{\theta}_1$ find $\theta_L = b^{-1}(\hat{\theta}_{obs})$ such that $\frac{\alpha}{2} = P_{\theta_L}(\hat{\theta}_1 \geq \hat{\theta}_{obs})$, which is equivalent to find θ_L such that 1 - $\frac{\alpha}{2} = P_{\theta_L}(\hat{\theta}_1 \leq \hat{\theta}_{obs})$. As $P_{\theta_1}[\hat{\theta}_1 \geq b]$ is a monotonically increasing function of θ_1, therefore $P_{\theta_1}[\hat{\theta}_1 \leq c]$ is a monotonically decreasing function of θ_1. Let c(θ) be a function such that $\frac{\alpha}{2}$ = $P_{\theta_1}[\hat{\theta}_1 \leq c(\theta_1)]$. It can be shown exactly as before that c(θ) is a decreasing function and therefore $c^{-1}(\theta)$ exists. Find $\theta_U = c^{-1}(\hat{\theta}_{obs})$, such that $\frac{\alpha}{2} = P_{\theta_U}(\hat{\theta}_1 \leq \hat{\theta}_{obs})$. Since the closed form expression of the function $b(\theta)$ or $c(\theta)$ is not possible, we need to use some iterative technique to get θ_L and θ_U. Note that we can get an exact $(1-\alpha)100\%$ confidence bound on θ_1, if we know θ_2. Since θ_2 is usually unknown, we need to estimate θ_2 and we get an approximate $(1-\alpha)100\%$ confidence bound on θ_1. The construction of the confidence bound of θ_1 is based on the assumption that $P_{\theta_1}[\hat{\theta}_1 \geq b]$ is a monotonically increasing function of θ. It is quite difficult to prove this assumption because of the complicated nature of the function. Some heuristic justification can be given as follows. Since $\hat{\theta}_1$ is the MLE of the mean life θ_1 due to cause 1, for fixed θ_2, the larger the parameter θ_1 is, the more probable it will be for its MLE to exceed a given value. Numerical values of $P_{\theta_1}[\hat{\theta}_1 \geq b]$ for various values of θ_1 and b confirms this conjecture.

Table 4.1
The tabulated values of $P_{\theta_1}[\hat{\theta}_1 \geq b]$, for $\theta_2 = 2$, b = 1 and when 10% of the data are incomplete.

n $\theta_1 =$	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
10	.475	.685	.815	.890	.933	.958	.973	.982	.987
20	.483	.766	.905	.962	.985	.994	.997	.999	.999
30	.486	.818	.948	.986	.996	.999	1.00	1.00	1.00
40	.488	.855	.971	.995	.999	1.00	1.00	1.00	1.00
50	.489	.883	.983	.998	1.00	1.00	1.00	1.00	1.00

Table 4.2
The tabulated values of $P_{\theta_1}[\hat{\theta}_1 \geq b]$, for $\theta_2 = 2$, b = 2 and when 10% of the data are incomplete.

n $\theta_1 =$	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
10	.045	.130	.246	.371	.486	.585	.667	.732	.784
20	.009	.058	.172	.329	.491	.630	.738	.817	.873
30	.002	.028	.125	.297	.492	.661	.785	.868	.920
40	.000	.014	.093	.271	.493	.686	.821	.903	.949
50	.000	.007	.070	.249	.494	.708	.849	.927	.966

Tables 4.1 and 4.2 indicate that $P_{\theta_1}[\hat{\theta}_1 \geq b]$ is an increasing function of θ_1

4.2: Asymptotic Confidence Bound:

In this subsection first we obtain the asymptotic distribution of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, using that we obtain the asymtotic distributon of $\hat{\theta}_1$ and $\hat{\theta}_2$. The Fisher Information matrix of the parameters λ_1 and λ_2 is $\mathbf{I}(\lambda_1, \lambda_2) = I_{ij}(\lambda_1, \lambda_2)$, for i = 1 and 2. Here

$$I_{ij}(\lambda_1, \lambda_2) = -E\left(\frac{\partial^2 log L(\lambda_1, \lambda_2)}{\partial \lambda_i \partial \lambda_j}\right),$$

and

$$\begin{aligned}
I_{11}(\lambda_1, \lambda_2) &= \frac{m\lambda_2 + n\lambda_1}{\lambda_1(\lambda_1 + \lambda_2)^2}, \\
I_{12}(\lambda_1, \lambda_2) &= I_{21}(\lambda_1, \lambda_2) = \frac{n - m}{(\lambda_1 + \lambda_2)^2}, \\
I_{22}(\lambda_1, \lambda_2) &= \frac{m\lambda_1 + n\lambda_2}{\lambda_2(\lambda_1 + \lambda_2)^2}.
\end{aligned}$$

The Fisher Information matrix of θ_1 and θ_2, $\mathbf{I}(\theta_1, \theta_2)$, can be obtained easily from $\mathbf{I}(\lambda_1, \lambda_2)$ by the Jacobian transformation. The Fisher Information matrix $\mathbf{I}(\theta_1, \theta_2)$ is $\mathbf{I}(\theta_1, \theta_2) =$

$((I_{ij}(\theta_1, \theta_2)))$ for i, j = 1 and 2, where

$$
\begin{aligned}
I_{11}(\theta_1, \theta_2) &= \frac{\theta_2(m\theta_1 + n\theta_2)}{\theta_1^2(\theta_1 + \theta_2)^2}, \\
I_{12}(\theta_1, \theta_2) &= I_{21}(\theta_1, \theta_2) = \frac{(n-m)}{(\theta_1 + \theta_2)^2}, \\
I_{22}(\theta_1, \theta_2) &= \frac{\theta_1(m\theta_2 + n\theta_1)}{\theta_2^2(\theta_1 + \theta_2)^2}.
\end{aligned}
$$

Therefore if $\theta = (\theta_1, \theta_2)$ and $\hat{\theta} = (\hat{\theta_1}, \hat{\theta_2})$, then from the asymptotic theory of the MLE's, see Miller (1981), we have

$$(\hat{\theta} - \theta) \rightarrow N_2(\mathbf{0}, \mathbf{I}^{-1}(\theta_1, \theta_2))$$

where $\mathbf{I}^{-1}(\theta_1, \theta_2) = I_{ij}^{-1}(\theta_1, \theta_2)$ for i,j = 1 and 2 and

$$
\begin{aligned}
I_{11}^{-1}(\theta_1, \theta_2) &= \frac{\theta_1^2(m\theta_2 + n\theta_1)}{mn\theta_2}, \\
I_{12}^{-1}(\theta_1, \theta_2) &= I_{21}(\theta_1, \theta_2) = \frac{\theta_1\theta_2(n-m)}{mn} \\
I_{22}^{-1}(\theta_1, \theta_2) &= \frac{\theta_2^2(m\theta_1 + n\theta_2)}{mn\theta_1},
\end{aligned}
$$

Note that here $\mathbf{I}(\theta_1, \theta_2)$ is the Fisher Information matrix for the whole sample. To obtain the confidence interval on θ_1, we substitute the true value of the parameters by the corresponding MLE's in the expression of $\mathbf{I}(\theta_1, \theta_2)$.

4.3: Bootstrap Confidence Intervals:

In this section we propose two bootstrap confidence intervals, the percentile bootstrap confidence intervals suggested by Efron (1982) and the bootstrap-t confidence intervals suggested by Hall (1988). Hall (1988) showed that the bootstrap-t confidence interval is better than the percentile bootstrap confidence intervals from an asymptotic point of view. Although the finite sample properties is not yet known.

We propose the following percentile confidence interval:

[1] From $(x_1, \delta_1), \ldots, (x_m, \delta_m)$ obtain the bootstrap sample $(x_1^*, \delta_1^*), \ldots, (x_m^*, \delta_m^*)$ by resampling with replacement and from the sample $(x_{m+1}, *), \ldots, (x_n, *)$ obtain the bootstrap sample $(x_{m+1}^*, *), \ldots, (x_n^*, *)$ again by resampling with replacement.

[2] From the bootstrap sample $(x_1^*, \delta_1^*) \ldots, (x_m^*, \delta_m^*), (x_{m+1}, *), \ldots, (x_n, *)$ obtain the bootstrap estimates of all the unknown parameters. For any unknown parameter, say θ, denote the bootstrap estimates of θ as $\hat{\theta}^*$.

[3] Repeat the process 1-2 NBOOT times.

[4] Let $C\hat{D}F(t) = P_*(\hat{\theta}^* \le t)$ be the cumulative distribution of $\hat{\theta}^*$, the bootstrap estimator of the parameter θ, then from the NBOOT $\hat{\theta}^*$ obtain the upper bound and the lower bound of the 100(1- 2α) % bootstrap confidence bound for θ as follows. For a given α define $\hat{\theta}_{boot}(\alpha) = C\hat{D}F^{-1}(\alpha)$, then the approximate 100(1- 2α) % confidence interval for θ is given by

$$(\hat{\theta}_{boot}(\alpha), \hat{\theta}_{boot}(1-\alpha))$$

We propose the following algorithm for computing the bootstrap-t confidence intervals:

[1] From $(x_1, \delta_1), \ldots, (x_m, \delta_m)$ obtain the bootstrap sample $(x_1^*, \delta_1^*), \ldots, (x_m^*, \delta_m^*)$ by resampling with replacement and from the sample $(x_{m+1}, *), \ldots, (x_n, *)$ obtain the bootstrap sample $(x_{m+1}^*, *), \ldots, (x_n^*, *)$ again by resampling with replacement.

[2] From the bootstrap sample $(x_1^*, \delta_1^*) \ldots, (x_m^*, \delta_m^*), (x_{m+1}, *), \ldots, (x_n, *)$ obtain the bootstrap estimates of all the unknown parameters. For any unknown parameter, say θ, denote the bootstrap estimates of θ as $\hat{\theta}^*$.

[3] For any unknown parameter θ, compute

$$T^* = \frac{\sqrt{n}(\hat{\theta}^* - \hat{\theta})}{\hat{\sigma}^*}$$

where $\hat{\theta}$ is the MLE of θ and $\hat{\sigma}^*$ is the estimated standard error of $\hat{\theta}^*$.

[4] Repeat the process 1-3 NBOOT times.

[5] From the NBOOT T^* obtain the upper bound and the lower bound of the 100(1- 2α) % bootstrap-t confidence bound for θ as follows. Let $C\hat{D}FN(\mathrm{t}) = P_*(\hat{\theta}^* \le t)$ be the cumulative distribution of T^*, then for a given α define

$$\hat{\theta}_{boot-t}(\alpha) = \hat{\theta} + n^{-\frac{1}{2}}\hat{\sigma}C\hat{D}FN^{-1}(\alpha)$$

The approximate 100(1- 2α) % confidence interval for θ is given by

$$(\hat{\theta}_{boot-t}(\alpha), \hat{\theta}_{boot-t}(1-\alpha))$$

5. WEIBULL FAILURE DISTRIBUTIONS:

5.1: Estimation of the Parameters:

In this section we assume that X_{ji}'s are Weibull random variable with parameters (α, λ_j) for j = 1 and 2 and for i = 1, 2, ..., n. The distribution function $F_j(.)$ of X_{ji} has the following form:

$$F_j(t) = (1 - e^{-\lambda_j t^{\alpha}}). \tag{5.1}$$

We assume that the lifetime distribution of the different causes follow the Weibull distributions which has different scale parameters but they have the same shape parameter, which is a quite practical assumption, see for example Rao et al. (1991). We introduce one more parameter in the model, which gives more flexibility in the hazard rate, which was constant in case of the exponential distribution. The hazard rate or the instanteneous dath rate due to cause 1 is given by

$$\frac{dF_1(t)}{\bar{F}_1(t)} = \alpha\lambda_1 t^{\alpha_1 - 1},$$

and the mean lifetime due to cause 1 is

$$E(X_{1i}) = \frac{1}{\lambda_1^{\frac{1}{\alpha}}}\Gamma\left(1 + \frac{1}{\alpha}\right).$$

It is well known that it can be increasing or decreasing depending on whether $\alpha > 1$ or $\alpha < 1$. For $\alpha = 1$, it has a constant hazard rate, because it becomes an exponential distribution. The relative risk rate, π, due to cause 1 is

$$\pi = P(X_{1i} < X_{2i}) = \int_0^{\infty} \alpha_1\lambda_1 x^{\alpha-1} e^{-\lambda_1 x^{\alpha}} e^{-\lambda_2 x^{\alpha}} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2},$$

which is independent of the shape parameter α and it is the same as the exponential case. The log likelihood function of the observed data as given in Section 3, becomes

$$\begin{aligned} ln(L) &= nln(\alpha) + r_1 ln(\lambda_1) + r_2 ln(\lambda_2) + (\alpha - 1)\sum_{i=1}^{n} ln(x_i) - (\lambda_1 + \lambda_2)\sum_{i=1}^{n} x_i^{\alpha} \\ &+ (n - m)ln(\lambda_1 + \lambda_2). \end{aligned}$$

Then taking the derivatives with respect to the unknown parameters α, λ_1 and λ_2 and equating them to zeros, we get

$$\hat{\lambda}_1(\alpha) = \frac{n}{m}\frac{r_1}{\sum_{i=1}^{n} x_i^{\alpha}} \qquad \hat{\lambda}_2(\alpha) = \frac{n}{m}\frac{r_2}{\sum_{i=1}^{n} x_i^{\alpha}}.$$

We put $\hat{\lambda}_1$ and $\hat{\lambda}_2$ in the expression of ln (L) above and maximize with respect to α. We do not have any explicit expresion for $\hat{\alpha}$. We obtain $\hat{\alpha}$ by maximizing

$$\begin{aligned} ln[L(\alpha)] &= nln(\alpha) + r_1 ln(\hat{\lambda}_1(\alpha)) + r_2 ln(\hat{\lambda}_2(\alpha)) + (\alpha - 1)\sum_{i=1}^{n} ln(x_i) \\ &- (\hat{\lambda}_1(\alpha) + \hat{\lambda}_2(\alpha))\sum_{i=1}^{n} x_i^{\alpha} + (n - m)ln(\hat{\lambda}_1(\alpha) + \hat{\lambda}_2(\alpha)), \end{aligned}$$

with respect to α. Once we obtain $\hat{\alpha}$, we obtain the maximum likelihood estimators of λ_1 and λ_2 as $\hat{\lambda}_1(\hat{\alpha})$ and $\hat{\lambda}_2(\hat{\alpha})$ respectively. From the invariance principle of the MLE's we can say that the MLE's of the relative risk rate due to cause 1 is

$$\hat{\pi} = \frac{\hat{\lambda}_1(\hat{\alpha})}{\hat{\lambda}_1(\hat{\alpha}) + \hat{\lambda}_2(\hat{\alpha})},$$

and also the MLE of the mean lifetime due to cause 1 is

$$\hat{\tau}_1 = \frac{1}{\hat{\lambda}_1^{\frac{1}{\hat{\alpha}}}}\Gamma\left(1 + \frac{1}{\hat{\alpha}}\right).$$

For known α the distributions of $\hat{\lambda}_1(\alpha)$ or $\hat{\lambda}_2(\alpha)$ can be obtained similarly as Section 3. But if α is unknown then the exact distribution of $\hat{\lambda}_1(\hat{\alpha})$ or $\hat{\lambda}_2(\hat{\alpha})$ is not possible to obtain. So we have to rely on the asymptotic distribution only.

5.2: Confidence Intervals:

In this section we provide the confidence intervals of the different parameters. Since the exact confidence intervals are not possible to obtain when the shape parameter is unknown, we propose the asymptotic confidence intervals and also two different bootstrap confidence intervals. The asymptotic result can be stated as follows:

$$(\hat{\alpha} - \alpha, \hat{\lambda}_1 - \lambda_1, \hat{\lambda}_2 - \lambda_2) \rightarrow N_3(\mathbf{0}, \mathbf{I}^{-1}(\alpha, \lambda_1, \lambda_2)).$$

Here $\mathbf{I}(\alpha, \lambda_1, \lambda_2)$ is the Fisher Information matrix for the parameters $(\alpha, \lambda_1, \lambda_2)$. The matrix $\mathbf{I} = ((I_{ij}))$ for i,j = 1,2 and 3 are as follows:

$$\begin{aligned}
I_{11}(\alpha, \lambda_1, \lambda_2) &= n\left[\frac{1}{\alpha^2} + (\lambda_1 + \lambda_2)V\right], \\
I_{12}(\alpha, \lambda_1, \lambda_2) &= nU = I_{21}(\alpha, \lambda_1, \lambda_2), \\
I_{13}(\alpha, \lambda_1, \lambda_2) &= nU = I_{31}(\alpha, \lambda_1, \lambda_2), \\
I_{22}(\alpha, \lambda_1, \lambda_2) &= \frac{m\lambda_2 + n\lambda_1}{\lambda_1(\lambda_1 + \lambda_2)^2}, \\
I_{33}(\alpha, \lambda_1, \lambda_2) &= \frac{m\lambda_1 + n\lambda_2}{\lambda_2(\lambda_1 + \lambda_2)^2}, \\
I_{23}(\alpha, \lambda_1, \lambda_2) &= \frac{(n - m)}{(\lambda_1 + \lambda_2)^2} = I_{32}(\alpha, \lambda_1, \lambda_2).
\end{aligned}$$

Here U = E(X^α ln(X)) and V = E(X^αln(X).ln(X)), where X is distributed as Weibull (α, $(\lambda_1 + \lambda_2)$). We propose to use bootstrap confidence intervals similarly as subsection 4.3.

6. NUMERICAL EXPERIMENTS:

In this section we present some numerical results to see how the different methods behave for small sample sizes and also for different parametric values. All these numerical works are performed at the University of New Brunswick using the HP workstation. We use the method of Press et al. (1986) for the random deviate generator. We consider the cases when the lifetimes are exponential and also when the lifetimes are Weibull. We mainly observe the behavior of the MLE's in terms of their biases and in terms of their variances. We also compare the performances of the different proposed confidence intervals in terms of the coverage percentages and also in terms of their average confidence lengths.

6.1: Exponential Case:

In this subsection we present the results, when the lifetimes are exponential. Since for the exponential lifetime, the biases and the asymptotic variances are functions of $\frac{\theta_2}{\theta_1}$ only, we keep $\theta_1 = 1$ fixed, and consider different values of $\theta_2 = 1.25, 1.50, \ldots, 2.50$. We take sample sizes n = 10, 20, 30, 40 and 50. In all the cases we assume 10% of the data are incomplete. We draw random samples for different values of n and θ_2 and compute the MLE's of θ_1 and θ_2, we also compute the four different 95% confidence intervals, namely (1) asymptotic (Asymp) (2) approximate (Approx) (3) percentile Bootstar (Boot-p) and (4) Bootstrap-t (Boot-t) confidence intervals. We replicate the process one thousand times and compute the average values of the MLE's, the variances, the biases and the absolute biases. For the different confidence intervals we compute the coverage percentages and also the average confidence lengths. The results of θ_1 and θ_2 are quite similar in nature so we present the results only of θ_1. The average values of $\hat{\theta}_1$, the variances, the absolute biases and the negative biases are reported in Table 6.1. The average error bounds (half of the confidence length) and the corresponding coverage percentages are reported within brackets for different methods in Table 6.2.

Some of the points are very clear from these experiments. From Table 6.1, it is clear that as sample size increases, the biases and the variances decrease. It implies that the MLE's are asymptotically unbiased and they are consistent estimators of the corresponding parameters. From the Table 6.1 and Table 3.1, it is observed that the theoretical biases and simulated biases are quite close to each other. As θ_2 increases, equivalently as $\frac{\theta_2}{\theta_1}$ increases, the biases, the variances of $\hat{\theta}_1$ decrease and the corresponding biases and variances of $\hat{\theta}_2$ increase (not reported here). It is not very surprising, because as $\frac{\theta_2}{\theta_1}$ increases the mean life due to cause 1 decreases compared to the mean life due to cause 2. It is expected that as $\frac{\theta_2}{\theta_1}$ increases, the sample consists more deaths due to cause 1 than due to cause 2. Therefore the sample has more information about θ_1 than θ_2.

From Table 6.2 it is clear that as sample size increases or $\frac{\theta_2}{\theta_1}$ increases, the average confidence lengths of θ_1 decrease for all the four methods. In case of θ_2 it is observed (not reported here) that as sample size increases the average confidence lengths decrease but as $\frac{\theta_2}{\theta_1}$ increases, the average confidence lengths increase. It is not very surprising as it has been mentioned previously. Among the different methods, it is clear that all the methods work quite well if the sample size is large and all of them are able to keep the nominal coverage percentage. Although for small sample sizes, mainly for n = 10 all the methods can not maintain the nominal coverage percentage. It is observed that for all the methods except the approximate one, the coverage perentages are slightly lower than the nominal level, where as for the approximate method the coverage percentages are slightly higher than the nominal levels. Between the approximate and the asymptotic method, the average lengths of the asymptotic methods are slightly lower than that of the approximate method. Comapring the percentile bootstrap method and the bootstrap-t method, it is observed that bootstrap-t is preferred in terms of the confidence lengths although their coverage percentages are almost equal in all the cases considered. Both the methods can't maintain the coverage percentages at least for small samples. Now comparing the computational complexities, the asymptotic method is the easiest to obtain. Approximate confidence interval can be obtained by equating two non-linear equation where as the bootstrap methods can be obtained by resampling from the original sample. It is observed that bootstrap methods take longer times than the approximate method. Considering all the points it is recommended that for small sample sizes, approximate confidence interval can be used where as for large sample the asymptotic confidence bound is preferred.

6.2: Weibull Case:

For the Weibull lifetime distribution, we mainly consider the MLE's of the α's and λ's for different sample sizes. We consider $\alpha = 1$, $\lambda_1 = 1$, $\lambda_2^{-1} = 1.25, 1.50, 1.75, 2.00, 2.25$ and 2.50. We take n = 10, 20, 30, 40 and 50. For a particular choice of α, n and λ_2 we draw a random sample from Weibull lifetime distribution and compute the MLE's of λ_1, λ_2 and α when 10% data are incomplete. We also compute the three different 95% confidence bounds namely the asymptotic one, the percentile bootstrap and the bootstrap-t confidence intervals. We replicate the process one thousand times and compute the average biases, absolute biases and the variances of the MLE's over one thousand replications. The average variances, the negative bises and the absolute biases of λ_1 are reported in Table 6.3. The average error bounds and the coverage percentages (within brackets) are reported in Table 6.4. Since the results of λ_2 and α are quite similar to that of λ_1 they are not reported here.

Some of the points are very clear from this experiment. From the Table 6.3 it is clear that as sample size increases the variances the biases and the absolute biases all decrease which indicates the consistency of the MLE's even in the Weibull case also. Comparing Table 6.1 and Table 6.3 it is observed that the biases, variances and the absolute biases of $\hat{\theta}_1$ are less than the corresponding biases, variances and the absolute biases of $\hat{\lambda}_1$, although both are estimating the same parameter $\lambda_1 = \theta_1^{-1} = 1$ in this case. Which is not very surprising because in the exponential case only two parameters to estimate whereas for the Weibull case there are three unknown parameters.

From Table 6.4, it is clear that none of the methods are able to maintain the coverage percentages for small sample sizes. Although for the large sample sizes all the three methods behave reasonably well. As far as the confidence lengths are concerned the confidence lengths due to the asymptotic methods have the marginally smallest size than the other two and coverage percentages are close to the nominal value at least for large sample sizes. Therefore for moderate or large sample sizes asymptotic method can be used.

7. DATA ANALYSIS:

In this section we consider one real life data set from Lawless (1982, page 491). It consists of failure or censoring times for 36 appliances subjected to an automatic life test. Failures were classified into 18 different modes, though among 33 observed failures only 7 modes are present and only modes 6 and 9 appear more than once. We are mainly interested in the failure mode 9. The data consist of two causes of failure $\delta = 1$ (failure mode 9) and $\delta = 2$ (all other failure modes. The data are given below, which indicates the failure times and the cause of failure if available.

Data Set: (11,2), (35,2), (49,2), (170,2), (329,2), (381,2), (708,2), (958,2), (1062,2), (1167,1), (1594,2), (1925,1), (1990,1), (2223,1), (2327,2), (2400,1), (2451,2), (2471,1), (2551,1), (2565,*), (2568,1), (2694,1), (2702,2), (2761,2), (2831,2), (3034,1), (3059,2), (3112,1), (3214,1), (3478,1), (3504,1), (4329,1), (6367,*), (6976,1), (7846,1), (13403,*)

Here we have n = 36, m = 33, $r_1 = 17$, $r_2 = 16$, $\sum_{i=1}^{36} = 99245$. Therefore using the exponential lifetime distribution, the ML estimators $\hat{\theta}_1 = 5351.45$ and $\hat{\theta}_2 = 5685.91$. The estimates of the mean life due to cause 1 and cause 2 become 5351.45 and 5685.91 respectively. The ML estimators of the relative risk rate due to cause 1 is $\hat{\pi} = .5152$ and due to cause 2 is 1 - $\hat{\pi} = .4848$. The following 95 % confidence intervals are obtained for θ_1 and θ_2 by using different methods.

Table 7.1

	θ_1		θ_2	
Methods	LB	UB	LB	UB
Asymp.	2862.73	7840.16	2956.68	8415.14
Approx.	4451.45	6251.28	4785.91	6585.22
Boot-p	3683.07	7407.33	2464.83	9406.65
Boot-t	3534.64	6992.28	3421.01	7103.94

Using Weibull lifetime, we obtain the estimates of $\hat{\theta}_1 = \hat{\lambda}_1^{-1} = 6980.22$, $\hat{\theta}_2 = \hat{\lambda}_2^{-1} = 7416.49$ and $\hat{\alpha} = 1.0321$. The 95 % confidence band of α becomes (.7625, 1.3016). Since this interval contains one, therefore we cannot reject the null hypothesis H_0: $\alpha = 1$ at the 95 % level. Therefore we conclude that the competing lifetimes are exponential and they have the constant failure rates. Since the simulation results indicate that the approximate method works quite well for the exponential case, we use the approximate confidence bands for the parameters θ_1 and θ_2 respectively, which are given in Table 7.1.

8. CONCLUSIONS:

In this paper we consider estimation of the parameters of the competing risks model when the data may not be complete. We consider two different lifetime distributions of the competing causes, namely exponential and Weibull. We obtain the exact distribution of the MLE's of the mean lifetime, when the lifetime distributions are assumed to be exponential. We propose approximate confidence bands for the mean lifetime and compare their performances with the asymptotic confidence bands and two other bootstrap confidence bands. It is observed that approximate confidence bands work quite well for the exponential case. When the lifetime distributions are Weibull, it is observed that MLE's behave reasonably well and similarly as the exponential case they also provide consistent estimates of the unknown parameters. To obtain the confidence bounds of the unknown parameters the asymptotic results can be used for moderate or large sample sizes but for small sample sizes more work is needed.

Another important aspect which is not addressed here is the analysis when the competing risks may not be independent and when some of the causes of failure are not known. It is a difficult problem. One way it can be handled through mixture model formulation as was suggested by Babu et al. (1992) or Kundu et al. (1992). It will be reported else where.

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Table 6.1

Average values of $\hat{\theta}_1$, Variance of $\hat{\theta}_1$, bias and absolute bias of the LSE's when 10% of the date are incomplete. The lifetimes are exponential.

n	Average	$\theta_2 = 1.25$	$\theta_2 = 1.50$	$\theta_2 = 1.75$	$\theta_2 = 2.00$	$\theta_2 = 2.25$	$\theta_2 = 2.50$
10	$\hat{\theta}_1$	1.1261	1.1018	1.0947	1.0741	1.0776	1.0615
	$V(\hat{\theta}_1)$	.4107	.3459	.2872	.2409	.2227	.1902
	Bias	.1261	.1018	.0947	.0741	.0776	.0615
	$\lvert$Bias$\rvert$	.4107	.3801	.3575	.3343	.3235	.3056
20	$\hat{\theta}_1$	1.0489	1.0411	1.0340	1.0327	1.0303	1.0291
	$V(\hat{\theta}_1)$	.1226	.1410	.1010	.0944	.0894	.0850
	Bias	.0489	.0411	.0340	.0327	.0303	.0291
	$\lvert$Bias$\rvert$	.2602	.2510	.2409	.2310	.2273	.2223
30	$\hat{\theta}_1$	1.0286	1.0266	1.0243	1.0228	1.0183	1.0173
	$V(\hat{\theta}_1)$	.0782	.0680	.0636	.0581	.0560	.0523
	Bias	.0286	.0266	.0243	.0228	.0183	.0173
	$\lvert$Bias$\rvert$	.2100	.1986	.1953	.1854	.1834	.1782
40	$\hat{\theta}_1$	1.0264	1.0215	1.0134	1.0175	1.0095	1.0097
	$V(\hat{\theta}_1)$	.0564	.0500	.0440	.0437	.0402	.0373
	Bias	.0264	.0215	.0134	.0175	.0095	.0097
	$\lvert$Bias$\rvert$	.1816	.1727	.1647	.1625	.1558	.1522
50	$\hat{\theta}_1$	1.0192	1.0151	1.0153	1.0116	1.0086	1.0078
	$V(\hat{\theta}_1)$	.0425	.0388	.0362	.0339	.0326	.0298
	Bias	.0192	.0151	.0153	.0116	.0086	.0078
	$\lvert$Bias$\rvert$	.1588	.1534	.1483	.1438	.1423	.1360

Table 6.2

Different confidence intervals of θ_1, when 10% of the date are incomplete. The lifetimes are exponential. The average length of the error bound and the corresponding coverage probability (within bracket) are reported. The nominal level is 95%.

n	Methods	$\theta_2 = 1.25$	$\theta_2 = 1.50$	$\theta_2 = 1.75$	$\theta_2 = 2.00$	$\theta_2 = 2.25$	$\theta_2 = 2.50$
10	Asymp	1.089(.917)	.979(.900)	.949(.898)	.908(.897)	.892(.867)	.865(.847)
	Approx	.703(.928)	.707(.938)	.701(.925)	.697 (.931)	.694(.910)	.692(.909)
	Boot-p	1.528(.901)	1.442(.894)	1.298(.863)	1.219(.870)	1.160(.861)	1.166(.873)
	Boot-t	1.140(.898)	1.076(.891)	.985(.873)	.948(.867)	.888(.862)	.919(.868)
20	Asymp	.656(.917)	.627(.923)	.596(.908)	.576(.911)	.559(.940)	.552(.918)
	Approx	.625(.952)	.601(.962)	.605(.940)	.590 (.962)	.522(.956)	.523(.955)
	Boot-p	.969(.933)	.841(.939)	.774(.923)	.683 (.935)	.691(.931)	.668(.922)
	Boot-t	.794(.935)	.719(.931)	.683(.926)	.617(.925)	.622(.928)	.608(.924)
30	Asymp	.516(.913)	.492(.918)	.471(.928)	.462(.933)	.454(.934)	.441(.938)
	Approx	.547(.952)	.529(.952)	.508(.960)	.494 (.956)	.485(.948)	.483(.960)
	Boot-p	.645(.940)	.604(.951)	.561(.944)	.535 (.932)	.514(.944)	.502(.938)
	Boot-t	.585(.939)	.558(.951)	.528(.934)	.506(.928)	.490(.945)	.482(.940)
40	Asymp	.444(.928)	.427(.944)	.408(.924)	.400(.946)	.392(.940)	.377(.928)
	Approx	.473(.950)	.449(.964)	.430(.953)	.424(.963)	.417(.953)	.408(.956)
	Boot-p	.524(.944)	.503(.938)	.466(.945)	.452 (.945)	.435(.950)	.431(.944)
	Boot-t	.492(.935)	.477(.941)	.447(.941)	.436(.939)	.421(.944)	.419(.942)
50	Asymp	.399(.946)	.378(.931)	.362(.948)	.352(.933)	.345(.931)	.340(.939)
	Approx	.424(.966)	.399(.963)	.384(.949)	.372 (.948)	.365(.957)	.359(.967)
	Boot-p	.466(.962)	.438(.948)	.415(.960)	.401 (.953)	.393(.949)	.386(.948)
	Boot-t	.445(.965)	.421(.947)	.404(.961)	.389 (.955)	.383(.949)	.369(.948)

Table 6.3

Average values of $\hat{\lambda}_1$, Variance of $\hat{\lambda}_1$, bias and absolute bias of the LSE's when 10% of the date are incomplete. The lifetimes are Weibull.

n	Average	$\lambda_2^{-1} = 1.25$	$\lambda_2^{-1} = 1.50$	$\lambda_2^{-1} = 1.75$	$\lambda_2^{-1} = 2.00$	$\lambda_2^{-1} = 2.25$	$\lambda_2^{-1} = 2.50$
10	$\hat{\lambda}_1$	1.2356	1.2138	1.1926	1.1994	1.1925	1.1385
	$V(\hat{\lambda}_1)$	.8067	.5599	.6956	.5110	.8079	.3775
	Bias	.2356	.2138	.4926	.1994	.1925	.1385
	|Bias|	.5006	.4530	.4334	.4315	.4260	.3742
20	$\hat{\lambda}_1$	1.0996	1.1116	1.0749	1.0844	1.0972	1.0781
	$V(\hat{\lambda}_1)$	.1553	.1318	.1250	.1278	.1077	.1241
	Bias	.0996	.1116	.0794	.0844	.0972	.0781
	|Bias|	.2859	.2769	.2696	.2649	.2499	.2556
30	$\hat{\lambda}_1$	1.0499	1.0426	1.0448	1.0488	1.0591	1.0357
	$V(\hat{\lambda}_1)$	.0793	.0713	.0623	.0661	.0764	.0627
	Bias	.0499	.0426	.0448	.0488	.0591	.0357
	|Bias|	.2117	.2030	.1962	.1982	.2059	.1925
40	$\hat{\lambda}_1$	1.0425	1.0429	1.0452	1.0312	1.0293	1.0250
	$V(\hat{\lambda}_1)$	.0542	.0502	.0470	.0428	.0457	.0410
	Bias	.0425	.0429	.0452	.0312	.0293	.0250
	|Bias|	.1819	.1755	.1670	.1608	.1672	.1591
50	$\hat{\lambda}_1$	1.0199	1.0274	1.0273	1.0267	1.0230	1.0180
	$V(\hat{\lambda}_1)$	.0404	.0408	.0370	.0372	.0324	.0310
	Bias	.0199	.0274	.0273	.0267	.0230	.0180
	|Bias|	.1595	.1577	.1518	.1502	.1414	.1412

Table 6.4

Different confidence intervals of λ_1, when 10% of the date are incomplete. The length of the confidence intervals and the corresponding coverage probability (within bracket) are reported. The lifetime distributions are Weibull.

n	Methods	$\lambda_2^{-1} = 1.25$	$\lambda_2^{-1} = 1.50$	$\lambda_2^{-1} = 1.75$	$\lambda_2^{-1} = 2.00$	$\lambda_2^{-1} = 2.25$	$\lambda_2^{-1} = 2.50$
	Asymp	1.131(.929)	1.063(.932)	1.012(.925)	.993 (.931)	.973(.893)	.893(.890)
10	Boot-p	1.282(.875)	1.273(.854)	1.176(.848)	1.136(.836)	1.087(.841)	1.026(.838)
	Boot-t	1.152(.865)	1.121(.840)	1.052(.834)	1.047(.832)	.972(.831)	.892(.806)
	Asymp	.675(.952)	.648(.969)	.610(.932)	.600(.947)	.593(.948)	.575(.939)
20	Boot-p	1.057 (.920)	.926 (.924)	.896 (.919)	.882 (.939)	.824 (.923)	.802 (.925)
	Boot-t	.767 (.902)	.730 (.920)	.699 (.911)	.685 (.947)	.657(.921)	.609 (.913)
	Asymp	.521(.938)	.496(.941)	.479(.956)	.471(.953)	.467(.937)	.449(.942)
30	Boot-p	.694 (.935)	.634 (.951)	.605(.945)	.598 (.931)	.572(.950)	.569(.939)
	Boot-t	.602 (.938)	.571 (.943)	.544(.938)	.540 (.943)	.529(.944)	.514(.941)
	Asymp	.446(.951)	.427(.952)	.416(.952)	.400(.948)	.392(.947)	.383(.943)
40	Boot-p	.554 (.934)	.527 (.940)	.497 (.942)	.493 (.941)	.476(.947)	.462(.940)
	Boot-t	.516 (.929)	.485 (.942)	.474 (.937)	.455 (.943)	.453(.939)	.437(.945)
	Asymp	.392(.957)	.377(.952)	.365(.943)	.356(.941)	.348(.950)	.341(.959)
50	Boot-p	.470 (.963)	.453 (.944)	.428 (.959)	.423 (.952)	.407(.948)	.406(.950)
	Boot-t	.454 (.942)	.429 (.940)	.417 (.939)	.416 (.946)	.399(.947)	.391(.948)