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ON THE THEORY OF UNIFORMLY LOADED ANISOTROPIC CANTILEVER BEAM

AND

ON THE DISPLACEMENTS IN THE PROBLEMS OF SAINT-VENANT, AND THE CENTER OF SHEAR AND THE CENTER OF TWIST

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ON THE THEORY OF UNIFORMLY LOADED ANISOTROPIC CANTILEVER BEAMS*

-Communist China-

[Following is the translation of an article by Hu Hai-ch'ang (5170 3189 2490), of the Institute of Mechanics, Academia Sinica, in Wu-li Hsüeh-pao (Journal of Physics), Vol. 12, No. 4, July 1956, pp. 339-349.]

The cylinder is a structural unit frequently dealt with in structural engineering. In dealing with cylinders the main problem, with respect to different load conditions, may be divided into the following three categories:

1. The problem of twist in cylinders;
2. the problem of transverse loads on the free end of a cantilever beam; and,
3. the problem of the uniformly distributed load on a cantilever beam.

Saint Venant was the first to establish a general theory on the first two problems mentioned above -- under isotropic conditions. The general theory on the third problem was first developed by J. H. Michell. What Michell has done was to resolve it into a problem of plane deformation. S. G. Lehnitzky has established a general theory on the first two problems under an arbitrary anisotropic condition. The objective of this paper is to establish a general theory on uniformly loaded anisotropic cantilever beams. We shall apply Lehnitzky's method and utilize his results to resolve our proposed problem into a more general problem of plane deformation, the general solution of which has been established by him.

Now let us take any uniformly loaded anisotropic cantilever beam and consider its equilibrium problem. Take a rectangular coordinate system (x, y, z) and make the xy plane coincide with the free end of the beam, and let the z-axis point toward the beam's interior, as shown in the

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Figure 1. The volume load X, Y, and the surface load X_n, Y_n, on the beam are co-efficients of x, y only and have no relation to the z coordinate.

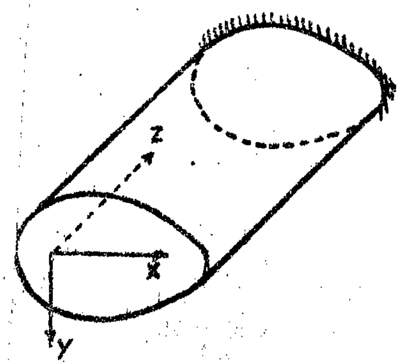

Figure 1.

Assume the resultant force P, acting on unit length of the external load along the z-axis, does not equal zero (in practical problems the case when P equals zero is very unlikely; if P could be zero, our problem would be much simpler). Then, the origin of the coordinates and the direction of the y-axis can be appropriately chosen, and we can make P pass through the z-axis, parallel with the y-axis. Thus,

$$\left.\begin{aligned} &\iint X\,dx\,dy + \int X_n\,ds = 0; \qquad \iint Y\,dx\,dy + \int Y_n\,ds = P, \\ &\iint (xY - yX)\,dx\,dy + \int (xY_n - yX_n)\,ds = 0. \end{aligned}\right\} \tag{1}$$

The area integral can be made to cover the whole area of the cross-section in the equation, and the linear integral to cover the entire perimeter of the cross-section.

The stress and displacement in the beam should satisfy Hooke's general law in the following manner:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy}, \\ \frac{\partial v}{\partial y} &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{xz} + a_{26}\tau_{xy}, \\ \frac{\partial w}{\partial z} &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy}, \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} &= a_{14}\sigma_x + a_{24}\sigma_y + a_{34}\sigma_z + a_{44}\tau_{yz} + a_{45}\tau_{xz} + a_{46}\tau_{xy}. \end{aligned}\right\} \tag{2}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = a_{15}\sigma_x + a_{25}\sigma_y + a_{35}\sigma_z + a_{45}\tau_{yz} + a_{55}\tau_{xz} + a_{56}\tau_{xy},$$
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_{16}\sigma_x + a_{26}\sigma_y + \sigma_{36}\sigma_z + a_{46}\tau_{yz} + a_{56}\tau_{xz} + a_{66}\tau_{xy}, \qquad (2)$$

And, the equilibration equation

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0,\\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0,\\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0.
\end{aligned}\right\} \qquad (3)$$

There will be a surface load X_n, Y_n acting on the sides of the beam, so the stresses should satisfy the following boundary condition (where n is the normal line on the sides of the beam, taking as positive the line pointed outwards).

On the sides:

$$\left.\begin{aligned}
&\sigma_x \cos(n,x) + \tau_{xy}\cos(n,y) = X_n,\\
&\tau_{xy}\cos(n,x) + \sigma_y \cos(n,y) = Y_n,\\
&\tau_{xz}\cos(n,x) + \tau_{yz}\cos(n,y) = 0.
\end{aligned}\right\} \qquad (4)$$

At the free end of the beam, we do not anticipate that every point will be free from the effects of external forces. By application of Saint Venant's principle, we only expect the resultant and the moment of resultant of the external forces acting on the free end of the beam to equal zero. Then the stresses on the free end need only satisfy the following boundary conditions:

At the free end of the beam

$$\left.\begin{aligned}
&\iint \sigma_z\,dx\,dy = 0, \quad \iint (x\tau_{yz} - y\tau_{xz})\,dx\,dy = 0,\\
&\iint \tau_{xz}\,dx\,dy = 0, \quad \iint y\sigma_z\,dx\,dy = 0,\\
&\iint \tau_{yz}\,dx\,dy = 0, \quad \iint x\sigma_z\,dx\,dy = 0.
\end{aligned}\right\} \qquad (5)$$

These boundary conditions can be generalized still further, so that they are not limited to the free end. Based upon conditions of equilibrium, for any cross-section $z = z$, the transverse force is $(-P_z)$, and the bending moment is $(-P/2 \cdot z^2)$. Therefore, the stresses on any cross-section should satisfy the following condition:

$$\left.\begin{aligned}
&\iint \sigma_z \, dx\, dy = 0, && \iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy = 0, \\
&\iint \tau_{xz}\, dx\, dy = 0, && \iint y\sigma_z\, dx\, dy = -\frac{P}{2} z^2, \\
&\iint \tau_{yz}\, dx\, dy = -Pz, && \iint x\sigma_z\, dx\, dy = 0.
\end{aligned}\right\} \quad (6)$$

So our problem is to find the stresses and displacement to fit equations (2) and (3), and the boundary conditions in equations (4) and (6).

Now let us consider the following stress and displacement conditions:

$$\left.\begin{aligned}
&\sigma_x' = \frac{\partial \sigma_x}{\partial z}, && \sigma_y' = \frac{\partial \sigma_y}{\partial z}, && \sigma_z' = \frac{\partial \sigma_z}{\partial z}, \\
&\tau_{yz}' = \frac{\partial \tau_{yz}}{\partial z}, && \tau_{xz}' = \frac{\partial \tau_{xz}}{\partial z}, && \tau_{xy}' = \frac{\partial \tau_{xy}}{\partial z}, \\
&u' = \frac{\partial u}{\partial z}, && v' = \frac{\partial v}{\partial z}, && w' = \frac{\partial w}{\partial z}.
\end{aligned}\right\} \quad (7)$$

Now differentiate equations (2) and (3), and the boundary conditions in equations (4) and (6), once, with respect to z; then, substituting equation (7) into it, we know that $\sigma_x', \sigma_y', \ldots, w'$ will satisfy the following equations and the boundary conditions:

$$\left.\begin{aligned}
\frac{\partial u'}{\partial x} &= a_{11}\sigma_x' + a_{12}\sigma_y' + a_{13}\sigma_z' + a_{14}\tau_{yz}' + a_{15}\tau_{xz}' + a_{16}\tau_{xy}', \\
\frac{\partial v'}{\partial y} &= a_{12}\sigma_x' + a_{22}\sigma_y' + a_{23}\sigma_z' + a_{24}\tau_{yz}' + a_{25}\tau_{xz}' + a_{26}\tau_{xy}', \\
\frac{\partial w'}{\partial z} &= a_{13}\sigma_x' + a_{23}\sigma_y' + a_{33}\sigma_z' + a_{34}\tau_{yz}' + a_{35}\tau_{xz}' + a_{36}\tau_{xy}', \\
\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z} &= a_{14}\sigma_x' + a_{24}\sigma_y' + a_{34}\sigma_z' + a_{44}\tau_{yz}' + a_{45}\tau_{xz}' + a_{46}\tau_{xy}', \\
\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x} &= a_{15}\sigma_x' + a_{25}\sigma_y' + a_{35}\sigma_z' + a_{45}\tau_{yz}' + a_{55}\tau_{xz}' + a_{56}\tau_{xy}', \\
\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y} &= a_{16}\sigma_x' + a_{26}\sigma_y' + a_{36}\sigma_z' + a_{46}\tau_{yz}' + a_{56}\tau_{xz}' + a_{66}\tau_{xy}'.
\end{aligned}\right\} \quad (8)$$

$$
\left.
\begin{aligned}
\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} &= 0\,, \\
\frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau'_{yz}}{\partial z} &= 0\,, \\
\frac{\partial \tau'_{xz}}{\partial x} + \frac{\partial \tau'_{yz}}{\partial y} + \frac{\partial \sigma'_z}{\partial z} &= 0\,.
\end{aligned}
\right\} \qquad (9)
$$

On the sides:

$$
\left.
\begin{aligned}
\sigma'_x \cos(n,x) + \tau'_{xy} \cos(n,y) &= 0\,, \\
\tau'_{xy} \cos(n,x) + \sigma'_y \cos(n,y) &= 0\,, \\
\tau'_{xz} \cos(n,x) + \tau'_{yz} \cos(n,y) &= 0\,.
\end{aligned}
\right\} \qquad (10)
$$

$$
\left.
\begin{aligned}
&\iint \sigma'_z \, dx\, dy = 0\,, && \iint (x\, \tau'_{yz} - y\, \tau'_{xz})\, dx\, dy = 0\,. \\
&\iint \tau'_{xz} \, dx\, dy = 0\,, && \iint y\, \sigma'_z \, dx\, dy = -Pz\,, \\
&\iint \tau'_{yz} \, dx\, dy = -P\,, && \iint x\, \sigma'_z \, dx\, dy = 0\,.
\end{aligned}
\right\} \qquad (11)
$$

Equations (8) and (9) indicate that $\sigma'_x, \ldots, w'$ is a state of stress deformation group which satisfies the continuity

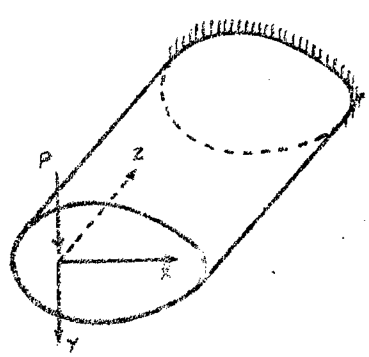

Figure 2.

and boundary conditions. However, the boundary conditions in equation (10) and (11) show that this state of stress deformation is developed when a cantilever beam has no external force on the sides, but the free end carries a transverse load P, as shown in Figure 2.

The problem of a cantilever beam with a transverse load on its free end has been thoroughly investigated by S. G. Lehnitzky. He established a general theory on this problem. By means of his theory he haw obtained many solutions to some concrete problems. It is beyond the scope of this paper to abstract some of his results, but it is necessary to point out that, with the exception of σ_z', the measurements of the rest of the tensile stresses are co-efficients unrelated to z. Henceforth, we shall assume that σ'_x, ..., w' were obtained by S. G. Lehnitzky's own method.

Integrate σ'_x, ..., w' once with respect to z, we obtain

$$\left.\begin{array}{l} \sigma_x = \int_0^z \sigma_x' \, dz + \sigma_x^\circ . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w = \int_0^z w' \, dz + w^\circ . \end{array}\right| \qquad (12)$$

where the superscript $^\circ$ on the right side of a letter indicates the value of that co-efficient when $z = 0$.

In order to solve σ_x, σ_y, ..., w', we must first solve σ_x°, σ_y°, ..., w°. Integrate, once, equations (8) and (9), and the boundary conditions in equations (10) and (11), with respect to z (from zero to z); then, use equations (2) and (3), and the boundary conditions in equations (4) and (6) to eliminate terms σ_x, ..., w, which appeared after integration, and we have

$$\left.\begin{array}{l}
\frac{\partial u^\circ}{\partial x} = a_{11}\sigma_x^\circ + a_{12}\sigma_y^\circ + a_{13}\sigma_z^\circ + a_{14}\tau_{yz}^\circ + a_{15}\tau_{xz}^\circ + a_{16}\tau_{xy}^\circ . \\
\frac{\partial v^\circ}{\partial x} = a_{12}\sigma_x^\circ + a_{22}\sigma_y^\circ + a_{23}\sigma_z^\circ + a_{24}\tau_{yz}^\circ + a_{25}\tau_{xz}^\circ + a_{26}\tau_{xy}^\circ . \\
(w')^\circ = a_{13}\sigma_x^\circ + a_{23}\sigma_y^\circ + a_{33}\sigma_z^\circ + a_{34}\tau_{yz}^\circ + a_{35}\tau_{xz}^\circ + a_{36}\tau_{xy}^\circ . \\
\frac{\partial w^\circ}{\partial y} + (v')^\circ = a_{14}\sigma_x^\circ + a_{24}\sigma_y^\circ + a_{34}\sigma_z^\circ + a_{44}\tau_{yz}^\circ + a_{45}\tau_{xz}^\circ + a_{46}\tau_{xy}^\circ . \\
(u')^\circ + \frac{\partial w^\circ}{\partial x} = a_{15}\sigma_x^\circ + a_{25}\sigma_y^\circ + a_{35}\sigma_z^\circ + a_{45}\tau_{yz}^\circ + a_{55}\tau_{xz}^\circ + a_{56}\tau_{xy}^\circ . \\
\frac{\partial v^\circ}{\partial x} + \frac{\partial u^\circ}{\partial y} = a_{16}\sigma_x^\circ + a_{26}\sigma_y^\circ + a_{36}\sigma_z^\circ + a_{46}\tau_{yz}^\circ + a_{56}\tau_{xz}^\circ + a_{66}\tau_{xy}^\circ .
\end{array}\right\} \qquad (13)$$

$$\begin{aligned} &\frac{\partial \sigma_x^\circ}{\partial x} + \frac{\partial \sigma_y^\circ}{\partial y} + X + \tau'_{xz} = 0, \\ &\frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + Y + \tau'_{yz} = 0, \\ &\frac{\partial \tau^\circ_{xz}}{\partial x} + \frac{\partial \tau^\circ_{yz}}{\partial y} + (\sigma'_z)^\circ = 0. \end{aligned} \qquad (14)$$

$$\begin{aligned} &\sigma_x^\circ \cos(n, x) + \tau_{xy}^\circ \cos(n, y) = X_n. \\ &\tau_{xy}^\circ \cos(n, x) + \sigma_y^\circ \cos(n, y) = Y_n. \\ &\tau_{xz}^\circ \cos(n, x) + \tau_{yz}^\circ \cos(n, y) = 0. \end{aligned} \qquad (15)$$

$$\begin{aligned} &\iint \sigma_z^\circ \, dx\, dy = 0, \qquad \iint (x \tau_{yz}^\circ - y \tau_{xz}^\circ)\, dx\, dy = 0, \\ &\iint y\, \sigma_z^\circ \, dx\, dy = 0. \qquad \iint \tau_{xz}^\circ \, dx\, dy = 0. \\ &\iint x\, \sigma_z^\circ \, dx\, dy = 0. \qquad \iint \tau_{yz}^\circ \, dx\, dy = 0. \end{aligned} \qquad (16)$$

From equations (1) and (11), the volume forces $(X+\tau'_{xz})$, $(Y+\tau'_{xz})$, $(\sigma'_z)^\circ$, are just right in order to maintain equilibrium with the surface forces X_n, Y_n in the boundary conditions in equation (15). From these groups of equations and boundary conditions we can see that σ°_x, σ°_y, ..., w°, are a group of stresses and displacements of generalized plane deformation problems.

From the third line in equation (13), we can solve σ°_z as follows:

$$\sigma_z^\circ = -\frac{1}{a_{33}} (a_{13} \sigma_x^\circ + a_{23} \sigma_y^\circ + a_{34} \tau_{yz}^\circ + a_{35} \tau_{xz}^\circ + a_{36} \tau_{xy}^\circ) + \frac{1}{a_{33}} (w')^\circ. \qquad (17)$$

Substituting this into the other lines in equation (13), we obtain the following:

$$
\left.\begin{aligned}
\frac{\partial u^\circ}{\partial x} &= \beta_{11}\sigma_x^\circ + \beta_{12}\sigma_y^\circ + \beta_{14}\tau_{yz}^\circ + \beta_{15}\tau_{xz}^\circ + \beta_{16}\tau_{xy}^\circ + \frac{a_{13}}{a_{33}}(w')^\circ, \\
\frac{\partial v^\circ}{\partial y} &= \beta_{12}\sigma_x^\circ + \beta_{22}\sigma_y^\circ + \beta_{24}\tau_{yz}^\circ + \beta_{25}\tau_{xz}^\circ + \beta_{26}\tau_{xy}^\circ + \frac{a_{23}}{a_{33}}(w')^\circ, \\
\frac{\partial v^\circ}{\partial x} + \frac{\partial u^\circ}{\partial y} &= \beta_{16}\sigma_x^\circ + \beta_{26}\sigma_y^\circ + \beta_{46}\tau_{yz}^\circ + \beta_{56}\tau_{xz}^\circ + \beta_{66}\tau_{xy}^\circ + \frac{a_{36}}{a_{33}}(w')^\circ, \\
\frac{\partial w^\circ}{\partial x} &= \beta_{15}\sigma_x^\circ + \beta_{25}\sigma_y^\circ + \beta_{45}\tau_{yz}^\circ + \beta_{55}\tau_{xz}^\circ + \beta_{56}\tau_{xy}^\circ + \frac{a_{35}}{a_{33}}(w')^\circ - (u')^\circ, \\
\frac{\partial w^\circ}{\partial y} &= \beta_{14}\sigma_x^\circ + \beta_{24}\sigma_y^\circ + \beta_{44}\tau_{yz}^\circ + \beta_{45}\tau_{xz}^\circ + \beta_{46}\tau_{xy}^\circ + \frac{a_{34}}{a_{33}}(w')^\circ - (v')^\circ.
\end{aligned}\right\} \qquad (18)
$$

where $\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}}, \qquad (i, j = 1, 2, 4, 5, 6)$.

The solution of equations (14) and (18), within the boundary conditions in equations (15) and (16) can be found by applying S. G. Lehnitzky's general solution for generalized plane deformation problems.
Let

$$
\Sigma_x = \int (X + \tau'_{xz})\,dx, \quad \Sigma_y = \int (Y + \tau'_{yz})\,dy, \quad T = \int (\sigma'_z)\,dx. \qquad (19)
$$

In order to satisfy the different quantities for the stresses in equation (14), we can use the two stress coefficients F and ψ to express the following:

$$
\left.\begin{aligned}
&\sigma_x^\circ = \frac{\partial^2 F}{\partial y^2} - \Sigma_x, \quad \sigma_y^\circ = \frac{\partial^2 F}{\partial x^2} - \Sigma_y, \quad \tau_{xy}^\circ = -\frac{\partial^2 F}{\partial x\,\partial y}, \\
&\tau_{xz}^\circ = \frac{\partial \psi}{\partial y} - T, \qquad \tau_{yz}^\circ = -\frac{\partial \psi}{\partial x}.
\end{aligned}\right\} \qquad (20)
$$

Substituting these values into equation (18), and eliminating u°, v°, w° we arrive at the proper equations for F and ψ as we see in the following:

$$
\begin{aligned}
L_4 F + L_3 \psi = &\left(\beta_{12}\frac{\partial^2}{\partial x^2} - \beta_{16}\frac{\partial^2}{\partial x\,\partial y} + \beta_{11}\frac{\partial^2}{\partial y^2}\right)\Sigma_x + \\
&+\left(\beta_{22}\frac{\partial^2}{\partial x^2} - \beta_{26}\frac{\partial^2}{\partial x\,\partial y} + \beta_{12}\frac{\partial^2}{\partial y^2}\right)\Sigma_y + \\
&+\left(\beta_{25}\frac{\partial^2}{\partial x^2} - \beta_{56}\frac{\partial^2}{\partial x\,\partial y} + \beta_{15}\frac{\partial^2}{\partial y^2}\right)T - \\
&-\left(\frac{a_{23}}{a_{33}}\frac{\partial^2}{\partial x^2} - \frac{a_{36}}{a_{33}}\frac{\partial^2}{\partial x\,\partial y} + \frac{a_{13}}{a_{33}}\frac{\partial^2}{\partial y^2}\right)(w')^\circ.
\end{aligned} \qquad (21a)
$$

$$L_3 F + L_2 \psi = -\left(\beta_{14} \frac{\partial}{\partial x} - \beta_{15} \frac{\partial}{\partial y}\right)\Sigma_x - \left(\beta_{24} \frac{\partial}{\partial x} - \beta_{25} \frac{\partial}{\partial y}\right)\Sigma_y -$$
$$-\left(\beta_{45} \frac{\partial}{\partial x} - \beta_{55} \frac{\partial}{\partial y}\right)T + \left(\frac{a_{34}}{a_{33}} \frac{\partial}{\partial x} - \frac{a_{35}}{a_{33}} \frac{\partial}{\partial y}\right)(w')^2 -$$
$$- \frac{\partial}{\partial x} (v')^2 + \frac{\partial}{\partial y} (u')^2 . \qquad \text{(21b)}$$

where L_2, L_3, L_4, are S. G. Lehnitzky's second, third, and fourth order of differentials.

$$\left.\begin{aligned}
L_2 &= \beta_{44} \frac{\partial^2}{\partial x^2} - 2\,\beta_{45} \frac{\partial^2}{\partial x\,\partial y} + \beta_{55} \frac{\partial^2}{\partial y^2} . \\
L_3 &= -\beta_{24} \frac{\partial^3}{\partial x^3} + (\beta_{25}+\beta_{46}) \frac{\partial^3}{\partial x^2\,\partial y} - (\beta_{14}+\beta_{56}) \frac{\partial^3}{\partial x\,\partial y^2} + \beta_{15} \frac{\partial^3}{\partial y^3} , \\
L_4 &= \beta_{22} \frac{\partial^4}{\partial x^4} - 2\beta_{26} \frac{\partial^4}{\partial x^3\,\partial y} + (2\beta_{12}+\beta_{66}) \frac{\partial^4}{\partial x^2\,\partial y^2} - 2\beta_{16} \frac{\partial^4}{\partial x\,\partial y^3} + \beta_{11} \frac{\partial^4}{\partial y^4} .
\end{aligned}\right\} \qquad \text{(22)}$$

Let F_o and ψ_o be one of the special solutions for equation (21), based upon S. G. Lehnitzky's results. All the different stress quantities may be expressed as in the following:

$$\left.\begin{aligned}
\sigma_x^o &= 2\,\mathrm{Re}\left\{\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2) + \mu_3^2 \lambda_3 \Phi_3'(z_3)\right\} + \frac{\partial^2 F_0}{\partial y^2} - \Sigma_x , \\
\sigma_y^o &= 2\,\mathrm{Re}\left\{\Phi_1'(z_1) + \Phi_2'(z_2) + \lambda_3 \Phi_3'(z_3)\right\} + \frac{\partial^2 F_0}{\partial x^2} - \Sigma_y , \\
\tau_{xy}^o &= -2\,\mathrm{Re}\left\{\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2) + \mu_3 \lambda_3 \Phi_3'(z_3)\right\} - \frac{\partial^2 F_0}{\partial x\,\partial y} , \\
\tau_{xz}^o &= 2\,\mathrm{Re}\left\{\mu_1 \lambda_1 \Phi_1'(z_1) + \mu_2 \lambda_2 \Phi_2'(z_2) + \mu_3 \Phi_3'(z_3)\right\} + \frac{\partial \psi_0}{\partial y} - T , \\
\tau_{yz}^o &= -2\,\mathrm{Re}\left\{\lambda_1 \Phi_1'(z_1) + \lambda_2 \Phi_2'(z_2) + \Phi_3'(z_3)\right\} - \frac{\partial \psi_0}{\partial x} .
\end{aligned}\right\} \qquad \text{(23)}$$

If s is the length of the boundary curve, let

$$\left.\begin{aligned}
f_1(s) &= -\int_0^s \left[Y_n + \Sigma_y \cos(n, y)\right] ds , \\
f_2(s) &= \int_0^s \left[X_n + \Sigma_x \cos(n, x)\right] ds , \\
f_3(s) &= \int_0^s T\, ds .
\end{aligned}\right\} \qquad \text{(24)}$$

Then, Φ, Φ, Φ, satisfy the following boundary conditions where c_1, c_2, c_3 are constants:

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[\Phi_1 + \Phi_2 + \lambda_3 \Phi_3] &= f_1(s) - \frac{\partial F_0}{\partial x} + c_1, \\ 2\,\mathrm{Re}\,[\mu_1 \Phi_1 + \mu_2 \Phi_2 + \mu_3 \lambda_3 \Phi_3] &= f_2(s) - \frac{\partial F_0}{\partial y} + c_2, \\ 2\,\mathrm{Re}\,[\lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \Phi_3] &= f_3(s) - \psi_0 + c_3. \end{aligned}\right\} \quad (25)$$

From this group of boundary conditions we can find $\Phi_1(z_1)$, $\Phi_2(z_2)$, $\Phi_3(z_3)$. Then, from equation (23), we can calculate σ_x^o, σ_y^o, ..., w^o. After we obtain the two stress groups, σ'_x, σ'_y, ..., and σ_x^o, σ_y^o, ..., we can easily find the true stresses in the beam by means of equation (12).

To sum up, the problem of shear in cantilever beams with uniformly distributed loads can be solved separately in two consecutive problems. The objective of the first problem is to solve for σ'_x, σ'_y, ..., w', which corresponds to the problem of a cantilever beam with a transverse load on the free end. The aim of the second problem is to solve for σ_x^o, σ_y^o, ... τ_{xy}^o, which corresponds to the problem of a generalized plane deformation.

If any point in the beam has an elastic, symmetrical surface perpendicular to the axis of the beam, then the problem can be much simplified. Because, at this time,

$$\left.\begin{aligned} &a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{46} = a_{56} = 0, \\ &\beta_{14} = \beta_{15} = \beta_{24} = \beta_{25} = \beta_{46} = \beta_{56} = 0. \end{aligned}\right\} \quad (26)$$

After a simple calculation, it can be proved that

$$\left.\begin{aligned} &\sigma'_x = 0, \quad \sigma'_y = 0, \quad \tau'_{xy} = 0, \quad \sigma'_z = -\frac{P}{I} y z, \\ &(u')^o = -\omega_2, \quad (v')^o = \omega_1, \quad (w')^o = a_{33}(Ax + By + C); \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} &\tau_{xz}^o = 0, \quad \tau_{yz}^o = 0, \quad \Phi_3(z_3) \equiv 0, \\ &\sigma_z^o = -\frac{1}{a_{33}}(a_{13}\sigma_x^o + a_{23}\sigma_y^o + a_{36}\tau_{xy}^o) + Ax + By + C, \\ &w^o = \omega_1 y - \omega_2 x + \omega_0. \end{aligned}\right\} \quad (28)$$

where A, B, C, and w_1, w_2, w_o, are undetermined constants. The remainder of the stresses σ_x^o, σ_y^o, τ_{xy}^o can be expressed by two co-efficients $\Phi_1(z_1)$, and $\Phi_2(z_2)$ in the following manner:

$$
\left.
\begin{aligned}
\sigma_x^0 &= 2\,\mathrm{Re}\left\{\mu_1^2\,\Phi_1'(z_1) + \mu_2^2\,\Phi_2'(z_2)\right\} + \frac{\partial^2 F_0}{\partial y^2} - \Sigma_x, \\
\sigma_y^0 &= 2\,\mathrm{Re}\left\{\Phi_1'(z_1) + \Phi_2'(z_2)\right\} + \frac{\partial^2 F_0}{\partial x^2} - \Sigma_y, \\
\tau_{xy}^0 &= -2\,\mathrm{Re}\left\{\mu_1\,\Phi_1'(z_1) + \mu_2\,\Phi_2'(z_2)\right\} - \frac{\partial^2 F_0}{\partial x\,\partial y}.
\end{aligned}
\right\} \qquad (29)
$$

However, $\phi_1(z_1)$, $\phi_2(z_2)$, satisfy the following boundary conditions:

$$
\left.
\begin{aligned}
2\,\mathrm{Re}\,[\Phi_1 + \Phi_2] &= f_1(s) - \frac{\partial F_0}{\partial x} + c_1, \\
2\,\mathrm{Re}\,[\mu_1\,\Phi_1 + \mu_2\,\Phi_2] &= f_2(s) - \frac{\partial F_0}{\partial y} + c_2.
\end{aligned}
\right\} \qquad (30)
$$

Now let us consider a specific problem in order that we might explain the general theoretical application discussed above. Assume that we want to find the stress produced when a cylindrical cantilever beam carries its own weight. Let the radius of the cross-section of the beam be 1 (this supposition evidently does not limit the generality of the problem), and the weight of the unit volume be W. Then the resultant force of the internal load on a unit length will be:

$$
P = \pi W. \qquad (31)
$$

In order to simplify this, let us assume that at some point in the beam there is an elastic, symmetrical surface perpendicular to the axis of the beam. Based upon S. G. Lehnitzky's results, we have

$$
\left.
\begin{aligned}
\tau_{xz}' &= 2Bxy + C(x^2 + 3y^2 - 1), \\
\tau_{yz}' &= -B(3x^2 + y^2 - 1) - 2Cxy + 2W(x^2 + y^2 - 1),
\end{aligned}
\right\} \qquad (32)
$$

where

$$
\left.
\begin{aligned}
B &= 2\cdot\frac{(a_{44} - 2a_{13})(a_{44} + 3a_{55}) - 2a_{45}(a_{36} + a_{45})}{(3a_{44} + a_{55})(a_{44} + 3a_{55}) - 4a_{45}^2}, \\
C &= 2\cdot\frac{2a_{45}(a_{44} - 2a_{13}) - (a_{36} + a_{45})(3a_{44} + a_{55})}{(3a_{44} + a_{55})(a_{44} + 3a_{55}) - 4a_{45}^2}.
\end{aligned}
\right\} \qquad (33)
$$

Therefore,

$$\left.\begin{aligned}\Sigma_x &= \int \tau'_{xz}\,dx = B(x^2 y + y^3 - y) + C\left(\frac{x^3}{3} + 3xy^2 - x\right);\\ \Sigma_y &= \int \tau'_{yz}\,dy = -B\left(3x^2y + \frac{y^3}{3} - y\right) - C(xy^2 + x^3 - x) + 2W\left(x^2y + \frac{y^3}{3} - \frac{y}{2}\right).\end{aligned}\right\} \quad (34)$$

F will satisfy the following equation:

$$\beta_{22}\frac{\partial^4 F}{\partial x^4} - \beta_{26}\frac{\partial^4 F}{\partial x^3\,\partial y} + (2\beta_{16} + \beta_{66})\frac{\partial^4 F}{\partial x^2\,\partial y^2} - \beta_{16}\frac{\partial^4 F}{\partial x\,\partial y^3} + \beta_{11}\frac{\partial^4 F}{\partial y^4} = \quad (35)$$
$$= \beta_{22}\,px + \beta_{11}\,qy,$$

where

$$\left.\begin{aligned} p &= \frac{1}{\beta_{22}}\left\{6\beta_{11}\,C + 2(3\beta_{26} - \beta_{16})\,B - 4\,\beta_{26}\,W\right\},\\ q &= \frac{1}{\beta_{11}}\left\{2(\beta_{26} - 3\beta_{16})\,C - 6\beta_{22}\,B + 4(\beta_{12} + \beta_{22})\,W\right\}.\end{aligned}\right\} \quad (36)$$

Therefore, one of the special solutions of F is

$$F_0 = \frac{p}{5!}\left(x^5 - \frac{5}{3}x^3 + \frac{5}{8}x\right) + \frac{q}{5!}\left(y^5 - \frac{5}{3}y^3 + \frac{5}{8}y\right). \quad (37)$$

At the boundary of the cross-section, x, y can be expressed in polar coordinates:

$$x = \cos\theta, \quad y = \sin\theta. \quad (38)$$

Hence, the length of the arc on the boundary is

$$ds = d\theta.$$

Since,

$$\left.\begin{aligned} f_1(\theta) &= -\int_0^\theta \Sigma_y \sin\theta\,d\theta = \frac{1}{6}\left(B + \frac{W}{2}\right)\left(\sin 2\theta - \frac{1}{2}\sin 4\theta\right),\\ f_2(\theta) &= \int_0^\theta \Sigma_x \cos\theta\,d\theta = -\frac{C}{6}\left(\sin 2\theta + \frac{1}{2}\sin 4\theta\right).\end{aligned}\right\} \quad (39)$$

Therefore, the boundary conditions of $\phi_k(z_k)$ can be solved as

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[\Phi_1+\Phi_2] &= \frac{1}{6}\left(B+\frac{W}{2}\right)\left(\sin 2\theta-\frac{1}{2}\sin 4\theta\right)-\frac{1}{8}\cdot\frac{p}{4!}\cos 4\theta, \\ 2\,\mathrm{Re}\,[\mu_1\Phi_1+\mu_2\Phi_2] &= -\frac{C}{6}\left(\sin 2\theta+\frac{1}{2}\sin 4\theta\right)-\frac{1}{8}\cdot\frac{q}{4!}\cos 4\theta, \end{aligned}\right\} \tag{40}$$

Let

$$\begin{aligned} \Phi_k = A_k\{(z_k-\sqrt{z_k^2-1-\mu_k^2})^2+(z_k+\sqrt{z_k^2-1-\mu_k^2})^2\}+ \\ +B_k\{(z_k-\sqrt{z_k^2-1-\mu_k^2})^4+(z_k-\sqrt{z_k^2-1-\mu_k^2})^4\}. \end{aligned} \tag{41}$$

Since, on the boundary of the cross-section

$$z_k = \cos\theta + \mu_k \sin\theta,$$

Therefore, since it is on the boundary of the cross-section, ϕ_k can be solved as

$$\begin{aligned} \Phi_k = A_k\{[(1-i\mu_k)^2+(1+i\mu_k)^2]\cos 2\theta+i\,[(1-i\mu_k)^2-(1+i\mu_k)^2]\sin 2\theta\}+ \\ +B_k\{[(1-i\mu_k)^4+(1+i\mu_k)^4]\cos 4\theta+i\,[(1-i\mu_k)^4-(1+i\mu_k)^4]\sin 4\theta\}. \end{aligned}$$

Substituting this into the boundary conditions in equation (40), and comparing the co-efficients of the corresponding terms, we obtain

$$\left.\begin{aligned} &2\,\mathrm{Re}\sum A_k\left[(1-i\mu_k)^2+(1+i\mu_k)^2\right]=0, \\ &2\,\mathrm{Re}\sum \mu_k A_k\left[(1-i\mu_k)^2+(1+i\mu_k)^2\right]=0, \\ &2\,\mathrm{Re}\sum iA_k\left[(1-i\mu_k)^2-(1+i\mu_k)^2\right]=\frac{1}{6}\left(B+\frac{W}{2}\right), \\ &2\,\mathrm{Re}\sum i\mu_k A_k\left[(1-i\mu_k)^2-(1+i\mu_k)^2\right]=-\frac{C}{6}, \end{aligned}\right\} \tag{42}$$

$$
\left.
\begin{aligned}
&2\,\mathrm{Re}\sum B_k\left[(1-i\mu_k)^4+(1+i\mu_k)^4\right]=-\frac{1}{8}\cdot\frac{p}{4!},\\
&2\,\mathrm{Re}\sum \mu_k B_k\left[(1-i\mu_k)^4+(1+i\mu_k)^4\right]=-\frac{1}{8}\cdot\frac{q}{4!},\\
&2\,\mathrm{Re}\sum iB_k\left[(1-i\mu_k)^4-(1+i\mu_k)^4\right]=-\frac{1}{12}\left(B+\frac{W}{2}\right),\\
&2\,\mathrm{Re}\sum i\mu_k B_k\left[(1-i\mu_k)^4-(1+i\mu_k)^4\right]=-\frac{C}{12}.
\end{aligned}
\right\} \quad (43)
$$

From equations (42) and (43), we can solve for A_1, $\overline{A}_1$, A_2, $\overline{A}_2$, and B_1, $\overline{B}_1$, B_2, $\overline{B}_2$.

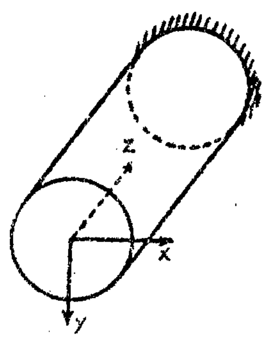

Figure 3.

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ON THE DISPLACEMENTS IN THE PROBLEMS OF SAINT-VENANT, AND THE CENTER OF SHEAR AND THE CENTER OF TWIST*

[Following is the translation of an article by Hu Hai-ch'ang (5170 3189 2490), of the Institute of Mechanics, Academia Sinica, in Wu-li Hsüeh-pao (Journal of Physics), Vol. 12, No. 4, July 1956, pp. 350-359.]

Abstract

Consider the equilibrium of an elastic cylinder fixed at one end and loaded at the other end. The solution of this problem according to Saint-Venant is well known. This solution gives a uniquely determined stress system, but the corresponding displacement contains an arbitrary rigid body motion. In this paper, we first relax the boundary conditions at the fixed end to six conditions by an energy consideration. These conditions determine the arbitrary rigid body motion in Saint-Venant's solution uniquely. Then the translations and rotations of any transverse cross-section is defined by a similar energy consideration. The center of twist is defined as the point which remains fixed during the twist of the cylinder. The center of shear is defined as such a point that when the resultant of transverse loads passes through it, transverse cross-sections have no rotations about the longitudinal axis. It is shown that these two centers have identical coordinates

$$x_0 = \frac{1}{I_x} \iint_S y\varphi \, dx\, dy, \qquad y_0 = -\frac{1}{I_y} \iint_S x\varphi \, dx\, dy, \qquad (21)$$

where φ is the warping function in the problem of torsion, and I_x, I_y are the principal moments of inertia of the cross-

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section. It is proved that for constant transverse loads with parallel directions, the one which passes through the center of shear produces minimum strain energy in the cylinder.

I Introduction

The problem of equilibrium on elastic cylinders is very important in its practical applications. A strictly-defined mathematical solution, however, is rather troublesome. Saint-Venant, in 1855, simplified the original problem by relaxing a portion of its boundary conditions, thus opened up an extensive possibility in its mathematical approach. In honor of his contribution, this simplified problem is widely referred to as the "Problem of Saint-Venant".

But the "Problem of Saint-Venant" cannot satisfy the original total stress boundary condition. Aside from which it has another shortcoming -- the inadequacy of the Saint-Venant method to determine the absolute displacement within the cylinder. In discussing the problem of bending on a cantilever, it is generally stated: One end of the beam is fixed, while the other end carries a transverse load. But there is no explanation as to what exactly is meant by 'fixed'. Strictly speaking, the meaning of 'fixed' should be satisfied by the following conditions:

$$u = 0, \quad v = 0, \quad w = 0. \tag{1}$$

where u, v, w, are the displacements at the cross-section of that end. Yet, there is no comparison between this definition and the "Problem of Saint-Venant", since the "Problem of Saint-Venant" cannot satisfy that many boundary conditions. Thus, we may raise the question: How can we clearly, accurately, and logically determine the meaning of 'fixed'?

At present, there are still few people discussing the question of displacement in the "Problem of Saint-Venant" in its totality. Because of the demands of practicality, however, quite a few scholars have already done some research on the question of locii concerning both the center of twist in the problem of twist and the center of bending in the problem of bending (used synonymously with 'shear'). Since they did not proceed from a consideration of the total displacement problem, they have isolated the closely related original problems. Quite a few authors have determined the locii of the center of twist and the center of shear more or less arbitrarily. Authors such as W. J. Ducan, D. L. Ellis, and C. Scruton thought that the locii of the center of twist and the center of shear could only be arbitrarily determined if in accordance with different substantial conditions. S. Timoshenko and J. N. Goodier still think that the center of twist has no fixed position. L. C. Leybenzon, S. Timoshenko, J. N. Goodier, I. S. Sokolnikoff, and A. S. Stevenson have determined that the center of shear is a point such that the average local rotation of the cross-section is equal to zero if the resultant of the external loads passes this point; that is to say,

$$\iint_S \omega_z \, dx \, dy = \iint_S \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dx \, dy = 0 . \tag{2}$$

But this definition is not too satisfactory, because the question will inevitably be raised: Why take into consideration the irrational average and not the rational? At the same time, even the practical value of this definition is doubtful. That is to say, even though the average local rotation is equal to zero this does not necessarily mean that there is no twisting action within the cylinder.

In 1935, E. Trefftz started out from the proposition that strain energy can be superimposed, and proved that the center of twist and the center of shear are virtually the same point. Its coordinates are

$$x_0 = \frac{1}{I_x} \iint_S y \varphi \, dx \, dy, \quad y_0 = -\frac{1}{I_y} \iint_S x \varphi \, dx \, dy . \tag{3}$$

in which φ is the co-efficient of warp in torsion.

The writer of this paper thinks that Trefftz's conclusion is valid. Due perhaps to the fact that Trefftz's presentation was not too convincing, only a few in this field agreed with his idea.

In 1947, A. Weinstein, basing his work upon the thinking of P. Cicala, explained Trefftz's formula. He proved that formula (3) could also be derived by the following conditions:

$$\iint_S w^2 \, dx \, dy = \min . \tag{4}$$

Weinstein's explanation, however, offers no help in respect to understanding formula (3); instead, it leads us even further away from the physical meaning of the center of twist and the center of shear.

There had been quite a number of different views concerning the locii of the center of twist and the center of shear in elastic thin-walled rods. In 1936, Fu-la-so-fu [transliteration of Russian name], V. Z., established a general theory of constrained twist in elastic thin-walled rods; hence, he logically determined the locii of the center of twist and the center of shear. The formula Fu-la-so-fu derived is the same as formula (3). Although his formula is the same as that derived by Trefftz, Fu-la-so-fu presents a far more convincing discussion pertaining to the thin-walled rod aspect.

We should understand that the purpose of this paper is not to discuss specifically the question of the center of twist or the center of shear, but to take the "Six Problems of Saint-Venant" as a whole and discuss the absolute displacement in the problems (the six problems are axial elongation, axial contraction, two-dimensional simple bending and twist, and two-dimensional shears).

First, we have determined the meaning of 'fixed' in the description of a cantilever beam: one end of the beam is fixed... Now we shall proceed to determine clearly and accurately the meaning of three-dimensional translations and rotations in any cross-section. Finally, based on these definitions, we can determine the locii of the center of twist and the center of shear, and prove that the two locii actually constitute the same point. The formulae -- derived from this paper -- for coordinates on the center of twist and the center of shear are the same as those used by Trefftz and Fu-la-so-fu.

The theorems in this paper can be applied directly to the anisotropic and irregular elastic cylinders in Saint-Venant's problems. Very little research has been done on the center of twist and the center of shear in these problems.

II The Meaning of Fixed Cantilever Beams

Now, assume we have an elastic cylindrical cantilever beam, one end of which is fixed, with no external load acting on the suspending sides; on the other end, the cross-section carries a distributed load.

Taking a rectangular coordinate system, x, y, z -- as shown in Figure 1 (see page 19) -- make the xy surface parallel to the cross-section of the fixed end, orient the z-axis toward the inside of the cylinder. Then, take the x, y axes, and have them coincide with the central major axes. If we establish this problem in strict accordance with the mechanics of elastic bodies, then it will be a problem of finding the stress and displacement of all the locii in the cylinder, and making them satisfy the equilibrium equation and the continuity conditions. Then they

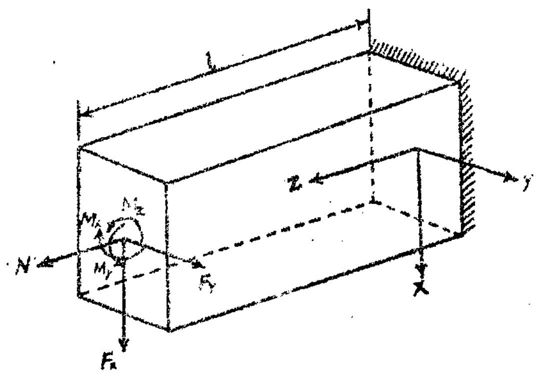

Figure 1.

should further satisfy the following boundary conditions:

On the sides:

$$\left.\begin{aligned} \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) = 0, \\ \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) = 0, \\ \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) = 0; \end{aligned}\right\} \tag{5}$$

On the cross-section of the end where force is applied (z = l):

$$\tau_{xz} = T_x(x, y), \quad \tau_{yz} = T_y(x, y), \quad \sigma_z = \Sigma(x, y); \tag{6}$$

On the fixed end (z = 0):

$$u = 0, \quad v = 0, \quad w = 0. \tag{7}$$

If we establish this problem according to Saint-Venant's method, then it will correspond to the strictly limited problem stated above, in equations (6) and (7), with two of the boundary conditions relaxed. Then, substituting with the following groups of assumptions and boundary conditions:

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{yz}}{\partial z} = 0, \tag{8}$$

On the cross-section of the loaded end (the end where force is applied; z=1):

$$\left.\begin{aligned}
&\iint_S \tau_{xz}\, dx\, dy = F_x = \iint_S T_x\, dx\, dy, \\
&\iint_S \tau_{yz}\, dx\, dy = F_y = \iint_S T_y\, dx\, dy, \\
&\iint_S \sigma_z\, dx\, dy = N = \iint_S \Sigma\, dx\, dy, \\
&\iint_S y\, \sigma_z\, dx\, dy = M_x = \iint_S y\, \Sigma\, dx\, dy, \\
&\iint_S x\, \sigma_z\, dx\, dy = -M_y = \iint_S x\, \Sigma\, dx\, dy, \\
&\iint_S (x\, \tau_{yz} - y\, \tau_{xz})\, dx\, dy = M_z = \iint_S (x\, T_y - y\, T_x)\, dx\, dy.
\end{aligned}\right\} \tag{9}$$

There is only one answer for the stress of any point within the cylinder -- if we seek a solution based upon the problem established by Saint-Venant. This solution has the following form:

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \tag{10}$$

$$\left.\begin{aligned}
&\sigma_z = \frac{N}{A} - \frac{M_y + F_x(l-z)}{I_y}\, x + \frac{M_x - F_y(l-z)}{I_x}\, y, \\
&\tau_{xz} = F_x \tau^1_{xz} + F_y \tau^2_{xz} + M_z \tau^3_{xz}, \\
&\tau_{yz} = F_x \tau^1_{yz} + F_y \tau^2_{yz} + M_z \tau^3_{yz}.
\end{aligned}\right\} \tag{11}$$

where A is the area of the cross-section; I_x, I_y are the major moments of inertia for the cross-section; the six co-efficients, τ^1_{xz}, τ^2_{xz}, ..., τ^3_{xz} bear a relationship only to x, y, and not to z; and, at the same time, the co-efficients are affected by the shape of the cross-section, but not by the condition of the load.

There is no discussion in Saint-Venant's problems pertaining to displacement, so we cannot accurately determine displacement from the two stress equation (10) and (11). The reason for this inaccuracy is the fact that the total motion of a cylinder consists of six undetermined constants. In order to determine the displacement, therefore, it is necessary to establish six related conditions of displacement at the fixed end to show that it is fixed. This will relax the meaning of 'fixed' so that there is no conflict between it and the existing idea of 'fixed' and common sense.

The rigid 'fixed' conditions of formula (7) can also be expressed in a different form in the following manner:

Cross-section at the fixed end, under any load condition:

$$\iint_S (u\,\tau_{xz} + v\,\tau_{yz} + w\,\sigma_z)\,dx\,dy = 0 . \qquad (12)$$

Since in a rigidly established problem $(\tau_{xz})_{z=0}$, $(\tau_{yz})_{z=0}$, $(\sigma_z)_{z=0}$ may assume any numerical value, therefore, we can derive Eq. (7) from Eq. (12). But the meaning of 'fixed' as determined by Eq. (12) may quite appropriately be applied to Saint-Venant's problem, in which arbitrary values cannot be assumed for $(\tau_{xz})_{z=0}$, $(\tau_{yz})_{z=0}$, $(\sigma_z)_{z=0}$, only the six quantities F_x, F_y, ... M_z, may be varied arbitrarily. Hence, when we substitute the general solution, Eq. (11), into Eq. (12), then let the six quantities F_x, F_y, ... M_z equal zero, we obtain the following meaning of 'fixed' in Saint-Venant's problem:

At the Fixed end:

$$\left.\begin{aligned}
&\iint_S w\,dx\,dy = 0, \quad \iint_S xw\,dx\,dy = 0, \quad \iint_S yw\,dx\,dy = 0, \\
&\iint_S (u\,\tau^1_{xz} + v\,\tau^1_{yz})\,dx\,dy = 0, \\
&\iint_S (u\,\tau^2_{xz} + v\,\tau^2_{yz})\,dx\,dy = 0, \\
&\iint_S (u\,\tau^3_{xz} + v\,\tau^3_{yz})\,dx\,dy = 0 .
\end{aligned}\right\} \qquad (13)$$

These six conditions are sufficient to determine the six unknown constants in displacement.

III The Translation and Rotation of Transverse Cross-Section

When the cantilever beam assumes external load, displacement developed at all points. Generally speaking, the locii which are on the same cross-section before the deformation will fall on a curved surface after deformation. Strictly speaking, therefore, the transformation on a transverse cross-section cannot be expressed by translation and rotation (i.e., the motion of a rigid body). But in Saint-Venant's problem we can reasonably determine the meaning of translation and rotation on a transverse cross-section -- so much so that it will not conflict with the established meaning as commonly understood, but will certainly maintain its practical aspects.

In the "Problems of Saint-Venant", the first factors to be determined are the resultant force and the moment of the resultant force. Generally, to find the corresponding displacement with existing forces, we can use the following definition:

Generalized forces X Generalized displacement = Work done. (14)

When we use this definition to determine the translation and rotation of a certain transverse cross-section $z = k$, we may draw a cross-section $z = k$ showing the stresses in it (actually the condition is practically the same as that shown in Figure 1, just change 1 to k). Then the displacement and rotation of the cross-section can be determined as follows:

Resultant force on the cross-section X the translation of the cross-section = total work done by the resultant force,

Moment of resultant force on the cross-section X the rotation of the cross-section = total work done by the amount of the resultant force. (15)

For instance, assume U is the translation of the transverse cross-section along the x-axis. Since the resultant force of the cross-section along the x-axis is F_x, then the total work done is:

$$\iint_s F_x(u\,\tau^1_{xz} + v\,\tau^1_{yz})\,dx\,dy .$$

Therefore, based upon Eq. (15), we get

$$U = \iint_s (u\,\tau^1_{xz} + v\,\tau^1_{yz})\,dx\,dy . \tag{16}$$

Similarly, we can prove that the translations V, W, of the cross-sections along y, z-axis determined by Eq. (15), and the rotations ψ_x, ψ_y, θ around x, y, z axes, are the following:

$$\left.\begin{aligned}
V &= \iint_s (u\,\tau^2_{xz} + v\,\tau^2_{yz})\,dx\,dy ,\\
W &= \frac{1}{A}\iint_s w\,dx\,dy ,\\
\psi_x &= \frac{1}{I_x}\iint_s yw\,dx\,dy ,\\
\psi_y &= -\frac{1}{I_y}\iint_s xw\,dx\,dy ,\\
\theta &= \iint_s (u\,\tau^3_{xz} + v\,\tau^3_{yz})\,dx\,dy .
\end{aligned}\right\} \tag{17}$$

These definitions are identical with the meaning of 'fixed' which was decided upon in the preceding section. Based upon equations (13), (16), and (17), we know that the so-called 'fixed' condition means that there is no translation or rotation at the transverse cross-section. At the same time, it may be proved that actually there is no conflict between this definition and the existing meaning as understood by common sense. For example, when a certain transverse cross-section remains a plane surface even after transformation, it is not difficult to prove that the translation and rotation determined by equations (16) and (17) are the same as the translation and rotation which are ordinarily understood.

Assume that U^0, V^0, ..., θ^0 are the displacements and rotations at the loaded end to which a force has been applied. According to definition, the work done by the external force is:

$$E = \frac{1}{2}\,(F_x U^0 + F_y V^0 + N W^0 + M_x \psi^0_x + M_y \psi^0_y + M_z \theta^0) . \tag{18}$$

And, based upon the principle of conservation of energy, we know that the above equation is also the equation for the strain energy of the cylinder.

IV Displacement and the Center of Twist in the Problem of Twist

Now assume that there is only one moment of twist, M_z, on the loaded end of a cantilever beam, and all other resultants and their moments are equal to zero. The Saint-Venant solution to this problem is well-known. The stresses and displacements of all locii in the cylinder can be expressed by a co-efficient of torsion $\varphi(x, y)$, as in the following:

$$\left.\begin{aligned} &\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0\,, \\ &\tau_{xz} = \frac{M_z}{D}\left(\frac{\partial\varphi}{\partial x} - y\right), \quad \tau_{yz} = \frac{M_z}{D}\left(\frac{\partial\varphi}{\partial y} + x\right), \\ &u = -\frac{M_z}{GD}(y - y_0)z\,, \quad v = \frac{M_z}{GD}(x - x_0)z\,, \\ &w = \frac{M_z}{GD}(\varphi - w_0 - y_0 x + x_0 y)\,. \end{aligned}\right\} \qquad (19)$$

where G is the shearing modulus of elasticity of the cylinder and D is the torsional rigidity of the cylinder. In the original problem x_o, y_o, w_o were indeterminate constants; here, x_o, y_o are the coordinates of the center of twist. These constants can be determined by application of the 'fixed' conditions of Eq. (13), which we have discussed in this paper. By substituting Eq. (19) into Eq. (13), we can see that the three latter conditions are satisfied and the first three can be formulated as follows:

$$\left.\begin{aligned} &\iint_S (\varphi - w_0 - x_0 y + y_0 x)\,dx\,dy = 0\,, \\ &\iint_S x(\varphi - w_0 - x_0 y + y_0 x)\,dx\,dy = 0\,, \\ &\iint_S y(\varphi - w_0 - x_0 y + y_0 x)\,dx\,dy = 0\,. \end{aligned}\right\} \qquad (20)$$

noting that x, y axes are the central major axes of the cross-section; and, from the above equations, we can find:

$$w_0 = \frac{1}{A}\iint_s \varphi \, dx\, dy, \quad x_0 = \frac{1}{I_x}\iint_s y\varphi \, dx\, dy \quad y_0 = -\frac{1}{I_y}\iint_s x\varphi \, dx\, dy. \tag{21}$$

After we have found the quantities of x_o, y_o, w_o, we can with complete accuracy determine the displacements of all points. It can easily be seen that the coordinates of the center of twist presented in this paper are the same as those given by Trefftz.

We notice that the torsional rigidity D can be given by the following equation:

$$D = \iint_s \left\{ x\left(\frac{\partial\varphi}{\partial y} + x\right) - y\left(\frac{\partial\varphi}{\partial x} - y\right)\right\} dx\, dy. \tag{22}$$

Thus, we can find the values of three translations and three rotations on any transverse cross-section by the following:

$$\left.\begin{array}{lll} U = \frac{M_z}{GD} y_0 z, & V = -\frac{M_z}{GD} x_0 z, & W = 0, \\ \psi_x = 0, & \psi_y = 0, & \theta = \frac{M_z}{GD} z. \end{array}\right| \tag{23}$$

Therefore, after a cylinder has undergone twist, there is rotation around the z-axis and translation along the x, y axes in the transverse cross-sections.

V The Center of Shear of the Cantilever Beams

Now assume that the cantilever beam shown in Figure 1, is loaded, simultaneously, on its free end with a transverse load F_x, F_y, and a moment of twist M_z, but not with any other load. Under these three loading actions, translations U^o, V^o, along the x, y axes, and rotation θ^o around the z-axis, will be developed at the cross-section of the free end. U^o, V^o, θ^o, should be the linear co-efficients of F_x, F_y, M_z; thus, they can be written in the following manner:

(Refer to Eq. 23):

$$\left.\begin{aligned} U^0 &= U^1 F_x + U^2 F_y + \frac{y_0 l}{GD} M_z, \\ V^0 &= V^1 F_x + V^2 F_y - \frac{x_0 l}{GD} M_z, \\ \theta^0 &= \theta^1 F_x + \theta^2 F_y + \frac{l}{GD} M_z. \end{aligned}\right\} \qquad (24)$$

Now, let us solve for θ^1 and θ^2.

From Eq. (18), the strain energy of the cylinder in this problem is equal to the work done by the external forces, as shown below:

$$E = \frac{1}{2}\left\{F_x\left(U^1 F_x + U^2 F_y + \frac{y_0 l}{GD} M_z\right) + F_y\left(V^1 F_x + V^2 F_y - \frac{x_0 l}{GD} M_z\right) + \right. \qquad (25)$$

$$\left. + M_z\left(\theta^1 F_x + \theta^2 F_y + \frac{l}{GD} M_z\right)\right\}.$$

Since the strain energy in an elastic body remains unchanged, even when we alter the sequence of the external forces applied, presumeably we could first apply the transverse load F_x and F_y, to the cantilever beam, then we could again apply the moment of twist M_z. In this manner, first applying F_x and F_y, the total work done by the external forces is:

$$E_1 = \frac{1}{2}\left\{F_x(U^1 F_x + U^2 F_y) + F_y(V^1 F_x + V^2 F_y)\right\}. \qquad (26)$$

Then, when we again apply M_z, the total work done by the external forces is:

$$E_2 = \frac{1}{2} M_z \cdot \frac{l}{GD} M_z + F_x \frac{y_0 l}{GD} M_z - F_y \frac{x_0 l}{GD} M_z. \qquad (27)$$

So, the strain energy in the beam is also equal to:

$$E = \frac{1}{2}\left\{F_x(U^1 F_x + U^2 F_y) + F_y(V^1 F_x + V^2 F_y) + \frac{l}{GD} M_z^2\right\} + \qquad (28)$$

$$+ F_x \frac{y_0 l}{GD} M_z - F_y \frac{x_0 l}{GD} M_z.$$

Comparing Eq. (25) and Eq. (28), we know that

$$M_z(\theta^1 F_x + \theta^2 F_y) = F_x \frac{y_0 l}{GD} M_z - F_y \frac{x_0 l}{GD} M_z . \tag{29}$$

This equation is established no matter what the values are for F_x, F_y, and M_z. Thus, the following must also be true:

$$\theta^1 = \frac{y_0 l}{GD}, \qquad \theta^2 = -\frac{x_0 l}{GD}. \tag{30}$$

Therefore, the rotation around the z-axis at the cross-section of the free end is:

$$\theta^0 = \frac{y_0 l}{GD} F_x - \frac{x_0 l}{GD} F_y + \frac{l}{GD} M_z . \tag{31}$$

From this equation we can see that, when

$$M_z = x_0 F_y - y_0 F_x \tag{32}$$

there will be no rotation around the z-axis at the cross-section. The condition of Eq. (32) indicates that the resultant of the external load passes through the point (x_0, y_0). From this, we know that the point (x_0, y_0) is the center of shear.

From the above discussion, we arrive at the following conclusion: the center of twist and the center of shear are one and the same point and their coordinates are $x = x_0$ and $y = y_0$. We should point out, however, that the reason why we can prove that the center of twist coincides with the center of shear is that when we determined the meaning of translation and rotation for the transverse cross-section, we based our work upon the relationship between force and work done so that we could utilize the Law of Conservation of Energy, from whence we derived Eq. (31). In fact, Eq. (30) is the well-known Theorem of Equality used in the mechanics of elastic bodies.

If we should use another method to determine the meaning of the translation and rotation of a transverse cross-section, we could not apply the Law of Conservation of Energy. Consequently, neither could we derive the Theorem of Equality. Even if we could determine the locii of the center of twist and the center of shear, the two might not necessarily coincide.

Insert Eq. (30) into the strain energy equation — Eq. (25) — and, after rearrangement, we obtain:

$$E = \frac{1}{2}\left\{\left(U^1 - \frac{y_0 l}{GD}\right)F_x^2 + (U^2 + V^1)F_x F_y + \left(V^2 - \frac{x_0 l}{GD}\right)F_y^2 + \right. \tag{33}$$

$$\left. + \frac{l}{GD}(M_z - x_0 F_y + y_0 F_x)^2\right\}.$$

From this equation we can see that if the resultant and the load orientation remain unchanged (i.e., if there is no change in F_x and F_y), but the position of the resultant is changed (i.e. , a change in M_z), then the strain energy of the cylinder will reach its minimum value when the resultant passes through the center of shear. The strain energy value can be used to indicate the average intensity of all the local stresses in the body. Hence, when an external force passes through the center of shear, the average stress intensity produced in the cylinder will be at a minimum. Therefore, determining the center of shear by the method developed in this paper not only provides a clear and accurate geometrical meaning, it also indicates the close relationship with the average stress intensity in the cylinder.

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END