

Final Report for ONR Project

Robust Gain-Scheduled Nonlinear Control Design for Stability and Performance

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Chapter 1

Introduction

The models of control systems encountered in several diverse naval applications are nonlinear; moreover, they are also time-varying, and have uncertainties affecting them. The underlying controller design problems, beyond requiring system stability, also typically require the optimization of some performance objectives. Ideally, the control law should have the properties that: (1) it adapts itself according to the nonlinearities; (2) it can handle several performance objectives (besides stability) in a uniform manner; and (3) it guarantees the performance of the closed-loop system in spite of the variations and the uncertainties.

The primary goal of the project was a numerical solution methodology for solving the general nonlinear controller design problem. Our approach is motivated by recent advances in control and optimization theory as well as the sustained growth in the available computing power over the past few years. The proposed controller architecture is *gain-scheduled* (i.e., the controller uses the measured nonlinearities and time-variations), and optimizes the worst-case performance—over the uncertainties—of the system. The search for the optimal controller parameters can be reformulated as convex optimization problems involving linear matrix inequalities (LMIs) in several important cases. The design approach lends itself naturally to the development of computer-aided design tools, and enjoys the advantages of widespread applicability, ease of implementation and cost effectiveness. *Another goal of this research project was to integrate the advances in gain-scheduled controller design with the project Intelligent Information and Autonomous Agents Applied Research on Unmanned Combat Air Vehicles, directed by Dr. Allen Moshfegh.*

A third goal of the project was the development of robust estimation techniques. Estimation problems arise in many naval applications in control, communications and signal processing areas. Traditional estimation algorithms are usually based on a nominal system model without uncertainty. However, in many cases, there exist uncertainties in model parameters and even model structures because of modeling errors from system identifications. These uncertainties may degrade the estimation performance of the algorithm without robustness.

In this report, we will describe the results from the pursuit of the goals listed above.

1.1 Analysis and gain-scheduled control

The plant model consists of a nominal system, which is affected by unmeasurable uncertainties Δ_{unmeas} . Measurable uncertainties such as parameter variations are accounted for by Δ_{meas} . y consists of sensor signals, u consists of actuator inputs, w consists of exogenous signals (noises, reference inputs etc), and z contains all the signals of interest. The map from w to z contains all input-output maps of interest. The objective is to design a control law $u = K(y)$ so that the closed-loop system map from w to z possesses “desirable” properties.

The controller architecture that we have studied is one where the controller itself is in a feedback configuration with a linear part, with the measured uncertainties Δ_{meas} appearing in a feedback loop. The closed-loop system architecture is shown in Fig. 1.1, with the controller enclosed in dotted lines. Thus, depending on the nature of Δ_{meas} , the controller may be LTI, linear time-varying, or even nonlinear.

We have solved a number of robustness analysis and design problems in this setting.

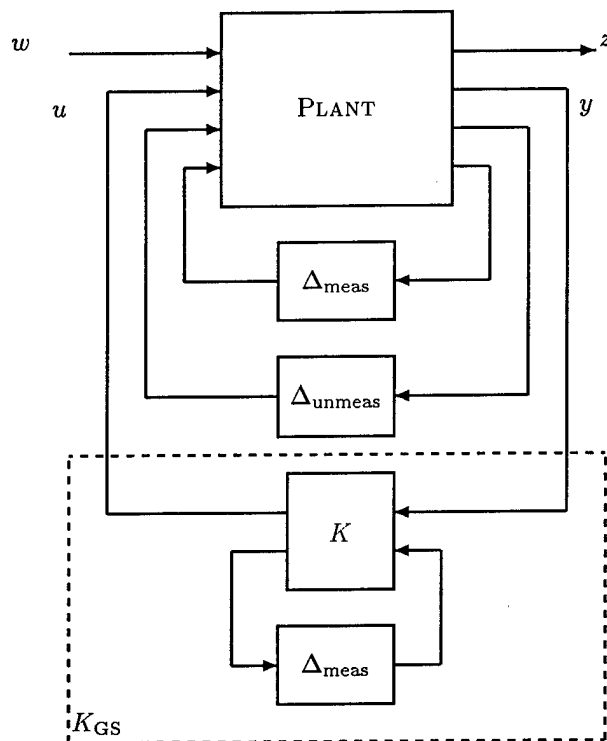


Figure 1.1: Closed-loop system. The gain-scheduled controller K_{GS} is shown enclosed in dashed lines.

List of research results

1. *Robust performance analysis using Lyapunov functions.*

The first step in realizing the project goals is that of robust performance analysis; the results achieved with this goal are summarized in Chapter 2.

2. *Robustness under bounded uncertainty with phase information.*

The special case where the uncertainties, in addition to being bounded, also satisfy constraints on their phase, is of particular interest. Robustness analysis of such systems is presented in Chapter 3.

3. *Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems.*

We were also able to substantially reduce the computational burden associated with LMI methods for establishing robust stability. In Chapter 4, we describe efficient computational methods for determining a guaranteed lower bound on the robust stability margin of uncertain systems.

4. *Improved stability analysis and gain-scheduled controller synthesis.*

The main results on gain-scheduled controller synthesis are described in Chapter 5.

1.2 Gain-scheduled control of the ONR UCAV

We have applied the results on gain-scheduled controller synthesis towards the control of the ONR Unmanned Intelligent Autonomous Air Vehicles (UCAVs). The models that we have used are from the research team from Texas A&M University, led by Professors Junkins, Ward and Valasek. The results from this effort are described in Chapter 6.

1.3 Robust estimation

A second research direction that is relevant to Dr. Moshfegh's project was identified during the January semi-annual program review, held in January 1999.

Estimation problems arise in many naval applications in control, communications and signal processing areas. Traditional estimation algorithms are usually based on a nominal system model without uncertainty. However, in many cases, there exist uncertainties in model parameters and even model structures because of modeling errors from system identifications. These uncertainties may degrade the estimation performance of the algorithm without robustness.

Figure 1.2 shows a block diagram of the problem setting. The subscript " Δ " is used to indicate that the plant parameters are affected by the uncertainties Δ . The objective is to design a filter for obtaining optimal estimates $\hat{z}$ of the signal of interest z , with different optimality criteria. Chapter 7 describes the results of our robust estimation efforts.

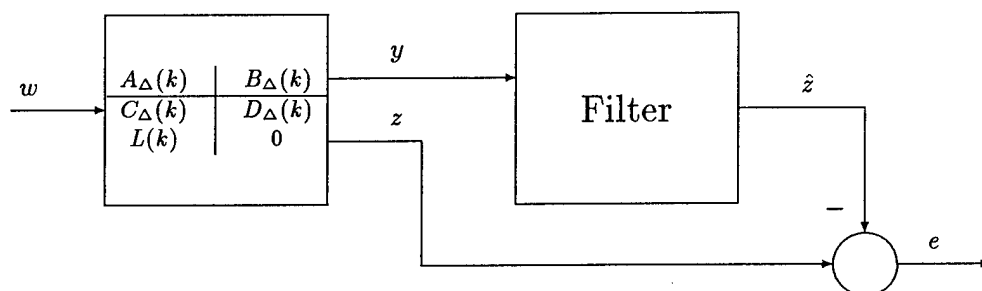


Figure 1.2: Estimation for uncertain linear time-varying systems.

1.4 Availability of results

The results from this project have been documented in several refereed publications.

1.4.1 Publications

Journal Publications

1. L. Vandenberghe and V. Balakrishnan. "Algorithms and Software for LMI Problems in Control". *IEEE Control Systems Magazine*, pages 89–95, October 1997.
2. A. L. Tits and V. Balakrishnan. "Small- μ Theorems with Frequency-Dependent Uncertainty Bounds". *Mathematics of Control, Signals, and Systems*, vol 11, no. 3, pages 220–243, 1998.
3. A. E. Rundell, R. A. DeCarlo, V. Balakrishnan and H. HogenEsch. "Investigations into Treatment Strategies Aiding the Immune Response to *Haemophilus Influenzae*". *IEEE Trans. Biomed. Eng.*, vol. 45, no. 4, pages 429–439, April 1998.
4. V. Balakrishnan and R. L. Kashyap. "Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities". *Journal of Optimization Theory and Applications*, vol. 100, no. 3, March 1999.
5. A. L. Tits, V. Balakrishnan and L. Lee. "Robustness under Bounded Uncertainty with Phase Information". *IEEE Trans. Aut. Contr.*, AC-44(1):50–65, January 1999.
6. Y. S. Chou, A. L. Tits and V. Balakrishnan. "Stability multipliers and μ upper bounds: Connections and implications for numerical verification of frequency domain conditions". *IEEE Trans. Aut. Contr.*, AC-44(5):906–913, May 1999.

7. V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". *IEEE Trans. Aut. Contr.*, AC-44(11):2185-2190, November 1999.
8. C-Y. Chang, A. A. Maciejewski, and V. Balakrishnan. "Fast Eigenspace Decomposition of Correlated Images." Accepted for publication in *IEEE Transactions on Image Processing*, 1999.

Conference papers published

1. V. Balakrishnan and L. Vandenberghe. "Linear Matrix Inequalities for signal processing. An overview". Invited for presentation at the *32nd Annual Conference on Information Sciences and Systems*, Department of Electrical Engineering, Princeton University, Princeton NJ, March 1998.
2. V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". In *Proc. IEEE Conf. on Decision and Control*, pages 4406-4407, Tampa, FL, December 1998.
3. F. Wang and V. Balakrishnan. "Improved Stability Analysis and Gain-Scheduled Controller Synthesis for Parameter-Dependent Systems". In *Proc. IEEE Conf. on Decision and Control*, pages 1771-1776, Tampa, FL, December 1998.
4. F. Wang and V. Balakrishnan. "Robust Adaptive Kalman Filters for Linear Time-Varying Systems with Stochastic Parametric Uncertainties." In *IEEE American Control Conf.*, pages 440-444, San Diego, CA, June 1999. **Finalist, best student paper award.**
5. F. Wang and V. Balakrishnan. "Robust Estimators for Systems with Both Deterministic and Stochastic Uncertainties." In *IEEE Conf. on Decision and Control*, Phoenix, Arizona, December 1999.
6. F. Wang and V. Balakrishnan. "Robustness analysis and gain-scheduled controller synthesis for rational parameter-dependent uncertain systems using parameter-dependent Lyapunov functions." In *IEEE Conf. on Decision and Control*, Phoenix, Arizona, December 1999.

1.4.2 Web-based dissemination of results

The web-sites

<http://www.ece.purdue.edu/~ragu/onr/gs-contr.html>

and

<http://www.ece.purdue.edu/~ragu/onr/rob-est.html>

have been established to disseminate the research results. The first web-site also has instruction for downloading matlab code.

Part I

Analysis and Gain-Scheduled Control

Chapter 2

Robust performance analysis using Lyapunov functions

A wide variety of problems in system and control theory can be formulated (or reformulated) as convex optimization problems involving linear matrix inequalities (LMIs), that is, constraints requiring an affine combination of symmetric matrices to be positive semidefinite. For a few very special cases, there are “analytical solutions” to these problems, but in general they can be solved numerically very efficiently. Thus, the reduction of a control problem to an optimization problem based on LMIs constitutes, in a sense, a “solution” to the original problem. The first objective of this chapter is to provide a tutorial on the application of optimization based on LMIs to robust control problems. The second objective is that of robust stability and performance analysis of uncertain systems using LMI optimization.

2.1 Introduction

A wide variety of problems in system and control theory can be reduced to a handful of standard convex and quasiconvex optimization problems that involve linear matrix inequalities or LMIs, that is constraints of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0, \quad (2.1)$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \dots, m$, are given. Though the form of the LMI (2.1) appears very specialized, it turns out that it is widely encountered in system and control theory. Examples include: multicriterion LQG, synthesis of linear state feedback for multiple or nonlinear plants (“multi-model control”), optimal state-space realizations of transfer matrices, norm scaling, synthesis of multipliers for Popov-like analysis of systems with unknown gains, robustness analysis and robust controller design, gain-scheduled controller design, and many others. For a few very special cases there are “analytical solutions” to LMI optimization problems, but in general they can be solved numerically very efficiently. Indeed, the recent and growing popularity of LMI optimization for control can be directly traced to the recent breakthroughs in interior point methods for LMI optimization (see for example, [NN94]–[VB96]). In many cases—for example, with multi-model control [BEFB94]—the LMIs encountered in systems and control theory have the form of simultaneous (coupled) Lyapunov or algebraic Riccati inequalities; using interior-point methods, such problems can be solved in a time that is roughly comparable to the time required to solve the same number of (uncoupled) Lyapunov or Algebraic Riccati equations [VB95]. Therefore the computational cost of extending current control theory that is based on the solution of algebraic Riccati equations to a theory based on the solution of (multiple, simultaneous) Lyapunov or Riccati inequalities is modest.

A number of publications can be found in the control literature that survey applications of LMI optimization to the solution of system and control problems. Perhaps the most comprehensive list can be found in the book [BEFB94]. Since its publication, a number of papers have appeared chronicling further applications of LMI optimization techniques in control; a few examples are [BE] and [VB97]. The growing popularity of LMI methods for control is also evidenced by the large number of publications in recent control conferences.

Our first objective in this chapter is to give a brief introduction to optimization based on LMIs. In Section 2.2.1, we describe a few “standard” convex and quasiconvex optimization problems involving LMIs. We make a few brief remarks about solving LMI-based optimization problems in Section 2.2.2. In Section 2.2.3, we present a brief history of LMIs in system and control theory.

Our second objective is that of robust stability and performance analysis of uncertain systems, with various assumptions on the nature of the uncertainties (sector-bounded nonlinear, linear time-invariant, parametric, etc.), as well as their structure (diagonal, block-diagonal, etc.). We first show, in Section 2.4, how the robust stability analysis of such systems can be performed in a unified manner using multiplier theory and LMI-based convex optimization. Not only does this provide a unification of several apparently-diverse robust stability tests, but it also paves the way for developing new stability tests. In addition, the multipliers used in the stability analysis can be shown to yield a convex parametrization of a subset of Lyapunov functions that provide a certificate of robust stability. In Section 2.5, we show how these Lyapunov functions can in turn be used to derive bounds on various robust performance measures for uncertain systems. We illustrate our approach with two specific robust performance analysis problems.

2.2 Optimization Based on Linear Matrix Inequalities

Recall the definition of a linear matrix inequality:

$$F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0,$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \dots, m$ are given. The set $\{x \mid F(x) > 0\}$ is convex, and need not have smooth boundary. (We have used strict inequality mostly for convenience; inequalities of the form $F(x) \geq 0$ are also readily handled.)

Multiple LMIs $F_1(x) > 0, \dots, F_n(x) > 0$ can be expressed as the single LMI

$$\text{diag}(F_1(x), \dots, F_n(x)) > 0.$$

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0, \quad (2.2)$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x , is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \quad (2.3)$$

In other words, the set of nonlinear inequalities (2.3) can be represented as the LMI (2.2).

The matrix norm constraint $\|Z(x)\| < 1$, where $Z(x) \in \mathbf{R}^{p \times q}$ and depends affinely on x , is represented as the LMI

$$\begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0$$

(since $\|Z\| < 1$ is equivalent to $I - ZZ^T > 0$). Note that the case $q = 1$ reduces to a general convex quadratic inequality on x .

The constraint

$$\text{Tr } S(x)^T P(x)^{-1} S(x) < 1, \quad P(x) > 0,$$

where $P(x) = P(x)^T \in \mathbf{R}^{n \times n}$ and $S(x) \in \mathbf{R}^{n \times p}$ depend affinely on x , is handled by introducing a new (slack) matrix variable $X = X^T \in \mathbf{R}^{p \times p}$, and the LMI (in x and X):

$$\text{Tr } X < 1, \quad \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0.$$

We often encounter problems in which the variables are matrices, e.g.,

$$A^T P + P A < 0, \quad (2.4)$$

where $A \in \mathbf{R}^{n \times n}$ is given and $P = P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables. Leaving LMIs in a condensed form such as (2.4), in addition to saving notation, leaves open the possibility of more efficient computation.

2.2.1 Some Standard LMI Optimization Problems

(i) LMI Feasibility Problems

Given an LMI $F(x) > 0$, the corresponding LMI feasibility problem is to find x^{feas} such that $F(x^{\text{feas}}) > 0$ or determine that the LMI is infeasible. (By duality, this means: find a nonzero $G \geq 0$ such that $\text{Tr } GF_i = 0$ for $i = 1, \dots, m$ and $\text{Tr } GF_0 \leq 0$.) Of course, this is a convex feasibility problem.

(ii) Eigenvalue Problems

The eigenvalue problem is to minimize the maximum eigenvalue of a matrix, subject to an LMI:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \lambda I - A(x) > 0, \quad B(x) > 0. \end{aligned}$$

Here, A and B are symmetric matrices that depend affinely on the optimization variable x . This is a convex optimization problem.

(iii) Generalized Eigenvalue Problems

The generalized eigenvalue problem is to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on the optimization variable, subject to an LMI constraint. The general form of a generalized eigenvalue problem is:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \lambda B(x) - A(x) > 0 \\ & B(x) > 0 \\ & C(x) > 0 \end{aligned}$$

where A , B and C are affine functions of x . This is a quasiconvex problem.

Note that when the matrices are all diagonal, this problem reduces to the general linear fractional programming problem. Many nonlinear quasiconvex functions can be represented in the form of a generalized eigenvalue problem with appropriate A , B , and C (see [BE93]).

2.2.2 Solving LMI-Based Problems

The most important point is:

LMI feasibility, eigenvalue and generalized eigenvalue problems are all tractable

in a sense that can be made precise from a number of theoretical and practical viewpoints. (This is to be contrasted with much less tractable problems, e.g., the general problem of robustness analysis for a system with real parameter perturbations.)

From a theoretical standpoint:

- We can immediately write down necessary and sufficient optimality conditions.
- There is a well-developed duality theory (for generalized eigenvalue problems, in a limited sense).
- These problems can be solved in polynomial time (indeed with a variety of interpretations of the term "polynomial-time").

The most important practical implication is that there are effective and powerful algorithms for the solution of these problems, that is, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Thus, on exit, the algorithms can prove that the global optimum has been obtained to within some prespecified accuracy.

There are a number of general algorithms for the solution of LMI problems, for example, the ellipsoid algorithm (see e.g., [BB91] and [BGT81]). The ellipsoid method has polynomial-time complexity, and works in practice for smaller problems, but can be slow for larger problems. Other algorithms specifically

for LMI-based problems are discussed in, e.g., [CDW75] and [Ove88]. More recently, various researchers ([Ali95]–[VB96]) have developed interior point methods for solving LMI-based problems, based on the work of Nesterov and Nemirovskii [NN94]. Numerical experience shows that these algorithms solve LMI problems with great efficiency. A survey of algorithms and software for LMI optimization can be found in [VB97].

A number of software packages are also available for solving LMI problems. The first implementation of an interior-point method for LMI problems was by Nesterov and Nemirovskii in [NN90], using the projective algorithm [NN94]. Matlab's LMI Control Toolbox [GNLC95] is based on the same algorithm, and offers a graphical user interface and extensive support for control applications. The code SP [VB94] is based on a primal-dual potential reduction method with the Nesterov and Todd scaling. The code is written in C with calls to BLAS and LAPACK and includes an interface to Matlab. SDPSOL [WB96] and LMITOOL [END95] offer user-friendly interfaces to SP that simplify the specification of LMI problems where the variables have matrix structure. The Induced-Norm Control Toolbox [Ber95] is a toolbox for robust and optimal control, in turn based on LMITOOL.

2.2.3 Brief History of LMIs in System and Control Theory

The history of linear matrix inequalities in the analysis of dynamical systems goes back more than 100 years, when Lyapunov published his seminal work introducing what we now call Lyapunov theory. He showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t) \quad (2.5)$$

is stable if and only if there exists a positive definite matrix P such that

$$A^T P + PA < 0. \quad (2.6)$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we now call a Lyapunov inequality on P , which is a special form of an LMI. Of course, we can solve this LMI (that is, find a suitable P) by solving a Lyapunov equation.

The next major development was in the 1940s when Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator [Lur57]. Although they did not explicitly form matrix inequalities, their stability criteria in fact have the form of LMIs. These inequalities were reduced to polynomial inequalities which were then checked "by hand" (for, needless to say, small systems).

Then, in the 1960s, Yakubovich, Popov, Kalman, and other researchers succeeded in reducing the solution of the LMIs that arose in the problem of Lur'e to simple graphical criteria, using what we now call the Kalman-Yakubovich-Popov (KYP) lemma. This resulted in the celebrated Popov criterion, Circle criterion, Tsytkin criterion, and many variations. These criteria could be applied to higher order systems, but did not gracefully or usefully extend to systems containing more than one nonlinearity. Thus, their contribution may be viewed—in the context of the history of LMIs in control theory—as showing how to solve a certain family of LMIs by a graphical method. We should note that the important role of LMIs in control theory was already recognized in the early 1960s, especially by Yakubovich [Yak67].

By 1971, researchers knew several methods for solving special types of LMIs: direct (for very small systems), graphical methods, and by solving Lyapunov or Riccati equations. In Willems' 1971 paper [Wil71] we find the following striking quote:

The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

Willems' suggestion that LMIs might have some advantages in computational algorithms became closer to being realized with the following observation:

The LMIs that arise in system and control theory can be formulated as *convex optimization problems*, and hence are amenable to computer solution.

This observation was made explicitly by several researchers: Pyatnitskii and Skorodinskii [PS82], and Horisberger and Belanger [HB76], to name just a few. This observation, coupled with the development of interior point methods that apply directly to convex problems involving matrix inequalities, by Nesterov and Nemirovskii in 1988, now mean that we can reliably and quickly solve many problems in systems and control

for which no “analytical solution” has been found (or is likely to be found), by reducing to them to LMI problems. The efficient solution of LMI optimization problems as well as the numerical solution of several diverse engineering problems via LMI optimization remain areas of active research.

2.3 Robust Stability and Performance Analysis of Uncertain Systems: Overview

Most control system models in use today explicitly incorporate in them “uncertainties” or “perturbations”. These uncertainties may model a number of factors, including: dynamics that are neglected to make the model tractable, as with large scale structures; nonlinearities that are either hard to model or too complicated; and parameters that are not known exactly, either because they are hard to measure or because of varying manufacturing conditions. A widely-used model for uncertain systems, shown in Fig. 2.4, consists of a nominal finite-dimensional, stable, linear time-invariant system, with a perturbation or uncertainty Δ in a feedback loop. The signal w represents exogenous inputs, and z represents all outputs of interest. Often additional information about Δ is either known or assumed; common examples are that Δ is diagonal or block-diagonal; sector-bounded memoryless, linear time-invariant or parametric; bounded in norm, passive, etc. The analysis of and design for such control system models is commonly referred to as “robust control with structured perturbations”.

One of the most fundamental questions concerning the system in Fig. 2.4 is that of stability: “Is the model stable irrespective of the perturbation Δ , that is, do all solutions of the system equations go to zero, irrespective of Δ ?” This is also referred to as the *robust stability* problem. Some of the approaches for solving this problem, with various assumptions on Δ , are the use of the small-gain, passivity or circle-criteria [DV75], the Popov criterion [LP44], and μ or K_m analysis methods ([Doy82]–[CS92]). Robust stability is but one desired feature of an uncertain system; of considerable importance are questions beyond stability, known broadly as *robust performance* problems. Robust performance analysis problems concern the computation of the worst possible value, over all uncertainties, of performance indices; these performance indices may be bounds on some state variables, norms of the map from w to z etc. An example of a robust performance analysis problem is the so-called H_2 problem, which is the computation of the largest possible RMS value of the output z , over all Δ , when the input w is unit-intensity white noise. This finds application where the average value of a certain signal is of interest, when the system is affected by an unpredictable input that is modeled as white noise. Another example is the computation of the largest possible RMS gain, over all uncertainties, from w to z ; this is also known as the H_∞ performance analysis problem.

It has been found recently that several stability analysis methods for control systems can be unified in the setting of multiplier theory ([DV75] and [SC93]–[GS95]), or more generally, in the framework of integral quadratic constraints or IQCs [MR97]. (While the framework of IQCs is more general than the multiplier-based framework, we have chosen to present the latter here for “historical” reasons.) As a consequence, several sufficient conditions for robust stability can be performed without frequency sampling, using convex optimization techniques based on LMIs [Bal95]; we describe this in Section 2.4. It also turns out that the multiplier techniques yield a convex parametrization of a set of Lyapunov functions that prove robust stability [Bal94a]. By imposing additional conditions on these Lyapunov functions, bounds on the robust performance for the system in Fig. 2.4 can be derived; thus Lyapunov functions can be used to “prove” robust performance as well. The “best” performance bounds can then be obtained by numerically optimizing these bounds, using LMI-based methods, over the set of Lyapunov functions that prove robust performance. We describe this approach in Section 2.5.

2.4 Robust Stability Using Multiplier Theory

Let the equations governing the system in Fig. 2.4 be:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad z(t) = C_z x(t), \quad (2.7a)$$

$$p(t) = -\Delta(q, t), \quad (2.7b)$$

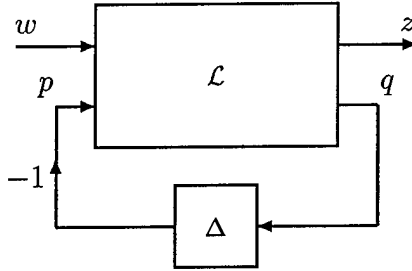
where $x(t) \in \mathbb{R}^n$, $p(t) \in \mathbb{R}^{n_p}$, $q(t) \in \mathbb{R}^{n_q}$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$. For convenience, we have assumed that there is no feed-through from w to z , w to q and p to z ; we will also assume that $n_p = n_q = m$. For

future reference, we let $\mathcal{L}$ denote the linear system (2.7a), and G the transfer function of the linear part of the system from p to q , i.e., $G(s) = C_q(sI - A)^{-1}B_p + D_{qp}$. $\mathcal{L}$ is assumed to be stable. Equations (2.7) can be interpreted either as representing a “family of systems”, each member corresponding to some Δ , or as an “uncertain system” with uncertainty Δ ; we will use these terms interchangeably. The perturbation Δ is in general nonlinear, and is assumed to be passive, that is, the following holds⁵:

For some $\epsilon \geq 0$ and $\beta \in \mathbf{R}$,

$$\int_0^T u(t)^T (\Delta u)(t) dt \geq \epsilon \int_0^T u(t)^T u(t) dt - \beta \text{ for all } T \geq 0 \text{ and all } u \in \mathbf{L}_2.$$

We let $\mathcal{D}$ denote the set of all passive operators Δ .



Remark 2.4.1 Another frequently encountered class of uncertainties are *bounded* uncertainties, that is, those Δ with an $\mathbf{L}_2$ gain that does not exceed some $\gamma > 0$. However, it is well-known (for example, see [And72]) that under standard assumptions, one may use loop transformations to transform the case of norm-bounded Δ to the case of passive Δ considered here.

We say that the family of systems (2.7) is robustly stable if with w identically zero, for every member of the family and for every initial condition, every solution $x(t)$ of (2.7) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$. Our first objective is to show how the robust stability analysis of the family of systems (2.7) can be performed using LMI optimization.

2.4.1 Robust Stability from the Passivity Theorem

Since Δ is known to be passive, the passivity theorem [DV75] can be invoked to establish robust stability of the system (2.7):

The uncertain system (2.7) is robustly stable if the (linear) map from p to q is strictly passive, that is, there exist $\epsilon > 0$ and $\beta \in \mathbf{R}$ such that for all $p \in \mathbf{L}_2$, the condition

$$\int_0^T p(t)^T q(t) dt \geq \epsilon \int_0^T p(t)^T p(t) dt - \beta \text{ for all } T \geq 0$$

holds, where p and q satisfy (2.7a) with w identically zero.

Strict passivity of the map from p to q is equivalent [DV75] to the frequency-domain condition that for some $\epsilon > 0$,

$$G(j\omega) + G(j\omega)^* \geq 2\epsilon I \quad \text{for all } \omega \in \mathbf{R}. \quad (2.8)$$

(If this holds, we will say with some abuse of terminology that “ $G(s)$ is strictly passive”.) This condition can be numerically verified (approximately) by checking that the minimum eigenvalue of $G(j\omega) + G(j\omega)^*$, sampled at a number of frequencies, is positive. An alternate condition for strict passivity that avoids frequency sampling is expressed in terms of an LMI.

⁵For precise technical definitions, mathematical preliminaries, and the notation used here, see [DV75] and [Vid93].

Lemma 2.4.1 Let (A, B_p, C_q, D_{qp}) be a state-space realization of a stable linear system with transfer function $G(s)$. Then, G satisfies (2.8) if and only if there exists $P = P^T > 0$ satisfying the LMI

$$\begin{bmatrix} A^T P + P A & P B_p - C_q^T \\ B_p^T P - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} < 0. \quad (2.9)$$

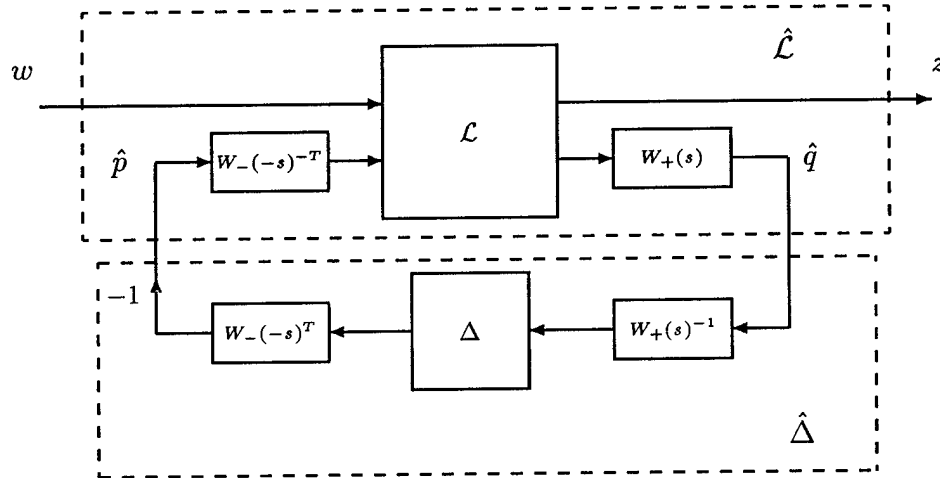
This is simply the Kalman-Yakubovich-Popov Lemma or the Positive-Real Lemma (see for example, [NT73], [Vid93], pages 474–478, and [AV73], Chapters 5–7; see also [Ran96]).

Remark 2.4.2 We also have the following extension of Lemma 2.4.1. Let (A, B_p, C_q, D_{qp}) be a realization of a not necessarily stable linear system with transfer function $G(s)$, with A having no eigenvalues on the imaginary axis. Then, G satisfies condition (2.8) if and only if there exists $P = P^T$ satisfying the LMI

$$\begin{bmatrix} A^T P + P A & P B_p - C_q^T \\ B_p^T P - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} < 0. \quad (2.10)$$

2.4.2 Multiplier Analysis

Suppose $\Delta \in \mathcal{D}$ also possesses additional properties, such as being diagonal or block-diagonal, sector-bounded memoryless, linear time-invariant or parametric, etc. Let $\mathcal{D}_{\text{struct}} \subseteq \mathcal{D}$ denote the set of Δ satisfying these additional properties. (We will see two specific cases for $\mathcal{D}_{\text{struct}}$ in Example 2.5.1 and Example 2.5.2.) Then the passivity theorem yields only a sufficient condition for robust stability. In this case, we can employ *multiplier theory* to utilize the additional information on Δ in order to obtain less conservative conditions for robust stability; see for example, [DV75] and [SC93]–[Bal95].



Consider the system in Fig. 2.4.2, where $W_+(s)$ and $W_-(-s)$ are transfer functions of some finite-dimensional LTI systems. Moreover, suppose that $W_+(s)$ and $W_-(-s)$ satisfy

$$W_+(s) \text{ and } W_-(-s) \text{ are stable with stable inverses.} \quad (2.11)$$

The equations governing the system in Fig. 2.4.2 are:

$$\frac{d}{dt} x(t) = A x(t) + B_p p(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad z(t) = C_z x(t), \quad (2.12a)$$

$$p(t) = -\Delta(q, t), \quad \hat{p}(t) = (w_- \star p)(t), \quad \hat{q}(t) = (w_+ \star q)(t), \quad (2.12b)$$

where w_- and w_+ are the inverse Laplace transforms of $W_-(-s)^T$ and $W_+(s)$ respectively, and “ $\star$ ” denotes convolution.

We have the obvious but important connection between the solutions of the equations (2.12) and (2.7).

Lemma 2.4.2 For every $\Delta \in \mathcal{D}_{\text{struct}}$, x satisfies (2.7) if and only if it satisfies (2.12) for some (every) W_+ and W_- satisfying condition (2.11). Therefore, system (2.7) is robustly stable if and only if system (2.12) is robustly stable, for some (every) W_+ and W_- satisfying condition (2.11).

Next, suppose that

$$\text{for every } \Delta \in \mathcal{D}_{\text{struct}}, W_-(-s)^T \circ \Delta \circ W_+(s)^{-1} \text{ is passive.} \quad (2.13)$$

Then from the passivity theorem, system (2.12) is stable if

$$W_+(s)G(s)W_-(-s)^{-T} \text{ is strictly passive.} \quad (2.14)$$

Thus, if we find *some* W_+ and W_- such that conditions (2.13) and (2.14) hold, we have then established the robust stability of system (2.7). Robust stability methods using multiplier theory involve systematically searching for W_+ and W_- satisfying (2.11) such that conditions (2.13) and (2.14) hold. Every pair of such W_+ and W_- is called a “multiplier pair proving robust stability”.

It turns out that searching directly for W_+ and W_- is numerically “hard”, as the set of multiplier pairs that prove robust stability is nonconvex in general.

Remark 2.4.3 Numerical counterexamples that establish that the set of constant multiplier pairs that prove robust stability is not necessarily convex are easy to find. Here is a simple one, with G being a linear system with no dynamics, i.e., with transfer matrix a constant:

$$G(s) = \begin{bmatrix} 1.30 & 0.30 & -0.40 \\ 0.20 & 0.50 & 0.10 \\ -0.10 & -0.60 & 0.80 \end{bmatrix}.$$

For this system, it is easily checked numerically that the multiplier factors with no dynamics

$$\left(\begin{bmatrix} 0.70 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.3 \end{bmatrix}, \begin{bmatrix} 0.70 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} 1.2 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \right)$$

both prove robust stability, but their average

$$\left(\begin{bmatrix} 0.95 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.80 \end{bmatrix}, \begin{bmatrix} 0.95 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.80 \end{bmatrix} \right)$$

does not.

2.4.3 Robust Stability via LMI Feasibility Tests

A simple change of variables enables the reformulation of the problem of search of multiplier pairs into a convex feasibility problem. The central idea underlying the reformulation is that we should regard the product $W(s) \triangleq W_-(s)W_+(s)$ as the variable; this product, which we shall denote by $W(s)$ is called the stability multiplier or just “multiplier”. We will say that the multiplier W “proves robust stability” if it can be factored into $W(s) = W_-(s)W_+(s)$, with the pair $\{W_-, W_+\}$ proving robust stability, i.e., such that conditions (2.11), (2.13) and (2.14) hold. It can be shown (see [Bal95] and [CTB97a] for details) that these conditions are equivalent, in several important cases, to the following conditions:

- W should have a certain *structure* (e.g., diagonal or block-diagonal) and *nature* (e.g., constant, Hermitian on the imaginary axis, etc.), depending on the additional information known about $\mathcal{D}_{\text{struct}}$. (For specific examples, see [Bal95].) (2.15)

- For some $\epsilon > 0$, $W(j\omega) + W(j\omega)^* \geq 2\epsilon I$ for all $\omega \in \mathbf{R}$. (2.16)

- For some $\epsilon > 0$, $W(j\omega)G(j\omega) + G(j\omega)^*W(j\omega)^* \geq 2\epsilon I$ for all $\omega \in \mathbf{R}$. (2.17)

Thus, establishing robust stability using multiplier theory involves numerically searching for a multiplier W that satisfies conditions (2.15)–(2.17) above. The important observation regarding this reformulation is the following.

Finding W that satisfies conditions (2.15)–(2.17) is a (possibly infinite-dimensional) convex feasibility problem.

Additionally restricting W , if necessary, to lie in a finite-dimensional subspace results in a *finite-dimensional* convex feasibility problem. Specifically, suppose that W is restricted to lie in the subspace

$$\mathcal{W} \triangleq \left\{ \sum_{i=1}^m \theta_i W_i \mid \theta \in \mathbf{R}^m \right\}, \quad (2.18)$$

where W_i are fixed transfer matrices, each satisfying the structure and nature condition that is to be satisfied by W (see (2.15)). Then, every $W \in \mathcal{W}$ has a realization $(A_W, B_W, C_W(\theta), D_W(\theta))$, with C_W and D_W being *linear* functions of θ . Using the remark following Lemma 2.4.1, we conclude that condition (2.16) is equivalent to the LMI in $P_W = P_W^T$ and θ :

$$\begin{bmatrix} A_W^T P_W + P_W A_W & P_W B_W - C_W(\theta)^T \\ B_W^T P_W - C_W(\theta) & -(D_W(\theta) + D_W(\theta)^T) \end{bmatrix} < 0. \quad (2.19)$$

Next, WG has a state-space realization $(A_{WG}, B_{WG}, C_{WG}, D_{WG})$ where

$$A_{WG} = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \quad B_{WG}(\theta) = \begin{bmatrix} B \\ B_W D \end{bmatrix}, \\ C_{WG} = [D_W(\theta) C \quad C_W(\theta)], \quad D_{WG}(\theta) = D_W(\theta) D.$$

Note that C_{WG} and D_{WG} are linear functions of θ . Then, checking condition (2.17) is equivalent to the LMI in $P = P^T$ and θ :

$$\begin{bmatrix} A_{WG}^T P + P A_{WG} & P B_{WG} - C_{WG}(\theta)^T \\ B_{WG}^T P - C_{WG}(\theta) & -(D_{WG}(\theta) + D_{WG}(\theta)^T) \end{bmatrix} < 0. \quad (2.20)$$

Thus sufficient conditions for robust stability, in several important cases, can be posed as the following feasibility problem:

$$\text{Find } P_W, P \text{ and } \theta \text{ such that the LMIs (2.19) and (2.20) hold.} \quad (2.21)$$

The above formulation of robust stability tests is important in several different ways:

- (i) Robust stability can be ascertained using state-space techniques and reliable convex optimization; there is no frequency sampling involved with LMI tests.
- (ii) A number of classical and modern robust stability tests can be unified in this setting (for example, several tests for absolute stability, as well as the modern μ tests; see [Bal95]).
- (iii) New tests can be devised, when other assumptions on the uncertainties are in effect (see for example, [TBL99], as well as [MR97]).

In addition, the LMI tests for robust stability can be shown to yield Lyapunov functions that “prove” robust stability (see for example [Bal94a]). These Lyapunov functions can in turn be used to derive bounds on robust performance. We describe this next.

2.5 Robust Performance Bounds from Lyapunov Functions

The multiplier-based robust stability condition (2.21) can be shown to yield a positive-definite function $V : \mathbf{R}^n \times \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that $dV(x, t)/dt < 0$ along the trajectories of (2.7). (This function V can be constructed from the LMI variables P , P_W and θ ; we will see a demonstration of this in Example 2.5.1 and Example 2.5.2.) Using standard arguments from Lyapunov theory, it can be shown that V is a Lyapunov function that provides a *certificate* or *guarantee* of robust stability. Additional conditions on the Lyapunov function can be imposed, yielding robust performance bounds as follows (see for example [AC84], Section 6, and [BEFB94]): For a given initial condition $x(0)$ for the linear part (2.7a), suppose that the Lyapunov function V satisfies

$$\frac{d}{dt} V(x, t) \leq -\phi_\alpha(x(t), w(t), z(t)) \text{ along the trajectories of (2.7).} \quad (2.22)$$

Then it is a simple exercise to show that

$$V(x(0), 0) \geq \int_0^\infty \phi_\alpha(x(t), w(t), z(t)) dt, \quad (2.23)$$

or we have an upper bound on the performance index

$$\mathcal{P}_\alpha \triangleq \int_0^\infty \phi_\alpha(x(t), w(t), z(t)) dt. \quad (2.24)$$

Thus, if we can find a Lyapunov function V satisfying (2.22), then we can obtain upper bounds on robust performance indices of the form (2.24), for system (2.7). (A variation of this technique can be used to obtain bounds on more general performance measures; see for example [BEFB94], Chapters 5–6.)

We will see in Example 2.5.1 and Example 2.5.2 that imposing condition (2.22) on the Lyapunov functions obtained from multiplier theory leads to LMI conditions in several important cases. Consequently, in all these cases, the problem of computing the optimal (or smallest) robust performance bound can be reformulated as an LMI optimization problem.

Example 2.5.1 Unstructured Passive Δ , Bound on the Output Energy

Consider the case when Δ is a general passive operator, with the performance analysis problem being:

With $w = 0$, find a bound on the energy of the output, i.e., $\int_0^\infty z(t)^T z(t) dt$, for a given initial condition $x(0)$ of the state of the linear part of the system.

In order to derive upper bounds on $\int_0^\infty z(t)^T z(t) dt$, we seek Lyapunov functions that satisfy condition (2.22) with $\phi_\alpha(x, w, z) = z^T z$, i.e.,

$$\frac{d}{dt} V(x, t) \leq -z(t)^T z(t). \quad (2.25)$$

Then $V(x(0), 0)$ yields an upper bound on the output energy.

Since Δ is a general passive operator, the only possible multiplier pair is given by $W_+ = W_- = I$. The corresponding Lyapunov functions turn out (see [Wil72a] and [Wil72b]) to be of the form

$$V(x, t) = x(t)^T P x(t) - 2 \int_0^t p(\tau)^T q(\tau) d\tau,$$

where $P > 0$. Condition (2.25) holds along the trajectories of system (2.7) if the following condition holds:

$$\begin{bmatrix} A^T P + P A + C_z^T C_z & P B_p - C_q^T \\ B_p^T P - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} \leq 0. \quad (2.26)$$

Thus, the best upper bound on the energy of z , for a given initial condition $x(0)$ of the linear part of the system, is simply

$$\begin{aligned} \min \quad & x(0)^T P x(0) \\ \text{s.t.} \quad & P > 0, \text{ and (2.26)} \end{aligned}$$

This is an “eigenvalue problem” (see Section 2.2.1).

Example 2.5.2 Diagonal, Time-Invariant, Sector-Bounded, Memoryless Nonlinearities Δ , Bound on the State Variables

Next, consider the case when: D_{qp} is zero; the uncertainty Δ is diagonal, i.e., $p_i(t) = -\delta_i(q_i(t))$; and each δ_i is a time-invariant, memoryless nonlinearity in sector $[0, \infty)$ (see [DV75] for the definitions of the various terms used here). For this case, the performance analysis problem considered is:

With exogenous input $w = 0$, find bounds on the state $x(t)$, $t \geq 0$, of the linear part of the system, for a given initial condition $x(0)$.

In other words, we seek an invariant set for the trajectories $x(t)$ of the linear part of the system. For every positive-definite function V for which $dV(x, t)/dt \leq 0$ holds along the trajectories of system (2.7), we have

$$V(x(t), t) \leq V(x(0), 0) \text{ for } t \geq 0.$$

This inequality can then be used to derive bounds on $x(t)$.

For the case of diagonal, time-invariant, sector-bounded, memoryless nonlinearities, the multiplier pairs for robust stability turn out to be $W_- = I$ and

$$W_+(s) = \mathbf{diag}((\lambda_1 + \mu_1 s), \dots, (\lambda_m + \mu_m s)),$$

where $\mu_i \geq 0$, $\lambda_i > 0$. (This corresponds to a multivariable version of the Popov criterion [DV75].) The corresponding Lyapunov functions are of the form

$$V(x, t) = x(t)^T P x(t) + 2 \sum_{i=1}^m \mu_i \int_0^{q_i(t)} \delta_i(\sigma) d\sigma - 2 \sum_{i=1}^m \lambda_i \int_0^t p_i(\tau) q_i(\tau) d\tau,$$

where $P > 0$, $\mu_i \geq 0$ and $\lambda_i > 0$. In this case, condition $dV(x, t)/dt \leq 0$ is equivalent to the LMI

$$\begin{bmatrix} A^T P + P A & P B_p - (\Lambda C_q + M C_q A)^T \\ B_p^T P - (\Lambda C_q + M C_q A) & -(M C_q B_p + B_p^T C_q^T M) \end{bmatrix} \leq 0 \quad (2.27)$$

where $P > 0$, $M = \mathbf{diag}(\mu_1, \dots, \mu_m) \geq 0$ and $\Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_m) > 0$. If condition (2.27) holds, then the ellipsoid

$$\mathcal{E} = \{ \psi \in \mathbf{R}^n \mid \psi^T P \psi \leq x(0)^T P x(0) \}$$

is an invariant ellipsoid for the state x .

The problem of finding the smallest invariant ellipsoid (using our techniques), i.e., one with the smallest major axis, can be solved by solving the “eigenvalue problem” (see Section 2.2.1)

$$\begin{aligned} \max \quad & \nu \\ \text{s.t.} \quad & P > \nu I, \quad x(0)^T P x(0) \leq 1, \quad (2.27), \\ & M = \mathbf{diag}(\mu_1, \dots, \mu_m) \geq 0, \quad \Lambda = \mathbf{diag}(\lambda_1, \dots, \lambda_m) > 0 \end{aligned}$$

With P_{opt} denoting the optimal P , the smallest invariant ellipsoid that contains the state $x(t)$ for all $t \geq 0$ is

$$\{ \psi \in \mathbf{R}^n \mid \psi^T P_{\text{opt}} \psi \leq 1 \}.$$

2.6 Conclusions

We have provided an introduction to optimization based on Linear Matrix Inequalities, and demonstrated its application in robust stability and performance analysis of uncertain systems.

Chapter 3

Robustness under Bounded Uncertainty with Phase Information

We consider uncertain linear systems where the uncertainties, in addition to being bounded, also satisfy constraints on their phase. In this context, we define the “phase-sensitive structured singular value” (PS-SSV) of a matrix, and show that sufficient (and sometimes necessary) conditions for stability of such uncertain linear systems can be rewritten as conditions involving PS-SSV. We then derive upper bounds for PS-SSV, computable via convex optimization. We extend these results to the case where the uncertainties are structured (diagonal or block-diagonal, for instance).

3.1 Introduction

Consider the model of a control system with uncertainty, shown in Fig. 3.1. Here $P(s)$ is the transfer function of a stable linear system, and Δ is a stable operator that represents the “uncertainties” that arise from various sources such as modeling errors, neglected or unmodeled dynamics or parameters, etc. As we noted in Section 2.3, such control system models have found wide acceptance in robust control.

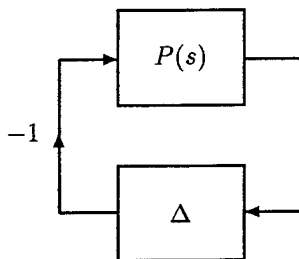


Figure 3.1: Closed-loop system

It is often the case that the uncertainty Δ is linear, time-invariant (LTI), and diagonal. In this case, unity-bounded L_2 -gain and passivity assumptions on Δ can be interpreted as knowledge on the frequency response of each diagonal entry δ_{ii} of Δ : the Nyquist plot of δ_{ii} lies inside the unit-disk and in the right-half complex plane, respectively. There are instances where it is appropriate to model δ_{ii} as having its Nyquist plot entirely contained within some acute sector, of aperture $2\theta < \pi$. Such a sector can be assumed, without loss of generality (via simple loop transformation), to be a proper subset of the right-half plane. This can occur when modeling is done from experimental data and the “Nyquist cloud” is better approximated by a sector portion of a disk than by a full disk; see Example 3 in §3.5. It can also occur when the uncertainty, due to several uncertain parameters, is “lumped” into a single dynamic uncertainty block; in this case, the approach presented in this chapter would result in significant computational savings in comparison with the

⁰Joint work with Professor A. L. Tits, University of Maryland, and Professor L. Lee, National Sun Yat-Sen Univ., Taiwan R.O.C.

direct approach; see Example 2 in §3.5. In both instances conservativeness can often be further reduced by allowing for frequency dependent sectors. Investigation of robust stability under such “uncertainty with phase information” is another objective of this chapter.

Uncertainty is often best represented by a set of full matrices (or block-diagonal matrices), and handling this situation in our framework necessitates a concept of “phase of a matrix”. Several authors have proposed such concepts. In [PEM81], the “principal phases” of a matrix are defined as the arguments of the eigenvalues of the unitary part of its polar decomposition, and a “small phase theorem” is derived that holds under rather stringent conditions. Hung and MacFarlane, in [HM82], propose a “quasi-Nyquist decomposition” in which the phase information of a transfer matrix is obtained by minimizing a measure of misalignment between the input and output singular vectors. Finally, Owens, in [Owe84], uses the numerical range to characterize phase uncertainty in multivariable systems. The concept of phase we adopt here is related to that of [Owe84]. Our definition not only serves to characterize phase uncertainty in multivariable systems, but also provides a practical and tractable way of using uncertainty phase information in robustness analysis.

Thus, in this chapter, we consider the robust stability of the system in Fig. 3.1 when Δ is a block-diagonal LTI uncertainty that simultaneously satisfies constraints on its norm, and on its “phase”. In §3.2, we define the phase-sensitive structured singular value (PS-SSV), defining in the process the phase of a matrix. We then derive a condition for robust stability of the system in Fig. 3.1 in terms of the PS-SSV. It turns out that when the uncertainty is scalar, or made of diagonal scalar blocks, the PS-SSV-based condition on robust stability is both necessary and sufficient. Computing the PS-SSV exactly turns out to be an NP-hard problem. We therefore concentrate on computing an upper bound on the PS-SSV, in §3.3. In §3.4, we show that computation of this upper bound can be reformulated as a quasi-convex optimization problem; we discuss some schemes for its solution. In §3.5, we demonstrate our results via three numerical examples, and we conclude with §3.6. Many of the ideas developed in this chapter were adapted from earlier work by two of the coauthors and M. K. H. Fan, see [LTF89, LT92b, LT92a, Lee92]. Results closely related to those of §3.1 were obtained independently by Eszter and Hollot [EH97] for the case when the phase bounds amount to a passivity constraint on the uncertainty.

Notation. $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_e$ denote the sets of real numbers, nonnegative real numbers, and $\mathbb{R} \cup \{\infty\}$ (one point-compactification of $\mathbb{R}$), respectively. $\mathbb{C}, \mathbb{C}_+$ and $\mathbb{C}_{+e}$ denote the set of complex numbers, complex numbers with positive real part (i.e., the open right-half complex plane), and $\mathbb{C}_+ \cup \{\infty\}$ respectively. $\mathbf{H}_\infty$ denotes the set of scalar- or matrix-valued functions that are analytic and bounded in the open right half plane, and $\mathbf{RH}_\infty$ denotes the set of functions in $\mathbf{H}_\infty$ that are real rational. H^* denotes the complex-conjugate transpose of $H \in \mathbb{C}^{n \times n}$. For $H \in \mathbb{C}^{n \times n}$ satisfying $H = H^*$, the notation $H \geq 0$ ($H > 0$) means that H is positive-semidefinite (positive-definite).

3.2 The phase-sensitive structured singular value

3.2.1 Scalar Case

Let us first consider the case when both the LTI system and the LTI uncertainty in Fig. 3.1 have a single input and a single output. Thus $P(s)$ and $\Delta(s)$ are scalar transfer functions, and to emphasize this, we rename them as $p(s)$ and $\delta(s)$, respectively.

Let $\phi : \mathbb{C} \rightarrow (-\pi, \pi]$ be the usual phase of a complex scalar, with $\phi(0)$ defined to be 0. Note that $|\phi(\cdot)|$ is lower semicontinuous (and continuous outside every neighborhood of the origin). Given a complex scalar m and a real scalar $\theta \in [0, \pi]$, let $\mu_\theta(m)$ be defined by

$$\mu_\theta(m) = (\inf \{|\gamma| : |\phi(\gamma)| \leq \theta, 1 + \gamma m = 0\})^{-1},$$

if the set over which the infimum is taken is nonempty, and $\mu_\theta(m) = 0$ otherwise (i.e., $\mu_\theta(m) = |m|$ if $|\phi(m)| \geq \pi - \theta$, and 0 otherwise). Note that $\mu_\theta(m)$ is upper semicontinuous in (θ, m) .

Theorem 1 below shows that various properties of the closed-loop system depicted in Fig. 3.1 can be assessed from the knowledge of $\mu_\theta(p(s))$ on the imaginary axis, under various assumptions on δ .

Theorem 1 *Let $p \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, let $\theta : \mathbb{R}_e \rightarrow [0, \pi]$, and let*

$$\Delta_\theta = \{\delta \in \mathbf{H}_\infty : \delta \text{ is continuous on } \overline{\mathbb{C}_{+e}}, \|\delta\|_\infty \leq 1, |\phi(\delta(j\omega))| \leq \theta(\omega) \quad \forall \omega \in \mathbb{R}_e\}.$$

Suppose that

(a) $\sup_{\omega \in \mathbb{R}_e} \mu_{\theta(\omega)}(p(j\omega)) < 1$.

Then $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \Delta_\theta$ and, if θ is upper semicontinuous,

$$\sup_{\delta \in \Delta_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty. \quad (3.1)$$

Moreover, if θ is constant, then statements (a), (b) and (c) are equivalent, where (b) and (c) are as follows:

(b) $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \Delta_\theta$ and $\sup_{\delta \in \Delta_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty$.

(c) $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \mathbf{RH}_\infty \cap \Delta_\theta$ and $\sup_{\delta \in \mathbf{RH}_\infty \cap \Delta_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty$.

Proof: We first prove by contradiction that (a) implies that $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \Delta_\theta$. Thus suppose that, for some $\delta \in \Delta_\theta$, $(1 + \delta p)^{-1} \notin \mathbf{H}_\infty$. It follows from Cauchy's Principle of the Argument, using a simple homotopy (see, e.g., Lemma 1 in [Tit95, TB97] for details) that there exists $\alpha \in (0, 1]$ and $\hat{\omega} \in \mathbb{R}_e$ such that

$$1 + \alpha \delta(j\hat{\omega})p(j\hat{\omega}) = 0.$$

Since $\delta \in \Delta_\theta$, it is clear that $|\alpha \delta(j\hat{\omega})| \leq 1$ and $|\phi(\alpha \delta(j\hat{\omega}))| \leq \theta(\hat{\omega})$. Thus $\mu_{\theta(\hat{\omega})}(p(j\hat{\omega})) \geq 1$, a contradiction. To complete the proof of the first claim, suppose that θ is upper semicontinuous and, proceeding again by contradiction, suppose that $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \Delta_\theta$ but that, given any $\epsilon > 0$ there exist $\delta_\epsilon \in \Delta$ and $\omega_\epsilon \in \mathbb{R}$ such that

$$|1 + \delta_\epsilon(j\omega_\epsilon)p(j\omega_\epsilon)| < \epsilon.$$

Let $\gamma_\epsilon = \delta_\epsilon(j\omega_\epsilon)$ and note that, since $\delta_\epsilon \in \Delta$,

$$|\phi(\gamma_\epsilon)| \leq \theta(\omega_\epsilon).$$

Since $|\gamma_\epsilon| \leq 1$ it follows from compactness of the complex unit disk, continuity of p on $j\mathbb{R}_e$, lower semicontinuity of $|\phi|$ and upper semicontinuity of θ that there exists $\hat{\gamma} \in \overline{\mathbb{C}_{+e}}$ and $\hat{\omega} \in \mathbb{R}_e$ such that $|\hat{\gamma}| \leq 1$, $|\phi(\hat{\gamma})| \leq \theta(\hat{\omega})$ and $1 + \hat{\gamma}p(j\hat{\omega}) = 0$. Thus $\mu_{\theta(\hat{\omega})}(p(j\hat{\omega})) \geq 1$, a contradiction. To complete the proof of the theorem, first note that the implication (b) $\Rightarrow$ (c) holds trivially. Suppose now that θ is constant. The implication (a) $\Rightarrow$ (b) has just been proven. It thus remains to show that (c) $\Rightarrow$ (a). We again use contradiction. Thus assume that

$$\sup_{\omega \in \mathbb{R}_e} \mu_\theta(p(j\omega)) \geq 1.$$

We show that, given any $\epsilon > 0$, there exists $\hat{\delta} \in \Delta_\theta \cap \mathbf{RH}_\infty$ and $\hat{\omega} \in \mathbb{R}_e$ such that $|1 + \hat{\delta}(j\hat{\omega})p(j\hat{\omega})| < \epsilon$, a contradiction. Let $\tilde{\omega} \in \mathbb{R}_e$ be such that $\mu_\theta(p(j\tilde{\omega})) \geq 1$ (since p is continuous on $j\mathbb{R}_e$ and $\mu_\theta(\cdot)$ is upper semicontinuous, such $\tilde{\omega}$ always exists). Thus, for some $\tilde{\gamma} \in \mathbb{C}_+$, with $|\tilde{\gamma}| \leq 1$ and $|\phi(\tilde{\gamma})| \leq \theta$, $1 + \tilde{\gamma}p(j\tilde{\omega}) = 0$. Note that, if $\theta = 0$, the claim holds trivially (take $\hat{\delta}(s) = \tilde{\gamma}$ for all s); thus assume that $\theta > 0$. Since p is continuous on $j\mathbb{R}_e$, there exist $\hat{\omega} \in \mathbb{R} \setminus \{0\}$ and $\hat{\gamma} \in \{\gamma \in \mathbb{C}_+ : |\gamma| < 1, |\phi(\gamma)| < \theta\}$ such that $|1 + \hat{\gamma}p(j\hat{\omega})| < \epsilon$. It is shown in Appendix A that, under these conditions, there exists $\hat{\delta} \in \Delta_\theta \cap \mathbf{RH}_\infty$ such that $\hat{\delta}(j\hat{\omega}) = \hat{\gamma}$. This completes the contradiction argument. $\square$

Remark 3.2.1 The upper semicontinuity assumption on θ is indeed needed in order for (3.1) (uniform robust stability) to follow, as shown by the following example. Let $p(s) = 2(s + 1)/(s + 2)$. Let $\hat{\omega}$ be the frequency at which the phase of $p(j\omega)$ is largest (in the first quadrant) and let $\hat{\phi}$ be its value. Define θ by $\theta(\hat{\omega}) = \pi/2$ and $\theta(\omega) = \pi - \hat{\phi}$ for all other ω . It is readily checked that $\mu_{\theta(\omega)}(p(j\omega)) = 0$ for all ω but that $1 + p(j\omega)\delta(j\omega)$ can be made arbitrarily close to 0 in the neighborhood of $\hat{\omega}$.

Remark 3.2.2 The sufficiency part of Theorem 1 can be extended to handle more general uncertainty sets. See remark immediately following Theorem 3.

We leave open the question of necessity of condition (a) of Theorem 1 under relaxed assumptions on θ . It may be necessary, e.g., to require that $\theta(\omega)$ not approach zero too fast.

3.2.2 The matrix case with structure

Phase and phase-sensitive structured singular value

As a first step toward extending the results of §3.2.1 to matrix-valued P and Δ , we propose a concept of phase of a matrix.

Given a complex matrix Γ , let $\mathcal{N}(\Gamma)$ be its numerical range, i.e.,

$$\mathcal{N}(\Gamma) = \{x^* \Gamma x : x \in \partial B\} \subset \mathbb{C}$$

where $\partial B = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$ and $\|\cdot\|_2$ is the Euclidean norm. This set is known to be convex. The following definition is a slight modification of that used in [Lee92].

Definition 1 Let $\Gamma \neq 0$ be a complex square matrix such that $0 \notin \text{int}\mathcal{N}(\Gamma)$. The median phase $\text{MP}(\Gamma)$ of Γ is the angle, with a range of $(-\pi, \pi]$, between the positive real axis and the ray bisecting the smallest sector containing $\mathcal{N}(\Gamma)$. The phase spread $\text{PS}(\Gamma)$ of Γ is half the angle of this sector (see Fig. 3.2). We define $\text{MP}(0)$ and $\text{PS}(0)$ to be 0.

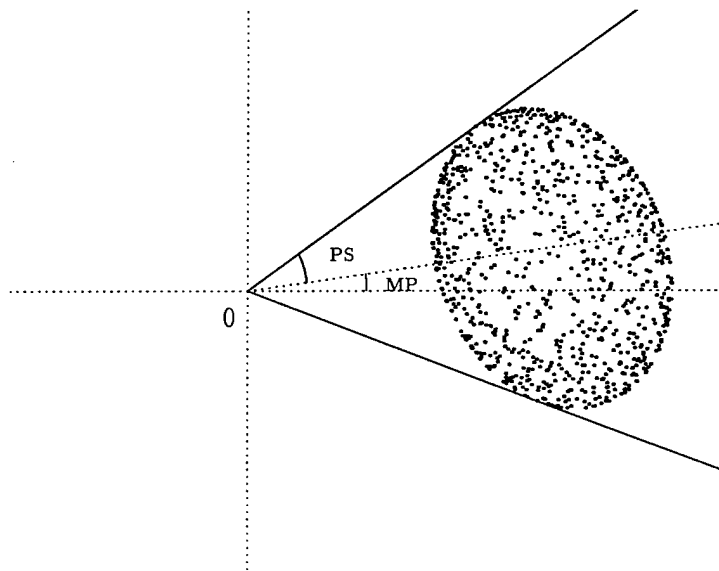


Figure 3.2: Numerical range, median phase and phase spread.

Thus $\text{MP}(\Gamma) \in (-\pi, \pi]$ and $\text{PS}(\Gamma) \in [0, \pi/2]$. Below we will refer to the pair $(\text{MP}(\Gamma), \text{PS}(\Gamma))$ as phase information of Γ . If $0 \in \text{int}\mathcal{N}(\Gamma)$, there is no phase information for Γ .

Note that in the case of a complex number $a = \rho e^{j\phi}$ with $\rho > 0$ and $\phi \in (-\pi, \pi]$, the phase information of a is $(\phi, 0)$. Also, the phase information of a matrix is invariant under multiplication of the matrix by a positive number, and if Γ is Hermitian positive semidefinite, its phase information is $(0, 0)$. Finally, the phase information of a matrix is invariant under unitary similarity transformations (since the numerical range is).

Median phase and phase spread are related to the concept of principal phases introduced by Postlethwaite et al. in [PEM81]. Namely, for any square complex matrix Γ ,

$$\text{MP}(\Gamma) - \text{PS}(\Gamma) \leq \psi_{\min}(\Gamma) \leq \psi_{\max}(\Gamma) \leq \text{MP}(\Gamma) + \text{PS}(\Gamma)$$

where $\psi_{\min}(\Gamma)$ and $\psi_{\max}(\Gamma)$ are the minimum and maximum principal phases of Γ , respectively. This result, stated differently, was obtained by Owens [Owe84] (who also used the term “phase spread”).

For any matrix Γ with $0 \notin \text{int}\mathcal{N}(\Gamma)$, $\mathcal{N}(e^{-j\text{MP}(\Gamma)}\Gamma) \subset \mathbb{C}_+$. In other words, we can rotate the numerical range of any matrix Γ for which $0 \notin \text{int}\mathcal{N}(\Gamma)$ so that it is contained in the right-half complex plane. With this in mind, we restrict our attention in the sequel to matrices Γ with $\Gamma + \Gamma^* \geq 0$ (or equivalently $\mathcal{N}(\Gamma) \subset \mathbb{C}_+$).

For such matrices, we next give alternate characterizations of the phase information; these will serve us well in our derivation of stability tests in the sequel.

Given Γ with $\Gamma + \Gamma^* \geq 0$, of particular interest is the smallest sector (i.e., one that subtends the smallest angle at the origin) in the right-half plane, symmetric about the real axis, that contains $\mathcal{N}(\Gamma)$. (The interest stems from the fact that in the sequel, we will consider uncertainties Δ whose numerical range is known to lie in such symmetric sectors at every frequency.) Let $2\Phi(\Gamma)$ be the angle subtended by this sector at the origin. Evidently (see Fig. 3.2),

$$\Phi(\Gamma) = \max \{ \text{MP}(\Gamma) + \text{PS}(\Gamma), -(\text{MP}(\Gamma) - \text{PS}(\Gamma)) \}. \quad (3.2)$$

We then have the following alternate characterization for $\Phi(\Gamma)$.

Lemma 3.2.1 *Let $\Gamma \in \mathbb{C}^{n \times n}$, with $\Gamma + \Gamma^* \geq 0$. Then,*

$$\Phi(\Gamma) = \cot^{-1} \left(\sup \left\{ b : \Gamma + \Gamma^* - \frac{\beta}{j}(\Gamma - \Gamma^*) \geq 0 \ \forall \beta \in \{-b, b\} \right\} \right). \quad (3.3)$$

Proof: From Fig. 3.2 it is clear that, for $b > \cot(\Phi(\Gamma))$ (i.e., $b \neq 0$ and $b^{-1} < \tan(\Phi(\Gamma))$), there exists $\hat{v} \in \mathbb{C}^n$ such that

$$\text{Re}(\hat{v}^* \Gamma \hat{v}) < b |\text{Im}(\hat{v}^* \Gamma \hat{v})|$$

i.e., for some $\beta \in \{-b, b\}$,

$$\Gamma + \Gamma^* - \frac{\beta}{j}(\Gamma - \Gamma^*) \not\geq 0.$$

Moreover, with $b = \cot(\Phi(\Gamma)) \in (-\infty, \infty)$, it is clear from the figure that, for all $v \in \mathbb{C}^n$,

$$\text{Re}(v^* \Gamma v) \geq \beta |\text{Im}(v^* \Gamma v)| \ \forall \beta \in \{-b, b\}$$

i.e.,

$$\Gamma + \Gamma^* - \frac{\beta}{j}(\Gamma - \Gamma^*) \geq 0 \ \forall \beta \in \{-b, b\}.$$

Finally, if $\Phi(\Gamma) = 0$ then $\mathcal{N}(\Gamma)$ is a subset of the negative imaginary axis, i.e., $\Gamma = \Gamma^* \geq 0$, and thus the matrix inequality in (3.3) holds for every finite β . $\square$

Remark 3.2.3 *It is easy to verify that for any Γ satisfying $\Gamma + \Gamma^* \geq 0$ with $\Phi(\Gamma) > 0$, we have*

$$\Gamma + \Gamma^* - \frac{\beta}{j}(\Gamma - \Gamma^*) \geq 0 \quad \text{for all } \beta \in [-\cot \Phi(\Gamma), \cot \Phi(\Gamma)].$$

Lemma 3.2.2 Φ is lower semicontinuous over $\{\Gamma : \Gamma + \Gamma^* \geq 0\}$.

Proof: Let $\theta \in [0, \pi/2]$ and let $\{\Gamma_k\} \rightarrow \hat{\Gamma}$, with $\Gamma_k + \Gamma_k^* \geq 0$ for all k , and $\limsup \Phi(\Gamma_k) \leq \theta$. We show that $\Phi(\hat{\Gamma}) \leq \theta$, proving the claim. If $\theta = \pi/2$ the result is obvious. Thus suppose $\theta \in [0, \pi/2)$ and let $\epsilon \in (0, \pi/2 - \theta]$. For k large enough, $\Phi(\Gamma_k) \leq \theta + \epsilon$, and thus, taking the cotangent of both side of (3.3) ($\theta + \epsilon > 0$), for k large enough,

$$\Gamma_k + \Gamma_k^* - \frac{\beta}{j}(\Gamma_k - \Gamma_k^*) \geq 0 \quad \forall \beta \in [-\cot(\theta + \epsilon), \cot(\theta + \epsilon)].$$

It follows that

$$\hat{\Gamma} + \hat{\Gamma}^* - \frac{\beta}{j}(\hat{\Gamma} - \hat{\Gamma}^*) \geq 0 \quad \forall \beta \in [-\cot(\theta + \epsilon), \cot(\theta + \epsilon)],$$

i.e., (again using (3.3)), that $\cot(\Phi(\hat{\Gamma})) \geq \cot(\theta + \epsilon)$, i.e., $\Phi(\hat{\Gamma}) \leq \theta + \epsilon$. Since this holds for arbitrarily small $\epsilon > 0$, the claim follows. $\square$

With an eye towards issues of robust stability with respect to possibly block-structured uncertainty, we now extend the definition of μ_θ to handle block-diagonal structures. Given positive integers k_i , $i = 1, \dots, \ell$, such that $\sum k_i = n$, we define the set of block-diagonal matrices with block sizes k_i as

$$\Gamma = \{\text{diag}(\Gamma_1, \dots, \Gamma_\ell) : \Gamma_i \in \mathbb{C}^{k_i \times k_i}\}. \quad (3.4)$$

We next define Γ_Θ as the following phase-constrained subset of Γ :

$$\Gamma_\Theta = \{\text{diag}(\Gamma_1, \dots, \Gamma_\ell) : \Gamma_i \in \mathbb{C}^{k_i \times k_i}, \Phi(\Gamma_i) \leq \theta_i\}, \quad (3.5)$$

where $\Theta = (\theta_1, \dots, \theta_\ell)$ with, for $i = 1, \dots, \ell$, $\theta_i \in [0, \pi/2]$. Note that $\Gamma + \Gamma^* \geq 0$ for all $\Gamma \in \Gamma_\Theta$.

Definition 2 The phase-sensitive structured singular value of $M \in \mathbb{C}^{n \times n}$ with respect to Γ_Θ is given by

$$\mu_{\Gamma_\Theta}(M) = (\inf\{\bar{\sigma}(\Gamma) : \Gamma \in \Gamma_\Theta, \det(I + \Gamma M) = 0\})^{-1}$$

if $\det(I + \Gamma M) = 0$ for some $\Gamma \in \Gamma_\Theta$, and $\mu_{\Gamma_\Theta}(M) = 0$ otherwise.

Properties of μ_{Γ_Θ}

Unlike the "standard" mixed μ , μ_{Γ_Θ} is clearly not invariant under change of sign of its argument. Thus, in particular, it is not always larger than the spectral radius ρ (complex μ) or the real spectral radius $\rho_{\mathbb{R}}$. On the other hand it is clear that

$$\rho_{\mathbb{R}_-}(M) \leq \mu_{\Gamma_\Theta}(M) \leq \mu(M), \quad (3.6)$$

where, for any complex matrix M ,

$$\rho_{\mathbb{R}_-}(M) = \max\{\lambda : -\lambda \text{ is a negative, real eigenvalue of } M\},$$

with $\rho_{\mathbb{R}_-}(M) = 0$ if M has no negative, real eigenvalues. This leads to the following easily derived characterization of μ_{Γ_Θ} . (Note that $\rho_{\mathbb{R}_-}$ is upper semicontinuous, which justifies the "max".)

Theorem 2

$$\mu_{\Gamma_\Theta}(M) = \max_{\Gamma \in \Gamma_\Theta, \bar{\sigma}(\Gamma) \leq 1} \rho_{\mathbb{R}_-}(\Gamma M) = \max_{\Gamma \in \Gamma_\Theta, \bar{\sigma}(\Gamma) \leq 1} \rho_{\mathbb{R}_-}(M\Gamma). \quad (3.7)$$

Like the standard mixed μ , μ_{Γ_Θ} is invariant under similarity scaling of its argument by matrices that commute with the elements of the uncertainty set, i.e., given any nonsingular matrix $D = \text{diag}(d_1 I_{k_1}, \dots, d_\ell I_{k_\ell})$,

$$\mu_{\Gamma_\Theta}(M) = \mu_{\Gamma_\Theta}(DM D^{-1}).$$

In general however, μ_{Γ_Θ} is clearly not invariant under pre- or post-multiplication of its argument by a unitary matrix in Γ_Θ .

Next, it is readily verified that $\mu_{\Gamma_\Theta}(M)$ is monotonic nondecreasing in each of the components of Θ and, using lower semicontinuity of Φ (Lemma 3.2.2), that $\mu_{\Gamma_\Theta}(M)$ is upper semicontinuous in (Θ, M) . Finally, the following result holds.

Proposition 1 Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, and let $\Theta \in [0, \pi/2]^\ell$. Then

$$\sup_{\omega \in \mathbb{R}_e} \mu_{\Gamma_\Theta}(P(j\omega)) = \sup_{s \in \overline{\mathbb{C}_{+e}}} \mu_{\Gamma_\Theta}(P(s)).$$

Proof: We show that the following statements are equivalent:

(a)

$$\sup_{\omega \in \mathbb{R}_e} \mu_{\Gamma_\Theta}(P(j\omega)) < 1,$$

(b)

$$(I + \Gamma P)^{-1} \in \mathbf{H}_\infty \quad \forall \Gamma \in \Gamma_\Theta, \bar{\sigma}(\Gamma) \leq 1,$$

(c)

$$\sup_{s \in \overline{\mathbb{C}_{+e}}} \mu_{\Gamma_\Theta}(P(s)) < 1.$$

Since μ_{Γ_Θ} is positive homogeneous, the claim then follows from the equivalence of (a) and (c). We first show by contradiction that (a) $\Rightarrow$ (b). Thus let $\hat{\Gamma} \in \Gamma_\Theta$, with $\bar{\sigma}(\hat{\Gamma}) \leq 1$, be such that $(I + \hat{\Gamma}P)^{-1} \notin \mathbf{H}_\infty$. As in the proof of Theorem 1, it follows from Cauchy's Principle of the Argument that there exist $\alpha \in (0, 1]$ and $\hat{\omega} \in \mathbb{R}_e$ such that

$$\det(I + \alpha \hat{\Gamma}P(j\hat{\omega})) = 0.$$

Since $\alpha \hat{\Gamma} \in \Gamma_\Theta$ and $\bar{\sigma}(\alpha \hat{\Gamma}) \leq 1$, this implies that $\mu_{\Gamma_\Theta}(P(j\hat{\omega})) \geq 1$, a contradiction. Concerning the implication (b) $\Rightarrow$ (c), if there exists $\hat{s} \in \overline{\mathbb{C}_{+e}}$ be such that $\mu_{\Gamma_\Theta}(P(\hat{s})) \geq 1$, then there exists $\hat{\Gamma} \in \Gamma_\Theta$, with $\bar{\sigma}(\hat{\Gamma}) \leq 1$, such that $\det(I + \hat{\Gamma}P(\hat{s})) = 0$, contradicting (b). Finally, the implication (c) $\Rightarrow$ (a) holds trivially. $\square$

The small- $\mu_{\mathbf{r}_\Theta}$ theorem

Given any $\Theta : \mathbb{R}_e \rightarrow [0, \pi/2]^\ell$, we define

$$\Delta_\Theta = \{\Delta \in \mathbf{H}_\infty : \Delta \text{ is continuous on } \overline{\mathbb{C}_{+e}}, \|\Delta\|_\infty \leq 1, \Delta(j\omega) \in \Gamma_{\Theta(\omega)} \quad \forall \omega \in \mathbb{R}_e\}.$$

Theorem 3 Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, let $\Theta : \mathbb{R}_e \rightarrow [0, \pi/2]^\ell$. Suppose that

(a) $\sup_{\omega \in \mathbb{R}_e} \mu_{\Gamma_{\Theta(\omega)}}(P(j\omega)) < 1$.

Then $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \Delta_\Theta$, and if Θ is upper semicontinuous, then

$$\sup_{\delta \in \Delta_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty.$$

Moreover, if Θ is constant, then (a) is equivalent to

(b) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \Delta_\Theta$ and $\sup_{\delta \in \Delta_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty$.

Proof: The implication (a) $\Rightarrow$ (b) is proved as in Theorem 1 with $\det(I + \Delta P)$ replacing $1 + \delta p$. Concerning the implication (b) $\Rightarrow$ (a), note that, if θ is constant and (a) does not hold, then (since P is continuous over $\overline{\mathbb{C}_{+e}}$ and $\mu_{\mathbf{r}_\Theta}$ is upper semicontinuous) there exists, among others, a constant (complex) $\Delta \in \Delta_\Theta$ and some $\omega \in \mathbb{R}_e$ such that $\det(I + \Delta P(j\omega)) = 0$, contradicting (b). $\square$

Remark 3.2.4 Again, the sufficiency part of Theorem 3 can be extended to handle more general uncertainty sets. For example, consider the uncertainty set

$$\tilde{\Delta}_\Theta = \{\Delta \in \mathbf{H}_\infty : \Delta \text{ is continuous on } \overline{\mathbb{C}_{+e}}, U(\omega)\Delta(j\omega) \in \Gamma_{\Theta(\omega)}, \bar{\sigma}(\Delta_i(j\omega)) \leq d_i(\omega), i = 1, \dots, \ell, \quad \forall \omega \in \mathbb{R}_e\},$$

where $d_i : \mathbb{R}_e \rightarrow [0, \infty)$, $i = 1, \dots, \ell$, $\Theta : \mathbb{R}_e \rightarrow [0, \pi/2]^\ell$, and $U(\omega) = \text{diag}(u_1(\omega)I_{k_1}, \dots, u_\ell(\omega)I_{k_\ell})$, with $u_i : \mathbb{R}_e \rightarrow \{z \in \mathbb{C} : |z| = 1\}$. Then, it is easy to show that if

$$\sup_{\omega \in \mathbb{R}_e} \mu_{\Gamma_{\Theta(\omega)}}(\text{diag}(d_i(\omega)I_{k_i})U(\omega)^*P(j\omega)) < 1,$$

then, $(1 + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \tilde{\Delta}_\Theta$; and that if, in addition, d_i and Θ are upper semicontinuous, and U is continuous, then

$$\sup_{\Delta \in \tilde{\Delta}_\Theta} \|(1 + \Delta P)^{-1}\| < \infty.$$

Again we will leave open the question of necessity of condition (a) of Theorem 3 under relaxed assumptions on Θ . On the other hand, even for constant Θ , it is unclear in general whether, if (a) does not hold, there exists $\Delta \in \Delta_\Theta$ real on the real axis (which must be the case if Δ is the transfer function of a real impulse response) such that $(I + \Delta P)^{-1} \notin \mathbf{H}_\infty$ or $\|(I + \Delta P)^{-1}\|_\infty$ is arbitrarily large. In the case of purely diagonal uncertainty structures, though, this is the case even with the additional requirement that Δ be rational. In other words the following holds.

Theorem 4 Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, let $\Theta \in [0, \pi/2]^\ell$, and suppose $k_i = 1$, $i = 1, \dots, \ell$ ($= n$), i.e., suppose that Γ_Θ consists of diagonal matrices. The following statements are equivalent.

(a) $\sup_{\omega \in \mathbb{R}_e} \mu_{\mathbf{r}_\Theta}(P(j\omega)) < 1$.

(b) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \Delta_\Theta$ and $\sup_{\Delta \in \Delta_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty$.

(c) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{RH}_\infty \cap \Delta_\Theta$ and $\sup_{\Delta \in \mathbf{RH}_\infty \cap \Delta_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty$.

(The proof of the implication (c) $\Rightarrow$ (a) is exactly along the lines of that of the corresponding implication in Theorem 1.)

3.3 Upper Bounds on $\mu_{\mathbf{r}_\Theta}$

So far, we have seen definitions of $\mu_{\mathbf{r}_\Theta}$, and how conditions on $\mu_{\mathbf{r}_\Theta}$ give sufficient (and sometimes necessary) conditions for uniform robust stability. In this section, we will concern ourselves with the numerical computation of $\mu_{\mathbf{r}_\Theta}$.

Computing $\mu_{\mathbf{r}_\Theta}$ exactly is equivalent to finding the global minimum of a nonconvex optimization problem, and we are not aware of any efficient solution methods for it. Therefore, we will not attempt to compute $\mu_{\mathbf{r}_\Theta}$ directly; instead, we will derive numerically computable upper bounds on $\mu_{\mathbf{r}_\Theta}$, which will give, in turn, sufficient conditions for robust stability.

3.3.1 The matrix case with structure

Computing $\mu_{\Gamma_\Theta}(m)$ for a scalar m is trivial. We then consider the problem of computing an upper bound on $\mu_{\Gamma_\Theta}(M)$, when M is a matrix, and Γ is assumed to have some structure, that is, it is required to belong to the set Γ_Θ . Let

$$\{S : S = \{\text{diag}(s_1 I_{k_1}, \dots, s_\ell I_{k_\ell}) : s_i > 0\},$$

and, given $\Theta = (\theta_1, \dots, \theta_\ell)$ with $\theta_i \in [0, \pi/2]$, $i = 1, \dots, \ell$, let

$$\mathcal{B}_\Theta \triangleq \left\{ B : \begin{array}{l} B = \text{diag}(\beta_1 I_{k_1}, \dots, \beta_\ell I_{k_\ell}), \beta_i \in \mathbb{R}, i = 1, \dots, \ell, \\ \text{with } \beta_i \in [-\cot \theta_i, \cot \theta_i] \text{ when } \theta_i > 0 \end{array} \right\}.$$

We then have the following lemmas.

Lemma 3.3.1 *Let $\Gamma \in \Gamma_\Theta$, with $\Gamma^* \Gamma \leq I$. Then, Γ satisfies, for every $R, S \in \mathcal{S}$, and $B \in \mathcal{B}_\Theta$,*

$$\begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} \geq 0. \quad (3.8)$$

Proof: Consider any $\Gamma \in \Gamma_\Theta$ satisfying $\Gamma^* \Gamma \leq I$. Since Γ commutes with every $R \in \mathcal{R}$, we have

$$R - \Gamma^* R \Gamma \geq 0. \quad (3.9)$$

Next, since $\Gamma \in \Gamma_\Theta$, in view of Lemma 3.2.1 and of the remark following it, and since $-B \in \mathcal{B}_\Theta$ whenever $B \in \mathcal{B}_\Theta$, we have for every $B \in \mathcal{B}_\Theta$,

$$\Gamma + \Gamma^* - j(B\Gamma - \Gamma^*B) \geq 0.$$

Since Γ , every $S \in \mathcal{R}$ and every $B \in \mathcal{B}$ commute with each other, we then have

$$S\Gamma + \Gamma^*S - j(BST - \Gamma^*SB) \geq 0. \quad (3.10)$$

From (3.9) and (3.10), we conclude that every $\Gamma \in \Gamma_\Theta$ with $\Gamma^* \Gamma \leq I$ satisfies, for every $R, S \in \mathcal{S}$ and $B \in \mathcal{B}_\Theta$,

$$R - \Gamma^* R \Gamma + (I - jB)S\Gamma + \Gamma^*S(I + jB) \geq 0,$$

which is equivalent to (3.8). $\square$

Theorem 5 *Let $\Gamma \in \Gamma_\Theta$, with $\Gamma^* \Gamma \leq I$. If there exists some $R \in \mathcal{S}$, $S \in \mathcal{S}$, and $B \in \mathcal{B}_\Theta$, such that*

$$M^*RM - R - (S(I + jB)M + M^*(I - jB)S) < 0, \quad (3.11)$$

then $\det(I + \Gamma M) \neq 0$.

Remark 3.3.1 *Theorem 5 constitutes a special case of the general stability theorem for systems with uncertainties described by integral quadratic constraints or IQCs [MR97, Theorem 1]. In particular, Theorem 5 can be viewed as a sufficient condition for the well-posedness of a feedback interconnection of a constant matrix with a constant phase- and norm-bounded uncertainty in the feedback loop. Since there are no dynamics involved, a direct linear algebraic proof can be given, which we present next for the sake of completeness.*

Proof: Rewriting (3.11) as

$$\begin{bmatrix} -M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} -M \\ I \end{bmatrix} < 0, \quad (3.12)$$

we now proceed by contradiction. Suppose that $\det(I + \Gamma M) = 0$. Then for some nonzero $v \in \mathbb{C}^n$, we have $(I + \Gamma M)v = 0$. Defining $u = Mv$, we have $v = -\Gamma u$. Now, from (3.12), we have

$$v^* \begin{bmatrix} -M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} -M \\ I \end{bmatrix} v < 0,$$

i.e.,

$$\begin{bmatrix} -u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} < 0.$$

But from Lemma 3.3.1, we must have

$$\begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} \geq 0,$$

which yields

$$(-u)^* \begin{bmatrix} I \\ \Gamma \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} I \\ \Gamma \end{bmatrix} (-u) \geq 0,$$

i.e.,

$$\begin{bmatrix} -u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - jB)S \\ S(I + jB) & -R \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} \geq 0,$$

which is a contradiction. $\square$

We can use Theorem 5 to derive an upper bound $\hat{\mu}_{r_\Theta}(M)$ on $\mu_{r_\Theta}(M)$. Suppose that for some $\gamma > 0$, R and S in $\mathcal{S}$, and some B in $\mathcal{B}_\Theta$, we have

$$M^*RM - \gamma^2 R - (S(I + jB)M + M^*(I - jB)S) < 0.$$

Then, it can be shown with a little algebra that γ is an upper bound on $\mu_{r_\Theta}(M)$. We therefore have the following upper bound on $\mu_{r_\Theta}(M)$.¹

Corollary 1 *Let $M \in \mathbb{C}^{n \times n}$. Then $\mu_{r_\Theta}(M) \leq \hat{\mu}_{r_\Theta}(M)$, where*

$$\hat{\mu}_{r_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^*RM - \gamma^2 R - S(I + jB)M - M^*(I - jB)S < 0 \\ \gamma > 0, R, S \in \mathcal{S}, B \in \mathcal{B}_\Theta \end{array} \right\}. \quad (3.13)$$

The conclusion of Corollary 1 represents one of the central contributions of the chapter—we now have an upper bound for μ_{r_Θ} , which, as we shall see in §3.4, can be numerically computed quite efficiently, using convex optimization techniques.

Remark 3.3.2 *It is easily shown that, for any scalar m , $\hat{\mu}_{r_\Theta}(m) = \mu_{r_\Theta}(m)$.*

3.3.2 An off-axis circle-criterion interpretation

As was done in [Doy85] and in §V of [FTD91] in the context of the “classical” mixed μ , it is possible to obtain the phase-sensitive μ upper bound by optimizing the complex μ upper bound over a set of disk uncertainties. Consider a “block-diagonal disk uncertainty set”, i.e., a set of block diagonal matrices such that each block ranges over a certain “hyperdisk”, namely over the image of $\{\Gamma_i : \bar{\sigma}(\Gamma_i) \leq 1\}$ under a certain linear fractional transformation. A “complex- μ ” type upper bound is readily obtained corresponding to such uncertainty blocks. Clearly, if the uncertainty set covers $\{\Gamma \in \Gamma_\Theta : \bar{\sigma}(\Gamma) \leq 1\}$, then this upper bound is also an upper bound for μ_{r_Θ} . Below we show that minimizing this bound over a certain family of such transformations yields precisely the bound given by Theorem 5 and Corollary 1.

Given $S \in \mathcal{S}$ and $B \in \mathcal{B}_\Theta$, let

$$T \triangleq \begin{bmatrix} F(I + F^*F)^{-1/2} & I \\ (I + F^*F)^{-1/2} & 0 \end{bmatrix},$$

where $F = S(I + jB)$. It is readily checked that the “lower” linear fractional transformation $F_l(T, -M)$ is well defined for any M , that the “upper” linear fractional transformation $F_u(T, \Gamma)$ is well defined whenever $\bar{\sigma}(\Gamma) \leq 1$, and that (provided $\bar{\sigma}(\Gamma) \leq 1$)

$$F_l(T, -M) = (F - M)(I + F^*F)^{-1/2},$$

$$F_u(T, \Gamma) = ((I + F^*F)^{1/2} - \Gamma F)^{-1} \Gamma.$$

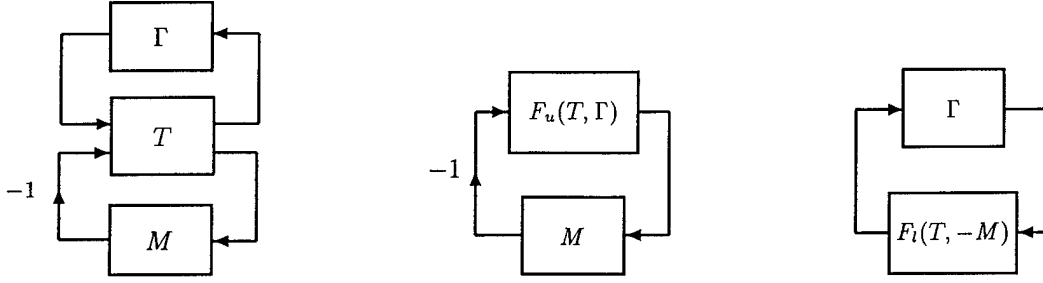


Figure 3.3: Three “equivalent” block diagrams.

Consequently, the systems in the three block diagrams of Fig. 3.3 are all equivalent in the sense that each one is well-posed if and only if the other two are.

For the sake of geometric intuition consider now the case of a diagonal (rather than block diagonal) structure, say, $F = \text{diag}(f_i)$, with $f_i = s_i(1 + j\beta_i)$, $s_i > 0$ and $|\beta_i| \leq \cot \theta_i$ when $\theta_i > 0$. Let $\mathbf{\Gamma}$ be the set of complex diagonal matrices Γ with $\bar{\sigma}(\Gamma) \leq 1$. The image of $\mathbf{\Gamma}$ under the linear fractional transformation $F_u(T, \cdot)$ is given by

$$F_u(T, \mathbf{\Gamma}) = \left\{ \text{diag} \left(\left(\gamma^{-1} \sqrt{1 + |f_i|^2} - f_i \right)^{-1} \right) : |\gamma| \leq 1 \right\}.$$

It is straightforward to check that each diagonal entry ranges over a circle of radius $\sqrt{1 + |f_i|^2} = \sqrt{1 + s_i^2(1 + \beta_i^2)}$ centered at $\bar{f}_i = s_i(1 - j\beta_i)$ (see Fig. 3.4). It follows that, for each $s_i > 0$ and each β_i with $|\beta_i| \leq \cot \theta_i$ when $\theta_i > 0$,

$$\{\Gamma \in \mathbf{\Gamma}_\Theta : \bar{\sigma}(\Gamma) \leq 1\} \subset F_u(T, \mathbf{\Gamma}), \quad (3.14)$$

which shows that each diagonal “disk” entry of $F_u(T, \mathbf{\Gamma})$ “covers” the corresponding entry in the uncertainty set of interest, $\{\Gamma \in \mathbf{\Gamma}_\Theta : \bar{\sigma}(\Gamma) \leq 1\}$. Conversely, it is easy to show that any disk that covers a diagonal entry in the uncertainty set $\{\Gamma \in \mathbf{\Gamma}_\Theta : \bar{\sigma}(\Gamma) \leq 1\}$ must be the corresponding diagonal entry of $F_u(T, \mathbf{\Gamma})$ for some T : it is easy to solve “backwards” for s_i and β_i , given the center and radius of the disk.

The same inclusion (3.14) holds in the general (block diagonal) case. Indeed

$$T^{-1} = \begin{bmatrix} 0 & (I + F^*F)^{1/2} \\ I & -F \end{bmatrix}.$$

and simple algebra shows that, for any Γ with $\bar{\sigma}(\Gamma) \leq 1$, $F_l(T^{-1}, \Gamma)$ is well defined and

$$F_u(T, F_l(T^{-1}, \Gamma)) = \Gamma,$$

and thus it is enough to show that

$$F_l(T^{-1}, \{\Gamma \in \mathbf{\Gamma}_\Theta : \bar{\sigma}(\Gamma) \leq 1\}) \subset \mathbf{\Gamma}.$$

To see that the latter inclusion holds, assume without loss of generality that $\ell = 1$ (full matrix uncertainty), i.e., $F = fI$, with $f = s(1 + j\beta)$, $s > 0$, $|\beta| \leq \cot \theta$ when $\theta > 0$, and let $\Gamma \in \mathbf{\Gamma}_\Theta$ with $\bar{\sigma}(\Gamma) \leq 1$. It remains to show that

$$\bar{\sigma}(F_l(T^{-1}, \Gamma)) \leq 1,$$

or, equivalently,

$$I - (1 + |f|^2)(I + \bar{f}\Gamma^*)^{-1}\Gamma^*\Gamma(I + f\Gamma)^{-1} \geq 0,$$

i.e., via a congruence transformation,

$$(I + \bar{f}\Gamma^*)(I + f\Gamma) - (1 + |f|^2)\Gamma^*\Gamma \geq 0,$$

¹This result was first reported, in a slightly different form, in [LTF89]. For the case when $\theta_i = \pi/2$, $i = 1, \dots, \ell$ (passive uncertainty), it is a special case of a result obtained independently by Eszter and Holot [EH97].

i.e.

$$(I - \Gamma^* \Gamma) + (f\Gamma + \bar{f}\Gamma^*) \geq 0.$$

Since this clearly holds for any $\Gamma \in \Gamma_\Theta$ with $\bar{\sigma}(\Gamma) \leq 1$, (3.14) holds in the general case as claimed.

A sufficient condition for

$$\det(I + \Gamma M) \neq 0 \quad \forall \Gamma \in \Gamma_\Theta, \bar{\sigma}(\Gamma) \leq 1,$$

i.e., for $\mu_{r_\Theta}(M) < 1$, is thus that

$$\det(I + \Gamma M) \neq 0 \quad \forall \Gamma \in F_u(T, \mathbf{B}\Gamma).$$

Since the second and third block diagrams in Fig. 3.3 are equivalent, the latter holds if and only if

$$\det(I - F_l(T, -M)\Gamma) \neq 0 \quad \forall \Gamma \in \mathbf{B}\Gamma,$$

i.e.,

$$\mu(F_l(T, -M)) < 1,$$

and a sufficient condition for this is that, for some $R \in \mathcal{S}$,

$$\bar{\sigma}(RF_l(T, -M)R^{-1}) < 1. \quad (3.15)$$

Since S and B commute, letting $M_R = RMR^{-1}$, we get

$$RF_l(T, -M)R^{-1} = (F - M_R)(I + F^*F)^{-1/2}.$$

It follows that (3.15) holds if, and only if

$$((F - M_R)(I + F^*F)^{-1/2})^*((F - M_R)(I + F^*F)^{-1/2}) < I,$$

i.e., if, and only if,

$$(F^* - M_R^*)(F - M_R) < I + F^*F,$$

which holds if, and only if,

$$M_R^*M_R - I - F^*M_R - M_R^*F < 0,$$

which is equivalent to the condition given in Theorem 5.

3.3.3 Some special cases

It is instructive to study the application of Theorem 5 and Corollary 1 to some special cases for the set Γ_Θ . These cases are encountered more often in practice; also, for some of these special cases, we can relate our results to those from literature.

Bounded passive uncertainty

We consider first the case when the Γ_Θ consists of unstructured or full matrices (i.e., $\ell = 1$) with a known bound on their maximum singular value, and whose phase is known to be $\pi/2$ or less. This situation arises when the uncertainty Δ is *passive* and *bounded*. If Δ were scalar (i.e., $k_1 = 1$), this would mean that the Nyquist plot of Δ is in a semicircle of known radius that lies in the right-half complex plane, shown in in Fig. 5(a).

In this case, $\mathcal{B}_\Theta = \{0\}$ and $\mathcal{S}$ consists of positive multiples of the identity matrix. Therefore, from Corollary 1, we have

$$\hat{\mu}_{r_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} rM^*M - \gamma^2 rI - s(M + M^*) < 0 \\ \gamma > 0, r > 0, s > 0 \end{array} \right\},$$

which further simplifies to

$$\hat{\mu}_{r_\Theta}(M) = (\max\{0, \inf\{\lambda_{\max}(M^*M - c(M + M^*)) : c > 0\}\})^{1/2},$$

where $\lambda_{\max}$ denotes the largest eigenvalue of the corresponding Hermitian matrix.

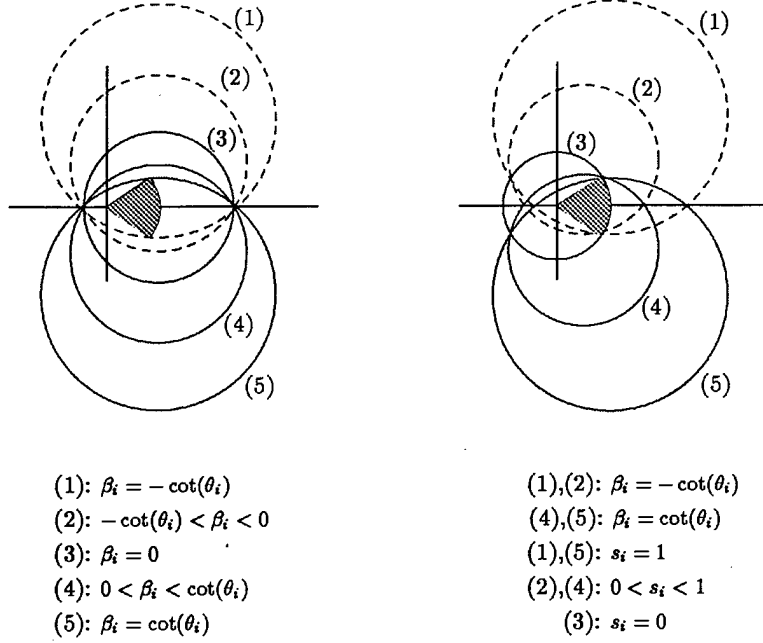


Figure 3.4: Covering the uncertainty with off-axis circles. The figure on the left shows the covering of the i th diagonal phase-bounded uncertainty with disk-uncertainties obtained by loop transforming the unit-disk with $f_i = (1 + j\beta_i)$ for various values of β_i . The figure on the right shows the covering with disk-uncertainties obtained by loop transforming the unit-disk with $f_i = s_i(1 \pm j \cot \theta_i)$ for various values of s_i .

Bounded, constant, Hermitian, positive-definite uncertainty

We next consider the case when Δ is a constant, Hermitian, positive-definite matrix, with a known bound on its maximum singular value. If the uncertainty were scalar (i.e., $k_1 = 1$), this would mean that the Nyquist plot of the uncertainty is simply a point in a sub-interval of the positive real axis, as shown in Fig. 5(b).

In this case the set $\mathcal{B}_\Theta$ consists of arbitrary real multiples of the identity, while $\mathcal{S}$ consists of positive multiples of the identity. Therefore, we have,

$$\hat{\mu}_{r_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} rM^*M - \gamma^2 rI - s(1 + jb)M - M^*(1 - jb)s < 0 \\ \gamma > 0, r > 0, s > 0, b \in \mathbb{R} \end{array} \right\},$$

which further simplifies to

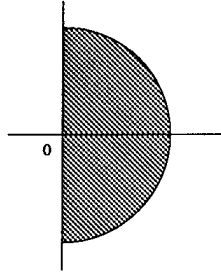
$$\hat{\mu}_{r_\Theta}(M) = (\max\{0, \inf\{\lambda_{\max}(M^*M - (c + jd)M - M^*(c - jd)) : c > 0, d \in \mathbb{R}\}\})^{1/2}.$$

It is instructive to consider other special cases of the instances considered above, when the uncertainty is *diagonal*, so that $k_1 = \dots = k_\ell = 1$.

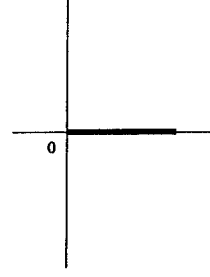
Diagonal bounded passive uncertainty

Suppose that the Nyquist plot of each of the diagonal uncertainties is known to lie a half-disk such as the one shown in Fig. 5(a). In other words the uncertainty is diagonal, passive and bounded. In this case, the set $\mathcal{B}_\Theta = \{0\}$ and the set $\mathcal{S}$ consists of diagonal positive-definite matrices. Here

$$\hat{\mu}_{r_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^*RM - \gamma^2 R - SM - M^*S < 0 \\ \gamma > 0, R, S > 0 \text{ and diagonal} \end{array} \right\}.$$



(a) Bounded passive uncertainty:
The Nyquist plot is known to lie in the shaded region.



(b) Positive, real uncertainty: The
Nyquist plot is known to lie in a sub-
interval of the positive real axis.

Figure 3.5: Various special cases

Diagonal, bounded, positive, constant real uncertainty

Finally, we consider the case when each of the diagonal uncertainties is a constant unknown parameter, known only to lie in some sub-interval of the positive real axis such as the one shown in Fig. 5(b). Such uncertainties are often called parametric uncertainties. Here, the set $\mathcal{B}_\Theta$ consists of arbitrary real diagonal matrices, while $\mathcal{S}$ consists of diagonal positive-definite matrices. Thus,

$$\hat{\mu}_{r_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^* R M - \gamma^2 R - S(I + jB)M - M^*(I - jB)S < 0 \\ \gamma > 0, R, S, B \text{ real and diagonal, } R > 0, S > 0 \end{array} \right\}. \quad (3.16)$$

Remark 3.3.3 This case of bounded diagonal real uncertainty is well-studied in the literature, usually under the name of “real- μ ” analysis; see for example, [FTD91, Bal95]. The problem considered in these references is the computation of $\mu_{\mathbb{R}}(M)$, which is defined as

$$\mu_{\mathbb{R}}(M) \triangleq \begin{cases} \left(\inf \left\{ \bar{\sigma}(\Gamma) : \begin{array}{l} \Gamma \text{ is diagonal and real,} \\ \det(I + \Gamma M) = 0 \end{array} \right\} \right)^{-1} & \text{if } \det(I + \Gamma M) = 0 \text{ for some} \\ & \text{diagonal and real } \Gamma, \\ 0 & \text{otherwise} \end{cases}$$

We point out that $\mu_{\mathbb{R}}(M)$ is different from $\mu_{r_\Theta}(M)$ with $\Theta = 0$ (for ease of reference, we will call the latter quantity $\mu_{\mathbb{R}_+}$ and its upper bound given in (3.16) by $\hat{\mu}_{\mathbb{R}_+}$). The difference between $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{R}_+}$ is that with $\mu_{\mathbb{R}_+}$, the uncertainty is required to be nonnegative, unlike with the definition of $\mu_{\mathbb{R}}$. For this reason, we will refer to $\mu_{\mathbb{R}}$ as “two-sided real- μ ”, while we will call $\mu_{\mathbb{R}_+}$ “one-sided real- μ ”.

The upper bound for the two-sided real- μ from [FTD91] and [Bal95] can be easily adapted via a loop transformation to yield an upper bound for the one-sided real- μ . This upper bound on $\mu_{\mathbb{R}_+}$ is just

$$\tilde{\mu}_{\mathbb{R}_+}(M) = \inf \left\{ \gamma : \begin{array}{l} -2\gamma S - S(I + jB)M - M^*(I - jB)S < 0 \\ \gamma > 0, S, B \text{ diagonal, } S > 0 \end{array} \right\}. \quad (3.17)$$

Remarkably, computing $\tilde{\mu}_{\mathbb{R}_+}$ using (3.17) has the same complexity as computing $\hat{\mu}_{\mathbb{R}_+}$ using (3.16). Extensive numerical simulations suggest that this upper bound is tighter than the bound (3.16). We should note however that the bound (3.17) does not extend to the case of general phase-bounded uncertainty considered in this chapter.

Finally, we note that it is possible to adapt $\hat{\mu}_{\mathbb{R}_+}$, the upper bound for the one-sided real μ , to yield an upper bound for the two-sided real μ . This upper bound on $\mu_{\mathbb{R}}$ turns out to be

$$\inf \left\{ \gamma : \begin{array}{l} M^* (3R + 2S) M - \gamma^2 R + \gamma((S - R + jB)M - M^*(2S - R - jB)) < 0 \\ \gamma > 0, S, B \text{ diagonal, } S > 0 \end{array} \right\}.$$

However, we know of no efficient way of computing this upper bound.

3.4 Computing $\sup_{\omega} \hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega))$

From Theorem 3 in §3.2.2, it should be clear that the computation of $\sup_{\omega \in \mathbb{R}_+} \mu_{\Gamma_{\Theta}(\omega)}(P(j\omega))$, which we shall denote by $\mathcal{M}_{\Theta}(P)$, is of considerable interest. For reasons pointed out at the beginning of §3.3, we will consider instead the problem of computing $\sup_{\omega} \hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega))$, which we shall denote by $\hat{\mathcal{M}}_{\Theta}(P)$. Since $\hat{\mathcal{M}}_{\Theta}(P) \geq \mathcal{M}_{\Theta}(P)$, computing $\hat{\mathcal{M}}_{\Theta}(P)$ will enable us to state sufficient conditions for the stability of the system in Fig. 3.1.

For each frequency ω , the quantity $\hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega))$, defined in Corollary 1, can be computed as the solution to a quasi-convex optimization problem. There are several ways of showing this; we will demonstrate one method. For convenience, we let $M = P(j\omega)$.

Recall that $\hat{\mu}_{\Gamma_{\Theta}}$ is given by (3.13). Let $T = BS$. Then the condition on B is equivalent to

$$\Lambda S > T > -\Lambda S,$$

where Λ is a constant diagonal matrix given by

$$\Lambda = \text{diag}(\cot(\theta_1(\omega)) I_{k_1}, \dots, \cot(\theta_{\ell}(\omega)) I_{k_{\ell}}).$$

Thus $\hat{\mu}_{\Gamma_{\Theta}}$ is given as the optimal value of γ obtained by solving the problem

$$\begin{aligned} & \text{minimize} && \gamma^2 \\ & \text{subject to} && \gamma^2 R > M^* R M - (S M + M^* S) - j(TM - M^* T) \\ & && \Lambda S > T > -\Lambda S, \\ & && R = \text{diag}(r_1 I_{k_1}, \dots, r_{\ell} I_{k_{\ell}}), \quad r_i > 0 \\ & && S = \text{diag}(s_1 I_{k_1}, \dots, s_{\ell} I_{k_{\ell}}), \quad s_i > 0 \\ & && T = \text{diag}(t_1 I_{k_1}, \dots, t_{\ell} I_{k_{\ell}}) \end{aligned} \tag{3.18}$$

With $\nu \triangleq \gamma^2$, the optimization variables in this problem are ν , R , S and T . Problem (3.18) is one of minimizing a linear objective ν , subject to constraints on ν , R , S and T that are convex (in fact, linear matrix inequalities²) in R , S and T for fixed ν , and vice versa. It can be shown that problem (3.18) is a quasi-convex optimization problem [BEFB94]. Much work has been done lately on problems such as (3.18): it is well-known that such problems have polynomial worst-case complexity; moreover, very efficient algorithms and software tools are available for their solution [GN95, GDN95].

Next, we have the following obvious lower bound on $\hat{\mathcal{M}}_{\Theta}(P)$.

Lemma 3.4.1 *Let $\Omega = \{\omega_0, \omega_1, \dots, \omega_N\}$ be a set of frequencies. Then, $\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega)$, defined as*

$$\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega) \triangleq \max_i \left\{ \hat{\mu}_{\Gamma_{\Theta}(\omega_i)}(P(j\omega_i)) \right\},$$

satisfies $\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega) \leq \hat{\mathcal{M}}_{\Theta}(P)$, i.e., it is a lower bound on $\hat{\mathcal{M}}_{\Theta}(P)$.

In order to compute $\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega)$, we need to solve $N + 1$ quasi-convex optimization problems of the form (3.18). Of course, the number and choice of frequencies comprising Ω determines how tight a bound $\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega)$ is.

Remark 3.4.1 *The lower bound given by Lemma 3.4.1 suffers from a possible shortcoming: It is known that in general, $\hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega))$ may be discontinuous as a function of ω . Specifically, $\hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega))$ might only be upper semicontinuous, and therefore we have no guarantees with the convergence of the lower bound $\hat{\mathcal{M}}_{\Theta}^{\text{lb}}(P, \Omega)$ to $\hat{\mathcal{M}}_{\Theta}(P)$ even if N , the number of elements of Ω , tends to ∞ (but a scheme analogous to that proposed in [LTD96] might be applicable). However, in most engineering applications (as we will see in §3.5), this does not pose a serious problem.*

²A linear matrix inequality or an LMI is a matrix inequality of the form $F(x) = F_0 + \sum_{i=1}^m x_i F_i > 0$ or $F(x) \geq 0$, where F_i are given Hermitian matrices, and the x_i s are the real optimization variables.

It is also possible to compute upper bounds on $\hat{\mathcal{M}}_{\Theta}(P)$ using state-space methods. The basic idea is this. $\hat{\mathcal{M}}_{\Theta}(P) \leq \gamma$ if and only if there exist $R : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, $S : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ and $T : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ such that for every $\omega \in \mathbb{R}_e$, the following constraints are satisfied (the dependence of Λ on ω is now made explicit).

$$\begin{aligned}
(i) \quad & \gamma^2 R(j\omega) > P(j\omega)^* R(j\omega) P(j\omega) - (S(j\omega) P(j\omega) + P(j\omega)^* S(j\omega)) - \\
& \quad j(T(j\omega) P(j\omega) - P(j\omega)^* T(j\omega)), \\
(ii) \quad & \Lambda(\omega) S(j\omega) > T(j\omega), \\
(iii) \quad & T(j\omega) > -\Lambda(\omega) S(j\omega), \\
(iv) \quad & R(j\omega) = \text{diag}(r_1(j\omega) I_{k_1}, \dots, r_\ell(j\omega) I_{k_\ell}), \quad r_i(j\omega) > 0 \\
(v) \quad & S(j\omega) = \text{diag}(s_1(j\omega) I_{k_1}, \dots, s_\ell(j\omega) I_{k_\ell}), \quad s_i(j\omega) > 0 \\
(vi) \quad & T(j\omega) = \text{diag}(t_1(j\omega) I_{k_1}, \dots, t_\ell(j\omega) I_{k_\ell})
\end{aligned} \tag{3.19}$$

It can be shown [CTB97b] that the constraints in (3.19) hold for some γ if and only if they hold for some real-rational transfer functions $\hat{R}$, $\hat{S}$ and $\hat{T}$. This fact can be combined with the Positive-Real (PR) lemma [Wil71, AV73] to write down LMIs whose feasibility is equivalent to conditions (i)–(v) (see for example, [Bal95, CTB97b] for an illustration of this procedure). Thus, a sufficient condition for the feasibility of problem (3.19) can be recast as an LMI feasibility problem. A bisection scheme can then be used to compute an upper bound for $\hat{\mathcal{M}}_{\Theta}(P)$. It is also possible to avoid the bisection scheme altogether, by recasting the upper bound computation problem as a single generalized eigenvalue minimization problem; see [BW].

3.5 Numerical examples

We demonstrate on a few examples the application of stability tests based on the PS-SSV.

3.5.1 Example 1: Stability of a flexible structure

We consider the stability of a planar truss structure, with a model adapted from the one presented in [IO91]. The truss structure has sixteen free nodes, each with two degrees of freedom; thus it exhibits thirty-two flexible modes. We assume that the first mode is exactly modeled as a linear time-invariant system, with transfer function p given by

$$p(s) = \frac{0.3927^2 s^2}{s^2 + 2(0.0075)(131)s + 131^2}.$$

The remaining modes are modeled as a linear time-invariant uncertainty, with transfer function denoted by $\delta(s)$. It is known that δ is stable, and satisfies

$$|\delta(j\omega)| \leq 0.3370, \quad \text{Re } \delta(j\omega) \geq 0, \quad \text{for all } \omega \in \mathbb{R}, \tag{3.20}$$

that is, δ is passive, and has an $\mathbf{H}_{\infty}$ norm bound of 0.3370. A linear time-invariant controller c with transfer function

$$c(s) = \frac{2.38s^5 + 33.18s^4 + 40842.00s^3 + 489341.01s^2 + 203926.51s + 489289.16}{s^5 + 15.15s^4 + 10927.81s^3 + 163193.36s^2 + 587196.79s + 434923.70}$$

has been designed to stabilize $p(s)$, placing the poles at -1 , -4 and -10 . The robust stability question then is whether the controller stabilizes $p + \delta$.

The block diagram of the system is shown in Fig. 6(a). The system redrawn in our analysis framework is shown in Fig. 6(b), where $g = c/(1 + pc)$. The magnitude and real part of g are shown in Fig. 3.7.

From an inspection of these plots, and the properties of δ given in (3.20), we conclude that:

- The small gain theorem does not prove stability of the system in Fig. 6(b), since the $\mathbf{H}_{\infty}$ norm of g exceeds $1/0.3370$.
- The passivity theorem does not prove stability of the system in Fig. 6(b), since g is not strictly passive (the real part of $g(j\omega)$ is nonpositive for some ω).

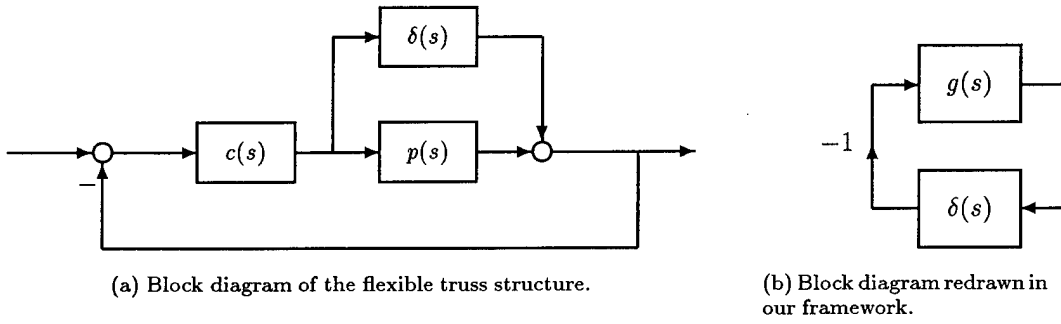


Figure 3.6: Example 1: Models of the flexible truss structure.

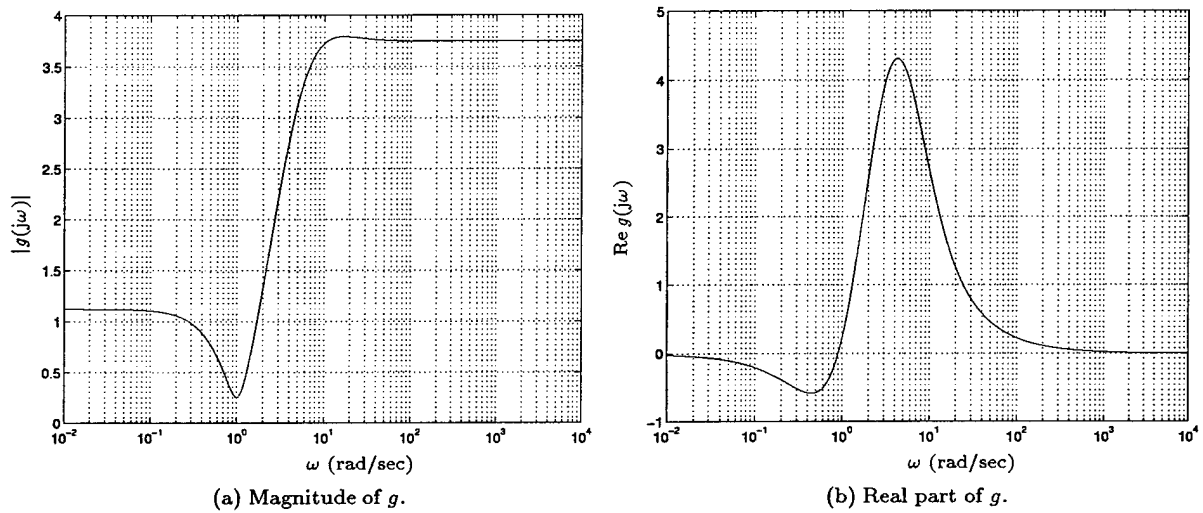


Figure 3.7: Example 1: Frequency response of g .

However, the analysis techniques presented in this chapter do prove uniform robust stability. A plot of $\mu_{\mathbf{r}_\Theta}(g(j\omega))$ is shown in Fig. 3.8. (Since g is a scalar transfer function, $\mu_{\mathbf{r}_\Theta}$ is trivial to compute.) Since $\sup_{\omega \in \mathbb{R}_+} \mu_{\mathbf{r}_\Theta}(g(j\omega)) < 1/0.3370$, the system in Fig. 6(b) is indeed uniformly robustly stable.

3.5.2 Example 2: Analysis of parametric systems

We next consider the problem of uniform robust stability of the closed-loop system shown in Fig. 9(a). P is the parameter-dependent plant, with transfer function given by

$$P(s) = \text{diag}(p_1(s), p_2(s)), \quad p_i(s) = \frac{a_i(s + b_i)}{s^2 + 2c_i s + 1}, \quad a_i \in [0, 1], \quad b_i \in [1, 2], \quad c_i \in [1, 2],$$

and C is the controller with the transfer function

$$C(s) = 0.3 \begin{bmatrix} \frac{s^2 + s - 1}{(s + 1)(s + 2)} & \frac{s + 1}{s + 2} \\ 1 & \frac{s + 1}{s + 10} \end{bmatrix}.$$

The problem now is to ascertain the stability of this system for all allowable values of the parameters.

Fig. 3.10 shows the values of the frequency response of p_i , over a number of allowable parameter values, at a sample list of frequencies. This figure indicates that each p_i is passive, and has a frequency response

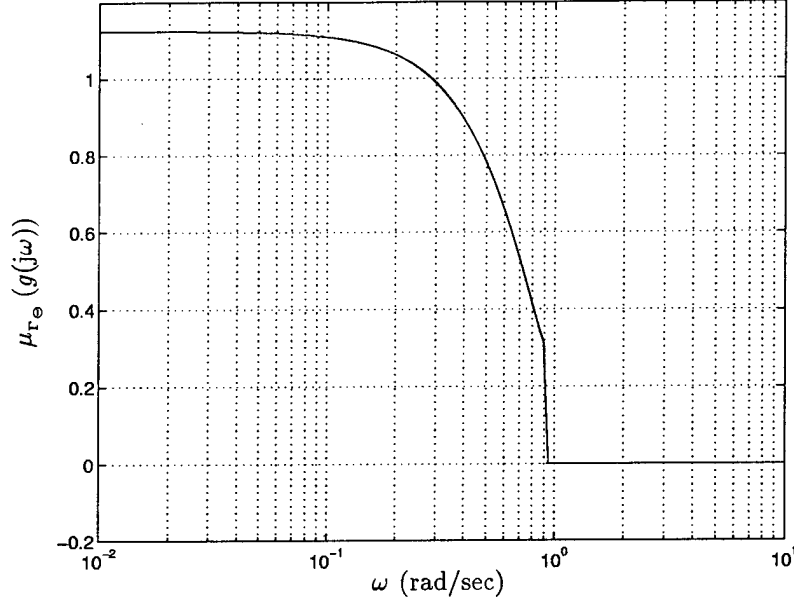


Figure 3.8: Example 1: The PS-SSV of $g(j\omega)$ versus ω .

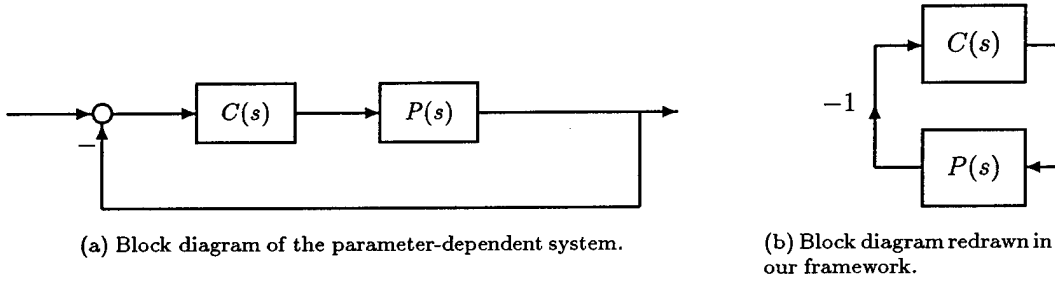


Figure 3.9: Example 2: Stability analysis of a parameter-dependent system.

which can be described as satisfying certain magnitude and phase constraints. Fig. 3.11 shows the magnitude and phase constraints on each of the terms.

This problem can be posed in our PS-SSV framework, as shown in Fig. 9(b). The uniform robust stability condition is

$$\sup_{\omega \in \mathbb{R}_+} \bar{\sigma}(P(j\omega)) \mu_{r_{\Theta(\omega)}} \left(e^{j\psi(\omega)} C(j\omega) \right) < 1, \quad (3.21)$$

where $\psi(\omega)$ and the entries of $\Theta(\omega)$ are plotted against ω in Figs. 11(b) and 11(c), respectively. For convenience we let $\tilde{C}(j\omega) = e^{j\psi(\omega)} C(j\omega)$. A plot of $\hat{\mu}_{r_{\Theta(\omega)}}(\tilde{C}(j\omega))$ is shown in Fig. 3.12, in solid lines. For reference, the optimally scaled maximum singular value of $\tilde{C}(j\omega)$ is shown in dotted lines; this is an upper bound on $\mu(\tilde{C}(j\omega))$, which can be thought of as an upper bound on PS-SSV that does not use the phase information. Since condition (3.21) holds, the system in Fig. 9(b) is indeed uniformly robustly stable. Note that, since $\bar{\sigma}(P(j0)) = 2$, the bound on PS-SSV that does not use the phase information does not yield this conclusion.

Remark 3.5.1 *There is a more direct method of analyzing parameter-dependent systems, namely “real- μ ” analysis (see [FTD91]). It is of interest to compare PS-SSV-based stability methods with real- μ methods.*

Let us consider the question of whether the system in Fig. 9(b) is uniformly robustly stable. The answer is affirmative in the PS-SSV framework if $\sup_{\omega \in \mathbb{R}_+} \bar{\sigma}(P(j\omega)) \hat{\mu}_{r_{\Theta(\omega)}}(\tilde{C}(j\omega))$ is less than one. Checking this numerically, from the discussion in §3.4 (in particular, Lemma 3.4.1), requires the solution of N LMI

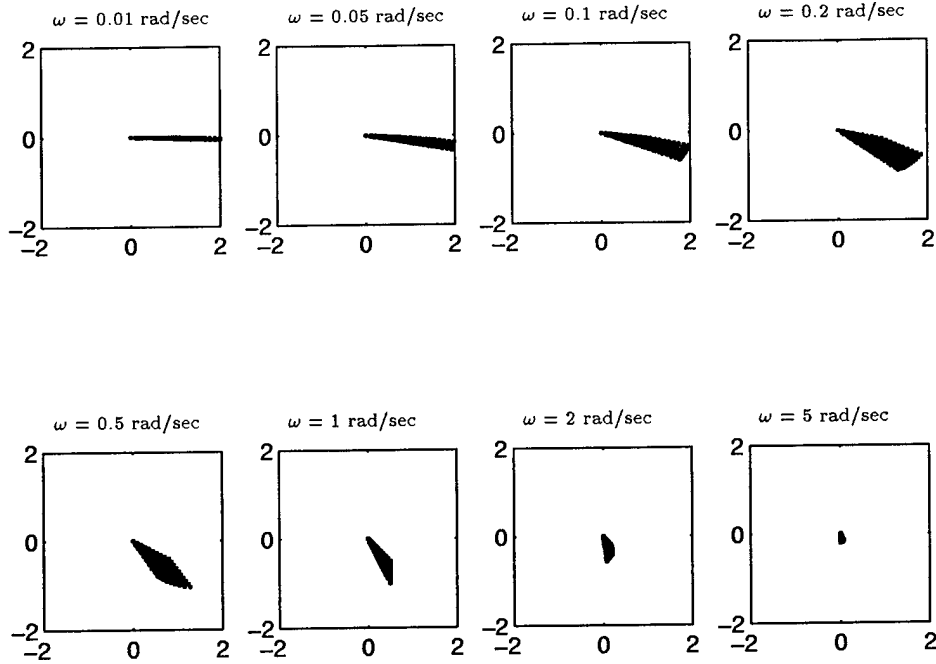


Figure 3.10: Example 2: Frequency response of each p_i at a number of frequencies.

feasibility problems, one for each frequency. Let us consider one such feasibility problem. The variables in this problem are diagonal 2×2 matrices R , S and T . Thus, the number of scalar variables is 6. There is one LMI constraint of size 2×2 , and 6 scalar constraints.

When the uniform robust stability of the same system is posed in the real- μ framework of [FTD91], we once again have to solve an LMI feasibility problem at each frequency. Here the variables in each problem are diagonal 6×6 matrices $D = D^T$ and $G = G^T$ (see [FTD91] for details); thus the number of scalar optimization variables is 12. There is one LMI constraint of size 6×6 , and 6 scalar constraints.

For the problem of uniform robust stability with parametric uncertainties, PS-SSV-based tests are likely to be more conservative than real- μ tests. However, it should be clear from the number of variables and constraints that the amount of computation required by PS-SSV-based methods is less than that required by real- μ methods. For our example, empirical studies indicate that the computation required by real- μ methods is approximately 12 times that required by PS-SSV-based methods [VB95].³ Thus, the PS-SSV approach can be useful in analyzing parameter-dependent systems, albeit more conservatively, when the number of parameters is large.

3.5.3 Example 3: Experimentally measured matrix phase information

We consider an uncertain system as in Fig. 3.1, where the plant P is strictly proper (i.e., $P(\infty) = 0$), has two inputs, two outputs, and a state-space realization (A, B, C) with

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \frac{1}{7} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

We assume that the two-input two-output LTI uncertainty Δ has been experimentally measured at a number of frequencies. A scatter-plot of the phase information of $\Delta(j\omega)$ at a number of frequencies is shown in Fig. 13(a); a scatter-plot of the norm of $\Delta(j\omega)$ at a number of frequencies is shown in Fig. 13(b).

³In general, for an LMI problem with k variables and L LMI constraints of size $n_i \times n_i$, the computation required is dominated by $O(k^2 \sum_{i=1}^L n_i(n_i + 1)/2)$.

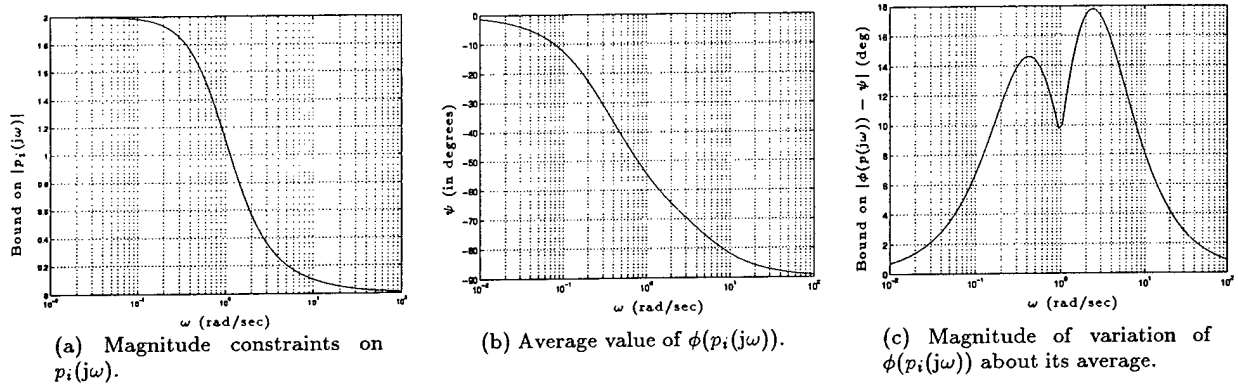


Figure 3.11: Example 2: Magnitude and phase constraints on $p_i(j\omega)$.

From the scatter plot shown in Fig. 13(a), we can determine continuous functions ϕ_{lb} and ϕ_{ub} such that for every frequency ω and Δ , the smallest sector containing $\mathcal{N}(\Delta(j\omega))$ is

$$\{z : z = re^{j\psi}, r \geq 0, \psi \in [\phi_{lb}(\omega), \phi_{ub}(\omega)]\}.$$

(These functions are shown in solid lines in Fig. 13(a).) Also, from Fig. 13(b), we can determine a function $d(\omega)$ such that for every frequency ω and Δ ,

$$\bar{\sigma}(\Delta(j\omega)) < d(\omega).$$

(This function is shown in a solid line in Fig. 13(b).)

Then, defining $\theta(\omega) = 0.5(\phi_{ub}(\omega) - \phi_{lb}(\omega))$ and $\psi(\omega) = 0.5(\phi_{ub}(\omega) + \phi_{lb}(\omega))$, we have that the system in Fig. 3.1 is uniformly robustly stable if

$$\sup_{\omega \in \mathbb{R}_+} \mu_{\Gamma_{\Theta}(\omega)} \left(e^{j\psi(\omega)} P(j\omega) \right) d(\omega) < 1,$$

where in the notation of §3.2.2, $k_1 = 2$, and $\Theta = (\theta)$. The upper bound $\hat{\mu}_{\Gamma_{\Theta}(\omega)}(P(j\omega)e^{j\psi(\omega)})$ from (3.13) is obtained for various ω by solving the optimization problem (3.18), and plotted in Fig. 3.14. Since $\sup_{\omega \in \mathbb{R}_+} \hat{\mu}_{\Gamma_{\Theta}(\omega)}(e^{j\psi(\omega)} P(j\omega)) d(\omega) < 1$, the system in Fig. 3.1 is indeed uniformly robustly stable.

3.6 Conclusions

The “phase-sensitive structured singular value” framework developed in this chapter provides an effective robustness analysis tool in various situations, e.g., in the case when the uncertainty, besides being (possibly block-structured and) small, is known to be passive.

Several issues have been left unresolved.

1. Under what “minimal” assumptions on $\theta(\cdot)$ are statements (a), (b) and (c) of Theorem 1 equivalent?
2. In the presence of non-scalar “full blocks”, and with Θ constant, are statements (a) and (b) in Theorem 3 equivalent to the analogue of statement (c) in Theorem 1, namely
(c) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{RH}_\infty \cap \Delta$ and $\sup_{\delta \in \mathbf{RH}_\infty \cap \Delta} \|(I + \Delta P)^{-1}\|_\infty < \infty$.
3. When is the upper bound $\hat{\mu}_{\Gamma_\Theta}$ defined in Corollary 1 of §3.3 equal to μ_{Γ_Θ} , in particular is it always equal to μ_{Γ_Θ} when $\ell = 1$ (full block uncertainty)?

The answer to some of these questions may be within reach.

The contributions in the chapter can be generally viewed as the following: When the uncertainty Δ in Fig. 3.1 is LTI, and when additional information on the phase of the frequency response of Δ is available, we have derived sufficient (and sometimes necessary) conditions for robust stability. A natural extension of

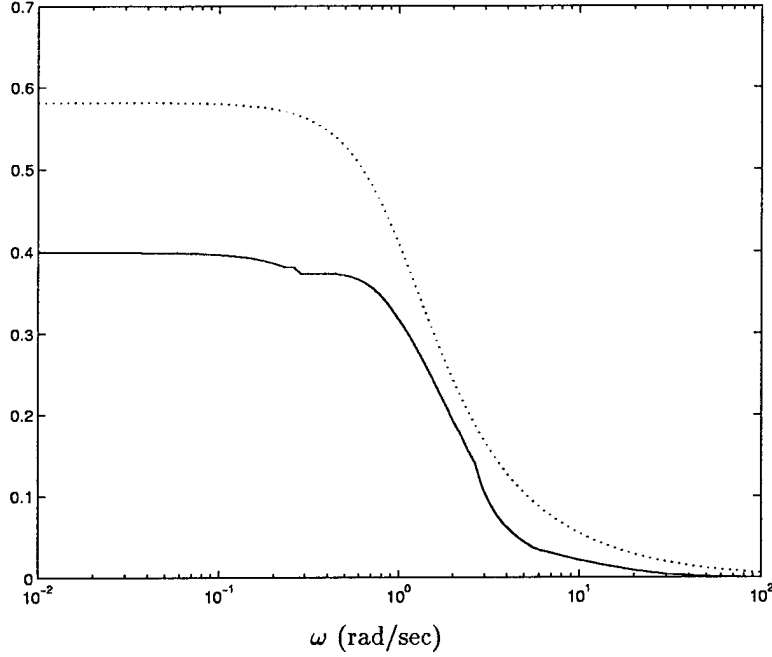


Figure 3.12: Example 2: The upper bound on $\mu_{\text{re}(\omega)}(\tilde{C}(j\omega))$ is plotted against ω in solid lines. The optimally scaled maximum singular value of $\tilde{C}(j\omega)$ is plotted against ω in dotted lines.

this problem considered in this chapter is the following. Consider for simplicity the case when Δ is a scalar uncertainty, and suppose that it is known that the Nyquist plot of Δ is restricted to lie in some region in the complex plane that can be described as the intersection of generalized disks (i.e., disks and half-spaces). Then, we can derive a sufficient robust stability condition by combining robust stability conditions for each generalized disk, just as we did to arrive at Theorem 5. As a further extension along these lines, consider the situation when the Nyquist plot of Δ is restricted to lie in some region in the complex plane that can be described as the union of sets which are themselves obtained as an intersection of generalized disks. (A classic example of such a region is the “butterfly” uncertainty set, described in [LT92a].) The techniques described in this chapter can be extended to handle these more general cases as well.

The focus of this chapter has been exclusively on uncertainties about which phase information is available. The techniques herein can be combined with other standard robustness analysis techniques such as complex or real- μ analysis, when phase information about only certain blocks of the uncertainty is available, leading to a new “mixed- μ ” paradigm. Finally, while the theory was developed for the continuous-time case, extension to discrete time is straightforward.

Appendix A

Proposition 2 *Let $\theta \in (0, \pi]$, let $\hat{\omega} \in \mathbb{R} \setminus \{0\}$, and let $\gamma \in \overline{\mathbb{C}_+}$ be such that $|\gamma| < 1$ and $|\phi(\gamma)| < \theta$. There exists $\delta \in \text{RH}_\infty$, continuous on $\overline{\mathbb{C}_{+e}}$, such that $\delta(j\hat{\omega}) = \gamma$ and such that $\|\delta\|_\infty < 1$ and $\sup_{\omega \in \mathbb{R}} |\phi(\delta(j\omega))| < \theta$.*

Proof: If $\gamma = 0$, simply let δ map $\overline{\mathbb{C}_+}$ to zero. Assume now $\gamma \neq 0$. Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{D}^\theta = \{z \in \mathcal{D} : \text{Re } z \geq 0, |\phi(z)| < \theta\}$. We first construct a non-rational mapping $\delta : \overline{\mathcal{D}} \rightarrow \mathbb{C}$, taking real values on the real axis, such that $\delta(\overline{\mathcal{D}})$ belongs to $\mathcal{D}^\theta$ and contains γ and $1/2 + j0$ in its interior. This map is selected from a one-parameter family of mappings $\tilde{\delta}^\lambda : \overline{\mathcal{D}} \rightarrow \mathbb{C}$, $\lambda \in (0, 1)$, constructed as the composition of two maps, i.e., $\tilde{\delta}^\lambda = \tilde{\delta}_2^\lambda \circ \tilde{\delta}_1^\lambda$.

First, for $\lambda \in (0, 1)$, the map $\tilde{\delta}_1^\lambda$, defined on $\overline{\mathcal{D}}$, is given by

$$\tilde{\delta}_1^\lambda(z) = -\frac{1 + \lambda z - (1 - 2\lambda z \cos \psi + (\lambda z)^2)^{1/2}}{1 + \lambda z + (1 - 2\lambda z \cos \psi + (\lambda z)^2)^{1/2}}, \text{ with } \sin(\psi/2) = \frac{\lambda}{2 - \lambda}.$$

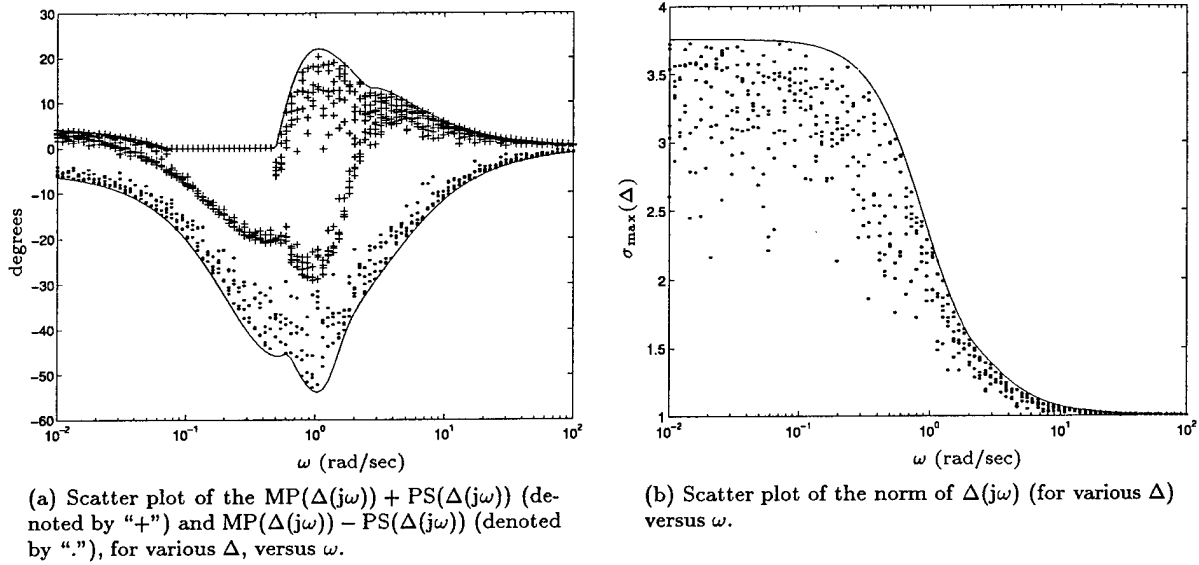


Figure 3.13: Example 3: Experimentally determined magnitude and phase characteristics of Δ .

For fixed λ , $\tilde{\delta}_1^\lambda$ maps $\mathcal{D}$ to the interior of a set such as the one depicted in Fig. 15(a).

Next $\tilde{\delta}_2^\lambda$, defined on $\tilde{\delta}_1^\lambda(\overline{\mathcal{D}})$, is given by

$$\tilde{\delta}_2^\lambda(w) = (w + (1 - \lambda))^{\theta/\pi}.$$

(In the definition of $\tilde{\delta}_1^\lambda$ and $\tilde{\delta}_2^\lambda$, given $\zeta = \rho e^{j\varphi}$, with $\rho \geq 0$, and $\varphi \in (-\pi, \pi]$, and given $p \in (0, 1]$, we set $\zeta^p = \rho^p e^{jp\varphi}$. In other words, the "cut" is taken along the negative real axis.) It is readily checked that, for every $\lambda \in (0, 1)$, $\tilde{\delta}^\lambda$ takes real values on the real axis. For fixed $\lambda \in (0, 1)$, $\tilde{\delta}^\lambda(\overline{\mathcal{D}})$ is as depicted in Fig. 15(b) (it belongs to $\mathcal{D}^\theta$).

As $\lambda \rightarrow 1$, the boundary of $\tilde{\delta}^\lambda(\overline{\mathcal{D}})$ uniformly approaches that of $\mathcal{D}^\theta$. As the next step, we select $\tilde{\delta} = \tilde{\delta}^{\lambda^*}$ where $\lambda^* \in (0, 1)$ is such that both γ and $1/2 + j0$ belong to the interior of $\tilde{\delta}^{\lambda^*}(\overline{\mathcal{D}})$. We next define $\hat{\delta}$ as a truncated Taylor series of $\tilde{\delta}$ about $1/2 + j0$, with the properties that $\hat{\delta}(\overline{\mathcal{D}})$ belongs to $\mathcal{D}^\theta$, and that γ belongs to $\hat{\delta}(\mathcal{D})$. The existence of such $\hat{\delta}$ is a direct consequence of the uniform convergence of the Taylor series. Since $\tilde{\delta}$ is real on the real axis, $\hat{\delta}$ is a polynomial with real coefficients. Further, a real-rational mapping $\check{\delta}$ is defined as the composition of the mapping $s \mapsto (s - 1)/(s + 1)$, which maps $\overline{\mathbb{C}_+}$ to $\overline{\mathcal{D}}$, with $\xi\hat{\delta}$, where $\xi \in (0, 1)$ is such that γ belongs to the boundary of $\xi\hat{\delta}(\mathcal{D})$. It is readily checked that the image under $\check{\delta}$ of the imaginary axis is this boundary. Also, since $\check{\delta}(\mathbb{C}_+)$ belongs to $\mathcal{D}^\theta$, it obviously is bounded in the right half plane. Finally, we let $\delta(s) = \check{\delta}(\frac{\omega}{\tilde{\omega}}s)$, where $\tilde{\omega} \in \mathbb{R} \cup \{\infty\}$ is such that $\check{\delta}(j\tilde{\omega}) = \gamma$. (In particular, if $\tilde{\omega} \in \{0, \infty\}$, γ is real (since $\check{\delta}$ is real rational) and $\delta(s) = \gamma$ for all s .) Clearly, δ has all the claimed properties. $\square$

Acknowledgement. The authors wish to thank Dr. Carlos Berenstein for his helpful suggestions in connection with Proposition 2, and Dr. Pramod Khargonekar for pointing out the non-obviousness of the necessity part of the small- μ theorem in the present context.

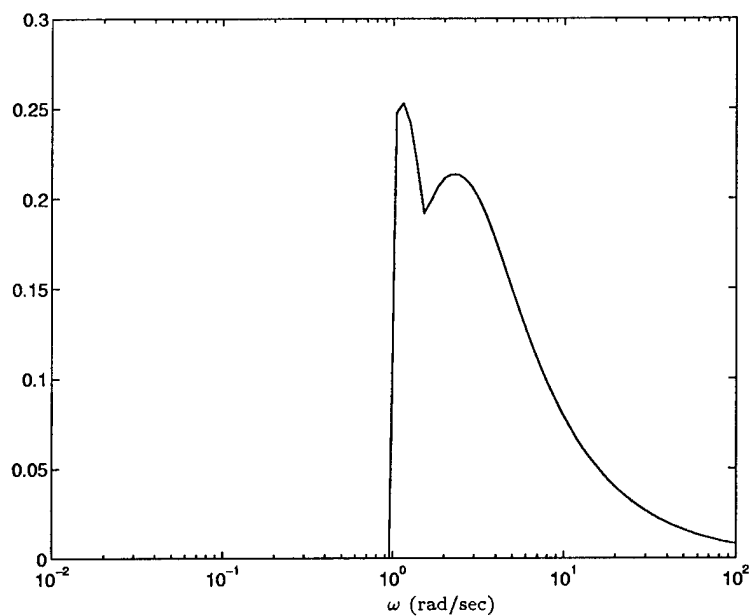
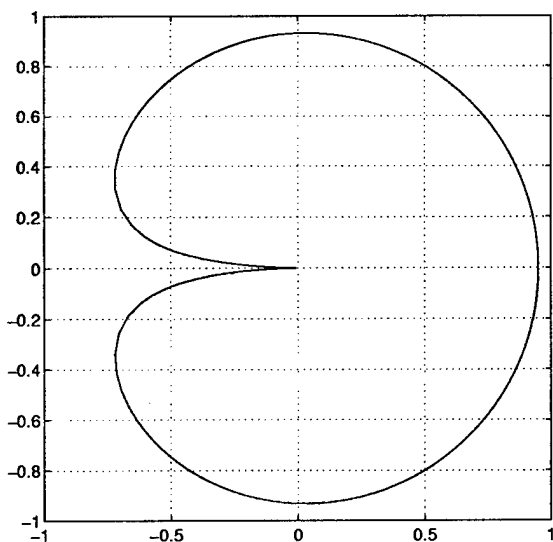
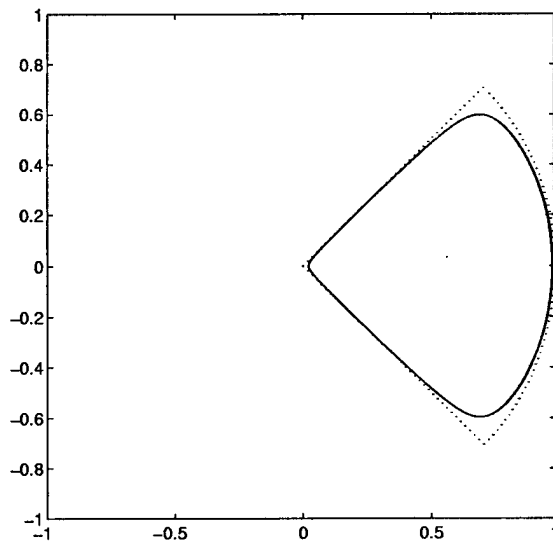


Figure 3.14: Example 3: Upper bound on $\mu_{r_{\Theta(\omega)}}(e^{j\psi(\omega)}P(j\omega))$ as a function of ω .



(a) The set $\tilde{\delta}_1^\lambda(\mathcal{D})$, for $\lambda = 0.9$.



(b) For $\lambda = 0.9$ and $\theta = \pi/4$, the boundary of the set $\tilde{\delta}^\lambda(\mathcal{D})$ is shown in solid lines and the boundary of $\mathcal{D}^\theta$ is shown in dotted lines.

Figure 3.15: An illustration of the mappings $\tilde{\delta}_1^\lambda(\cdot)$ and $\tilde{\delta}^\lambda(\cdot)$.

Chapter 4

Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin

Sufficient conditions for the robust stability of a class of uncertain systems, with several different assumptions on the structure and nature of the uncertainties, can be derived in a unified manner in the framework of integral quadratic constraints. These sufficient conditions, in turn, can be used to derive lower bounds on the robust stability margin for such systems. The lower bound is typically computed with a bisection scheme, with each iteration requiring the solution of a linear matrix inequality feasibility problem. We show how this bisection can be avoided altogether by reformulating the lower bound computation problem as a single generalized eigenvalue minimization problem, which can be solved very efficiently using standard algorithms. We illustrate this with several important, commonly-encountered special cases: Diagonal, nonlinear uncertainties; diagonal, memoryless, time-invariant sector-bounded (“Popov”) uncertainties; structured dynamic uncertainties; and structured parametric uncertainties. We also present a numerical example that demonstrates the computational savings that can be obtained with our approach.

4.1 Introduction

We revisit the interconnection of a linear system with transfer function $H(s)$ and an uncertainty or perturbation Δ , shown in Fig. 4.1, and described by

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad w = e + \gamma p, \quad p = \Delta q, \quad q = f + z, \quad (4.1)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $z(t) \in \mathbb{R}^{n_z}$, A , B , C and D are real matrices of appropriate sizes, and $\Delta : \mathcal{L}_2^{n_z}[0, \infty) \rightarrow \mathcal{L}_2^{n_w}[0, \infty)$. We assume that all the eigenvalues of A have negative real parts. We also assume that $n_z = n_w = m$; the results herein can be extended with little difficulty to cover the more general cases. Finally, we assume that system (4.1) is well-posed.

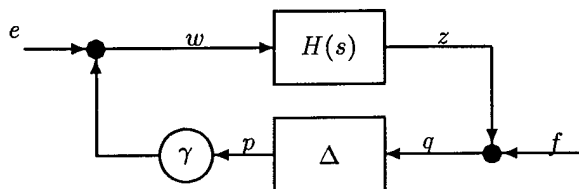


Figure 4.1: A standard framework for robustness analysis

⁰Joint work with Fan Wang, Ph.D., School of Electrical and Computer Engineering, Purdue University.

The framework that we use in this chapter for modeling the uncertainty is based on the notion of *Integral Quadratic Constraints* or *IQCs*; see [MR97] and the references therein. Borrowing notation and terminology from [MR97], two signals $p \in \mathbf{L}_2^m[0, \infty)$ and $q \in \mathbf{L}_2^n[0, \infty)$, with Fourier Transforms $\hat{p}$ and $\hat{q}$ respectively, are said to “satisfy the IQC defined by Π ”, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{q}(j\omega) \\ \hat{p}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{q}(j\omega) \\ \hat{p}(j\omega) \end{bmatrix} d\omega \geq 0,$$

where $\Pi : j\mathbf{R} \rightarrow \mathbf{C}^{2m \times 2m}$ is a measurable Hermitian function, bounded on the imaginary axis. We also say that $\Delta : \mathbf{L}_2^m[0, \infty) \rightarrow \mathbf{L}_2^n[0, \infty)$ “satisfies the IQC defined by Π ”, if for every $q \in \mathbf{L}_2^n[0, \infty)$, q and Δq satisfy the IQC defined by Π .

With the above terminology, we assume that Δ lies in the set

$$\Delta = \{ \Delta \mid \text{For every } \Pi \in \mathbf{\Pi}, \text{ any } \tau\Delta \text{ satisfies the IQC defined by } \Pi, \text{ where } \tau \in [0, 1] \}, \quad (4.2)$$

where $\mathbf{\Pi}$ is some specified set. (We will consider a number of special cases for $\mathbf{\Pi}$ in the sequel.) The set $\mathbf{\Pi}$ can be thought of as summarizing all the information known about Δ . We will make the following assumption about $\mathbf{\Pi}$: Partitioning any $\Pi \in \mathbf{\Pi}$ as $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$, we assume that:

$$\text{For some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, \Pi_{11}(j\omega) \geq 2\epsilon \text{ and } \Pi_{22}(j\omega) \leq -2\epsilon. \quad (4.3)$$

We shall see in §4.3 that for a number of commonly encountered uncertainty descriptions, the set $\mathbf{\Pi}$ defining the corresponding IQCs satisfies this assumption; thus it is not very restrictive.

Given some $\gamma > 0$, we say that system (4.1) is robustly stable if it is $\mathbf{L}_2$ -stable (see [DV75]) for every $\Delta \in \Delta$. The quantity of interest, in this chapter, is the *robust stability margin* γ_m of system (4.1), which is defined as the largest γ such that system (4.1) is robustly stable. The quantity γ_m is very useful in practice, as it has the interpretation of the largest uncertainty size for which the H - Δ interconnection is robustly stable. It is well-known that computing γ_m exactly is an NP-hard problem in several important and commonly-encountered situations [BYDM94]. (Roughly speaking, this means that the computational effort required to compute γ_m to within a given accuracy grows more than polynomially with the problem size.) Therefore, we will be content with computing lower bounds on γ_m .

For fixed γ , a number of sufficient conditions for the robust stability of system (4.1) exist, depending on Δ . When Δ can be *any* operator satisfying an $\mathbf{L}_2$ -gain bound, the small-gain theorem provides a necessary and sufficient condition for robust stability. When Δ is structured—say diagonal—the small gain condition is no longer necessary for stability; diagonal scalings can then be used to derive less conservative robust stability conditions [Doy82, Saf82]. In addition, if Δ is a memoryless time-invariant sector-bounded nonlinearity, the celebrated Popov criterion yields a sufficient condition for robust stability (see for example, [DV75]). When Δ is LTI or parametric, the well-known μ analysis and K_m analysis methods provide sufficient conditions for robust stability [BDG⁺91, CS92, FTD91]. It has been noted recently that several of these stability criteria can be unified in the setting of stability analysis using IQCs [MR97]. These stability criteria can be used to define a *guaranteed* lower bound on γ_m (as the largest γ for which robust stability can be proved using the IQC framework). The computation of the lower bound on γ_m thus defined requires bisection schemes, with each iteration requiring the solution of a convex feasibility problem, typically a linear matrix inequality (LMI) feasibility problem [J96, SC93, HH93b, LSC94, Bal95, Hel95].

The main contribution of this chapter is to show how bisection can be avoided altogether, by reformulating the lower bound computation problem as a *single* generalized eigenvalue minimization problem (GEVP)¹. This is a quasiconvex optimization problem over LMIs, and can be solved very efficiently using standard algorithms and software (see, for example, [BEFB94, BE93] and [GN95]). We also present examples that illustrate the computational improvement obtained with our approach.

The organization of the chapter is as follows. In §4.2, we very briefly review the robust stability analysis of system (4.1) using the IQC framework. We then show how to recast the robust stability margin lower bound computation problem as a GEVP. In §4.3, we illustrate our approach on several important commonly-encountered special cases for the set of uncertainties Δ . In §4.4, we compare the computational effort of the GEVP and bisection schemes with a simple numerical example.

¹ A similar reduction of the stability margin calculation to a GEVP was also made in [J96], however with severe restrictions on the system matrices.

4.2 Robust stability margin bound via generalized eigenvalue minimization

We review a robust stability criterion, taken from [MR97], for systems with uncertainties described by IQCs. (For all the uncertainties considered here, this stability criterion can be derived via an application of the passivity theorem with multipliers, see for example, [DV75, HH93b, SC93, Bal95].)

Given some $\gamma > 0$, a sufficient condition for the stability of system (4.1) for all $\Delta \in \Delta$ is given by the following lemma [MR97, Theorem 1].

Lemma 4.2.1 *Suppose that the interconnection of $H(j\omega)$ and $\tau\Delta$ in Fig. 4.1 is well-posed for any $\tau \in [0, \gamma]$ and any $\Delta \in \Delta$. Then, if there exist $\Pi \in \Pi$ and $\epsilon > 0$ such that*

$$\begin{bmatrix} \gamma H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \gamma H(j\omega) \\ I \end{bmatrix} \leq -2\epsilon I, \quad \text{for all } \omega \in \mathbf{R}, \quad (4.4)$$

system (4.1) is robustly stable for all $\Delta \in \Delta$.

The above sufficient condition for the robust stability of system (4.1) yields a lower bound for the robust stability margin via the following optimization problem:

$$\begin{aligned} \text{Maximize: } & \gamma \\ \text{Subject to: } & \text{There exist } \Pi \in \Pi \text{ and } \epsilon > 0 \text{ such that (4.4) holds.} \end{aligned} \quad (4.5)$$

We now describe the current, commonly used technique for the numerical solution of Problem (4.5) (see for example [J96]). In general, Π —the set defining the IQCs corresponding to Δ —is not described by a finite number of variables. In order to reduce the number of optimization variables to a finite number, a subset of Π is defined as

$$\Pi_{\text{fin}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} W(j\omega)^* R_{11} W(j\omega) & W(j\omega)^* R_{12} W(j\omega) \\ W(j\omega)^* R_{12}^T W(j\omega) & -W(j\omega)^* R_{22} W(j\omega) \end{bmatrix}, \\ W(j\omega) = \begin{bmatrix} C_W(j\omega I - A_W)^{-1} B_W \\ D_W \end{bmatrix}, \quad \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega, \\ \text{for some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, \\ W(j\omega)^* R_{11} W(j\omega) \geq 2\epsilon I, \quad W(j\omega)^* R_{22} W(j\omega) \geq 2\epsilon I \end{array} \right. \right\}, \quad (4.6)$$

where $A_W \in \mathbf{R}^{n_W \times n_W}$, $B_W \in \mathbf{R}^{n_W \times m}$, $C_W \in \mathbf{R}^{K \times n_W}$, $D_W \in \mathbf{R}^{L \times m}$, and Ω is an appropriately chosen subspace of $\mathbf{R}^{2(K+L) \times 2(K+L)}$. (We will see specific examples in §4.3.)

Remark 4.2.1 *It is typically computationally more efficient to parameterize the various blocks Π_{ij} of Π in (4.6) as $\Pi_{ij}(j\omega) = W_i(j\omega)^* R_{ij} W_j(j\omega)$, where W_i and W_j are different transfer functions. However, for simplicity of presentation, we will continue with the definition of Π_{fin} as in (4.6), noting that the development herein can be readily extended to the more general case.*

With Π restricted to lie in Π_{fin} , we have another lower bound on the robust stability margin via the following finite-dimensional optimization problem:

$$\begin{aligned} \text{Maximize: } & \gamma \\ \text{Subject to: } & \text{There exist } \Pi \in \Pi_{\text{fin}} \text{ and } \epsilon > 0 \text{ such that (4.4) holds.} \end{aligned} \quad (4.7)$$

For a fixed γ , checking if there exists $\Pi \in \Pi_{\text{fin}}$ such that condition (4.4) holds can be reformulated as a convex feasibility problem with LMI constraints (see Lemma 4.2.3 below). Current techniques take advantage of this observation to solve Problem (4.7) using a bisection scheme; see [J96, Bal95, SC93, Hel95].

There are a number of problems associated with using the bisection scheme to solve Problem (4.7). First, upper and lower bounds on the optimal value γ_{opt} need to be determined to initialize the bisection; such bounds may be known *a priori* or may have to be determined. The quality of these bounds will of course

affect the efficiency of the bisection scheme in computing γ_{opt} . Moreover, the bisection scheme does not take advantage of the fact that Problem (4.7) is a quasiconvex optimization problem (see for example, [BEFB94] and the references therein).

We now show how the drawbacks with the bisection scheme can be avoided altogether, by reformulating Problem (4.7) as a Generalized Eigenvalue Minimization Problem or GEVP. A GEVP is an optimization problem of the form

$$\text{Minimize: } \lambda$$

$$\text{Subject to: } \lambda B(x) - A(x) > 0, \quad B(x) > 0, \quad C(x) > 0.$$

Here $x \in \mathbf{R}^p$ is the optimization variable, and $A(x)$, $B(x)$ and $C(x)$ are symmetric matrices that are affine functions of x , i.e.,

$$A(x) = A_0 + x_1 A_1 + \cdots + x_p A_p, \quad B(x) = B_0 + x_1 B_1 + \cdots + x_p B_p, \quad C(x) = C_0 + x_1 C_1 + \cdots + x_p C_p,$$

where A_i , B_i and C_i are given symmetric matrices. GEVPs are quasiconvex optimization problems based on linear matrix inequalities, and can be solved very efficiently using standard algorithms (see for example, [BEFB94, BE93, GN95]). In particular, as we will demonstrate in §4.4, the solution of Problem (4.7) as a GEVP leads to considerable computational savings over the solution via a bisection scheme.

The following restatement of the positive-real lemma, taken from [Ran96], plays a central role in the reformulation.

Lemma 4.2.2 *Let $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $M = M^T \in \mathbf{R}^{(m+n) \times (m+n)}$, with A having no eigenvalues on the imaginary axis. Then, the following statements are equivalent.*

1. *For some $\epsilon > 0$,*

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \geq 2\epsilon I, \quad \text{for all } \omega \in \mathbf{R}.$$

2. *There exists a symmetric matrix $P = P^T$ such that*

$$\begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} < M.$$

Given $\gamma > 0$, Lemma 4.2.2 then enables us to restate the condition that (4.4) is feasible for some $\Pi \in \Pi_{\text{fin}}$ as an LMI feasibility condition.

Lemma 4.2.3 *Let $H(s)$ has the state space realization (A, B, C, D) with A having no eigenvalues on the imaginary axis. Let*

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A_W & B_W C & 0 \\ 0 & A & 0 \\ 0 & 0 & A_W \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_W D \\ B \\ B_W \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \end{bmatrix}, \\ \tilde{D} &= \begin{bmatrix} 0 \\ D \\ 0 \\ I \end{bmatrix}, \quad E = \begin{bmatrix} C_W & 0 \\ 0 & D_W \end{bmatrix}. \end{aligned} \tag{4.8}$$

Then, given $\gamma > 0$, there exist some $\Pi \in \Pi_{\text{fin}}$ and $\epsilon > 0$ such that condition (4.4) holds if and only if there exist symmetric matrices $P = P^T$, $Q_1 = Q_1^T$ and $Q_2 = Q_2^T$, and $\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega$ such that

$$M_1 = E^T R_{11} E - \begin{bmatrix} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{bmatrix} > 0, \tag{4.9a}$$

$$M_2 = E^T R_{22} E - \begin{bmatrix} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{bmatrix} > 0, \quad (4.9b)$$

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} \gamma M_1 & E^T R_{12} E \\ E^T R_{12}^T E & -\frac{1}{\gamma} M_2 \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \quad (4.9c)$$

Proof: Consider the condition that given $\gamma > 0$, there exist $\Pi \in \Pi_{\text{fin}}$ and $\epsilon > 0$ such that condition (4.4) holds, i.e., there exists $\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega$ such that with

$$W(j\omega) = \begin{bmatrix} C_W(j\omega I - A_W)^{-1} B_W \\ D_W \end{bmatrix},$$

we have:

$$\text{For some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, W(j\omega)^* R_{11} W(j\omega) \geq 2\epsilon I \text{ and } W(j\omega)^* R_{22} W(j\omega) \geq 2\epsilon I. \quad (4.10a)$$

For some $\epsilon > 0$, for all $\omega \in \mathbf{R}$,

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \gamma W(j\omega)^* R_{11} W(j\omega) & W(j\omega)^* R_{12} W(j\omega) \\ W(j\omega)^* R_{12}^T W(j\omega) & -\frac{1}{\gamma} W(j\omega)^* R_{22} W(j\omega) \end{bmatrix} \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \leq -2\epsilon I. \quad (4.10b)$$

Using Lemma 4.2.2, it is easily argued that condition (4.10a) is equivalent to the existence of $Q_1 = Q_1^T$ and $Q_2 = Q_2^T$ such that

$$M_1 = E^T R_{11} E - \begin{bmatrix} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{bmatrix} > 0,$$

and

$$M_2 = E^T R_{22} E - \begin{bmatrix} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{bmatrix} > 0,$$

where E is defined in (4.8); moreover, it is easily verified that for all $\omega \in \mathbf{R}$,

$$\begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix}^* M_1 \begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix} = W(j\omega)^* R_{11} W(j\omega),$$

and

$$\begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix}^* M_2 \begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix} = W(j\omega)^* R_{22} W(j\omega).$$

Then, with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ as given by (4.8), condition (4.10b) can be rewritten as:

$$\text{For some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, \quad \left(\tilde{C}(j\omega I - \tilde{A})^{-1} \tilde{B} + \tilde{D} \right)^* \begin{bmatrix} \gamma M_1 & E^T R_{12} E \\ E^T R_{12}^T E & -\frac{1}{\gamma} M_2 \end{bmatrix} \left(\tilde{C}(j\omega I - \tilde{A})^{-1} \tilde{B} + \tilde{D} \right) \leq -2\epsilon I. \quad (4.11)$$

Condition (4.11), using Lemma 4.2.2, is equivalent to the existence of $P = P^T$ such that

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} \gamma M_1 & E^T R_{12} E \\ E^T R_{12}^T E & -\frac{1}{\gamma} M_2 \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \quad (4.12)$$

This completes the proof. $\square$

Lemma 4.2.3 immediately enables the reformulation of Problem (4.7) as a GEVP, which is the central result of this chapter.

Theorem 6 Let κ_{opt} be the optimal value of the GEVP

Minimize: κ

Subject to: $P = P^T, Q_1 = Q_1^T, Q_2 = Q_2^T, X = X^T > 0, Y = Y^T > 0,$

$$\begin{aligned} & \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega, \\ & M_1 = E^T R_{11} E - \begin{bmatrix} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{bmatrix} > 0, \\ & M_2 = E^T R_{22} E - \begin{bmatrix} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{bmatrix} > 0, \\ & \begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} X & E^T R_{12} E \\ E^T R_{12}^T E & -Y \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0, \\ & \kappa \begin{bmatrix} X & 0 \\ 0 & M_2 \end{bmatrix} - \begin{bmatrix} M_1 & 0 \\ 0 & Y \end{bmatrix} > 0, \end{aligned} \quad (4.13)$$

where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, and E are defined in (4.8). Then, the optimal value of Problem (4.7) is $1/\kappa_{\text{opt}}$.

Proof: Follows directly from Lemma 4.2.3, with the introduction of “slack” variables X and Y and the change of variable $\kappa = 1/\gamma$. $\square$

4.3 Specified structured uncertainties

With the preliminaries in §4.2, we now consider a number of special cases for Δ . In each case, the corresponding set Π defining the IQCs satisfies assumption (4.3), so that the results of §4.2 apply.

4.3.1 Diagonal nonlinearities

Suppose that

$$\Pi^{\text{DNL}} = \left\{ \Pi \mid \Pi(j\omega) = \begin{bmatrix} W & 0 \\ 0 & -W \end{bmatrix} \text{ for all } \omega \in \mathbf{R}, W \in \mathbf{R}^{m \times m}, W > 0 \text{ and diagonal} \right\}. \quad (4.14)$$

Every “diagonal” Δ with an L_2 gain that does not exceed one, satisfies every IQC from Π^{DNL} . Note that Π^{DNL} is already described by a finite number of variables so that $\Pi^{\text{DNL}} = \Pi_{\text{fin}}^{\text{DNL}}$ and is defined by (4.6), where A_W, B_W and C_W are vacuous, $D_W = I$, and $\Omega = \Pi^{\text{DNL}}$.

From Theorem 6, Problem (4.7) is equivalent to the GEVP

Minimize: κ

Subject to: $P = P^T, X = X^T > 0, Y = Y^T > 0, W \in \mathbf{R}^{m \times m}$ and diagonal, $W > 0,$

$$\begin{aligned} & \begin{bmatrix} A^T P + P A & P B \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0, \\ & \kappa \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix} - \begin{bmatrix} W & 0 \\ 0 & Y \end{bmatrix} > 0, \end{aligned} \quad (4.15)$$

with $1/\kappa_{\text{opt}}$ being the optimal lower bound on γ_m . This GEVP formulation is exactly the same one derived in [EBFB92] for minimizing the H_∞ norm of $H(s) = C(sI - A)^{-1}B + D$ over diagonal similarity scalings (also see [GA94]).

4.3.2 Diagonal, memoryless, time-invariant, sector-bounded nonlinearities

Suppose that $D = 0$, that is, H is strictly proper, and

$$\Pi^{\text{Popov}} = \left\{ \Pi \mid \Pi(j\omega) = \begin{bmatrix} \Lambda & -j\omega\Gamma \\ j\omega\Gamma & -\Lambda \end{bmatrix} \text{ for all } \omega \in \mathbf{R}, \Lambda, \Gamma \in \mathbf{R}^{m \times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

The set of uncertainties satisfying every IQC from Π^{Popov} contains the set of all diagonal, memoryless, time-invariant nonlinearities, with the diagonal entries in sector $[-1, 1]$; see [DV75, MR97]. These are often referred to as diagonal “Popov-type” uncertainties.

Note that the elements of $\Pi_{\text{fin}}^{\text{Popov}}$ are not bounded on the imaginary axis. But a simple change of variables enables us to address this problem. (This is a standard technique, used to derive the Popov criterion; see [DV75, MR97].) We simply define $\tilde{H}(s) = H(s)(1+s) = (C+CA)(sI-A)^{-1}B+CB$, and $\tilde{\Delta} = \Delta \circ 1/(1+s)$, and observe that the stability of the $\tilde{H}$ - $\tilde{\Delta}$ interconnection is equivalent to that of the H - Δ interconnection. Now, $\tilde{\Delta}$ satisfies every IQC from

$$\tilde{\Pi}^{\text{Popov}} = \left\{ \tilde{\Pi} \mid \tilde{\Pi}(j\omega) = \begin{bmatrix} \frac{1}{1+\omega^2}\Lambda & \frac{-j\omega}{1-j\omega}\Gamma \\ \frac{j\omega}{1+j\omega}\Gamma & -\Lambda \end{bmatrix} \text{ for all } \omega \in \mathbf{R}, \right. \\ \left. \Lambda, \Gamma \in \mathbf{R}^{m \times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

The set $\tilde{\Pi}^{\text{Popov}}$ is described by a finite number of variables so that $\tilde{\Pi}^{\text{Popov}} = \tilde{\Pi}_{\text{fin}}^{\text{Popov}}$, and is defined by (4.6) with $A_W = -I$, $B_W = C_W = D_W = I$, and

$$\Omega^{\text{Popov}} = \left\{ \begin{bmatrix} \Lambda & 0 & 0 & -\Gamma \\ 0 & 0 & 0 & \Gamma \\ 0 & 0 & 0 & 0 \\ -\Gamma & \Gamma & 0 & -\Lambda \end{bmatrix} \mid \Lambda, \Gamma \in \mathbf{R}^{m \times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

Then, applying Theorem 6 to the $\tilde{H}$ - $\tilde{\Delta}$ interconnection, we get the following GEVP.

Minimize: κ

Subject to: $P = P^T$, $X = X^T > 0$, $Y = Y^T > 0$, Λ and Γ are diagonal, $\Lambda > 0$,

$$\begin{bmatrix} \tilde{A}^T P + P \tilde{A} & P \tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} X & 0 & 0 & -\Gamma \\ 0 & 0 & 0 & \Gamma \\ 0 & 0 & 0 & 0 \\ -\Gamma & \Gamma & 0 & -Y \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0, \\ \kappa \begin{bmatrix} X & 0 \\ 0 & \Lambda \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & Y \end{bmatrix} > 0,$$

where

$$\tilde{A} = \begin{bmatrix} -I & C+CA & 0 \\ 0 & A & 0 \\ 0 & 0 & -I \end{bmatrix}, \tilde{B} = \begin{bmatrix} CB \\ B \\ I \end{bmatrix}, \tilde{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C+CA & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \tilde{D} = \begin{bmatrix} 0 \\ CB \\ 0 \\ I \end{bmatrix}.$$

The optimal lower bound on γ_m is given by $1/\kappa_{\text{opt}}$.

4.3.3 Parametric uncertainties

Suppose Δ is a constant real matrix with a specified block-diagonal structure, and with a spectral norm that does not exceed one: $\Delta = \text{diag}(D_1, \dots, D_M, d_1 I_{\ell_1}, \dots, d_N I_{\ell_N})$, $D_i \in \mathbf{R}^{k_i \times k_i}$, $i = 1, \dots, M$, $d_i \in \mathbf{R}$, $i = 1, \dots, N$, with $\sigma_{\max}(\Delta) \leq 1$. (Note that $\sum k_i + \sum \ell_i = m$.)

Then, Δ satisfies every IQC from

$$\Pi^{\text{par}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ -Y(j\omega) & -X(j\omega) \end{bmatrix} = \Pi(j\omega)^*, \quad X(j\omega) \text{ and } Y(j\omega) \in \mathcal{W}, \\ \text{for some } \epsilon > 0, X(j\omega) \geq 2\epsilon I, \text{ for all } \omega \in \mathbf{R} \end{array} \right. \right\},$$

where

$$\mathcal{W} = \left\{ \text{diag}(w_1 I_{k_1}, \dots, w_M I_{k_M}, W_1, \dots, W_N) \left| \begin{array}{l} w_i \in \mathbf{C}, i = 1, \dots, M \\ W_i \in \mathbf{C}^{\ell_i \times \ell_i}, i = 1, \dots, N \end{array} \right. \right\}. \quad (4.16)$$

(See for example [Bal95, MR97].)

For such uncertainties, Theorem 6 can be immediately used to obtain GEVPs that yield a guaranteed lower bound on the robust stability margin. Let $\mathcal{P}$ denote the subset of real matrices that lie in $\mathcal{W}$, i.e., $\mathcal{P} = \mathcal{W} \cap \mathbf{R}^{m \times m}$.

Then, a subset $\Pi_{\text{fin}}^{\text{par}}$ of Π^{par} , described by a finite number of variables, can be defined as follows. Let $W^{(1)}, \dots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Let

$$\Theta^{\text{par}} = \left\{ \left[\begin{array}{cccc} \theta_{11} & \theta_{12} & \cdots & \theta_{1N} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N1} & \theta_{N2} & \cdots & \theta_{NN} \end{array} \right] \left| \theta_{ij} \in \mathcal{P} \right. \right\}. \quad (4.17)$$

Then, a subset $\Pi_{\text{fin}}^{\text{par}}$ of Π^{par} described by a finite number of variables is given by (4.6), where (A_W, B_W, C_W) is any state space realization of $[W^{(1)}(s)^T \dots W^{(N-1)}(s)^T]^T$, $D_W = I$, and

$$\Omega^{\text{par}} = \left\{ \left[\begin{array}{cc} \Theta + \Theta^T & \Phi - \Phi^T \\ \Phi^T - \Phi & -(\Theta + \Theta^T) \end{array} \right] \left| \Theta, \Phi \in \Theta^{\text{par}} \right. \right\}. \quad (4.18)$$

Thus, the problem of computing lower bounds on the robust stability margin of systems with parametric uncertainties can be efficiently solved as a GEVP.

Note that the choice of $W^{(i)}$ is *ad hoc*, and the value of κ_{opt} will certainly depend on this choice. However, for any choice of $W^{(i)}$, the inverse of the optimal value κ_{opt} obtained from GEVP (4.13) is a *guaranteed* lower bound on the robust stability margin γ_m . Moreover, it can be shown (see [CTB97a]) that the actual choice of the $W^{(i)}$ is immaterial, provided the set of $W^{(i)}$ s is chosen to be “rich enough”.

4.3.4 Structured dynamic uncertainties

Suppose Δ is a dynamic block-structured uncertainty with an $\mathbf{L}_2$ -gain that does not exceed one (i.e., it is nonexpansive): For all $\omega \in \mathbf{R}$, $\Delta(j\omega) = \text{diag}(D_1, \dots, D_M, d_1 I_{\ell_1}, \dots, d_N I_{\ell_N})$, $D_i \in \mathbf{C}^{k_i \times k_i}$, $i = 1, \dots, M$, $d_i \in \mathbf{C}$, $i = 1, \dots, N$, with $\sigma_{\max}(\Delta(j\omega)) \leq 1$. (Note that $\sum k_i + \sum \ell_i = m$.)

Then, Δ satisfies every IQC from

$$\Pi^{\text{LTI}} = \left\{ \Pi \left| \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix} = \Pi(j\omega)^*, \quad X(j\omega) \in \mathcal{W}, \\ \text{for some } \epsilon > 0, X(j\omega) \geq 2\epsilon I, \text{ for all } \omega \in \mathbf{R} \end{array} \right. \right\},$$

where $\mathcal{W}$ is defined in (4.16) (see for example [Bal95, MR97]). Similarly to the development in §4.3.3, $\Pi_{\text{fin}}^{\text{LTI}} \subset \Pi^{\text{LTI}}$ can be described by a finite number of variables. Let $W^{(1)}, \dots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Then, a subset $\Pi_{\text{fin}}^{\text{LTI}}$ of Π^{LTI} described by a finite number of variables is given by (4.6), where (A_W, B_W, C_W) is any state space realization of $[W^{(1)}(s)^T \dots W^{(N-1)}(s)^T]^T$, $D_W = I$, and

$$\Omega^{\text{LTI}} = \left\{ \left[\begin{array}{cc} \Theta + \Theta^T & 0 \\ 0 & -(\Theta + \Theta^T) \end{array} \right] \left| \Theta \in \Theta^{\text{par}} \right. \right\}. \quad (4.19)$$

Thus, the problem of computing lower bounds on the robust stability margin of systems with structured dynamic uncertainties can be efficiently solved as a GEVP.

4.4 A numerical example

We present an application of the results of this chapter on a simple example. Consider an instance of the H - Δ interconnection system with

$$H(s) = \begin{bmatrix} \frac{0.2}{s^2 + 0.1s + 0.7} & \frac{-1.5}{s^2 + 0.1s + 0.7} & \frac{-s^2 + 0.9s - 0.2}{s^3 + 0.4s^2 + 0.73s + 0.21} \\ \frac{s}{s^2 + 0.1s + 0.7} & \frac{-7.5s}{s^2 + 0.1s + 0.7} & \frac{10s^2 + 3s + 3.5}{s^3 + 0.4s^2 + 0.73s + 0.21} \\ 0 & 0 & \frac{-2}{s + 0.3} \end{bmatrix}.$$

With Δ assumed to be diagonal in addition to satisfying various IQCs, we now demonstrate that significant computational savings accrue when the lower bound on the robust stability margin is computed using the GEVP formulation from Theorem 6, as compared to a bisection scheme.

In implementing a bisection scheme to solve Problem (4.7), we used $\kappa = 1/\gamma$ as the optimization variable. Upper and lower bounds on the optimal value κ_{opt} that are required to initialize the bisection can be computed using different methods; and the performance of the bisection scheme can be made arbitrarily poor by choosing the bounds to be far enough apart. We avoided introducing any such bias against the bisection scheme as follows. In all cases that we consider, Δ satisfies the IQC with $\Pi = \text{diag}(I, -I)$, and therefore $\|H\|_\infty$ is an upper bound on κ_{opt} ; this can be computed very efficiently using the algorithms in [BB90]. A lower bound on κ_{opt} is simply zero. The upper bound can also be readily incorporated into the GEVP (4.13). We denote the GEVP with this additional linear constraint as GEVPWB.

Table 4.1 shows a comparison of the performance of the bisection and the GEVP schemes. In every case, κ_{opt} was computed to an relative accuracy of 1%. In the case of diagonal parametric uncertainties, $\Pi_{\text{fn}}^{\text{par}}$ was given by (4.6), (4.17) and (4.18), with the following choices: $A_W = \text{diag}(-10, -10, -10)$, $B_W = C_W = D_W = I$, and

$$\Theta^{\text{par}} = \left\{ \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{bmatrix} \mid \theta_{ij} \in \mathbf{R}^{3 \times 3} \text{ and diagonal} \right\}.$$

In the diagonal dynamic uncertainties, $\Pi_{\text{fn}}^{\text{LTI}}$ was given similarly, with the only difference being in the definition of Ω , which was given by (4.19). All LMI computations were performed using the LMI Toolbox of MATLAB [GN95], and computation times on a Sparc 20 are reported here². The numerical results show that GEVP is always considerably more efficient than the bisection scheme. In addition, this example suggests that *a priori* knowledge of an upper bound on κ_{opt} makes little difference to the performance of the GEVP.

Uncertainty Type	$1/\kappa_{\text{opt}}$	Bisection (sec)	GEVP (sec)	GEVPWB (sec)
General nonlinear	1.2896×10^{-2}	5.6800	0.8200	0.8000
Dynamic	1.2899×10^{-2}	62.5700	20.3000	18.9200
Popov-type	1.3264×10^{-2}	99.5700	26.5300	26.1200
Parametric	1.3278×10^{-2}	51.8400	19.8300	19.1700

Table 4.1: A comparison of the bisection and GEVP schemes. All uncertainties are assumed to be *diagonal*.

4.5 Conclusion

We have shown that a guaranteed lower bound on the robust stability margin for a number of commonly encountered uncertain systems can be computed via generalized eigenvalue minimization. Examples show that the GEVP reformulation of the robust stability margin lower bound computation leads to considerable computational savings. The results presented herein also apply to many other uncertain systems, besides the special cases considered in §4.3; see for example, [MR97] and [TBL96].

²Though the values of the time themselves are not very meaningful as they depend on the hardware used, the *ratio* of the computation time of the bisection scheme to the GEVP scheme still provides a meaningful basis for comparison.

Chapter 5

Gain-Scheduled Controller Synthesis

We present new algorithms for the robust stability analysis and gain-scheduled controller synthesis for linear systems affected by time-varying parametric uncertainties. The new techniques can also be applied to parameter-dependent nonlinear systems with real rational nonlinearities. Sufficient conditions for robust stability as well as conditions for the existence of a robustly stabilizing gain-scheduled controller are given in terms of a finite number of Linear Matrix Inequalities; explicit formulae for constructing robustly stabilizing gain-scheduled controllers are given in terms of the feasible set of these LMIs. Our approach is proven to be in general less conservative than the existing methods for stability analysis and gain-scheduled controller synthesis for parameter-dependent linear systems and real rational nonlinear systems; numerical examples are presented to show that our approach offers significant improvement in practice as well.

5.1 Introduction

Consider the parameter-dependent system

$$\dot{x} = A(\theta(t))x + B(\theta(t))u, \quad (5.1a)$$

$$y = C(\theta(t))x + D(\theta(t))u, \quad (5.1b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$, and A , B , C and D are real-valued rational functions of the time-varying parameter vector $\theta(t) = [\theta_1(t) \cdots \theta_m(t)]^T \in \mathbb{R}^m$, which for all $t > 0$ is restricted to lie in a polytope $\Theta \subset \mathbb{R}^m$ containing the origin. (When Θ is not a polytope, the results developed herein can still be applied by replacing Θ with some polytope $\Theta_{\text{poly}} \supset \Theta$.) For convenience, from now on we will often drop the dependence of θ on t . The signals u and y have the interpretation of the control input and the measured output respectively. We assume that the parameters $\theta(t)$ are unknown a priori, but can be measured in real-time, so that they can be incorporated, if possible, in a “gain-scheduled” control strategy. System (5.1) models a wide variety of commonly-encountered parameter-dependent systems; see, e.g., [Doy82, ZKSN92, AGB95, DG98].

We consider questions of stability analysis and stabilizing controller synthesis for system (5.1):

- (P1) With u identically zero, does the state x of system (5.1) satisfy $\lim_{t \rightarrow \infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say the system is “robustly stable over Θ ”.
- (P2) Does there exist a control law $u = K(y, \theta, t)$ such that the state x of system (5.1) satisfies $\lim_{t \rightarrow \infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say the system is “robustly stabilizable over Θ ”.

In addition, for a uncertainty set Θ , we define the *robust stability margin* of system (5.1) as

$$\sigma_m = \sup \{ \sigma \mid \text{for any } \gamma \in [0, \sigma], \text{ system (5.1) is robustly stable over } \gamma\Theta \} \quad (5.2)$$

and the *robust stabilizability margin* as

$$\rho_m = \sup \{ \rho \mid \text{for any } \gamma \in [0, \rho], \text{ system (5.1) is robustly stabilizable over } \gamma\Theta \}. \quad (5.3)$$

⁰Joint work with Fan Wang, Ph.D., School of Electrical and Computer Engineering, Purdue University.

Let us first consider the question (P1), that of robust stability over Θ . A number of numerally tractable sufficient conditions are available for robust stability, depending on the assumptions on the function $\mathbf{A}(\theta(t))$. One class of sufficient conditions is based on the notion of quadratic stability: The system $\dot{x} = \mathbf{A}(\theta(t))x$ is said to be quadratically stable if and only if there exists a single quadratic Lyapunov function $V(x) = x^T P x$ whose derivative is negative along every trajectory of the system, or equivalently, there exists $P = P^T$ such that

$$P > 0, \quad P\mathbf{A}(\theta) + \mathbf{A}(\theta)^T P < 0 \text{ for all } \theta(t) \in \Theta. \quad (5.4)$$

For the simplest case when $\mathbf{A}(\theta(t))$ is an *affine* function of $\theta(t)$ (this is the so-called “polytopic system” [BEFB94]), a necessary and sufficient condition for quadratic stability can be given in terms of a finite number of LMIs, one for each vertex of the polytope Θ [HB76, BY89]. The use of more general Lyapunov functions offers the potential for improved robust stability analysis. One such technique uses parameter-dependent Lyapunov functions; for systems with slowly time-varying parameters, stability analysis using parameter-dependent Lyapunov functions usually leads to less conservative robust stability conditions than the analysis based on quadratic Lyapunov functions [GAC96, GN95].

The approach towards answering question (P1), for systems having more general uncertainties than polytopic ones, uses a *linear fractional representation* (LFR) for parameter-dependent systems [BBB91, KG96]. Here, the parameter-dependent system is represented as an LTI system, with the uncertain parameters appearing in the feedback loop as a diagonal uncertainty Δ (see Fig. 5.1). Then, scaling matrices can be used in conjunction with the small-gain theorem to yield sufficient conditions for robust stability of system (5.1); see for example, [BDG⁺91, FTD91, MR97]. In a special case when the uncertainties are time-invariant, the most sophisticated (and least conservative) technique available to date uses the framework on multiplier theory [DV75, HH93a, SC93, Bal94a], or equivalently in this case, the framework of integral quadratic constraints [MR97]; these approaches correspond to searching for general Lyapunov functions [Bal94b].

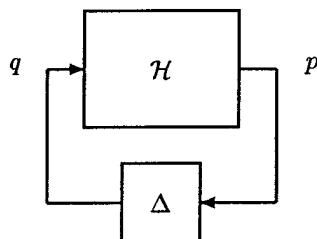


Figure 5.1: LFR of parametric uncertain systems

The problem of controller synthesis (P2), has turned out to be considerably harder. While the problem of robustly stabilizing constant state-feedback synthesis for polytopic systems as well as LFR systems with real constant scalings has turned out to be convex, no convex reformulations are known for the problem of even constant output feedback synthesis for even polytopic systems. This makes the general output feedback controller synthesis problem very hard to solve numerically. The approach that holds the most promise for output feedback synthesis appears to be that of *gain-scheduled* controller synthesis¹: Since in many applications the parameters $\theta(t)$ can be measured [AGB95, DG98, SA92], we can design feedback schemes that are themselves parameter-dependent. Designing gain-scheduled output feedback controllers for polytopic systems or LFR systems, using a single quadratic Lyapunov function can be reduced to finding the feasible set of a finite number of LMIs [Bec93, AGB95, AG95, Sch96a, CG95]. Gain-scheduled controller synthesis for polytopic systems using parameter-dependent Lyapunov functions is also studied in the literature, see for example [AA98, WYPB94]. In this case however, one usually needs to grid the parameter space made up of the uncertainties and their first derivatives [AA98, WYPB94]; thus, in a sense, one needs to check an infinite number of LMIs. Although parameter gridding can be avoided in some cases, it requires either more restrictive assumptions on the system matrices [AA98], or the use of a more conservative cover for the set of uncertainties [YS97].

¹Our use of the term “gain-scheduled” refers to the framework of LMI-based gain-scheduling techniques [AA97, EGS96, Bec93, Wu95]. This is different from classical gain-scheduling controller synthesis techniques, where several controllers are designed for the system under different operating conditions, with the actual control law “switching” between the locally designed controllers using some “scheduling” scheme [SA90, SA92].

Our approach uses the framework of quadratic (or parameter-dependent Lyapunov function based) stability and the LFR representation as do [Bec93, AGB95, AG95]. However, our method for robust stability analysis is less conservative than the conventional methods using constant scalings, but enjoys the same advantages:

- Our method results in a finite number of LMIs;
- Our method can be extended to the synthesis of gain-scheduled controllers;
- Our method is based on quadratic (or parameter-dependent) Lyapunov functions, and therefore can be extended to robust performance analysis and synthesis.

Indeed, we will show that the conventional constant scaling method can be viewed as a special case in our approach.

Our approach can be interpreted as combining vertex-type quadratic stability results for polytopic systems with conventional scaling techniques; thus our approach can effectively take advantage of the knowledge of polytopic covers that describe the uncertainties more accurately than conventional norm bounds. In addition, we will demonstrate that even when only a norm-bound condition on the uncertainties (or equivalently, only a hypercube cover) is available, our approach enables the use of *unstructured* scalings, as opposed to the usual *structured* scaling techniques. As a consequence, our robust stability analysis conditions are always less conservative than the stability analysis using structured scalings. Perhaps more significantly, our approach also readily leads to tractable LMI conditions for the existence of a gain-scheduled output feedback controller. In addition, the controller designed in our approach can be easily implemented in real-time.

The organization of this chapter is as follows. In Section 5.2, we outline the improved stability criterion for parameter-dependent systems derived using the LFR framework. We extend this robust stability analysis technique to solve the problem of gain-scheduled output feedback controller synthesis in Section 5.3. In Section 5.4, we demonstrate through numerical examples that the new approach offers significant improvement over existing constant scaling techniques.

5.2 Robustness analysis using Lyapunov functions

5.2.1 Quadratic stability analysis

Consider the state-trajectories of the system (5.1) with u identically zero:

$$\dot{x} = \mathbf{A}(\theta(t))x. \quad (5.5)$$

Since $\mathbf{A}(\theta(t))$ is a real-valued rational function of $\theta(t)$, we have

$$\mathbf{A}(\theta(t)) = A + B_q \Delta(\theta(t))(I - D_{pq} \Delta(\theta(t)))^{-1} C_p$$

for some appropriate matrices² A , B_q , C_p and D_{pq} . Then, an equivalent linear-fractional representation [BBB91, EGS96, KG96] of the autonomous system (5.5) is given by

$$\dot{x} = Ax + B_q q, \quad p = C_p x + D_{pq} q, \quad q = \Delta(\theta)p, \quad \Delta(\theta) = \text{diag}(\theta_1 I_{s_1}, \dots, \theta_m I_{s_m}), \quad (5.6)$$

where $x \in \mathbf{R}^n$, $q \in \mathbf{R}^d$, $p \in \mathbf{R}^d$ and A , B_q , C_p , D_{pq} are real matrices of appropriate sizes, with A being Hurwitz, i.e., with all its eigenvalues having negative real parts. The quantity $\max(s_1, \dots, s_m)$ will be termed the *LFR degree* of system (5.6).

Define

$$\Delta = \{\Delta(\theta) \mid \theta(t) \in \Theta\}. \quad (5.7)$$

Obviously Δ is a polytope as well. Let Δ_i , $i = 1, \dots, r$ be the vertices of Δ . The following theorem gives a sufficient condition for the system (5.6) to be quadratically stable.

²Here we assume the LFR is well-posed, i.e., $\det(I - D_{pq} \Delta(\theta(t))) \neq 0$, $\forall \theta(t) \in \Theta$.

Theorem 7 System (5.6) is quadratically stable if there exists $P \in \mathbf{R}^{n \times n}$ with $P = P^T > 0$ such that for every $\Delta \in \Delta$, there exist $G_\Delta \in \mathbf{C}^{n \times d}$ and $H_\Delta \in \mathbf{C}^{d \times d}$ satisfying

$$\begin{bmatrix} PA + A^T P + G_\Delta C_p + C_p^T G_\Delta^* & P(B_q \Delta) + G_\Delta(D_{pq} \Delta) - G_\Delta + C_p^T H_\Delta^* \\ (B_q \Delta)^T P + (D_{pq} \Delta)^T G_\Delta^* - G_\Delta^* + H_\Delta C_p & H_\Delta(D_{pq} \Delta) + (D_{pq} \Delta)^T H_\Delta^* - H_\Delta - H_\Delta^* \end{bmatrix} < 0. \quad (5.8)$$

Moreover, $V(\psi) = \psi^T P \psi$ is a Lyapunov function that proves the quadratic stability of system (5.6).

Proof: Suppose there exist $P = P^T > 0$, $G_\Delta \in \mathbf{C}^{n \times d}$ and $H_\Delta \in \mathbf{C}^{d \times d}$ satisfying (5.8) for all $\Delta \in \Delta$. Consider system (5.6) for some $\Delta \in \Delta$. The equations governing the system can be rewritten as

$$\dot{x} = Ax + (B_q \Delta)p, \quad (5.9a)$$

$$p = C_p x + (D_{pq} \Delta)p. \quad (5.9b)$$

Now, for any $G_\Delta \in \mathbf{C}^{n \times d}$ and $H_\Delta \in \mathbf{C}^{d \times d}$, we have

$$\begin{aligned} x^T G_\Delta p &= x^T G_\Delta C_p x + x^T G_\Delta (D_{pq} \Delta)p, \\ p^T H_\Delta p &= p^T H_\Delta C_p x + p^T H_\Delta (D_{pq} \Delta)p, \end{aligned}$$

or equivalently

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} G_\Delta C_p + C_p^T G_\Delta^* & G_\Delta(D_{pq} \Delta) - G_\Delta + C_p^T H_\Delta^* \\ (D_{pq} \Delta)^T G_\Delta^* - G_\Delta^* + H_\Delta C_p & H_\Delta(D_{pq} \Delta) + (D_{pq} \Delta)^T H_\Delta^* - H_\Delta - H_\Delta^* \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = 0. \quad (5.10)$$

Then, we have

$$\begin{aligned} \frac{d}{dt}(x(t)^T P x(t)) &= \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} PA + A^T P & P(B_q \Delta) \\ (B_q \Delta)^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ &= \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} PA + A^T P & P(B_q \Delta) \\ (B_q \Delta)^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ &+ \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} G_\Delta C_p + C_p^T G_\Delta^* & G_\Delta(D_{pq} \Delta) - G_\Delta + C_p^T H_\Delta^* \\ (D_{pq} \Delta)^T G_\Delta^* - G_\Delta^* + H_\Delta C_p & H_\Delta(D_{pq} \Delta) + (D_{pq} \Delta)^T H_\Delta^* - H_\Delta - H_\Delta^* \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ &< 0. \end{aligned}$$

(This is simply an application of the $\mathcal{S}$ -procedure; see, e.g., [BEFB94] and the references therein.) $\square$

Theorem 7 implies that the quadratic stability of system (5.6) can be established by checking condition (5.8) for all Δ in the polytope (5.7). With no restrictions on G_Δ and H_Δ —these matrices may depend on Δ —this entails verifying infinite number of matrix inequalities. This issue can be addressed by simply restricting G_Δ and H_Δ to be of special forms such that the left hand side of inequalities (5.8) is convex in Δ . It is then sufficient to check that inequality (5.8) holds for Δ_i , $i = 1, \dots, r$, the vertices of the polytope (5.7). One such choice for G_Δ and H_Δ is described in the following Corollary; this choice is interesting in that it can be interpreted as an “unstructured scaling” technique.

Corollary 2 System (5.6) is quadratically stable if there exist $P = P^T > 0$ and $M = M^T > 0$ such that

$$\begin{bmatrix} A^T P + PA + C_p^T M C_p & P B_{q,i} + C_p^T M D_{pq,i} \\ B_{q,i}^T P + D_{pq,i}^T M C_p & -M + D_{pq,i}^T M D_{pq,i} \end{bmatrix} < 0, \quad i = 1 \dots r, \quad (5.11)$$

where $B_{q,i} = B_q \Delta_i$ and $D_{pq,i} = D_{pq} \Delta_i$.

Proof: Condition (5.8), with $G_\Delta = C_p^T M / 2$ and $H_\Delta = ((D_{pq} \Delta)^T + I)M / 2$, reduces to (5.11). The conclusion of the corollary then follows from Theorem 7. $\square$

Remark 5.2.1 Corollary 2 can be viewed as an extension of the well-known structured scaling methods for robustness analysis. Consider the special case when $\Theta = [-\gamma, \gamma]^m$ with $\gamma > 0$, i.e., when it is a hypercube. In this case, the structured scaling techniques [EBFB92, GA94] can be shown to be equivalent to condition (5.11), with the additional restriction that $M = M^T > 0$ has such a structure that it commutes with Δ . (In our condition (5.11), M has no other constraints other than being positive definite.) More significantly, our approach can also effectively take advantage of the knowledge of polytopic covers that describe the uncertainties more accurately than conventional norm bounds.

While the quadratic stability condition (5.11) is less conservative than the conventional structured scaling methods, this comes at the expense of an increased number of optimization variables, owing to M being unstructured. In addition, the number of LMIs in our condition (5.11) equals r , the number of vertices of Δ . This reflects the added price to pay for the improvement in the robustness analysis [AGB95, BEFB94].

Other special cases for G_Δ and H_Δ are listed below:

1. Let $G_\Delta = G$ and $H_\Delta = H$ be any unstructured real constant matrices. Then (5.8) is feasible if

$$\begin{bmatrix} PA + A^T P + GC_p + C_p^T G^T & PB_{q,i} + GD_{pq,i} - G + C_p^T H^T \\ B_{q,i}^T P + D_{pq,i}^T G^T - G^T + HC_p & HD_{pq,i} + D_{pq,i}^T H^T - H - H^T \end{bmatrix} < 0, \quad i = 1, \dots, r, \quad (5.12)$$

where $B_{q,i} = B_q \Delta_i$ and $D_{pq,i} = D_{pq} \Delta_i$.

2. Let $G_\Delta = C_p^T M/2$ and $H_\Delta = ((D_{pq} \Delta)^T + I)M/2 + \Delta^T S^*$, with $\Delta = \Delta^T$, $M = M^T > 0$ being a real constant matrix and $S = -S^*$, and with both M and S commuting with Δ . Then (5.8) is feasible if

$$\begin{bmatrix} A^T P + PA + C_p^T M C_p & PB_q \Delta + C_p^T M D_{pq} \Delta + C_p^T S \Delta \\ \Delta^T B_q^T P + \Delta^T D_{pq}^T M C_p - \Delta^T S C_p & -M + \Delta^T D_{pq}^T M D_{pq} \Delta + \Delta^T D_{pq}^T S \Delta - \Delta^T S D_{pq} \Delta \end{bmatrix} < 0. \quad (5.13)$$

If $S = -S^T$ is real and skew-symmetric, (5.13) yields the stability criterion derived in [EGS96], which is also the stability criterion for systems affected by time-varying real uncertainties, obtained in the IQC framework [MR97]. If $S = jN$ with $N = N^T$ being real and symmetric, (5.13) gives the constant-scaling version of the the stability criterion from real- μ analysis [FTD91].

3. Consider the special case when $D_{pq} = 0$. Let $G_\Delta = (QC_p)^T/2$ and $H_\Delta = (2S\Delta + Q)^T/2$, where $Q = Q^T$ and S are real matrices. Then (5.8) is feasible if

$$\begin{bmatrix} A^T P + PA + C_p^T Q C_p & PB_{q,i} + C_p^T S_i \\ B_{q,i}^T P + S_i^T C_p & -(Q + S_i + S_i^T) \end{bmatrix} < 0, \quad i = 1 \dots r, \quad (5.14)$$

where $S_i = S \Delta_i$. Condition (5.14) is sufficient and necessary for the quadratic stability of the system $\dot{x} = (A + B_q \Delta(\theta) C_p) x$ over Θ , see [MR97]. Also note that in (5.14), P is not necessary to be positive definite. However, since we are interested in the quadratic stability, we require $P > 0$ in this chapter. Indeed in our case, we can always define $S_i = -Q = I$ without arising more conservatism (See Section 5.6).

For another special case when the LFR degree of system (5.6) is one, further reduction in the conservatism of condition (5.11) is possible. Here, let $G_\Delta = C_p^T M_\Delta/2$ and $H_\Delta = ((D_{pq} \Delta)^T + I)M_\Delta/2$, where M_Δ depends on Δ . Then, (5.8) is equivalent to

$$\begin{bmatrix} A^T P + PA + C_p^T M_\Delta C_p & P(B_q \Delta) + C_p^T M_\Delta (D_{pq} \Delta) \\ (B_q \Delta)^T P + (D_{pq} \Delta)^T M_\Delta C_p & -M_\Delta + (D_{pq} \Delta)^T M_\Delta (D_{pq} \Delta) \end{bmatrix} < 0, \quad \Delta \in \text{Co}\{\Delta_1, \dots, \Delta_r\}. \quad (5.15)$$

Since M_Δ is a variable depending on Δ , condition (5.15) is less conservative than condition (5.11). While condition (5.15) is not tractable in general as it includes infinitely many inequalities, for the special case when the LFR degree of system (5.6) is one, condition (5.15) is equivalent to checking a finite number of LMIs, as the following shows.

Corollary 3 Suppose the LFR degree of system (5.6) is one. If there exist real positive matrices $P = P^T > 0$ and $M_i = M_i^T > 0$ such that

$$\begin{bmatrix} A^T P + P A + C_p^T M_i C_p & P B_{q,i} + C_p^T M_i D_{pq,i} \\ B_{q,i}^T P + D_{pq,i}^T M_i C_p & -M_i + D_{pq,i}^T M_i D_{pq,i} \end{bmatrix} < 0, \quad i = 1, \dots, r, \quad (5.16)$$

where $B_{q,i} = B_q \Delta_i$ and $D_{pq,i} = D_{pq} \Delta_i$, Δ_i is a vertex of the polytope (5.7), then the system is quadratically stable.

Proof: Inequality (5.16) implies that there exists a quadratic Lyapunov function $V(\psi) = \psi^T P \psi$, such that $P > 0$ and

$$P(A + B_q \Delta_i (I - D_{pq} \Delta_i)^{-1} C_p) + (A + B_q \Delta_i (I - D_{pq} \Delta_i)^{-1} C_p)^T P < 0, \quad i = 1, \dots, r.$$

(This follows from an argument exactly along the line of the proof of Lemma 7.) Now, since the LFR degree of system (5.5) is one, it turns out (see [BY89]) that

$$\begin{aligned} & \text{Co} \{ A + B_q \Delta_i (I - D_{pq} \Delta_i)^{-1} C_p \mid i = 1, \dots, r \} \\ &= \{ A + B_q \Delta (I - D_{pq} \Delta)^{-1} C_p \mid \Delta \in \text{Co} \{ \Delta_1, \dots, \Delta_r \} \}. \end{aligned}$$

Therefore,

$$P(A + B_q \Delta (I - D_{pq} \Delta)^{-1} C_p) + (A + B_q \Delta (I - D_{pq} \Delta)^{-1} C_p)^T P < 0,$$

for any $\Delta \in \text{Co} \{ \Delta_1, \dots, \Delta_r \}$, and system (5.5) is quadratically stable. $\square$

Condition (5.16) is clearly less stringent than condition (5.11), as it allows for different scaling matrices M_i for different vertices. We will therefore refer to the application of Corollary 3 as a vertex-dependent scaling method or simply “vertex scaling”.

5.2.2 Robustness analysis using parameter-dependent Lyapunov functions

In Section 5.2.1, the robustness analysis is based on a single quadratic Lyapunov function. This approach can be easily applied in the robust performance analysis. More significantly, this approach leads to convex conditions in solving synthesis problems, which will be shown in Section 5.3.

If system (5.5) is affinely parameter-dependent or polytope type parameter uncertain systems, and if the time varying rate of the uncertainty θ is bounded and lies in the set Φ (specially $\Phi = \{0\}$ when θ is a real constant uncertainty), [GAC96] shows that the quadratic Lyapunov function method can be extended by using parameter-dependent Lyapunov functions: The system $\dot{x} = \mathbf{A}(\theta)x$ is stable if there exists a parameter-dependent Lyapunov function $V(x) = x^T P(\theta)x$ such that for all $\theta(t) \in \Theta$,

$$P(\theta) > 0, \quad \dot{P}(\theta) + P(\theta)\mathbf{A}(\theta) + \mathbf{A}(\theta)^T P(\theta) < 0. \quad (5.17)$$

Since the knowledge of the time varying rate of the uncertainty is incorporated in the analysis, Condition (5.17) is evidently less conservative than condition (5.4). However, condition (5.17) consists of an infinite number of inequalities, even in the simple case when $\mathbf{A}(\theta)$ and $P(\theta)$ are restricted to be affine functions of θ . Multi-convexity techniques are proposed in [GAC96, GN95, AT98] in order to derive a sufficient condition, consisting of a finite number of LMIs, for (5.17) to hold.

Robust stability condition (5.17) has been applied to design gain-scheduled controllers, using multi-convexity techniques [AA98, AT98], or using a more conservative cover of the set of uncertainties [YS97]. The use of parameter-dependent Lyapunov function to solve the problems at hand, i.e., when the state space matrices are real-valued rational functions of $\theta(t)$, remains essentially unexplored.

Our approach is combining the polytope type system analysis method and conventional constant scaling techniques to solve the robustness problem of LFR uncertain systems. Following the same line as Theorem 7, we derive a sufficient condition for robust stability using parameter dependent Lyapunov functions; in addition, the associated controller synthesis problem can also be reformulated as a convex optimization problem. The stability condition using parameter dependent Lyapunov functions is summarized in the following corollary.

Corollary 4 Let $\Theta_i = [\Theta_{i1} \cdots \Theta_{im}]^T$, $i = 1, \dots, r$ be the vertices of Θ and $\Delta_i = \Delta(\Theta_i)$. Let $\Phi_k = [\Phi_{k1} \cdots \Phi_{km}]^T$, $k = 1, \dots, v$ be the vertices of Φ . System (5.6) is stable for all θ satisfying $(\theta, \dot{\theta}) \in \Theta \times \Phi$ if there exist $Q_j = Q_j^T \in \mathbb{R}^{n \times n}$ and $M = M^T > 0$ such that

$$\begin{bmatrix} E_{11,ik} & E_{12,i} \\ E_{12,i}^T & E_{22,i} \end{bmatrix} < 0 \quad \text{and} \quad Q_0 + \sum_{j=1}^m \Theta_{ij} Q_j > 0, \quad i = 1, \dots, r, \quad k = 1, \dots, v, \quad (5.18)$$

where

$$\begin{aligned} E_{11,ik} &= A \left(Q_0 + \sum_{j=1}^m \Theta_{ij} Q_j \right) + \left(Q_0 + \sum_{j=1}^m \Theta_{ij} Q_j \right) A^T + B_{q,i} M B_{q,i}^T - \sum_{j=1}^m \Phi_{kj} Q_j, \\ E_{12,i} &= \left(Q_0 + \sum_{j=1}^m \Theta_{ij} Q_j \right) C_p^T + B_{q,i} M D_{pq,i}^T, \\ E_{22,i} &= -M + D_{pq,i} M D_{pq,i}^T, \end{aligned}$$

and $B_{q,i} = B_q \Delta_i$, $D_{pq,i} = D_{pq} \Delta_i$, $i = 1 \dots r$, $k = 1 \dots v$.

Proof: It is easy to establish using routine matrix algebra that if conditions (5.18) hold for all the vertices (Θ_i, Φ_i) , they also hold for any uncertainty θ with $(\theta, \dot{\theta}) \in \Theta \times \Phi$. Following the same line as the proof of Theorem 7, it can be shown that the affinely parameter-dependent Lyapunov function

$$V(x) = x^T \left(Q_0 + \sum_{i=1}^m \theta_i(t) Q_i \right)^{-1} x$$

provides a guarantee for the stability of system (5.6) for all θ satisfying $(\theta, \dot{\theta}) \in \Theta \times \Phi$. $\square$

When $Q_i = 0$, $i = 1, \dots, m$, condition (5.18) reduces to the robust stability conditions based on a single quadratic Lyapunov function, given in Corollary 2, where the information on the rate of variation of the uncertainty is not incorporated into the stability analysis. Thus Corollary 4 offers the potential for an improved stability analysis when information on the rate of variation of the uncertainty is available. However, this comes with the added price that an LMI needs to be checked at every vertex of $\Theta \times \Phi$.

If the set $\Theta \times \Phi$ is symmetric around the origin, i.e., $(\theta, \dot{\theta}) \in \Theta \times \Phi$ implies $(-\theta, -\dot{\theta}) \in \Theta \times \Phi$, it can be verified that if condition (5.18) is feasible for some Q_i , then the system must be quadratically stable, i.e., $Q_i = 0$, $i = 1, 2, \dots$. Therefore in this case Corollary 4 offers no improvement over quadratic stability condition. However, for many cases when set of uncertainties is not symmetric around the origin, Corollary 4 can be strictly less conservative than the quadratic stability condition (see the numerical example).

Note that Corollary 4 offers a direct approach for the use of parameter-dependent Lyapunov functions in the analysis of systems that exhibit a rational dependence on the parameters. This in contrast with indirect techniques such as using a (conservative) polytopic cover for

$$\{(\theta, \dot{\theta}, \theta_1 \theta_1, \dots, \theta_i \theta_j, \dots, \theta_m \theta_m) | \theta \in \Theta, \dot{\theta} \in \Phi, i \leq j\}$$

as in [GN95, YS97], or using a conservative multi-convexity method [AA98, GAC96, AT98].

5.3 Gain-scheduled output feedback synthesis

We next consider the problem of designing a gain-scheduled output feedback control strategy $u = K(y, \theta(t))$ such that system (5.1) is robustly stable. In particular, we show that the sufficient condition for robust stability that we stated in Corollary 2 and Corollary 4 for system (5.5) can be directly extended to designing a gain-scheduled controller $K(y, \theta(t))$ that is guaranteed to stabilize system (5.1). As noted in Section 5.2, our analysis technique guarantees a larger stability margin over conventional constant structured scaling methods; therefore, the corresponding gain-scheduled controller will come with a larger guaranteed closed-loop stability margin as well³.

³In this context, we cite the works [SNB97, Sch96b], in which full block scalings method is used to solve gain-scheduled controller synthesis problems. The robust stability condition using full block scalings is less conservative than condition (5.11). However, the procedure to construct controllers using full block scalings method is numerically delicate.

Consider system (5.1). Since A , B , C and D are real-valued rational functions of θ , we have

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \begin{bmatrix} A & B_u \\ C_y & D_{yu} \end{bmatrix} + \begin{bmatrix} B_q \\ D_{yq} \end{bmatrix} \Delta(\theta) (I - D_{pq} \Delta(\theta))^{-1} \begin{bmatrix} C_p & D_{pu} \end{bmatrix} \quad (5.19)$$

for some appropriate matrices A , B_q , B_u , C_p , C_y , D_{yq} , D_{yu} , D_{pq} and D_{pu} . Then, an equivalent linear-fractional representation of system (5.1) is described by

$$\begin{aligned} \dot{x} &= Ax + B_q q + B_u u, & p &= C_p x + D_{pq} q + D_{pu} u, & y &= C_y x + D_{yq} q + D_{yu} u. \\ q &= \Delta(\theta) p, & \Delta(\theta) &= \text{diag}(\theta_1 I_{s_1}, \dots, \theta_m I_{s_m}), \end{aligned} \quad (5.20)$$

where $x \in \mathbb{R}^n$, $q \in \mathbb{R}^d$, $p \in \mathbb{R}^d$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$. We will henceforth assume that $D_{yu} = 0$ and $D_{yq} = 0$. The former is a standard assumption and can always be satisfied via loop transformations, while the latter is a technical assumption that implies that there is no uncertainty in the measured output.

The controller that we will design consists of a parameter-dependent linear system with unity feedback (see Fig. 5.2). This, while incurring no loss of generality, will be important in establishing LMI conditions for the existence of a stabilizing gain-scheduled controller. Thus, the gain-scheduled controller $\mathcal{K}$ is described by the state equations

$$\begin{aligned} \dot{x}_k &= A_{k\Delta} x_k + B_{k\Delta_1} y + B_{k\Delta_2} v, & u &= C_{k\Delta_1} x_k + D_{k\Delta_{11}} y + D_{k\Delta_{12}} v, \\ w &= C_{k\Delta_2} x_k + D_{k\Delta_{21}} y + D_{k\Delta_{22}} v, & v &= w, \end{aligned} \quad (5.21)$$

where $x_k(t) \in \mathbb{R}^{n_k}$, $v \in \mathbb{R}^d$ and $w \in \mathbb{R}^d$.

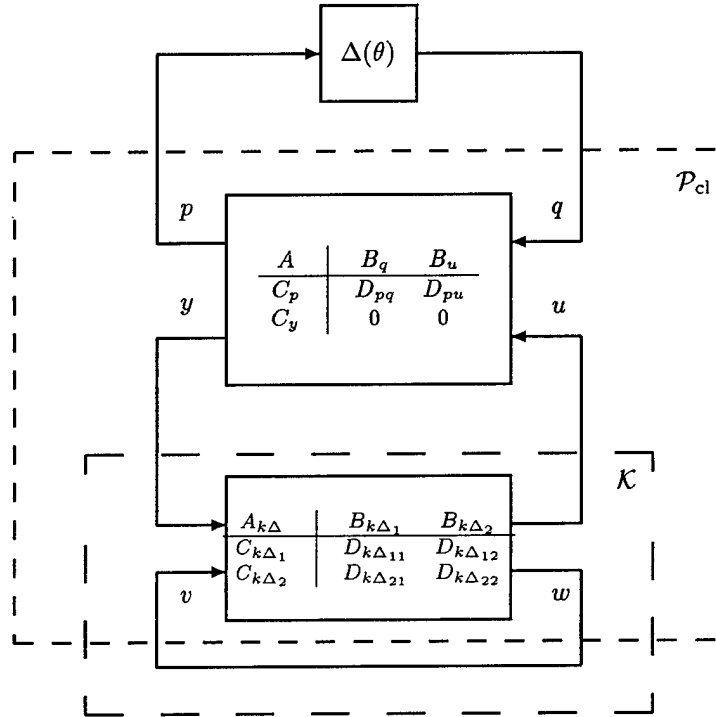


Figure 5.2: Gain-scheduled output feedback controller

All the state-space matrices in (5.21) are functions of the time-varying matrix $\Delta(\theta(t))$; hence their subscript Δ . (Their exact dependence on Δ will become clear later.) Then, with the notation

$$\Omega(\theta) = \begin{bmatrix} A_{k\Delta} & B_{k\Delta_1} & B_{k\Delta_2} \\ C_{k\Delta_1} & D_{k\Delta_{11}} & D_{k\Delta_{12}} \\ C_{k\Delta_2} & D_{k\Delta_{21}} & D_{k\Delta_{22}} \end{bmatrix} \quad (5.22)$$

the state space equations governing the closed-loop system $\mathcal{P}_{cl}$ are

$$\begin{aligned}\begin{bmatrix} \dot{x} \\ \dot{x}_k \end{bmatrix} &= A_{cl}(\theta) \begin{bmatrix} x \\ x_k \end{bmatrix} + B_{cl}(\theta) \begin{bmatrix} v \\ q \end{bmatrix}, \\ \begin{bmatrix} w \\ p \end{bmatrix} &= C_{cl}(\theta) \begin{bmatrix} x \\ x_k \end{bmatrix} + D_{cl}(\theta) \begin{bmatrix} v \\ q \end{bmatrix}, \\ \begin{bmatrix} v \\ q \end{bmatrix} &= \begin{bmatrix} I & \\ & \Delta(\theta) \end{bmatrix} \begin{bmatrix} w \\ p \end{bmatrix},\end{aligned}$$

where

$$A_{cl}(\theta) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_u & 0 \\ I & 0 & 0 \end{bmatrix} \Omega(\theta) \begin{bmatrix} 0 & I \\ C_y & 0 \\ 0 & 0 \end{bmatrix}, \quad (5.23a)$$

$$B_{cl}(\theta) = \begin{bmatrix} 0 & B_q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_u & 0 \\ I & 0 & 0 \end{bmatrix} \Omega(\theta) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}, \quad (5.23b)$$

$$C_{cl}(\theta) = \begin{bmatrix} 0 & 0 \\ C_p & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & I \\ 0 & D_{pu} & 0 \end{bmatrix} \Omega(\theta) \begin{bmatrix} 0 & I \\ C_y & 0 \\ 0 & 0 \end{bmatrix}, \quad (5.23c)$$

$$D_{cl}(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & D_{pq} \end{bmatrix} + \begin{bmatrix} 0 & 0 & I \\ 0 & D_{pu} & 0 \end{bmatrix} \Omega(\theta) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}. \quad (5.23d)$$

Next, we state two lemmas that play an important role in the subsequent development.

Lemma 5.3.1 (Elimination Lemma [BEFB94, GA94]) *Given $G \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times p}$, $V \in \mathbb{R}^{n \times q}$, there exists a matrix $\Omega \in \mathbb{R}^{p \times q}$ such that*

$$G + U\Omega V^T + V\Omega^T U^T > 0$$

if and only if

$$U_{\perp}^T G U_{\perp} > 0 \quad \text{and} \quad V_{\perp}^T G V_{\perp} > 0,$$

where $U_{\perp}$ and $V_{\perp}$ are the orthogonal complements of U and V respectively.

Lemma 5.3.2 (Completion Lemma [Pac94]) *Let $0 < X^T = X \in \mathbb{R}^{n \times n}$ and $0 < Y = Y^T \in \mathbb{R}^{n \times n}$. There exist $X_2 \in \mathbb{R}^{n \times r}$, $X_3 \in \mathbb{R}^{r \times r}$, $Y_2 \in \mathbb{R}^{n \times r}$ and $Y_3 \in \mathbb{R}^{r \times r}$ such that*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0 \quad \text{and} \quad \text{rank} \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \leq n + r.$$

5.3.1 Gain-scheduled controller design using quadratic Lyapunov function

We first design quadratically stabilizing gain-scheduled output feedback controllers by applying Corollary 2. The following theorem provides a sufficient condition for the existence of a full order robustly stabilizing gain-scheduled controller.

Theorem 8 *Consider the closed-loop system in Fig. 5.2. Let $n_k = n$. Then, given $\gamma > 0$, there exist $P = P^T > 0$ and $M = M^T > 0$ such that for every $\theta(t) \in \gamma\Theta$, there exists $\Omega(\theta(t))$ satisfying*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + C_{cl}^T M C_{cl} & P B_{cl} + C_{cl}^T M D_{cl} \\ B_{cl}^T P + D_{cl}^T M C_{cl} & -M + D_{cl}^T M D_{cl} \end{bmatrix} < 0, \quad (5.24)$$

if and only if there exist $R \in \mathbf{R}^{n \times n}$ and $S \in \mathbf{R}^{n \times n}$, $L \in \mathbf{R}^{d \times d}$ and $J \in \mathbf{R}^{d \times d}$ such that the following matrix inequalities hold:

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_p^T & \gamma B_{q,i} J \\ C_p R & -J & \gamma D_{pq,i} J \\ \hline \gamma J B_{q,i}^T & \gamma J D_{pq,i}^T & -J \end{array} \right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.25a)$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^T S + SA & \gamma S B_{q,i} & C_p^T L \\ \gamma B_{q,i}^T S & -L & \gamma D_{pq,i}^T L \\ \hline LC_p & \gamma L D_{pq,i} & -L \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.25b)$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad i = 1, \dots, r.$$

where $B_{q,i} = B_q \Delta_i$, $D_{pq,i} = D_{pq} \Delta_i$, Δ_i , $i = 1, \dots, r$ are the vertices of the polytope Δ , N_R and N_S are matrices whose columns comprise the bases of the null spaces of $[B_u^T \ D_{pu}^T]$ and $[C_y \ 0]$ respectively.

Proof: Since $\theta \in \gamma\Theta$, we have $\Delta \in \gamma\Delta$. Using Schur complements, observe that condition (5.24) is equivalent to

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} & C_{cl}^T \\ B_{cl}^T P & -M & D_{cl}^T \\ C_{cl} & D_{cl} & -M^{-1} \end{bmatrix} < 0. \quad (5.26)$$

Let

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0(\theta) = \begin{bmatrix} 0 & B_q(\theta) \\ 0 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 \\ C_p & 0 \end{bmatrix}, \quad D_0(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & D_{pq}(\theta) \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & B_u & 0 \\ I & 0 & 0 \end{bmatrix}, \quad D_{pu} = \begin{bmatrix} 0 & 0 & I \\ 0 & D_{pu} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & I \\ C_y & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{yq} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}.$$

where $B_q(\theta) = B_q \Delta(\theta)$ and $D_{pq}(\theta) = D_{pq} \Delta(\theta)$ affinely depend on θ . Then using (5.23), inequality (5.26) can be written as

$$X + U^T \Omega(\theta) V + V^T \Omega(\theta)^T U < 0, \quad (5.27)$$

where

$$X = \begin{bmatrix} A_0^T P + P A_0 & P B_0 & C_0^T \\ B_0^T P & -M & D_0^T \\ C_0 & D_0 & -M^{-1} \end{bmatrix}, \quad U = [B^T P \ 0 \ D_{pu}^T] \text{ and } V = [C \ D_{yq} \ 0].$$

Partition P , P^{-1} , M and M^{-1} as

$$P = \begin{bmatrix} S & \star \\ \star & \star \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} R & \star \\ \star & \star \end{bmatrix}, \quad M = \begin{bmatrix} \star & \star \\ \star & L \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \star & \star \\ \star & J \end{bmatrix}.$$

Using this partitioning of P and M and the elimination Lemma and the completion Lemma, it is straight forward to verify that inequality (5.27) is feasible for some $\Omega(\Delta(\theta))$ if and only if

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_p^T & (B_q \Delta) J \\ C_p R & -J & (D_{pq} \Delta) J \\ \hline J(B_q \Delta)^T & J(D_{pq} \Delta)^T & -J \end{array} \right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.28a)$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^T S + SA & S(B_q \Delta) & C_p^T L \\ (B_q \Delta)^T S & -L & (D_{pq} \Delta)^T L \\ \hline LC_p & L(D_{pq} \Delta) & -L \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} \geq 0. \quad (5.28b)$$

Note that the left hand sides of (5.28a) are affine in Δ . Therefore conditions (5.28a) hold if and only if they hold for each vertex of $\gamma\Delta$. Finally it is easily argued that if (5.28) is feasible with nonstrict inequality, it is also feasible with strict inequality. Thus we get condition (5.25). $\square$

The main implication of Theorem 8 is that we now have a sufficient condition for the existence of a robustly stabilizing gain-scheduled controller for system (5.1). In contrast with the gain-scheduled controller designed in [AG95] and [EGS96], there are no structure constraints on M in Theorem 8; consequently, even in the case when Θ is a hypercube, our design is at most as conservative as the design using structured scalings. Of course, as with the stability analysis, our design can also directly take advantage of the knowledge of more accurate polytopic covers for the uncertainties.

Remark 5.3.1 *Note that since the gain-scheduled controller depends on the uncertain parameters $\theta(t)$, the LFR degree of the closed loop system is always greater than one. Thus we may not use vertex scaling (Corollary 3) in designing gain-scheduled controllers. However for general output feedback synthesis problems (see [ACGB93]), we may still apply Corollary 3 to design output feedback controllers that guarantee larger closed-loop stability margins.*

A direct consequence of Theorem 8 is that a lower bound of the robust stabilizability margin ρ_m can be computed by solving the following Generalized Eigenvalue Minimization Problem (GEVP).

Minimize: κ

Subject to:

$$\begin{aligned} & \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_p^T & B_{q,i}J \\ C_p R & -X & D_{pq,i}J \\ \hline JB_{q,i}^T & JD_{pq,i}^T & -X \end{array} \right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \\ & \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^T S + SA & SB_{q,i} & C_p^T L \\ B_{q,i}^T S & -Y & D_{pq,i}^T L \\ \hline LC_p & LD_{pq,i} & -Y \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \\ & \begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad i = 1, \dots, r. \\ & X < \kappa J, \quad Y < \kappa L \end{aligned} \tag{5.29}$$

where $B_{q,i} = B\Delta_i$, $D_{pq,i} = Dpq\Delta_i$, R , S , L , and J are optimization variables, $\rho_m = 1/\kappa$ is the robustly stabilizability margin.

For a given $\gamma > 0$, we have thus far only derived conditions for the existence of a quadratically stabilizing gain-scheduled output feedback controller over $\gamma\Theta$. We now describe an algorithm for explicitly constructing a family of gain-scheduled output feedback controllers that are guaranteed to stabilize the system over $\gamma\Theta$.

Step 1. Design controllers corresponding to each vertex of the polytope $\gamma\Delta$.

Let (R, S, L, J) be a feasible solution to (5.25). With $S > R^{-1}$, define $P_{12} = (S - R^{-1})^{1/2}$ and $Q_{12} = -RP_{12}$, and

$$P = \begin{bmatrix} S & P_{12} \\ P_{12}^T & I \end{bmatrix}. \tag{5.30a}$$

Then,

$$P^{-1} = \begin{bmatrix} R & Q_{12} \\ Q_{12}^T & I - P_{12}^T Q_{12} \end{bmatrix}. \tag{5.30b}$$

Next, with $M_{12} = (L - J^{-1})^{1/2}$ and $N_{12} = -JM_{12}$, define

$$M = \begin{bmatrix} I & M_{12}^T \\ M_{12} & L \end{bmatrix}.$$

Then,

$$M^{-1} = \begin{bmatrix} I - M_{12}^T N_{12} & N_{12}^T \\ N_{12} & J \end{bmatrix}.$$

With P , M and M^{-1} defined above, for each Δ_i , $i = 1, \dots, r$, the LMI (5.27) must be feasible from the equivalence between (5.27) and (5.25). By solving the LMI feasibility problem (5.27) for $\Omega_{\Delta_i} = \Omega(\Delta_i(\theta))$, we get a controller Ω_{Δ_i} corresponding to the vertex Δ_i .

Step 2. Design the gain-scheduled controller.

For any $\Delta(\theta(t)) \in \gamma\Delta$, solve the set of linear equations $\Delta(\theta(t)) = \sum_{i=1}^r \alpha_i(t) \gamma \Delta_i$ to get $\alpha_i(t)$. Define

$$\Omega(\theta(t)) = \sum_{i=1}^r \alpha_i(t) \gamma \Omega_{\Delta_i}. \quad (5.31)$$

Then, (5.31) gives the state space matrices of a gain-scheduled controller that is guaranteed to quadratically stabilize the system.

Note that in controller (5.31), every system matrix of the controller in (5.22) is “scheduled”, that is, depends on $\theta(t)$ (unlike with [AGB95, EGS96] where only B_{k_2} , $D_{k_{12}}$ and $D_{k_{22}}$ are scheduled).

5.3.2 Gain-scheduled controller design using parameter-dependent Lyapunov functions

For affinely parameter-dependent systems, the gain-scheduled controller design based on a single quadratic Lyapunov function can be improved by using parameter-dependent Lyapunov functions [AA98]. In this section, we consider the use of parameter-dependent Lyapunov functions for designing gain-scheduled controllers for the more general parameter-dependent systems described by the LFR framework.

Theorem 9 Consider the parameter-dependent system with a full order output feedback controller in Fig. 5.2, i.e., $n_k = n$. Then the parameter-dependent system is robustly stabilizable for all θ satisfying $(\theta, \dot{\theta}) \in \Theta \times \Phi$ if there exist $R_j \in \mathbb{R}^{n \times n}$ and $S_j \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{d \times d}$ and $J \in \mathbb{R}^{d \times d}$ such that the following matrix inequalities hold:

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} E_{11} & E_{12} & E_{13} \\ E_{12}^T & -J & E_{23} \\ \hline E_{13}^T & E_{23}^T & -J \end{array} \right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.32a)$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} F_{11} & F_{12} & F_{13} \\ F_{12}^T & -X & F_{23} \\ \hline F_{13}^T & F_{23}^T & -L \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.32b)$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad (5.32c)$$

$$\begin{bmatrix} L & \Delta(\theta(t))X \\ X\Delta(\theta(t)) & X \end{bmatrix} > 0, \quad (5.32d)$$

where

$$R = R_0 + \sum_{j=1}^m \theta_j(t) R_j, \quad S = S_0 + \sum_{j=1}^m \theta_j(t) S_j,$$

$$E_{11} = AR + RA^T - \sum_{j=1}^m \dot{\theta}_j(t) R_j,$$

$$E_{12} = RC_p^T, \quad E_{13} = B_q(\theta)J, \quad E_{23} = D_{pq}(\theta)J,$$

$$F_{11} = A^T S + SA + \sum_{j=1}^m \dot{\theta}_j(t) S_j,$$

$$F_{12} = SB_q, \quad F_{13} = C_p^T L, \quad F_{23} = D_{pq}^T L.$$

$B_q(\theta) = B_q \Delta(\theta(t))$, $D_{pq}(\theta) = D_{pq} \Delta(\theta(t))$, N_R and N_S are matrices whose columns comprise the bases of the null spaces of $[B_u^T \ D_{pu}^T]$ and $[C_y \ 0]$ respectively.

Note that conditions (5.32) are linear matrix inequalities. Therefore we only need to check the feasibility of these LMIs on the vertices Δ_i and Φ_k . Moreover, when $R_j = 0$ and $S_j = 0$, $j = 1, \dots, m$, conditions (5.32) yield the quadratically stabilizing gain-scheduled controller synthesis condition (5.25); this corresponds to the situation when the rate of variation of the uncertainty cannot be measured in real time or is unbounded. Conditions (5.32) offer the potential of improved gain-scheduled controller design when information on the rate of variation of the uncertainty is available.

Proof: Condition

$$\begin{bmatrix} L & \Delta(\theta(t))X \\ X\Delta(\theta(t)) & X \end{bmatrix} > 0$$

implies $L > \Delta X \Delta$. Multiplying the first inequality in (5.32b) on the left and right by $\text{diag}(I, \Delta, I)$, we get

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} F_{11} & \hat{F}_{12} & F_{13} \\ \hline \hat{F}_{12}^T & -L & \hat{F}_{23} \\ \hline F_{13}^T & \hat{F}_{23}^T & -L \end{array} \right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \quad (5.33)$$

where $\hat{F}_{12} = \left(S_0 + \sum_{j=1}^m \theta_j(t) S_j \right) B_q(\theta)$ and $\hat{F}_{23} = D_{pq}(\theta)^T L$. Therefore (5.32b) holds for all $(\theta, \dot{\theta}) \in \Theta \times \Phi$

if and only if (5.33) holds for all $(\theta, \dot{\theta}) \in \Theta \times \Phi$. Using a similar argument as the one in the proof of Theorem 8, it can be checked that

$$\begin{bmatrix} A_{cl}(\theta)^T P(\theta) + P(\theta) A_{cl}(\theta) + C_{cl}(\theta)^T M C_{cl}(\theta) + \dot{P}(\theta) & P(\theta) B_{cl}(\theta) + C_{cl}(\theta)^T M D_{cl}(\theta) \\ B_{cl}(\theta)^T P(\theta) + D_{cl}(\theta)^T M C_{cl}(\theta) & -M + D_{cl}(\theta)^T M D_{cl}(\theta) \end{bmatrix} < 0,$$

where

$$P(\theta) = \begin{bmatrix} S & -(S - R^{-1}) \\ -(S - R^{-1}) & S - R^{-1} \end{bmatrix}, \quad S = S_0 + \sum_{j=1}^m \theta_j(t) S_j, \quad R = R_0 + \sum_{j=1}^m \theta_j(t) R_j,$$

and

$$M = \begin{bmatrix} I & M_{12}^T \\ M_{12} & L \end{bmatrix} > 0, \quad M_{12} = (L - J^{-1})^{1/2}, \quad N_{12} = -J M_{12}.$$

Thus the system is robustly stable and $P(\theta)$ defines parameter-dependent Lyapunov functions which guarantee the stability. $\square$

We now present an algorithm for explicitly constructing a family of gain-scheduled output feedback controllers that are guaranteed to stabilize the system.

Suppose that the synthesis conditions (5.32) are feasible for some matrix variables L , J , and R_j, S_j , $j = 1, \dots, m$.

Step 1. For a uncertainty measured in real time, $(\theta, \dot{\theta}) \in \Theta \times \Phi$, construct $P(\theta)$ and $P(\theta)^{-1}$.

Let

$$S = S_0 + \sum_{j=1}^m \theta_j(t) S_j, \quad R = R_0 + \sum_{j=1}^m \theta_j(t) R_j.$$

Then

$$\dot{S} = \sum_{j=1}^m \dot{\theta}_j(t) S_j, \quad \dot{R} = \sum_{j=1}^m \dot{\theta}_j(t) R_j.$$

Define

$$P = \begin{bmatrix} S & -(S - R^{-1}) \\ -(S - R^{-1}) & S - R^{-1} \end{bmatrix},$$

and

$$M = \begin{bmatrix} I & M_{12}^T \\ M_{12} & L \end{bmatrix} > 0,$$

where $M_{12} = (L - J^{-1})^{1/2}$ and $N_{12} = -JM_{12}$. Thus

$$P^{-1} = \begin{bmatrix} R & R \\ R & (S - R^{-1})^{-1}SR \end{bmatrix},$$

and

$$M^{-1} = \begin{bmatrix} I - M_{12}^T N_{12} & N_{12}^T \\ N_{12} & J \end{bmatrix} > 0.$$

Then

$$\dot{P} = \begin{bmatrix} \dot{S} & -(\dot{S} + R^{-1}\dot{R}R^{-1}) \\ -(\dot{S} + R^{-1}\dot{R}R^{-1}) & \dot{S} + R^{-1}\dot{R}R^{-1} \end{bmatrix}.$$

Since conditions (5.32) are feasible with L , J , and $R_j, S_j, j = 1, \dots, m$, it can be checked using the Elimination Lemma 5.3.1 that

$$X(\theta, \dot{\theta}) + U^T \Omega(\theta) V + V^T \Omega(\theta)^T U < 0, \quad (5.34)$$

must be feasible for some $\Omega(\theta)$, where

$$X(\theta, \dot{\theta}) = \begin{bmatrix} A_0^T P + P A_0 + \dot{P} & P B_0 & C_0^T \\ B_0^T P & -M & D_0^T \\ C_0 & D_0 & -M^{-1} \end{bmatrix},$$

$$U = \begin{bmatrix} B^T P & 0 & D_{pu}^T \end{bmatrix}, \quad V = \begin{bmatrix} C & D_{yq} & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0(\theta) = \begin{bmatrix} 0 & B_q(\theta) \\ 0 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 \\ C_p & 0 \end{bmatrix}, \quad D_0(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & D_{pq}(\theta) \end{bmatrix}.$$

Step 2. Design the gain-scheduled controller.

Solve the LMI feasibility problem (5.34) for $\Omega(\theta)$, which comprises the state-space matrices of the gain-scheduled controller that stabilizes the system. Moreover, $V(x, \theta) = x^T P(\theta)x$ is a parameter-dependent Lyapunov function that guarantees the stability of the closed loop system.

Note that in the above algorithm, since $P(\theta)$ is not an affine function on θ , solving (5.34) for a stabilizing gain-scheduled controller is not as easy as designing a gain-scheduled quadratic stabilizing controller in Section 5.3.1 (where $\Omega(\theta)$ is simply a convex combination of the controllers Ω_i that each corresponds to a vertex of Δ). In order to construct the gain-scheduled stabilizing controller using parameter-dependent Lyapunov functions, we need to solve the LMI feasibility problem (5.34) in real time. One approach is to solve LMI feasibility problem (5.34) numerically using an LMI solver [GN95, GDN95]; another approach, which requires less on line computation time, is to find a feasible solution of (5.34) analytically (see [GA94]).

5.4 Numerical Examples

In Section 5.2 and Section 5.3, we observed that our approach is in general less conservative than structured scaling methods. We now illustrate this point through numerical examples.

5.4.1 Improved robustness analysis

The objective of the first example is to demonstrate that our approach yields significantly better results for robust stability analysis, as compared to the structured constant scaling methods. Consider a second order differential equation with parametric uncertainties

$$\ddot{x} + (1 - r(t) \cos \phi + r(t) \sin \phi + 0.5r(t)^2 \sin 2\phi)\dot{x} + x = 0, \quad (5.35)$$

where $r(t)$ is a bounded uncertain time-varying parameter and ϕ is a (uncertain) angle lying in the sector $[0, \pi/4]$. A state-space realization of (5.35) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 + r(t) \cos \phi - r(t) \sin \phi - 0.5r(t)^2 \sin 2\phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Note that the state matrix is not a real rational function on the uncertainties. However, with a simple change of variables $\theta_1(t) = r(t) \cos \phi$ and $\theta_2(t) = r(t) \sin \phi$, we obtain the following LFR with $\Delta(t) = \text{diag}(\theta_1(t), \theta_2(t))$:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} q \\ p &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} q \\ q &= \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} p. \end{aligned} \quad (5.36)$$

This LFR is always well-posed. Define

$$\Theta = \{(\cos \phi, \sin \phi) \mid \phi \in [0, \pi/4] \cup [\pi, 5\pi/4]\}.$$

This set is shown shaded in Fig. 5.3. Note that Θ is not a polytope; we therefore use its polytopic covers in order to apply the results of Sections 5.2 and 5.3. The stability margin σ_m is the largest σ such that the stability of the system (5.36) can be guaranteed for any $r \in [0, \sigma]$.

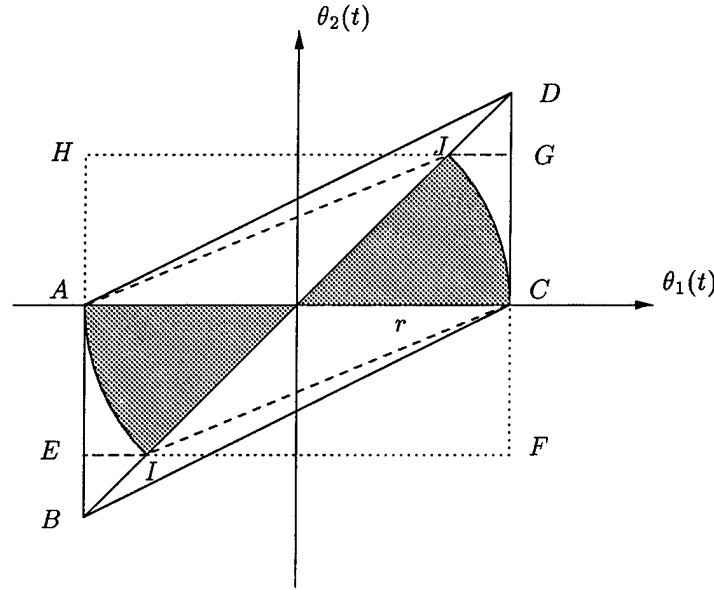


Figure 5.3: Uncertainty set $(\theta_1(t), \theta_2(t))$ and its polytopic cover

We will compare the following robust stability analysis methods, using as the basis for comparison the robust stability margin that they can guarantee for system (5.36):

1. *Hypercube cover, diagonal scaling* [EBFB92, BEFB94, DPZ91]. This is equivalent to covering Θ by a rectangle $EFGHE$ (dotted line).
2. *Polytopic cover, unstructured scaling (Corollary 2)* This can be interpreted as covering Θ by the polytope $ABCD A$ (solid line).
3. *More accurate polytopic cover, unstructured scaling (Corollary 2)*. This can be interpreted as covering Θ by the polytope $AEICGJA$ (dashed line).
4. *Vertex scaling (Corollary 3)*. Note that the LFR degree of system (5.36) is one, and therefore Corollary 3 can be applied with both polytopic covers $ABCD A$ and $AEICGJA$ for Θ .

In this example, the *exact* stability margin turns out to be one. The lower bounds on the robust stability margin computed⁴ using the four approaches described above are given in Table 5.1. The results illustrate

⁴The actual calculation was performed by reformulating the lower bound calculation problem as a GEVP; see [BW98].

the observation that for linear systems affected by time-varying parameters, the approach described in this chapter offers significant improvement for robustness analysis over the traditional structured scaling methods.

Polytope	No. of LMIs in (5.11)	Constant scaling	Variable scaling
<i>EFGHE</i>	1	0.4874	—
<i>ABCD A</i>	2	0.8082	1.0000
<i>AEICGJA</i>	3	0.9024	1.0000

Table 5.1: A comparison of the stability analysis by using structured scaling and unstructured scalings

5.4.2 Improved gain-scheduled controller synthesis

We have seen in Section 5.4.1 that our approach offers significant improvement over conventional structured scaling techniques for robustness analysis. We now demonstrate that similar improvements accrue with gain-scheduled controller synthesis as well.

Consider the system in Example 1 with an additional control input u and a measured output y . The linear fractional representation of the system is

$$\begin{aligned}
 \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
 p &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} q, \\
 y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
 q &= \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} p.
 \end{aligned} \tag{5.37}$$

Table 5.2 shows a comparison of the performance of different synthesis methods, based on the robust stabilizability margin that they can guarantee using quadratic Lyapunov functions for system (5.37). It can be seen that our approach using unstructured scalings yields significantly improved stabilizability margin than the design using structured scalings.

Type of scaling	Polytope	No. of LMIs in (5.25a)	Constant scaling
Diagonal	<i>EFGHE</i>	2	0.9985
Unstructured	<i>ABCD A</i>	4	1.3646
Unstructured	<i>AEICGJA</i>	6	1.8609

Table 5.2: A comparison of gain-scheduled controller synthesis by using structured scaling and unstructured scalings

5.4.3 Improved stability analysis with parameter dependent Lyapunov function

In this example, we show that the robust stability criterion by using parameter-dependent Lyapunov functions (Corollary 4) is strictly less conservative than the quadratic stability criterion (Corollary 2).

Consider the system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1.1447 & -2.2358 \\ 2.5930 & 1.7060 \end{bmatrix} q, \\ p &= \begin{bmatrix} 1.0416 & 2.3899 \\ -0.5656 & -3.3902 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} q, \\ q &= \Delta p, \end{aligned} \quad (5.38)$$

where $\Delta = \text{diag}(\theta_1(t), \theta_2(t))$, $\theta_1(t) = r(t) \cos \phi$ and $\theta_2(t) = r(t) \sin \phi$, ϕ is an uncertain parameter and lies in the set $[0, \pi/4]$. $r(t)$ is a bounded time-varying uncertain parameter and satisfies $r(t) \in [0, r]$. The rate of variation of $r(t)$ is also bounded by k , i.e., $|\dot{r}(t)| \leq k$. With different k , Fig. 5.4 shows the lower bounds of r , under which the robust stability of the system can be guaranteed by using parameter-dependent Lyapunov functions. The results show that the information of the bounds on the rate of variation of the uncertainties can be incorporated in the robustness analysis to reduce the conservatism.

Similarly, the gain-scheduled controller design using parameter dependent Lyapunov functions is also less conservative than the gain-scheduled controller design using a single quadratic Lyapunov function. However, the improvement comes with the cost of a larger number of LMIs and optimization variables.

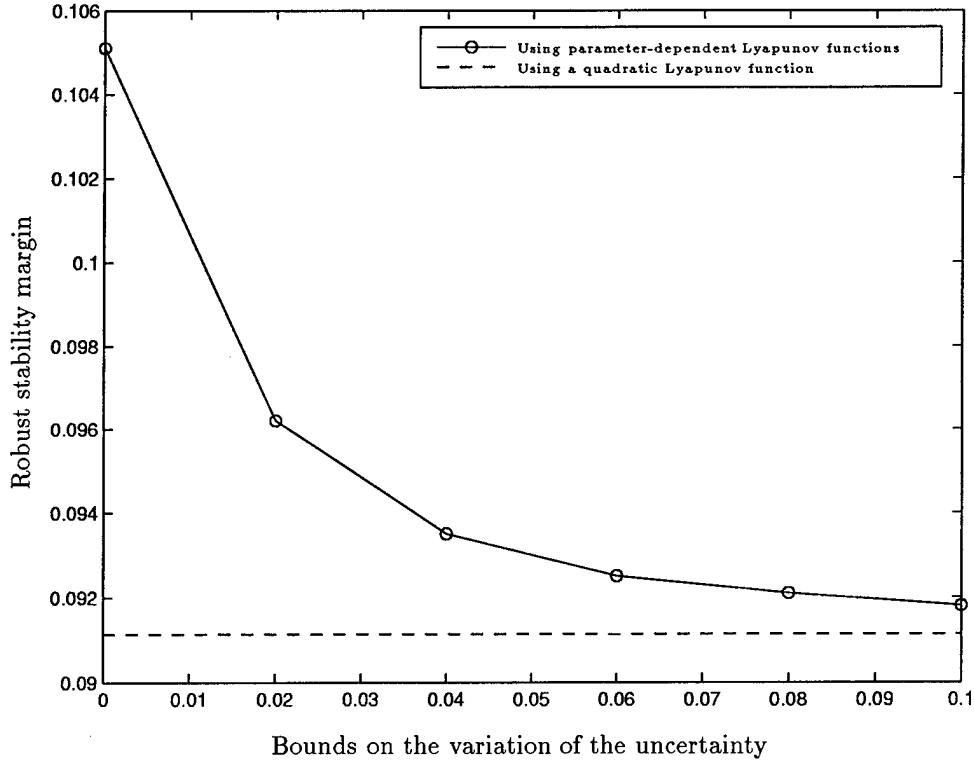


Figure 5.4: Robust stability analysis of system (5.38) with different bounds on the rate of variation of the uncertainty

5.4.4 Robust stability analysis of a nonlinear system

In this example, we show that the approach we proposed in this chapter can also be applied to solve the stability analysis problems of nonlinear systems with real rational nonlinearities. Consider an autonomous nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 - \frac{3.2x_1x_2}{(1-x_1)(1-x_2)} & \frac{1+2.2x_2}{1-x_2} \\ -\frac{2.4x_1x_2}{(1-x_1)(1-x_2)} & \frac{-10+12.4x_2}{1-x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (5.39)$$

The linear fractional representation of this second order nonlinear system is

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 3.2 \\ 0 & 2.4 \end{bmatrix} q, \\ p &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q, \\ q &= \Delta p,\end{aligned}\tag{5.40}$$

where $\Delta = \text{diag}(x_1, x_2)$ is the “uncertainty”. We study the local stability around the origin, with the approach of [EGS96] by using a quadratic Lyapunov function. First, we assume that the states are bounded, i.e., $|x_i| \leq \sigma$, $i = 1, 2$. Then, with the following LMI condition

$$\begin{bmatrix} \sigma^2 & e_i^T \\ e_i & P \end{bmatrix} > 0, \quad i = 1, 2,\tag{5.41}$$

where P is from the quadratic Lyapunov function $V(\psi) = \psi^T P \psi$, which guarantees the robust stability of system (5.40) when $|x_i| \leq \sigma$, $i = 1, 2$, it can be shown that $\mathcal{E}_P \triangleq \{x \mid x^T P x \leq 1\}$ is an invariant set (ellipsoid) around the origin (see [EGS96] for details).

With a fixed σ , the size of the invariant set $\mathcal{E}_P$ can be maximized. Here with $\sigma = 0.5$, we maximize the trace of P , which has the effect of maximizing the largest sum of the squared semi-axis lengths of the invariant ellipsoid $\mathcal{E}_P$.

There are several methods to search for the quadratic Lyapunov function $V(\psi) = \psi^T P \psi$. Our conditions from Theorem 7 make use of unstructured scalings and is guaranteed less conservative than the conventional method using diagonal scalings. In Table 5.3, we list the largest traces of P , which can be guaranteed by conventional method using diagonal scalings, Corollary 2, and Corollary 3 respectively.

Quadratic stability criterion	Largest $\text{Tr}(P)$
Conventional method	0.1249
Corollary 2	0.4994
Corollary 3	0.4996

Table 5.3: Comparison of the largest invariant sets guaranteed by Theorem 7 and by conventional method using diagonal scalings

5.4.5 Stabilizing nonlinear systems using gain-scheduled approach

In this example, we show that our approach can be applied to the design of output feedback controllers for parameter-dependent nonlinear systems with real rational nonlinearities.

Consider the following nonlinear system:

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2.5588 & 6.6086 \\ 4.8928 & 5.5663 \end{bmatrix} q + \begin{bmatrix} 3.6443 \\ 5.5731 \end{bmatrix} u, \\ p &= \begin{bmatrix} 7.7256 & 7.3275 \\ 8.0556 & 6.9718 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q, \\ y &= \begin{bmatrix} 3.9401 & 12.8006 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ q &= \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} p.\end{aligned}\tag{5.42}$$

We design an output feedback controller (see [EGS96] for details) such that:

1. $\mathcal{E}_P \triangleq \{x \mid x^T P x \leq 1\}$ is an invariant set (ellipsoid) inside the box $\mathcal{B}_\sigma \triangleq \{x \mid |x_i| \leq 0.5, i = 1, 2\}$, where P defines a quadratic Lyapunov function which gives a guarantee for the local stability around the origin.

2. The upper bound on output peak $y_{\max}$ is minimized. Note that this requirement can be enforced by the optimization

$$\text{Minimize: } y_{\max}^2, \quad \text{Subject to: } y_{\max}^2 \geq c_y R c_y^T, \quad (5.43)$$

where R is the left upper block of P^{-1} , defined in (5.30).

Note that there are several methods to search for the quadratic Lyapunov function corresponding to $\mathcal{E}_p$. By combining conditions in Theorem 8 and LMIs (5.41) and (5.43), we can design an output feedback controller with a guaranteed upper bound of output peak of 4.6465 when the initial condition $x_0 \in \mathcal{E}_p$. By using conventional method with structured scalings [AG95, EGS96], however, the best upper bound is 7.8314. The improvement by using our method is up to 40%.

5.5 Conclusion

We have presented new algorithms for the stability analysis and gain-scheduled controller synthesis for linear systems affected by parametric uncertainties. We have also established that these algorithms offer significant improvement over existing methods. The analysis and synthesis conditions are in the form of linear matrix inequalities; therefore, our algorithms can be very efficiently implemented numerically. The techniques presented in this chapter can be applied to parameter-dependent nonlinear systems with real rational nonlinearities, using the approach of [EGS96]. In addition, several of the techniques proposed in this chapter can be extended to the solution of robust performance problems.

5.6 Proofs

Theorem 10 *System (5.6) is quadratically stable if and only if there exists $P = P^T > 0$ such that for any uncertainty $\Delta \in \Delta$ which is defined in (5.7), the following condition holds*

$$\begin{bmatrix} A^T P + P A - C_p^T C_p & P(B_q \Delta) + C_p^T - C_p^T (D_{pq} \Delta) \\ (B_q \Delta)^T P + C_p - (D_{pq} \Delta)^T C_p & (D_{pq} \Delta) + (D_{pq} \Delta)^T - (D_{pq} \Delta)^T (D_{pq} \Delta) - I \end{bmatrix} < 0. \quad (5.44)$$

Moreover, if $D = 0$, condition (5.44) is equivalent to (5.14) with $S_i = -Q = I$.

Proof: The connection between (5.44) and (5.14) is very straight forward when $D = 0$. Therefore, we only need to show condition (5.44) holds for general case.

Sufficiency

By Schur's Complements Lemma, condition (5.44) is equivalent to for all $\Delta \in \Delta$

$$P(A + B_q \Delta(I - D_{pq} \Delta)^{-1} C_p) + (A + B_q \Delta(I - D_{pq} \Delta)^{-1} C_p)^T P + P B_q \Delta(I - D_{pq} \Delta)^{-1} (I - D_{pq} \Delta)^{-T} \Delta^T B_q^T P < 0.$$

Therefore the quadratic stability of system (5.6) follows immediately. Also note that condition (5.44) implies that system (5.6) is well posed.

Necessity

Suppose system (5.6) is quadratically stable. Then, there exists $P = P^T > 0$ such that

$$L(\Delta) = P(A + B_q \Delta(I - D_{pq} \Delta)^{-1} C_p) + (A + B_q \Delta(I - D_{pq} \Delta)^{-1} C_p)^T P < 0, \quad \forall \Delta \in \Delta. \quad (5.45)$$

Let

$$\eta = \sup_{\Delta \in \Delta} (\lambda_{\max}(L(\Delta))),$$

where $\lambda_{\max}$ denotes the largest eigenvalue. Obviously $\eta \leq 0$. We shall show that $\eta < 0$. This follows from the fact that Δ is a compact set and $\lambda_{\max}(\cdot)$ is a continuous function. If $\eta = 0$, there must exist some $\Delta \in \Delta$ such that $\lambda_{\max}(L(\Delta)) = \eta = 0$, which contradicts the assumption (5.45).

Since the system is well posed (Otherwise, it can not be quadratically stable), $\det(I - D_{pq} \Delta) \neq 0$ for all $\Delta \in \Delta$. This implies $(I - D_{pq} \Delta)(I - D_{pq} \Delta)^T > 0$ for all $\Delta \in \Delta$. Following the same argument as before, we

get $\inf_{\Delta \in \Delta} \lambda_{\min}((I - D_{pq}\Delta)(I - D_{pq}\Delta)^T) = \tau > 0$, or equivalently $\|(I - D_{pq}\Delta)^{-T}\|^2 \leq 1/\tau$ for all $\Delta \in \Delta$. Define $\tilde{P} = \xi P$ where

$$0 < \xi < -\frac{\tau\eta}{\lambda_{\max}(PB_qB_q^TP)\mu}, \quad \|\Delta\|^2 \leq \mu, \quad \mu > 0.$$

It is easy to check that $\tilde{P}$ is a feasible solution for inequality (5.44). Thus we showed that (5.44) is a necessary condition for quadratic stability. $\square$

Part II

**Gain-Scheduled Control of the ONR
UCAV**

Chapter 6

Gain-scheduled control of the ONR UCAV

The dynamics of Unmanned Combat Air Vehicles (UCAVs) undergoing aggressive maneuvers are highly nonlinear and time-varying. One approach towards modeling UCAVs involves linearizing the dynamics of the UCAV around various points in the flight envelope. Then, the composite model of the UCAV can be given as a linear parameter-dependent model, with the parameters being the flight conditions. A comprehensive effort towards identifying such models is currently being undertaken at Texas A & M University.

We describe here research efforts towards a systematic design procedure for controller synthesis for the linear parameter-dependent ONR UCAV models. Traditional techniques for controller synthesis consist of designing a single controller that is intended to function across varying flight conditions. These include constant state-feedback, as well as the celebrated LQR/LQG and H_∞ controllers. While these control techniques can be sometimes proven to work (i.e., stabilize the system or provide acceptable performance), they can be quite conservative, especially when the flight conditions vary considerably.

Gain-scheduled controller design offers the potential much more aggressive control design. The basic idea is to synthesize a series of dynamic controllers, one for each linearized model of the UCAV around a flight condition, and then "schedule" these controllers according to the actual flight condition. An ad hoc implementation of such a scheme is not guaranteed to work; however, it is possible to develop a gain-scheduling scheme, using Lyapunov functions, that is guaranteed to work across various flight regimes.

Another advantage of an approach based on Lyapunov functions is that it can be extended to handle constraints other than mere stability. Stability requires the Lyapunov function to decrease along the trajectories of the parameter-varying system. Additional constraints on the Lyapunov function can be used to design controllers with guaranteed performance. Examples of performance measures are the energy in the state vector, and peak values of signals of interest; in several instances, it is desirable that these performance measures be small. And gain-scheduled controllers that minimize upper bounds on these performance measures can be designed.

The technical approach behind gain-scheduled control was described in Part I of this report, especially in Chapter 5. We now describe briefly some example problems in the control of the ONR UCAV that we solved using our techniques.

The complete results of this study, along with the matlab codes are available at the web site <http://www.ece.purdue.edu/~ragu/onr/gs-contr.html>

⁰Joint work with Qing Su, School of Electrical and Computer Engineering, Purdue University.

6.1 Example: Plant varies linearly with flight condition

6.1.1 Stability

The UCAV6 linear models are based on the unmanned, V/STOL UCAV6 model. The online documentation from the Texas A & M website

<http://aero.tamu.edu/ucav/>

provides more information on the model. The following table lists the inputs and states for the longitudinal model.

Inputs	Input	Units	Description
	d_e	%	Elevator stick input
	d_t	%	Throttle input
	d_n	deg	Nozzle angle input

States	State	Units	Description
	u	ft/s	Velocity along x -axis of aircraft
	w	ft/s	Velocity along z -axis of aircraft
	q	deg/s	Pitch rate
	θ	deg	Euler angle rotation of aircraft reference frame about inertial y -axis

We address the problem of designing a stabilizing controller that is "scheduled" by the flight condition. The design requirement is that with zero exogenous inputs, the states of the closed-loop system must converge to zero, irrespective of the initial condition.

The flight condition V varies between 10 and 25 knots as shown in Figure 6.1. We assume that the system model varies linearly with the flight condition V between the two linear models of the system corresponding to the flight conditions $V = 10$, and 25 knots (taken from the Texas A & M website).

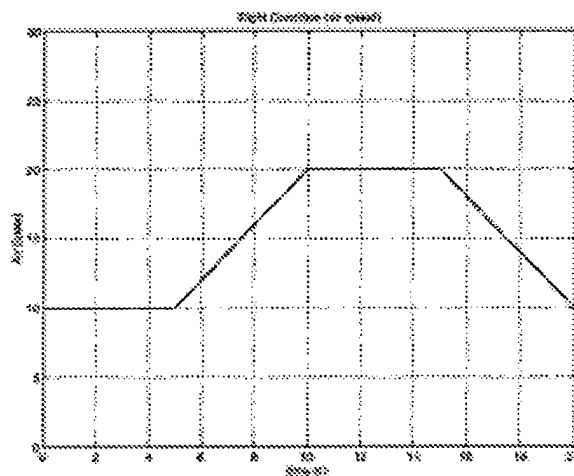


Figure 6.1: Flight condition variation.

Using the techniques outlined in Chapter 5, we designed a gain-scheduled controller to stabilize the system. The plots of the state variables for the closed-loop system (with our gain-scheduled controller in place), and the control inputs are shown in Figure 6.2.

We note that our techniques are *guaranteed* to stabilize the system, unlike some *ad hoc* gain-scheduling methods.

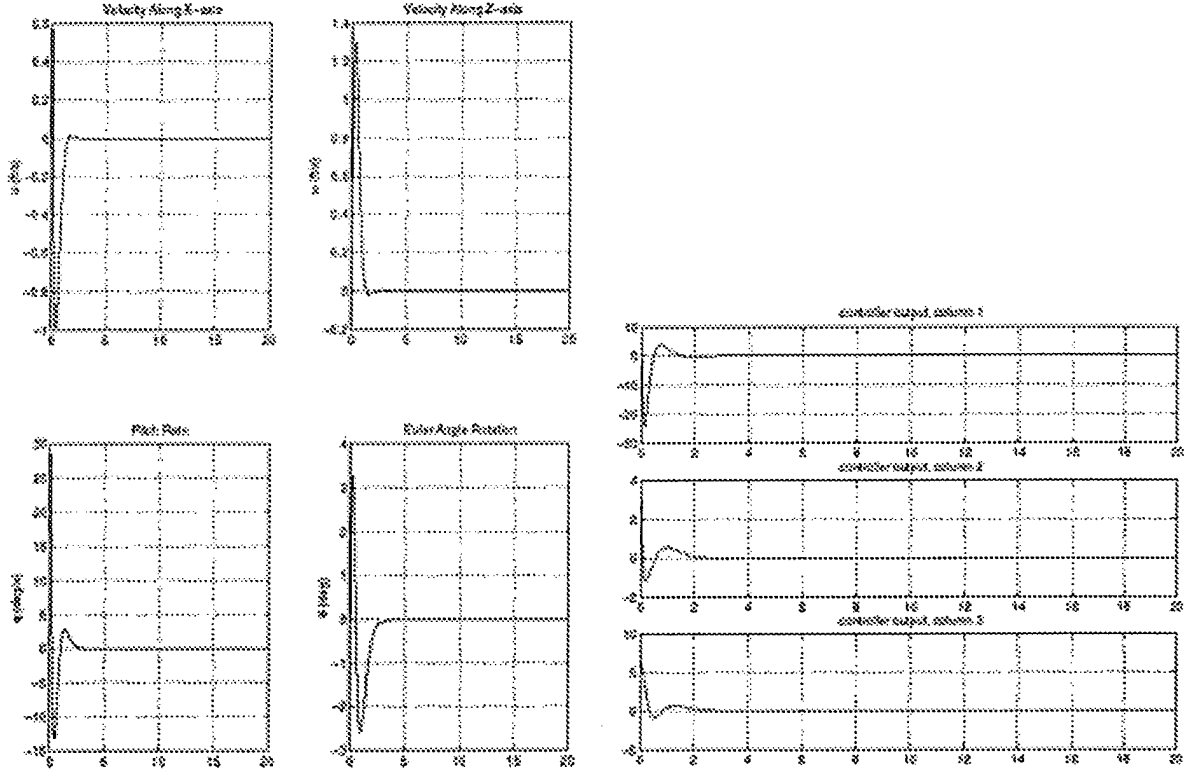


Figure 6.2: The longitudinal states of the closed-loop system are shown on the left. The control inputs are shown on the right.

6.1.2 Minimizing L_2 gain from disturbance to output

With the same flight condition profile as in Section 6.1.1, we introduce disturbances in all three control inputs, as shown in Figure 6.3.

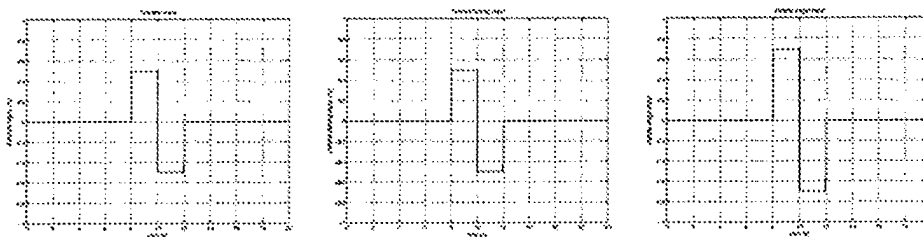


Figure 6.3: Disturbances introduced at the control input.

We now consider the problem of designing a gain-scheduled controller that minimizes an upper bound on the worst-case L_2 gain from disturbance (introduced at the control inputs) to the state. The basic idea here is to search for Lyapunov functions that satisfy the constraint

$$\frac{d}{dt}V(x, t) \leq \gamma^2 w(t)^T w(t) - z(t)^T z(t), \quad (6.1)$$

with the smallest possible γ , where w and z denote the disturbance input and output of interest respectively. (See Example 2.5.1, and inequality (2.25) in Chapter 2.)

With this gain-scheduled controller in place, Figure 6.4 shows the various state trajectories corresponding to the disturbance input in Figure 6.3, as well as the various control inputs.

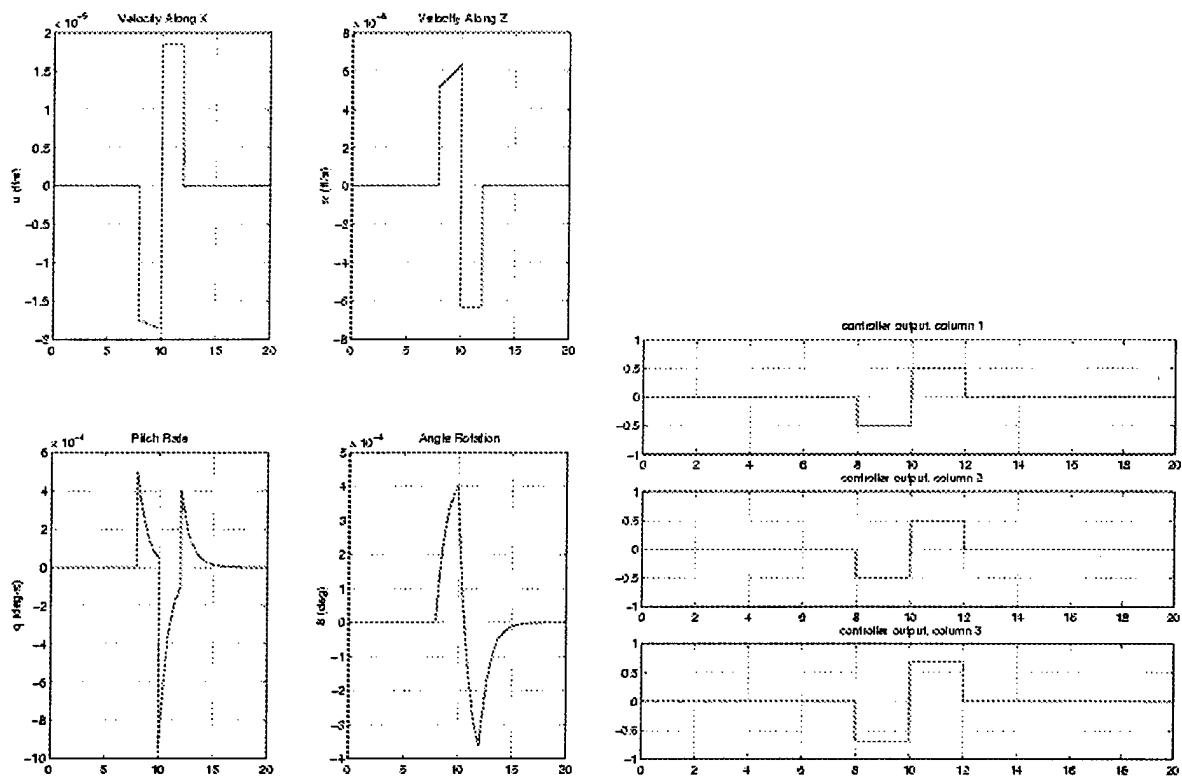


Figure 6.4: The state trajectories with the gain-scheduled controller that minimizes the L_2 gain from disturbance to output are shown on the left. The control inputs are shown on the right.

6.1.3 Minimizing peak value of the first state-component

With the same flight condition profile as in Section 6.1.1, and with the same disturbances as in Section 6.1.2, we now consider the problem of designing a gain-scheduled controller that minimizes an upper bound on the worst-case peak value of the first component of the state. The basic idea here is to search for Lyapunov functions that satisfy the constraint

$$\frac{d}{dt}V(x, t) \leq 0, \quad (6.2)$$

and use the invariant sets that arise to bound the state variables (cf. example 2.5.2 in Chapter 2).

With this gain-scheduled controller in place, Figure 6.5 shows the various state trajectories corresponding to the disturbance input in Figure 6.3, as well as the various control inputs.

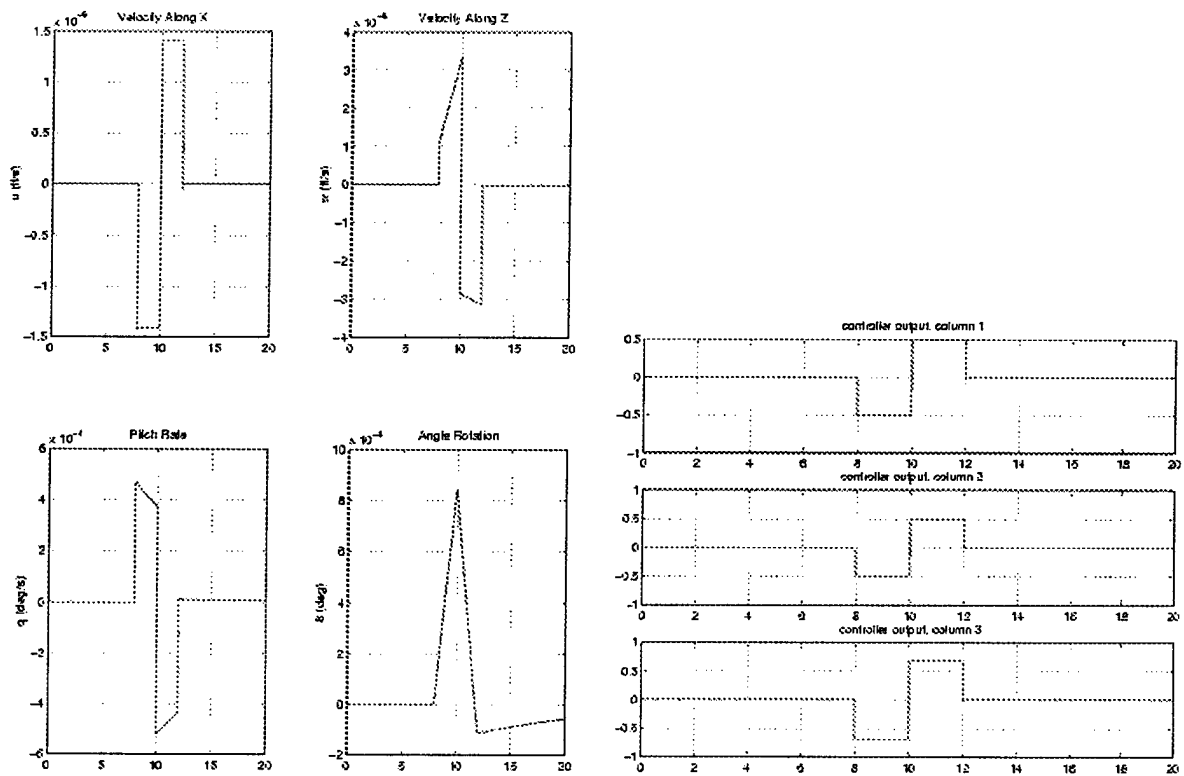


Figure 6.5: The state trajectories with the gain-scheduled controller that minimizes the worst-case peak value of the first state component are shown on the left. The control inputs are shown on the right.

6.2 Other example problems

We have performed a comprehensive study, employing:

- Different design objectives.
- Different UCAV models (taken from the Texas A & M database).
- Different flight condition variation scenarios.

Hierarchical description of problems studied

- At the highest level, the problems are classified according to how the plant parameter vary with the flight condition: *linear* and *polynomial*. Our techniques extend to the case when the plant parameters are rational functions of the flight condition as well.
- At the next level, we have studied six models for the linear variation case:
 - Longitudinal UCAV6 model.
 - Lateral/directional UCAV6 model.
 - Longitudinal VSTOL1 model.
 - Lateral/directional VSTOL1 model.
 - Longitudinal VSTOL2 model.
 - Lateral/directional VSTOL2 model.

For the polynomial variation case, we have only considered the longitudinal and lateral/directional UCAV6 model.

- At the last level, for each scenario, we have studied the following problems:
 - Single objective problems:
 - * L_2 gain bound of 10000 from disturbance to output.
 - * Minimize L_2 gain from disturbance to output.
 - * Minimize L_2 gain from disturbance to first output.
 - * Minimize L_2 gain from disturbance to second output.
 - * Minimize L_2 gain from disturbance to third output.
 - * Minimize L_2 gain from disturbance to fourth output.
 - * Minimize peak value of the state Euclidean norm.
 - * Minimize peak value of the first state component.
 - * Minimize peak value of the second state component.
 - * Minimize peak value of the third state component.
 - * Minimize peak value of the fourth state component.
 - Multi-objective problems:
 - * L_2 gain bound of 10000 from disturbance to output + minimizing upper bound on the peak value of the Euclidean norm of the state vector.
 - * L_2 gain bound of 10000 from disturbance to output + minimizing upper bound on the peak value of the first state component.
 - * L_2 gain bound of 10000 from disturbance to output + minimizing upper bound on the peak value of the second state component.
 - * L_2 gain bound of 10000 from disturbance to output + minimizing upper bound on the peak value of the third state component.
 - * L_2 gain bound of 10000 from disturbance to output + minimizing upper bound on the peak value of the fourth state component.

Part III

Robust Estimation

Chapter 7

Robust estimation for systems with parametric uncertainties

We present an adaptive robust Kalman filtering algorithm that addresses estimation problems that arise in linear time-varying systems with stochastic parametric uncertainties. The filter has the one-step predictor-corrector structure and minimizes the mean square estimation error at each step, with the minimization reduced to a convex optimization problem based on linear matrix inequalities. The algorithm is shown to converge when the system is mean square stable and the state-space matrices are time-invariant. A numerical example, consisting of equalizer design for a communication channel, demonstrates that our algorithm offers considerable improvement in performance when compared to standard Kalman filtering techniques.

7.1 Introduction

In this chapter, we consider the robust estimation problems for the uncertain system

$$\begin{aligned}x(k+1) &= A_{\Delta}(k)x(k) + B_{\Delta}(k)w(k), \\y(k) &= C_{\Delta}(k)x(k) + D_{\Delta}(k)w(k), \quad z(k) = Lx(k).\end{aligned}\tag{7.1}$$

When system (7.1) contains mixed deterministic and stochastic uncertainties, we consider two problems of optimal estimation. The first is the design of a filter that minimizes an upper bound on the worst-case mean energy gain (MEG) between the noise affecting the system and the estimation error. The second is the design of a filter that minimizes an upper bound on the worst-case asymptotic mean square estimation error (MSEE) when the plant is driven by a white noise process. We present filtering algorithms that solve each of these problems, with the filter parameters determined via convex optimization based on linear matrix inequalities. We demonstrate the performance of these robust algorithms on numerical examples consisting of the design of equalizers for communication channels.

When system (7.1) is time-varying and contains stochastic parametric uncertainties (See (7.29)), we present an adaptive robust Kalman filtering algorithm. The filter has the one-step predictor-corrector structure and minimizes the mean square estimation error at each step, with the minimization reduced to a convex optimization problem based on linear matrix inequalities. The algorithm is shown to converge when the system is mean square stable and the state-space matrices are time-invariant. A numerical example, consisting of adaptive equalizer design for a communication channel, demonstrates that our algorithm offers considerable improvement in performance when compared to standard adaptive Kalman filtering techniques.

The organization of this chapter is as follows. In Section 7.2, we propose linear time-invariant robust MMEG and MMSE estimators for uncertain systems containing mixed deterministic and stochastic uncertainties. In Section 7.3, we propose an adaptive Kalman filtering algorithm for linear time-varying systems with stochastic parametric uncertainties.

^oJoint work with Fan Wang, Ph.D., School of Electrical and Computer Engineering, Purdue University.

7.2 Robust estimators for systems with mixed deterministic and stochastic uncertainties

In this section, we consider a robust estimation setting where both deterministic and stochastic uncertainties are present. For such systems (see Fig. 7.3), we design:

A *robust MMEG filter*, i.e., , one that minimizes an upper bound on the largest value (over all possible values of the deterministic uncertainties and all possible realizations of the mean energy bounded noise input w) of the mean energy gain (with the expectation taken over the statistics of the stochastic uncertainties) from the noise input w to the estimation error e .

A *robust MMSE filter*, i.e., , one that minimizes an upper bound on the largest value (over all possible values of the deterministic uncertainties) of the asymptotic mean square value (with the expectation taken over the statistics of the stochastic uncertainties and the white noise input w) of the estimation error e .

In each case, the robust filter design problem is reduced to a convex optimization problem based on linear matrix inequalities.

7.2.1 Preliminaries

Consider the parametric uncertain system

$$\begin{aligned} x(k+1) &= A_{\Delta}(k)x(k) + B_{\Delta}(k)w(k), \\ y(k) &= C_{\Delta}(k)x(k) + D_{\Delta}(k)w(k), \quad z(k) = Lx(k), \end{aligned} \quad (7.2a)$$

where

$$\begin{aligned} A_{\Delta}(k) &= A_0 + \sum_{t=1}^n A_t^d \zeta_t^d(k) + \sum_{j=1}^m A_j^s \zeta_j^s(k), \quad B_{\Delta}(k) = B_0 + \sum_{t=1}^n B_t^d \zeta_t^d(k) + \sum_{j=1}^m B_j^s \zeta_j^s(k), \\ C_{\Delta}(k) &= C_0 + \sum_{t=1}^n C_t^d \zeta_t^d(k) + \sum_{j=1}^m C_j^s \zeta_j^s(k), \quad D_{\Delta}(k) = D_0 + \sum_{t=1}^n D_t^d \zeta_t^d(k) + \sum_{j=1}^m D_j^s \zeta_j^s(k). \end{aligned} \quad (7.2b)$$

The system has mixed deterministic and stochastic uncertainties. It is said to be *mean square stable* if with $w(k) = 0$ and for all $k \in \mathbf{Z}_+$, we have

$$\lim_{k \rightarrow \infty} E[x(k)x(k)^T] = 0,$$

regardless of the initial condition $x(0)$.

Let

$$\Omega_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad i = 1, \dots, 2^n \quad (7.3)$$

be the vertices of the polytope

$$\Omega = \left\{ \begin{bmatrix} A_0 + \sum_{t=1}^n A_t^d \zeta_t^d & B_0 + \sum_{t=1}^n B_t^d \zeta_t^d \\ C_0 + \sum_{t=1}^n C_t^d \zeta_t^d & D_0 + \sum_{t=1}^n D_t^d \zeta_t^d \end{bmatrix} \middle| |\zeta_t^d| \leq 1 \right\}.$$

The following lemma gives a sufficient condition for the mean square stability of system (7.2). This lemma is an extension of the corresponding result for systems with stochastic parametric uncertainties in [BEFB94, Chapter 9] to systems with mixed deterministic and stochastic uncertainties.

Lemma 7.2.1 *System (7.2) is mean square stable if there exists a matrix $Q > 0$ such that*

$$A_i^T Q A_i - Q + \sum_{j=1}^m (A_j^s)^T Q (A_j^s) < 0, \quad i = 1, \dots, 2^n. \quad (7.4)$$

Moreover, the quadratic Lyapunov function $V(x(k)) = E[x(k)^T Q x(k)]$ satisfies $V(x(k)) \searrow 0$ as $k \rightarrow \infty$.

Proof: Let $X(k) = E[x(k)^T x(k)]$ and $A(k) = A_0 + \sum_{t=1}^n A_t^d \zeta_t^d(k)$. Suppose that there exists a matrix $Q > 0$ such that condition (7.4) holds. By convexity, condition (7.4) implies that

$$A(k)^T Q A(k) - Q + \sum_{j=1}^m (A_j^s)^T Q (A_j^s) < 0, \quad k = 0, 1, 2, \dots \quad (7.5)$$

With $w(k) = 0, k = 0, 1, \dots$, the correlation of the states of system (7.2) satisfies the recursion

$$X(k+1) = A(k)X(k)A(k)^T + \sum_{j=1}^m A_j^s X(k)(A_j^s)^T.$$

Define a Lyapunov function $V(x(k)) = E[x(k)^T Q x(k)]$. If $X(k) \neq 0$, then $V(x(k)) > 0$, and

$$V(x(k+1)) - V(x(k)) = \text{Tr}(X(k) \left(A(k)^T Q A(k) - Q + \sum_{i=1}^m (A_i^s)^T Q (A_i^s) \right)).$$

From (7.5), we get $V(x(k+1)) - V(x(k)) < 0$. On the other hand,

$$V(x(k+1)) - V(x(k)) = \text{Tr}((X(k+1) - X(k))Q).$$

Since $Q > 0$, we get $X(k+1) < X(k)$. Thus $X(k)$ is monotonically decreasing as $k \rightarrow \infty$. Moreover, it can be argued that $X(k+1) < \eta X(k)$ with $\eta \in [0, 1)$. Therefore we have $\lim_{k \rightarrow \infty} X(k) = 0$. $\square$

Lemma 7.2.1 gives a sufficient condition for the mean square stability of system (7.2). If there is no deterministic uncertainty, i.e., $\delta_i^d(k) = 0, i = 1, \dots, n, k = 0, 1, \dots$, then condition (7.4) is also necessary for the system to be mean square stable [BEFB94, pp. 136–137]. However, in more general cases, condition (7.4) is not necessary since it requires the use of a *single* quadratic Lyapunov function to prove the mean square stability of system (7.2) for all possible choices of state matrices $A(k)$. If there exists a matrix $Q > 0$ such that condition (7.4) holds, system (7.2) is said to be *mean square quadratically stable*.

Our objective here is to design an LTI signal estimator

$$x_f(k+1) = A_f x_f(k) + B_f y(k), \quad \hat{z}(k) = C_f x_f(k), \quad (7.6)$$

where $x_f(k) \in \mathbb{R}^{n_{x_f} \times n_{x_f}}$.

We may write down a state space realization for the interconnection in Fig. 7.3:

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ x_f(k+1) \end{bmatrix} &= \begin{bmatrix} A_\Delta(k) & 0 \\ B_f C_\Delta(k) & A_f \end{bmatrix} \begin{bmatrix} x(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_\Delta(k) \\ B_f D_\Delta(k) \end{bmatrix} w(k), \\ e(k) &= \begin{bmatrix} L & -C_f \end{bmatrix} \begin{bmatrix} x(k) \\ x_f(k) \end{bmatrix}. \end{aligned} \quad (7.7)$$

We will focus on two design objectives:

- **Robust MMEG filter design**

With $x(0) = 0$ almost surely, we wish to determine the estimator parameters $\{A_f, B_f, C_f\}$ to solve the following problem:

Minimize: γ_∞

Subject to: $\sum_{k=0}^{\infty} E[\|w(k)\|^2] \leq 1,$
 $\sum_{k=0}^{\infty} E[\|e(k)\|^2] \leq \gamma_\infty.$

This has the interpretation of minimizing an upper bound on the largest value (over all possible values of the deterministic uncertainties and all possible realizations of mean energy bounded noise input w) of the mean energy gain (with the expectation taken over the statistics of the stochastic uncertainties) from the noise input w to the estimation error e .

- **Robust MMSE filter design**

We wish to determine the estimator parameters $\{A_f, B_f, C_f\}$ to solve the following problem:

Minimize:

$$\gamma_2$$

Subject to: $\exists N$ s.t. $E[\|e(k)\|^2] \leq \gamma_2, \forall k \geq N$,

where w is zero-mean unit variance white noise. This has the interpretation of minimizing an upper bound on the largest (over all possible values of the deterministic uncertainties) asymptotic mean square value (with the expectation taken over the statistics of the stochastic uncertainties and the white noise input w) of the estimation error e .

7.2.2 Robust MMEG filter design

In this section, we consider the robust MMEG filter design problem. For system (7.2), with $x(0) = 0$ almost surely, the quantity

$$\sup_{\left\{ \sum_{k=0}^{\infty} E[\|w(k)\|^2] \leq 1, \zeta^d \right\}} \sum_{k=0}^{\infty} E[\|y(k)\|^2] \quad (7.8)$$

will be referred to as the worst-case mean energy gain (worst-case MEG). The following lemma gives a sufficient condition for the worst-case MEG to be less than a level γ_{∞} .

Lemma 7.2.2 *The worst-case MEG of system (7.2) is less than γ_{∞} if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} \mathcal{P} & \mathcal{A}_i P & \mathcal{B}_i & 0 \\ P \mathcal{A}_i^T & P & 0 & P \mathcal{C}_i^T \\ \mathcal{B}_i^T & 0 & \gamma_{\infty} I & \mathcal{D}_i^T \\ 0 & \mathcal{C}_i P & \mathcal{D}_i & I \end{bmatrix} > 0, \quad i = 1, \dots, 2^n, \quad (7.9)$$

where $\mathcal{P} = \text{diag} P, \dots, P$ and

$$\begin{aligned} \mathcal{A}_i &= [(A_i)^T, (A_1^s)^T, \dots, (A_m^s)^T]^T, & \mathcal{B}_i &= [(B_i)^T, (B_1^s)^T, \dots, (B_m^s)^T]^T, \\ \mathcal{C}_i &= [(C_i)^T, (C_1^s)^T, \dots, (C_m^s)^T]^T, & \mathcal{D}_i &= [(D_i)^T, (D_1^s)^T, \dots, (D_m^s)^T]^T. \end{aligned}$$

Proof: First, let us define a Lyapunov function $V(x(k)) = E[x(k)^T Q x(k)]$ with $Q > 0$. If

$$V(x(k+1)) - V(x(k)) \leq \gamma_{\infty} E[\|w(k)\|^2] - E[\|y(k)\|^2], \quad (7.10)$$

we then have

$$\sum_{k=0}^{\infty} E[\|y(k)\|^2] \leq \gamma_{\infty} \sum_{k=0}^{\infty} E[\|w(k)\|^2].$$

Following the same argument as in the proof of Lemma 7.2.1, it can be checked that condition (7.10) is equivalent to LMI condition

$$\begin{bmatrix} Q & 0 \\ 0 & \gamma_{\infty} I \end{bmatrix} - \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix} > 0, \quad (7.11)$$

where $Q = \text{diag} Q, \dots, Q$. With a change of variable $P = Q^{-1}$, and standard matrix manipulations, it follows that condition (7.11) is equivalent to (7.9). $\square$

The upper bound on the worst-case MEG given by (7.9) can be conservative in general. However, if there exist no deterministic uncertainties, it turns out that the upper bound is tight, i.e., it equals the worst-case MEG. We also note that if the worst-case MEG is bounded, then the system must be mean square quadratically stable. This simply follows from the fact that (7.9) implies that

$$\begin{bmatrix} \mathcal{P} & \mathcal{A}_i P \\ P \mathcal{A}_i^T & P \end{bmatrix} > 0, \quad i = 1, \dots, 2^n,$$

which is equivalent to condition (7.4) with $Q = P^{-1}$.

We are now ready to state our main result on the robust MMEG filtering, that a robust MMEG filter for system (7.2) can be designed by solving a set of linear matrix inequalities.

Theorem 11 *For the uncertain system (7.2), there exists a full order LTI filter (7.6) such that the worst-case MEG from the input noise w to the estimation error e is less than γ_∞ if there exist $Z = Z^T$, $Y = Y^T$, $H \in \mathbb{R}^{n_x \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, and $G \in \mathbb{R}^{n_x \times n_x}$ such that*

$$\begin{bmatrix} M_{11} & M_{12,i} & M_{13,i} & 0 \\ M_{12,i}^T & Z & 0 & M_{24} \\ M_{13,i}^T & 0 & \gamma_\infty I & 0 \\ 0 & M_{24}^T & 0 & I \end{bmatrix} > 0, \quad i = 1, \dots, 2^n, \quad (7.12)$$

where

$$M_{11} = \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix}, \quad M_{12,i} = \begin{bmatrix} \mathcal{A}_{f,0}^i \\ \mathcal{A}_{f,1}^i \\ \vdots \\ \mathcal{A}_{f,m}^i \end{bmatrix}, \quad M_{13,i} = \begin{bmatrix} \mathcal{B}_{f,0}^i \\ \mathcal{B}_{f,1}^i \\ \vdots \\ \mathcal{B}_{f,m}^i \end{bmatrix},$$

$$M_{24} = \begin{bmatrix} L^T - G^T \\ L^T \end{bmatrix}, \quad Z = \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}, \quad (7.13)$$

$$\mathcal{A}_{f,0}^i = \begin{bmatrix} ZA_i & ZA_i \\ YA_i + FC_i + H & YA_i + FC_i \end{bmatrix}, \quad \mathcal{B}_{f,0}^i = \begin{bmatrix} ZB_i \\ YB_i + FD_i \end{bmatrix},$$

$$\mathcal{A}_{f,j}^s = \begin{bmatrix} ZA_j^s & ZA_j^s \\ YA_j^s + FC_j^s & YA_j^s + FC_j^s \end{bmatrix}, \quad \mathcal{B}_{f,j}^s = \begin{bmatrix} ZB_j^s \\ YB_j^s + FD_j^s \end{bmatrix}.$$

If there is no deterministic uncertainty, the smallest γ_∞ such that (7.12) holds is the exact value of MEG.

Proof: The central part of the proof is a change-of-variable technique due to [GBG98]. From Lemma 7.2.2, we have

$$\sum_{k=0}^{\infty} E[\|e(k)\|^2] / \sum_{k=0}^{\infty} E[\|w(k)\|^2] < \gamma_\infty,$$

if there exists a matrix $P > 0$ such that

$$\begin{bmatrix} \mathcal{P} & \bar{\mathcal{A}}_i P & \bar{\mathcal{B}}_i & 0 \\ P \bar{\mathcal{A}}_i^T & P & 0 & P \bar{\mathcal{C}}_i^T \\ \bar{\mathcal{B}}_i^T & 0 & \gamma_\infty I & 0 \\ 0 & \bar{\mathcal{C}}_i P & 0 & I \end{bmatrix} > 0, \quad i = 1, \dots, 2^n, \quad (7.14)$$

where

$$\bar{\mathcal{A}}_i = \begin{bmatrix} A_i & 0 \\ B_f C_i & A_f \\ \hline A_1^s & 0 \\ B_f C_1^s & 0 \\ \vdots & \vdots \\ A_m^s & 0 \\ B_f C_m^s & 0 \end{bmatrix}, \quad \bar{\mathcal{B}}_i = \begin{bmatrix} B_i \\ B_f D_i \\ \hline B_1^s \\ B_f D_1^s \\ \vdots \\ B_m^s \\ B_f D_m^s \end{bmatrix}, \quad \bar{\mathcal{C}}_i = [L \quad -C_f].$$

Let P be partitioned as

$$P = \begin{bmatrix} X & U \\ U^T & \star \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} Y & V \\ V^T & \star \end{bmatrix}, \quad (7.15)$$

where $X, Y \in \mathbf{R}^{n_x \times n_x}$, $U, V \in \mathbf{R}^{n_x \times n_{x_f}}$. According to the matrix completion lemma (see for example [Pac94]), by requiring $n_{x_f} = n_x$, such a decomposition is feasible for some fixed X and Y if and only if X and Y satisfy $X \geq Y^{-1} > 0$.

Define new variables

$$H = V A_f U^T X^{-1}, \quad F = V B_f, \quad G = C_f U^T X^{-1}, \quad (7.16)$$

and $Z = X^{-1}$. Note that we may always require V being nonsingular in (7.15). If V is singular, we may add some perturbations on P to enforce this requirement. Multiplying by $\text{diag} T^T, \dots, T^T, I, I$ and $\text{diag} T, \dots, T, I, I$ from both sides of (7.14), where

$$T = \begin{bmatrix} Z & Y \\ 0 & V^T \end{bmatrix},$$

we get the linear matrix inequality (7.12). Since T is nonsingular, (7.12) and (7.14) are equivalent. By Schur's complement lemma, the condition $X > Y^{-1} > 0$ is implied by the LMI (7.12).

If there is no deterministic uncertainties, the set of uncertainties Ω becomes a fixed point. According to the bounded real lemma for discrete time systems, the second part of the theorem is proved immediately. This completes the proof. $\square$

Condition (7.12) is sufficient for the existence of an LTI filter such that the worst-case MEG from w to e of system (7.2) is less than γ_∞ . By minimizing γ_∞ , we can design an optimal robust MMEG filter. This becomes the following semidefinite programming problem ([VB96]):

$$\text{Minimize: } \gamma_\infty, \quad \text{Subject to: } (7.12). \quad (7.17)$$

We summarize the various steps comprising the construction of a robust MMEG filter using the feasible solution of LMI (7.12).

Robust MMEG Filtering Algorithm

1. Solve the semidefinite programming problem (7.17) and find the optimal values of Z, Y, H, F , and G .
2. Define $V = (Y - Z)^{1/2}$ and $U = -Z^{-1}V$. Note that (7.12) implies $Y > Z$ and that V is nonsingular.
3. Construct the state-space matrices of the optimal MMEG estimator

$$\begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H & F \\ G & 0 \end{bmatrix} \begin{bmatrix} -V^{-1} & 0 \\ 0 & I \end{bmatrix}. \quad (7.18)$$

We also note that $E[x(k)Qx(k)]$, where

$$Q = \begin{bmatrix} Y & (Y - Z)^{1/2} \\ (Y - Z)^{1/2} & I \end{bmatrix},$$

is a quadratic Lyapunov function that guarantees that irrespective of ζ_d , we have

$$\sum_{k=0}^{\infty} E[\|e(k)\|^2] / \sum_{k=0}^{\infty} E[\|w(k)\|^2] < \gamma_\infty.$$

7.2.3 Robust asymptotic MMSE filter

Suppose the input w is a zero-mean white noise process, and satisfies the condition $E[w(i)w(j)^T] = \delta(i-j)I$. We now consider the problem of designing a linear time invariant filter (7.6) that minimizes an upper bound on the largest (over all possible values of the deterministic uncertainties) asymptotic mean square value (with the expectation taken over the statistics of the stochastic uncertainties and the white noise input w) of the estimation error e :

$$\begin{aligned} &\text{Minimize:} && \gamma_2, \\ &\text{Subject to:} && \exists N \text{ s.t. } E[\|e(k)\|^2] \leq \gamma_2 \text{ for } k \geq N. \end{aligned} \quad (7.19)$$

We begin with the following lemma.

Lemma 7.2.3 Consider system (7.2). Suppose there exists a matrix $P > 0$ such that for $i = 1, \dots, 2^n$,

$$A_i P A_i^T - P + B_i B_i^T + \sum_{j=1}^m ((A_j^s P (A_j^s)^T + B_j^s (B_j^s)^T) < 0. \quad (7.20)$$

Then there exists $N > 0$ such that if $k \geq N$, then $E[x(k)x(k)^T] \leq P$. Moreover, if $P \geq E[x(0)x(0)^T]$, then $E[x(k)x(k)^T] \leq P$ for every $k \geq 0$.

Proof: Suppose that there exists $P > 0$ such that (7.20) holds. Let $A(k) = A_0 + \sum_{i=1}^n A_i^d \zeta_i^d(k)$ and $B(k) = B_0 + \sum_{i=1}^n B_i^d \zeta_i^d(k)$. Since $\{A(k), B(k)\} \in \text{Co}A_i, B_i, i = 1, \dots, 2^n$, where A_i, B_i are defined in (7.3), we have

$$A(k) P A(k)^T - P + B(k) B(k)^T + \sum_{j=1}^m (A_j^s P (A_j^s)^T + B_j^s (B_j^s)^T) < -\epsilon I,$$

for some $\epsilon > 0$.

By (7.2), $X(k) = E[x(k)x(k)^T]$ satisfies the recursion

$$X(k+1) = A(k)X(k)A(k)^T + B(k)B(k)^T + \sum_{j=1}^m (A_j^s X(k)(A_j^s)^T + B_j^s (B_j^s)^T).$$

We then have

$$P - X(k+1) > A(k)(P - X(k))A(k)^T + \sum_{j=1}^m A_j^s (P - X(k))(A_j^s)^T + \epsilon I.$$

It can be verified that $P - X(k+1) > M(k+1) + \epsilon I$, where

$$M(k+1) = A(k)M(k)A(k)^T + \sum_{j=1}^m A_j^s M(k)(A_j^s)^T,$$

and $M(0) = P - X(0)$. Then, from Lemma 7.2.1, we have $\lim_{k \rightarrow \infty} M(k) = 0$. Therefore, there exists N such that if $k \geq N$, then $P > X(k)$. To prove the second claim, suppose that $P > X(0)$. It can be then immediately verified that $P > X(k)$ for $k \geq 0$. $\square$

Lemma 7.2.3 provides a solution to the asymptotic MMSE filtering problem.

Theorem 12 For the system (7.2), there exists a full order LTI filter (7.6) such that $E[\|e(k)\|^2] \leq \gamma_2$ when k is large enough if there exist $Z = Z^T$, $Y = Y^T$, $H \in \mathbb{R}^{n_x \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, and $G \in \mathbb{R}^{n_x \times n_z}$ such that

$$\text{Tr}(W) \leq \gamma_2,$$

$$\begin{bmatrix} Z & Z & L^T - G^T \\ Z & Y & L^T \\ L - G & L & W \end{bmatrix} > 0, \quad \begin{bmatrix} Z & M_{12,i} & M_{13,i} \\ M_{12,i}^T & M_{22} & 0 \\ M_{13,i}^T & 0 & I \end{bmatrix} > 0, \quad i = 1, \dots, 2^n, \quad (7.21)$$

where

$$M_{22} = \begin{bmatrix} Z & & \\ & \ddots & \\ & & Z \end{bmatrix}, \quad M_{12,i} = [A_{f,0}^i \quad A_{f,1} \quad \dots \quad A_{f,m}],$$

$$M_{13,i} = [B_{f,0}^i \quad B_{f,1} \quad \dots \quad B_{f,m}],$$

with $Z, A_{f,0}^i, A_{f,1}, \dots, A_{f,m}, B_{f,0}^i, B_{f,1}, \dots, B_{f,m}$ defined in (7.13).

Proof: The proof of Theorem 12 is similar to that of Theorem 11.

First, from Lemma 7.2.3, there exists a filter (7.6) such that $E[\|e(k)\|^2] \leq \gamma_2$ when k is large enough, if

$$\text{Tr}(W) < \gamma_2, \quad \begin{bmatrix} L & -C_f \end{bmatrix} P \begin{bmatrix} L^T \\ -C_f^T \end{bmatrix} < W, \quad \begin{bmatrix} P & \bar{A}_i P & \bar{B}_i \\ P A_i^T & \bar{P} & 0 \\ \bar{B}_i^T & 0 & I \end{bmatrix} > 0, \quad (7.22)$$

where $\mathcal{P} = \text{diag}P, \dots, P$, and

$$\bar{\mathcal{A}}_i = \left[\begin{array}{cc|cc|ccc} A_i & 0 & A_1^s & 0 & \cdots & A_m^s & 0 \\ B_f C_i & A_f & B_f C_1^s & 0 & \cdots & B_f C_m^s & 0 \end{array} \right],$$

and

$$\bar{B}_i = \left[\begin{array}{cc|cc|ccc} B_i & B_f D_i & B_1^s & B_f D_1^s & \cdots & B_m^s & B_f D_m^s \end{array} \right].$$

Using the same change-of-variable technique as in the proof of Theorem 11, the claim made in the theorem follows. $\square$

The following algorithm constructs a robust asymptotic MMSE filter for system (7.2) based on Theorem 12.

Robust Asymptotic MMSE Filtering Algorithm

1. Solve the semidefinite programming problem

$$\begin{aligned} \text{Minimize:} \quad & \gamma_2, \\ \text{Subject to:} \quad & (7.21), \end{aligned} \tag{7.23}$$

and find the optimal values of Z , Y , H , F , and G .

2. Define $V = (Y - Z)^{1/2}$ and $U = -Z^{-1}V$.

3. Construct the desired asymptotic MMSE filter by

$$\begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H & F \\ G & 0 \end{bmatrix} \begin{bmatrix} -V^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{7.24}$$

In addition, the Lyapunov function $E[x(k)Qx(k)]$, where

$$Q = \begin{bmatrix} Y & (Y - Z)^{1/2} \\ (Y - Z)^{1/2} & I \end{bmatrix},$$

guarantees that $E[\|e(k)\|^2] \leq \gamma_2$ when k is large enough.

Our robust steady state filtering algorithm extends the results in [GBG98, PM96] by incorporating stochastic uncertainties in the system model. In addition, our filtering algorithm also has the advantage that the so-called mixed performance filtering problem [KRB96] can be solved easily by combining the corresponding LMI conditions. This will be illustrated with the numerical example that we describe next.

7.2.4 A numerical example: LTI equalizer for communication channels

We present an application of the robust MMEG and MMSE filtering techniques on designing a linear time invariant equalizer for a communication channel. Consider the following IIR model of a communication channel

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.1 & 0.5 \\ 0 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s(k), \\ y(k) &= \begin{bmatrix} 1 + \zeta(k) \\ 1 + \zeta(k) \end{bmatrix}^T x(k) + (5 + \zeta(k))s(k) + 0.3w(k). \end{aligned} \tag{7.25}$$

$s(k)$ is the signal that is transmitted through the channel, and w is an additive measurement noise. The channel model (7.25) is affected by time-varying uncertainties ζ that are a combination of mixed deterministic and stochastic parametric uncertainties (see for example, [MC88]). Specifically, $\zeta(k) = 0.1\zeta_d(k) + 0.5\zeta_s(k)$, where: $\zeta_d(k)$ is deterministic, satisfies $|\zeta_d(k)| < 1$ for all k , and can be measured in real-time; ζ_s is a zero-mean white noise process with a unit variance, and is independent of w and s .

For the channel (7.25), we will design an LTI equalizer to estimate the input signal $s(k)$ (Fig. 7.1). We will allow ourselves a unit delay in equalization.

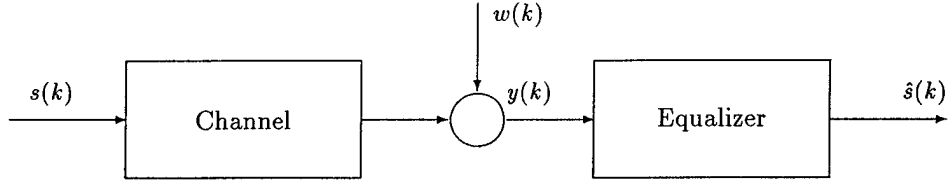


Figure 7.1: A communication channel with an equalizer

In order to apply the technique proposed in Sections 7.2.2 and 7.2.3 to equalize the channel (7.25), we need to add one more state variable in (7.25). The unit delay in equalization adds another state variable, resulting in the following description of the channel model.

$$\begin{aligned}
 x(k+1) &= \begin{bmatrix} 0.1 & 0.5 & 1 & 0 \\ 0 & 0.1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s(k+1), \\
 y(k) &= \begin{bmatrix} 1 + \zeta(k) \\ 1 + \zeta(k) \\ 5 + \zeta(k) \\ 0 \end{bmatrix}^T x(k) + 0.3w(k), \\
 z(k) &= [0 \ 0 \ 0 \ 1] x(k).
 \end{aligned}$$

If the signal s and the noise w are white noise processes, then the robust MMSE filtering algorithm presented in Section 7.2.3 can be used to design an estimator that minimizes an upper bound of the worst-case MSE of the estimation. We will impose an additional requirement on the estimator that it should also satisfy a worst-case MEG requirement [KRB96]; this can be thought as a safeguard against the possibility that the statistics of w are not actually white. Specifically, with a prescribed mean energy gain bound of γ_∞ in effect, we will design an optimal steady state MMSE filter. The mixed performance filtering problem is:

$$\begin{aligned}
 \text{minimize:} \quad & \gamma_2, \\
 \text{Subject to:} \quad & (7.12), (7.13) \text{ and } (7.21).
 \end{aligned} \tag{7.26}$$

The dashed line in Fig. 7.2 shows the best bound on the asymptotic MMSE of the estimation with no MEG constraint. This is simply the optimal answer from the semidefinite programming problem (7.23). The solid line shows the optimal values of problem (7.26) as a function of the MEG constraint γ_∞ . Corresponding to every point on the tradeoff curve represented by the solid line, an LTI equalizer can be constructed that is guaranteed to yield: (i) a worst-case MEG from w and s to e that is less than γ_∞ ; and (ii) with unit variance white noise processes w and s , an asymptotic worst-case MSE of the estimation that is less than γ_2 .

7.3 Robust adaptive Kalman filters for linear time-varying systems with stochastic parametric uncertainties

In this section, we consider the robust estimation problem for linear time-varying systems with stochastic parametric uncertainties. For such systems, we present an adaptive robust Kalman filtering algorithm.

7.3.1 Preliminary

First, let us consider a linear time-varying system

$$x(k+1) = A(k)x(k) + B(k)w(k), \quad y(k) = C(k)x(k) + D(k)u(k), \tag{7.27}$$

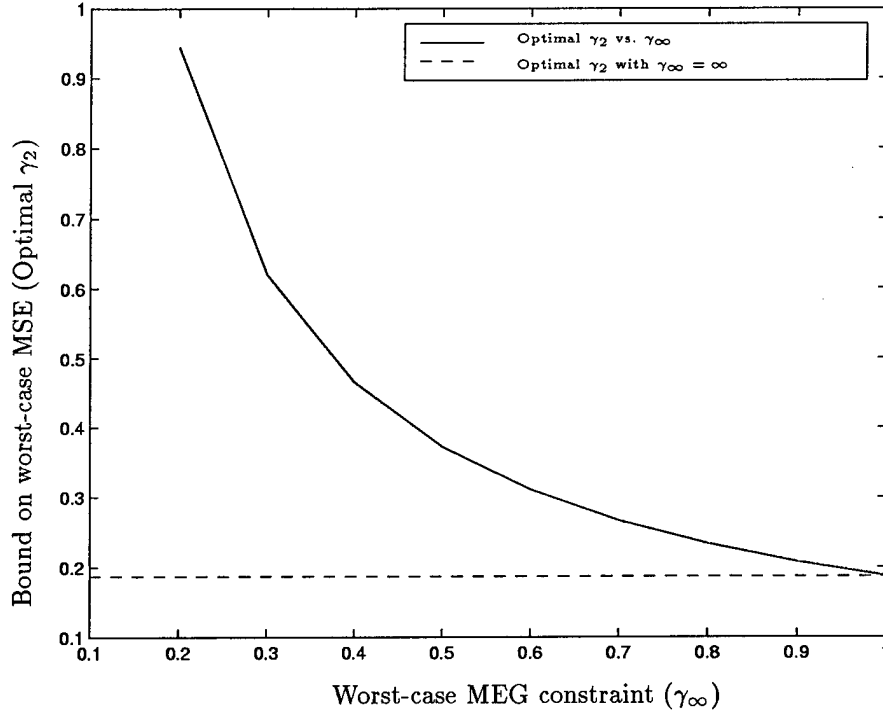


Figure 7.2: Tradeoff between the MEG and MSE performance constraints

where $k \in \mathbf{Z}_+$, $x(k) \in \mathbf{R}^n$ is the state, $w(k) \in \mathbf{R}^{n_w}$ is the input noise, $y(k) \in \mathbf{R}^{n_y}$ is the measured output, and $u(k) \in \mathbf{R}^{n_u}$ is the measurement noise. w and u are independent white noise random processes. Equations (7.27) model systems which are affected by both actuator and sensor noises (w and u respectively). A fundamental problem associated with such systems is that of *state estimation*, i.e., the “optimal” estimation of the state $x(k)$ from the noisy measurements $\{y(i), i = 0, 1, \dots, k\}$; the corresponding state estimation is denoted $\hat{x}(k|k)$. Such estimation problems arise in several applications in control, communications and signal processing; see for example, [Men95, GKN⁺74] and the references therein.

Recursive¹ minimum mean square estimators (MMSE) form an important class of optimal state estimators for system (7.27) [GKN⁺74, MC88, Pro95, OPG96]. They minimize the expected value of the square of the estimation error, i.e., $E[(x(k) - \hat{x}(k|k))^T(x(k) - \hat{x}(k|k))]$, at each k . When the random processes w and u are Gaussian, it turns out that the MMSE estimator is a linear filter whose coefficients can be determined by solving a Riccati difference equation. (This is the celebrated Kalman filter.) When w and u are not Gaussian, the Kalman filter yields the best *linear* MMSE estimator. An important (and desirable) property of the Kalman filtering algorithm is that it converges when system (7.27) is stable and time-invariant [Koc79].

The Kalman filter consists of the following two parts:

1. One-step prediction update:

$$\begin{aligned} x_f(k) &= A(k-1)x_f(k-1) + K(k-1)(C(k-1)x_f(k-1) - y(k-1)), \\ \hat{x}(k|k-1) &= x_f(k), \end{aligned} \quad (7.28a)$$

2. Filtered estimation update:

$$\hat{x}(k|k) = \hat{x}(k|k-1) + F(k)(C(k)\hat{x}(k|k-1) - y(k)). \quad (7.28b)$$

When the matrices $A(k)$, $B(k)$, $C(k)$, and $D(k)$ in (7.27) can be measured exactly, computing the Kalman gains $K(k+1)$ and $F(k)$ in (2) is equivalent to a quadratic optimization problem, one that can be solved analytically [Koc79]. However, in many cases, there exist uncertainties in model parameters

¹In the following, we will use the terms “recursive” and “adaptive” interchangeably.

and/or model structure because of errors from system identification routines or model reductions, see for example [MC88, CC94, OPG96, KG96]. The performance of estimators designed without accounting for these uncertainties can be severely degraded, and sometimes even unacceptable [Hay89]. Thus, robust Kalman estimators must be designed with graceful performance degradation in the presence of modeling errors.

Another issue in Kalman filtering design is the initialization of the adaptive algorithm. The standard adaptive Kalman filter is initialized in an *ad hoc* manner, leaving room for improvement in its transient performance.

Here we consider MMSE estimation problems for linear time-varying systems affected by stochastic uncertainties, with a view towards optimizing the “transient performance” of the estimation. The stochastic uncertainties that we consider affect the system matrices; in addition, we assume that the correlation of the state initial condition is known only to lie in a polytope². For such systems, starting with the standard one-step predictor-corrector filter structure (2), we develop a recursive estimation algorithm where at each step, an upper bound of the worst-case value, over all possible initial values of the state correlation, of the mean square of the estimation error, is minimized. The minimization is performed via numerical convex optimization over Linear Matrix Inequalities (LMIs). As the system matrices are allowed to be time-varying, the algorithm can readily incorporate measured deterministic parametric uncertainties. We demonstrate through an example that the robust adaptive algorithm we develop leads to much improved transient behavior than the standard adaptive Kalman filtering algorithm. We will see that as a by-product of our filtering algorithm, we obtain a method to optimally initialize any recursive filtering algorithm, for instance the standard adaptive Kalman filter. While convergence of the recursive algorithm that we present is not guaranteed in general—as with the standard recursive Kalman filter for time-varying systems—we prove convergence in the steady state, when applied towards estimation in a linear time-invariant system affected by stochastic uncertainties, provided the uncertain system is mean square stable.

7.3.2 Problem setup

We consider the linear time-varying system

$$\begin{aligned} x(k+1) &= A_{\Delta}(k)x(k) + B_{\Delta}(k)w(k), \\ y(k) &= C_{\Delta}(k)x(k) + D_{\Delta}(k)w(k), \quad z(k) = L(k)x(k), \end{aligned} \quad (7.29a)$$

where

$$\begin{aligned} A_{\Delta}(k) &= A(k) + \sum_{i=1}^m A_i^s(k)\zeta_i^s(k), & B_{\Delta}(k) &= B(k) + \sum_{i=1}^m B_i^s(k)\zeta_i^s(k), \\ C_{\Delta}(k) &= C(k) + \sum_{i=1}^m C_i^s(k)\zeta_i^s(k), & D_{\Delta}(k) &= D(k) + \sum_{i=1}^m D_i^s(k)\zeta_i^s(k), \end{aligned} \quad (7.29b)$$

which contains stochastic parametric uncertainties. Our objective is to design a worst-case optimal robust adaptive filter of the form given in (2) with $\hat{z}(k) = L(k)\hat{x}(k|k)$ to estimate $z(k)$ (see Fig. 7.3). (The case $L(k) = I$ corresponds to state estimation.) Specifically, since at each k the correlation of the state $X(k)$ depends on the correlation of the initial condition $X(0)$, $X(k)$ is uncertain when $X(0)$ is uncertain. We wish to find the optimal Kalman gains $K(k-1)$ and $F(k)$ so as to minimize the maximum value of the mean square of the estimation error, $E[\|z(k) - \hat{z}(k)\|^2]$, over all allowable values for $X(k)$.

7.3.3 Robust adaptive Kalman filtering

We begin by rewriting equations (7.29) in an equivalent form (see for example [MC88, CC94]). Let

$$v_x(k) = \sum_{i=1}^m A_i^s(k)\zeta_i^s(k)x(k) + \left(B(k) + \sum_{i=1}^m B_i^s(k)\zeta_i^s(k) \right) w(k)$$

²Our framework is perhaps related closest with the one in [CC94], where the “uncertainties” affect the noise moments; a game-theoretic argument is used to establish the existence of an optimal recursive scheme. Another approach to solve this problem is to include the uncertain parameters in the state vector and use *Extended Kalman Filtering* (EKF) algorithm. Extended Kalman filter is an approximate filter based on the first-order linearization of the nonlinear system and is in general biased [Men95, Lju79]. The order of the filter depends on the number of uncertain parameters.

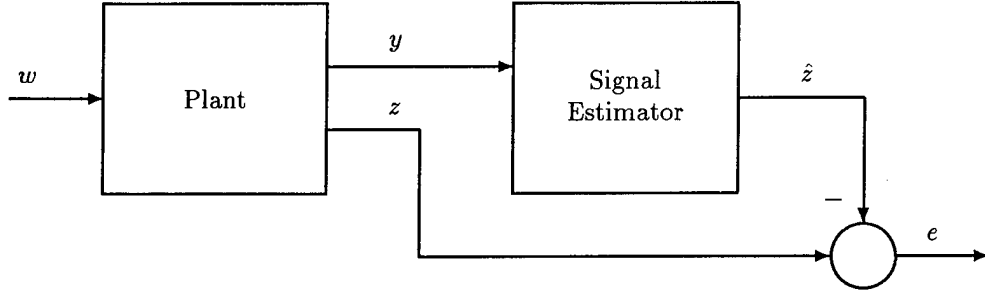


Figure 7.3: Block diagram describing the estimation framework

and

$$v_y(k) = \sum_{i=1}^m C_i^s(k) \zeta_i^s(k) x(k) + \left(D(k) + \sum_{i=1}^m D_i^s(k) \zeta_i^s(k) \right) w(k).$$

v_x and v_y are random processes. Since ζ_i^s , $i = 1, \dots, m$ and w are independent zero-mean white-noise processes, v_x and v_y satisfy the following first order and second order moment conditions for every $k \in \mathbf{Z}_+$:

$$E[v_x(k)] = 0, \quad E[v_y(k)] = 0,$$

$$E[v_x(k)v_x(k)^T] = B(k)B(k)^T + \sum_{i=1}^m (A_i^s(k)X(k)A_i^s(k)^T + B_i^s(k)B_i^s(k)^T),$$

$$E[v_y(k)v_y(k)^T] = D(k)D(k)^T + \sum_{i=1}^m (C_i^s(k)X(k)C_i^s(k)^T + D_i^s(k)D_i^s(k)^T),$$

$$E[v_x(k)v_y(k)^T] = B(k)D(k)^T + \sum_{i=1}^m (A_i^s(k)X(k)C_i^s(k)^T + B_i^s(k)D_i^s(k)^T),$$

$$E[v_y(k)v_x(k)^T] = D(k)B(k)^T + \sum_{i=1}^m (C_i^s(k)X(k)A_i^s(k)^T + D_i^s(k)B_i^s(k)^T),$$

where $X(k) = E[x(k)x(k)^T]$. We also note that v_x and v_y are white-noise random processes, so that when $i \neq j$, $E[v_x(i)v_x(j)^T] = 0$, $E[v_y(i)v_y(j)^T] = 0$ and $E[v_x(i)v_y(j)^T] = 0$. In addition, $E[x(i)v_x(j)^T] = 0$ and $E[x(i)v_y(j)^T] = 0$ for $i \leq j$. Therefore, system (7.29) can be represented equivalently as

$$x(k+1) = A(k)x(k) + \hat{B}(k)v(k), \quad y(k) = C(k)x(k) + \hat{D}(k)v(k), \quad z(k) = L(k)x(k), \quad (7.30)$$

where

$$\hat{B}(k) = [A_1^s(k)X(k)^{1/2} \quad \dots \quad A_m^s(k)X(k)^{1/2} \quad B(k) \quad B_1^s(k) \quad \dots \quad B_m^s(k)]$$

and

$$\hat{D}(k) = [C_1^s(k)X(k)^{1/2} \quad \dots \quad C_m^s(k)X(k)^{1/2} \quad D(k) \quad D_1^s(k) \quad \dots \quad D_m^s(k)],$$

v is a zero-mean unit-variance white noise random process.

If the variance $X(0) = E[x(0)x(0)^T]$ is known, then it is easily verified that $X(k)$ can be uniquely determined by the following recursion:

$$\begin{aligned} X(k+1) = h(X(k)) &\triangleq A(k)X(k)A(k)^T + B(k)^T B(k) \\ &+ \sum_{j=1}^m (A_j^s(k)X(k)A_j^s(k)^T + B_j^s(k)B_j^s(k)^T). \end{aligned} \quad (7.31)$$

However, $X(0)$ is usually not known. In our setting, $X(0)$ is assumed to lie in a polytope $\text{Co}X_1(0), \dots, X_p(0)$. In this case, it can be easily shown that $X(k)$ also lies in a polytope $\text{Co}X_1(k), \dots, X_p(k)$, where

$$X_i(k+1) = A(k)X_i(k)A(k)^T + B(k)B(k)^T + \sum_{j=1}^m (A_j^s(k)X_i(k)A_j^s(k)^T + B_j^s(k)B_j^s(k)^T). \quad (7.32)$$

If the system is mean square stable, the polytope converges to a fixed point as $k \rightarrow \infty$.

While (7.30) is similar to the setting for the standard recursive Kalman filtering problem, we note that the noise $v(k)$ is *state-dependent* and its second moment is uncertain. For this reason, the optimal linear recursive MMSE filter design problem is considerably harder than the standard Kalman filtering problem, which can be solved analytically [Koc79]. In particular, at each k , we are given:

(N1) Measurements of the system matrices $A(k)$, $B(k)$, $C(k)$, $D(k)$ and $L(k)$.

(N2) Measurement of the noisy output $y(k)$.

(N3) From (7.32), the vertices $X_1(k)$, $\dots$, $X_p(k)$ of the polytope where $X(k)$ lies in.

We then need to solve the following minimax optimization problem for the optimal Kalman gains $K(k-1)$ and $F(k)$ that determine the filter (2):

$$\begin{aligned} \text{Minimize:} \quad & E [\|z(k) - \hat{z}(k)\|^2] \\ \text{Subject to:} \quad & \text{Data (N1-N3),} \\ & \text{Equations (2) and (7.30).} \end{aligned} \tag{7.33}$$

Our approach towards the solution of Problem (7.33) is as follows: We will first define an upper bound on the objective function, i.e., the mean square of the estimation error over all possible values for the corresponding state correlation matrix $X(k)$. We will then reformulate the problem of determining $K(k-1)$ and $F(k)$ so as to minimize this upper bound as a convex optimization problem.

We now proceed with deriving an upper bound on the objective function of Problem (7.33). Suppose that at each k , we have available a matrix $P(k-1)$ with

$$P(k-1) \geq E [(x(k-1) - \hat{x}(k-1|k-2))(x(k-1) - \hat{x}(k-1|k-2))^T].$$

Then,

$$\begin{aligned} & E [(x(k) - \hat{x}(k|k-1))(x(k) - \hat{x}(k|k-1))^T] \\ & \leq (A(k-1) + K(k-1)C(k-1))P(k-1)(A(k-1) + K(k-1)C(k-1))^T \\ & \quad + (\hat{B}(k-1) + K(k-1)\hat{D}(k-1))(\hat{B}(k-1) + K(k-1)\hat{D}(k-1))^T \end{aligned} \tag{7.34}$$

$$\triangleq f(P(k-1), X(k-1)). \tag{7.35}$$

Then, we can obtain $P(k) \geq E [(x(k) - \hat{x}(k|k-1))(x(k) - \hat{x}(k|k-1))^T]$, simply by requiring that

$$f(P(k-1), X(k-1)) \leq P(k), \quad \forall X(k-1) \in \text{Co}X_1(k-1), \dots, X_p(k-1). \tag{7.36}$$

Next, we have

$$\begin{aligned} & E [(x(k) - \hat{x}(k|k))(x(k) - \hat{x}(k|k))^T] \\ & \leq (I + F(k)C(k))P(k)(I + F(k)C(k))^T + (F(k)\hat{D}(k))(F(k)\hat{D}(k))^T \end{aligned} \tag{7.37}$$

$$\triangleq g(F(k), X(k)). \tag{7.38}$$

Then, for any matrix $M(k)$ that satisfies

$$\begin{aligned} g(F(k), X(k)) & \leq M(k), \quad X(k) = h(X(k-1)), \\ \forall X(k-1) & \in \text{Co}X_1(k-1), \dots, X_p(k-1), \end{aligned} \tag{7.39}$$

we have

$$E [(z(k) - \hat{z}(k))(z(k) - \hat{z}(k))^T] \leq L(k)M(k)L(k)^T,$$

which implies that $\text{Tr}(L(k)M(k)L(k)^T)$ is an upper bound of the objective function in (7.33).

We can now formulate the problem of determining $K(k-1)$ and $F(k)$ so as to minimize an upper bound on the the mean square of the estimation error over all possible values for the corresponding state correlation matrix as follows:

$$\begin{aligned} \text{Minimize: } & \text{Tr}(L(k)M(k)L(k)^T) \\ \text{Subject to: } & \text{Conditions (7.36) and (7.39),} \end{aligned} \quad (7.40)$$

where $P(k)$, $M(k)$, $K(k-1)$ and $F(k)$ are optimization variables. While this minimax problem has no analytical solution in general, we establish via the following theorem that problem (7.40) is a convex optimization problem, which can be reduced to minimizing a linear objective subject with Linear Matrix Inequality (LMI) constraints. This problem can be solved numerically very efficiently using standard algorithms [NN94, GN95, GDN95, WB96], so that Theorem 13 provides for the efficient and effective numerical solution of problem (7.40); for details on LMIs, we refer the reader to the book [BEFB94] and the references therein.

Theorem 13 *The minimax problem (7.40) is equivalent to the following LMI optimization problem with variables $M(k) = M(k)^T \in \mathbf{R}^{n \times n}$, $Q(k) = Q(k)^T \in \mathbf{R}^{n \times n}$, $Y(k) \in \mathbf{R}^{n \times n_y}$ and $F(k) \in \mathbf{R}^{n \times n_y}$,*

$$\text{Minimize: } \text{Tr}(L(k)M(k)L(k)^T)$$

Subject to:

$$\begin{aligned} & \begin{bmatrix} Q(k) & T_{12}(k-1) & T_{13,i}(k-1) \\ T_{12}(k-1)^T & Q(k-1) & 0 \\ T_{13,i}(k-1)^T & 0 & I \end{bmatrix} \geq 0, \quad Q(k) > 0, \\ & \begin{bmatrix} M(k) & I + F(k)C(k) & F(k)\hat{D}_i(k) \\ I + C(k)^T F(k)^T & Q(k) & 0 \\ \hat{D}_i(k)^T F(k)^T & 0 & I \end{bmatrix} \geq 0, \quad i = 1, \dots, p, \end{aligned} \quad (7.41)$$

where

$$T_{12}(k-1) = Q(k)A(k-1) + Y(k)C(k-1),$$

$$T_{13,i}(k-1) = Q(k)\hat{B}_i(k-1) + Y(k)\hat{D}_i(k-1). \quad (7.42)$$

$$\hat{B}_i(k) = [A_1^s(k)X_i(k)^{1/2} \quad \dots \quad A_m^s(k)X_i(k)^{1/2} \quad B(k) \quad B_1^s(k) \quad \dots \quad B_m^s(k)],$$

$$\hat{D}_i(k) = [C_1^s(k)X_i(k)^{1/2} \quad \dots \quad C_m^s(k)X_i(k)^{1/2} \quad D(k) \quad D_1^s(k) \quad \dots \quad D_m^s(k)],$$

$X_i(k)$ is defined in (7.32), $P(k) = Q(k)^{-1}$ and

$$P(k) \geq E[(x(k) - \hat{x}(k|k-1))(x(k) - \hat{x}(k|k-1))^T]$$

for any $X(k-1) \in \text{Co}X_1(k-1), \dots, X_p(k-1)$, $K(k-1) = Q(k)^{-1}Y(k)$ and $F(k)$ are the optimal Kalman gains, and the upper bound of the mean square of the estimation error $E[\|z(k) - \hat{z}(k)\|^2]$ is $\text{Tr}(L(k)M(k)L(k)^T)$.

Proof: Let $X(k-1) = \sum_{i=1}^p \lambda_i X_i(k-1)$, where $\lambda_i \in [0, 1]$ and $\sum_{i=1}^p \lambda_i = 1$. Since $h(X(k-1))$ is a linear function in $X(k-1)$, we have $X(k) = \sum_{i=1}^p \lambda_i X_i(k)$ for the same set of λ_i , $i = 1, \dots, p$. By Schur's complements lemma, condition (7.36) is equivalent to

$$\sum_{i=1}^p \lambda_i \begin{bmatrix} P(k) & T_{12}(k-1) & T_{13,i}(k-1)\mathcal{X}_i \\ T_{12}(k-1)^T & P(k-1)^{-1} & 0 \\ T_{13,i}(k-1)^T \mathcal{X}_i & 0 & \mathcal{X}_i^2 \end{bmatrix} \geq 0, \quad (7.43)$$

and condition (7.39) is equivalent to

$$\sum_{i=1}^p \lambda_i \begin{bmatrix} M(k) & I + F(k)C(k) & F(k)\hat{D}_i(k)\mathcal{X}_i \\ I + C(k)^T F(k)^T & P(k)^{-1} & 0 \\ \hat{D}_i(k)^T F(k)^T \mathcal{X}_i & 0 & \mathcal{X}_i^2 \end{bmatrix} \geq 0, \quad (7.44)$$

where $\mathcal{X}_i = \text{diag} X_i(k)^{1/2}, \dots, X_i(k)^{1/2}, I, \dots, I$.

Defining new variables $Q(k) = P(k)^{-1} > 0$ and $Y(k) = Q(k)K(k-1)$, it is straight forward to verify that conditions (7.43) and (7.44) are equivalent to the LMI constraints in (7.41), concluding the proof. $\square$

Theorem 13 paves the way for a recursive robust Kalman filtering algorithm. To start the algorithm, we need to initialize $Q(0)$, the process which we describe next. Let $x_f(0) = 0$.³ Then we have $Q(0) = E[x(0)x(0)^T]^{-1} = X(0)^{-1}$, where $X(0) \in \text{Co}X_1(0), \dots, X_p(0)$. Using Theorem (13), $Q(1)$ can be computed as an optimizer to the following problem:

$$\begin{aligned} & \text{Minimize: } \text{Tr}(L(1)M(1)L(1)^T) \\ & \text{Subject to:} \\ & \begin{bmatrix} Q(1) & T_{12}(0) & T_{13,i}(0) \\ T_{12}(0)^T & X(0)^{-1} & 0 \\ T_{13,i}(0)^T & 0 & I \end{bmatrix} \geq 0, \quad Q(1) > 0 \\ & \begin{bmatrix} M(1) & I + F(1)C(1) & F(1)\hat{D}_i(1) \\ I + C(1)^T F(1)^T & Q(1) & 0 \\ \hat{D}_i(1)^T F(1)^T & 0 & I \end{bmatrix} \geq 0, \\ & X(0) \in \text{Co}X_1(0), \dots, X_p(0), \quad i = 1, \dots, p. \end{aligned} \quad (7.45)$$

The matrix inequalities in (7.45) are not linear in all the variables, because the first LMI constraint has the term $X(0)^{-1}$. However, it is easily shown that (7.45) is equivalent to the LMI

$$\begin{aligned} & \text{Minimize: } \text{Tr}(L(1)M(1)L(1)^T) \\ & \text{Subject to:} \\ & \begin{bmatrix} Q(1) & T_{12}(0)X_i(0) & T_{13,i}(0) \\ X_i(0)T_{12}(0)^T & X_i(0) & 0 \\ T_{13,i}(0)^T & 0 & I \end{bmatrix} \geq 0, \quad Q(1) > 0, \\ & \begin{bmatrix} M(1) & I + F(1)C(1) & F(1)\hat{D}_i(1) \\ I + C(1)^T F(1)^T & Q(1) & 0 \\ \hat{D}_i(1)^T F(1)^T & 0 & I \end{bmatrix} \geq 0, \\ & X(0) \in \text{Co}X_1(0), \dots, X_p(0), \quad i = 1, \dots, p, \end{aligned} \quad (7.46)$$

where $T_{12}(0)$, $T_{13,i}(0)$, $\hat{D}_i(1)$ are defined in (7.42), and $K(0) = Q(1)^{-1}Y(1)$.

Remark 7.3.1 The solution to the optimization problem (7.45) can also be applied towards “optimally” initializing other recursive algorithms, for instance the standard adaptive Kalman filtering algorithms, with an attendant improvement in transient performance of such routines.

We now summarize the various steps in our robust adaptive Kalman filtering algorithm:

Step 1. Solve (7.46) to initialize $Q(1)$, $M(1)$, $K(0)$ and $F(1)$. Let $k = 1$.

Step 2. At time $k + 1$, let

$$\begin{aligned} X_i(k+1) &= A(k)X_i(k)A(k)^T + B(k)B(k)^T \\ &+ \sum_{j=1}^m (A_j^s(k)X_i(k)A_j^s(k)^T + B_j^s(k)B_j^s(k)^T), \quad i = 1, \dots, p. \end{aligned} \quad (7.47)$$

Step 3. Solve the eigenvalue minimization problems (7.41) for $Q(k)$, $M(k)$ and $K(k-1)$, $F(k)$.

Step 4. Repeat Steps 2 and 3.

³If $E[x(0)] = a$ is known and $a \neq 0$, we may define $x_f(0) = a$ and $Q(0)$ becomes $(X(0) - aa^T)^{-1}$. The initialization process (7.46) can still be applied by replacing $X(0)$ by $X(0) - aa^T$ and $X_i(0)$ by $X_i(0) - aa^T$ respectively.

We have the following observations on the robust Kalman filtering algorithm:

1. If the correlation $X(0)$ is exactly known in advance, the polytope is a fixed point at each k . The eigenvalue minimization problem (7.41) turns out to be equivalent to the optimization problem encountered in the standard adaptive Kalman filtering problem, and can be solved analytically.
2. If we have $x_f(0) = E[x(0)]$ in estimator (2), then it can be readily checked that the state estimation algorithm is unbiased.
3. If system (7.29) is mean square stable and the state-space matrices (the matrices on the right-hand sides of (7.29b)) are constant, then the state estimation using robust adaptive Kalman filter converges to a steady state estimator; a proof is given in Section 7.5.

7.3.4 A numerical example: Adaptive equalizer for communication channels

We present an application of the proposed robust adaptive Kalman filtering techniques towards the design of equalizers for a communication channel. Consider the following IIR model of a communication channel

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0.9 & 0.5 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} s(k), \\ y(k) &= [1 + \zeta(k) \quad 1 + \zeta(k)] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + (5 + \zeta(k))s(k) + w(k). \end{aligned} \quad (7.48)$$

s is the signal which is transmitted through the channel, w is the -10 dB white noise that corrupts the received signal y . The channel model (7.48) is affected by time-varying uncertainties ζ that are a combination of mixed deterministic and stochastic parametric uncertainties (see for example, [MC88]). Specifically, $\zeta(k) = 0.1\zeta_d(k) + 0.5\zeta_s(k)$, where: $\zeta_d(k)$ is deterministic, satisfies $|\zeta_d(k)| < 1$ for all k , and can be measured in real-time; ζ_s is a zero-mean white noise process with a unit variance, and is independent of w and s . The initial conditions of the state vectors, $x_1(0)$ and $x_2(0)$, are random variables and satisfy the second moment conditions

$$E[x_1(0)x_1(0)] \leq 1, \quad E[x_2(0)x_2(0)] \leq 1, \quad E[x_1(0)x_2(0)] = 0, \quad E[x_2(0)x_1(0)] = 0.$$

For the channel (7.48), we design an equalizer to estimate the input signal $s(k)$ (Fig. 7.1).

Assuming a zero mean, uncorrelated white noise model for the input signal s , we design an equalizer using the robust adaptive Kalman filtering. The first case considered is when the channel does not have any uncertainty, i.e., $\zeta_d = \zeta_s = 0$; this is the ideal channel, and the corresponding equalizer is a standard adaptive Kalman filter. In Fig. 7.4, we compare the standard Kalman filter with optimal initialization (obtained by solving (7.46)), with the standard Kalman filter with an *ad hoc* initialization scheme [Men95] ($P(0) = \eta I$, where $\eta = 0.1, 1$ and 10). The Mean Square Error (MSE) estimates are obtained by averaging 200 runs. The results indicate that the transient performance of the standard Kalman filtering algorithm can be improved by solving (7.46) to optimally initialize the recursion.

In the second example, we compare the performance of the standard Kalman filtering algorithm and the proposed robust adaptive Kalman filtering algorithm in the presence of uncertainties; Fig. 7.5 shows a comparison of the experimentally obtained mean square error values, averaged over 200 runs, obtained with the standard adaptive Kalman filter with optimal initialization by solving (7.46), and our robust adaptive Kalman filter. It is clear that the adaptive Kalman filtering techniques presented herein offer significantly improved estimation performance over the standard adaptive Kalman filter in the presence of stochastic parametric uncertainties.

7.4 Conclusion

First, we have developed robust minimum mean energy gain (MMEG) and asymptotic minimum mean square error (MMSE) filtering algorithms for systems with mixed deterministic and stochastic uncertainties.

Secondly, we have developed a robust adaptive Kalman filtering algorithm for linear time-varying systems with stochastic parametric uncertainties. If the system is mean square stable and the state-space matrices are time-invariant, we have established that the robust adaptive filter converges to a steady state estimator.

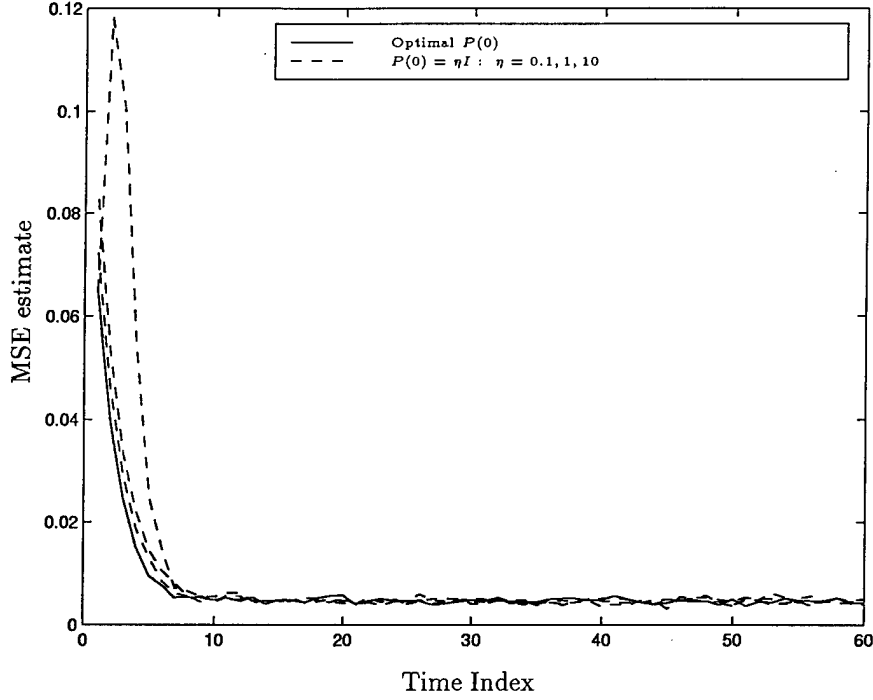


Figure 7.4: Illustration of the performance improvement of the standard Kalman filter with optimal initialization

The filtering algorithm is formulated as an optimization problem based on linear matrix inequalities, and can be implemented numerically efficiently.

We have illustrated, via numerical examples, the application of these filtering techniques towards designing equalizers for communication channels.

7.5 Appendix: Proof of the convergence of the robust adaptive Kalman filtering algorithm

Proposition 3 Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_y \times n}$, and $D \in \mathbb{R}^{n_y \times n_w}$. Suppose that:

1. A has all of its eigenvalues inside the unit disk.
2. $DD^T > 0$.
3. The Riccati equation

$$P = APA^T + BB^T - (BD^T + APC^T)(CPC^T + DD^T)^{-1}(DB^T + CPA^T) \quad (7.49)$$

has a unique positive semidefinite solution P^* .

4. The matrix K^* defined by

$$K^* = -(AP^*C^T + BD^T)(CP^*C^T + DD^T)^{-1}$$

satisfies

$$(K^*D + B)(K^*D + B)^T > 0.$$

Under these conditions, consider the following recursion:

$$P^*(k+1) = \min_{K(k)} f(P^*(k), K(k)),^4 \quad \text{and} \quad K^*(k) = \operatorname{argmin}_{K(k)} f(P^*(k), K(k)), \quad (7.50a)$$

⁴Though the set of positive semidefinite matrices is only a partially ordered set, it is well-known that the problem $\min_{K(k)} f(P^*(k), K(k))$ is well defined; see [Koc79].

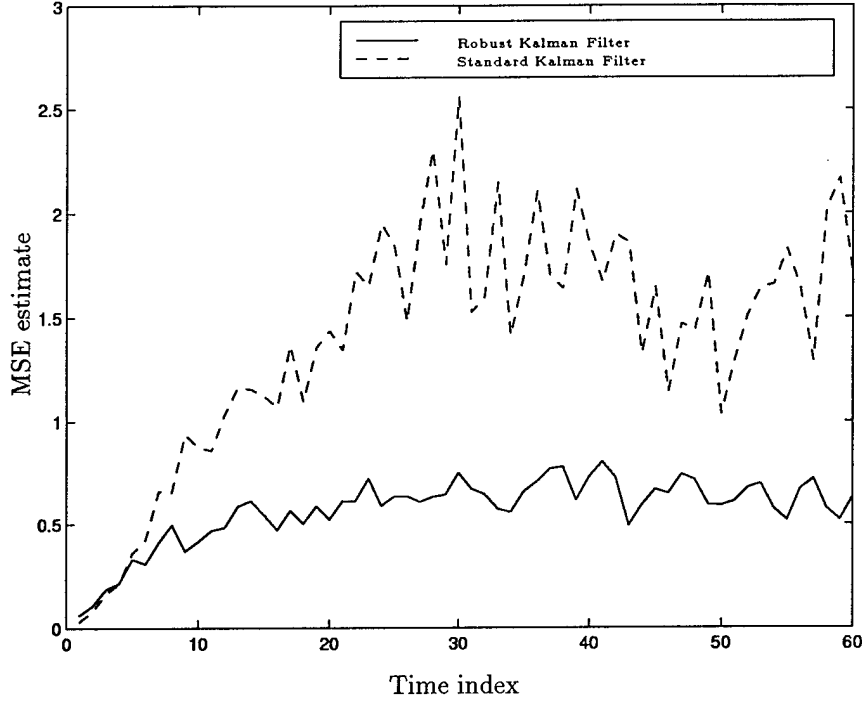


Figure 7.5: Performance comparison between the robust and standard Kalman filters

where

$$f(P(k), K(k)) \triangleq (A + K(k)C)P(k)(A + K(k)C)^T + (K(k)D + B)(K(k)D + B)^T. \quad (7.50b)$$

Then, for any initial condition $P(0) \geq 0$, the sequence of solutions $\{P^*(k), K^*(k)\}$ to the recursion (7.50) satisfies

$$\lim_{k \rightarrow \infty} P^*(k) = P^* \quad \text{and} \quad \lim_{k \rightarrow \infty} K^*(k) = K^*.$$

Remark 7.5.1 Condition (2) is a standard assumption with Kalman filtering, see for example [Koc79, Men95, GKN⁺74].

For the special case $B^T D = 0$, the solution of the recursion (7.50) yields the standard recursive Kalman filter; see for example [Koc79, Men95, GKN⁺74]. In this case, conditions (3) and (4) in the theorem statement are automatically satisfied.

Condition (4) ensures that $A + K^*C$ is stable.

Proof: Let $P(0) \geq 0$ be a given initial condition. The proof of the theorem consists of the following steps.

1. Let $\{P_1^*(k), K_1^*(k)\}$ and $\{P_2^*(k), K_2^*(k)\}$ be two sequences consisting of the solutions of (7.50) for two different initial conditions. If $P_1^*(k_0) \geq P_2^*(k_0)$, then $P_1^*(k) \geq P_2^*(k)$ for any $k \geq k_0$.

This follows inductively from

$$\begin{aligned} P_2^*(k_0 + 1) &= f(P_2^*(k_0), K_2^*(k_0)) \leq f(P_2^*(k_0), K_1^*(k_0)) \\ &\leq f(P_1^*(k_0), K_1^*(k_0)) = P_1^*(k_0 + 1). \end{aligned}$$

2. Let $\{P^*(k), K^*(k)\}$ be the sequence consisting of the solutions of (7.50) for an initial condition $P(0) \geq 0$. Then, $\{P^*(k)\}$ is bounded.

Since A is stable, the sequence $\{P(k)\}$ obtained via the recursion

$$P(0) = P(0); \quad P(k+1) = f(P(k), 0)$$

is bounded. (Indeed this sequence converges to the controllability Gramian of (A, B) .) Next,

$$\begin{aligned} P^*(1) &= f(P(0), K^*(0)) \leq f(P(0), 0) = P(1), \\ P^*(2) &= f(P^*(1), K^*(1)) \leq f(P^*(1), 0) \leq f(P(1), 0) = P(2). \end{aligned}$$

Inductively, we get $P^*(k) \leq P(k)$. Therefore the sequence $\{P^*(k)\}$ is bounded.

3. The sequence of solutions $\{P_0^*(k), K_0^*(k)\}$ obtained by solving (7.50) with $P(0) = 0$ satisfies $P_0^*(k+1) \geq P_0^*(k)$, $k \in \mathbb{Z}_+$, with $\lim_{k \rightarrow \infty} P_0^*(k) = P^*$.

First note that $P_0^*(1) \geq P(0) = 0$. Next, define $\tilde{P}^*(k) = P_0^*(k+1)$ and $\tilde{K}^*(k) = K_0^*(k+1)$, $k = 0, 1, 2, \dots$. Then, clearly $\{\tilde{P}^*(k), \tilde{K}^*(k)\}$ is a sequence consisting of the solutions of (7.50) for the initial condition $\tilde{P}(0) = P_0^*(1)$. Since $\tilde{P}(0) \geq P_0(0) = 0$, an application of Step 1 of the proof yields $\tilde{P}^*(k) \geq P_0^*(k)$, or $P_0^*(k+1) \geq P_0^*(k)$, $k = 0, 1, 2, \dots$. From Step 2, the sequence $\{P_0^*(k)\}$ is bounded, and therefore converges to some limit $\underline{P}$. Since $\underline{P} \geq 0$, and satisfies (7.49), $\underline{P} = P^*$.

4. Given any $\rho > 0$, there exists $P(0) > \rho I$ such that the resulting sequence of solutions $\{P^*(k), K^*(k)\}$ of (7.50) satisfies $P^*(k+1) \leq P^*(k)$, $k \in \mathbb{Z}_+$, with $\lim_{k \rightarrow \infty} P^*(k) = \bar{P}$.

Since A is stable, there exists some $Q > 0$ such that $AQA^T - Q + BB^T < 0$ and $Q > \rho I$. Let $P(0) = Q$. We then have

$$P(1) = f(P(0), K^*(0)) \leq f(P(0), 0) < P(0).$$

Following the same argument as in Step (3), it is easily shown that $P(k)$ is a nonincreasing matrix function of k . Since the sequence $\{P(k)\}$ is bounded below by zero, it converges to some limit $\bar{P}$. Since $\bar{P} \geq 0$, and satisfies (7.49), $\bar{P} = P^*$.

Now for any initial condition $P(0) \geq 0$, there exists $P_2(0)$ large enough such that $P(0) \leq P_2(0)$. Let $\{P_1^*(k), K_1^*(k)\}$ and $\{P_2^*(k), K_2^*(k)\}$ be the sequences consisting of the solutions of (7.50) for the initial conditions $P_1(0) = 0$ and $P_2(0)$ respectively. Since both $\{P_1(k)\}$ and $\{P_2(k)\}$ converge to P^* , the sequence $\{P(k)\}$ converges to P^* as well. Convergence of $\{K^*(k)\}$ to K^* follows immediately from the observation (see for example, [Koc79]) that $P^*(k)$ and $K^*(k)$ are related by

$$K^*(k) = -(AP^*(k)C^T + BD^T)(CP^*(k)C^T + DD^T)^{-1}.$$

□

Proposition 4 Let $\{X(k)\}$ be the sequence consisting of the solutions to the recursive equation

$$X(k+1) = AX(k)A^T + BB^T + \sum_{j=1}^m (A_j^s X(k)(A_j^s)^T + B_j^s (B_j^s)^T), \quad k = 1, 2, \dots, \quad (7.51)$$

with initial condition $X(0) \geq 0$, where A , B , A_j^s , B_j^s are constant matrices. The sequence $\{X(k)\}$ is convergent and the limit is independent of $X(0)$ if and only if there exists some matrix $Q > 0$ such that

$$A^T Q A - Q + \sum_{j=1}^m (A_j^s)^T Q (A_j^s) < 0, \quad (7.52)$$

i.e., the system is mean square stable.

Proof: We first show that the sequence $\{X(k)\}$ obtained from (7.51) is convergent and the limit is independent of $X(0)$ if and only if the sequence $\{\hat{X}(k)\}$ obtained from

$$\hat{X}(k+1) = A\hat{X}(k)A^T + \sum_{j=1}^m A_j^s \hat{X}(k)(A_j^s)^T \quad (7.53)$$

is convergent and $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$.

Suppose the limit of the sequence $\{\hat{X}(k)\}$ exists and $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$. For any $X(0)$, define $\hat{X}(k) = X(k+1) - X(k)$. It is easily checked that $\hat{X}(k)$ satisfies the recursion (7.53). Since $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$, $\{X(k)\}$ is convergent.

To show that the limit of $\{X(k)\}$ is unique, suppose $\{X_1(k)\}$ and $\{X_2(k)\}$ are two sequences of solutions to (7.51) corresponding to two different initial conditions $X_1(0)$ and $X_2(0)$, with $\lim_{k \rightarrow \infty} X_1(k) = X_1$ and $\lim_{k \rightarrow \infty} X_2(k) = X_2$. Then, it is easily verified that with $\hat{X}(k) = X_1(k) - X_2(k)$, the sequence $\{\hat{X}(k)\}$ consists of solutions of the recursive equation (7.53); then $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$ implies $X_1 = X_2$.

Now suppose the limit of the sequence $\{X(k)\}$ exists and is independent of $X(0)$. We show that $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$.

For any $\hat{X}(0)$, let $X_1(0) = \hat{X}(0)$, $X_2(0) = 0$ and $\{X_i(k)\}$, $i = 1, 2$ be the corresponding sequences from recursion (7.51). By recursion (7.53), we have $\hat{X}(k) = X_1(k) - X_2(k)$. Since $\lim_{k \rightarrow \infty} X_1(k) = \lim_{k \rightarrow \infty} X_2(k)$, we get $\lim_{k \rightarrow \infty} \hat{X}(k) = 0$.

Finally, with Lemma 7.2.1 that condition (7.52) is necessary and sufficient for $\{\hat{X}(k)\}$ to converge to zero, the statement in the proposition is established. $\square$

In the following, we define $\lim_{k \rightarrow \infty} X(k) = X$,

$$\bar{B} = [A_1^s X \quad \dots \quad A_m^s X \quad B \quad B_1^s \quad \dots, B_m^s],$$

and

$$\bar{D} = [C_1^s X \quad \dots \quad C_m^s X \quad D \quad D_1^s \quad \dots \quad D_m^s].$$

Proposition 5 Let $P(0)$ be given. Let $\{P^*(k), K^*(k)\}$ be the sequence consisting of the solutions to the optimization problem (7.50). Let $\{\tilde{P}^*(k), \tilde{K}^*(k), \tilde{F}^*(k)\}$ be the sequence consisting of the solutions to the optimization problem

$$\min_{K(k-1), F(k)} \text{Tr}(M(k)), \quad k = 1, 2, \dots, \quad (7.54a)$$

where

$$f(P(k-1), K(k-1)) \triangleq (A + K(k-1)C)P(k-1)(A + K(k-1)C)^T + (K(k-1)D + B)(K(k-1)D + B)^T, \quad (7.54b)$$

$$P(k) = f(P(k-1), K(k-1)), \quad (7.54c)$$

$$M(k) = (I + F(k)C)P(k)(I + F(k)C)^T + F(k)DD^T F(k)^T, \quad (7.54d)$$

and DD^T is assumed to be positive definite.

Then:

$$P^*(k) = \tilde{P}^*(k) \quad \text{and} \quad K^*(k) = \tilde{K}^*(k), \quad k = 1, 2, \dots$$

Proof: Starting with the sequence $\{P^*(k), K^*(k)\}$ consisting of the solutions to the optimization problem (7.50), it is clear that

$$\tilde{P}^*(k) = P^*(k) \quad \text{and} \quad \tilde{K}^*(k) = K^*(k), \quad k = 1, 2, \dots,$$

must be (possibly nonunique) optimizers for problem (7.54) (this follows immediately from the equation (7.54d)). Thus, it remains to show that $\tilde{P}^*(k) = P^*(k)$ and $\tilde{K}^*(k) = K^*(k)$ are the only candidates as optimizers for problem (7.54). In other words, it suffices to show that for a given $\tilde{P}^*(k-1)$, the optimal solutions $\tilde{P}^*(k)$ and $\tilde{K}^*(k)$ in (7.54) are unique. However, note that given $\tilde{P}^*(k-1)$ and $\tilde{K}^*(k-1)$, $\tilde{P}^*(k)$ is given simply as $f(\tilde{P}^*(k-1), \tilde{K}^*(k-1))$, thus we only need to show that given $\tilde{P}^*(k-1)$, $\tilde{K}^*(k-1)$ that solves (7.54) is unique.

Now, let $\{\tilde{P}^*(k), \tilde{K}^*(k), \tilde{F}^*(k)\}$ be the sequence consisting of the solutions to the optimization problem (7.54). Assuming an optimal value for $\tilde{P}^*(k-1)$, we consider the optimization problem (7.54). For convenience, we introduce new notation for $M(k)$:

$$g(K(k-1), F(k)) = (I + F(k)C)f(P(k-1), K(k-1))(I + F(k)C)^T + F(k)DD^T F(k)^T.$$

We also denote

$$g_{\text{opt}}(K(k-1)) = \min_{F(k)} g(K(k-1), F(k))$$

and

$$F_{\text{opt}}(K(k-1)) = \arg \min g(K(k-1), F(k)).$$

Note that $\tilde{F}^*(k) = F_{\text{opt}}(\tilde{K}^*(k-1))$.

Now for any $P(k) = f(P(k-1), K(k-1)) \geq 0$, since $(CP(k)C^T + DD^T) > 0$, it can be easily verified that if $F(k) \neq F_{\text{opt}}(K(k-1))$, then

$$g(K(k-1), F(k)) \geq g(K(k-1), F_{\text{opt}}(K(k-1)))$$

and

$$g(K(k-1), F(k)) \neq g(K(k-1), F_{\text{opt}}(K(k-1))).$$

Therefore

$$\text{Tr}(g(K(k-1), F(k))) > \text{Tr}(g(K(k-1), F_{\text{opt}}(K(k-1)))).$$

Thus, $F_{\text{opt}}(K(k-1))$ is uniquely determined from $K(k-1)$.

To show that $\tilde{K}^*(k-1)$ is unique, it suffices to show that for any $K(k-1) \neq \tilde{K}^*(k-1)$, $\text{Tr}(g(K(k-1), F_{\text{opt}}(K(k-1)))) > \text{Tr}(g(\tilde{K}^*(k-1), F_{\text{opt}}(\tilde{K}^*(k-1))))$. Let $P_1(k) = f(\tilde{P}(k-1), K(k-1))$ and $P_2(k) = f(\tilde{P}(k-1), \tilde{K}^*(k-1))$. Then we have $P_1(k) \geq P_2(k)$ and $P_1(k) \neq P_2(k)$. Then, we consider two cases:

1. If $F_{\text{opt}}(K(k-1)) \neq F_{\text{opt}}(\tilde{K}^*(k-1))$, then

$$g(K(k-1), F_{\text{opt}}(K(k-1))) \geq g(\tilde{K}^*(k-1), F_{\text{opt}}(\tilde{K}^*(k-1)))$$

and

$$g(K(k-1), F_{\text{opt}}(K(k-1))) \neq g(\tilde{K}^*(k-1), F_{\text{opt}}(\tilde{K}^*(k-1))).$$

Therefore

$$\text{Tr}(g(K(k-1), F_{\text{opt}}(K(k-1)))) > \text{Tr}(g(\tilde{K}^*(k-1), F_{\text{opt}}(\tilde{K}^*(k-1)))).$$

2. If $F_{\text{opt}}(K(k-1)) = F_{\text{opt}}(\tilde{K}^*(k-1))$, then

$$P_1(k)C^T(CP_1(k)C^T + DD^T)^{-1} = P_2(k)C^T(CP_2(k)C^T + DD^T)^{-1}.$$

After simple manipulations, we get

$$DD^T(CP_1(k)C^T + DD^T)^{-1} = DD^T(CP_2(k)C^T + DD^T)^{-1}.$$

Since $DD^T > 0$ and $P_1(K) \geq P_2(K)$, we have $C(P_1(k) - P_2(k)) = 0$. By (7.54d), this implies that

$$\text{Tr}(g(K(k-1), F_{\text{opt}}(K(k-1)))) > \text{Tr}(g(\tilde{K}^*(k-1), F_{\text{opt}}(\tilde{K}^*(k-1)))).$$

This completes the proof. $\square$

Now we are ready to show that the robust Kalman filtering algorithm is convergent.

Theorem 14 Consider the state estimation problem ($L = I$) of system (7.29), with all the state-space matrices being time invariant and DD^T being positive definite. The robust adaptive Kalman filtering algorithm converges to a steady state LTI estimator, i.e., for any $P(0) \geq 0$, we have $\lim_{k \rightarrow \infty} P^*(k) = \bar{P}^*$, $\lim_{k \rightarrow \infty} K^*(k) = \bar{K}^*$, and $\lim_{k \rightarrow \infty} F^*(k) = \bar{F}^*$ if:

1. There exists $Q = Q^T$ with $Q > 0$ such that

$$A^TQA - Q + \sum_{j=1}^m (A_j^s)^T Q (A_j^s) < 0. \quad (7.55)$$

2. The Riccati equation

$$\bar{P} = A\bar{P}A^T + \bar{B}\bar{B}^T - (\bar{B}\bar{D}^T + A\bar{P}C^T)(C\bar{P}C^T + \bar{D}\bar{D}^T)^{-1}(\bar{D}\bar{B}^T + C\bar{P}A^T)$$

has a unique positive semidefinite solution $\bar{P}^*$, and the matrix $\bar{K}^*$ defined by

$$\bar{K}^* = -(A\bar{P}^*C^T + \bar{B}\bar{D}^T)(C\bar{P}^*C^T + \bar{D}\bar{D}^T)^{-1}$$

satisfies

$$(\bar{K}^*\bar{D} + \bar{B})(\bar{K}^*\bar{D} + \bar{B})^T > 0.$$

Proof: With all the state-space matrices in system (7.29) being time invariant, we have $A(k) = A$, $B(k) = B$ and

$$\begin{aligned}\hat{B}(k) &= [A_1^s X(k)^{1/2} \quad \dots \quad A_m^s X(k)^{1/2} \quad B \quad B_1^s \quad \dots \quad B_m^s], \\ \hat{D}(k) &= [C_1^s X(k)^{1/2} \quad \dots \quad C_m^s X(k)^{1/2} \quad D \quad D_1^s \quad \dots \quad D_m^s].\end{aligned}$$

The correlation of the state $X(k) = E[x(k)x(k)^T]$ satisfies the recursion

$$X(k+1) = AX(k)A^T + BB^T + \sum_{j=1}^m (A_j^s X(k)(A_j^s)^T + B_j^s (B_j^s)^T), \quad k = 0, 1, \dots$$

By Proposition 4, we have $\lim_{k \rightarrow \infty} X(k) = X$, where X is unique and independent of the initial condition $X(0) \in \text{Co}X_1(0), \dots, X_p(0)$.

With $\lim_{k \rightarrow \infty} \hat{B}(k) = \bar{B}$ and $\lim_{k \rightarrow \infty} \hat{D}(k) = \bar{D}$, we now define another set of recursive equations

$$\min_{\bar{K}(k-1), \bar{F}(k)} \text{Tr}(\bar{M}(k)), \quad (7.56a)$$

$$\begin{aligned}\bar{f}(\bar{P}(k-1), \bar{K}(k-1)) &\triangleq (A + \bar{K}(k-1)C)\bar{P}(k-1)(A + \bar{K}(k-1)C)^T \\ &\quad + (\bar{K}(k-1)\bar{D} + \bar{B})(\bar{K}(k-1)\bar{D} + \bar{B})^T, \quad (7.56b)\end{aligned}$$

$$\bar{P}(k) = \bar{f}(\bar{P}(k-1), \bar{K}(k-1)), \quad (7.56c)$$

$$\bar{M}(k) = (I + \bar{F}(k)C)\bar{P}(k)(I + \bar{F}(k)C)^T + \bar{F}(k)\bar{D}\bar{D}^T\bar{F}(k)^T. \quad (7.56d)$$

Using arguments similar to those in the proof of Theorem 13, we conclude that (7.56) is equivalent to the LMI problem

$$\begin{aligned}\text{Minimize:} & \quad \text{Tr}(\bar{M}(k)) \\ \text{Subject to:} & \quad \begin{bmatrix} \bar{Q}(k) & \bar{Q}(k)A + \bar{Y}(k)C & \bar{Q}(k)\bar{B} + \bar{Y}(k)\bar{D} \\ (\bar{Q}(k)A + \bar{Y}(k)C)^T & \bar{Q}(k-1) & 0 \\ (\bar{Q}(k)\bar{B} + \bar{Y}(k)\bar{D})^T & 0 & I \end{bmatrix} \geq 0, \\ & \quad \begin{bmatrix} \bar{M}(k) & I + \bar{F}(k)C & \bar{F}(k)\bar{D} \\ I + C^T\bar{F}(k)^T & \bar{Q}(k) & 0 \\ \bar{D}^T\bar{F}(k)^T & 0 & I \end{bmatrix} \geq 0, \quad \bar{Q}(k) > 0, \quad (7.57)\end{aligned}$$

where $\bar{Q}(k) = \bar{P}(k)^{-1}$ and $\bar{Y}(k) = \bar{Q}(k)\bar{K}(k-1)$.

Let $\bar{K}^*(k)$ and $\bar{F}^*(k)$ denote the optimal Kalman gains of (7.56). From Proposition 5, equation (7.56) has the same optimal solution of $\bar{P}(k)$ and $\bar{K}^*(k)$ as that of equation (7.50). From Proposition 3, we have $\lim_{k \rightarrow \infty} \bar{K}^*(k) = \bar{K}^*$ and $(A + \bar{K}^*C)$ is stable. Thus

$$\left\| \sum_{j=1}^i \prod_{k=j}^i (A + \bar{K}^*(k)C)(A + \bar{K}^*(k)C)^T \right\| \leq T_1(\bar{P}(0)),$$

where $T_1(\bar{P}(0))$ is a uniform bound over $i > 0$ and depends only on $\bar{P}(0)$.

$\bar{f}(\bar{P}(k), \bar{K}(k))$ is a continuous and strictly convex function of $\bar{P}(k)$ and $\bar{K}(k)$. From LMIs (7.35) and (7.57), it follows that for any small number ϵ , there exists N , such that whenever $k \geq N$, we have

$$\|f(P(k), K^*(k)) - \bar{f}(P(k), \bar{K}^*(k))\| \leq T_2(P(k))\epsilon,$$

where $K^*(k)$ is the optimal solution of LMI (7.41) and $T_2(P(k))$ is a constant that depends on $P(k)$. $T_2(P(k))$ is finite if $P(k)$ is bounded.

By setting $P(N) = \bar{P}(N)$, we get $\|P(N+1) - \bar{P}(N+1)\| \leq T_2(P(N))\epsilon$. At time $N+1$, we have

$$\begin{aligned}\|P(N+2) - \bar{P}(N+2)\| &= \|f(P(N+1), K^*(N+1)) - \bar{f}(\bar{P}(N+1), \bar{K}^*(N+1))\| \\ &\leq (T_2(P(N))\|(A + \bar{K}^*(N+1)C)(A + \bar{K}^*(N+1)C)^T\| + T_2(P(N+1)))\epsilon.\end{aligned}$$

Recursively for $i = 2, 3, \dots$, we get

$$\begin{aligned}
& \|P(N+i) - \bar{P}(N+i)\| \\
&= \|f(P(N+i-1), K^*(N+i-1)) - \bar{f}(\bar{P}(N+i-1), \bar{K}^*(N+i-1))\| \\
&\leq T_2(P(N+i-1))\epsilon \\
&\quad + \left\| \sum_{j=1}^{i-1} T_2(P(N+j-1)) \prod_{k=N+j}^{N+i-1} (A + \bar{K}^*(k)C)(A + \bar{K}^*(k)C)^T \right\| \epsilon \\
&\leq \left(T_2(P(N+i-1)) + \max_{j=0, \dots, i-2} T_2(P(N+j)) T_1(P(N)) \right) \epsilon.
\end{aligned}$$

Using an argument along the lines of Step (2) in Proposition 3, we obtain that $\{P(k)\}$ is also bounded, with the bound depending only on $P(0)$. Therefore, there exist finite constants $T_3(P(0))$ and $T_4(P(0))$, that depend on $P(0)$, such that for any $k \geq 0$, $T_1(P(k)) \leq T_3(P(0))$ and $T_2(P(k)) \leq T_4(P(0))$. Thus, we conclude that $\|P(N+i) - \bar{P}(N+i)\|$ is bounded and that

$$\|P(N+i) - \bar{P}(N+i)\| \leq [T_4(P(0))T_3(P(0)) + T_4(P(0))]\epsilon.$$

Since ϵ can be arbitrarily small when N is large enough, we obtain

$$\lim_{k \rightarrow \infty} P(k) = \lim_{k \rightarrow \infty} \bar{P}(k) = \bar{P}^*.$$

Similar arguments establish that

$$\lim_{k \rightarrow \infty} K(k) = \lim_{k \rightarrow \infty} \bar{K}(k) = \bar{K}^* \quad \text{and} \quad \lim_{k \rightarrow \infty} F(k) = \lim_{k \rightarrow \infty} \bar{F}(k) = \bar{F}^*.$$

□

Bibliography

- [AA97] P. Apkarian and R. J. Adams. Advanced Gain-Scheduled Techniques for Uncertain Systems. In *Proc. American Control Conf.*, pages 3331–3335, Albuquerque, New Mexico, 1997.
- [AA98] P. Apkarian and R. J. Adams. Advanced Gain-Scheduled Techniques for Uncertain Systems. *IEEE Trans. Control Sys. Tech.*, 6(1):21–32, January 1998.
- [AC84] J. P. Aubin and A. Cellina. *Differential Inclusions*, volume 264 of *Comprehensive Studies in Mathematics*. Springer-Verlag, 1984.
- [ACGB93] P. Apkarian, J. P. Chretien, P. Gahinet, and J. M. Biannic. μ synthesis by $D - K$ iterations with constant scaling. In *Proc. American Control Conf.*, pages 3192–3196, 1993.
- [AG95] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled H_∞ controllers. *IEEE Transactions on Automatic Control*, 40(5):853–864, May 1995.
- [AGB95] P. Apkarian, P. Gahinet, and G. Becker. Self-Scheduled H_∞ Control of Linear Parameter-Varying Systems: A Design Example. *Automatica*, 31(9):1251–1261, 1995.
- [Ali95] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM Journal on Optimization*, 5(1):13–51, February 1995.
- [And72] B. D. O. Anderson. The small-gain theorem, the passivity theorem and their equivalence. *J. Franklin Inst.*, 293(2):105–115, 1972.
- [AT98] P. Apkarian and H. D. Tuan. Parameterized LMIs in Control Thoery. In *Proc. IEEE Conf. on Decision and Control*, pages 152–157, Tampa, Florida, December 1998.
- [AV73] B. Anderson and S. Vongpanitlerd. *Network analysis and synthesis: a modern systems theory approach*. Prentice-Hall, 1973.
- [Bal94a] V. Balakrishnan. Construction of Lyapunov functions in robustness analysis with multipliers. In *Proc. IEEE Conf. on Decision and Control*, volume 3, pages 2021–2025, Orlando, Florida, December 1994. Invited session *LMIs in Control Theory*.
- [Bal94b] V. Balakrishnan. Robust performance analysis using convex optimization and multipliers. 1994. In preparation.
- [Bal95] V. Balakrishnan. Linear matrix inequalities in robustness analysis with multipliers. *Syst. Control Letters*, 25(4):265–272, 1995.
- [BB90] S. Boyd and V. Balakrishnan. A regularity result for the singular values of a transfer matrix and a quadratically convergent algorithm for computing its L_∞ -norm. *Syst. Control Letters*, 15:1–7, 1990.
- [BB91] S. Boyd and C. Barratt. *Linear Controller Design: Limits of Performance*. Prentice-Hall, 1991.
- [BBB91] V. Balakrishnan, S. Boyd, and S. Balemi. Branch and bound algorithm for computing the minimum stability degree of parameter-dependent linear systems. *Int. J. of Robust and Nonlinear Control*, 1(4):295–317, October–December 1991.

- [BDG⁺91] G. Balas, J. C. Doyle, K. Glover, A. Packard, and Roy Smith. *μ -analysis and synthesis*. MUSYN, inc., and The Mathworks, Inc., 1991.
- [BE] V. Balakrishnan and E. Feron (Eds). *Linear Matrix Inequalities in Control Theory and Applications*. special issue of the *International Journal of Robust and Nonlinear Control*, vol. 6, no. 9/10, pp. 896–1099, November-December, 1996.
- [BE93] S. Boyd and L. El Ghaoui. Method of centers for minimizing generalized eigenvalues. *Linear Algebra and Applications, special issue on Numerical Linear Algebra Methods in Control, Signals and Systems*, 188:63–111, July 1993.
- [Bec93] G. Becker. *Quadratic Stability and Performance of Linear Parameter Dependent Systems*. PhD thesis, University of California, Berkeley, 1993.
- [BEFB94] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.
- [Ber95] E. Beran. *The Induced Norm Control Toolbox. User's manual*, 1995. Available from <http://www.iau.dtu.dk/Staff/ebb/INCT>.
- [BGT81] R. G. Bland, D. Goldfarb, and M. J. Todd. The ellipsoid method: A survey. *Operations Research*, 29(6):1039–1091, 1981.
- [BW] V. Balakrishnan and F. Wang. Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems. Submitted to the *IEEE Trans. Aut. Control*, 1998.
- [BW98] V. Balakrishnan and F. Wang. Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems. In *Proc. IEEE Conf. on Decision and Control*, pages 4406–4407, Tampa, FL, December 1998.
- [BY89] S. Boyd and Q. Yang. Structured and simultaneous Lyapunov functions for system stability problems. *Int. J. Control*, 49(6):2215–2240, 1989.
- [BYDM94] R. D. Braatz, P. M. Young, J. C. Doyle, and M. Morari. Computational complexity of μ calculation. *IEEE Trans. Aut. Control*, AC-39:1000–1002, May 1994.
- [CC94] Y. Chen and B. Chen. Minimax Robust Deconvolution Filters under Stochastic Parametric and Noise Uncertainties. *IEEE Trans. Signal Processing*, 42(1):32–45, January 1994.
- [CDW75] J. Cullum, W. Donath, and P. Wolfe. The minimization of certain nondifferentiable sums of eigenvalues of symmetric matrices. *Math. Programming Study*, 3:35–55, 1975.
- [CG95] M. Chilali and P. Gahinet. h_∞ design with pole placement constraints: An lmi approach. *IEEE Trans. Aut. Control*, 40(3):358–367, March 1995.
- [CS92] R. Chiang and M. G. Safonov. *Robust Control Toolbox*. The Mathworks, Inc., 1992.
- [CTB97a] Y. S. Chou, A. L. Tits, and V. Balakrishnan. Absolute stability theory, theory, and state-space verification of frequency-domain conditions: Connections and implications for computation. Technical Report TR 97-23, Institute for Systems Research, University of Maryland, 1997. Accepted for publication in *IEEE Trans. Aut. Control*, 1998.
- [CTB97b] Y. S. Chou, A. L. Tits, and V. Balakrishnan. On the relationship between stability multipliers and scaling matrices in mixed μ analysis. 1997. Under review, *IEEE Trans. Aut. Control*, 1997.
- [DG98] S. Dussy and L. El Ghaoui. Measurement-Scheduled Control for the RTAC Problem: an LMI approach. *Int. J. Robust and Nonlinear Control*, 8:377–400, April 1998.
- [Doy82] J. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

- [Doy85] J. C. Doyle. Structured uncertainty in control system design. In *acc*, Fort Lauderdale, Florida, June 1985.
- [DPZ91] J. Doyle, A. Packard, and K. Zhou. Review of LFT's, LMI's and μ . In *Proc. IEEE Conf. on Decision and Control*, volume 2, pages 1227–1232, Brighton, December 1991.
- [DV75] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.
- [EBFB92] L. El Ghaoui, V. Balakrishnan, E. Feron, and S. Boyd. On maximizing a robustness measure for structured nonlinear perturbations. In *Proc. American Control Conf.*, volume 4, pages 2923–2924, Chicago, June 1992.
- [EGS96] L. El Ghaoui and G. Scorletti. Control of rational systems using Linear-Fractional Representations and Linear Matrix Inequalities. *Automatica*, 32(9):1273–84, September 1996.
- [EH97] E.G. Eszter and C.V. Hollot. An IQC for uncertainty satisfying both norm-bounded and passivity constraints. *Automatica*, 33(8):1545–1548, 1997.
- [END95] L. El Ghaoui, R. Nikoukha, and F. Delebecque. *LMITool: a front-end for LMI optimization, user's guide*, 1995. Available via anonymous ftp to [ftp.ensta.fr](ftp://ftp.ensta.fr/pub/elghaoui/lmitool) under [/pub/elghaoui/lmitool](ftp://ftp.ensta.fr/pub/elghaoui/lmitool).
- [FTD91] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, AC-36(1):25–38, January 1991.
- [GA94] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H^∞ control. *Int. J. Robust and Nonlinear Control*, 4:421–448, 1994.
- [GAC96] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter-dependent Lyapunov functions for real parameter uncertainty. *IEEE Trans. Aut. Control*, 41(3):436–442, March 1996.
- [GBG98] J. C. Geromel, J. Bernussou, and G. Garcia. H_2 and H_∞ Robust Filtering for Discrete-Time Linear Systems. In *Proc. IEEE Conf. on Decision and Control*, Tampa, Florida, December 1998. Submitted.
- [GDN95] L. El Ghaoui, F. Delebecque, and R. Nikoukhah. *LMITool: A User-friendly Interface for LMI Optimization*. ENSTA/INRIA, 1995. Software available via anonymous FTP from [ftp.inria.fr](ftp://ftp.inria.fr), under directory [pub/elghaoui/lmitool](ftp://ftp.inria.fr/pub/elghaoui/lmitool).
- [GKN⁺74] A. Gelb, J. F. Kasper, R. A. Nash, C. F. Price, and A. A. Sutherland. *Applied Optimal Estimation*. The M.I.T. Press, 1974.
- [GN95] P. Gahinet and A. Nemirovskii. *The LMI Control Toolbox*. The MathWorks, Inc., 1995.
- [GNLC95] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. *LMI Control Toolbox*. The MathWorks, Inc., 1995.
- [GS95] K.-C. Goh and M. G. Safonov. Robust analysis, sectors, and quadratic functionals. In *Proc. IEEE Conf. on Decision and Control*, volume 2, pages 1988–1993, New Orleans, LA, 1995.
- [Hay89] S. Haykin. *Modern Filters*. MacMillan, New York, 1989.
- [HB76] H. P. Horisberger and P. R. Bélanger. Regulators for linear, time invariant plants with uncertain parameters. *IEEE Trans. Aut. Control*, AC-21:705–708, 1976.
- [Hel95] A. Helmersson. *Methods for Robust Gain Scheduling*, volume 406 of *Linköping Studies in Science and Technology, Dissertations*. Linköping University Dept. of Electrical Engineering, Linköping, Sweden, 1995.

- [HH93a] J. P. How and S. R. Hall. Connections between the Popov stability criterion and bounds for real parametric uncertainty. In *Proc. American Control Conf.*, volume 2, pages 1084–1089, June 1993.
- [HH93b] J. P. How and S. R. Hall. Connections between the Popov stability criterion and bounds for real parameter uncertainty. In *Proc. American Control Conf.*, San Francisco, California, June 1993.
- [HM82] Y. S. Hung and A. G. J. MacFarlane. *Multivariable feedback: a quasi-classical approach*. Springer-Verlag, 1982.
- [IO91] A. İftar and Ü. Özgüner. Modeling of uncertain dynamics for robust controller design in state space. *Automatica*, 1991.
- [J96] U. Jönsson. *Robustness Analysis of Uncertain and Nonlinear Systems*. PhD thesis, Lund Institute of Technology, Sweden, 1996.
- [KG96] K. Zhou with J. Doyle and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.
- [Koc79] K. R. Koch. *B. D. O. Anderson and J. B. Moore*. Prentice Hall, Englewood Cliffs, NJ, 1979.
- [KRB96] P. Khargonekar, M. Rotea, and E. Baeyens. Mixed H_2/H_∞ Filtering. *Int. J. Robust and Nonlinear Control*, 6:313–330, 1996.
- [Lee92] L. Lee. *Robustness Study of Systems with Phase-Informed Uncertainty*. PhD thesis, Department of Electrical Engineering, University of Maryland, 1992.
- [Lju79] Lennart Ljung. Asymptotic Behavior of the Extended Kalman Filter as a Parameter Estimator for Linear Systems. *IEEE Trans. Aut. Control*, AC-24(1):36–50, February 1979.
- [LO96] A. S. Lewis and M. L. Overton. Eigenvalue optimization. *Acta Numerica*, pages 149–190, 1996.
- [LP44] A. I. Lur'e and V. N. Postnikov. On the theory of stability of control systems. *Prikl. Mat. i Mekh.*, (*Applied mathematics and mechanics*), 8(3):3–13, 1944. In Russian.
- [LSC94] J. H. Ly, M. G. Safonov, and R. Y Chiang. Real/complex multivariable stability margin computation via generalized Popov multiplier — LMI approach. In *Proc. American Control Conf.*, 1994.
- [LT92a] L. Lee and A. Tits. Mixed one- and two-sided real uncertainty. a vertex result. In *Proc. American Control Conf.*, pages 2197–2199, Chicago, June 1992.
- [LT92b] L. Lee and A. L. Tits. Linear fractional transformations for the approximation of various uncertainty sets. In S.P. Bhattacharyya and L. Keel, editors, *Control of Uncertain Systems*, pages 53–62. CRC Press, 1992.
- [LTD96] C. T. Lawrence, A. L. Tits, and P. Van Dooren. A fast algorithm for the computation of an upper bound on the μ -norm. In *Proceedings of the 13th IFAC World Congress, Volume H*, pages 59–64, 1996.
- [LTF89] L. Lee, A. L. Tits, and M. K. H. Fan. Robustness under uncertainty with phase information. In *Proc. IEEE Conf. on Decision and Control*, pages 2315–2316, Tampa, Florida, December 1989.
- [Lur57] A. I. Lur'e. *Some Nonlinear Problems in the Theory of Automatic Control*. H. M. Stationery Off., London, 1957. In Russian, 1951.
- [MC88] B. Mulgrew and C. F. N. Cowan. *Adaptive Filters and Equalizers*. Kluwer Academic Publishers, 1988.
- [Men95] J.M. Mendel. *Lessons in Estimation Theory for Signal Processing, Communications, and Control*. Prentice Hall Signal Processing Series. Prentice Hall PTR, 1995.
- [MR97] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Control*, 42(6):819–830, June 1997.

- [NN90] Yu. Nesterov and A. Nemirovsky. *Optimization over positive semidefinite matrices: Mathematical background and user's manual*. USSR Acad. Sci. Centr. Econ. & Math. Inst., 32 Krasikova St., Moscow 117418 USSR, 1990.
- [NN94] Yu. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.
- [NT73] K. S. Narendra and J. H. Taylor. *Frequency domain criteria for absolute stability*. Electrical science. Academic Press, New York, 1973.
- [OPG96] J. M. Omid, S. Pasupathy, and P. G. Gulak. Joint data and Kalman estimation of fading channel using a generalized Viterbi algorithm. In *Proceedings of the 1996 IEEE International Conference on Communications*, volume 2, Dallas, TX, 1996.
- [Ove88] M. Overton. On minimizing the maximum eigenvalue of a symmetric matrix. *SIAM J. on Matrix Analysis and Applications*, 9(2):256–268, 1988.
- [Owe84] A. Owen. A neighbourhood-based classifier for LANDSAT data. *The Canadian Journal of Statistics*, 12:191–200, 1984.
- [Pac94] A. Packard. Gain scheduling via linear fractional transformations. *Syst. Control Letters*, 22:79–92, 1994.
- [PEM81] I. Postlethwaite, J. M. Edmunds, and A. G. J. MacFarlane. Principal gains and principal phases in the analysis of linear multivariable feedback systems. *IEEE Trans. Aut. Control*, 1981.
- [PM96] L. Petersen and D. McFarlane. Optimal Guaranteed Cost Filtering for Uncertain Discrete-time Linear Systems. *Int. J. Robust and Nonlinear Control*, 6:267–280, 1996.
- [Pro95] John G. Proakis. *Digital Communications*. McGraw-Hill, New York, 1995.
- [PS82] E. S. Pyatnitskii and V. I. Skorodinskii. Numerical methods of Lyapunov function construction and their application to the absolute stability problem. *Syst. Control Letters*, 2(2):130–135, August 1982.
- [Ran96] A. Rantzer. On the Kalman-Yacubovich-Popov lemma. *Syst. Control Letters*, 28(1):7–10, 1996.
- [SA90] J. S. Shamma and Michael Athans. Analysis of Gain Scheduled Control for Nonlinear Plants. *IEEE Trans. Aut. Control*, 35(8):898–907, August 1990.
- [SA92] J. S. Shamma and Michael Athans. Gain Scheduling: Potential Hazards and Possible Remedies. *IEEE Control Syst. Mag.*, pages 101–107, June 1992.
- [Saf82] M. G. Safonov. Stability margins of diagonally perturbed multivariable feedback systems. *IEE Proc.*, 129-D:251–256, 1982.
- [SC93] M. G. Safonov and R. Y. Chiang. Real/complex K_m -synthesis without curve fitting. In C. T. Leondes, editor, *Control and Dynamic Systems*, volume 56, pages 303–324. Academic Press, New York, 1993.
- [Sch96a] C. Scherer. Mixed H_2/H_∞ control for time-varying and linear parametrically-varying systems. *Int. J. Robust and Nonlinear Control*, 6(9/10):929–952, Nov–Dec 1996.
- [Sch96b] C. Scherer. Robust generalized H_2 control for uncertain and LPV systems with general scalings. In *Proc. IEEE Conf. on Decision and Control*, pages 3970–3975, Kobe, Japan, December 1996.
- [SNB97] C. W. Scherer, R.G.E. Njio, and S. Bannani. Parametrically varying flight control system design with full block scalings. In *Proc. IEEE Conf. on Decision and Control*, pages 1510–1515, San Diego, CA, December 1997.
- [TB97] A. L. Tits and V. Balakrishnan. Small- μ theorems with frequency-dependent uncertainty bounds. 1997. Under review, *MCSS*, 1997.

- [TBL96] A. L. Tits, V. Balakrishnan, and L. Lee. Robustness under bounded uncertainty with phase information. Technical Report TR 96-14, Institute for Systems Research, University of Maryland, 1996. To appear in *IEEE Trans. Aut. Control*, 1998.
- [TBL99] A. L. Tits, V. Balakrishnan, and L. Lee. Robustness under bounded uncertainty with phase information. *IEEE Trans. Aut. Control*, AC-44(1):50–65, January 1999.
- [Tit95] A. L. Tits. The small μ theorem without real-rational assumption. In *Proceedings of the 1995 American Control Conference*, pages 2870–2872, 1995.
- [VB94] L. Vandenberghe and S. Boyd. SP: *Software for Semidefinite Programming. User's Guide, Beta Version*. Stanford University, October 1994. Available at <http://www-is1.stanford.edu/people/boyd>.
- [VB95] L. Vandenberghe and S. Boyd. A primal-dual potential reduction method for problems involving matrix inequalities. *Mathematical Programming*, 69(1):205–236, July 1995.
- [VB96] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, March 1996.
- [VB97] L. Vandenberghe and V. Balakrishnan. Algorithms and software for LMI problems in control. *IEEE Control Syst. Mag.*, pages 89–95, October 1997.
- [Vid93] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall, Englewood Cliffs, second edition, 1993.
- [WB96] S.-P. Wu and S. Boyd. SDPSOL: *A Parser/Solver for Semidefinite Programming and Determinant Maximization Problems with Matrix Structure. User's Guide, Version Beta*. Stanford University, June 1996.
- [Wil71] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Aut. Control*, AC-16(6):621–634, December 1971.
- [Wil72a] J. C. Willems. Dissipative dynamical systems. Part I: General theory. *Archive for Rational Mechanics and Analysis*, 45:321–351, 1972.
- [Wil72b] J. C. Willems. Dissipative dynamical systems. Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45:352–393, 1972.
- [Wu95] F. Wu. *Control of Linear Parameter Varying Systems*. PhD thesis, University of California, Berkeley, 1995.
- [WYPB94] F. Wu, X. H. Yang, A. Packard, and G. Becker. Induced L_2 -norm control for LPV system with bounded parameter variation rates. *Int. J. Robust and Nonlinear Control*, December 1994.
- [Yak67] V. A. Yakubovich. The method of matrix inequalities in the stability theory of nonlinear control systems, *I, II, III*. *Automation and Remote Control*, 25-26(4):905–917, 577–592, 753–763, April 1967.
- [YS97] J. Yu and A. Sideris. H_∞ control with parametric Lyapunov functions. *Syst. Control Letters*, 30:57–69, 1997.
- [ZKSN92] K. Zhou, P. P. Khargonekar, J. Stoustrup, and H. H. Niemann. Robust stability and performance of uncertain systems in state space. In *Proc. IEEE Conf. on Decision and Control*, pages 662–667, Tucson, AZ, December 1992.

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1 Introduction

The models of control systems encountered in several diverse naval applications are nonlinear; moreover, they are also time-varying, and have uncertainties affecting them. The underlying controller design problems, beyond requiring system stability, also typically require the optimization of some performance objectives. Ideally, the control law should have the properties that: (1) it adapts itself according to the nonlinearities; (2) it can handle several performance objectives (besides stability) in a uniform manner; and (3) it guarantees the performance of the closed-loop system in spite of the variations and the uncertainties.

The overall goal of the project is a numerical solution methodology for solving the general nonlinear controller design problem. Our approach is motivated by recent advances in control and optimization theory as well as the sustained growth in the available computing power over the past few years. The proposed controller architecture is *gain-scheduled* (i.e., the controller uses the measured nonlinearities and time-variations), and optimizes the worst-case performance—over the uncertainties—of the system. The search for the optimal controller parameters can be reformulated as convex optimization problems involving linear matrix inequalities (LMIs) in several important cases. The design approach lends itself naturally to the development of computer-aided design tools, and enjoys the advantages of widespread applicability, ease of implementation and cost effectiveness.

The progress achieved towards overcoming some of the technical roadblocks was outlined in the following reports:

- Progress report for the period May 1997 - October 1997
- Annual report for the period May 1997 - April 1998
- Progress report for the period May 1998 - October 1998

Since then, efforts have been directed towards applying the research results towards the *Basic Research on Intelligent Information and Autonomous Agents*, and *Applied Research on Unmanned Combat Air Vehicles*, directed by Dr. Allen Moshfegh, ONR.

1.1 Gain-scheduled control of the Texas A&M UCAV models

The dynamics of Unmanned Combat Air Vehicles (UCAVs) undergoing aggressive maneuvers are highly nonlinear and time-varying. One approach towards modeling UCAVs involves linearizing the dynamics of the UCAV around various points in the flight envelope. Then, the composite model of the UCAV can be given as a linear parameter-dependent model, with the parameters being the flight conditions. A comprehensive effort towards identifying such models is currently being undertaken at Texas A&M University.

We have directed research efforts towards a systematic design procedure for controller synthesis for the linear parameter-dependent ONR UCAV models. Traditional techniques for controller synthesis consist of designing a single controller that is intended to function across varying flight conditions. These include constant state-feedback, as well as the celebrated LQR/LQG and H_∞ controllers. While these control techniques can be sometimes proven to work (i.e., stabilize the system or provide acceptable performance), they can be quite conservative, especially when the flight conditions vary considerably.

Gain-scheduled controller design offers the potential much more aggressive control design. The basic idea is to synthesize a series of dynamic controllers, one for each linearized model of the UCAV around a flight condition, and then “schedule” these controllers according to the actual flight condition. An *ad hoc* implementation of such a scheme is not guaranteed to work; however, it is possible to develop a gain-scheduling scheme, using Lyapunov functions, that is guaranteed to work across various flight regimes.

Another advantage of an approach based on Lyapunov functions is that it can be extended to handle constraints other than mere stability. Stability requires the Lyapunov function to decrease along the trajectories of the parameter-varying system. Additional constraints on the Lyapunov function can be used to design controllers with guaranteed performance. Examples of performance measures are the energy in the state vector, and peak values of signals of interest; in several instances, it is desirable that these performance measures be small. And gain-scheduled controllers that minimize upper bounds on these performance measures can be designed. The technical details behind synthesizing improved gain-scheduled controllers can be found in [1, 2].

The fore-mentioned ideas have been integrated into Dr. Moshfegh’s project:

The simulation results of applying gain-scheduled control techniques on the Texas A&M UCAV models, preprints outlining the technical approach, and software have been made available on the website

<http://dynamo.ecn.purdue.edu/~ragu/onr/gs-contr.html>

A snapshot of the web-page is shown in the appendix.

1.2 Robust estimation

A second research direction that is relevant to Dr. Moshfegh’s project was identified during the January semi-annual program review, held in January 1999.

Estimation problems arise in many naval applications in control, communications and signal processing areas. Traditional estimation algorithms are usually based on a nominal system model without uncertainty. However, in many cases, there exist uncertainties in model parameters and even model structures because of modeling errors from system identifications. These uncertainties may degrade the estimation performance of the algorithm without robustness. One application area of robust estimation in Dr. Moshfegh’s project is in the automated landing of UCAV with passive sensors.

In recent years, with the advances of robust control theory, many results on robust estimation have become available. Our efforts have been directed towards deriving Kalman-type estimators for linear time varying systems with stochastic uncertainties. Specifically, we have derived an adaptive robust Kalman filtering algorithm that addresses estimation problems that arise in linear time-varying systems with stochastic parametric uncertainties. The filter has the one-step predictor-corrector structure and minimizes the mean square estimation error at each step, with the minimization reduced to a convex optimization problem based on linear matrix inequalities. The algorithm is shown to converge when the system is mean square stable and the state-space matrices are time-invariant. A numerical example, consisting of equalizer design for a communication channel, demonstrates that our algorithm

offers considerable improvement in performance when compared to standard Kalman filtering techniques.

The technical details can be found in [3]; a preprint of this paper is available on the website

<http://dynamo.ecn.purdue.edu/~ragu/onr/rob-est.html>

1.3 Publications

Journal papers published/accepted for publication

- A. L. Tits and V. Balakrishnan. "Small- μ Theorems with Frequency-Dependent Uncertainty Bounds". *Mathematics of Control, Signals, and Systems*, vol 11, no. 3, pages 220-243, 1998.
- V. Balakrishnan and R. L. Kashyap. "Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities". *Journal of Optimization Theory and Applications*, vol. 100, no. 3, March 1999.
- A. L. Tits, V. Balakrishnan and L. Lee. "Robustness under Bounded Uncertainty with Phase Information". *IEEE Trans. Aut. Contr.*, AC-44(1):50-65, January 1999.
- Y. S. Chou, A. L. Tits and V. Balakrishnan. "Absolute Stability Theory, μ Theory, and State-Space Verification of Frequency-Domain Conditions: Connections and Implications for Computation". *IEEE Trans. Aut. Contr.*, AC-44(5):906-913, May 1999.
- V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". Accepted for publication in *IEEE Trans. Aut. Contr.*, 1999.

Conference papers accepted/published

- V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". In *Proc. IEEE Conf. on Decision and Control*, pages 4406-4407, Tampa, FL, December 1998.
- F. Wang and V. Balakrishnan. "Improved Stability Analysis and Gain-Scheduled Controller Synthesis for Parameter-Dependent Systems". In *Proc. IEEE Conf. on Decision and Control*, pages 1771-1776, Tampa, FL, December 1998.
- F. Wang and V. Balakrishnan. "Robust Adaptive Kalman Filters for Linear Time-Varying Systems with Stochastic Parametric Uncertainties." In *IEEE American Control Conf.*, pages 440-444, San Diego, CA, June 1999. **Finalist, best student paper award.**
- F. Wang and V. Balakrishnan. "Robust Estimators for Systems with Both Deterministic and Stochastic Uncertainties." Accepted for presentation in *IEEE Conf. on Decision and Control*, Phoenix, Arizona, December 1999.

- F. Wang and V. Balakrishnan. "Robustness analysis and gain-scheduled controller synthesis for rational parameter-dependent uncertain systems using parameter-dependent Lyapunov functions." Accepted for presentation in *IEEE Conf. on Decision and Control*, Phoenix, Arizona, December 1999.

2 Research Plan for the next period

2.1 Multi-objective gain-scheduled controller synthesis for UCAV models

The simulation results, obtained from the Texas A&M UCAV models, show considerable promise. Therefore, further efforts will be directed to study controller design with other performance measures. In particular, we intend to systematically study the tradeoff between competing design requirements.

2.2 Further robust estimation efforts

The preliminary efforts towards robust estimation have concentrated on Kalman-type filtering problems. We intend to direct further effort towards other robust estimation problems, such as deriving H_∞ -type estimators. We will also direct our efforts towards applying some of these research results on the specific problems that arise in Dr. Moshfegh's project, such as the automated landing of UCAV with passive sensors.

References

- [1] V. Balakrishnan and F. Wang. Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems. In *Proc. IEEE Conf. on Decision and Control*, pages 4406–4407, Tampa, FL, December 1998.
- [2] F. Wang and V. Balakrishnan. Robustness analysis and gain-scheduled controller synthesis for rational parameter-dependent uncertain systems using parameter-dependent Lyapunov functions. In *Proc. IEEE Conf. on Decision and Control*, Phoenix, AZ, December 1999.
- [3] F. Wang and V. Balakrishnan. Robust Adaptive Kalman Filters for Linear Time-Varying Systems with Stochastic Parametric Uncertainties. In *Proc. American Control Conf.*, pages 440–444, June 1999. Finalist, Best Student Paper Award.

Snapshot of the webpage

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Gain-scheduled control of the ONR UCAV



Introduction

The dynamics of Unmanned Combat Air Vehicles (UCAVs) undergoing aggressive maneuvers are highly nonlinear and time-varying. One approach towards modeling UCAVs involves linearizing the dynamics of the UCAV around various points in the flight envelope. Then, the composite model of the UCAV can be given as a linear parameter-dependent model, with the parameters being the flight conditions. A comprehensive effort towards identifying such models is currently being undertaken at Texas A & M University .



We describe here research efforts towards a systematic design procedure for controller synthesis for the linear parameter-dependent ONR UCAV models. Traditional techniques for controller synthesis consist of designing a single controller that is intended to function across varying flight conditions. These include constant state-feedback, as well as the celebrated LQR/LQG and H-infinity controllers. While these control techniques can be sometimes proven to work (i.e., stabilize the system or provide acceptable performance), they can be quite conservative, especially when the flight conditions vary considerably.

Gain-scheduled controller design offers the potential much more aggressive control design. The basic idea is to synthesize a series of dynamic controllers, one for each linearized model of the UCAV around a flight condition, and then "schedule" these controllers according to the actual flight condition. An ad hoc implementation of such a scheme is not guaranteed to work; however, it is possible to develop a gain-scheduling scheme, using Lyapunov functions, that is guaranteed to work across various flight regimes.

Another advantage of an approach based on Lyapunov functions is that it can be extended to handle constraints other than mere stability. Stability requires the Lyapunov function to decrease along the trajectories of the parameter-varying system. Additional constraints on the Lyapunov function can be used to design controllers with guaranteed performance. Examples of performance measures are the energy in the state vector, and peak values of signals of interest; in several instances, it is desirable that these performance measures be small. And gain-scheduled controllers that minimize upper bounds on these performance measures can be designed.

The technical details behind synthesizing improved gain-scheduled controllers can be found in the following papers, and the references therein:

F. Wang and V. Balakrishnan, "Improved Stability Analysis and Gain-Scheduled Controller Synthesis for Parameter-Dependent Systems". In Proc. IEEE Conf. on Decision and Control, pages 1771--1776, Tampa, FL, December 1998.

F. Wang and V. Balakrishnan, "Robustness analysis and gain-scheduled controller synthesis for rational parameter-dependent uncertain systems using parameter-dependent Lyapunov functions". Accepted for presentation in the IEEE Conf. on Decision and Control, December 1999.

Gain-scheduled controllers for some UCAV models

We present the application of gain-scheduled controller design techniques on the models for

UCAVs developed at Texas A & M University:

- | | |
|---|-------------|
| ● Longitudinal VSTOL1 model, stability | MATLAB code |
| ● Longitudinal VSTOL1 model, performance | MATLAB code |
| ● Lateral/directional VSTOL1 model, stability | MATLAB code |
| ● Lateral/directional VSTOL1 model, performance | MATLAB code |
| ● Longitudinal UCAV6 model, stability | MATLAB code |
| ● Longitudinal UCAV6 model, performance | MATLAB code |
| ● Lateral/directional UCAV6 model, stability | MATLAB code |
| ● Lateral/directional UCAV6 model, performance | MATLAB code |

PROGRESS REPORT FOR THE PERIOD MAY 1998 - OCTOBER 1998

Project title: Robust Gain-Scheduled Nonlinear Control Design
for Stability and Performance
Principal Investigator: Venkataramanan Balakrishnan, Purdue University
Contract No: N00014-97-1-0640 (Young Investigator Award)
Program Director: Allen Moshfegh, Ph.D.

The research goals outlined in the original proposal were:

- Deriving models for uncertain nonlinear systems that lend themselves to the use of robust control techniques.
- Robust stability analysis of nonlinear systems using optimization based on linear matrix inequalities (LMIs).
- Robust performance analysis of nonlinear systems using LMI methods.
- Gain-scheduled controller design.

The progress made towards achieving some of these goals were reported in two earlier reports:

- Progress report for the period May 1997 - October 1997
- Annual report for the period 1 May 1997 - 30 April 1998

Progress since these reports is described below:

Stability Multipliers and μ Upper Bounds: Connections and Implications for Computation

The technical approach underlying the project uses convex optimization over Linear Matrix Inequalities (LMIs) in the framework on Integral Quadratic Constraints (IQCs) to obtain sufficient conditions for the robust stability of systems with structured uncertainties. While this approach applies to a wider class of uncertain systems than those handled by the traditional μ methods of robust control, a fundamental question is how it compares with μ analysis. We have answered this question by establishing the equivalence between the two approaches. In particular, the relationship between the stability multipliers used in the IQC framework and the scaling matrices used in the μ methods has been explicitly characterized. The development hinges on the derivation of certain properties of a parameterized family of complex LMIs, a result of independent interest. The derivation also suggests a general computational framework for checking the feasibility of a broad class of frequency-dependent conditions, based on which bisection schemes can be devised to reliably compute several quantities of interest for robust control.

The results from this work have been reported in publication [1].

Small- μ Theorems with Frequency-dependent Upper Bounds

As noted before, the technical approach underlying the project is equivalent to the traditional scaling methods of μ analysis when applied to linear time-invariant systems affected by structured linear time-invariant uncertainties. The result of such analyses are the so-called small- μ theorems, typically stated when the bounds on the uncertainty norms are frequency-independent. Of course, this is usually not the case in engineering: The uncertainty sizes typically tail off at high frequencies, necessitating the use of frequency-dependent norm bounds. We have derived conditions, some necessary and some sufficient, valid under weak assumptions, for robust stability and uniform robust stability of uncertain linear time-invariant systems with linear time-invariant uncertainties that are block-diagonal, with known frequency-dependent norm bounds on the diagonal blocks. Small- μ theorems with frequency-independent uncertainty bounds are recovered as special cases.

The results from this work have been reported in publication [2].

Robust Adaptive Kalman Filtering

Often, it is useful to model uncertain systems as linear time-varying systems with stochastic parametric uncertainties. For such models, state estimation is the first step in designing control strategies. An ongoing effort is the design of adaptive robust Kalman filtering algorithms that address such estimation problems. Specifically, we are focusing on the design of Kalman filters, with the one-step predictor-corrector structure, that minimize the mean square estimation error at each step, with the aim of reformulating the minimization as a convex optimization problem based on linear matrix inequalities. Preliminary results indicate that our algorithm offers considerable improvement in performance when compared to standard Kalman filtering techniques.

The results from this work have been submitted for presentation at the 1999 American Control Conference.

REFERENCES

(The publications cited herein all acknowledge support from ONR Award No. N00014-97-1-0640.)

1. Y. S. Chou, A. L. Tits and V. Balakrishnan. "Stability Multipliers and μ Upper Bounds: Connections and Implications for Numerical Verification of Frequency Domain conditions". To appear in the IEEE Trans. Aut. Contr., 1999.
2. A. L. Tits and V. Balakrishnan. "Small- μ Theorems with Frequency-Dependent Uncertainty Bounds". To appear in Mathematics of Control, Systems and Signals, 1999.
3. F. Wang and V. Balakrishnan. "Robust Adaptive Kalman Filters for Linear Time-Varying Systems with Stochastic Parametric Uncertainties". Submitted for presentation at the 1999 American Control Conference.

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1 Introduction

The models of control systems encountered in several diverse naval applications are nonlinear; moreover, they are also time-varying, and have uncertainties affecting them. The underlying controller design problems, beyond requiring system stability, also typically require the optimization of some performance objectives. Ideally, the control law should have the properties that: (1) it adapts itself according to the nonlinearities; (2) it can handle several performance objectives (besides stability) in a uniform manner; and (3) it guarantees the performance of the closed-loop system in spite of the variations and the uncertainties.

The goal of the project is a numerical solution methodology for solving the general nonlinear controller design problem. Our approach is motivated by recent advances in control and optimization theory as well as the sustained growth in the available computing power over the past few years. The proposed controller architecture is *gain-scheduled* (i.e., the controller uses the measured nonlinearities and time-variations), and optimizes the worst-case performance—over the uncertainties—of the system. The search for the optimal controller parameters can be reformulated as convex optimization problems involving linear matrix inequalities (LMIs) in several important cases. The design approach lends itself naturally to the development of computer-aided design tools, and enjoys the advantages of widespread applicability, ease of implementation and cost effectiveness.

2 Summary of Results to Date

2.1 Robust stability and performance analysis using multiplier theory and LMI optimization

The first step in realizing the project goals is that of *robustness analysis* of nonlinear systems. In order to achieve this, we began with a linear fractional representation of uncertain nonlinear systems, shown in the block-diagram in Fig. 1.

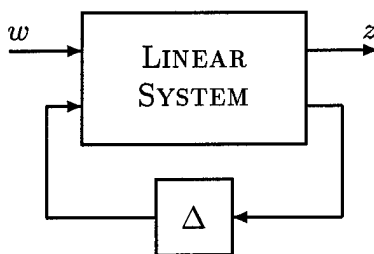


Figure 1: Robustness analysis framework.

In Fig. 1, Δ consists of:

- uncertainties affecting the system;
- non-rational nonlinearities; and
- appropriate copies of the state-vector, so as to realize any *rational* nonlinearities.

With the uncertain nonlinear system represented as in Fig. 1, we performed its robust stability and performance analysis in a unified manner using multiplier theory [1, 2, 3, 4, 5], or more generally, in the framework of integral quadratic constraints or IQCs [6]. The merit of this approach is that several sufficient conditions for robust stability can be performed without frequency sampling, using convex optimization techniques based on LMIs. We showed that multiplier techniques yield a convex parametrization of a set of Lyapunov functions that prove robust stability. By imposing additional conditions on these Lyapunov functions, we derived bounds on the robust performance for the system in Fig. 1; thus we used Lyapunov functions to “prove” robust performance. The “best” performance bounds were obtained by numerically optimizing these bounds, using LMI-based methods, over the set of Lyapunov functions that prove robust performance.

The results were documented in the publication [7].

2.2 Robustness under bounded uncertainty with phase information

The special case of systems considered in Section 2.1 where there are no nonlinearities, and where the uncertainties, in addition to being bounded, also satisfy constraints on their phase, is of particular interest. In this context, we have defined the “phase-sensitive structured singular value” (PS-SSV) of a matrix, and have shown that sufficient (and sometimes necessary) conditions for stability of such uncertain linear systems can be rewritten as conditions involving PS-SSV. We have also derived upper bounds for PS-SSV, computable via convex optimization. We have extended these results to the case where the uncertainties are structured (diagonal or block-diagonal, for instance).

The results of this research will appear in [8].

2.3 Modeling and control of the humoral immune response

In order to evaluate the efficacy of the approach described in Section 2.1, we considered a nonlinear model of the humoral immune response. The original immune system model was a second-order predator-prey model with a time-delay element. We obtained an approximate finite-state nonlinear model next, with the nonlinearities being rational functions of the state vector elements. This nonlinear system was then cast in the robust control framework, by using linear fractional transformations; the resulting model consisted of a fixed linear system with multiple copies of the state vector appearing in a feedback loop. Optimization based on LMIs was then applied to derive a nonlinear controller (i.e., an intravenous drug delivery strategy) for this system so as to stabilize the system, and in addition to minimize a measure of the drug delivered. Through simulations, the proposed intravenous drug strategy was shown to shorten patient recovery time, to lower peak drug concentrations and to decrease the total drug administered, when compared to standard antibiotic strategies.

The results from this work have been reported in publication [9].

2.4 Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems

We were also able to substantially reduce the computational burden associated with LMI methods for establishing robust stability. Returning to the system shown in Fig. 1, consider the *robust stability margin* of the system, defined as the largest “size” of Δ against which the system is guaranteed to be stable. (This quantity is of considerable importance in robust control; approaches such as μ analysis [10] can be viewed as obtaining lower bounds on the robust stability margin.) When lower bounds on the robust stability margin are obtained through multiplier methods using the theory of Integral Quadratic Constraints, bisection schemes are typically used, with each iteration requiring the solution of an LMI feasibility problem. We have been able to avoid the bisection scheme altogether by reformulating the lower bound computation problem as a single generalized eigenvalue minimization problem, which can be solved very efficiently using standard algorithms. We have illustrated this with several important, commonly-encountered special cases: Diagonal, nonlinear uncertainties; diagonal, memoryless, time-invariant sector-bounded (“Popov”) uncertainties; structured dynamic uncertainties; and structured parametric uncertainties. We also have generated a numerical example that illustrates our approach. The following table shows typical computational improvement that can be expected.

Uncertainty Type	Stab. Margin	Bisect. (sec)	Our scheme (sec)
General nonlinear	1.2896×10^{-2}	5.6800	0.8200
Dynamic	1.2899×10^{-2}	62.5700	20.3000
Popov-type	1.3264×10^{-2}	99.5700	26.5300
Parametric	1.3278×10^{-2}	51.8400	19.8300

The results from this work have been reported in publication [?].

2.5 Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems

We have derived an algorithm for the robust stability analysis and gain-scheduled controller synthesis for linear systems affected by time-varying parametric uncertainties. (This represents an important first step in our ultimate goal of deriving effective gain-scheduled control strategies for nonlinear systems.) Sufficient conditions for robust stability as well as conditions for the existence of a robustly stabilizing gain-scheduled controller are given in terms of a finite number of LMIs; explicit formulae for constructing robustly stabilizing gain-scheduled controllers are given in terms of the feasible set of these LMIs. We have proved that our approach is in general less conservative than the existing methods for stability analysis and gain-scheduled controller synthesis for parameter-dependent linear systems; numerical examples have shown that our approach offers significant improvement in practice as well.

The results from this work have been reported in publication [11].

2.6 Algorithms and software

One of the tasks that is of fundamental importance to the proposed project is a survey of optimization algorithms and software for the solution of LMI problems. A comprehensive overview of the state of the art of numerical algorithms for LMI problems, and of the available software was reported in publication [12].

2.7 Application of results to signal processing problems

The models underlying many signal processing systems are exactly the same as those of control systems. Motivated by this fact, we explored the solution of some signal processing problems using convex optimization using LMIs, and reported the results in publication [13].

2.8 Publications

Journal papers published/accepted for publication

1. V. Balakrishnan and R. L. Kashyap. "Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities". To appear in *Journal of Optimization Theory and Applications*, vol. 100, no. 3, March 1999. 20 pages.
2. A. L. Tits, V. Balakrishnan and L. Lee. "Robustness under Bounded Uncertainty with Phase Information". To appear in the *IEEE Trans. Aut. Contr.*, 1998. 13 pages.
3. A. E. Rundell, R. A. DeCarlo, V. Balakrishnan and H. HogenEsch. "Investigations into Treatment Strategies Aiding the Immune Response to *Haemophilus Influenzae*". *IEEE Trans. Biomed. Eng.*, vol. 45, no. 4, pages 429–439, April 1998.
4. V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". To appear after revision in *IEEE Trans. Aut. Contr.*, 1998.
5. L. Vandenberghe and V. Balakrishnan. "Algorithms and Software for LMI Problems in Control". *IEEE Control Systems Magazine*, pages 89–95, October 1997.

Conference papers published

1. V. Balakrishnan and L. Vandenberghe. "Linear Matrix Inequalities for signal processing. An overview". Invited for presentation at the *32nd Annual Conference on Information Sciences and Systems*, Department of Electrical Engineering, Princeton University, Princeton NJ, March 1998.

Pending conference papers

1. V. Balakrishnan and F. Wang. "Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin of Uncertain Systems". Under review for presentation at the *IEEE Conf. on Decision and Control*, December 1998.

2. F. Wang and V. Balakrishnan. "Improved Stability Analysis and Gain-Scheduled Controller Synthesis for Parameter-Dependent Systems". Under review for presentation at the *IEEE Conf. on Decision and Control*, December 1998.

3 Research Plan for the next period

3.1 Gain-scheduled controller synthesis for nonlinear systems

The research results reported in Section 2.5 describe the synthesis of gain-scheduled controllers for linear systems affected by time-varying parameters. The next step is to extend this approach to the case of *nonlinear* parameter-dependent systems. The underlying technical problem is to handle, in a non-conservative fashion, multiple copies of the state vector that appear in Δ in Fig. 1. One approach that is promising is to introduce constant scaling matrices that take advantage of the structure of Δ . Less conservative methods that use dynamic scalings will also be investigated.

3.2 Tuning our approach towards the control of Unmanned Intelligent Autonomous Air Vehicles

While our research efforts thus far have tended to address the fundamentals of robust nonlinear control, we will next focus on the specific problem of the control of Unmanned Intelligent Autonomous Air Vehicles (UIAAVs). To this end, we will collaborate with other ONR researchers in this area. In particular, we will test our algorithms on the system models gathered from other researchers.

3.3 Development of CAD tools

The proposed analysis and design techniques are applicable to other nonlinear control problems besides the control of UIAAVs. To enable such applications, CAD tools with graphical user interfaces can be built, which will serve as a valuable addition to the library of performance evaluation and controller design software for nonlinear systems. This CAD tool will take as (graphical) input the block-diagram of the nonlinear system, and will generate controller parameters, and a Lyapunov function guarantee for the design.

3.4 Efficient solution of large-scale linear matrix inequalities

The linear matrix inequalities encountered with our approach tend to be so large (both in the number of variables, as well as the number of constraints) so as to test the limits of available computing software. We intend to explore alternative ways towards solving such large-scale LMI problems efficiently.

References

- [1] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.
- [2] M. G. Safonov and R. Y. Chiang. Real/complex K_m -synthesis without curve fitting. In C. T. Leondes, editor, *Control and Dynamic Systems*, volume 56, pages 303–324. Academic Press, New York, 1993.
- [3] J. P. How and S. R. Hall. Connections between the Popov stability criterion and bounds for real parametric uncertainty. In *Proc. American Control Conf.*, volume 2, pages 1084–1089, June 1993.
- [4] V. Balakrishnan. Linear matrix inequalities in robustness analysis with multipliers. *Syst. Control Letters*, 25(4):265–272, 1995.
- [5] K.-C. Goh and M. G. Safonov. Robust analysis, sectors, and quadratic functionals. In *Proc. IEEE Conf. on Decision and Control*, volume 2, pages 1988–1993, New Orleans, LA, 1995.
- [6] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Control*, 42(6):819–830, June 1997.
- [7] V. Balakrishnan and R. L. Kashyap. Robust stability and performance analysis of uncertain systems using linear matrix inequalities. *Journal of Optimization Theory and Applications*, 100(3), March 1999.
- [8] A. L. Tits, V. Balakrishnan, and L. Lee. Robustness under bounded uncertainty with phase information. To appear in *IEEE Trans. Aut. Control*, 1998.
- [9] A. Rundell, R. DeCarlo, V. Balakrishnan, and H. HogenEsch. Investigations into treatment strategies aiding the immune response to *haemophilus influenzae*. *IEEE Trans. Biomed. Eng.*, 45(4):429–439, April 1998.
- [10] G. Balas, J. C. Doyle, K. Glover, A. Packard, and Roy Smith. *μ -analysis and synthesis*. MUSYN, inc., and The Mathworks, Inc., 1991.
- [11] F. Wang and V. Balakrishnan. Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems. In *Proc. IEEE Conf. on Decision and Control*, pages 1771–1776, Tampa, FL, December 1998.
- [12] L. Vandenberghe and V. Balakrishnan. Algorithms and software for LMI problems in control. *IEEE Control Syst. Mag.*, pages 89–95, October 1997.
- [13] V. Balakrishnan and L. Vandenberghe. Linear Matrix Inequalities for signal processing: An overview. In *Proceedings of the 32nd Annual Conference on Information Sciences and Systems*, Princeton, NJ, March 1998.

PROGRESS REPORT FOR THE PERIOD MAY 1997 - OCTOBER 1997

Project title: Robust Gain-Scheduled Nonlinear Control
Design for Stability and Performance
Principal Investigator: Venkataramanan Balakrishnan
Contract No: N00014-97-1-0640 (Young Investigator Award)

The research goals outlined in the original proposal were:

- * Deriving models for uncertain nonlinear systems that lend themselves to the use of robust control techniques.
- * Robust stability analysis of nonlinear systems using optimization based on linear matrix inequalities (LMIs).
- * Robust performance analysis of nonlinear systems using LMI methods.
- * Gain-scheduled controller design.

The progress made towards achieving some of these goals is outlined below:

ALGORITHMS AND SOFTWARE

One of the tasks that is of fundamental importance to the proposed project is a survey of optimization algorithms and software for the solution of LMI problems. This work was performed in collaboration with Professor Vandenberghe of UCLA, and a comprehensive overview of the state of the art of numerical algorithms for LMI problems, and of the available software was reported in publication [1].

ROBUST STABILITY AND PERFORMANCE ANALYSIS USING MULTIPLIER THEORY AND LMI OPTIMIZATION

Our proposed approach for robust nonlinear control design builds on the following: The robust stability and performance analysis of uncertain systems, with various assumptions on the nature of the uncertainties (sector-bounded nonlinear, linear time-invariant, parametric, etc.), as well as their structure (diagonal, block-diagonal, etc.), can be performed in a unified manner using multiplier theory and LMI-based convex optimization. Not only does this provide a unification of several apparently-diverse robust stability tests, but it also paves the way for developing new stability tests. In addition, the multipliers used in the stability analysis can be shown to yield a convex parametrization of a subset of Lyapunov functions that provide a certificate of robust stability. These Lyapunov functions can in turn be used to derive bounds on various robust performance measures for uncertain systems. A tutorial outlining this approach was prepared for publication [2].

MODELING AND CONTROL OF THE HUMORAL IMMUNE RESPONSE

In order to test of the efficacy of our proposed methods, a nonlinear model of the humoral immune was considered. (This represents ongoing work, performed in collaboration with Professor DeCarlo and Ms. Rundell at the School of Electrical and Computer Engineering at Purdue.) The original immune system model is a second-order predator-prey model with a time-delay element. An approximate finite-state nonlinear model was next obtained, with the nonlinearities being rational functions of the state vector elements. This nonlinear system was then cast in the robust control framework, by using linear fractional transformations; the resulting model consisted of a fixed linear system with multiple copies of the state vector appearing in a feedback loop. Optimization based on LMIs was then applied to derive a nonlinear controller for this system. The results from this work have been reported in publication [3].

REFERENCES

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- [1] L. Vandenberghe and V. Balakrishnan. "Algorithms and Software for LMI Problems in Control". In IEEE Control Systems Magazine, pages 89-95, October 1997
- [2] V. Balakrishnan and R. Kashyap. "Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities". Under review for publication in Journal of Optimization Theory and Applications, 1997.
- [3] A. E. Rundell, R. A. DeCarlo, V. Balakrishnan and H. HogenEsch. "Investigations into Treatment Strategies Aiding the Immune Response to Haemophilus Influenzae". Accepted for publication in the IEEE Trans. Biomed. Eng., 1997. 9 pages.