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Abstract

Space-time coordinate transformations valid for arbitrarily long coordinate time are derived from global Minkowski coordinates to the Fermi coordinates of an observer moving in a circle in three-dimensional space. The metric for the Fermi coordinates is calculated directly from the tensor transformation rule. The Fermi coordinates are used in an examination (from the observer's reference frame) of the detailed behavior of ideal clocks.

A complicated relation exists between Fermi coordinate time and proper time on stationary clocks (in the Fermi frame) and between proper time on satellite clocks that orbit the observer. For portable clocks that orbit the Fermi coordinate origin, an orbital Sagnac-like effect exists. The coordinate speed of light is isotropic but is periodic in time and varies with Fermi coordinate position. In a numerical illustration of the magnitudes of these kinematic effects, this report uses parameters relevant to clocks carried aboard satellites orbiting the Earth.

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1. Background

The U.S. Army is developing precision-guided munitions intended to be used against the threat of armored vehicles. These munitions have a coupled GPS (Global Positioning System) and inertial guidance system. The GPS portion of the system is intended to give precise updates to the inertial system during mid-course flight. Such precision-guided munitions are required to have a terminal accuracy of better than 1 m (corresponding to the typical scale dimension of an armored vehicle). This means that the accuracy of GPS position updates must be significantly better than 1 m in order for the total system to perform at 1 m.

The Department of Defense (DoD) and the civilian population have numerous other interests in precise navigation, such as aircraft instrument approaches. Today, the stand-alone GPS is used routinely for nonprecision instrument approaches. It would be a great advantage if the stand-alone GPS could also be used for precision approaches.* Depending on the future course of development of the GPS, such approaches could be feasible for unmanned as well as manned aircraft, if the GPS were sufficiently accurate and had a sufficient level of receiver autonomous integrity monitoring (RAIM).†

In addition to these applications, DoD has numerous other needs for precise navigation, such as mapping safe corridors and positioning space-based interferometers for surveillance of the ground. Another area where the GPS needs ultra-high accuracy is for precise time transfer, which is necessary for applications in communications encryption.

The current stand-alone GPS does not meet the accuracies needed in many of the above-mentioned areas. At present, it is well known that small anomalies exist in position and time computed from GPS data [1–3]. The origin of these anomalies is not understood. In particular, GPS time transfer data from the U.S. Naval Observatory (USNO) indicate that GPS time (for all satellites) is *periodic* with respect to the Master Clock, which is the most

*The FAA (Federal Aviation Administration) system known as WAAS (Wide Area Augmentation System), which is to provide precision approaches to most airports in the U.S., is a differential GPS system. I am referring to the possibility of a precision approach system based upon the stand-alone GPS, with no differential GPS corrections.

†For example, one could obtain an improvement in integrity over the present stand-alone GPS by increasing the number of GPS satellites in view.

accurate source of official time for DoD. Furthermore, the time obtained from all GPS satellites appears to speed up and slow down in-phase [3]. In 1997, the periodicity of this effect was approximately equal to the sidereal day and had a peak-to-peak amplitude of $\approx$20 ns, which translates to 20 ft ($\approx$6 m) of error in light travel time [3]. The existence of these anomalies in GPS data motivates the theoretical investigation reported here.

The purpose of this work is to explore the possibility that the periodicity in time between the GPS and the USNO Master Clock is a relativistic kinematic effect that is due to the way in which the GPS is implemented. If such an explanation were to prove correct, then this effect could be taken into account, and the accuracy of the GPS would be improved. The work reported here is only a first step in the analysis, because gravitational effects have been neglected. As elaborated below, the gravitational field is an essential ingredient in the analysis, but the formalism is sufficiently complicated that it is useful to first explore the effects without the contribution of gravity.

The GPS is implemented in an Earth-centered quasi-inertial (ECQI) reference frame. This reference frame is not a relativistic space-time coordinate system (as is required by a fully relativistic theory); instead, it is a spatial reference system plus a time scale [4]. From the point of view of relativity theory, the GPS satellite clocks keep coordinate time in a modified Schwarzschild metric [5], not the ECQI system of coordinates.

In this report, the ECQI frame coordinates are replaced by a Fermi-Walker transported system of coordinates. The apparent behavior of real clocks (i.e., the GPS satellites) is examined from the point of view of this coordinate system.

2. Introduction

It is well known that the choice of a particular space-time coordinate system can lead to apparent effects that vanish when an alternative coordinate system is used [6–9]. In this work, I report a detailed calculation of the apparent behavior of ideal clocks as observed from a system of coordinates whose origin moves in a circle in three-dimensional (3-d) space and whose axes maintain their orientation with respect to the distant stars. Such a system of coordinates is similar to that used to implement an Earth-centered inertial frame of reference, as in astrodynamics applications [10,11]. However, the case treated here differs from that of the Earth-centered inertial frame, in that gravitational effects are neglected. In this work, the origin of coordinates is kept in circular motion by a force (such as that provided by rocket engines), so the system of coordinates is accelerated but nonrotating. I take a Fermi coordinate system as the closest relativistic analogue to a nonrotating system of local Cartesian coordinates used in experiments [6,10,11]. In particular, a Fermi coordinate system is a nonrotating system of coordinates where Coriolis forces are absent and light travels essentially along a straight line. I take the Fermi coordinate origin to move in a circle in 3-d space, and therefore, the observer's tetrad is Fermi-Walker transported along a helical time-like world line.

In the tetrad formalism, laboratory measurements are interpreted as projections of tensors on the tetrad basis vectors. These projections are invariant quantities under transformations of the space-time coordinates. However, these projections depend on the world line of the observer and the choice of local Cartesian axes used by the observer [6]. The need for the tetrad formalism to relate experiment to theory, as well as the problem of measurable quantities in general relativity, is extensively discussed by Pirani [12], Synge [6], Soffel [10], Brumberg [11], and more recently Guinot [9].

The tetrad formalism was initially investigated for the case of inertial observers that move on geodesics [12–17]. However, many observers are terrestrially based or are based on noninertial platforms. The general theory for the case of noninertial observers has been investigated by Synge [6], who considered nonrotating observers moving along a time-like world line, and by others [18–23], who considered accelerated, rotating observers. For arbitrary observer motions, the effects seen by the observer are indeed very complicated, and the general theory [6,12–23] gives limited physical insight.

In this report, I work out in detail a simple model problem. I consider the particular case of an observer moving in a circle in 3-d space. I use the tetrad formalism to obtain the transformation from an inertial coordinate system in Minkowski space-time to the accelerated, nonrotating system of Fermi coordinates of an observer moving in a circle in 3-d space. I compute the apparent behavior of ideal clocks as observed with respect to these Fermi (laboratory) coordinates. The results are given here as expansions in low velocity of the observer, compared with the speed of light. However, the coordinate transformations derived here are not expanded in a power series in time, and hence they are valid for arbitrarily long times. Knowing the correct long-time behavior of the coordinate transformations allows the study of the apparent long-time behavior of clocks as observed with respect to the Fermi coordinates of the observer. In a numerical example of the magnitude of the effects, I use parameters relevant to clocks carried aboard GPS satellites orbiting the Earth [24].

3. Fermi-Walker Transport Differential Equations

Consider a time-like world line C of an observer moving in a circle in 3-d space, given in Minkowski space-time coordinates by

$$\begin{aligned} x^0 &= x_0^0 + u \,, \\ x^1 &= x_0^1 + a\cos\left(\omega u/c\right) \,, \\ x^2 &= x_0^2 + a\sin\left(\omega u/c\right) \,, \\ x^3 &= x_0^3 \,, \end{aligned} \tag{1}$$

where x_0^i are constants, u is a parameter along the world line, and c is the speed of light in an inertial frame in vacuum. The observer travels in a circle in the x-y plane (see fig. 1). Along his world line, the observer travels with a 4-velocity $u^i = dx^i/ds$ and acceleration

$$w^i = \frac{\delta u^i}{\delta s} = \frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k \,, \tag{2}$$

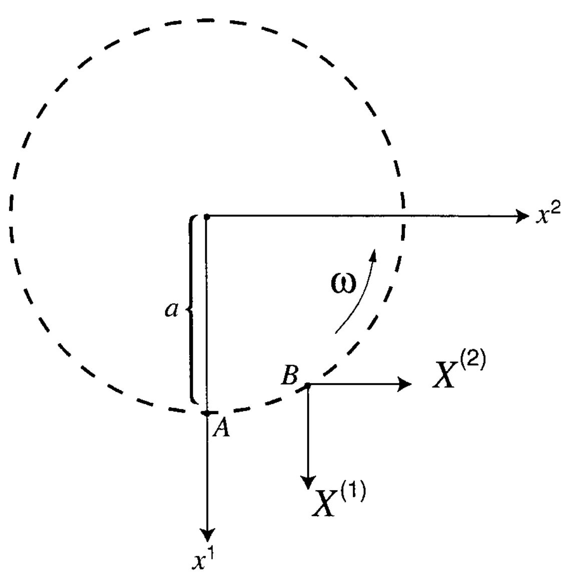


Figure 1. Observer's circular 3-d path is shown against background of global inertial Minkowski coordinates x^i. At proper time $s = 0$, observer is at point A. At proper time $s > 0$, observer is at point B. Origin of Fermi coordinates is at current position of observer. Orientation of Fermi coordinate axes $X^{(a)}$ is shown schematically at a given proper time s.

where Γ^i_{jk} is the affine connection. The normalization of the 4-velocity, $u^i u_i = -1$, provides the relation between the arc length, $s = c\tau$, where τ is the proper time, and the parameter $u = \gamma s$, where $\gamma = (1-\nu^2)^{-1/2}$. (See the appendix for the conventions used here.) The parameter ν is the dimensionless velocity, given by $\nu = a\omega/c$. Since the affine connection components all vanish in Minkowski space, the explicit expressions for the 4-velocity and acceleration are

$$u^i(s) = \gamma \left(1, -\frac{a\omega}{c}\sin\left(\frac{\gamma\omega}{c}s\right), \frac{a\omega}{c}\cos\left(\frac{\gamma\omega}{c}s\right), 0\right), \tag{3}$$

$$w^i = -a\gamma^2 \left(\frac{\omega}{c}\right)^2 \left(0, \cos\left(\frac{\gamma\omega}{c}s\right), \sin\left(\frac{\gamma\omega}{c}s\right), 0\right). \tag{4}$$

As the observer moves on the time-like world line C, he carries with him an ideal clock and three gyroscopes. At some initial coordinate time x^0, the observer is at point P_0 at proper time $\tau = s/c = 0$. On his world line, the observer carries with him three orthonormal basis vectors $\lambda^i_{(\alpha)}$, where $\alpha = 1, 2, 3$ labels the vectors and $i = 0, 1, 2, 3$ labels the components of these vectors in Minkowski coordinates. These vectors form the basis for his measurements [12] (see fig. 2). The orientation of each basis vector is held fixed with respect to each of the gyroscopes' axes of rotation [25]. The fourth basis vector is taken to be the observer 4-velocity, $\lambda^i_{(0)} = u^i$. The four unit vectors $\lambda^i_{(a)}$ form the observer's tetrad, which is an orthonormal set of vectors at P_0,

$$g_{ij}\, \lambda^i_{(a)}\, \lambda^j_{(b)} = \eta_{(ab)}, \tag{5}$$

where the matrix $\eta_{(ab)} = g_{ab}$ is the Minkowski metric (see the appendix).

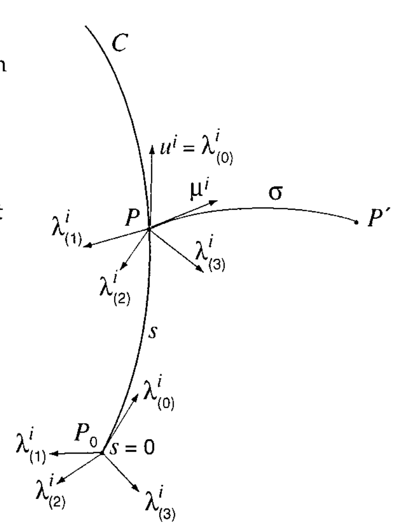
At a later time $s = c\tau$, the observer is at a point P. The observer's orthonormal set of basis vectors is related to his tetrad basis at P_0 by Fermi-Walker transport. Fermi-Walker transport preserves the lengths and relative angles of the transported vectors. For an arbitrary vector with contravariant components f^i, its components at P are related to its components at P_0 by the Fermi-Walker transport differential equations [6]:

$$\frac{\delta f^i}{\delta s} = W^{ij} f_j, \tag{6}$$

where

$$W^{ij} = u^i w^j - w^i u^j. \tag{7}$$

Figure 2. Observer's world line C is shown with initial tetrad basis vectors $\lambda^i_{(\alpha)}$ at $s = 0$ at point P_0. Fermi transported tetrad basis vectors at finite proper time s are shown at point P.



When we use equation (6) to transport a vector f^i that is orthogonal to the 4-velocity, $u^i f_i = 0$, the second term in equation (7) does not contribute. We refer to transport of such space-like basis vectors as Fermi transport, and $W^{ij} \rightarrow \tilde{W}^{ij} = u^i w^j$. For an arbitrary vector f^i perpendicular to the 4-velocity, the explicit form of the Fermi transport differential equations for the world line in equation (1) is

$$\frac{df^0}{d\xi} = -\gamma^3 \nu \cos(\gamma\xi) f^1 - \gamma^3 \nu \sin(\gamma\xi) f^2 , \tag{8}$$

$$\frac{df^1}{d\xi} = \gamma^3 \nu^2 \cos(\gamma\xi) \sin(\gamma\xi) f^1 + \gamma^3 \nu^2 \sin^2(\gamma\xi) f^2 , \tag{9}$$

$$\frac{df^2}{d\xi} = -\gamma^3 \nu^2 \cos^2(\gamma\xi) f^1 - \gamma^3 \nu^2 \cos(\gamma\xi) \sin(\gamma\xi) f^2 , \tag{10}$$

$$\frac{df^3}{d\xi} = 0 , \tag{11}$$

where the components f^i are functions of ξ and the dimensionless proper time is given by $\xi = \omega s / c$. The differential equations (8) to (11) are identical to the equations that describe the Thomas precession of an electron's spin vector as it moves in a circular orbit around the nucleus [26]. The solution of equations (8) to (11) is given by

$$f^0 = -\gamma \nu A \cos\left(\gamma^2 \xi + \alpha\right) + \beta , \tag{12}$$

$$f^1 = A \cos(\gamma\xi) \cos\left(\gamma^2 \xi + \alpha\right) + A\gamma \sin(\gamma\xi) \sin\left(\gamma^2 \xi + \alpha\right) , \tag{13}$$

$$f^2 = A \sin(\gamma\xi) \cos\left(\gamma^2 \xi + \alpha\right) - A\gamma \cos(\gamma\xi) \sin\left(\gamma^2 \xi + \alpha\right) , \tag{14}$$

$$f^3 = \delta , \tag{15}$$

where A, α, β, and δ are integration constants.

4. Determination of the Tetrad

At the point P_0, the tetrad basis vectors satisfy equation (5). These equations constitute 12 relations (since eq (5) is symmetric in α and β) for 16 components $\lambda^i_{(a)}$. However, the zeroth member of the tetrad is the 4-velocity, which is assumed known,

$$\lambda^i_{(0)} = u^i = \gamma\left(1, v^1, v^2, v^3\right) , \tag{16}$$

and is given by equation (3). Equation (5) can be rewritten as two equations:

$$g_{ij}\lambda^i_{(\alpha)}\lambda^j_{(0)} = 0, \qquad \alpha = 1, 2, 3 , \tag{17}$$

$$g_{ij}\lambda^i_{(\alpha)}\lambda^j_{(\beta)} = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3 , \tag{18}$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and 0 otherwise. Substituting the explicit form of the Minkowski metric into equation (17) gives a condition on the zeroth components of a tetrad vector if the spatial components are known (Greek indices take values α, κ =1, 2, 3):

$$\lambda^0_{(\alpha)} = \delta_{\beta\kappa}v^\beta\lambda^\kappa_{(\alpha)} . \tag{19}$$

The use of equation (19) to eliminate the time components $\lambda^0_{(\alpha)}$ from equation (18) leads to an orthogonality relation for only the spatial components of the tetrad vectors:

$$\delta_{\mu\nu}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)} - \delta_{\mu\nu}\delta_{\kappa\gamma}v^\mu v^\kappa\lambda^\nu_{(\alpha)}\lambda^\gamma_{(\beta)} = \delta_{\alpha\beta} . \tag{20}$$

The general idea is to solve equation (20) for the spatial components of the tetrad at $s = 0$, and then to substitute the spatial components into equation (19) to obtain the time components of each tetrad vector. Thus, I obtain the tetrad vectors at the initial time corresponding to $s = 0$. For each tetrad vector, I use these components as initial conditions in equations (12) to (15) to determine the tetrad components at point P at finite s.

The exact solution of equation (20) can be obtained if we note that for $v^\alpha = 0$, the spatial components of the tetrad vectors are orthonormal, and the solution is given by

$$\lambda^\mu_{(\alpha)} = \delta^\mu_\alpha . \tag{21}$$

I assume that the tetrad unit vectors are oriented approximately parallel to the observer's local Cartesian x, y, and z axes. For small $v^\alpha \ll 1$, the solution of equation (20) can be obtained as a triple power series in v^α, by iteration, starting with equation (21) as the first approximation. Having solved for the tetrad components as a power series, it is easy to guess the exact solution to be

$$\lambda^0_{(\alpha)} = \gamma \delta_{\alpha\beta} v^\beta \,, \tag{22}$$

$$\lambda^\mu_{(\alpha)} = \delta^\mu_\alpha + \frac{\gamma - 1}{\nu^2} \delta_{\kappa\alpha} v^\kappa v^\mu \,, \tag{23}$$

$$\lambda^i_{(0)} = u^i = \gamma \left(1, v^1, v^2, v^3\right) \,. \tag{24}$$

where the components of dimensionless velocity v^α are given in equation (3). Equations (22) to (24) give the components of the observer's tetrad basis vectors at point P_0. At point P, the arc length is $s > 0$, and equations (22) to (24) give the initial components for each tetrad vector. These initial components are used in the general solution of a Fermi-Walker transported vector, given in equations (12) to (15). The tetrad components at point P at finite s are given by

$$\lambda^i_{(0)} = \{\gamma, -\gamma\nu \sin(\gamma\xi), \gamma\nu \cos(\gamma\xi), 0\} \,, \tag{25}$$

$$\begin{aligned} \lambda^i_{(1)} = \{&-\gamma\nu \sin(\gamma^2\xi), \cos(\gamma\xi)\cos(\gamma^2\xi) + \gamma \sin(\gamma\xi)\sin(\gamma^2\xi), \cos(\gamma^2\xi)\sin(\gamma\xi) \\ &- \gamma \cos(\gamma\xi)\sin(\gamma^2\xi), 0\} \,, \end{aligned} \tag{26}$$

$$\begin{aligned} \lambda^i_{(2)} = \{&\gamma\nu \cos(\gamma^2\xi), -\gamma \cos(\gamma^2\xi)\sin(\gamma\xi) + \cos(\gamma\xi)\sin(\gamma^2\xi), \gamma \cos(\gamma\xi)\cos(\gamma^2\xi) \\ &+ \sin(\gamma\xi)\sin(\gamma^2\xi), 0\} \,, \end{aligned} \tag{27}$$

$$\lambda^i_{(3)} = \{0, 0, 0, 1\} \,. \tag{28}$$

Equations (25) to (28) are valid for all proper times s, because they are exact solutions of equations (8) to (11) and they satisfy the orthogonality relations in equation (5) along the observer's world line C.

5. Fermi Coordinates

Fermi coordinates are defined by the geometric construction shown in figure 2. Every event P' in space-time has coordinates $x^{i\,\prime}$ in the global inertial coordinate system. According to the observer moving in circular motion along a time-like world line, the same event has the Fermi coordinates $X^{(a)}$, $a = 0, 1, 2, 3$. The first Fermi coordinate, $X^{(0)} = s$, is just the proper time (in units of length) associated with the event P'. The proper time for P' is defined as the value of arc length s such that a space-like geodesic from point P passes through event P', where the tangent vector of this geodesic, μ^i, is orthogonal to the observer 4-velocity at P:

$$\mu^i\, u_i|_P = 0\,. \tag{29}$$

For the simple case of Minkowski space, the geodesic connecting the points P and P' is simply a straight line,

$$\mu^i = N(x^{i\,\prime} - x^i(\gamma s))\,, \tag{30}$$

where $x^i(u)$ and $x^{i\,\prime}$ are the coordinates of P and P', respectively, in the global inertial frame, and N is a normalization constant that makes μ^i a unit vector. The orthogonality condition in equation (29) is

$$g_{ij}\, \mu^i(s)\, \lambda^j_{(0)}(s) = 0 \tag{31}$$

and gives an implicit equation for s for a given P'. This orthogonality condition gives the first Fermi coordinate of the point P',

$$X^{(0)} = s\,, \tag{32}$$

and equation (31) gives s as an implicit equation:

$$\gamma s = x^{0\,\prime} - x_0^0 + \nu\left[\left(x^{1\,\prime} - x_0^1\right)\,\sin\left(\frac{\gamma\omega s}{c}\right) - \left(x^{2\,\prime} - x_0^2\right)\,\cos\left(\frac{\gamma\omega s}{c}\right)\right]\,. \tag{33}$$

In the limit of small speeds of the observer, $\nu \to 0$, equation (33) gives $s = x^{0\,\prime} - x_0^0$. For $\nu \ll 1$, equation (33) can be solved for s by iteration, resulting in a power series in ν:

$$
\begin{aligned}
X^{(0)} = s = {} & \Delta x^0 + \left[\Delta x^1 \sin\left(\frac{\Delta x^0\omega}{c}\right) - \Delta x^2 \cos\left(\frac{\Delta x^0\omega}{c}\right)\right]\nu - \frac{1}{2}\Delta x^0\nu^2 \\
& - \frac{1}{2a}\left[-a + 2\Delta x^1 \cos\left(\frac{\Delta x^0\omega}{c}\right) + 2\Delta x^2 \sin\left(\frac{\Delta x^0\omega}{c}\right)\right] \\
& \times \left[\Delta x^2 \cos\left(\frac{\Delta x^0\omega}{c}\right) - \Delta x^1 \sin\left(\frac{\Delta x^0\omega}{c}\right)\right]\nu^3 + O(\nu)^4 , \qquad (34)
\end{aligned}
$$

where $\Delta x^i = x^{i\,\prime} - x^i_0$. For future reference, I label the coordinates of P, P', and P_0 in the global inertial frame by $P' = (x^{0\,\prime}, x^{1\,\prime}, x^{2\,\prime}, x^{3\,\prime})$, $P = (x^0, x^1, x^2, x^3)$, and $P_0 = (x^0_0, x^1_0, x^2_0, x^3_0)$.

The contravariant spatial Fermi coordinates, $X^{(\alpha)}$, $\alpha = 1, 2, 3$, are defined as [6]

$$
X^{(\alpha)} = \sigma\mu^i\lambda_i^{(\alpha)} = g_{ij}\,\sigma(s)\,\mu^i(s)\,\eta^{(\alpha\beta)}\,\lambda^j_{(\beta)}(s) , \qquad (35)
$$

where I used the Minkowski definition of space-time distance σ, along the space-like geodesic between P and P':

$$
\sigma^2 = g_{ij}(x^{i\,\prime} - x^i)(x^{j\,\prime} - x^j) , \qquad (36)
$$

where g_{ij} is the Minkowski metric and $\eta^{(ij)} = \eta_{(ij)} = \eta_{ij}$ is an invariant matrix, which is numerically equal to the Minkowski metric. Since the parameter s in equation (35) is a function of P' coordinates $x^{i\prime}$, s must be eliminated by equation (34), so that $X^{(\alpha)}$ are explicit functions of the global inertial coordinates $x^{i\prime}$. I have calculated the resulting coordinate transformation correct to fourth order in ν. However, since the expressions are complicated, I write them here correct only to third order in ν:

$$
\begin{aligned}
X^{(1)} = {} & \Delta x^1 - a\cos\left(\frac{\Delta x^0\omega}{c}\right) + \frac{1}{4}\left[\Delta x^1 - \Delta x^1 \cos\left(\frac{2\Delta x^0\omega}{c}\right) - \Delta x^2 \sin\left(\frac{2\Delta x^0\omega}{c}\right)\right]\nu^2 \\
& + \frac{1}{2a}\Delta x^0\left[-\Delta x^2 + a\sin\left(\frac{\Delta x^0\omega}{c}\right)\right]\nu^3 + O(\nu)^4 , \qquad (37) \\
X^{(2)} = {} & \Delta x^2 - a\sin\left(\frac{\Delta x^0\omega}{c}\right) + \frac{1}{4}\left[\Delta x^2 + \Delta x^2 \cos\left(\frac{2\Delta x^0\omega}{c}\right) - \Delta x^1 \sin\left(\frac{2\Delta x^0\omega}{c}\right)\right]\nu^2 \\
& + \frac{1}{2a}\Delta x^0\left[\Delta x^1 - a\cos\left(\frac{\Delta x^0\omega}{c}\right)\right]\nu^3 + O(\nu)^4 , \qquad (38) \\
X^{(3)} = {} & \Delta x^3 . \qquad (39)
\end{aligned}
$$

The inverse transformation can be obtained from equations (37) to (39) by iteration and some tedious algebra:

$$x^{0\prime} - x_0^0 = X^{(0)} + \left[X^{(2)} \cos\left(\frac{\omega X^{(0)}}{c}\right) - X^{(1)} \sin\left(\frac{\omega X^{(0)}}{c}\right)\right] \nu + \frac{1}{2} X^{(0)} \nu^2 + O(\nu)^3 \,, \quad (40)$$

$$\begin{aligned} x^{1\prime} - x_0^1 &= X^{(1)} + a \cos\left(\frac{\omega X^{(0)}}{c}\right) \\ &+ \frac{1}{4}\left[X^{(1)} - X^{(1)} \cos\left(\frac{2\,\omega X^{(0)}}{c}\right) - X^{(2)} \sin\left(\frac{2\,\omega X^{(0)}}{c}\right)\right] \nu^2 + O(\nu)^3 \,, \quad (41) \\ x^{2\prime} - x_0^2 &= X^{(2)} + a \sin\left(\frac{\omega X^{(0)}}{c}\right) \\ &+ \frac{1}{4}\left[X^{(2)} + X^{(2)} \cos\left(\frac{2\,\omega X^{(0)}}{c}\right) - X^{(1)} \sin\left(\frac{2\,\omega X^{(0)}}{c}\right)\right] \nu^2 + O(\nu)^3 \,, \quad (42) \\ x^{3\prime} - x_0^3 &= X^{(3)} \,. \quad (43) \end{aligned}$$

6. Metric in Fermi Coordinates

The space-time interval in the Fermi coordinate system of the observer is

$$ds^2 = -G_{(ij)}\, dX^{(i)}\, dX^{(j)}\,, \tag{44}$$

where $G_{(ij)}$ are the metric tensor components when the $X^{(i)}$ are used as coordinates. A direct calculation of the metric tensor components from the tensor transformation rule,

$$G_{(ij)} = g_{kl}\, \frac{\partial x^k}{\partial X^{(i)}}\, \frac{\partial x^l}{\partial X^{(j)}}\,, \tag{45}$$

using the transformation equations (40) to (43) and the Minkowski metric for g_{ij}, gives $G_{(\alpha\beta)} = \delta_{\alpha\beta}$, $G_{(0\alpha)} = 0$, and

$$G_{(00)} = -(1+\zeta)^2\,, \tag{46}$$

where ζ is given by

$$\zeta = \left[1 - \frac{\Delta x^1}{a}\cos\left(\frac{\omega \Delta x^0}{c}\right) - \frac{\Delta x^2}{a}\sin\left(\frac{\omega\Delta x^0}{c}\right)\right]\nu^2 + O(\nu^4)\,. \tag{47}$$

Using the transformation equations (40) to (43) to express ζ as a function of the Fermi coordinates, I obtain

$$\zeta = -\frac{1}{a}\left[X^{(1)}\cos\left(\frac{\omega}{c}X^{(0)}\right) + X^{(2)}\sin\left(\frac{\omega}{c}X^{(0)}\right)\right]\nu^2 + O(\nu^4)\,. \tag{48}$$

The result in equation (46) agrees with the general theory of Synge [6], specialized to flat space. Synge shows that

$$\zeta = X_{(\beta)}\, w^{(\beta)} = \eta_{(\alpha\beta)}\, X^{(\alpha)}\, w^{(\beta)} = \eta_{(\alpha\beta)}\, X^{(\alpha)}\, w^i\, \lambda_i^{(\beta)}\,. \tag{49}$$

To third order in ν, equation (48) agrees with the general theory given by Synge. The quantities $X_{(\beta)} = \eta_{(\alpha\beta)}\, X^{(\alpha)}$ are the covariant Fermi coordinates, and $w^{(\beta)} = w^i\, \lambda_i^{(\beta)}$ are the components of the observer's 4-acceleration in the Fermi coordinate system. The 4-acceleration w^i is related to the ordinary 3-d acceleration a^β, $\beta = 1, 2, 3$, by

$$w^i = \frac{\gamma^2}{c^2}\,\frac{d^2x^i}{dt^2} = \frac{\gamma^2}{c^2}\, a^i\,. \tag{50}$$

Using this relation, we can write ζ in terms of the global inertial coordinates of P and P' as

$$\zeta = \frac{\gamma^2}{c^2}\,\delta_{\alpha\beta}\left(x^{\alpha\,\prime} - x^{\alpha}(s)\right) a^{\beta}\,. \tag{51}$$

For the circular motion treated here, from equation (48), the three spatial components of the observer's acceleration in Fermi coordinates are given by

$$a^{(\beta)} = -\frac{\nu^2}{a}\left(\cos\left(\frac{\omega}{c}X^{(0)}\right), \sin\left(\frac{\omega}{c}X^{(0)}\right), 0\right) \tag{52}$$

where I have dropped fourth-order terms in ν. To third order in ν, the observer's acceleration is just the classical centripetal acceleration for an observer moving in a circle.

The quantity $c^2\zeta$ is a time-dependent potential that determines the rate of proper time with respect to coordinate time in Fermi coordinates. Proper time is kept by an ideal clock. Coordinate time depends on the definition of the reference frame and coordinates used within that frame. The quantity ζ, given in equation (51), depends on acceleration a^{β} and the difference in coordinates, $x^{\alpha\,\prime} - x^{\alpha}(s)$, from the spatial origin of the Fermi coordinates. Away from the spatial origin of Fermi coordinates, the acceleration produces a periodic time-dependent effective gravitational potential field $c^2\zeta$, which leads to a change in the relation of coordinate time to proper time, through the metric components $G_{(ij)}$.

7. Apparent Behavior of Clocks in Fermi Coordinates

The significance of proper time is that it is the quantity that ideal clocks keep, that it may be related to time on real clocks, and that it is closely related to measurable quantities. The significance of coordinate time is that it enters into the theory that defines the space-time grid. In order to analyze experiments with clocks, we must relate what is measured (i.e., proper time intervals between space-time events) to coordinate positions and times of these events.

Consider an ideal clock whose world line is given in Fermi coordinates by $X^{(i)} = X^{(i)}(X^{(0)})$. At coordinate times $X_A^{(0)}$ and $X_B^{(0)}$, the clock is at spatial positions $X_A^{(\alpha)}$ and $X_B^{(\alpha)}$, respectively. The elapsed proper time between these two events, $\Delta\tau = \Delta s/c$, is given by the integral along the world line of the clock:

$$\Delta s = \int_A^B \left(-G_{(ij)} \frac{dX^{(i)}}{dX^{(0)}} \frac{dX^{(j)}}{dX^{(0)}} \right)^{1/2} dX^{(0)} \, . \tag{53}$$

7.1 Stationary Clock

For the case of a stationary clock, located at constant Fermi coordinate position $X^{(\alpha)}$, we obtain the relation between proper time τ and coordinate time $X^{(0)}$ by taking $dX^{(\alpha)} = 0$ in equation (44). The *rate* of proper time with respect to coordinate time is then given by

$$\frac{ds}{dX^{(0)}} = 1 + \zeta \left(X^{(0)}, X^{(\alpha)} \right) \, . \tag{54}$$

Equation (54) is a special case of the well-known result that the rate of proper time depends on the location of the clock [31]. For the simple case of circular motion of the observer with ζ given by equation (48), substitution in equation (54) and integration from $X_A^{(0)}$ to $X_B^{(0)}$ gives

$$\begin{aligned} \Delta s &= X_B^{(0)} - X_A^{(0)} - \frac{a\omega}{c} \left\{ X^{(1)} \left[\sin\left(\frac{\omega}{c} X_B^{(0)}\right) - \sin\left(\frac{\omega}{c} X_A^{(0)}\right) \right] \right. \\ &- \left. X^{(2)} \left[\cos\left(\frac{\omega}{c} X_B^{(0)}\right) - \cos\left(\frac{\omega}{c} X_A^{(0)}\right) \right] \right\} \, , \end{aligned} \tag{55}$$

where I have dropped terms of $O(\nu^4)$. Equation (55) shows that there is a complicated relation between elapsed proper time on a clock located at

Fermi coordinate position $(X^{(1)}, X^{(2)})$ and Fermi coordinate time $X^{(0)}$. For example, a clock on the $X^{(1)}$ axis (with $X^{(2)} = 0$) has a periodic time with respect to the observer's Fermi coordinate time, given by $\Delta s = X_B^{(0)} - (a\omega/c)\, X^{(1)} \sin(\omega X_B^{(0)}/c)$. The amplitude and sign of this periodicity depend on $X^{(1)}$, the clock's spatial Fermi coordinate. The initial conditions in equation (55) determine complicated phase relations between proper time and coordinate time. This complicated behavior can, of course, be understood if we consider the clock's complicated motion in the global Minkowski coordinate system.

7.2 Clock in Circular Motion

Now consider an ideal clock that is a distance b from the origin of Fermi coordinates and that executes circular motion in Fermi coordinates in the $X^{(1)}$-$X^{(2)}$ plane. I take the world line for this motion to be given by

$$X^{(1)} = b \cos\left(\frac{\Omega}{c} X^{(0)}\right) . \tag{56}$$

$$X^{(2)} = b \sin\left(\frac{\Omega}{c} X^{(0)}\right) , \tag{57}$$

$$X^{(3)} = 0 . \tag{58}$$

The clock moves in the $X^{(1)}$-$X^{(2)}$ plane, which coincides with the plane defined by the circular motion of the origin of Fermi coordinates (see fig. 2). Substitution of the satellite clock's world line into equation (53) leads to the following relation between elapsed proper time observed on the clock and Fermi coordinate time:

$$\begin{aligned}\Delta s = &\left[1 - \frac{1}{2}\left(\frac{b\Omega}{c}\right)^2\right]\left(X_B^{(0)} - X_A^{(0)}\right) \\ &- \frac{ab}{c}\frac{\omega^2}{\Omega - \omega}\left[\sin\left(\frac{\Omega - \omega}{c} X_B^{(0)}\right) - \sin\left(\frac{\Omega - \omega}{c} X_A^{(0)}\right)\right] .\end{aligned} \tag{59}$$

The *rate* of proper time with respect to coordinate time is

$$\frac{ds}{dX^{(0)}} = -\frac{1}{2}\left(\frac{b\Omega}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 ab \cos\left(\frac{\Omega - \omega}{c} X^{(0)}\right) . \tag{60}$$

The proper time has a constant rate offset from coordinate time, represented by the first term in equation (59). In addition, the proper time has a periodic component with respect to coordinate time, given by the second term in equation (59).

To demonstrate the magnitude of these effects, I use parameters that correspond to the Earth orbiting the sun and to GPS satellites orbiting the Earth [24]. I compute magnitudes of the following terms with the values shown in table 1:

$$\frac{1}{2}\left(\frac{b\Omega}{c}\right)^2 = 8.348 \times 10^{-11} , \tag{61}$$

$$\frac{1}{2}\left(\frac{b\Omega}{c}\right)^2 \times \frac{2\pi}{\Omega} = 3.597 \times 10^{-6}\ \mathrm{s} . \tag{62}$$

Equation (61) gives the constant rate offset of the moving clock with respect to Fermi coordinate time, due to time dilation. In the GPS, this term leads to a net slowing down of GPS clocks by approximately 7 μs per day [5], which results from multiplying the value given in equation (62) for a single revolution by a factor of 2 (since the GPS satellites make approximately two revolutions per Earth day).

The second term in equation (59) is a periodicity of the proper time with respect to Fermi coordinate time, with amplitude

$$\frac{ab}{c^2}\frac{\omega^2}{\Omega - \omega} = 12.05 \times 10^{-9}\ \mathrm{s} , \tag{63}$$

which corresponds to an amplitude in the rate of

$$ab\left(\frac{\omega}{c}\right)^2 = 1.755 \times 10^{-12} \ . \tag{64}$$

The proper time given by equation (59) is periodic with respect to Fermi coordinate time with the difference frequency $\Omega - \omega$, which is the difference frequency between satellite and observer rotations. This is a kinematic effect due to the type of coordinate system used (Fermi coordinates) and the fact that the observer (the Fermi coordinate frame) is moving along an arc of a circle, and therefore experiences an acceleration. Note that the periodic term vanishes when the observer angular velocity $\omega = 0$. If the observer were moving in a straight line and hence had zero acceleration, $w^i = 0$,

Table 1. Numerical values of constants.

Constant	Definition	Value
a	Earth orbital semi-major axis	1.496×10^{11} m
ω	Earth orbital angular velocity	$1.99238 \times 10^{-7}\ \mathrm{s}^{-1}$
b	GPS satellite semi-major axis	2.656177×10^{7} m
Ω	GPS satellite angular velocity	$1.45842 \times 10^{-4}\ \mathrm{s}^{-1}$
c	Vacuum speed of light	2.997924×10^{8} m/s

the periodic effect would also be absent, as can be seen from the general equation (49). Although these kinematic effects are present in the GPS, one must be careful in applying these results to GPS because gravitational effects have been neglected in this calculation and including them would significantly modify the results [17,21].

7.3 Sagnac-Like Effect for Portable Clocks

Sagnac demonstrated that there is a phase shift between two counterpropagating light beams on a rotating platform and that this phase shift depends on the angular frequency of rotation of the platform [27,28,30]. In the Fermi coordinates of the observer moving in a circle, there is a Sagnac-like effect for two portable clocks that are orbiting in opposite directions.

Consider a clock that moves eastward along a circle of radius b in the equatorial plane of the Fermi coordinates with world line given by equations (56) to (58) (see fig. 2). The speed of the clock in Fermi coordinates is $b\Omega/c$. Assuming that the clock begins its journey at point A at time $X_A^{(0)} = 0$ and ends it at $X_B^{(0)} = 2\pi c/\Omega$ (which corresponds to one revolution), the proper time that elapses on the clock is given by equation (59) as

$$\Delta s_+ = \left[1 - \frac{1}{2}\left(\frac{b\Omega}{c}\right)^2\right]\frac{2\pi}{\Omega}c + \frac{ab}{c}\frac{\omega^2}{\Omega - \omega}\sin\left(2\pi\frac{\omega}{\Omega}\right) . \tag{65}$$

The first term in equation (65) is the time dilation due to the speed $b\Omega/c$ of the clock. The second term depends on the path traversed.

Next, consider an identical clock that starts at point A at time $X_A^{(0)} = 0$ but moves westward. The world line of this clock is then given by equations (56) to (58) but with $\Omega = -|\Omega|$. The proper time on this clock is given by

$$\Delta s_- = \left[1 - \frac{1}{2}\left(\frac{b\Omega}{c}\right)^2\right]\frac{2\pi}{|\Omega|}c - \frac{ab}{c}\frac{\omega^2}{|\Omega| + \omega}\sin\left(2\pi\frac{\omega}{|\Omega|}\right) . \tag{66}$$

When the east-moving and west-moving clocks have each made one revolution and they have returned to their starting positions at point A, the difference in their times is given by $\Delta\tau_+ - \Delta\tau_- = (\Delta s_+ - \Delta s_-)/c$. For the case $b \ll a$ and $\omega \ll |\Omega|$ (as is the case with Earth satellites), the difference of proper time on the two clocks is given by

$$\begin{aligned} \tau_+ - \tau_- &= 2\frac{b}{a}\left(\frac{a\omega}{c}\right)^2 \frac{|\Omega|}{\Omega^2 - \omega^2}\sin\left(2\pi\frac{\omega}{|\Omega|}\right) \\ &= 4\pi\frac{b}{a}\left(\frac{a\omega}{c}\right)^2\frac{\omega}{\Omega^2} + O\left(\frac{\omega^2}{\Omega^2}\right) . \end{aligned} \tag{67}$$

where in the last line I have dropped terms of $O(\omega^2/\Omega^2)$. This Sagnac-type effect is due to the noninertial nature of the Fermi coordinate system. The origin of this effect can be understood from the point of view of the global inertial coordinate system. If the Fermi coordinate origin moved along a straight line in 3-d space (rather than a circle), then there would be no acceleration and $\zeta = 0$. In this case, the east-moving and west-moving clocks would have zero time difference after one revolution. However, since the origin of the Fermi coordinates moves along a helix, there is an acceleration that picks out a direction in the Fermi coordinate space-time, and the symmetry between the two clock world lines is broken. Consequently, the elapsed proper time is different on the two clocks when they are compared after one revolution. Note that the time difference on the two clocks increases with increasing angular velocity of rotation ω, analogous to the effect observed by Sagnac [27]. This increase arises because the breaking of symmetry is due to the magnitude of the acceleration $w^{(\beta)}$ (see eq (49)). For values of the parameters appropriate to GPS satellites, given in table 1, I find the magnitude of the time difference on the two clocks after one revolution to be

$$\tau_+ - \tau_- = 2.066 \times 10^{-10}\,\mathrm{s}\,. \tag{68}$$

This is a small effect, which vanishes when the origin of Fermi coordinates moves along a straight line or when $\omega = 0$.

8. Coordinate Speed of Light

The 3-d space of Minkowski space-time is homogeneous and isotropic, and the speed of light is independent of position and direction of propagation. However, for the case of Fermi coordinates of an observer moving in a circle, the symmetry of the 3-d space is broken by an acceleration, and the coordinate speed of light depends on position in the 3-d space.

For a general space-time metric g_{ij}, the coordinate speed of light v_c in the direction of a unit vector e^α is given by Möller [29]:

$$v_c = \frac{\sqrt{-g_{00}}}{1 + \gamma_\alpha e^\alpha}\, c\,. \tag{69}$$

where

$$\gamma_\alpha = \frac{g_{0\alpha}}{\sqrt{-g_{00}}} \tag{70}$$

and the unit vector satisfies the relation $\gamma_{\alpha\beta}\, e^\alpha\, e^\beta = 1$, where the 3-d spatial metric is given by [31]

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}\,. \tag{71}$$

Equation (69) shows that for a general space-time metric, the speed of light depends on direction e^α and on position and time, through the metric components. A nonzero off-diagonal metric component $g_{0\alpha}$ leads to an anisotropy (directional dependence) of the coordinate speed of light. The metric for Fermi coordinates given in equation (46) has vanishing off-diagonal terms, $\gamma_\alpha = 0$, and consequently the coordinate speed of light is isotropic. However, the metric component $G_{(00)}$ has a nontrivial position and time dependence, and equation (69) gives the coordinate speed of light as $v_c = c\,(1 + \zeta)$, where ζ is a function of Fermi coordinate position and time, given by equation (48). Therefore, the coordinate dependence of the speed of light is

$$\frac{\Delta c}{c} = -\frac{\nu^2}{a}\left[X^{(1)} \cos\left(\frac{\omega}{c} X^{(0)}\right) + X^{(2)} \sin\left(\frac{\omega}{c} X^{(0)}\right)\right] + O(\nu^4)\,. \tag{72}$$

This variable speed of light is a kinematic effect due to the noninertial nature of the Fermi coordinate system. Note that at any given position

$\mathbf{X} = \{X^{(1)}, X^{(2)}, X^{(3)}\}$, the speed of light depends on time in a periodic way, with the orbital period of the reference frame, ω. This expression can be rewritten as

$$\frac{\Delta c}{c} = a\left(\frac{\omega}{c}\right)^2 \mathbf{X}\cdot\hat{\mathbf{n}} + O(\nu^4)\,, \tag{73}$$

where the unit vector $\hat{\mathbf{n}} = \left(-\cos\left(\frac{\omega}{c}X^{(0)}\right), -\sin\left(\frac{\omega}{c}X^{(0)}\right), 0\right)$ is in the observer's orbital plane and points in the direction of acceleration, toward the center of the circle (see fig. 3). As the Fermi reference frame moves around the circle, the vector $\hat{\mathbf{n}}$ always points toward the circle's center in Fermi coordinates. At any given time $X^{(0)}$, equation (73) shows that to second order in ν, in the plane given by

$$\mathbf{X}\cdot\hat{\mathbf{n}} = 0\,, \tag{74}$$

the speed of light does not differ from its vacuum inertial frame value c. In the 3-d half-space containing positions $\mathbf{X}$ closer to the center of the circle, the speed of light is increased. In contrast, in the 3-d half-space containing positions $\mathbf{X}$ that are farther from the center of the circle, the speed of light is decreased. As time $X^{(0)}$ increases, the dividing plane, given by equation (74), rotates at angular frequency ω around the circle, with the origin of Fermi coordinates remaining the point at which the plane is tangent to the circle.

The magnitude of the inhomogeneous variation in the speed of light in equation (72) is small. For the kinematics of Earth satellites such as GPS,

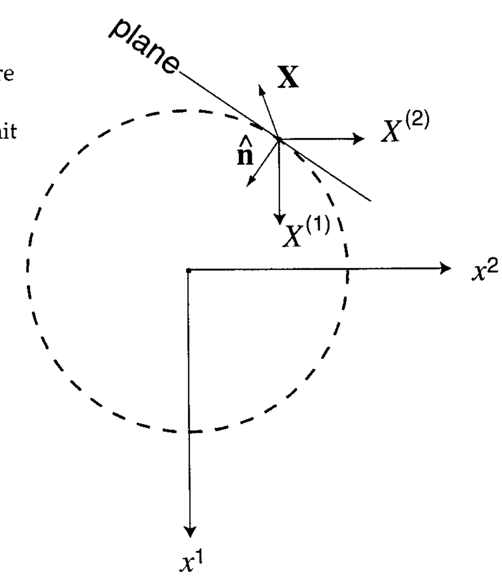


Figure 3. Global inertial coordinate axes x^i and Fermi coordinates $X^{(a)}$ are shown in relation to coordinate position $\mathbf{X}$, unit vector $\hat{\mathbf{n}}$, and plane $\mathbf{X}\cdot\hat{\mathbf{n}} = 0$, at a given coordinate time $X^{(0)}$.

with the numbers given in table 1 and the coordinates $X^{(1)}$, $X^{(2)} \sim b$, the effect has a magnitude

$$\frac{\nu^2}{a} b = 1.755 \times 10^{-12} \tag{75}$$

and has a complicated position and time dependence, given by equation (72). Recently, GPS data have been used [32] in obtaining an upper bound on the *anisotropy* of the speed of light: $\Delta c/c < 10^{-9}$. Given the present accuracy of GPS [24], the small inhomogeneity of the speed of light given by equation (72) is probably not measurable.

9. Summary

In this report I present a detailed calculation of the coordinate transformation from global Minkowski coordinates to the Fermi coordinates of an observer moving in a plane around a circle in 3-d space. The transformation is valid for arbitrarily long coordinate time (not a power series in time). The observer's axes are Fermi-Walker transported along his helical world line. I considered two cases: stationary clocks (with respect to Fermi coordinates) and satellite clocks in circular orbit with respect to Fermi coordinates. The proper time interval between two events on the stationary clock, given by equation (55), is a complicated periodic function of the clock's Fermi coordinate time and coordinate position in the plane perpendicular to the rotation axis. I considered next a clock that orbits the Fermi coordinate origin (similar to a satellite) in the equatorial plane of rotation and computed the relation of proper time to coordinate time. I find that the proper time on the clock has time dilation (constant rate offset) plus a periodic dependence on Fermi coordinate time at the difference frequency, $\Omega - \omega$, between satellite clock orbital frequency Ω and Fermi coordinate frame orbital frequency ω.

Next, I looked at the time difference of two counter-orbiting clocks in the equatorial plane of rotation, one west-moving and the other east-moving. After one revolution, the difference in proper time on the clocks shows a small Sagnac-like effect, increasing with the angular frequency of revolution of the Fermi frame, ω, given by equation (67) .

Finally, I computed the coordinate speed of light in the Fermi coordinates. The speed of light is isotropic but depends on Fermi coordinate position and time. Equation (73) gives the change in the speed of light from its vacuum value in an inertial frame. In order to illustrate the magnitude of these effects, I have used numbers relevant to GPS satellite orbits and Earth orbital frequency.

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Appendix. Conventions

I use Roman indices, $i, a = 0, 1, 2, 3$, on space-time coordinates x^i and Greek indices, $\alpha = 1, 2, 3$, for spatial coordinates. The proper time interval between two events in Minkowski space-time is $d\tau = ds/c$, where

$$ds^2 = -g_{ij}\, dx^i\, dx^j \; ; \tag{A-1}$$

here, $g_{00} = -1$, $g_{\alpha\beta} = \delta_{\alpha\beta}$, and $g_{0\alpha} = 0$. In Fermi coordinates, $X^{(a)}$, $a = 0, 1, 2, 3$, the interval is

$$ds^2 = -G_{(ab)}\, dX^{(a)}\, dX^{(b)} \; . \tag{A-2}$$

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13. ABSTRACT *(Maximum 200 words)*

Space-time coordinate transformations valid for arbitrarily long coordinate time are derived from global Minkowski coordinates to the Fermi coordinates of an observer moving in a circle in three-dimensional space. The metric for the Fermi coordinates is calculated directly from the tensor transformation rule. The Fermi coordinates are used in an examination (from the observer's reference frame) of the detailed behavior of ideal clocks.
A complicated relation exists between Fermi coordinate time and proper time on stationary clocks (in the Fermi frame) and between proper time on satellite clocks that orbit the observer. For portable clocks that orbit the Fermi coordinate origin, an orbital Sagnac-like effect exists. The coordinate speed of light is isotropic but is periodic in time and varies with Fermi coordinate position. In a numerical illustration of the magnitudes of these kinematic effects, this report uses parameters relevant to clocks carried aboard satellites orbiting the Earth.

14. SUBJECT TERMS
GPS, navigation, time transfer, timing, relativity clocks, Fermi coordinates, reference frame, Sagnac effect

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