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THREE-DIMENSIONAL, UNSTEADY-LIFT PROBLEMS IN HIGH-SPEED FLIGHT - BASIC CONCEPTS

By Harvard Lomax, Max. A. Heaslet, and Franklyn B. Fuller

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SUMMARY

The problem of the build-up of lift on two- and three-dimensional wings flying at high speeds is discussed as a boundary-value problem for the classical wave equation. Kirchhoff's formula is applied so that the analysis is reduced, just as in the steady state, to an investigation of sources and doublets. Some simple applications of this method are considered, including the determination of the starting lift of a three-dimensional wing and the potential functions for some types of unsteady vortex motion.

INTRODUCTION

The usual idealizations introduced in the development of linearized aerodynamic theory describe a frictionless, perfectly elastic, model fluid. As is well known, the effect of small disturbances in such a fluid can be analyzed by means of the familiar wave equation which, in terms of the perturbation velocity potential φ, can be written

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \frac{1}{a_o^2} \varphi_{t't'} \tag{1}$$

where

a_o speed of pressure propagation

t' time

x,y,z spatial coordinates

Equation (1) applies to a flow field which is stationary at large distances from the disturbance region; furthermore, the axial system is stationary relative to the fluid infinitely distant so that, if a moving wing is being analyzed, the wing moves with respect to the x,y,z axes.

It is possible to use a transformation which will bring the wing to rest with respect to a new axial system, but this process leads generally to a different equation for the perturbation potential. The case of a wing moving at subsonic speed constitutes an exception, for here the Lorentz transformation, familiar in modern physics, will fix the origin of the axial system on the wing while leaving the wave equation invariant. Such is not the case when the wing travels at supersonic speeds, so that, if it is desired to fix the axial system to the wing, the wave equation must be abandoned. This latter alternative has been the more extensively studied in aerodynamics, but, because of the large body of knowledge concerning equation (1) which is available in mathematical physics, the present report is based upon the wave-equation approach to problems of unsteady motion.

The solution to equation (1) known as Kirchhoff's formula (see reference 1) is found to be of considerable use in unsteady-motion problems involving thin wings with supersonic edges. The problem is reduced to one of summing elementary solutions, analogous to sources and doublets in steady flow, over a region determined by the position of the wing as well as its traversed path. The theoretical development leads naturally to the concepts (defined later) of, first, inverse sound waves, which have a counterpart in the Mach forecones used in steady-state wing theory; second, acoustic plan forms; and, third, homogeneous flow, which reduces in part to the familiar conical flow as the wing approaches a steady supersonic velocity.

In connection with boundary-value problems involving noninteracting surfaces, a theorem will be given which makes it possible to build up a three-dimensional supersonic-edged wing by superposition of elements of two-dimensional wings. This result is an extension to unsteady flow of a theorem given previously by Lagerstrom and Van Dyke (reference 2). By means of this theorem, problems involving supersonic-edged wings performing any prescribed maneuver can be solved, provided only that the behavior of a two-dimensional wing executing the same maneuver is known. Thus, the relatively large amount of material available for two-dimensional unsteady motion at supersonic speeds can be carried over directly to the supersonic-edged wing.

In the last part of the report, some simple applications of the general methods are considered, including the determination of the starting lift of a three-dimensional wing and the potential functions for some types of unsteady vortex motion.

LIST OF IMPORTANT SYMBOLS

a_o free-stream speed of sound

c_o wing root chord, or maximum chord of a wing

c wing chord

c_{l_α} section lift-curve slope

C_L lift coefficient

C_{l_r} coefficient of rolling moment due to yawing $\left[\frac{\partial C_l}{\partial(rb/2V_o)}\right]$

L lift

M_o free-stream Mach number

$\frac{\Delta p}{q}$ loading coefficient (pressure on the lower surface minus pressure on the upper surface divided by dynamic pressure)

q free-stream dynamic pressure $\left(\frac{1}{2}\rho_o V_o^2\right)$

t' time

t $a_o t'$

V_o free-stream velocity

w vertical component of perturbation velocity

x,y,z Cartesian coordinates

α angle of attack

β $\sqrt{|1 - M_o^2|}$

λ slope of stream surface in free-stream direction $\left(\frac{w}{V_o}\right)$

ρ_o free-stream density

φ perturbation velocity potential

Δ discontinuity in the quantity in question across the $z = 0$ plane

Subscript

u value on upper surface ($z = 0$ plane) of a wing

THEORETICAL DEVELOPMENTS

Basic Equation and Solutions

The partial differential equation which forms the basis of the present study has been written as equation (1). It is possible to put this equation in a more convenient form by introducing the notation

$$t = a_o t' \tag{2}$$

so that the dimension of t is length just as are the dimensions of the geometric variables x, y, and z. Equation (2), together with equation (1), yields the canonical form of the wave equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \varphi_{tt} \tag{3}$$

and it is this form which will be considered.

The first task is to study the relation between the motion of the wing and the coordinate system. As has already been mentioned, equation (3) is valid for a flow field produced by a wing moving relative to a fixed coordinate system. It is pertinent to consider the possibility of finding a transformation which will (1) fix the origin of the axial system on the wing and, at the same time, (2) retain the wave equation as the governing equation of the flow. Certainly the first of these requirements is simple to fulfill if the second is neglected. However, because of the simplicity of the wave equation and, what is more important, because of the great amount of developmental study that has been expended on it, the second requirement is not without justification.

The following transformation (known in relativity theory as a Lorentz transformation, or as a hyperbolic rotation)

$$\left.\begin{aligned}
\xi &= \frac{x - M_o t}{\sqrt{1 - M_o^2}} \\
\eta &= y \\
\zeta &= z \\
\tau &= \frac{t - M_o x}{\sqrt{1 - M_o^2}}
\end{aligned}\right\} \tag{4}$$

where $M_o = \frac{V_o}{a_o} < 1$, will satisfy both conditions (1) and (2). For example, suppose that the wing is moving along the x axis with velocity V_o. Application of equation (4) to equation (3) makes the origin of the new axial system also travel along the x axis with a velocity V_o. This is seen to be so since ξ is always zero when $x = V_o t' = M_o t$. Hence, the ξ axis is "fixed" on the wing. As to the second condition, a straightforward exercise in partial differentiation yields

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} + \varphi_{\zeta\zeta} = \varphi_{\tau\tau} \tag{5}$$

so that, in going from x,y,z to ξ,η,ζ space, the wave equation remains invariant; consequently, both the requirements mentioned have been fulfilled.

It is instructive to consider briefly the consequences of applying the Lorentz transformation. Although the wave equation remains invariant, such physical quantities as length and pressure do not. For example, a wing with a chord c_o in the x,y,z space has, according to equation (4), a chord $c_o/\sqrt{1-M_o^2}$ in the ξ,η,ζ space. Furthermore, the loading coefficient which, on the basis of linearized theory, is given in the x,y,z,t space by

$$\frac{\Delta p}{q} = \frac{4}{V_o M_o}\frac{\partial\varphi}{\partial t} \tag{6}$$

becomes for the ξ, η, ζ, τ space

$$\frac{\Delta p}{q} = \frac{4}{V_o\sqrt{1-M_o^2}}\left(\frac{1}{M_o}\frac{\partial\varphi}{\partial\tau} - \frac{\partial\varphi}{\partial\xi}\right) \tag{7}$$

If the wing motion is steady and there are no transient effects, equations (5) and (7) are independent of time and together with the resulting length transformations become

$$\left.\begin{aligned} &\varphi_{\xi\xi} + \varphi_{\eta\eta} + \varphi_{\zeta\zeta} = 0 \\ &\frac{\Delta p}{q} = -\frac{4\varphi_\xi}{V_o\sqrt{1-M_o^2}} \\ &\xi = \frac{x}{\sqrt{1-M_o^2}}, \ \eta = y, \ \zeta = z \end{aligned}\right\} \tag{8}$$

These are immediately recognized to be, respectively, Laplace's equation for incompressible flow and the familiar Prandtl-Glauert compressibility corrections.[1]

The preceding discussion has an important qualification, however, in the fact that the velocity of the moving axial system cannot exceed the speed of sound. A glance at equation (4) serves to verify this statement since that equation shows M_o must be less than 1 in order that ξ and τ be real for real x and t. In fact, it has been shown that there is no transformation which will fix the moving axial system in a wing traveling at a uniform supersonic speed away from the original fixed axes and still keep the wave equation invariant. Therefore, for analyzing a wing in supersonic flight, it is necessary to abandon one of the two proposed requirements: either the axial system cannot be fixed in the wing, or the field equation must be modified. The latter of these two alternatives has been studied by several authors (e.g., references 3, 4, 5, 6) but it is the former which will be considered in the present analysis. Further, since the axes cannot be made to travel as fast as the wing, they will not be made to move at all and equation (3) will be adopted throughout as the basic equation.

Having decided upon the form of the partial differential equation, the boundary conditions must next be established. For any given time these conditions are similar to those studied in steady-state thin-airfoil problems; namely, either that the given slope of the wing surface is proportional to the vertical induced velocity φ_z over the region occupied by the wing in the $z = 0$ plane,[2] or that the prescribed surface pressure is proportional to the timewise gradient φ_t in velocity potential over the same region. The addition of time simply means that this region moves about in the $z = 0$ plane in conformity with the known direction and velocity of the wing.

The additional condition is imposed that the induced velocities fall to zero on the surface which is formed by the envelopes of the spherical sound waves originating from the surface of the wing.

The solution to equation (3), subject to the boundary conditions just mentioned, can be expressed by a formula which may be regarded as

[1]The equation $\xi = x/\sqrt{1-M_o^2}$ would first read $\xi = (x-M_o t_o)/\sqrt{1-M_o^2}$ where t_o is a constant representing the time required for the motion to reach its steady state. However, the x coordinate can always be translated to any fixed position without affecting any of the equations for potential, loading, etc. Such a translation is assumed to have been made in equation (8).

[2]The $z = 0$ plane is assumed to be the "plane of the wing"; that is, if the angle of attack were zero and the wing had no thickness it would lie entirely in the $z = 0$ plane.

requiring either the evaluation of a double integral or the solution of a double-integral equation, depending upon whether a boundary-value problem of first or second kind is considered. This solution is known as Kirchhoff's formula. (See reference 1.) It may be written in a form convenient for aerodynamic applications as follows:

$$\varphi = -\frac{1}{4\pi}\iint_{S_a}\left[\frac{1}{r_o}\Delta\frac{\partial}{\partial z_1}\varphi(x_1,y_1,z_1,t-r_o) - \Delta\frac{\partial}{\partial z_1}\frac{\varphi(x_1,y_1,0,t-r)}{r}\right]dS \quad (9)$$

where the Δ indicates the jump (value on the upper surface minus value on the lower surface when applied to φ or $\frac{\partial\varphi}{\partial z}$) of the function in passing through the $z_1=0$ plane, $r = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$, $r_o=\sqrt{(x-x_1)^2+(y-y_1)^2+z^2}$, and where the area of integration S_a will be discussed in more detail later.

The terms in the integrand of equation (9) can be shortened by introducing the following notation:

$$\frac{1}{r_o}\Delta\frac{\partial}{\partial z_1}\varphi(x_1,y_1,z_1,t-r_o) = \frac{1}{r_o}\Delta\left[\frac{\partial\varphi}{\partial z_1}\right]_r \quad (10a)$$

$$\Delta\frac{\partial}{\partial z_1}\frac{\varphi(x_1,y_1,0,t-r)}{r} = \Delta\left[\frac{\partial r}{\partial z_1}\frac{\partial}{\partial r}\frac{\varphi(x_1,y_1,0,t-r)}{r}\right] = -\frac{\partial r_o}{\partial z}\Delta\frac{\partial}{\partial r}\left[\frac{\varphi}{r}\right] \quad (10b)$$

In this notation, the subscript r in equation (10a) means that r is to be held constant in the differentiation, and the prefix Δ obviates the necessity of indicating that the functions considered are to be evaluated for $z_1=0$, since it indicates that the difference of the values of the function across the $z_1=0$ plane is to be taken. The right-hand side of equation (10a) can be recognized as a term representing a source located in the $z_1=0$ plane, and the right-hand side of equation (10b) is seen to represent a doublet located in and with axis normal to the $z_1=0$ plane. The brackets [ ] about the functions in equations (10a) and (10b) have a special meaning which is defined in the following way: if f is a function whose value at a fixed point P depends upon the coordinates x_1,y_1,z_1,t of a moving point Q, so that

$$f = f(x_1,y_1,z_1,t)$$

then

$$[f] = f(x_1,y_1,z_1,t-r) \quad (11)$$

where r is the distance from P to Q. As an example, consider the potential φ at a point P due to a moving source, the location of which at any time is Q. Then φ satisfies the condition just mentioned that it depends on the coordinates x_1,y_1,z_1,t of Q. The brackets [ ] indicate that the potential $[\varphi]$ depends not upon the source strength now at "time" t, but rather upon the source strength that existed "time" r ago.[3] For convenience, $[\varphi]$ is referred to as the retarded value of φ.

The expression for a doublet, equation (10b), is usually expanded as follows:

$$-\frac{\partial r_0}{\partial z}\Delta\frac{\partial}{\partial r}\left[\frac{\varphi}{r}\right] = -\frac{\partial r_0}{\partial z}\frac{\partial}{\partial r_0}\left(\frac{1}{r_0}\right)[\Delta\varphi] - \frac{\partial r_0}{\partial z}\frac{1}{r_0}\left[\Delta\frac{\partial\varphi}{\partial r}\right]$$

$$= -\frac{\partial}{\partial z}\left(\frac{1}{r_0}\right)[\Delta\varphi] + \frac{1}{r_0}\frac{\partial r_0}{\partial z}\left[\Delta\frac{\partial\varphi}{\partial t}\right] \tag{12}$$

Finally, equation (9) becomes

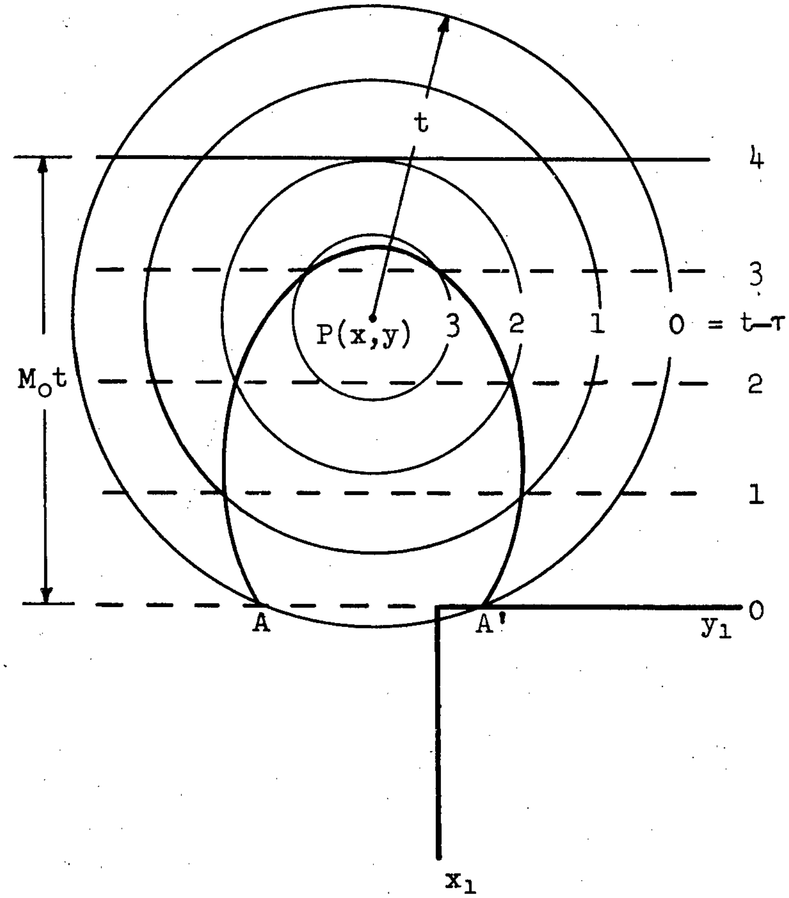
$$\varphi = -\frac{1}{4\pi}\int\int_{S_a}\left\{\frac{1}{r_0}\left[\frac{\partial\Delta\varphi}{\partial z_1}\right] + [\Delta\varphi]\frac{\partial}{\partial z}\left(\frac{1}{r_0}\right) - \frac{1}{r_0}\frac{\partial r_0}{\partial z}\left[\frac{\partial\Delta\varphi}{\partial t}\right]\right\}dS \tag{13}$$

The application of equation (13) awaits only a discussion of the area S_a over which the integration is to be made. This discussion is important enough, however, to merit consideration in some detail and will be given in the following section.

The Acoustic Plan Form

Suppose that a line of sources is placed along the y axis and that the strength of these sources is zero for $t < 0$. At $t = 0$ they are "turned on" and, at the same time, start moving along the negative x_1 axis with the velocity V_0. After time t' has passed, the source line has traveled a distance $M_0 t$ as shown in the accompanying sketch (which is drawn for the case $M_0 > 1$).

[3] Quotes are used around the word time since the dimension of t is actually length, not time. It is convenient, however, to refer to t as "time," and, since the actual value of time is simply t multiplied by the constant a_0, this should cause no confusion.



Suppose next that there are two sensing elements, or detectors, placed at the point $P(x,y)$ located somewhere ahead of the y_1 axis; one of these detectors is responsive to light and the other to sound. Now, the light detector at any given time will show the sources lying in a straight line just as they would appear visually at each particular instant. The situation is entirely different, however, for the sound detector. First it is necessary to understand the nature of a spherical sound wave. Such a wave travels outward from its origin at a velocity a_0, so that in the time t' it has traveled a distance t. Before the wave reaches a point, the point is completely unaware[4] of its existence and, further, after the wave has passed, the point remains subsequently unaware of its existence. Hence the only points disturbed by the wave are those momentarily on the spherical surface itself. (In this connection see reference 1, pp. 1 ff.)

[4]Unaware is used in the sense that an instrument will record no change in any of the physical properties of the air at the point in question.

The sound detector, therefore, can only "hear" sources which are so located that their spherical sound waves are just, at the given instant of time, reaching the detector. The locus of all the points which, at a time t ago, emitted sound waves that are just now reaching the point $P(x,y)$ is itself a sphere and for convenience this sphere will be referred to as an inverse sound wave.[5] The traces of these inverse sound waves in the $z = 0$ plane are drawn in the sketch as concentric circles about the point $P(x,y)$. The intersection of an inverse sound wave of radius $t-\tau$ with the line representing the position of the sources at a time $t'-\tau'$ ago gives the position of the sources which are just now signaling their presence to the sound detector at P. For example, when the source line started, it was lying along the y_1 axis. With reference to the present time t this was a_0t' removed. Hence the intersection of the circle about P of radius t with the y_1 axis fixes the two points A and A', the spherical sound waves of which are now reaching P. A continuation of this process yields, for the locus of all points from which waves emanated that are just now touching the point P, the part of the ellipse shown in the sketch.

In the sense that the light detector, because of the very large velocity of light, is "seeing" a straight line of sources, the sound detector, because of the relatively slow velocity of sound, is "hearing" an elliptic line of sources. Extending this concept to include a sheet of sources distributed over the surface of the wing, one can refer to the outline of that part of the wing which generates disturbances which can be measured by the light detector as the plan form (i.e., the visual plan form), and to the outline of that part of the wing which affects the sound detector as the acoustic plan form. In a mathematical sense, the acoustic plan form is the area S over which the integration of equation (13) is to be made.

The equation for the acoustic plan form can be formulated by means of the two equations

$$(x-x_1)^2 + (y-y_1)^2 + z^2 = (t-\tau)^2 \tag{14}$$

$$f(y_1,x_1,\tau) = 0 \tag{15}$$

where

x,y,z coordinates of the point at which the induced effects are to be measured

[5] The inverse sound wave has for its analogue in steady lifting-surface theory the Mach forecone. In that theory a disturbance outside the Mach forecone cannot affect the values of the induced velocities at the point where they are being measured. Similarly, in the present study, a source located outside the inverse sound wave of radius t cannot affect the values of any measurement made at the point P.

x_1,y_1 variable points of the sources

t "time" now

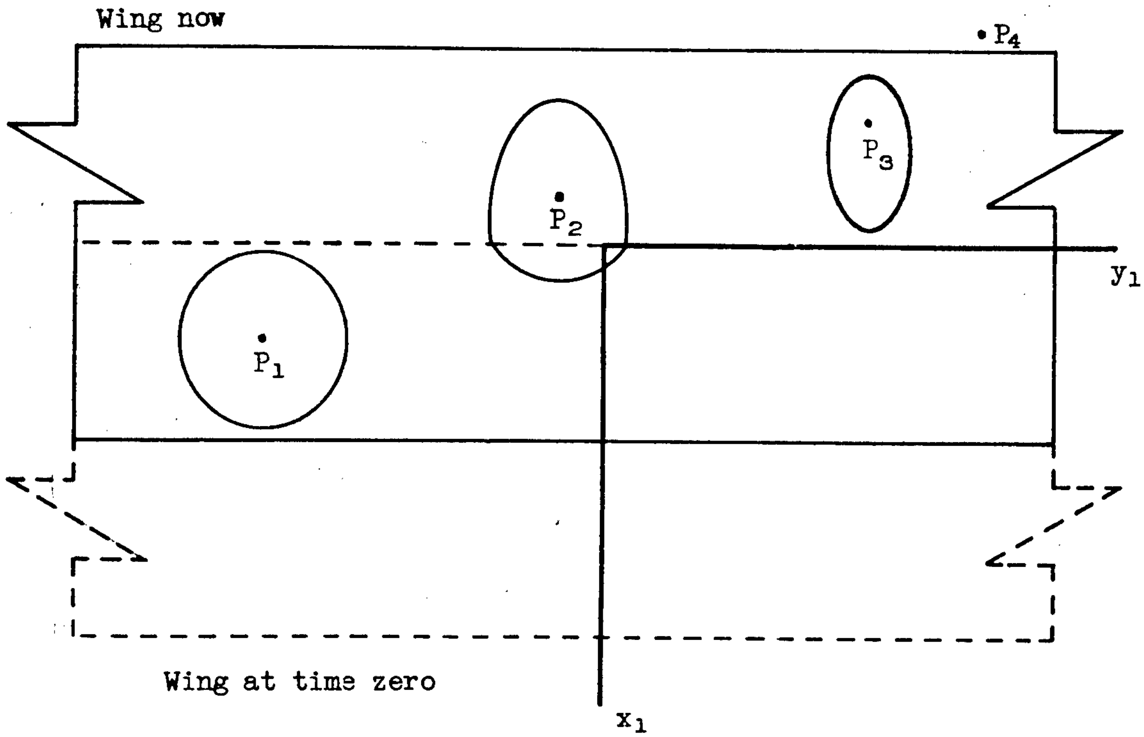
$t-\tau$ "time" ago

Equation (14) is that of the inverse sound waves and equation (15) represents the position of the visual plan form at a time $t-\tau$ past. It is necessary to include the region behind the wing covered by the vortex wake as part of the visual plan form. In case the vorticity in the wake vanishes, as in the thickness problem, the wake may still be considered as part of the visual plan form, but the strength of the source-doublet distribution over that part of the acoustic plan form corresponding to the wake will vanish. If $t^2-z^2<0$, the acoustic plan form does not exist since, for such a case, no source on the x_1,y_1 plane has had time to transmit its effect to the point x,y,z. On the other hand, if the circle (in the x_1,y_1 plane) given by the equation

$$(x-x_1)^2 + (y-y_1)^2 = t^2-z^2$$

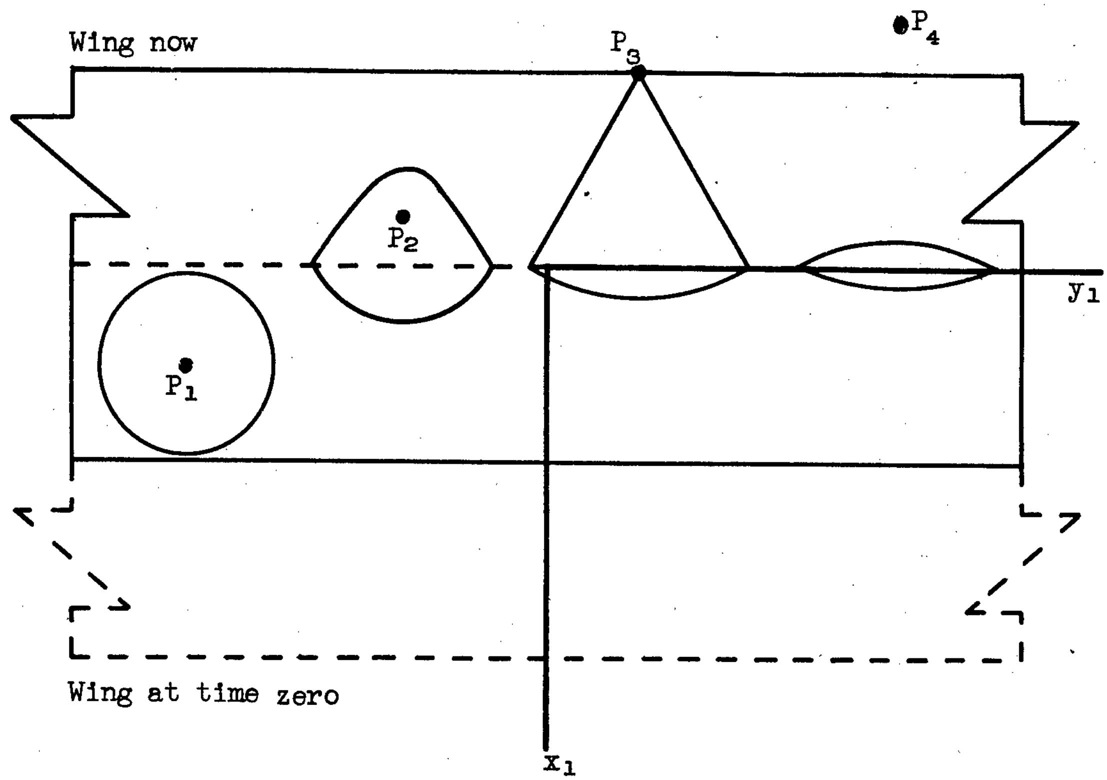
lies entirely within the area occupied by the visual plan form of the wing at the beginning of the motion, the acoustic plan form is just this circle itself. For any other situation the acoustic plan form is formed in part, or in whole, by the curve found by eliminating τ from equations (14) and (15) and in part, or not at all, by an arc of the circle $(x-x_1)^2 + (y-y_1)^2 = t^2-z^2$. One of the principal advantages of the acoustic-plan-form concept arises in the study of problems involving source or doublet distributions having constant strength. In such cases, the retarded values of the potential and its gradient appearing in equation (13) are constant and can be taken outside the integral signs. The problem is thereby reduced to the integration of a simple geometric variable over S_a.

A few examples will serve to fix the idea of the acoustic plan form. Consider first a two-dimensional, unswept wing moving at a constant supersonic speed in the negative x_1 direction. At time zero the leading edge of the wing was along the y_1 axis and now, at time equal to t', the wing has moved so that the leading edge coincides with the line $x_1 = -M_0t$. Choose three points that are now lying on the wing. Let one point have its x coordinate in the range $c_0-\sqrt{t^2-z^2} \geq x \geq \sqrt{t^2-z^2}$ (where c_0 is the chord of the wing), the second in the interval $\sqrt{t^2-z^2} > x > -\sqrt{t^2-z^2}$, and the third is the range $-\sqrt{t^2-z^2} \geq x \geq -M_0t$. Designating these points by P_1,P_2, and P_3 (see sketch), it can be shown that their acoustic plan forms are, respectively, a complete circle, a part circle and part ellipse, and a complete ellipse.



The points P are at the centers of the circles and at focal points of the ellipses. Since, moreover, the circular plan form about P_1 receives no signals from sources on the leading or trailing edge, conditions at P_1 are consequently completely independent of the actual (visual) plan form of the wing. The elliptical plan form about P_3, on the other hand, depends entirely on the shape of the leading edge; and finally the mixed plan form about P_2 is in certain regions (the circular portion) independent of the leading edge, and in other regions (the elliptic portion) entirely dependent upon it. Since the wing is traveling at supersonic speeds, the trailing edge and vortex wake can have no effect on the measurements taken on the wing and, in the same way, a point ahead of the wing leading edge, P_4 in the sketch, is undisturbed.

Next consider a wing moving at a constant subsonic speed in the negative x_1 direction. As before, the leading edge was on the y_1 axis at $t' = 0$ and has traveled a distance $-M_0t$. Choose now three points P_1, P_2, and P_3 on the wing and unaffected by the wing tips. The acoustic plan forms for these points are combinations of circles and hyperbolas as contrasted with the circle-ellipse combination in the supersonic case. Just as in the supersonic case, however, there is a certain region represented by P_1 in which the acoustic plan form is a complete circle and is independent of the visual shape of the wing (see sketch).



Point P_2 is surrounded by a plan form which is part hyperbolic and part circular, the point itself being the center of the circle and the focus of the hyperbola. Point P_3 is a limiting value of P_2; it lies on the leading edge of the wing and the hyperbolic sides of its plan form have degenerated into straight lines. Finally, P_4 lies ahead of the wing; its plan form is still a combination of a hyperbola and a circle, but P_4 is now the focal point lying ahead of the hyperbolic branch used.

The sketch was constructed so that the portion of the visual plan form behind the trailing edge had no effect on the potential at the various points P_1, etc. If these points had been chosen at positions where the wake could signal its effect, one of two acoustic configurations would result. First, if the wing is symmetric about the $z = 0$ plane, no lift is developed and the vorticity in the wake is zero so that the visual plan form need not include the wake, but effectively ends at the trailing edge. In this case, the leading edge of the acoustic plan form is then determined as before, while its modified trailing edge may be made up, in part, of circular arcs formed by the primary wave and, in part, by an arc of the hyperbola formed by the (acoustic) intersection of the straight visual trailing edge with the primary wave (such an arc being identical with the leading edge of the acoustic plan form but displaced backwards). On the other hand, if the wing has no thickness but is inclined to the free stream, it develops lift and the vorticity in the wake does not vanish; the acoustic plan form has a trailing edge made up entirely of an arc of the primary inverse sound wave. The space between this arc and the acoustic trace of the visual trailing edge is covered by a sheet of doublets, the strength of which is determined by the vorticity distribution of the vortex wake.

It is interesting to notice the conversion of terminology which arises in the analysis of unsteady lift problems. In the study of steady-state wings, it is customary (because of the nature of the governing partial differential equation) to speak of the subsonic problems as elliptic and the supersonic problems as hyperbolic. Yet the acoustic plan forms just presented involved ellipses for the supersonic wing and hyperbolas for the subsonic case.

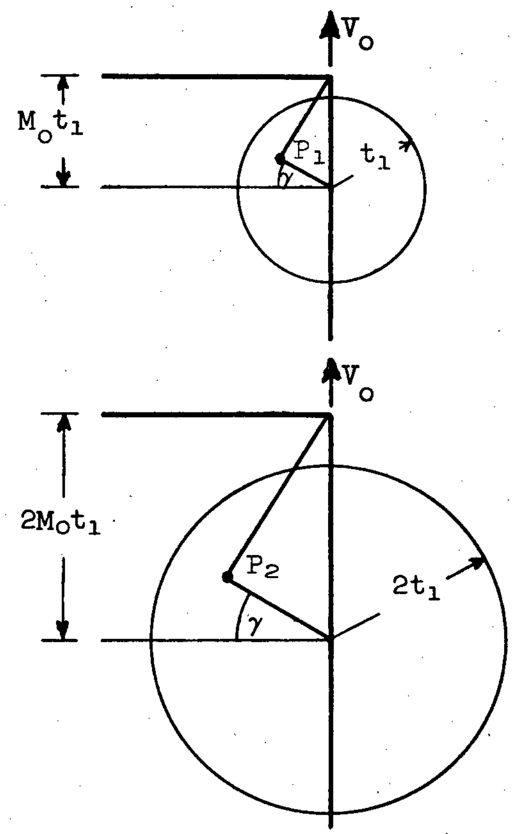
To complete the remark, it can be observed that when the velocity of the wing is sonic the steady-state partial differential equation becomes parabolic and, in this case, the acoustic plan form of a straight-edged wing also involves parabolas. When the leading edge is linear and normal to the stream direction, the eccentricity of the conic sections bounding the acoustic plan form for a point on the wing is equal to $1/M_o$. From this relation it is apparent that for M_o less than, equal to, and greater than 1 the sections are, respectively, hyperbolas, parabolas, and ellipses. As might be presumed, the value of the eccentricity satisfies simple sweep theory so that, for an infinitely long straight leading edge, the eccentricity of the acoustic plan form is $1/M_o \cos \Lambda$ where Λ is the angle of sweepback. The principal axes of the conic sections are always normal and parallel to straight leading edges.

Homogeneous Boundary-Value Problems

Kirchhoff's solution to the wave equation can be applied to arbitrary wing plan forms undergoing arbitrary maneuvers. The boundary values for such general problems, however, usually lead to the development of double integral equations which are difficult to solve. As is usual in such cases, there are many special types of plan forms and maneuvers which lead to boundary-value problems that are simpler to analyze. An important class of these simplified problems is that arising from homogeneous boundary conditions.

Let $\varphi(x,y,z,t)$ be a solution to equation (3). In certain special cases this can be written $\varphi = (t)^n \varphi_o(\frac{x}{t}, \frac{y}{t}, \frac{z}{t})$ in which case φ is called a homogeneous function of degree n. The number of variables affecting φ_o is only three as compared to the four which are necessary to determine φ. If, therefore, a partial differential equation can be set up for φ_o, it will contain one less mathematical "dimension" than the equation for φ. Following this observation it is necessary to proceed in two directions; one to find the partial differential equation for φ_o, and the other to find the physical problem and consequent boundary values leading to a homogeneous flow field. The latter path will be first explored.

First, consider an example of a homogeneous boundary-value problem. Suppose that a rectangular flat plate at an angle of attack starts suddenly from rest and moves forward at a supersonic Mach number M_o. At "time" t_1 the initial spherical wave generated by the forward right-hand corner has traveled outward to a radius t_1 and, at "time" $2t_1$, to a radius $2t_1$. The sketch indicates the traces of these spheres in the z = 0 plane together with the original and present position of the wing leading edge. Let the points P_1 and P_2 be located on the same rays through the origin of the circles and the wing corners. The problem is to find the pressures at P_1 and P_2.



It is apparent that, if every dimension in the figure involving P_2 is divided by $2t_1$ and every dimension in the figure involving P_1 is divided by t_1, the two figures will be similar in every respect and point P_1 will coincide with point P_2. Since the vertical velocity w_o is constant over the plan form, a simple change in scale has made the boundary conditions for both problems identical. But this means that the solutions at P_1 and P_2 are identical since the wave equation is invariant to change in scale. Hence, in regions of a rectangular wing unaffected by the waves from the trailing edge, the pressure can be written

$$\frac{\Delta p}{q}(x,y,z,t) = \frac{\Delta p}{q}\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) \tag{16}$$

and the pressure is a homogeneous function of degree zero. A generalization of this example is contained in the following statement:

(1) The pressure in any region affected by only two intersecting edges of a straight-sided flat plate traveling at a uniform subsonic or supersonic speed is homogeneous and of degree zero (i.e., satisfies equation (16)).

Consider as another example the case of a flat rectangular wing traveling forward at a subsonic or supersonic speed and rolling about one edge, taken to be coincident with the x axis. The argument follows the same lines as before, and again a change in scale, proportional to the time, makes the geometry of the wing-wave combinations identical (in regions affected by only two intersecting edges) for different times. The boundary values over the wings will not be the same, however, unless the slope of the w_o distribution is adjusted in each case. But w_o can be adjusted by reducing it an amount proportional to the distance from the axis of rotation. The boundary-value problems are then similar for different values of time. Finally, therefore, the pressure can be written

$$\frac{\Delta p}{q}(x,y,z,t) = y\,\frac{\Delta p}{q}\left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) \tag{17}$$

which is a homogeneous function of degree one. A generalization of this example is expressed as follows:

(2) The pressure in any region affected by only two intersecting edges of a straight-sided flat plate traveling at a uniform subsonic or supersonic speed and rotating at a constant rate of pitch or roll is homogeneous and of degree one (i.e., satisfies equation (17)).

It should be noted that both (1) and (2) are equally true for the steady-state case when all transient effects have disappeared. In supersonic wing theory they lead to conical and quasi-conical flows, respectively, while in the subsonic case they lead to flows about wings having infinite chordwise extent. In general, homogeneous flow occurs when the boundary conditions after a change in scale are proportional to their original values.

Consider next the modification of the basic partial differential equation (equation (3)) under the assumption that the flow is homogeneous. If the pressure is given by a function that is homogeneous and of degree zero, then, by equation (6), the velocity-potential function will be homogeneous and of degree one. If the notation

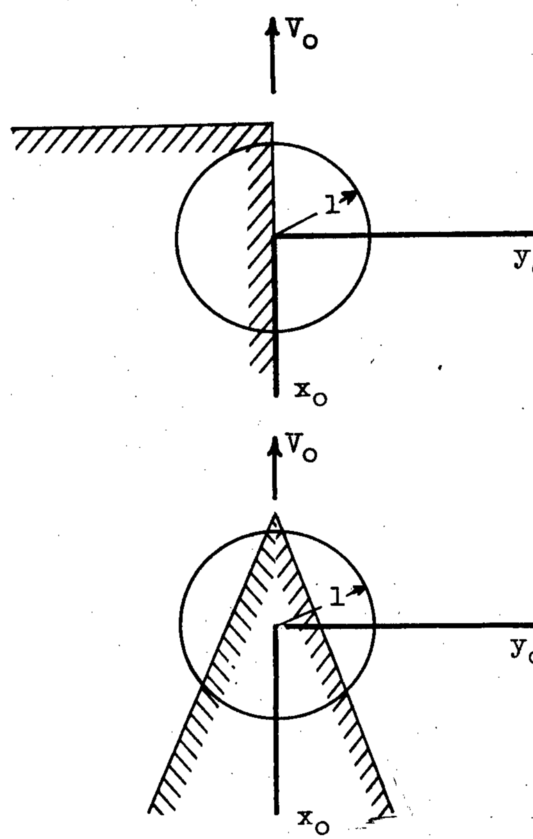
$$\left.\begin{aligned} \frac{x}{t} = x_o, \ \frac{y}{t} = y_o, \ \frac{z}{t} = z_o \\ \varphi\,(x,y,z,t) = t\,\Phi\,(x_o,\ y_o,\ z_o) \end{aligned}\right\} \tag{18}$$

is used, then equation (3) becomes

$$\begin{aligned} (1-x_o^2)\Phi_{x_o x_o} + (1-y_o^2)\Phi_{y_o y_o} + (1-z_o^2)\Phi_{z_o z_o} - 2x_o y_o\,\Phi_{x_o y_o} - \\ 2x_o z_o\,\Phi_{x_o z_o} - 2y_o z_o\,\Phi_{y_o z_o} = 0 \end{aligned} \tag{19}$$

and a linear partial differential equation with three independent variables is therefore obtained.

In the general theory of partial differential equations of second order, the character of an equation is determined from the geometric nature of a related quadric surface. The character of equation (19) can be shown from such considerations to depend on the sign of the expression $1-x_o^2-y_o^2-z_o^2$. It is immediately apparent, however, that within the unit sphere in the x_o,y_o,z_o space the sign of $1-x_o^2-y_o^2-z_o^2$ is everywhere positive and outside the sign is everywhere negative. It follows that outside the unit sphere equation (19) is hyperbolic and inside the unit sphere it is elliptic.



The character of equation (19) is of particular interest since the difficulties inherent in the determination of the solutions can be estimated without actually obtaining the solutions. For example, consider the two configurations shown in the sketch. These wings started moving at $t = 0$ with the foremost portion of their leading edges on the y_o axis and have by now traveled forward at a supersonic speed to attain the positions represented by the sketch, the unit circle being in each case the trace of the primary wave from the vertex on the $z = 0$ plane. Outside the unit sphere, the governing equation is hyperbolic and the behavior of the flow is similar to that in steady-state supersonic-wing problems; closed solutions, therefore, can be expected in many cases. Inside the unit sphere, on the other hand, the character of equation (19) is elliptic. The canonical form of an elliptic partial differential equation in three dimensions is Laplace's equation which is the governing equation in steady-state subsonic-wing theory. Judging from the complexity of subsonic, three-dimensional wing problems, it can be concluded that a closed solution within the unit sphere will be very difficult to find.

It is instructive to notice that this entire development has a direct analogue in the study of three-dimensional, steady state, supersonic wings. In that case the original equation is the three-dimensional wave equation

$$\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \tag{20}$$

By considering the velocity potential to be homogeneous and of degree one, Busemann in reference 7 was able to introduce the transformations

$$y_o = \frac{y}{x} \qquad z_o = \frac{z}{x}$$

$$\varphi(x,y,z) = x\,\Phi(y_o,z_o)$$

and transform equation (20) to the form

$$(1-y_o^2)\,\Phi_{y_oy_o} + (1-z_o^2)\,\Phi_{z_oz_o} - 2y_oz_o\,\Phi_{y_oz_o} = 0 \tag{21}$$

which is the two-dimensional form of equation (19). Flows governed by equation (21) have become known as conical flows. A study of equation (21) shows it to be elliptic inside and hyperbolic outside the unit circle. In this case, however, the equation has only two independent variables so that once the equation has been transformed to the two-dimensional form of Laplace's equation[6] solutions are not difficult to find.

[6]The Tschapligin transformation transforms equation (21) into Laplace's equation.

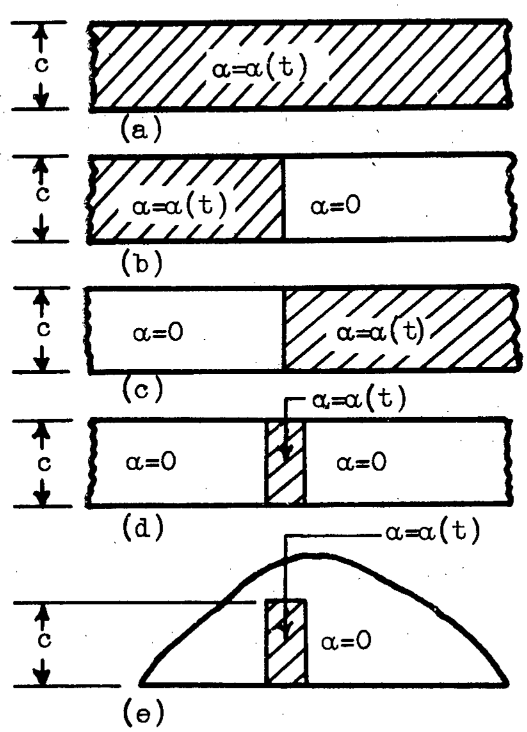
Boundary-Value Problems Involving Noninteracting Surfaces

The simplification of the four-dimensional theory brought out by the introduction of homogeneous flow was more apparent than real since the resulting partial differential equation, although containing one less dimension, was unwieldy. Another class of wing problem which is simplified both in theory and in practice by reasoning from physical knowledge of the flow behavior is that in which the wing has a supersonic edge (i.e., an edge which is traveling with a supersonic normal component of velocity).

When the acoustic plan form is affected only by a supersonic edge, it is obvious that the flow on the upper surface of the wing is independent of that on the lower surface. Hence the solution to such problems can always be written in terms only of sources as follows:

$$\varphi = -\frac{1}{2\pi}\iint_{S_a} \frac{1}{r_o}\left[\frac{\partial\varphi}{\partial z_1}\right] dS \tag{22}$$

where $\partial\varphi/\partial z_1 = w_u(x_1,y_1) = V_o\ \lambda_u(x_1,y_1)$, λ_u being the local slope of the surface in the direction of V_o. Since equation (22) is equally valid for symmetrical nonlifting surfaces and lifting plates, its value and simplicity is evident.



If the wing plan form is further specialized by having not only supersonic leading edges, but also having a straight trailing edge perpendicular to the direction of motion, additional simplifications can be used.[7] Consider, for example, the two-dimensional wing (a) in the sketch. Let this wing have an angle of attack $\alpha(t)$ which varies with time in an arbitrary manner. There results from such an angle-of-attack variation a certain lift which also varies with time. Hence, if L' represents the total lift on an airfoil of

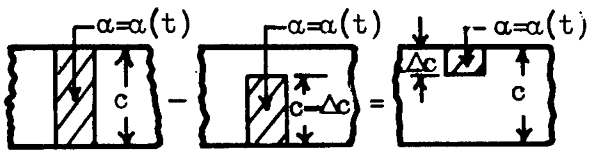
[7]The following method simply extends, to include the effects of unsteady motion, a theorem given by Lagerstrom and Van Dyke. (See reference 2.)

very high aspect ratio and $c_{l_\alpha}{'}$ represents the section lift-curve slope, then[8]

$$\frac{L'}{q\alpha} = c \quad c_{l_\alpha}{'}\ (c,t)(\text{span})$$

Next, it is clear by reason of symmetry that the total lift on wings (b) and (c) in the sketch are equal. Then, since the analysis is based on a linear partial differential equation, by superposition principles the total lift on wing (b) or (c) is just half of that on wing (a). In another sense, the lift coefficient for the whole wing based on the deflected area is the same for all three cases. A suitable superposition of wings (a), (b), and (c) will give wing (d), which then has the same lift coefficient based on deflected area. Finally, because of the supersonic stream, wing (e) can be obtained from (d), hence it also has the lift coefficient common to the other wings. It is of course necessary that the variation of α with time be the same in each case.

The preceding process can be extended one step farther to the development of the lift due to a single deflected element. By considering the sketch it can be seen that



$$\frac{L'}{q\alpha} = \frac{c \quad c_{l_\alpha}{'}(c,t) - (c-\Delta c)c_{l_\alpha}{'}\ (c-\Delta c,t)}{\Delta c}\,\Delta S$$

where

ΔS area of the deflected element

c distance from ΔS to trailing edge

By the usual limiting process the latter equation becomes

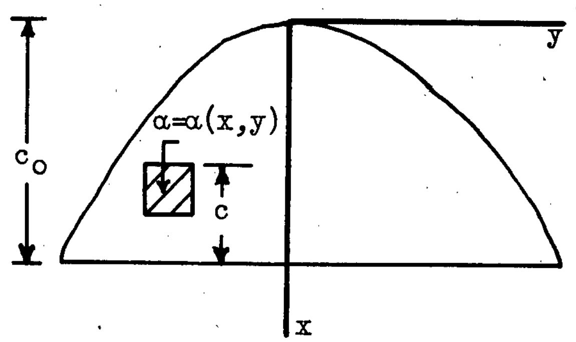
$$\frac{L'}{q\alpha} = \frac{\partial}{\partial c}\left[c\ c_{l_\alpha}{'}\ (c,t)\right]dS$$

[8]The primes on quantities indicate that two-dimensional values are taken, or, that a high-aspect-ratio wing is considered and tip effects are neglected.

Finally, if a wing is composed of a distribution of these elements, then, for an axial system centered at the apex of the wing leading edge, there results

$$C_L = -\frac{1}{S}\int_S\int \alpha(x,y)\,\frac{\partial}{\partial x}\left[(c_o-x)\; c_{l_\alpha}{}'\;(c_o-x,t)\right]dydx \tag{23}$$

where c_o is the maximum chord (see sketch). In the development of equation (23) each element is assumed to have the same variation of motion with time.



Notice that when the wing is a flat plate flying at a steady speed so that all transient effects have disappeared, $c_{l_\alpha}{}'$ is independent of t and of the chord length, being, in fact, equal to $4/\beta$. Then equation (23) becomes

$$C_L = \frac{4}{\beta S}\int_S\int \alpha(x,y)dxdy = \frac{4\bar{\alpha}}{\beta}$$

where $\bar{\alpha}$ is the average angle of attack of the wing. This result has already been obtained in reference 2. When $\alpha(x,y)$ is independent of x (as for a flat wing sinking or rolling), equation (23) becomes

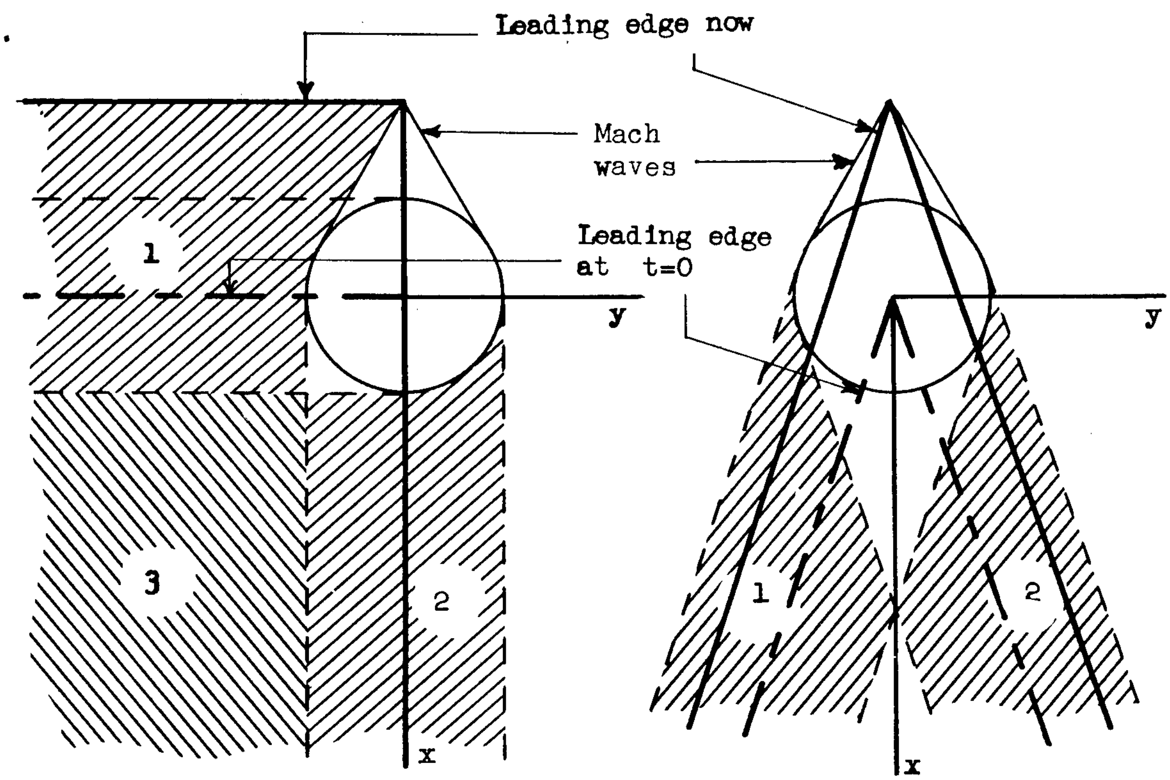
$$C_L = \frac{1}{S}\int_{-\frac{b}{2}}^{\frac{b}{2}} \alpha(y)\; c\; c_{l_\alpha}{}'\;(c,t)\;dy \tag{24}$$

where c is the local chord which is, in general, a function of y. Equation (24) simply indicates that longitudinal-strip theory is exact for such wings.

Finally, notice that the calculation of the unsteady lift on three-dimensional wings with supersonic leading edges and straight trailing edges perpendicular to the free-stream direction has been reduced to an integration involving the relatively simple results for a two-dimensional wing undergoing the same unsteady motion. For example, the lift on a flat triangular wing rising and sinking with a harmonic motion can be computed from a single integration of the results presented in reference 3. Such a calculation can be carried out numerically quite rapidly.

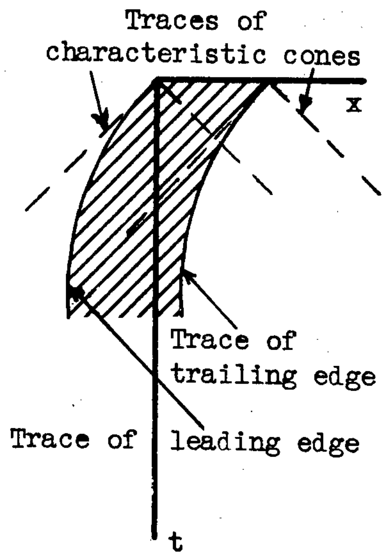
Two-Dimensional Boundary-Value Problems

The simplification brought about when the flow is independent of one dimension is again obvious. In such cases, the three-dimensional wave equation (3) reduces immediately to the two-dimensional wave equation. Typical examples of this type of problem can be constructed by considering flat plates which start suddenly at $t = 0$ and travel thereafter at constant supersonic velocities. Two examples, one a corner of a rectangular wing and the other a triangular flat-plate wing are shown in the sketch. After time $t = 0$, the edges of the wings send out cylindrical waves and the outer boundaries of these waves at time t are shown as dashed lines parallel to the edges in question.



Since points in regions 1 and 2 are affected only by a single edge, the wave phenomena in these regions are cylindrical, and the physical quantities are in both cases independent of distance parallel to the edge which acts as their generator. Hence, the flow field in these regions may be regarded as two dimensional. (Region 3, incidentally, is independent of distance in both the x and the y directions and is, therefore, one-dimensional.)

Solutions to the two-dimensional unsteady problems are sometimes especially easy to find because of the analogy they have with three-dimensional, steady-state, lifting-surface problems. (See, e.g., reference 8.) For example, consider an infinitely long unyawed wing which starts from rest and travels forward at a velocity V which may or may not be a function of time. The trace of this wing in the x,t plane is like that shown in the sketch. (In the sketch shown, the wing velocity is always less than the speed of sound and varying.) The boundary conditions are that φ_z is specified over the shaded area and the loading $\Delta\varphi_t$ is zero everywhere except within the shaded area. But if x is replaced by y and t by x, these boundary conditions are exactly the same as those for a plate of known camber and angle of attack, with a plan form as indicated by the shaded area, placed in a free stream directed along the positive x axis at a Mach number equal to $\sqrt{2}$. The solution for the one problem may be used, therefore, as a solution to the other with only a change in notation.

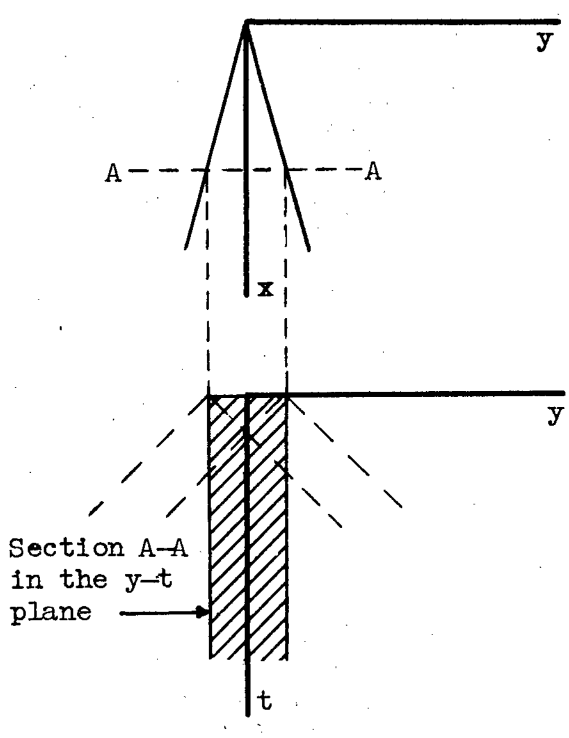


Boundary Conditions for Very Slender Wings

When the wing plan form is slender in the sense that its length in the streamwise direction is large compared to its span, an estimation of the loading on it can be obtained by neglecting in the partial differential equation the gradient of the induced velocity component in the stream direction. Thus, if the wing is moving in the negative x direction, equation (3) reduces to

$$\varphi_{yy} + \varphi_{zz} = \varphi_{tt} \qquad (25)$$

which is again the wave equation but in two space dimensions. Since equation (25) is independent of x, study can be made independently of the boundary conditions on each spanwise strip. This is an extension of steady-state slender-wing theory. (See, e.g., reference 9.) The sketch shows a



typical section in the yt plane. If the wing is a flat plate starting from rest and traveling forward at a uniform speed, the boundary condition is that φ_z should be constant over the shaded area in the sketch and $\Delta\varphi_t$ should be zero everywhere except across this area. The solution to this problem leads directly to the solution for the thin triangular wing.[9]

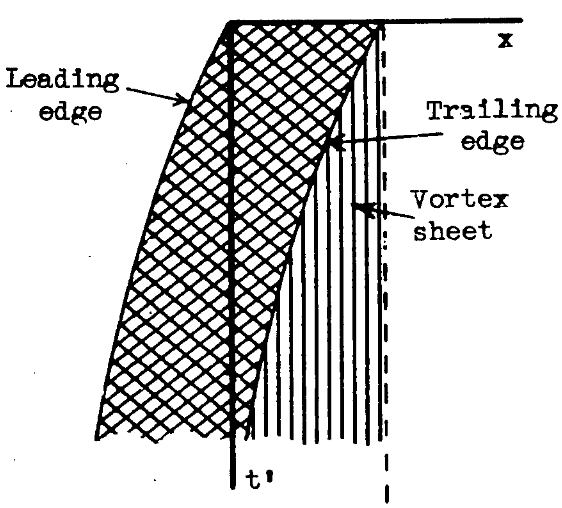
Two-Dimensional Unsteady Incompressible Flow

The analogy between two-dimensional unsteady and three-dimensional steady flow provides an interesting viewpoint for two-dimensional unsteady problems in incompressible flow. In this case the boundary conditions are independent of the lateral coordinate and equation (1) becomes

$$\varphi_{xx} + \varphi_{zz} = \frac{1}{a_o^2}\varphi_{t't'}$$

Since the speed of pressure propagation a_o is assumed infinite, the basic equation thus becomes

$$\varphi_{xx} + \varphi_{zz} = 0$$



where, however, time still appears in the boundary conditions and in the expression

$$\frac{\Delta p}{q} = \frac{4}{V_o^2}\frac{\partial\varphi}{\partial t'}$$

which is equivalent to equation (6). The technique involved in the solution of such problems is analogous to that in which one term of the three-dimensional steady-state equation vanishes by virtue of the fact that free-stream Mach number approaches one.

If the two-dimensional wing starts from rest and travels forward at a speed V, the trace of the wing is as shown in the sketch. The essential difference between this problem and the more general case of two-dimensional compressible flow lies in the fact that the traces of

[9]It is obvious that the analogous problem in steady-state wing theory is that of a low-aspect-ratio, rectangular, flat plate in a free stream having a Mach number equal to $\sqrt{2}$.

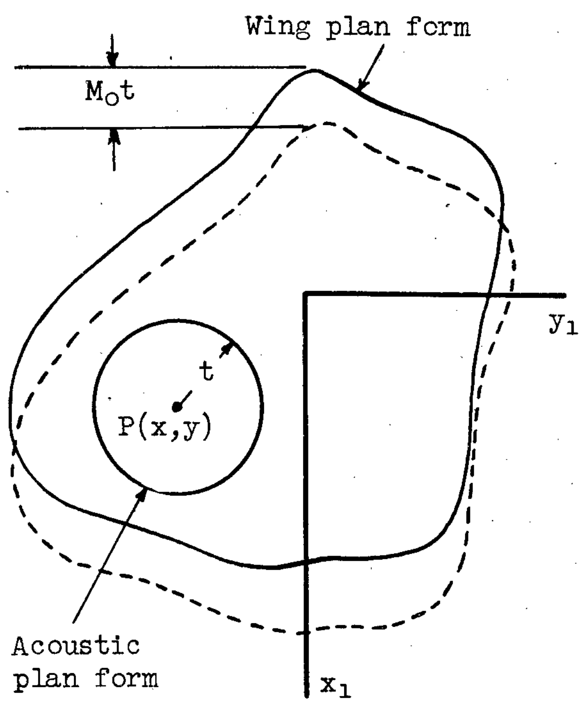
the characteristic cones are normal to the t' axis. The boundary conditions are therefore satisfied along lateral strips and, in lifting-surface terminology, the analysis corresponds to slender-wing theory. These latter methods are well established and in reference 9, for example, the manner in which the Kutta condition is imposed is discussed in some detail. The trailing vortex sheet for the lifting wing has the same distribution of vorticity that exists behind the unsteady two-dimensional airfoil and the rolling up of the vortex sheet can be studied from either standpoint.

APPLICATIONS

Starting Lift of a Wing

One of the simplest and yet most general results which can be derived on the basis of the present theory is the initial value of pressure on a wing surface starting impulsively from rest with a velocity V_o. The discussion will be made for a wing without thickness although the results will be seen to apply to the thickness case as well.

Consider a surface with a plan from as indicated in the sketch. The acoustic plan form of a point $P(x,y)$ on the surface is a small circle of radius t. Since no point on the wing outside this circle can influence the pressure at P, the upper surface is independent of the lower surface, except for a band of width t around the edge of the wing. It is, therefore, evident that the boundary-value problem to be solved has been treated in the section Boundary-Value Problems Involving Noninteracting Surfaces. The solution follows directly from equation (22) and can be written



$$\varphi = -\frac{1}{2\pi}\iint_{S_a} \frac{1}{r_o}\left[\frac{\partial\varphi}{\partial z_1}\right]dS$$

Using a polar coordinate system defined by

$$x - x_1 = r \cos \theta$$
$$y - y_1 = r \sin \theta$$
$$dx_1\, dy_1 = r dr d\theta$$

there results[10]

$$\varphi \approx -\frac{w_u(x,y)}{2\pi} \int_0^{2\pi} d\theta \int_0^t dr = -t w_u(x,y)$$

so that

$$\left(\frac{\Delta p}{q}\right)_{t=0} = +\frac{4}{V_o M_o}\left(\frac{\partial \varphi}{\partial t}\right)_{t=0} = -\frac{4 w_u(x,y)}{V_o M_o}$$

If $\alpha = -w_u(x,y)/V_o$ is the local slope of the wing, the expression for load coefficient becomes

$$\left(\frac{\Delta p}{q}\right)_{t=0} = \frac{4\alpha(x,y)}{M_o} \tag{26}$$

The starting value of lift coefficient can, therefore, be written

$$C_L = \frac{4\bar{\alpha}}{M_o} \tag{27}$$

where $\bar{\alpha}$ is the average angle of attack of the surface defined by

$$\bar{\alpha} = \frac{1}{S} \iint_S \alpha \, dx \, dy$$

S being the area of the wing plan form.

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[10]The mean value theorem gives

$$\varphi = -\frac{w_u(\xi,\eta)}{2\pi} \iint_{S_a} \frac{dS}{r_o}$$

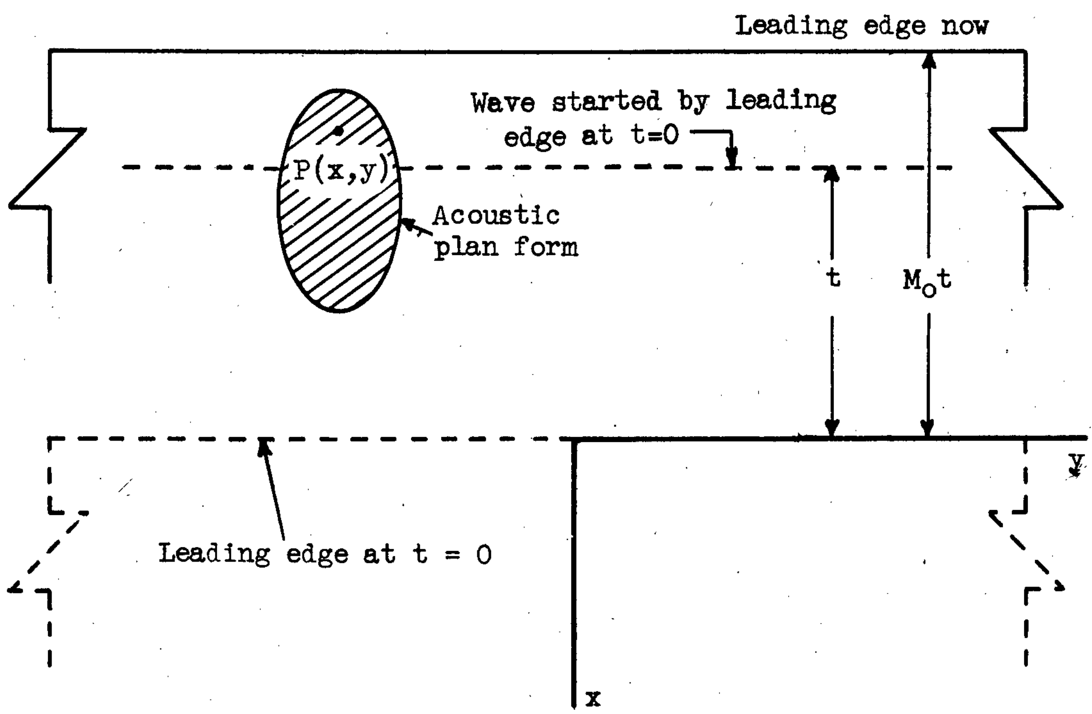
where ξ and η lie somewhere in S_a. Hence, as t approaches 0, ξ and η approach x and y, respectively.

Supersonic Steady-State Lift

In order to illustrate some of the basic concepts, consider next an example for which the result is well known. The problem will be to derive, by the methods outlined in this report, the steady-state loading on a two-dimensional flat plate traveling at a constant supersonic speed.

Since the upper and lower surfaces are noninteracting, the solution stems from equation (22). In the plane of the wing this equation becomes

$$\varphi = -\frac{1}{2\pi}\iint_{S_a}\frac{w_u dx_1\, dy_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}} \tag{28}$$



The accompanying sketch shows the positions of the wing in the xy plane. The wing has constant speed for $t>0$. The point $P(x,y)$ is chosen on the wing and ahead of the wave which started at time zero; therefore, $P(x,y)$ lies in the region which has attained its steady-state value. Further, the value of w_u is constant over the acoustic plan form in such a region. This constancy reduces the problem to one of integrating $[(x-x_1)^2+(y-y_1)^2]^{-1/2}$ over the ellipse representing the acoustic plan form.

The equation for S_a can be determined from equations (14) and (15). In this case, equation (15) becomes simply

$$x_1 = -M_o\tau$$

Eliminate τ between this equation and equation (14) and there results in the $z = 0$ plane

$$(x-x_1)^2 + (y-y_1)^2 = \left(t + \frac{x_1}{M_o}\right)^2 \tag{29}$$

That equation (29) is the equation of an ellipse with one focal point at x,y can be readily verified. It is more convenient, however, to change to a polar coordinate system with origin at P. Hence set

$$\begin{aligned} x-x_1 &= r \cos\theta \\ y-y_1 &= r \sin\theta \\ dy_1\,dx_1 &= r\,dr\,d\theta \end{aligned}$$

Then equation (29) becomes

$$M_o r = M_o t + x - r\cos\theta$$

or

$$r = \frac{x + M_o t}{M_o + \cos\theta} \tag{30}$$

and, therefore, equation (28) becomes simply

$$\varphi = -\frac{w_u}{2\pi}\int_0^{2\pi} d\theta \int_0^{\frac{x + M_o t}{M_o + \cos\theta}} dr$$

$$= -\frac{w_u}{2\pi}(x + M_o t)\int_0^{2\pi}\frac{d\theta}{M_o + \cos\theta}$$

The integral is not difficult to evaluate so

$$\varphi = -\frac{w_u}{2\pi}(x+M_o t)\left(\frac{2\pi}{\sqrt{M_o^2-1}}\right)$$

and, finally, by equation (6)

$$\frac{\Delta p}{q} = \frac{4}{V_o M_o} \left(- \frac{w_u M_o}{\sqrt{M_o^2 - 1}} \right) = \frac{4\alpha}{\sqrt{M_o^2 - 1}} \qquad (31)$$

which is the familiar Ackeret value for the loading on a two-dimensional flat plate. The lift coefficient, of course, follows immediately as

$$C_L = \frac{4\alpha}{\sqrt{M_o^2 - 1}} \qquad (32)$$

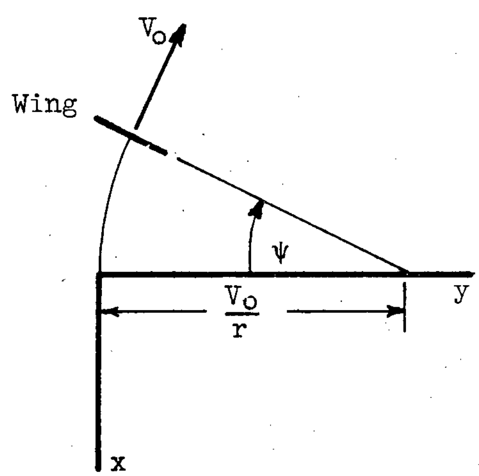
Stability Derivative C_{l_r}

Consider the case of a high-aspect-ratio (so that the tip effects can be neglected) rectangular wing yawing at a uniform rate r radians per second and traveling forward along its flight path at a supersonic velocity V_o. The equation of the leading edge of the wing, using the coordinates indicated by the sketch, is given by the relation

$$x = \left(y - \frac{V_o}{r} \right) \tan \psi$$

and since $\psi = rt'$ this becomes

$$x = \left(y - \frac{V_o}{r} \right) \tan rt' \qquad (33)$$



The boundary conditions (when the stability derivative is to be referred to the body axis) are that w_u, the vertical induced velocity in the region occupied at any given time by the plan form, shall be constant. Further, the steady-state condition is determined when the intersection of the leading-edge trace with the inverse sound wave forms a closed curve. Hence, the acoustic plan form is given by eliminating τ from the equations

$$\left.\begin{aligned} (x-x_1)^2 + (y-y_1)^2 &= (t-\tau)^2 \\ x_1 &= \left(y_1 - \frac{V_o}{r}\right)\tan r\tau' \end{aligned}\right\} \qquad (34)$$

It is usual to assume r to be small and to retain only the first power of r in a series expansion. Therefore, since it can be shown that the equation of the acoustic plan form is given correctly to the first power in r by assuming $rt' = \tan rt'$, equation (34) becomes

$$\left.\begin{aligned} R &= t - \tau \\ x - R\cos\theta &= \left(y - R\sin\theta - \frac{M_o}{r^*}\right) r^*\tau \end{aligned}\right\} \qquad (35)$$

where, as in the preceding example, the polar coordinates

$$\left.\begin{aligned} x-x_1 &= R\cos\theta \\ y-y_1 &= R\sin\theta \\ dx_1\,dy_1 &= RdRd\theta \end{aligned}\right\} \qquad (36)$$

have been introduced together with the notation

$$r^* = \frac{r}{a_o} \qquad (37)$$

solving equations (35) for R, there results

$$R = \frac{-A_o + \sqrt{A_o^2 - 4\,B_o r^* \sin\theta}}{2r^* \sin\theta}$$

where

$$A_o = \cos\theta - r^*y + M_o - r^*t\sin\theta$$

$$B_o = r^*ty - x - M_o t$$

and expanding in powers of r^* gives, neglecting powers higher than the first,

$$R = -\frac{B_o}{A_o} - \frac{B_o^2}{A_o^3} r^* \sin\theta$$

or

$$R = \frac{x+M_o t}{M_o + \cos\theta} - r^*\left[\frac{ty}{M_o + \cos\theta} - \frac{(x+M_o t)(y+t\sin\theta)}{(M_o+\cos\theta)^2} + \frac{(x+M_o t)^2}{(M_o+\cos\theta)^3}\sin\theta\right]$$

The solution for the velocity potential is given, just as in the preceding example, by the equation

$$\varphi = -\frac{w_u}{2\pi}\int_0^{2\pi} R d\theta$$

and since

$$\int_0^{2\pi}\frac{\sin\theta}{(M_o+\cos\theta)^3}d\theta = \int_0^{2\pi}\frac{\sin\theta}{(M_o+\cos\theta)^2}d\theta = 0$$

$$\int_0^{2\pi}\frac{d\theta}{M_o+\cos\theta} = \frac{2\pi}{\sqrt{M_o^2-1}}$$

$$\int_0^{2\pi}\frac{d\theta}{(M_o+\cos\theta)^2} = \frac{2\pi M_o}{(M_o^2-1)^{3/2}}$$

then finally

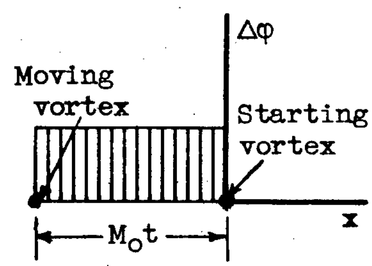
$$\varphi = -\frac{w_u}{\sqrt{M_o^2-1}}\left(M_o t + x + r^* y\,\frac{t+M_o x}{M_o^2-1}\right) \tag{38}$$

The loading can be obtained from equation (38) by using equation (6) hence

$$\frac{\Delta p}{q} = \frac{4\alpha}{\sqrt{M_o^2-1}} + \frac{4\alpha}{(M_o^2-1)^{3/2}}\left(\frac{ry}{V_o}\right) \tag{39}$$

This result has been obtained previously in reference 10 by a different method.

Moving Two-Dimensional Vortex



A further interesting application of the methods presented here is the development of the moving rectilinear vortex, where a vortex is represented by a line of constant circulation. The moving vortex is associated with a lifting surface[11] which starts impulsively at $t = 0$ and travels at a uniform velocity away from a fixed axial system. Since a vortex cannot end in a fluid (Helmholtz's fundamental result), the moving vortex must be adjoined by two trailing vortices and they, in turn, by a final starting vortex (see sketch) which remains in the vicinity of the position

[11] Associated in the sense that a distribution of these vortices will satisfy the boundary conditions for the lifting surface. The moving vortex is often referred to as the bound vortex.

at which the motion commenced. For the present, the assumption is made that the trailing vortices are far enough away so that they do not affect the flow around the center of the moving and starting vortices (i.e., the flow is two dimensional); and second, that the starting vortex remains fixed on the line along which the moving vortex first appeared. With these assumptions, the boundary conditions can be determined from the definition that the vortex system described is equivalent to a distribution of doublets of constant strength over the region of the $z = 0$ plane bounded by the moving vortex, the stationary vortex, and the vortices joining them at their extremities. Such a sheet of doublets gives rise to a constant potential difference, $\Delta\varphi$, across the sheet, while the normal derivative is continuous so that $\Delta(\partial\varphi/\partial z) = 0$. The solution is then given by equation (13). Thus

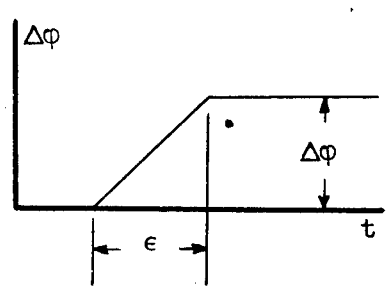
$$\varphi = \varphi_1 + \varphi_2 \tag{40}$$

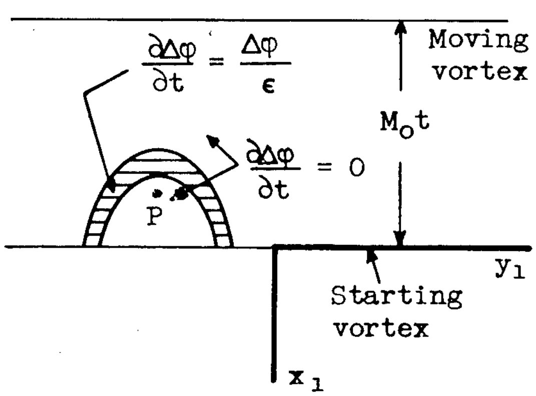
where

$$\varphi_1 = -\frac{\Delta\varphi}{4\pi}\iint_{S_a}\left(\frac{\partial}{\partial z}\frac{1}{r_o}\right)dS \tag{41}$$

and

$$\varphi_2 = \frac{1}{4\pi}\iint_{S_a}\frac{1}{r_o}\left(\frac{\partial\Delta\varphi}{\partial t}\right)\frac{\partial r_o}{\partial z}\,dS \tag{42}$$





The evaluation of φ_1 imposes no difficulties except in integration but the calculation of φ_2 requires some special consideration. Since $\Delta\varphi$ is a constant, $\partial\Delta\varphi/\partial t$ is zero everywhere except along the line of the moving vortex and there it is infinite. In order to evaluate such an indeterminate form, assume that the potential difference rises linearly to its value $\Delta\varphi$ in the time interval ϵ (see sketch). Then the value of $\partial\Delta\varphi/\partial t$ is $\Delta\varphi/\epsilon$ in the narrow strip between the two ellipses formed by the intersection of the inverse sound wave

$$(x-x_1)^2 + (y-y_1)^2 + z^2 = (t-\tau)^2$$

with the lines

$$x_1 = -M_o\tau$$

$$x_1 = -M_o(\tau-\epsilon)$$

When the equations for these two ellipses are formulated and placed in equation (42) there results an equation which can be represented by the expression

$$\varphi_2 = \frac{\Delta\varphi}{4\pi\epsilon}\left[F(t)-F(t-\epsilon)\right]$$

where

$$F(t) = \iint_{S_a} \frac{1}{r_o}\frac{\partial r_o}{\partial z}\, dS$$

and S_a is the same acoustic plan form used in equation (41). But since

$$\lim_{\epsilon \to 0} \frac{F(t)-F(t-\epsilon)}{\epsilon} = \frac{\partial F}{\partial t}$$

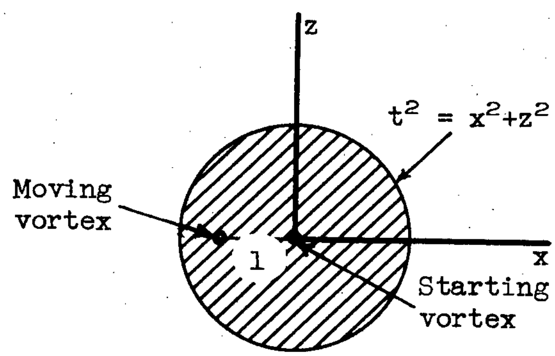
the equation for φ_2 becomes

$$\varphi_2 = \frac{\Delta\varphi}{4\pi}\frac{\partial}{\partial t}\iint_{S_a} \frac{1}{r_o}\frac{\partial r_o}{\partial z}\, dS$$

The calculation of φ_1 and φ_2 over the acoustic plan form given by the area within the curve

$$(x-x_1)^2 + (y-y_1)^2 + z^2 = \left(t + \frac{x_1}{M_o}\right)^2 \tag{43}$$

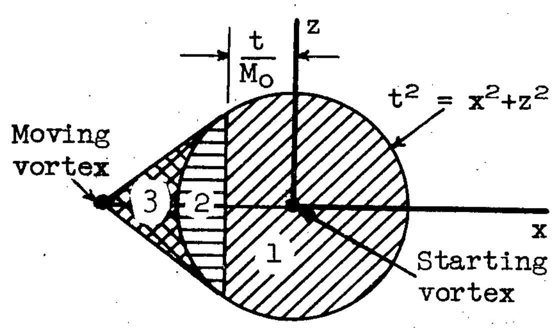
or between this curve and the y_1 axis gives for the regions defined by the sketch



Subsonic

Region (1)

$$\varphi = \frac{\Delta\varphi}{2\pi}\left[\text{arc tan}\ \frac{(x+M_o t)\sqrt{t^2-x^2-z^2}}{z(M_o x+t)} - \text{arc tan}\ \frac{x}{zt}\sqrt{t^2-x^2-z^2}\right] \tag{44}$$



Supersonic

Region (2)

$$\varphi = \frac{\Delta\varphi}{2\pi}\left[\pi\frac{z}{|z|} - \text{arc tan}\ \frac{x}{zt}\sqrt{t^2-x^2-z^2} + \text{arc tan}\ \frac{(x+M_o t)\sqrt{t^2-x^2-z^2}}{z(M_o x+t)}\right] \tag{45}$$

Region (3)

$$\varphi = \frac{\Delta\varphi}{2}\frac{z}{|z|} \tag{46}$$

It is to be noted that equation (44) gives the potential everywhere within the cylinder in which disturbances can be affected in subsonic flow as well as over a large portion of the disturbed region in supersonic flow.

It is apparent that the results given in equations (44), (45), and (46) can also be obtained independently by the methods indicated in the section entitled Two-Dimensional Boundary-Value Problems. From this viewpoint, the problem of the unsteady two-dimensional vortex pair has for its analogue in lifting-surface theory the case of a triangular wing yawed so that one side is parallel to the free-stream direction and the other side is either a subsonic or a supersonic edge, depending on the case being considered. Since $\Delta\varphi$ is specified a constant over the entire plan form, the solution is relatively easy. In reference 11, for example, the solution for the subsonic case is given in terms of the variables used in steady-state analysis.

The foregoing results for $M_o < 1$ reduce to more familiar expressions when their asymptotic expressions for large values of time are determined. A study of the two terms in equation (44) shows that as t becomes large the expression becomes

$$\varphi \rightarrow \frac{\Delta\varphi}{2\pi}\left(\frac{\pi}{2}\frac{z}{|z|} - \text{arc tan}\,\frac{x}{z}\right)$$

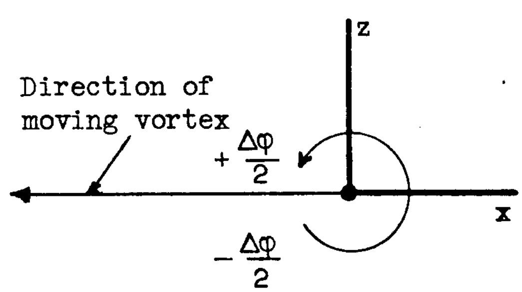
This result provides the velocity potential for a stationary vortex in an incompressible medium, the jump in the value of the potential occurring along the plane of the vortex motion (see sketch).

If the axes are fixed in the moving vortex, new coordinates ξ, ζ, τ need to be introduced as defined by the equations

$$\xi = x + M_o t$$

$$\zeta = z$$

$$\tau = t$$



As τ becomes large, equation (44) reduces to the form

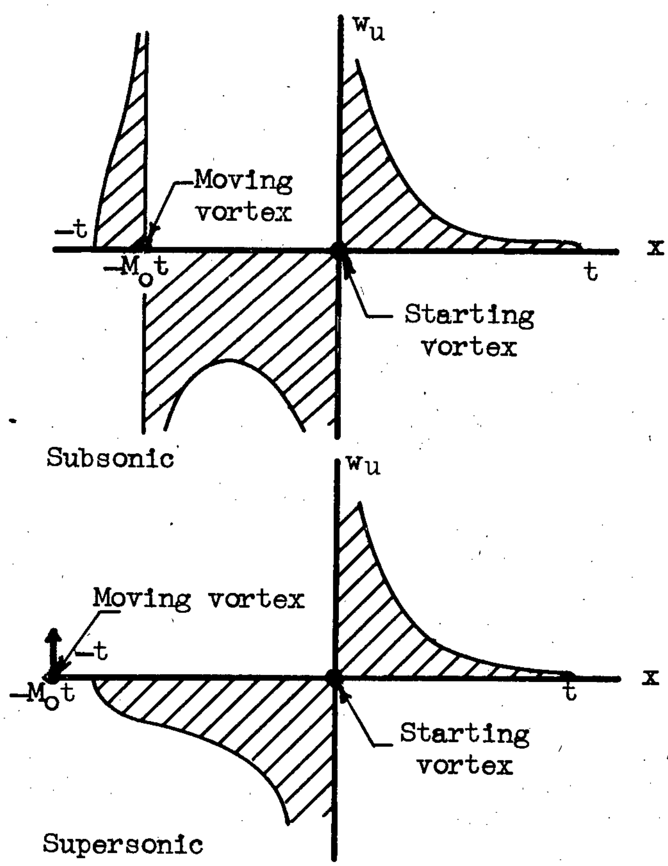
$$\varphi \rightarrow \frac{\Delta\varphi}{2\pi}\left(-\frac{\pi}{2}\frac{\zeta}{|\zeta|} + \text{arc tan}\,\frac{\xi}{\zeta\sqrt{1-M_o{}^2}}\right)$$

This result agrees with that of Glauert (reference 12) for a vortex obtained from the steady-state linearized theory of compressible flow, so that in the vicinity of the moving vortex the induced velocity field is identical to the perturbation field of a fixed vortex in a subsonic free stream.

It is interesting to inspect the vertical velocity induced by the vortex pair in the $z = 0$ plane. From equations (44) and (45) it follows that for $x^2<t^2$ the vertical velocity w_u for both subsonic and supersonic speeds is given by the relation

$$\left.\begin{aligned} w_u &= -\frac{\Delta\varphi}{2\pi}\left[\left(\frac{M_o x+t}{x+M_o t} - \frac{t}{x}\right) - \frac{1}{\sqrt{t^2-x^2}}\right] \\ &= \frac{\Delta\varphi}{2\pi}\frac{\sqrt{t^2-x^2}}{x(x+M_o t)}M_o \end{aligned}\right\} \qquad (47)$$

The sketch shows the variation of w_u along the x axis for a moving vortex traveling at both subsonic and supersonic speeds. For both cases in the vicinity of the starting vortex w_u behaves as it does near a two-dimensional, steady-state, fixed vortex, having large values of downwash on one side and large values of upwash on the other. The velocity distribution around the moving vortex is quite different, however, depending on whether its Mach number is greater or less than one. In the subsonic case the distribution is similar to that for the starting vortex, but in the supersonic case the velocity is just a pulse existing at the position of the vortex. Again in both cases the velocity falls to zero at the primary wave front (i.e., at $x = \pm t$ in the sketch).



A final remark can be made concerning equation (44) in connection with incompressible theory. In this case, the value of a_o becomes infinitely large and, if equation (44) is written in the form

$$\varphi = \frac{\Delta\varphi}{2\pi}\left[\text{arc tan}\,\frac{(x+V_0t')\sqrt{a_0{}^2t'^2-x^2-z^2}}{z\left(\frac{V_0x}{a_0}+a_0t'\right)} - \text{arc tan}\,\frac{x\sqrt{a_0{}^2t'^2-x^2-z^2}}{za_0t'}\right]$$

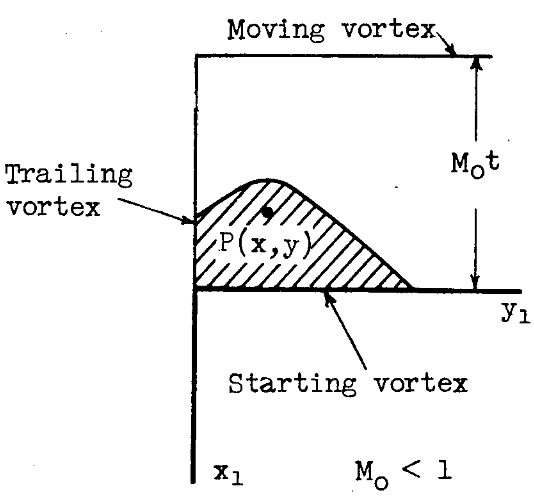
it is apparent that, as $a_0 \to \infty$ the velocity potential becomes

$$\varphi = \frac{\Delta\varphi}{2\pi}\left(\text{arc tan}\,\frac{x+V_0t'}{z} - \text{arc tan}\,\frac{x}{z}\right) \tag{48}$$

Equation (48) gives the velocity potential for a moving and a fixed vortex pair in an incompressible fluid.

Moving Three-Dimensional Vortex

In three-dimensional subsonic[12] unsteady lift theory an important role is played by the loop vortex composed of a moving bound vortex of fixed span, a stationary starting vortex of opposing strength, and two trailing vortices connecting the end-points of the other two and completing the vortex loop. The boundary condition for such a configuration requires that the jump in potential $\Delta\varphi$ is a constant over the interior of the loop. The calculation of the induced field can be resolved essentially into the determination of the velocity potential for two semi-infinite vortices and a trailing vortex (see sketch) since, by superposition methods, two of these latter configurations can be combined with the two-dimensional figure of the last section to give the required loop. The present example is concerned with the calculation of the velocity potential in the tip region of the semi-infinite case.



As in the two-dimensional case just discussed the potential can be divided into two parts φ_1 and φ_2 such that

[12] The extension to include the three-dimensional supersonic loop is obvious.

$$\varphi = \varphi_1 + \varphi_2$$

$$\varphi_1 = + \frac{\Delta\varphi z}{4\pi} \iint_{S_a} \frac{dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]^{3/2}}$$

$$\varphi_2 = + \frac{\Delta\varphi z}{4\pi} \frac{\partial}{\partial t} \iint_{S_a} \frac{dx_1 dy_1}{[(x-x_1)^2+(y-y_1)^2+z^2]}$$

where S_a is the acoustic plan form, being the area bounded by the lines

$$(x-x_1)^2 + (y-y_1)^2 + z^2 = \left(t + \frac{x_1}{M_o}\right)^2$$

$$y_1 = 0$$

$$x_1 = 0$$

It is convenient to use the notation

$$k = x+M_o t$$

$$\xi = x-x_1$$

$$\eta = y-y_1$$

Then the potential can be written

$$\varphi = \frac{z\Delta\varphi}{4\pi} \int_{-\sqrt{t^2-x^2-z^2}}^{y} d\eta \int_{x}^{\frac{1}{\beta^2}\left(k-M_o\sqrt{k^2+\beta^2 z^2+\beta^2\eta^2}\right)} \frac{d\xi}{(\xi^2+\eta^2+z^2)^{3/2}} +$$

$$\frac{z\Delta\varphi}{4\pi} \frac{\partial}{\partial t} \int_{-\sqrt{t^2-x^2-z^2}}^{y} d\eta \int_{x}^{\frac{1}{\beta^2}\left(k-M_o\sqrt{k^2+\beta^2 z^2+\beta^2\eta^2}\right)} \frac{d\xi}{\xi^2+\eta^2+z^2}$$

which becomes, after one integration,

$$\varphi = \frac{z\Delta\varphi}{4\pi}\int_{-\sqrt{t^2-x^2-z^2}}^{y}\left[\frac{-M_o}{k^2+z^2+\eta^2} - \frac{x}{(\eta^2+z^2)\sqrt{x^2+z^2+\eta^2}} + \right.$$

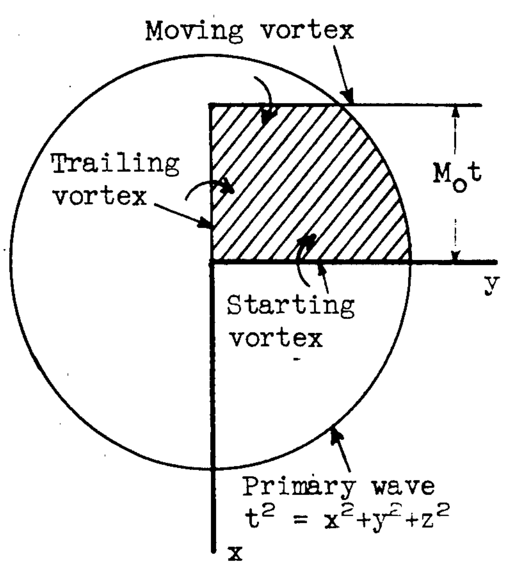
$$\left.\frac{k}{\sqrt{k^2+\beta^2 z^2+\beta^2\eta^2}}\left(\frac{1}{\eta^2+z^2} - \frac{{M_o}^2}{k^2+\eta^2+z^2}\right)\right] d\eta +$$

$$\frac{z\Delta\varphi}{4\pi}\int_{-\sqrt{t^2-x^2-z^2}}^{y}\left[\frac{M_o}{k^2+z^2+\eta^2} + \frac{{M_o}^2 k}{(k^2+z^2+\eta^2)\sqrt{k^2+\beta^2 z^2+\beta^2\eta^2}}\right] d\eta$$

Combining and integrating again there is obtained, for $M_o < 1$,

$$\varphi = \frac{\Delta\varphi}{4\pi}\left[\text{arc tan }\frac{(x+M_o t)\sqrt{t^2-x^2-z^2}}{z(M_o x+t)} - \text{arc tan }\frac{x\sqrt{t^2-x^2-z^2}}{tz} + \right.$$

$$\left.\text{arc tan }\frac{y(x+M_o t)}{z\sqrt{(x+M_o t)^2+\beta^2(y^2+z^2)}} - \text{arc tan }\frac{xy}{z\sqrt{x^2+y^2+z^2}}\right] \tag{49}$$



which is the potential within the primary spherical wave shown in the sketch. Notice that across the shaded area the jump in potential given by equation (49) is equal to $\Delta\varphi$ and elsewhere the potential is continuous. Along the portion of the sphere $t^2 = y^2+z^2+x^2$ for which $y > 0$, the potential is equal to that derived for the two-dimensional vortex; and on the rest of the sphere, φ equals 0.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., October 12, 1950.

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Flow, Compressible 1.1.2

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Three-Dimensional, Unsteady-Lift Problems in High-Speed Flight - Basic Concepts

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(Abstract on reverse side)

Wings, Complete - Theory 1.2.2.1

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