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Abstract for F49620-95-1-0296

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Nonlinear Robust Control: Theory and Applications
Final Report for AFOSR Grant F49620-95-1-0296

William M. McEneaney

1. Introduction

This final report is focused on methods for the reduction of the effects of unwanted disturbances on aerospace systems.

The approach taken was that of H_∞, or more generally Robust, control. This is an approach where one views the disturbances in the system as antagonistic to our goal and attempts to attenuate their effect on the system. These disturbances are assumed to have finite energy (L_2 processes). One may contrast this with the methods of stochastic control where the disturbances in the system are random rather than antagonistic, but more importantly are typically diffusion processes. Diffusion processes are driven by Brownian motion which has infinite energy. This contrast between Robust Control and control of diffusion processes in terms of energy of the disturbances has important numerical ramifications. In particular, the dynamic programming equations (DPE's) associated with Robust Control are first–order while those associated with stochastic control are second–order. We make use of fast numerical techniques for first–order equations which are not applicable to the second–order case.

We dealt primarily with nonlinear systems. H_∞ control was developed first for linear systems in the frequency domain. Later, a state space formulation was found. This state–space formulation allowed one to generalize to nonlinear systems and to other paradigms (such as Robust Escape). In the state–space formulation, Robust/H_∞ control formulations take the form of differential games. The corresponding DPE's are Isaacs equations (or in the case of a fixed feedback control, Hamilton–Jacobi–Bellman equations). For many problems, it has been shown that the correct weak solution of such a DPE is the viscosity solution. The solution of the DPE's is at the heart of our control approach.

There are two philosophies regarding computation of controls. The first is to pre–compute the feedback controls and store them in look–up tables which are read in real–time. The second is to generate the optimal (or sub–optimal) controls in real–time by some reasonably fast algorithm. Both approaches have merit. The latter approach has the advantage that the control system may adapt to changing parameters in real–time, whereas this may not be feasible for the former if the parameter space becomes too large. With nonlinear control, the computation of controls in real–time has always been a critical question. We used methods which may be used in the first approach (off–line) or for lower dimensional systems in the second approach (real–time algorithm). It is a fundamental point that we believe that these algorithms are applicable in real–time for appropriate nonlinear systems. This allows us to overcome a hurdle which has often stood in the way of nonlinear control and the dynamic programming approach to nonlinear control in particular. The algorithms use a generalization of the concept of characteristics which is only available for first–order PDE's (not applicable to stochastic control). The use of characteristics allows us to solve the DPE locally rather than over the whole state space. This technique bears some resemblance to the Pontryagin Maximum Principle, but is applicable to Robust/H_∞ control problems.

The bulk of the work is broken down into three areas: Nonlinear Robust Escape Formulation, Nonlinear H_∞ Control, and Robust Filtering. For Robust Escape, we discuss the current state of the theory, present examples and demonstrate some initial software which has been applied to compute controls for the Compressor Stall problem. We then discuss nonlinear H_∞ Control in general, and present an application to Aerospace Plane Ascent where we verify in a simulation that our algorithm does indeed produce an H_∞ controller. Lastly, we discuss a Robust Filter. This work was initially started in response to the partially–observed H_∞ control problem. However, it quickly became clear that the concept of attenuating disturbances in a robust fashion for tracking applications has several advantages similar to those presented in standard Robust/H_∞ Control. Since tracking absolutely requires real–time algorithms, our approach appears quite fruitful.

Since the document is quite long, we indicate briefly here the main areas.

Nonlinear Robust Escape:

We are currently using a method of characteristics for Robust Escape software. Some results for this method were obtained for the H_∞ case but not for Robust Escape. An application to the compressor stall problem is given.

Nonlinear H_∞ :

We have examined the possible multiple solutions to the H_∞ DPE, and have developed a method for isolating the "correct" solution in the case of fixed feedback control. This is extended to some active control cases as well. We also have some results on the connection between the characteristics and viscosity solutions for H_∞ control. We have employed the method on the Aerospace Plane Ascent problem and verified that it does indeed work.

Nonlinear Robust Tracking:

We are generating software for Robust Tracking of nonlinear systems with state spaces of several dimensions. There are some open questions regarding the numerics in this case. In particular, in the presence of very high quality measurements, the state–space components of the characteristics diverge rather rapidly.

2. Robust/H_∞ Control of Nonlinear Systems

Robust/H_∞ control techniques attempt to bound the system cost by some measure of the disturbance size. The most common measure of disturbance size is the L_2 norm. To be more specific, consider a system of the form

$$\frac{dX}{dt} = f(X,u) + \sigma(X)w \qquad (2.1)$$
$$X(0) = x$$

where X is the state, u is the control and w is the disturbance. Let us specifically assume throughout that X takes values in $\Re^n$, u takes values in some compact set U, and w takes values in $\Re^m$. One typically wants to obtain a control that bounds the cost to the controller from above by some function of the initial condition, x, and the L_2–norm of $w(\cdot)$. In the H_∞ set–up, this typically takes the form

$$\int_0^T C_1|X(t)|^2 + C_2|u(t)|^2\,dt \le \gamma^2 \int_0^T |w(t)|^2\,dt + W(x) \qquad \forall T \in (0,\infty) \qquad (2.2)$$

where $W(0) = 0$. In this case, W is referred to as a storage function, and γ is referred to as the disturbance attenuation constant. One wishes to find a controller u^* such that (2.2) holds for all $w \in L_2^{loc}$ where

$$L_2^{loc} = \{w : [0,\infty) \to \Re^n : \int_0^T |w(t)|^2\,dt < \infty \quad \forall T < \infty\}. \qquad (2.3)$$

Note that in the absence of a disturbance ($w \equiv 0$), (2.2) implies that $X(t) \to 0$ as $t \to \infty$. (This is referred to as internal stability.)

Another robust paradigm of significant interest (to be described more fully below) is the Robust Escape Problem. In this case, there is some set G that we would like to keep the state $X(t)$ in as long as possible. In this case, assume $x \in G$. Let τ be the first time the state leaves the set G. The robust criterion takes the form

$$\tau \ge \frac{W(x)}{\theta + \frac{1}{2}\frac{1}{\tau}\int_0^\tau |w(t)|^2\,dt} \qquad (2.4)$$

where θ is a design parameter analogous to the H_∞ disturbance attenuation constant γ. Consequently, if the average power $\frac{1}{\tau}\int_0^\tau |w|^2 < P$, then we have the cost bound

$$\tau \ge \frac{W(x)}{\theta + \frac{1}{2}P}.$$

This is the escape form of the Robust/H_∞ approach. This form is also quite similar to the Finite Power Gain approach recently being considered by James and Dower. [DJ].

There are other systems for which one can apply Robust techniques such as nonlinear finite time horizon systems [Mcr], and Markov Chain models [FHH], [FGM].

In the following subsections, we explore Robust Escape theory and applications (Compressor Stall), H_∞ control theory and applications, and recent numerical methods for these and other problems.

2.1 Nonlinear Robust Escape Problems

We first discuss the theory including examples and analysis of numerical methods. Then we turn to a specific application to compressor stall control.

2.1.1 General Theory

We briefly review the general theory of Robust Escape control.

Consider a system of the form (2.1) where $x \in G$. Suppose that G (the set we wish to keep the state in) is open and that the closure of G, $\overline{G}$, is compact. Suppose that $\overline{G}$ satisfies an exterior sphere condition, that is for any x in the boundary of G there exists a $y \in \overline{G}^c$ and an $r > 0$ such that $B_r(y) \cap \overline{G} = x_0$ where $B_r(y)$ is the closed sphere of radius r centered at y. (Any convex set or set with smooth boundary certainly satisfies this condition.) The Robust Escape problem may be formulated as a zero–sum, deterministic, differential game where u represents the control for the minimizing player, and w represents the control for the maximizing player. The payoff (which these players are trying to control) takes the form

$$P(x, u(\cdot), w(\cdot)) = -\int_0^{\tau} \left[\theta + \frac{1}{2}|w|^2\right] dt. \tag{2.5}$$

Let $\mathcal{U}$ be the set of measurable controls taking values in U. Let $\mathcal{W} = L_2^{loc}(\Re; \Re^m)$ be the space of disturbances. The Elliott–Kalton value of the game is used ([EK], [ES]). To be specific, a strategy for the minimizing player (our control) is a mapping from $\mathcal{W}$ to $\mathcal{U}$ which is non–anticipative. That is, $\phi : \mathcal{W} \to \mathcal{U}$ is a strategy for the minimizing player if for any $t \in [0, T]$ and any $w, \widehat{w} \in \mathcal{W}$ such that $w(r) = \widehat{w}(r)$ for all $r \in [0, t]$, one has $\phi[w](r) = \phi[\widehat{w}](r)$ for all $r \in [0, t]$. This condition guarantees that the strategy for the minimizing player will not have foresight of the behavior of the maximizing player. Let this set of strategies be denoted by Φ. The lower value of the game is given by

$$W(x) = \inf_{\phi \in \Phi} \sup_{w \in \mathcal{W}} P(x, \phi[w], w). \tag{2.6}$$

An analogous definition holds for the upper value. If the upper value and the lower value are identical, then the game has value in the Elliott–Kalton sense.

The Isaacs equation corresponding to this game is given by

$$\begin{aligned} \theta &= H(x, \nabla W) \qquad x \in G \\ W(x) &= 0 \qquad x \in \partial G \end{aligned} \tag{2.7}$$

where ∂G is the boundary of G and H is given by

$$\begin{aligned} H(x,p) &= \min_{u \in U}[f(x,u) \cdot p] + \max_{w \in \Re^n} \left[(\sigma(x)w)^T p - \frac{1}{2}|w|^2\right] \\ &= \min_{u \in U} f(x,u) \cdot p + \frac{1}{2} p^T \sigma(x) \sigma^T(x) p. \end{aligned}$$

Suppose that f and σ are Lipschitz and that $\sigma\sigma^T$ is uniformly non–degenerate, i.e. there exists $\mu > 0$ such that

$$\xi^T \sigma(x) \sigma^T(x) \xi \geq \mu |\xi|^2 \qquad \forall x \in \overline{G}, \ \forall \xi \in \Re^n.$$

Then the game has value, V, and this value is the unique continuous viscosity solution of the Isaacs equation [DMc], [McD].

Let us clarify the robust interpretation of this game described above. Suppose, for instance, that there exists an optimal feedback control for the minimizing player in this game. This implies that

$$W(x) \geq -\int_0^{\tau} \left[\theta + \frac{1}{2}|w|^2\right] dt \qquad \forall w \in L^2. \tag{2.8a}$$

Let $Q(x) = -W(x)$ and

$$\mathcal{W}^P = \left\{ w \in L^2[0,\infty) : \frac{1}{T}\int_0^T |w|^2\, dt \leq P \quad \forall 0 \leq T < \infty \right\}.$$

Then the (2.8a) has the interpretation

$$\tau \geq \frac{Q(x)}{\theta + \frac{1}{2}P} \qquad \forall w \in \mathcal{W}^P. \tag{2.8b}$$

This is a lower bound on the escape time as a function of the average power of the input noise. It is directly analogous to the attenuation bound of H_∞ control, as well as to finite power gain control [DJ].

We now provide some simple examples as a means of illustrating these concepts. (The Compressor Stall application will be discussed in the next section.) The absolutely simplest example is a one-dimensional system of the form

$$\frac{dX}{dt} = u(t) + w(t)$$
$$X(0) = x \in G.$$

Let $G = (-1, 1)$, that is we wish to keep the state between -1 and 1. Let the control take values in the set $[-1, 1]$, so that we are limited by 1 in the magnitude of our control effort. The payoff is simply

$$P(x, u(\cdot), w(\cdot)) = -\int_0^{\tau} (\theta + w^2(t))\, dt,$$

and we take the design parameter $\theta = 1$. Then the Isaacs equation (2.7) takes the form

$$\begin{aligned} 0 &= \min_{u\in[-1,1]}[uW_x] + \max_{w\in\Re}[wW_x - \frac{1}{2}w^2] \\ &= -|W_x| + \frac{1}{2}W_x^2 \\ W(-1) &= W(1) = 0 \end{aligned}$$

which has the viscosity solution

$$W(x) = 2(|x| - 1).$$

The optimal control is given by

$$u^*(t) \in \text{argmin}_{v\in[-1,1]}[vW_x(X(t))] = \begin{cases} -1 & \text{if } X(t) > 0 \\ 1 & \text{if } X(t) < 0 \end{cases}$$

and $u*(t)$ can take any value in $[-1,1]$ when $X(t)=0$. This optimal control is obvious of course; you always push as hard as you can away from the closer boundary. We obtain a bang–bang control due to the fact that we do not put any cost on the control.

A slightly more complex example is as follows. Consider a two–dimensional system of the form

$$\frac{dX}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} u_t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{pmatrix} w_t$$

which represents non–degenerate linear dynamics plus a constant drift in the $(1,1)$ direction. Let the set we wish to keep it in be the rectangular region

$$G = (0,2) \times (0,1),$$

and let the control take values in the disk

$$U = \{v \in \Re^2 : |v| \leq 4\}$$

so that we are combining a variety of geometrical forms here. Let the design parameter be $\theta = 1$. This problem is still analytically solvable, although the algebra gets a bit more tedious. The solution is piece–wise planar. This solution, W, is depicted in figure 2.1. It is plotted as a function over the region G given above. (The scalloping is due to the plotting software; the true solution looks like four planes descending from the four boundaries of the set G.) In each of the four planar regions, the optimal control is given by

$$u^*(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \nabla W(X(t)).$$

Also. along the edges where these planar regions meet, the optimal control can be chosen to be any value in the subdifferential at those points.

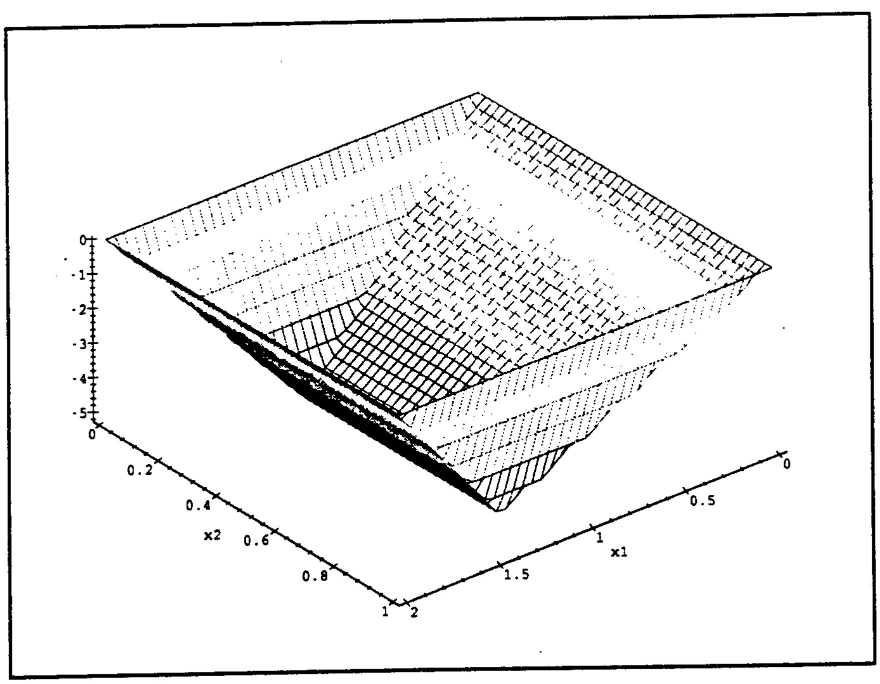


Figure 2.1: Two–dimensional example

As a last example, we consider a degenerate problem. In real–world systems, the state often represents a vector of objects such as position and velocity. In such a case, some of the component equations are exact (eg. the derivative of position is velocity), and one would not expect to have disturbances affecting those components of the system. Such systems are often referred to as degenerate (or not uniformly elliptic), and the theory is often not fully developed in those cases. For general degenerate problems, the solutions of the Isaacs equations may not be continuous, however for degeneracies of this form we have shown that they are continuous. Consequently, one does not need the tremendously difficult machinery of discontinuous viscosity solutions. We consider a simple such example. Let the system be given by

$$\frac{dX}{dt} = V$$
$$\frac{dV}{dt} = u + w$$

where X represents position and V represents velocity. Let the control magnitude be bounded as

$$U = \{v \in \Re : |v| \leq 4\}.$$

Let the region we are trying to keep the state in be

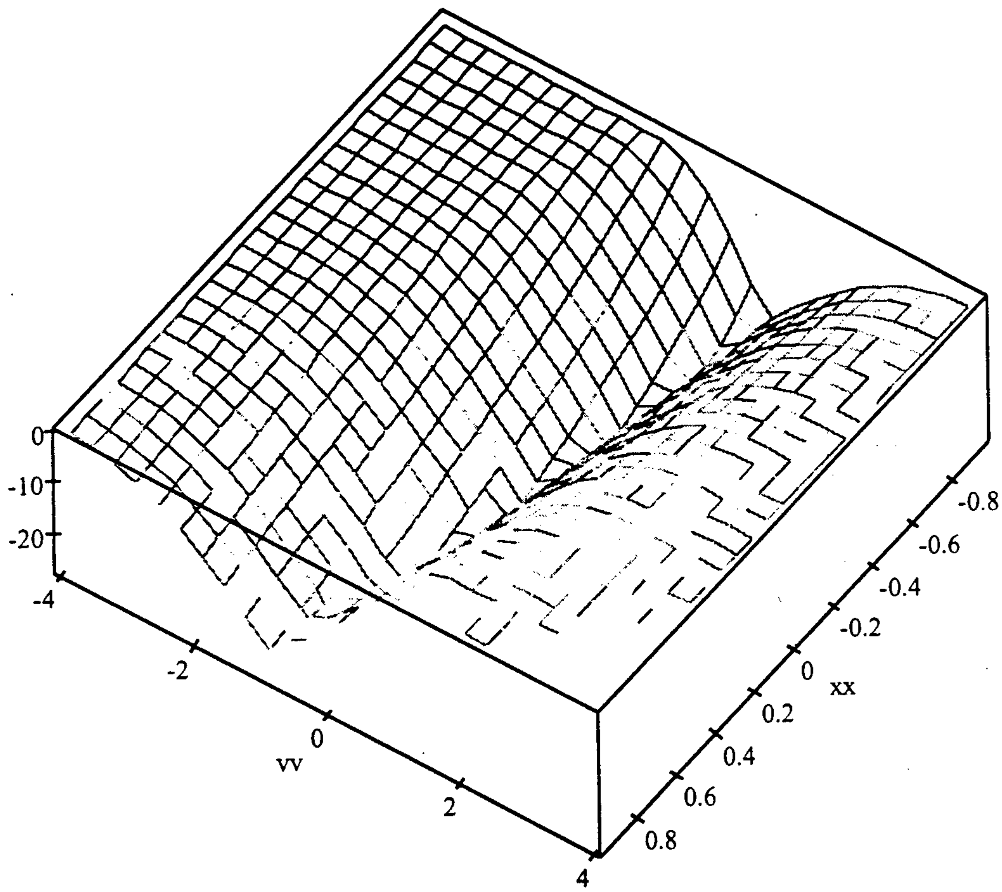
$$G = (-1, 1) \times (-4, 4).$$

This is analogous to a tracking system where one needs to keep the state in a field–of–view (in this case $X \in (-1, 1)$). Further, in such a tracking situation, the state cannot be allowed to move too quickly across the field–of–view; otherwise the image does not dwell long enough on individual detectors to register. Consequently we include a bound $V \in (-4, 4)$ in our example region. We again take the design parameter $\theta = 1$ for simplicity. In this case, the Isaacs equation takes the form

$$\begin{aligned} 0 &= -\theta + vW_x + \min_{|u| \leq 4} [uW_v] + \max_{w \in \Re} [wW_v - |w|^2] \\ &= -1 + vW_x - 4|W_v| + \frac{W_v^2}{4}. \end{aligned}$$

We again computed an analytical solution, although in this case a computer algebra system (Maple) was used as an aid. The solution is depicted in figure 2.2. Note that the solution is still continuous, although it does have a cusp along a curve across the region. The optimal control is given by $u^*(t) = \pm 4$ depending on which side of this curve the state lies on at a given moment.

Figure 2.2: Second–order example appears on next page


0
-10
-20
-4
-2
0
vv
2
4
-0.8
-0.6
-0.4
-0.2
0
xx
0.2
0.4
0.6
0.8

2.1.2 Application to Compressor Stall

The compressor stall problem is one which may be particularly suited to a Robust Escape formulation. The specific aspect of compressor stall that makes Robust Escape appropriate is the fact that entry of the state into rotating stall is catastrophic in the sense that it may require shut–down and restart of the system. One could alternatively employ nonlinear H_∞ control to stabilize the system around some point different from stall. However, that does not capture the essence of avoiding the specific region around the rotating stall equilibrium. Consequently, we have chosen to work with a Robust Escape formulation.

We use a compressor model adapted from [MG] in the form

$$\frac{dR}{dt} = \sigma R(1-\phi^2-R) \tag{2.9a}$$

$$\frac{d\phi}{dt} = -\psi+\psi_{c0}+1+\frac{3}{2}\phi-\frac{1}{2}\phi^3-3\phi R \tag{2.9b}$$

$$\frac{d\psi}{dt} = \frac{1}{\beta^2}\left[\phi+1-\gamma\sqrt{\psi}\right] \tag{2.9c}$$

where

$$R = \frac{J}{\tilde{\phi}}$$

$$\phi = \frac{\tilde{\phi}}{H}-1$$

$$\psi = \frac{\tilde{\psi}}{H}$$

where $\tilde{\phi}$ represents the circumferentially–averaged axial flow coefficient, $\tilde{\psi}$ represents the pressure rise, J represents the squared amplitude of the angular variation, and the parameter H is taken from [B] as $H = 0.32$. Further, the time variable, t, is scaled from real–time, ζ, by $t = \frac{H}{l_c W}\zeta$ where [B] $l_c = 6$ and $W = 0.18$. The remaining coefficients are given by

$$\sigma = \frac{3al_c}{(1+ma)}$$

$$\frac{1}{\beta^2} = \frac{W}{4B^2H^2}$$

$$\psi_{c0} = \frac{\tilde{\psi}_{c0}}{H}$$

$$\gamma = \tilde{\gamma}\sqrt{H}$$

where from [B], we take $a = 1/3$, $m = 2$, $\tilde{\psi}_{c0} = 0.23$, $B = 0.1$ and $\tilde{\gamma} \in (0,1)$. Consequently, we find

$$\sigma = 3.6$$

$$\frac{1}{\beta^2} = 44.$$

$$\psi_{c0} = 0.72$$

$$\gamma = \tilde{\gamma}\sqrt{0.32}.$$

The corresponding compressor characteristic

$$f(\phi) \doteq \psi_{c0}+1+\frac{3}{2}\phi-\frac{1}{2}\phi^3$$

is depicted in figure 2.3. We consider a fixed value of the throttle setting $\gamma = 0.5$. For this value of γ, there is a rotating stall equilibrium of

$$R = 0.923, \qquad \phi = -0.278, \qquad \psi = 2.083$$

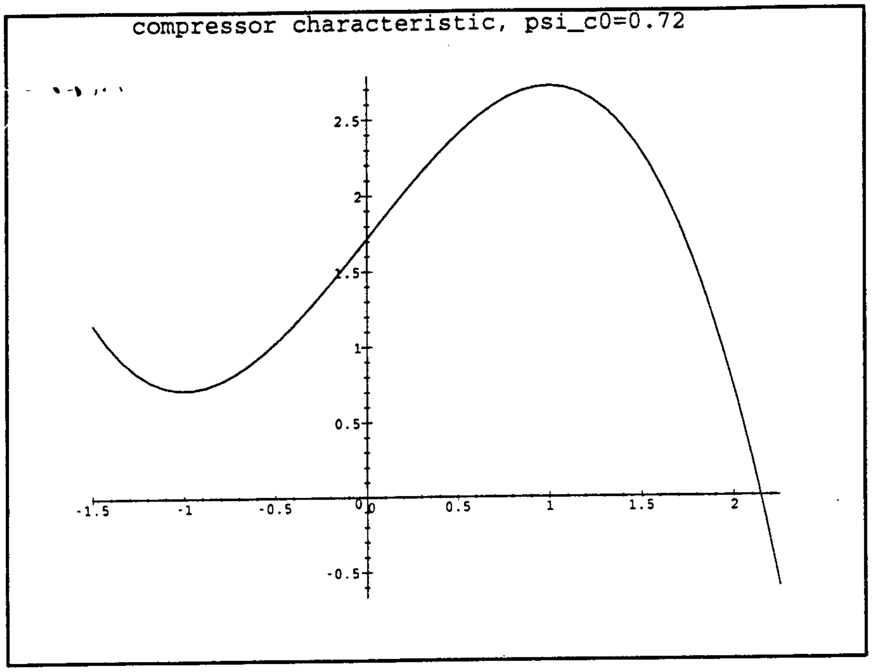


Figure 2.3: Compressor characteristic

For a specific example, we allow disturbance to affect all components of (2.9), and allow a control of the second component (2.9b). In this case, one has a model of the form

$$\frac{dR}{dt} = \sigma R(1 - \phi^2 - R) + w_1 \tag{2.10a}$$

$$\frac{d\phi}{dt} = -\psi + \psi_{c0} + 1 + \frac{3}{2}\phi - \frac{1}{2}\phi^3 - 3\phi R + u + w_2 \tag{2.10b}$$

$$\frac{d\psi}{dt} = \frac{1}{\beta^2}\left[\phi + 1 - \gamma\sqrt{\psi}\right] + w_3. \tag{2.10c}$$

Letting $X = (X_1, X_2, X_3) \doteq (R, \phi, \psi)$ and $\vec{w} \doteq (w_1, w_2, w_3)$, this can be rewritten as

$$\frac{dX}{dt} = F(X) + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} + \vec{w} \doteq \begin{pmatrix} \sigma X_1(1 - X_2^2 - X_1) \\ -X_3 + \psi_{c0} + 1 + \frac{3}{2}X_2 - \frac{1}{2}X_2^3 - 3X_2X_3 \\ \frac{1}{\beta^2}X_2 + 1 - \gamma\sqrt{X_3} \end{pmatrix} + \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} + \vec{w}. \tag{2.11}$$

This particular model has been chosen simply as an example. It is not intended to suggest that this is in any way the optimal choice of models. Continuing, to be specific, we allow the control $u(\cdot)$ to take values only in the interval $U \doteq [0, 2]$.

Finally, we let the region of interest be given by

$$G \doteq \{X \in \Re^3 : X_3 \geq 3\}, \tag{2.12}$$

i.e. the set of states where $\psi \geq 3$. This region was chosen so as to be as simple as possible and yet not include the rotating stall equilibrium computed above as being at $(0.923, -0.278, 2.083)$.

For this problem, the Isaacs equation is (2.7) with the particular Hamiltonian

$$\begin{aligned} H(x,p) &= \min_{u\in[0,2]}\{[F(x)+(0,u,0)^T]^T p\} + \max_{w\in\Re^3}\{w^T p + \frac{1}{2}|w|^2\} \\ &= \min_{u\in[0,2]}\{[F(x)+(0,u,0)^T]^T p\} + \frac{1}{2}|p|^2. \end{aligned} \tag{2.13}$$

In the theory of the previous section, it was indicated that the value of the game was the viscosity solution of the Isaacs equation in the case that G was compact. Here we have an unbounded G. It remains to prove that the result remains true in this case. Assuming that this is in fact true, we see by examining (2.13) that once the solution has been computed, the optimal control will take the bang–bang form

$$u^*(X(t)) = \begin{cases} 2 & \text{if } W_{X_2}(X(t)) < 0 \\ 0 & \text{if } W_{X_2}(X(t)) \geq 0 \end{cases} \tag{2.14}$$

where W_{X_2} is the partial of the value with respect to the second component (ϕ).

In [McDa], it is shown that under certain conditions, the viscosity supersolution of the Isaacs equation corresponding an H_∞ control problem can be obtained by propagating the characteristic equations over the state–space, and then, for each point in the state–space, taking the minimum value over all the characteristics which pass through that point. We expect this algorithm to hold (with maximum replaced by minimum) for the case of Robust Escape problems as well. Specifically, for this problem the characteristic equations are the 6–dimensional set of ODE's

$$\begin{aligned} \frac{dX}{dt} &= H_p(X,P) = F(X(t)) + (0, u_m(t), 0)^T + P(t) \\ \frac{dP}{dt} &= -H_x(X,P) = -F_x^T(X(t))P \end{aligned} \tag{2.15}$$

where $u_m(t) = \text{argmax}_{u\in U}[uP_2(t)]$, and F_x is the matrix of partials of F. One must specify the initial conditions for (2.15). In particular, one must propagate (2.15) starting from a sufficiently dense network of initial points on the boundary of G to obtain an approximation of the value. Thus, one has $X_3(0) = 3$ by (2.12), and then we let $X_1(0) = x_1$ and $X_2(0) = x_2$ range over various points on the boundary. Since the value is identically zero on the boundary, this implies that the partials in the plane of the boundary are zero so that $P_1(0) = 0$ and $P_2(0) = 0$. Finally, given all of the other initial conditions, one can obtain $P_3(0)$ from (2.7), (2.13). In particular, with the above, (2.7) becomes

$$0 = -\theta + \frac{1}{\beta^2}(x_2 + 1 - \gamma\sqrt{3})P_3(0) + \frac{1}{2}P_3^2(0)$$

which is simply a quadratic equation. This yields

$$P_3(0) = -\frac{1}{\beta^2}\left[(x_2 + 1 - \gamma\sqrt{3}) + \sqrt{(x_2+1-\gamma\sqrt{3})^2 + 2\beta^4(3+1)}\right]$$

where we have taken the positive root in order to obtain the characteristic trajectories which enter the set G. The value along each characteristic path, which we will denote by $V(t;X(0),P(0))$, is given by

$$V(t) = \int_0^t \left[P^T \frac{dX}{dt} \right] dt. \tag{2.16}$$

The value at each point $x \in G$ is obtained by considering all the characteristic paths such that $X(t) = x$ for some $t \geq 0$ (and some initial condition $(X(0),P(0))$), and their corresponding values $V(t;X(0),P(0))$. The maximum such value yields the value function at x. (In the actual software, the region G is divided into cubes, and the maximum value over all trajectories passing through that cube is assigned to be the value for that cube.)

This method has been employed for the above problem. To complete the specification of the problem, the value of the design parameter θ was taken to be simply $\theta = 1$. The value function was computed over a portion of the space by the above method. The subset of G over which the value was computed was $(R,\phi,\psi) \in [0.5,2] \times [-1,2] \times [3,7]$. The computation time on a SPARCstation 5 was less than a minute. Since the value is defined over a 3–dimensional region, it is difficult to depict graphically. We display Q, that is the value multiplied by -1. (Recall the robust interpretation (2.8b).) This is displayed over a slice of the region given by $R = 0.5$, and is depicted in figure 2.4. Recall that G was the region where $\psi \geq 3$, and so the value is zero at $\psi = 3$ and positive above that. Recall from (2.14) that the optimal control is 2 where the partial of $Q = -W$ with respect to ϕ is positive and 0 elsewhere. Considering figure 2.4, one immediately sees that in this case the control would be 2 for most values and 0 in the corner where ϕ is large and ψ small. One might also note that in figure 2.4, the value tends to separate from the boundary value $W(R,\phi,3) = 0$ for larger values of ϕ. This is due to the fact that the worst–case trajectories starting out at say $(0.5, 0.5, 3.1)$ would not escape from a nearby point on the boundary but instead circle upwards and then back down and out at a place further down on the boundary.

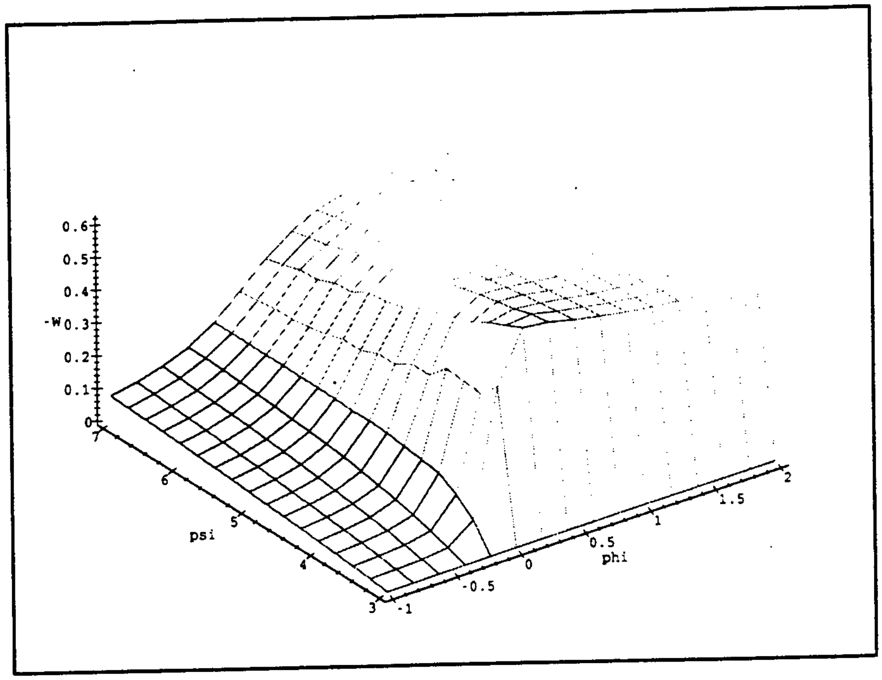


Figure 2.4: Solution

The results depicted in figure 2.4 should not be interpreted as a final (or even serious) solution to the Robust Escape approach to compressor stall. They have been computed merely to indicate that initial software for this problem has been generated, and is generating reasonable results. However, the model considered above is insufficient in several respects. First of all, the relative sizes of the noise in the various components of the dynamics has not been addressed at all. To produce more serious results, one would need to communicate with those modeling the dynamics so as to obtain a reasonable disturbance model. The overall expected size of the noise is also unknown (as well as the controls), so that it is not possible to tune the system design parameters to a reasonable set. Lastly, the region, G was chosen to be a half-space for simplicity. In a more reasonable model, one would need to consider other boundaries as well; certainly one would not want the state to move arbitrarily far in any direction.

One can use the controls obtained in the above manner in a simulation to verify the robustness of the controller. This has been delayed for the compressor stall problem until we have better noise and control models. To see how this may be done, refer to the end of the Aerospace Plane Ascent control discussion in Section 2.2.2.

In this section, we discuss some theoretical/numerical results concerning nonlinear H_∞ control and some remaining open questions. We also indicate how the theory and numerics can be applied in a particular case – Aerospace Plane Ascent.

2.2.1 General Theory

Once again, we suppose the dynamics are of the form (2.1). Let $L(x,u)$ be our cost criterion. In nonlinear H_∞ control, we choose a disturbance attenuation constant, γ, and look for a feedback control $u^*(x)$ such that

$$\int_0^T L(X(t), u^*(X(t)))\, dt \le \gamma^2 \int_0^T |w(t)|^2\, dt + W(x) \tag{2.17}$$

for some function of initial condition $W(x)$ for all $w \in L_2(0,T)$ for all $T < \infty$. Ideally one would like γ to be nearly as small as possible. However, it is difficult to find the smallest feasible γ, and further in some cases this leads to controls which are undesirable for other reasons. Consequently, one most often chooses a value of γ for which a solution exists, and searches for controls corresponding to that attenuation level γ.

Consider the differential game with dynamics (2.1) and payoff and value given by

$$\begin{aligned} P(x,u,w,T) &= \int_0^T L(X(t), \theta[w](t)) - \gamma^2 |w_t|^2\, dt \\ \overline{W}(x) &= \inf_{\phi\in\Phi} \sup_{w\in\mathcal{W}} \sup_{T<\infty} P(x, \phi[w], w) \end{aligned} \tag{2.18}$$

where the definitions of Φ and $\mathcal{W}$ can be found in Section 2.1.1. If one obtains an optimal feedback control $u^*(x)$ for the game, then

$$\overline{W}(x) = \sup_{w\in\mathcal{W}} \sup_{T<\infty} \int_0^T L(X(t), u^*(X(t))) - \gamma^2 |w_t|^2\, dt$$

which implies that (2.17) is satisfied. More generally, fix a feedback $\widetilde{u}(x)$, and consider

$$\widetilde{W}(x) = \sup_{w\in\mathcal{W}} \sup_{T<\infty} \int_0^T L(X(t), \widetilde{u}(X(t))) - \gamma^2 |w_t|^2\, dt. \tag{2.19}$$

If this problem has finite value $\widetilde{W}$, then (2.17) is satisfied with attenuation γ and storage $\widetilde{W}$.

The Isaacs equation corresponding to (2.18) is

$$\begin{aligned} 0 &= \inf_{u\in U} \sup_{w\in\Re^n} \left\{ [f(x,u) + \sigma(x)w]^T \nabla W + L(x,u) - \gamma^2 |w|^2 \right\} \\ &= \inf_{u\in U} \left\{ f^T(x,u)\nabla W + L(x,u) \right\} + \frac{1}{4\gamma^2} \nabla W^T \sigma(x) \sigma^T(x) \nabla W \\ &\doteq H(x, \nabla W). \end{aligned} \tag{2.20}$$

In the case where we take a feedback $\widetilde{u}(x)$, we can rewrite the dynamics as

$$\dot{X} = \overline{f}(X(t)) + \sigma(X(t))w(t) \doteq f(X(t), \widetilde{u}(X(t)) + \sigma(X(t))w(t)$$

and the cost as $\overline{L}(x) = L(x, \tilde{u}(x))$. Then, in this case, (2.20) is replaced by

$$0 = \{\overline{f}(x)\nabla W + \overline{L}(x)\} + \frac{1}{4\gamma^2}\nabla W^T \sigma(x)\sigma^T(x)\nabla W \tag{2.21}$$
$$\doteq \widetilde{H}(x, \nabla W).$$

Note that (2.20), (2.21) do not involve W itself, only its gradient. Consequently, we normalize the solutions by requiring $W(0) = 0$ as part of the solution. (Otherwise, if $W(\cdot)$ is a solution, then so is $W(\cdot) + k$ for any $k \in \Re$.)

Typically, one considers quadratic cost criteria such as

$$L(x, u) = |x|^2 + |u|^2. \tag{2.22}$$

If the nonlinear theory can handle such cost criteria (with Lipschitz dynamics), then it can subsume the linear–quadratic theory. One would like to compute the control by solving the Isaacs equation (2.20), or in the case of a trial control $\tilde{u}$, compute the solution of (2.21) to determine if $\tilde{u}$ is indeed an H_∞ controller (which will be true if the solution exists and is nonnegative [So]).

Unfortunately, under quadratic growth assumptions such as (2.22), there exists multiple solutions to (2.20) and (2.21). For instance, consider the simple one–dimensional case where $\overline{f}(x) = -x$, $\overline{L}(x) = x^2$ and $\sigma(x) \equiv 1$. Let $\gamma > 1$. (For $\gamma < 1$, there does not exist a solution, and one does not have H_∞ control for this system with such low attenuation.) Then (2.21) becomes

$$0 = \frac{W_x^2}{4\gamma^2} - xW_x + x^2. \tag{2.23}$$

This has two classical solutions with $W(0) = 0$:

$$W^1(x) = \gamma^2(1 - \sqrt{1 - 1/\gamma^2})x^2 \tag{2.24a}$$
$$W^2(x) = \gamma^2(1 + \sqrt{1 - 1/\gamma^2})x^2 \tag{2.24b}$$

and an infinite number of viscosity solutions Consequently it is important to be able to distinguish the correct one in any real computational approach. Soravia [So] indicated a connection between the value or storage $\overline{W}$ for (2.19) and the smallest, nonnegative viscosity supersolution. However, this condition (i.e. smallest) is difficult to establish in practice. In [Mcel], we demonstrate that there exists a unique viscosity solution to (2.21) satisfying a certain quadratic growth condition ($W(x) \leq k|x|^2$ for a particular k), and that this is the correct solution for our purposes. In the above example, this solution is the classical solution given by (2.24a). Under certain conditions, we have also shown this for Isaacs equation (2.20); however a general proof in that case remains an open question. (We should note that some authors (eg. [vdS]) have demonstrated for certain nonlinear systems that on some arbitrarily small region around the origin, the correct solution corresponds to the stable manifold of the Hamiltonian system; this corresponds to our solution. However, our solution is global not just local on some possibly very small neighborhood.)

Once one is able to separate correct solutions from artifacts, one is still left with the question of actual computation. Since (2.20), (2.21) are first–order PDEs, a possible approach is a generalization of the method of characteristics. Note that there does not exist a boundary or an initial time from which to begin propagating these characteristics; in this H_∞ situation, one needs to propagate the characteristics outward from the origin backward in time (i.e. backward along the stable manifold). In [McDa], we demonstrate under rather strong assumptions that a generalized method of characteristics yields a viscosity supersolution to the Isaacs equation. Specifically, one takes (2.20) or (2.21) and forms the characteristic equations

$$\dot{X} = -H_p(X, P) \tag{2.25a}$$

$$\dot{P} = H_x(X, P) \tag{2.25b}$$

to which we append

$$\dot{U} = P \cdot \dot{X} = -P \cdot H_p(X, P). \tag{2.25c}$$

In an ideal situation, one would have the solution given by

$$W(x) = U(X(t))$$

for the characteristic such that $X(t) = x$. That is, $U(t)$ would carry the value of the solution at $X(t)$ along each characteristic. This is what occurs in the classical case. However, for our problems, one cannot guarantee that there is a unique characteristic whose projection passes through a given point in the state space. (In fact, merely guaranteeing that *any* characteristic has such a projection is non–trivial!) Let us index a collection of solutions of (2.25) by some abstract index set $b \in \mathcal{B}$, that is let $(X(\cdot; b), P(\cdot; b), U(\cdot; b))$ be a solution to (2.25) satisfying some initial condition. Let $\mathcal{B}$ be some index collection such that for all $b \in \mathcal{B}$ one has

$$\begin{aligned} &H(X(t; b), P(t; b)) = 0 \qquad \forall t \in \Re \\ &\lim_{t\to\infty} X(t; b) = 0, \qquad \lim_{t\to\infty} P(t; b) = 0 \\ &\lim_{t\to\infty} U(t; b) = 0. \end{aligned} \tag{2.26}$$

Under certain conditions, a viscosity supersolution is given by

$$W(x) \doteq \inf\{U(t; b) : b \in \mathcal{B}, t \in \Re \text{ s. t. } X(t; b) = x\}. \tag{2.27}$$

(Note that (2.27) is similar to formulae appearing in [Sua] and [Mi] for optimal control problems.)

One can then apply the controller obtained from this approach in a simulation to demonstrate that one has, in fact, achieved a true, global, nonlinear H_∞ controller. This is discussed for a particular problem in the next section.

2.2.2 Application to Aerospace Plane Ascent

The first part of this section (up to equation (2.31)) recaps the discussion appearing in [McM1].

Under reasonable assumptions, the translational motion of the Aerospace Plane center-of-mass in the post-transonic portion of atmospheric ascent is given by

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{V(T-D)}{m} \\
\frac{dm}{dt} &= -\frac{T}{g_E I_{sp}} \\
\frac{dr}{dt} &= V \sin\gamma \\
\frac{d\gamma}{dt} &= \frac{L}{mV} - \left(\frac{\mu}{r^2} - \frac{V^2}{r}\right)\frac{\cos\gamma}{V}
\end{aligned}
$$

where

$r =$ radial distance of the vehicle from Earth center,

$V =$ vehicle speed,

$E = \frac{V^2}{2} - \frac{\mu}{r} =$ specific energy,

$\mu =$ gravitational constant of the Earth,

$\gamma =$ vehicle flight path angle,

$m =$ vehicle mass,

$T =$ propulsive force,

$D =$ drag force,

$L =$ lift force,

$g_E =$ gravitational acceleration at sea level,

and

$I_{sp} =$ specific impulse of fuel.

The two time-scale approach of Mease et al. ([MVB1], [MK]) will be used to separate the dynamics. In particular, E and m change only rather slowly, while r and γ change more rapidly. One may determine a fuel optimal trajectory for the slow variables (E, m). Then, with respect to the fast variables (r, γ), the values of (E, m) may be assumed to be at some steady-state values (E_s, m_s) at any given time. The control objective for the fast variables is then to continually drive the state back to the desired slow manifold while it is being perturbed by noise. A nonlinear H_∞ approach to this problem for the fast dynamics follows.

Let E_s, m_s, r_s, γ_s be the desired steady-state (relative to the fast dynamics) values of a point on the slow manifold. Let (r, γ) be the fast dynamics values at some time, and let $\Delta r = r - r_s$ and $\Delta\gamma = \gamma - \gamma_s$. Let V and V_s satisfy

$$E_s = \frac{V^2}{2} - \frac{\mu}{r}$$

and

$$E_s = \frac{V_s^2}{2} - \frac{\mu}{r_s},$$

respectively. Then the relevant dynamics become (see [MVB1])

$$\begin{aligned}
\frac{d\Delta r}{dt} &= V \sin\gamma - V_s \sin\gamma_s \equiv f^{(1)} \\
\frac{d\Delta\gamma}{dt} &= -\left(\frac{\mu}{r^2} - \frac{V^2}{r}\right)\frac{\cos\gamma}{V} + \left(\frac{\mu}{r_s^2} - \frac{V_s^2}{r_s}\right)\frac{\cos\gamma_s}{V_s} \\
&\quad - \frac{L_s}{m_s V_s} + \frac{L_s}{m_s V} + \frac{L_f}{m_s V} + \sigma w \\
&\equiv f^{(2)} + \frac{L_f}{m_s V} + \sigma w
\end{aligned} \tag{2.28}$$

where

$L_s =$ the desired value of the lift for the slow manifold,

$L_f =$ the lift control for the fast dynamics,

$w =$ the perturbing noise,

$\sigma =$ the noise coefficient.

For a first attempt, a payoff for the nonlinear H_∞ game (see (2.18)) was chosen to be a simple quadratic of the form

$$P((\Delta r, \Delta\gamma), L_f, w, T) = \int_0^T \Delta r^2 + C_1 \Delta\gamma^2 + C_2 L_f^2 - \theta^2 w^2 \, dt. \tag{2.29}$$

(The case where additional terms are included in P to account for undesirable regions in the state space has also been examined. However, for the sake of brevity, such results are not included here.) To further simplify the first attempt, it has been assumed that the control may be unbounded. Note that if one does not place a bound on the size of the noise, then one must allow possibly unbounded controls as well in order for the H_∞ problem to have a solution. The resulting Hamiltonian for (2.20) then becomes

$$\begin{aligned}
H(\Delta r, \Delta\gamma, W_1, W_2) &= \min_{L_f} \max_{w} \Big\{ W_1 f^{(1)} + W_2 \left(f^{(2)} + \frac{L_f}{m_s V} + \sigma w \right) \\
&\qquad + \Delta r^2 + C_1 \Delta\gamma^2 + C_2 L_f^2 - \theta^2 w^2 \Big\} \qquad (2.30) \\
&= W_1 f^{(1)} + W_2 f^{(2)} + \frac{1}{4}\left[\frac{\sigma^2}{\theta^2} - \frac{1}{C_2 m_s^2 V^2}\right] W_2^2 + \Delta r^2 + C_1 \Delta\gamma^2. \qquad (2.31)
\end{aligned}$$

In order for the problem to have a feasible solution, we must choose the coefficients such that $\frac{\sigma^2}{\theta^2} - \frac{1}{C_2 m_s^2 V^2} < 0$.

The generalized method of characteristics described in the previous section is used to generate the solution. Then, one obtains the H_∞ feedback controller from

$$L_f = -\frac{W_2(\Delta r, \Delta\gamma)}{2C_3 m_s V(\Delta r)}. \tag{2.32}$$

The possible real–time algorithm is as follows (refer to figure 2.6). Given a particular value of $(\Delta r, \Delta\gamma)$, one may search for the characteristic whose state component propagates outward from the origin to $(\Delta r, \Delta\gamma)$. One may store some characteristics such as those depicted in the figure. Then for any $(\Delta r, \Delta\gamma)$, one has bounds on the correct values of (W_1, W_2) which cause a characteristic propagating backwards from $(\Delta r, \Delta\gamma, W_1, W_2)$ to return to the origin. The bounds are obtained by examining the characteristics on either side of $(\Delta r, \Delta\gamma)$. (That is, one has bounds on, say W_2, and then one obtains W_1 from Δr, $\Delta\gamma$, W_2 and the PDE.) Then one applies a shooting technique to find the correct starting value for W_2 and, consequently, W_1. This is analogous to a standard approach for finding the initial values of the costate variables in optimal control theory.

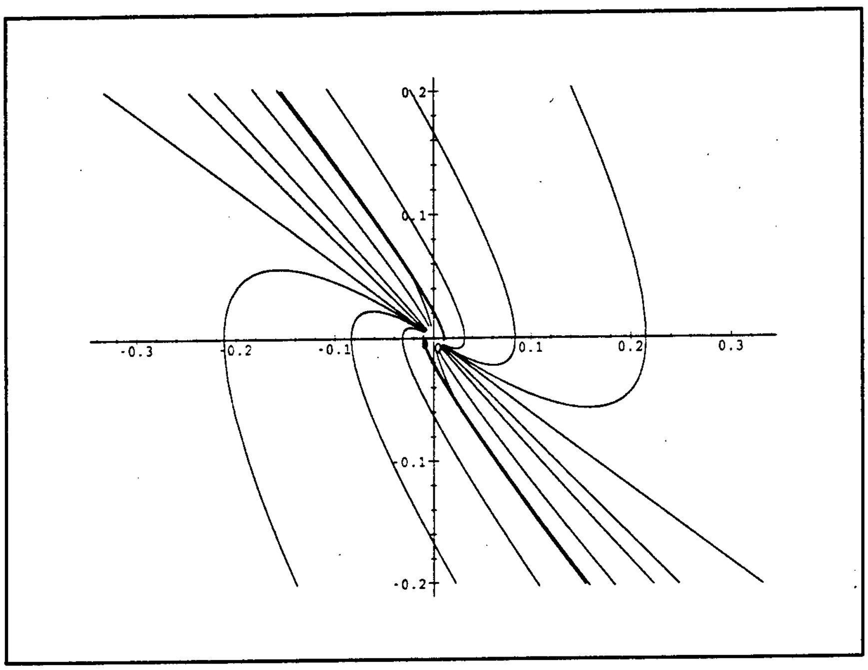


Figure 2.6: Projection of characteristics

This approach was embedded in a simulation, and the disturbance attenuation property was verified.

3. Nonlinear Robust Filtering

In this section, we discuss some recent advances we have made in nonlinear Robust Filtering. We describe the theory and some numerical methods. As additional support for this viewpoint on filtering, we also present a short discussion of the Robust Filter as the limit of the stochastic Risk–Sensitive Filter.

3.1 Motivation and Theoretical Development

Robust Filtering is the natural extension of Robust/H_∞ control methods to the filtering and estimation situation. The approach can be formulated as a deterministic game where the disturbance is chosen by an antagonisitc player trying to degrade our filter performance.

The fact that the disturbances are chosen by an antagonistic player (trying to degrade our filter estimate quality) rather than being random leads naturally to a worst-case based strategy. This is especially appropriate for systems where one desires to track a non-cooperative target [BWGH]. In such a case the state would not be expected to follow a nominal path with diffusion-type disturbances around that trajectory.

The robust/H_∞ approach uses a dynamical model of the form

$$\begin{aligned} \frac{dX}{dt} &= f(X) + \sigma(X)w \\ X(0) &= x_0 \end{aligned} \tag{3.1}$$

(where we are not assuming x_0 known a priori). X is the state, f represents the nominal dynamics, w is a deterministic, but (a priori) unknown L_2 process, and σ is some multiplier on the disturbance. This model is in contrast to the diffusion model

$$dX_t = f(X)\,dt + \sigma(X)\,dB_t$$

where $B_.$ is a Brownian motion. The fact that the disturbance in the robust model is an L_2 process rather than a Brownian motion has important numerical ramifications. Specifically, in the case of a stochastic model, the nonlinear filter takes the form of a second-order PDE often referred to as a Zakai or Kushner equation. On the other hand, the Robust Filter will lead to a first-order PDE similar to those discussed in the previous section. One may take advantage of the first-order nature in ways which are not possible for second-order equations. In particular, the information is carried along characteristics which propagate at a finite rate, thus allowing one to use numerical approaches such as the one described below. In contrast, for a second-order equation, a small change in the initial conditions at one point propagates at infinite speed throughout the system. Thus in that case, one is required to solve the PDE on the whole space (or some approximating large set) rather than just on some narrow manifold containing the optimal estimate. This difference is critical to the question of feasible real-time estimation and tracking.

We use an analogous measurement model of the form

$$z(t) = g(X(t)) + \rho(X(t))v(t) \tag{3.2}$$

where z is the measurement process taking values in $\Re^l$, g is the nominal measurement model, $v \in L_2$ is a finite-energy noise signal taking values in $\Re^k$ (where $k \geq l$), and ρ is some multiplier on the noise. One can view w and v together as antagonistic to our goal, and attempt to attenuate their effect. This will be called Robust (or H_∞) Filtering.

We also note that some of this work harks back to Mortensen [Mo] and Hijab [H], but is now set in the framework of nonlinear H_∞ control and viscosity solutions. In particular, the information state is quite close. However, this information state is interpreted in an H_∞-framework which then leads to a different estimate. This new estimate is, in fact, the estimate which corresponds to the risk-sensitive limit of the stochastic model. This work is also closely related to the study of H_∞ control under partial observations [JBE]. However, in this case, the chosen criterion leads to a finite-dimensional dynamic programming equation as opposed to the infinite-dimensional PDE of [JB].

In analogy with H_∞ control, the Robust Filter will provide an estimate meeting a disturbance attenuation bound of the form

$$|x_T - e_T|^2 \leq \gamma^2 \left[p_0(x_0) + \|w\|^2 + \|v\|^2\right]$$

where x_T is the true state at time T and e_T is our robust estimate at time T. This is a bound on the estimate error in terms of the L_2-norm of the disturbances.

We assume that f, σ, g and ρ are all continuously differentiable. We assume that f, g and σ are globally Lipschitz in x, that σ is bounded, and that $a \doteq \sigma\sigma^T$ is uniformly non-degenerate. We also assume that Range$(\rho(x)) = \Re^l$ for all $x \in \Re^n$ which guarantees that for any z, x there exists some v satisfying (3.2) which should certainly be the case if our measurement model is properly constructed. Finally, we define ρ^{-1} by

$$\rho^{-1}(x)b = \operatorname{argmin}\{|v| : \rho(x)v = b\}. \tag{3.3}$$

Assume that ρ^{-1} is uniformly bounded, that is, there exists $C_\rho < \infty$ such that $|\rho^{-1}(x)b| \leq C_\rho|b|$ for all $x \in \Re^n$ and $b \in \Re^l$. Note also that these assumptions imply that if we view the integral version of (2.1)

$$x_T \doteq X(T) = x_0 + \int_0^T f(X(t)) + \sigma(X(t))w(t)\,dt \tag{3.4}$$

as a mapping from x_0 to x_T then this mapping is one-to-one and onto for any $w \in L_2$, thereby ensuring that we may invert it.

Let $p_0(x_0)$ be a measure of our uncertainty about the initial state x_0, and let it have at most quadratic growth, so that $0 \leq p_0(x) \leq C(1 + |x|^2)$ for all $x \in \Re^n$ for some $C < \infty$. Suppose we wish to estimate the state at time T. Consider a cost criterion of the form

$$P(T, x_T, w(\cdot)) = -p_0(x_0) - \int_0^T |w(t)|^2 + |v(t)|^2\,dt \tag{3.5}$$

$$= -p_0(x_0) - \int_0^T |w(t)|^2 + |\rho^{-1}(X(t))[z(t) - g(X(t))]|^2\,dt \tag{3.6}$$

where x_0 is given by (3.4) for any particular w. Note that v no longer appears in (3.6) since it is fixed for a given choice of disturbance w, measurement time-history z and specific terminal state x_T. Similarly, x_0 is fixed given x_T and w as well. Therefore we define the value function by

$$\overline{W}(T, x_T) = \sup_{w \in L_2} P(T, x_T, w); \tag{3.7}$$

this is also known as the information state (see, for instance, [JB], [JBE]).

The Hamilton–Jacobi equation corresponding to (3.1), (3.6), (3.7) is

$$\begin{aligned} 0 &= -W_t + H(t, x, \nabla_x W) \qquad t > 0 \\ W(0, x) &= -p_0(x) \qquad t = 0 \end{aligned} \tag{3.8}$$

where ∇_x represents the gradient with respect to the space variable and

$$H(t, x, q) = \sup_{w \in \Re^m} \{-[f(x) + \sigma(x)w]^T q - |w|^2 - |\rho^{-1}(x)(z(t) - g(x))|^2\} \tag{3.9a}$$

$$= -f^T(x)q + \frac{1}{4} q^T a(x) q - |\rho^{-1}(x)(z(t) - g(x))|^2. \tag{3.9b}$$

It can be shown that $\overline{W}$ is a continuous viscosity solution of (3.8) [Mcrf].

We will now indicate how the solution to (3.8) may be used to obtain a robust state estimate. First we will indicate a direct approach, and then we will combine the above with a quadratic estimation criterion to obtain the Robust Filter. For the direct approach, note that $-\overline{W}(T, x)$ represents, in some sense, the minimum disturbance energy needed for the target state at time T to be x for a given measurement path $z(\cdot)$. The Mortensen estimate is

$$e_T = \operatorname{argmax}_{x \in \Re^n} P(T, x). \tag{3.10}$$

(See [H], [Mo] for earlier work in this vein.) We can assert the existence of the argmax in (3.10) under the quite reasonable assumption that there exists $C_p > 0$ such that $p_0(x) \geq C_p |x|^2$ for all x [Mcrf].

Although the direct estimate is of some interest, the following robust estimate will have desirable properties (analogous to H_∞ control) as discussed below, and will be the limit of a risk-sensitive filter. Specifically, the information state can be combined with a quadratic estimate error criterion to obtain the Robust Filter Let the dynamics and measurement models be those given above in (3.1) and (3.2), and make the same assumptions. However, now let the cost criterion take the form

$$\widetilde{P}(T, e, x_T, w) = -\gamma^2 \left[p_0(x_0) + \int_0^T |w(t)|^2 + |v(t)|^2 \, dt \right] + |x_T - e|^2$$

where we again note that x_0 and $v(\cdot)$ are given by x_T and $w(\cdot)$. Let

$$\widetilde{W}(T, e) = \sup_{x_0 \in \Re^n} \sup_{w \in \mathcal{W}} \widetilde{P}(T, e, x_T, w) = \sup_{x_T \in \Re^n} \sup_{w \in \mathcal{W}} \widetilde{P}(T, e, x_T, w). \tag{3.11}$$

Note that

$$\widetilde{W}(T, e) = \sup_{x_T \in \Re^n} \left[\gamma^2 \overline{W}(T, x_T) + |x_T - e|^2 \right]. \tag{3.12}$$

It is clear here that γ must be large enough so that this supremum will be finite and achieved at some x_T. Given the above assumptions we have this; that is, there exists γ_0 such that the supremum in (3.12) is finite.

This lower bound on γ is directly analogous to the optimal disturbance attenuation parameter in H_∞ control below which the supremum in that problem becomes unbounded. Note that in a computational system, given a $\overline{W}$, one can choose a γ corresponding to this $\overline{W}$ such that the supremum is finite.

Now note that since $\widetilde{W}(T,e)$ is a supremum of functions which are convex in e, it is also convex in e. Further, $\widetilde{W}(T,e) \to \infty$ as $|e| \to \infty$. Consequently, the minimum over e is obtained at some point, and we define the Robust Filter estimate at time T as:

$$e_T \doteq \operatorname{argmin}_{e\in\Re^n} \widetilde{W}(T,e). \tag{3.13}$$

Further, one may choose γ large enough so that $\min_{e\in\Re^n} W(T,e) \leq 0$, in which case one obtains the error estimate

$$|x_T - e_T|^2 \leq \gamma^2 \left[p_0(x_0) + \|w\|^2 + \|v\|^2\right] \tag{3.14}$$

for all $x_0 \in \Re^n$ and all $v, w \in L_2$ where $\|\cdot\|$ represents the L_2–norm over $[0,T]$. This is the robust bound on the estimate error in terms of the energy of the disturbance, and is analogous to H_∞ disturbance attenuation bounds.

Note that the information state computations are recursive. That is, having obtained an estimate at time T, one can obtain an estimate at time $\widehat{T} > T$ by extending the solution of (3.8) from T to $\widehat{T}$.

Finally, we note that this approach also bears some similarities to the recent work of Krener and Duarte [KD]

3.2 Numerical Methods

In the previous subsection, we introduced the Robust Filter. We also indicated that it had the desirable property of disturbance attenuation (see (3.14)) analogous to H_∞ control. We also mentioned that a secondary advantage of Robust Filtering for nonlinear systems is that the corresponding PDE is first–order (as opposed to the second–order nature of the Zakai equation). This is an advantage in that one may use numerical methods which are computationally much quicker than those available for second–order PDEs. In particular, one can use the generalized method of characteristics described in Section 2 as well as grid methods based on the characteristic flow. We will describe this briefly below. We will also indicate the Robust version of the operator splitting method which Rozovskii [LRR] employs in the stochastic filter case.

3.2.1 Generalized Method of Characteristics

The generalized method of characteristics described in Section 2 works here in more–or–less the same way. However, in this case, there is time–dependency in the Hamiltonian (3.9) via the measurement process $z(t)$. Thus there is another component of the characteristic system of ODE's. We denote this component by Q; it corresponds to W_t.

$$\frac{dX}{dt} = -H_p(t, X, P) \tag{3.15a}$$

$$\frac{dP}{dt} = H_x(t, X, P) \tag{3.15b}$$

$$\frac{dQ}{dt} = H_t(t, X, P) \tag{3.15c}$$

$$\frac{dU}{dt} = Q - P^T H_p(t, X, P) \tag{3.15d}$$

The characteristics are propagated forward in time from the initial time to the time at which we are attempting to estimate the state (the current time). Characterisitics are propagated forward from various initial state points $X(0) = x_0$. The initial conditions of the other components are given by

$$U(0) = -p_0(x_0)$$
$$P(0) = -\frac{dP_0}{dx}(x_0)$$
$$Q(0) = H(0, x_0, P(0)).$$

Index the characteristics by the initial state component (i.e. $X(t; x_0)$, $P(t; x_0)$, etcetera). Then the value is given by

$$W(t, x) = \max\{U(t; x_0) : X(t; x_0) = x_0\}.$$

In [Mcrf] it is shown that this does indeed yield the information state, that is $\overline{W}(t, x) = W(t, x)$ for all $t > 0$ and $x \in \Re^n$.

Note that for the Robust Filter, it is not necessary to solve the PDE over the whole space (as in stochastic nonlinear filtering), but only in some small region around the optimal estimate. This can be used to greatly reduce the required computation. We roughly outline the concept. Suppose one desires to estimate the state at times $t > 0$. Suppose one propagates a characteristic forward to some time $t_1 > 0$. yielding $U(t_1; x_0)$. Then, turn to another characteristic starting from x_1 at $t = 0$. If one is propagating this new characteristic, and the value falls significantly below $U(t_1; x_0)$ at time $t_2 < t_1$ (i.e. $U(t_2; x_1) < U(t_1; x_0)$) then one can stop propagating the characteristic for an estimate at time t_1 since it cannot be a point where the optimal estimate occurs. Taking this concept further, one obtains a classic branch–and–bound algorithm. The ramification is that one only propagates the characteristics forward over a very small region of the state space.

Now, we indicate a computation of the information state by the characteristic method for a simple, one–dimensional state example. (We will not use the branch–and–bound approach, but simply propagate all the characteristics forward to the time where we choose to estimate the state. This will produce a clearer graphical image.) Once one computes the information state to the desired accuracy, the computation of the Robust Filter estimate (3.13) is quite straight forward. The purpose of this example is simply to indicate that this method can provide an efficient means for obtaining Robust Filter estimates for a highly nonlinear system.

Let the dynamics and (discrete–time) measurement models be given by

$$\dot{X} = 4\sin(X) + \sigma w$$
$$z_i = \frac{X(t_i)}{2} + [X(t_i)]^3 + \rho v_i$$

with a measurement sample period of 0.025 units. The solution is propagated forward 1 time unit (i.e. 40 measurement updates). The noise multipliers were taken to be $\sigma = 1$ and $\rho = 5$. The (true) initial state is $X(0) = 0$. The initial information is described by $p_0(x) = x^2$ which corresponds to a correct initial state estimate. Note that the dynamics are unstable in a neighborhood of the origin. The disturbances were chosen to be

$$w(t) = \frac{1}{4} + \sin\left(\frac{-\pi}{2}t\right) \quad \text{and} \quad v_i = \frac{3}{20}\sin\left(\frac{\pi}{2}t_i\right)$$

so that these disturbances represent a slowly varying, biased noise signal. (Note that the Robust Filter does not require a statistical interpretation of these noise signals.) The resulting information state appears in figure 3.1.

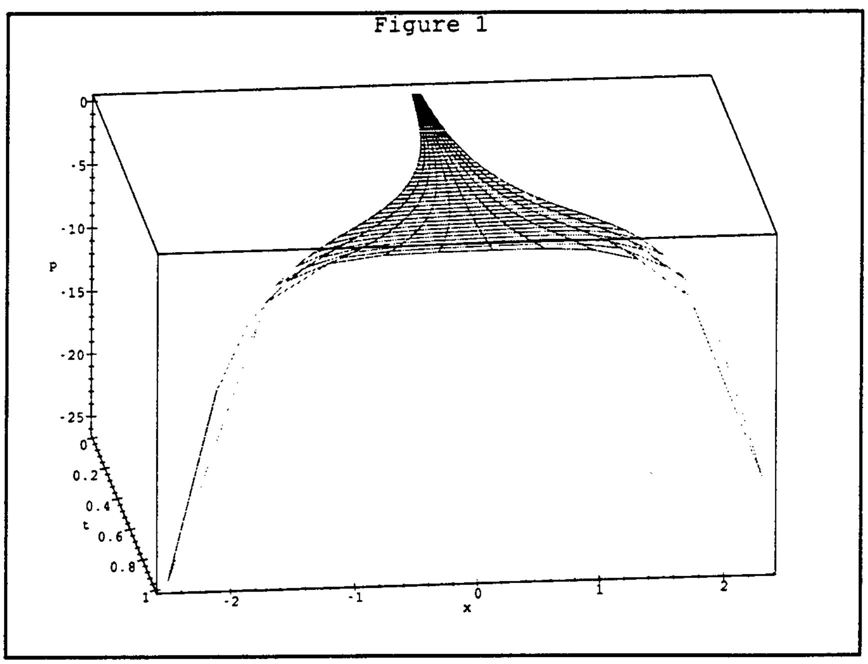


Figure 3.1: A simple information state computation

3.2.3 Splitting Method

A rather interesting approach which might be computationally competitive with a characteristic based method is an extension of the approach of Rozovskii (see for instance [LRR]) to a robust framework. In the Robust Filter, this approach leads to a very surprising and quite interesting technique. For this technique, we will suppose that the dynamics remain as a continuous–time model (3.1), but that the measurements occur at discrete times t_i.

$$z_i = g(X(t_i)) + \rho(X(t_i))v_i. \tag{3.16}$$

Suppose that we wish to estimate the state at time T, and that there have been N measurements in $(0, T]$. Then we may adapt the cost criterion (3.6) and value function (3.7) as follows. Let our cost criterion be

$$P(T, x_T, w(\cdot)) = -p_0(x_0) - \int_0^T |w(t)|^2\, dt - \sum_{i=1}^{N} |v_i|^2, \tag{3.17}$$

and let the value be given by

$$\overline{W}(T, x_T) = \sup_{w \in L_2} P(T, x_T, w). \tag{3.18}$$

Due to the discrete nature of the measurements, it is helpful recall the form of the dynamic programming principle for such a system. Let $\overline{W}(t_j^+, x)$ be the value at time t_j just after measurement j, and $\overline{W}(t_j^-, x)$ be the value at time t_j just before the measurement. If T is not a measurement time, and $t \in (t_{k-1}, t_k)$, we have

$$\overline{W}(T, x) = \sup_{w \in L_2([t,T];\Re^m)} \left\{ \overline{W}(t, X(t)) - \int_t^T |w(r)|^2\, dr - \sum_{j=k}^{N} |\rho^{-1}(X(t_j))[z_j - g(X(t_j))]|^2 \right\}.$$

and when T occurs at the time of measurement j, we have

$$\overline{W}(T^+, x) = \overline{W}(T^-, x) - |\rho^{-1}(x)[z_j - g(x)]|^2. \tag{3.19}$$

Let $S_{\Delta t}$ be the solution operator which propagates the solution forward from the time t_i to the time t_{i+1}, that is $\overline{W}(t_{i+1}^-, x) = S_{\Delta t}[\overline{W}(t_i^+, x)]$. Then we can show that this solution operator for the Robust Filter has the interesting property that

$$S_{\Delta t}[\max_{k \le K}\{\phi_k(x) + c_k\}] = \max_{k \le K}\{S_{\Delta t}[\phi_k(x)] + c_k\} \tag{3.20}$$

for any choice of constants c_k. This is analogous to a linear operator, L, in which case one has

$$L[\sum_{k \le K} c_k \phi_k(x)] = \sum_{k \le K} c_k L[\phi_k(x)]. \tag{3.21}$$

More specifically, the addition of a constant in (3.20) is analogous to multiplication by a constant in (3.21), and maximization in (3.20) is analogous to summation in (3.21). The operations in (3.20) pass through the operator $S_{\Delta t}$ in the same way that the operations in (3.21) pass through the linear operator L. Thus, $S_{\Delta t}$ provides a very interesting analogy to linear operators which is particularly suited to robust problems. And, in particular, (3.20) will provide a computationally effective method for solving the Robust Filter problem.

We approximate a function, $W(x)$ in a way analogous to the standard basis function approach. Let $\phi_k(x)$ be a set of concave–down functions which have higher concavity than W such that we can approximate the function W by $\max_{k\leq K}\{\phi_k(x)+c_k\}$ so that

$$\lim_{K\to\infty} |W(x) - \max_{k\leq K}\{\phi_k(x)+c_k\}| = 0$$

for all x in some region. For simplicity, we do not include a display of such functions ϕ_k, but simply note that for the problem at hand, a series of concave–down, quadratic functions is sufficient. We also note that to obtain the c_k, one may take $c_k \doteq \min_x[W(x)-\phi_k(x)]$. In this way, one has $W(x)\geq \phi_k(x)+c_k$ for all x and there exists x_0 such that $W(x_0)=\phi_k(x_0)+c_k$.

Suppose we approximate $\overline{W}(t_i^+,\cdot)$ by $\max_{k\leq K}\{\phi_k(\cdot)+c_k\}$. Then we have the approximation at $t=t_{i+1}^-$ given by

$$\overline{W}(t_{i+1}^-,x) \simeq S_{\Delta t}[\max_{k\leq K}\{\phi_k(\cdot)+c_k\}] = \max_{k\leq K}\{S_{\Delta t}[\phi_k(\cdot)]+c_k\}. \tag{3.22}$$

The point is that one may pre-compute the action of the solution operator on the basis functions ϕ_k up to some order $K\leq\infty$, that is, the $S_{\Delta t}[\phi_k]$ are precomputed and stored for all the basis functions ϕ_k, $k\leq K$. We then obtain an approximation to $\overline{W}(t_{i+1}^+,\cdot)$ by (see (3.19))

$$\overline{W}(t_{i+1}^+,x) \simeq \max_{k\leq K}\{S_{\Delta t}[\phi_k(\cdot)]+c_k\} - |\rho^{-1}(x)[z_j-g(x)]|^2. \tag{3.23}$$

One then expands this approximation of $\overline{W}(t_{i+1}^+,x)$ in basis functions, and repeats the process (3.23) to yield the approximation at the next time step and so on.

This method has the same advantages that the method of Rozovskii has for the stohastic problem. Of course our PDE is simpler to begin with due to the first–order nature, so it is not clear at this time whether this method is computationally competitive with characteristic methods.

3.3 Robust Filter as a Risk–Sensitive Limit

At the top of this section (Section 3), we indicated two arguments in support of the Robust Filter. The first being that, if one is tracking a target which is deliberately evading the tracking, one would not expect diffusion–type behavior of the target state. A more appropriate model is an L_2 disturbance which is chosen antagonistically to our tracking objective. Secondly, the nonlinear robust problem leads to a nonlinear first–order PDE which (we maintain) is easier to solve than the second–order Zakai equation.

As further support of the Robust Filter (and in particular, the above form as opposed to other forms suggested in [BJP], [MED]), we briefly present an outline of the demonstration that this filter is the limit of the Risk–Sensitive Stochastic Filter. The following discussion represents joint work with W. H. Fleming. It is extracted from the more complete version appearing in [FMcf].

In place of (3.1), (3.2), one employs the stochastic model

$$dX_t = f(X_t)\,dt + \sqrt{\frac{\epsilon}{2}}\sigma(X_t)\,dB_t \tag{3.24}$$

$$dZ_t = g(X_t)\,dt + \sqrt{\frac{\epsilon}{2}}\rho\,d\widetilde{B}_t \qquad Z_0 = 0 \tag{3.25}$$

where $B_{\cdot}$, $\widetilde{B}_{\cdot}$ are independent Brownian motion processes. The initial state X_0 is independent of $B_{\cdot}$, $\widetilde{B}_{\cdot}$, and has density $k_\epsilon \exp(-\epsilon^{-1}p_0(x)]$ for some constant k_ϵ. The parameter ϵ will be important in the risk–sensitive limit.

Let q^ϵ denote the unnormalized conditional density, which is a solution to the Zakai stochastic PDE with initial data $q^\epsilon(0,x) = \exp(-\epsilon^{-1}p_0(x))$. Let $V^\epsilon = -\epsilon \log q^\epsilon$.

For fixed T, let

$$\psi^\epsilon(e) = \int_{\Re^n} \exp\left[\frac{\mu}{\epsilon}|x-e|^2\right] q^\epsilon(x,T)\,dx, \tag{3.26}$$

where μ is another parameter similar to γ. The risk sensitive filtering problem is to find an estimate $\hat{e}^\epsilon_T$ which minimizes $\psi^\epsilon(e)$. (Note that if one deleted the exponentiation from (3.26). then one would have the standard stochastic filter.)

Minimizing $\psi^\epsilon(e)$ is equivalent to finding a $\mathcal{F}^Z_T$–measurable estimator e_T which minimizes the criterion $E\exp\left[\frac{\mu}{\epsilon}|X_T - e_T|^2\right]$, where $\mathcal{F}^Z_T$ is the σ–algebra generated by the accumulated observations Z_t for $0 \le t \le T$. Other authors consider instead the problem of finding $\mathcal{F}^Z_T$–measurable estimators e_t for $0 \le t \le T$ such that $E\exp\left[\frac{\mu}{\epsilon}\int_0^T |X_t - e_t|^2\,dt\right]$ is minimized. That formulation can be considered as a special case of an output–feedback, risk sensitive optimal control problem in which the estimate e_t has the role of a minimizing control. See Boel–James–Petersen [BJP], and Moore–Elliott–Dey [MED], and references cited therein. In that formulation, the optimal risk sensitive estimator involves the solution to an infinite–dimensional PDE for the dynamics of the information state. By considering the exponential terminal cost criterion (3.26), the analysis remains at the level of finite–dimensional PDE's.

In order to compare Risk-Sensitive Filtering and Robust Filtering, we must suppose that they view the same particular measurement path. Consequently, we let $Z_{\cdot}$ be the particular observation path that we see. If this is to be a path that could be viewed by the Robust Filter, then it must be differentiable (assumption of finite energy in the noise process in Robust Filtering – a reasonable assumption). Let $z(t) \doteq \dot{Z}_t$ be the time–derivative of this given observation path. Then V^ϵ satisfies the following PDE. For simplicity, we assume that σ is the identity, and that ρ is simply a scalar.

$$\begin{aligned} V^\epsilon_T &= \frac{\epsilon}{4}\nabla V^\epsilon - f\cdot\nabla V^\epsilon - \frac{1}{4}|\nabla V^\epsilon|^2 + \epsilon f_x - \frac{2}{\rho^2}z(T)\cdot g + \frac{1}{\rho^2}|g|^2 \\ V^\epsilon(0,x) &= p_0(x). \end{aligned} \tag{3.27}$$

As $\epsilon \downarrow 0$, V^ϵ tends formally to a limit V^0 which should satisfy (in a viscosity sense) the corresponding PDE with $\epsilon = 0$

$$\begin{aligned} 0 &= -V_T^0 - f \cdot \nabla V^0 - \frac{1}{4}|\nabla V^0|^2 - \frac{2}{\rho^2} z(T) \cdot g + \frac{1}{\rho^2}|g|^2 \\ &= -V_T^0 + \inf_{w \in \Re^n} \left[(-f - w) \cdot \nabla V^0 + w^2\right] + \frac{1}{\rho^2}|z(T) - g|^2 - \frac{1}{\rho^2}|z(T)|^2 \qquad (3.28) \\ V^0(0,x) &= p_0(x) \end{aligned}$$

Note that by relpacing W in (3.8) with $-V^0$ (and multiplying both sides by -1) and subtracting the last term in (3.28), one obtains (3.28) from (3.8). This suggests the following result which we can prove by the method of viscosity solutions. As $\epsilon \downarrow 0$, $V^\epsilon \to V^0$ uniformly on compact sets, where

$$V^0(T,x) = \overline{W}(T,x) - \frac{1}{\rho^2}\int_0^T |z(t)|^2\, dt.$$

That is, the $-V^\epsilon$ converge to the information state of the Robust Filter minus an additional term which is independent of x. Consequently, the Risk-Sensitive estimate $\hat{e}_T^\epsilon$ converges to the robust estimate e_T as $\epsilon \downarrow 0$. Thus, the Robust Filter described in the previous section is the limit of a stochastic filter.

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