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# CONTROL OF NONLINEAR SYSTEMS

AFOSR-49620-95-1-0101, Final Report, January 1998

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## ABSTRACT

This Final Report summarizes work in the above grant, which dealt with several topics related to the control of nonlinear systems, including: input to state stability and new related notions (such as input/output to state stability), discontinuous feedback techniques based on nonsmooth analysis, universal inputs, with applications to numerical methods, and control-Lyapunov functions.

## 1 Introduction

The control of highly nonlinear systems is of central importance to the Air Force mission, as nonlinear effects cannot be ignored in high-performance aircraft. This report focuses on the development of basic mathematical foundations for finite-dimensional nonlinear control theory.

### 1.1 ISS-Related Notions

Methods of feedback design are undergoing an exceptionally rich period of progress and maturation, fueled to a great extent by (1) the discovery of certain basic conceptual notions, and (2) the identification of classes of systems for which systematic decomposition approaches can result in effective and easily computable control laws. These two aspects are complementary, since the latter approaches are, typically, based upon the inductive verification of the validity of the former system properties under compositions.

One of the notions instrumental in the new developments is that of *input to state stability (ISS)*, introduced by the PI in 1989. Another one is that of *control-Lyapunov function (CLF)*, formalized and completely characterized in independent and complementary papers by Artstein and the PI in the early 1980s. These notions have been incorporated into the main current books used as texts and references for advanced nonlinear control courses, such as the books by Krstić, M., I. Kanellakopoulos, and P. V. Kokotović, Isidori, and Khalil's (latest editions).

*The ultimate goal of the PI's work in this area is highly ambitious: the complete reformulation of the foundations of nonlinear control based on ISS-like ideas.* This goal is very long-term, but definitely worth the effort: the payoff will be the development of a consistent, elegant, and ultimately design-oriented, systematic approach to the subject. After briefly surveying the concept of ISS, this report will describe the dual notion of IOSS (input/output to state stability), which corresponds to "detectability" of nonlinear systems and of IOS (input to output stability), which should provide the appropriate setting for i/o behaviors and regulation problems.

### 1.2 Discontinuous Feedback Design

It is known that, in general, in order to control nonlinear systems one must use switching (discontinuous) mechanisms of various types. Of course, time-optimal solutions for even linear systems often involve such discontinuities, but, for linear systems, most control problems usually admit also (perhaps suboptimal) continuous solutions. However, when dealing with arbitrary systems, discontinuities are unavoidable even when no optimality objectives are imposed. Faced with the theoretical impossibility of having feedback laws of the form  $u = k(x)$ ,  $k$  a continuous function, various alternatives have been proposed. (Objectives other than state stabilization, such as tracking and disturbance rejection, output feedback, or adaptive and robust variants, also lead to the necessity of discontinuous controllers; the study of stabilization is merely the first step in the search for general controller structures.)

One way of overcoming the limitations of continuous static feedback is to look for dynamic feedback laws, that is to say, to allow for additional (memory) variables in controllers. This approach, suggested by the Pi and Sussmann in 1980, and developed deeply by Coron in the late 1980s, works for certain restricted classes of systems, and is closely related to the use of classical gradient descent and Newton numerical algorithms for control, as shown in [8]. However, in general no such dynamic or time-varying solutions are known, so the interest in discontinuous feedback laws  $u = k(x)$  arises.

In [13], we introduced a new formulation of discontinuous feedback, and, using this notion, we showed that every asymptotically controllable system can be stabilized. One of the contributions was in defining precisely the meaning of stabilization when the feedback rule is not continuous. This work solved a longstanding open question in nonlinear control theory, concerning the relationship between asymptotic controllability to the origin and the existence of feedback controls. The main tools used were: (a) control-Lyapunov functions, (b) methods of nonsmooth analysis, especially proximal subgradients, and (c) techniques from positional differential games.

### 1.3 Other Subjects

The PI's invited presentation and paper at the latest International Congress of Mathematicians [17] dealt with spaces of observables as a unifying concept in nonlinear control. The systematic study of such spaces and their properties allows a deep understanding of important aspects of system structure, including the relationship between input/output equations and realization, and the existence and characterization of "universal" controls of various types, which in turn form the basis of numerical techniques.

The paper [8]) explained the PI's "generic nonsingular loop" method for the steering of nonlinear systems without drift. The path-planning problem for such systems has been of great interest during the last few years, and several methods, all based on a nontrivial analysis of the structure of the Lie algebra of vector fields generated by the system, have been proposed. Our method does not make any structural assumptions, and is based on a general transversality result showing that generic controls are universally nonsingular, in the sense that linearizations along the ensuing trajectories are controllable, for all initial states. Very recently, a student at Rutgers completed an implementation of this technique (as a set of MATLAB M-files), in a manner which allows easy generation of trajectories and with adaptive stepsize improvements.

### 1.4 Scope and Organization of the report

Given the range of subjects treated, and the different methodologies employed in each, lack of space prevents giving details on all subtopics. Thus we have chosen only a few selected subtopics, especially those represented in very recent work and hence possibly less easily available to readers. Overall, however, most of the discussion in this report is informal, with references to the literature for precise technical points. In addition, the Web page

<http://www.math.rutgers.edu/~sontag/>

provides access to a substantial number of papers by the PIs, and these may be consulted for many more details and a mathematically rigorous presentation.

## 2 Details

In this part of the report, we discuss the following subjects:

- ISS and Related Stability Concepts
- Nonsmooth Methods
- Universal Controls

As mentioned earlier, the Web page <http://www.math.rutgers.edu/~sontag/> provides access to a substantial number of papers by the PIs, to be consulted for details and a mathematically rigorous presentation.

### 2.1 Stability Notions: ISS

We consider general finite-dimensional systems, in the standard sense of nonlinear control:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)). \quad (1)$$

(As usual, dot indicates derivative, and one omits the time argument  $t$ . To keep the exposition elementary, we restrict to states in an Euclidean space  $\mathbb{R}^n$ , though the study of control systems evolving in manifolds is not only possible but more natural in many applications.)

There are two conceptually very different ways to formulate the notion of stability of control systems. One of them, which we may call the *input/output approach*, relies on operator-theoretic techniques. Among the main contributions to this area, one may cite the foundational work by Zames, Sandberg, Desoer, Safanov, Vidyasagar, and others. In this approach, a “system” is a causal operator  $F = F_{x(0)} : u(\cdot) \rightarrow y(\cdot)$  between spaces of signals (for fixed initial states), and “stability” is taken to mean that  $F$  maps bounded inputs into bounded outputs, or finite-energy inputs into finite-energy outputs. More stringent typical requirements in this context are that the gain of  $F$  be finite (in more classical mathematical terms, that the operator be bounded), or that it have finite incremental gain (mathematically, that it be globally Lipschitz). The input/output approach has been extremely successful in the robustness analysis of linear systems subject to nonlinear feedback and mild nonlinear uncertainties, and in general in the area that revolves around the various versions of the small-gain theorem. Moreover, geometric characterizations of robustness (gap metric and the like) are elegantly carried out in this framework. Finally, i/o stability provides a natural setting in which to study the classification and parameterization of dynamic controllers.

On the other hand, there is the model-based, or *state-space approach* to systems and stability, where the basic object is the forced dynamical system (1). In this approach, there is a standard notion of stability, namely Lyapunov asymptotic stability of the unforced system. Associated to such a system, there is the above-mentioned operator  $F$  mapping inputs (forcing functions) into state trajectories (or into outputs, if partial measurements on states are of interest). It becomes of interest then to ask to what extent Lyapunov-like stability notions for a state-space system are related to the stability, in the senses discussed in the previous paragraph, of the associated operator  $F$ . It is well-known that, in contrast to the case of linear systems, where there is—subject to mild technical assumptions—an equivalence between state-space and i/o stability, for nonlinear systems the two types of properties are not so closely related. Even

for the very special and comparatively simple case of “feedback linearizable” systems, this relation is far more subtle than it might appear at first sight: if one first linearizes a system and then stabilizes the equivalent linearization, in terms of the original system one does not in general obtain a closed-loop system that is input/output stable in any reasonable sense.

This leads one to focus on the study of the dependence of state trajectories on the size of inputs, a study which is especially relevant when the inputs in question represent disturbances acting on a system. For not necessarily linear systems, there is no complete agreement as yet regarding what are the most useful formulations of system stability with respect to input perturbations. (For linear systems, similar considerations led to the development of gains and the operator-theoretic approach, including the formulation, when using  $L^2$  norms, of  $H^\infty$  control.) One candidate for such a formulation is the property called “input to state stability” (ISS), introduced in a 1989 paper by the PI. This property has proved to be a very useful paradigm in the study of nonlinear stability for systems subject to external effects, as evidenced by the wide number of current papers and textbooks that use the notion in a fundamental manner. The notion differs fundamentally from the operator-theoretic ones, first of all because it takes account of initial states in a manner fully compatible with Lyapunov stability. Second, boundedness (finite gain) is far too strong a requirement for general nonlinear operators, and it must be replaced by “nonlinear gain estimates,” in which the norms of output signals are bounded by a nonlinear function of the norms of inputs; the definition of ISS incorporates such gains in a natural way. In very informal terms, the ISS property translates into the statement that “no matter what is the initial state, if the inputs are small, then the state must eventually be small” (the precise definition is recalled below).

The ISS notion was originally introduced in a 1989 paper by the PI and has since been employed by several authors in deriving results on control of nonlinear systems. It can be stated in several equivalent manners, which indicates that it is a mathematically natural concept: dissipation, robustness margins, and classical Lyapunov-like definitions. The dissipation characterizations are closely related to the pioneering work of Willems and Hill and Moylan, who introduced an abstract concept of energy dissipation in order to unify i/o and state space stability, and in particular with the purpose of understanding conceptually the meaning of Kalman-Yakubovich positive-realness (passivity), and frequency-domain stability theorems in a general nonlinear context.

### ISS: Some technical details\*

We review here some of the main characterizations of the ISS property. Euclidean norm in  $\mathbb{R}^n$  or  $\mathbb{R}^m$  is denoted simply as  $|\cdot|$ . (Often, for instance in the context of studying stability of invariant sets, or when studying so-called “practical stability”, one defines properties relative to a subset  $A$  of  $\mathbb{R}^n$ ; given such a set  $A$ , one may define instead  $|x| = d(x, A)$ , and similar results are proved.) The map  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  in (1) is assumed to be locally Lipschitz continuous, and  $f(0, 0) = 0$ ,  $h(0) = 0$ .

By a control or input we mean a measurable and locally essentially bounded function  $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ . Given any control  $u$  and any  $\xi \in \mathbb{R}^n$ , there is a unique maximal solution of the initial value problem  $\dot{x} = f(x, u)$ ,  $x(0) = \xi$ , denoted by  $x(\cdot, \xi, u)$ . The trajectory  $x(t, \xi, u)$ , and consequently also  $y(t, \xi, u) = h(x(t, \xi, u))$ , are defined on some maximal interval  $[0, t_{\max})$ , where  $t_{\max} = t_{\max}(\xi) \leq +\infty$ . The  $L^\infty$ -norm (possibly infinite) of a control  $u$  is denoted by  $\|u\|$ .

A function  $F : S \rightarrow \mathbb{R}$  defined on a subset  $S$  of  $\mathbb{R}^n$  containing 0 is said to be positive definite if  $F(x) > 0$  for all  $x \in S$ ,  $x \neq 0$ , and  $F(0) = 0$ , and it is proper if the preimage  $F^{-1}(-D, D)$  is bounded, for each  $D > 0$ . A function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{N}$  if it is continuous and nondecreasing; it is of

\*this subsection can be skipped without loss of continuity.

class  $\mathcal{N}_0$  if in addition it satisfies  $\gamma(0) = 0$ , of class  $\mathcal{K}$  if it is continuous, positive definite, and strictly increasing, and of class  $\mathcal{K}_\infty$  if it is also unbounded. Finally,  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be a function of class  $\mathcal{KL}$  if for each fixed  $t \geq 0$ ,  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  and for each fixed  $s \geq 0$ ,  $\beta(s, t)$  decreases to zero as  $t \rightarrow \infty$ .

For simplicity of statements, we assume here that the system is forward complete, that is to say, for each input and each  $\xi$ , the solution  $x(t, \xi, u)$  is defined on the entire interval  $\mathbb{R}_{\geq 0}$ .

We define next several natural properties of control systems and characterize ISS in alternative manners. The original definition of (ISS) given in the 1989 paper by the PI was as follows:

$$\begin{aligned} \exists \gamma \in \mathcal{K}, \beta \in \mathcal{KL} \text{ st : } \forall \xi \in \mathbb{R}^n \forall u(\cdot) \forall t \geq 0 \\ |x(t, \xi, u)| \leq \max\{\beta(|\xi|, t), \gamma(\|u\|)\}. \end{aligned} \quad (2)$$

(A sum can be used instead of “max” and an equivalent concept results.) It was shown in [4] that a system is ISS if and only if it satisfies a dissipation inequality, that is to say, there exists a smooth  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  (an “ISS-Lyapunov function”) and there are functions  $\alpha_i \in \mathcal{K}_\infty$ ,  $i = 1, 2, 3$  and  $\sigma \in \mathcal{K}$  so that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \quad (3)$$

and

$$\nabla V(\xi)f(\xi, v) \leq \sigma(|v|) - \alpha_3(|\xi|) \quad (4)$$

for each  $\xi \in \mathbb{R}^n$  and  $v \in \mathbb{R}^m$ . A very useful modification of this characterization due to Praly and the PI's former student Wang is the fact that the ISS property is also equivalent to the existence of a smooth  $V$  satisfying (3) and Equation (4) replaced by an estimate of the type

$$\nabla V(\xi)f(\xi, v) \leq -V(\xi) - \alpha_3(|v|).$$

This can be understood as: “for some positive definite and proper functions  $y = V(x)$  and  $v = W(u)$  of states and outputs respectively, along all trajectories of the system we have  $\dot{y} = -y + v$ ”.

If all trajectories of this system with initial conditions satisfying  $|x(0)| < \rho$ , for some  $\rho > 0$ , satisfy  $\lim_{t \rightarrow \infty} |x(t, \xi, \underline{0})| \rightarrow 0$ , where  $\underline{0}$  denotes the control which is identically equal to zero on  $\mathbb{R}_{\geq 0}$ , one says that the system has the 0-local attraction property. The 0-local stability property means that for each  $\varepsilon > 0$  there is a  $\delta > 0$  so that  $|\xi| < \delta$  implies that  $|x(t, \xi, \underline{0})| < \varepsilon$  for all  $t \geq 0$ . The conjunction of these two properties is the usual local asymptotic stability (0-AS). Next we introduce several new concepts. The *limit property* holds if every trajectory must at some time get to a ball around the origin which is a function of the magnitude of the input:

$$\begin{aligned} \exists \gamma \in \mathcal{N}_0 \text{ st : } \forall \xi \in \mathbb{R}^n \forall u(\cdot) \\ \inf_{t \geq 0} |x(t, \xi, u)| \leq \gamma(\|u\|). \end{aligned} \quad (\text{LIM})$$

The *asymptotic gain property* holds if every trajectory must ultimately stay not far from zero, depending on the magnitude of the input:

$$\begin{aligned} \exists \gamma \in \mathcal{N}_0 \text{ st : } \forall \xi \in \mathbb{R}^n \forall u(\cdot) \\ \overline{\lim}_{t \rightarrow +\infty} |x(t, \xi, u)| \leq \gamma(\|u\|). \end{aligned} \quad (\text{AG})$$

The *uniform asymptotic gain property* holds if the above limsup is attained uniformly with respect to initial states in compacts and all  $u$ :

$$\begin{aligned} \exists \gamma \in \mathcal{N}_0 \forall \varepsilon > 0 \forall \kappa > 0 \exists T = T(\varepsilon, \kappa) \geq 0 \text{ st : } \forall |\xi| \leq \kappa \\ \sup_{t \geq T} |x(t, \xi, u)| \leq \gamma(\|u\|) + \varepsilon \quad \forall u(\cdot). \end{aligned} \quad (\text{UAG})$$

The *boundedness property* holds if bounded initial states and controls produce uniformly bounded trajectories:

$$\exists \sigma_1, \sigma_2 \in \mathcal{N} \text{ st : } \forall \xi \in \mathbb{R}^n \forall u(\cdot)$$

$$\sup_{t \geq 0} |x(t, \xi, u)| \leq \max \{ \sigma_1(\|\xi\|), \sigma_2(\|u\|) \}. \quad (\text{BND})$$

(This is sometimes called the “UBIBS” or “uniform bounded-input bounded-state” property.) The *global stability property* holds if in addition small initial states and controls produce uniformly small trajectories:

$$\begin{aligned} & \exists \sigma_1, \sigma_2 \in \mathcal{N}_0 \text{ st : } \forall \xi \in \mathbb{R}^n \quad \forall u(\cdot) \\ & \sup_{t \geq 0} |x(t, \xi, u)| \leq \max \{ \sigma_1(\|\xi\|), \sigma_2(\|u\|) \}. \end{aligned} \quad (\text{GS})$$

It is shown in [12] that the ISS property is equivalent to each of the following:

(LIM) and (0-AS)

(AG) and (GS)

(UAG)

(LIM) and (GS)

(Several more equivalences are provided as well.) It is remarkable that several of the properties do not involve uniformity with respect to inputs; since no linearity assumptions are made regarding the dependence of the right hand side on inputs, there is no obvious way to use directly any sort of compactness (e.g., in the weak topology), so the proofs are quite delicate.

## 2.2 Stability Notions: IOSS

Given the central role often played in control theory by the duality between input/state and state/output behavior, one may reasonably ask what concept obtains if outputs are used instead of inputs in the ISS definition (2). This corresponds roughly to asking that “no matter the initial state, if the outputs are small, then the state must be eventually small”. For linear systems, the notion that arises is that of detectability. Thus, it would appear that this dual property, which we called in [14] *output to state stability* (OSS), is a natural candidate as a concept of nonlinear detectability.

Using a naive dualization of the dissipation characterization (4), one could also ask what is the notion that emerges when we ask that there be some function  $V$  such that, along all possible trajectories,  $V$  decreases if the outputs are not too large compared to the present states. For linear systems, one obtains again detectability. For nonlinear systems, variations of this property have very often been suggested as a notion of detectability as well, as mentioned later. It is easy to see that this dissipation property, once rigorously formulated, implies OSS. The main contribution of [14] was to prove a certain converse implication. However, the converse proof does not provide a smooth  $V$ , which is required for “backstepping” and other design arguments. (We conjecture that the full converse (smooth  $V$ ) holds.) We believe that this will be a most useful foundational result for further research in input/output stability, probably as useful as those involving ISS. Combining ISS and OSS there results the notion of input/output-to-state stability (IOSS), whose study was also started in that paper. In this case, we have only proved the easy implication (dissipation property implies IOSS) but we conjecture that the converse holds as well. The notion of IOSS is closely connected to the possibility of stabilizing a partially observed system using only output measurements, and is implied by the existence of observers. It is also implied by strict passivity. (Some further details are provided below.)

The IOSS property is connected with the possibility of stabilization by output feedback, and also to the existence of (one-sided) estimators of the norm of the state (see below). These are essentially properties connected with “zero-detectability”, meaning being able to asymptotically distinguish any given state  $\xi$  from the zero state. For linear systems, this 0-detectability

property is equivalent to detectability: being able to asymptotically distinguish *every pair* of states. But for general, nonlinear, systems, these properties are very different. Estimates of arbitrary states are not necessarily required if the objective is merely to drive the state to the special state “zero”.

One possible definition of “observer” for (1) is that of a dynamical system which processes inputs and outputs of (1) and produces an estimate  $\hat{x}(t)$  of the state  $x(t)$ . The estimation condition would be that  $x(t) - \hat{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$  and that this difference (the estimation error) should be small if it starts small. However, it is far more natural in the context of ISS-type notions to require that the estimation error  $x(t) - \hat{x}(t)$  be small even if the measurements of inputs and outputs taken by the observer are “noisy”. Writing  $u_d$  and  $y_d$  for the input and output measurement noises, we have the situation in Figure 1, which is formalized in the technical-details section to follow, in the special case of full state observers.

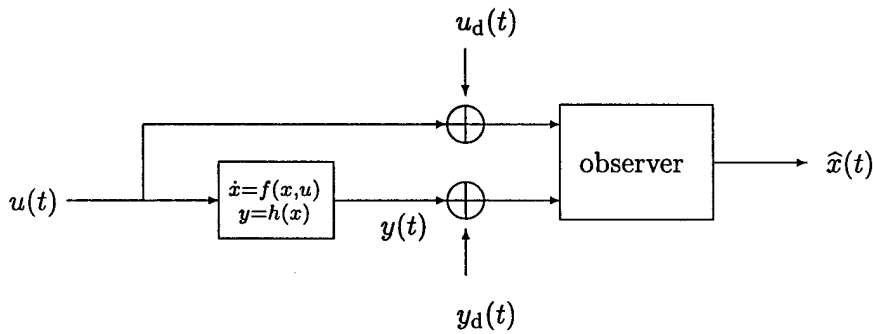


Figure 1: Observer with noise  $u_d$  and  $y_d$

The ISS-style definition insures that the error  $x(t) - z(t)$  converges to zero when there is no noise in the measurements taken by the observer, but also that in general it degrades gracefully as a function of the magnitude of such disturbances. This seems most natural, and we intend to seriously study the implications of this definition.

### IOSS: Some technical details\*

The system (1) is *input/output-to-state stable* (IOSS) if there exist some  $\beta \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that

$$|x(t, \xi, u)| \leq \max \{ \beta(|\xi|, t), \gamma_1(\|u|_{[0,t]}\|), \gamma_2(\|y_{\xi, u}|_{[0,t]}\|) \} \quad (5)$$

for every initial state  $\xi$  and control  $u$  and all  $t \in [0, t_{\max})$ ,  $t_{\max} = t_{\max}(\xi, u)$ . As in page 4,  $x(t, \xi, u)$  denotes the trajectory that results from initial state  $\xi$  and input  $u$ , and  $y_{\xi, u}(t) = y(t, \xi, u) = h(x(t, \xi, u))$ .

This property has appeared before in the literature. It represents a natural combination of the notions of “strong” observability and ISS. When there are no controls, that is, we are dealing with a system  $\dot{x} = f(x), y = h(x)$ , we call this property simply *output to state stability* (OSS). Note the formal duality with ISS: we have an estimate  $|x(t, \xi)| \leq \max \{ \beta(|\xi|, t), \gamma(\|y_{\xi}|_{[0,t]}\|) \}$  compared to  $|x(t, \xi, u)| \leq \max \{ \beta(|\xi|, t), \gamma(\|u|_{[0,t]}\|) \}$ . (the restriction of the control makes no difference, by causality). Unfortunately, this apparent duality does not seem to help at all in proving results regarding OSS starting from ISS results (there is no “semigroup property” for states with respect to outputs).

An *IOSS-Lyapunov function* for system (1) is any function  $V$  so that there exist  $\mathcal{K}_{\infty}$ -functions  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n, \quad (6)$$

\*this subsection can be skipped without loss of continuity.

$V$  is absolutely continuous along trajectories, for all initial states and controls, and there exist  $\mathcal{K}_\infty$ -functions  $\alpha$ ,  $\sigma_1$ , and  $\sigma_2$  such that for every trajectory  $x(t, \xi, u)$ , and almost all  $t \in [0, t_{\max})$ ,

$$\frac{d}{dt}V(x(t, \xi, u)) \leq -\alpha(|x(t, \xi, u)|) + \sigma_1(|u(t)|) + \sigma_2(|y(t, \xi, u)|). \quad (7)$$

In the very recent paper [14] and ongoing work, we have shown that for systems with no controls the existence of an IOSS-Lyapunov function is equivalent to the IOSS property, and a paper in preparation extends to the full problem. As mentioned earlier, the OSS property can be thought of as a definition of detectability. Indeed, variations of this notion can be found at various places in the literature. The definition involving comparison functions implies in particular that the system is "zero detectable". This means that under zero inputs, states whose outputs are identically zero should form an asymptotically stable subsystem (that is,  $y \equiv 0$  implies  $x \rightarrow 0$ , plus a local stability condition). That definition relates to the current definition (which says in addition that  $y \rightarrow 0$  implies  $x \rightarrow 0$  and that  $y$  small implies  $x$  small) in exactly the same way that global asymptotic stability of an unperturbed system  $\dot{x} = f(x, 0)$  relates to the ISS property. The definition which involves Lyapunov functions had also appeared in restricted forms. In several papers, one finds detectability defined by the requirement that there should exist a (differentiable) storage function  $V$  satisfying our Equations (6) and (7), but with the special choice  $\sigma(y) := |y|^2$ . A variation of this is to weaken (7) to require merely

$$x \neq 0 \Rightarrow \frac{d}{dt}V(x(t, \xi)) < \sigma(|y(t, \xi)|)$$

as done for instance in the definition of detectability given in recent work by Morse.

There is another type of relationship between the notions of OSS and ISS, different from the obvious formal duality. Consider a system in cascade form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_2) \end{aligned}$$

where  $n = n_1 + n_2$  and the variables  $x_i$  have sizes  $n_1$  and  $n_2$  respectively. Assume that the output is  $y = x_2$ . If we interpret the  $n_2$ -dimensional subsystem  $\dot{z} = g(z)$  as a generator of input signals ("exosystem") for the  $n_1$ -dimensional system  $\dot{x} = f(x, u)$ , then the OSS property amounts to a version of the input to state stability property for this first subsystem, but only with respect to the signals so generated.

In feedback control, the concept of passivity has been widely used. System (1) with  $m = p$  is said to be strictly passive if there exist a continuous nonnegative function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , called a storage function, and a positive definite function  $\alpha$ , called the dissipation rate, such that for any  $\xi \in \mathbb{R}^n$  and any input  $u$ ,

$$V(x(t, \xi, u)) - V(\xi) \leq -\int_0^t \alpha(|x(s, \xi, u)|) ds + \int_0^t y(s, \xi, u)u(s) ds \quad (8)$$

for all  $t \geq 0$ . If, as usually supposed,  $V$  is differentiable, positive definite, and proper, and if  $\alpha$  in (8) is also proper, then  $V$  is an IOSS-Lyapunov function, and the system is IOSS. (If  $V(x(t))$  does not have a derivative along trajectories, one obtains an integral equation version of (7); it is possible to show that the existence of a  $V$  with such a property also implies that the system is IOSS.)

There is a remarkable relationship between IOSS and the possibility of estimating the norms of states "on-line". By a (one-sided) "state-norm estimator" for the system (1) we mean a system

$$\dot{z} = g(z, u, y) \quad (9)$$

whose inputs are pairs  $(u(t), y(t))$  consisting of inputs and outputs of (1), which is itself ISS (with inputs  $(u, y)$ ), and which has the property that, for some function  $\rho$  of class  $\mathcal{K}$  and a function  $\beta$  of class  $\mathcal{KL}$ , for each initial states  $\xi$  and  $\zeta$  for (1) and (9) respectively, and each input  $u(\cdot)$ ,

$$|x(t, \xi, u)| \leq \beta(|\xi| + |\zeta|, t) + \rho(|z(t, \zeta, u, y_{\xi, u})|)$$

for all  $t \in [0, t_{\max})$ . (That is, the  $z$  equation provides an upper bound  $\rho(z(t))$  on the true state  $x(t)$ , with an error which is initially small if  $\xi$  and  $\zeta$  are small, and which in any case decays to zero as  $t \rightarrow \infty$ .) We conjecture that the existence of a norm-estimator for the system is equivalent to the IOSS property; for systems with no inputs, this was already proved in the recent paper [14].

Finally, a few details about observers. A (full-order state) *observer* for the system (1) is a system defined by equations  $\dot{z} = g(z, v, w)$  evolving in the same space  $\mathbb{R}^n$  as (1), driven by inputs  $v$  and  $w$  of dimensions equal to the dimension of the input and output value spaces of (1), respectively, and so that the following properties hold: There exist functions  $\beta \in \mathcal{KL}$ , and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that, for each initial states  $\xi$  and  $\zeta$  of the composite system consisting of (1) and

$$\dot{z} = g(z, u + u_d, y + y_d) \quad (10)$$

and each (measurable locally essentially bounded) inputs  $u, u_d, y_d$ , if  $[0, t_{\max})$  is the maximal interval of existence of  $x(t) = x(t, \xi, u)$  then the solution  $z(t)$  of (10) with  $z(0) = \zeta$ ,  $y(t) = h(x(t))$ , and the same  $u, u_d, y_d$  is also defined on  $[0, t_{\max})$  and there holds on  $[0, t_{\max})$  the estimate

$$|x(t) - z(t)| \leq \max \{ \beta(|\xi - \zeta|, t), \gamma_1 (\|u_d|_{[0,t]}\|), \gamma_2 (\|y_d|_{[0,t]}\|) \}.$$

It is easy to see that the equations of any observer (10) must have the ("output injection") form

$$\dot{z} = f(z, u + u_d) + L(z, u + u_d, y - h(z) + y_d)$$

where the vector field  $L$  satisfies that  $L(a, b, 0) = 0$  for all  $a, b$ .

Let's call a system (1) *incrementally (Lipschitz) input/output-to-state stable (i-IOSS)* if there exists some  $\beta \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that, for every two initial states  $\xi_1$  and  $\xi_2$ , and any two controls  $u_1$  and  $u_2$ ,

$$|x(t, \xi_1, u_1) - x(t, \xi_2, u_2)| \leq \max \{ \beta(|\xi_1 - \xi_2|, t) \gamma_1 (\|(u_1 - u_2)|_{[0,t]}\|), \gamma_2 (\|(y_{\xi_1, u_1} - y_{\xi_2, u_2})|_{[0,t]}\|) \}$$

for all  $t$  in the common domain of definition.

One can then see that if an observer for (1) exists, the system is i-IOSS, and thus also IOSS. In general, however, the IOSS property does not guarantee the existence of an observer. To see this, it suffices to give an example of a system that is IOSS but not i-IOSS: the one-dimensional system  $\dot{x} = u, y = x^2$  is one such example.

The output-injection form of the observer, in the special case  $x \equiv 0, u \equiv 0$ , and  $y_d = y$ , allows concluding that the system  $\dot{x} = f(x, u) + L(x, u, -h(x))$  is ISS, which generalizes the classical notion of output injection for linear systems.

### 2.3 Stability Notions: IOS

We complete the discussion of stability notions by returning to yet another concept introduced in the PI's by now well-known 1989 ISS paper: *input to output stability* (IOS, for short). In informal terms, this means that the output (instead of the full state) must be eventually small, no matter what the initial state. When there are no inputs, one has a generalization of the classical concept of partial stability. In general, this is the appropriate notion to study in the context of regulation problems. It turns out, however, that the notion of IOS given in the 1989 paper is not exactly the appropriate one for modeling the situation typical in regulation or in robust and adaptive control, where a condition of boundedness of internal variables is required in addition to asking that outputs become small. The concept must be modified somewhat. There is a fairly obvious connection between the various concepts introduced until now: a system is ISS if and only if it is both IOSS and IOS. This fact generalizes the linear systems theory result "internal stability is equivalent to detectability plus external stability" and its proof follows by routine arguments.

A long-term objective is to obtain necessary and sufficient characterizations of the solvability of complete regulation problems in this framework. In other approaches, stability with no disturbances is considered as a totally separate property from regulation (in particular, problems such as tracking); in the current context, these different aspects are taken into account simultaneously.

### IOS: Some technical details\*

Recall the GS property defined in page 6. Note that for a system with no controls,

$$\dot{x} = f(x), \quad y = h(x),$$

the GS property reduces to a “bounded state” estimate:

$$|x(t, \xi)| \leq \sigma(|\xi|), \quad \forall t \geq 0, \forall \xi \in \mathbb{R}^n$$

This property amounts to (neutral) stability plus what is sometimes called “uniform boundedness”.

The main property that we are now considering is as follows.: The system (1) is *input to output stable* (IOS) if it is GS and, in addition, there exist a  $\mathcal{KL}$ -function and a  $\mathcal{K}$ -function  $\gamma$  such that

$$|y(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|), \quad \forall t \geq 0, \quad (11)$$

holds for all  $u$  and all  $\xi \in \mathbb{R}^n$ . For an autonomous system, we say simply *output stable* (OS). That is, such a system is OS if it satisfies an a bounded-state estimate and there is some  $\mathcal{KL}$ -function  $\beta$  such that

$$|y(t, \xi)| \leq \beta(|\xi|, t) \quad \forall t \geq 0,$$

holds for all  $\xi \in \mathbb{R}^n$ . Clearly, if system (1) is IOS, then the associated 0-input system  $\dot{x} = f(x, 0)$  is OS.

When  $h$  is the identity, IOS and ISS coincide, and OS is exactly the same as global asymptotic stability. Observe that the GS property is redundant in that case, as it is obviously implied by the decay estimate (11), letting  $\sigma = \min\{\beta(\cdot, 0), \gamma\}$ . We can prove that a system is IOS if and only if it is GS and there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\chi \in \mathcal{K}$ , and  $\alpha_3 \in \mathcal{KL}$ , such that for each  $r \geq 0$ , there is a smooth function  $V$  so that:

- $\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|)$  for all  $\xi \in \mathbb{R}^n$ ;
- $DV(\xi)f(\xi, \mu) \leq -\alpha_3(V(\xi), |\xi|)$  for all  $\xi$  such that  $V(\xi) \geq \chi(r)$  and all  $|\mu| \leq r$ .

This Lyapunov condition is far more complicated than one would like. It says in essence that IOS is equivalent, for each bound on controls, to the existence of an Lyapunov-like function  $V$  which vanishes only when the output vanishes, and whose derivative along trajectories is negative (unless either the function is already zero or the current input is large). Moreover, the rate of decay of  $V(x(t))$  depends on the state and on the value of  $V(x(t))$  (the main role of  $\alpha_3$  is to allow for slower convergence if  $V(x(t))$  is very small or if  $x(t)$  is very large; the inequality can be restated in various alternative ways). We do not yet know if IOS implies the existence of one  $V$  (independent of the input level  $r$ ) with the stated properties.

For the case of no inputs, the above applied with  $r = 0$  provides the following, apparently new, result for the OS property: OS is equivalent to the boundedness condition plus the existence of  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and  $\alpha_3 \in \mathcal{KL}$ , and a smooth function  $V$ , so that:

- $\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|)$  for all  $\xi \in \mathbb{R}^n$ ;
- $DV(\xi)f(\xi) \leq -\alpha_3(V(\xi), |\xi|)$  for all  $\xi \in \mathbb{R}^n$ .

For this case, no inputs, the above Lyapunov property had already appeared in the literature, but only as a *sufficient* condition.

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\*this subsection can be skipped without loss of continuity.

## 2.4 Discontinuous Feedback

A longstanding open question in nonlinear control theory concerns the relationship between asymptotic controllability to the origin in  $\mathbb{R}^n$  of a nonlinear system (1) by an “open loop” control  $u : [0, +\infty) \rightarrow \mathbb{R}^m$  and the existence of a stabilizer, that is to say a system which processes the outputs  $y = h(x)$  and provides appropriate control signals  $u$  which drive the state to zero. A particular version of this problem is the (memoryless) state feedback problem: find a map  $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$  which stabilizes trajectories of the system

$$\dot{x} = f(x, k(x)) \quad (12)$$

with respect to the origin. (As earlier remarked, objectives other than state stabilization, such as tracking and disturbance rejection, output feedback, or adaptive and robust variants, are of even more interest, and stabilization is merely the first step in the development of a general theory.)

For the special case of linear control systems  $\dot{x} = Ax + Bu$ , this relationship between open-loop and closed-loop notions is well understood: asymptotic controllability is equivalent to the existence of a continuous (even linear) stabilizing feedback law. But it is well-known that continuous feedback laws may fail to exist even for simple asymptotically controllable nonlinear systems. This is especially easy to see for one-dimensional ( $m = n = 1$ ) systems (1): in that case asymptotic controllability is equivalent to the property “for each  $x \neq 0$  there is some value  $u$  so that  $xf(x, u) < 0$ ”, but it is easy to construct examples of functions  $f$ , even analytic, for which this property is satisfied but for which no possible continuous section  $k : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  exists so that  $xf(x, k(x)) < 0$  for all nonzero  $x$ .

These negative results led to the search for feedback laws which are not necessarily of the form  $u = k(x)$ ,  $k$  a continuous function. One possible approach is to look for dynamical feedback laws, where additional “memory” variables are introduced into a controller, and as a very special case, time-varying (even periodic) continuous feedback  $u = k(t, x)$ . Such time-varying laws were shown by the Pi and Sussmann in 1980 to be always possible in the case of one-dimensional systems, and in the major work of Coron in 1992 it was shown that they are also always possible when the original system is completely controllable and has “no drift”, meaning essentially that  $f(x, 0) = 0$  for all states (see also [8] for an alternative proof of the time-varying result for analytic systems). However, for the general case of asymptotically controllable systems with drift, no dynamic or time-varying continuous solutions are known. Because of this, and also because it is in any case the obvious natural mathematical question, one may ask about the existence of *discontinuous* feedback laws  $u = k(x)$ . Such feedbacks are often obtained when solving optimal-control problems, for example, so it is interesting to search for general theorems insuring their existence. Unfortunately, allowing nonregular feedback leads to an immediate difficulty: how should one define the meaning of *solution*  $x(\cdot)$  of the differential equation (12) with discontinuous right-hand side?

One of the best-known candidates for the concept of solution of (12) is that of a *Filippov solution*, which is defined as the solution of a certain differential inclusion with multivalued right-hand side which is built from  $f(x, k(x))$ . However, it follows from recent results of Coron, Rosier, and Ryan, that the existence of a discontinuous stabilizing feedback in the Filippov sense implies the same Brockett necessary conditions as the existence of a continuous stabilizing feedback does. Moreover, the existence of a stabilizing feedback in the Filippov sense is equivalent to the existence of a continuous stabilizing one, in the case of systems affine in controls. In conclusion, there is no hope of obtaining general results if one insists on the use of Filippov solutions.

In the major recent paper [13], done in collaboration with Yuri Ledyev (who is at Rutgers as a long-term visitor) and Clarke and Subbotin, we developed a concept of solution of (12) for arbitrary feedback  $k(x)$  which has (a) a clear and reasonable physical meaning (perhaps even more so than the definitions derived from differential inclusions), and (b) allows proving the desired general theorem. Our notion is borrowed from the theory of positional differential games, and it was systematically studied in that context by Krasovskii and Subbotin.

There had been several other papers dealing with rather general theorems on discontinuous stabilization. One of the best known is Sussmann's work, which provided piecewise analytic feedback laws for analytic systems which satisfy controllability conditions. The definition of "feedback" given in that paper involves a specification of "exit rules" for certain lower-dimensional submanifolds, and these cannot be expressed in terms of a true feedback law (even in the sense of [13]). *Sampling* is a strategy commonly used in digital control. Sampling is not true feedback, in that one typically uses a fixed sampling rate, or perhaps a predetermined sampling schedule, and intersample behavior is not accounted for. One may interpret our solutions as representing the behavior of sampling, with a fixed feedback law being used, as the sampling periods tend to zero – indeed, such a speedup of sampling is essential as we approach the target state, to avoid an overshoot during the sampling interval, as well as far from the target, due to possible explosion times in the dynamics.

This work has opened a whole new research direction, having to do with the solution of all control design problems in the new framework. More recent work, e.g. [29] and continuing work, has elucidated the relationship to effects of observation and actuator errors.

### Nonsmooth Feedback: Some technical details\*

We call any infinite sequence  $\pi = \{t_i\}_{i \geq 0}$  consisting of numbers  $0 = t_0 < t_1 < t_2 < \dots$  with  $\lim_{i \rightarrow \infty} t_i = \infty$  a partition (of  $[0, +\infty)$ ), and the number  $d(\pi) := \sup_{i \geq 0} (t_{i+1} - t_i)$  its diameter. Let's call any locally bounded (for each compact, image is relatively compact) function  $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$  a *feedback*. We next define the trajectory associated to a feedback  $k(x)$  and any given partition  $\pi$  as the solution obtained by means of the following procedure: on each interval  $[t_i, t_{i+1}]$ , the initial state is measured,  $u_i = k(x(t_i))$  is computed, and then the constant control  $u \equiv u_i$  is applied until time  $t_{i+1}$ , when a new measurement is taken: Given an  $x_0 \in \mathbb{R}^n$ , for each  $i$ ,  $i = 0, 1, 2, \dots$ , recursively solve

$$\dot{x}(t) = f(x(t), k(x(t_i))), \quad t \in [t_i, t_{i+1}] \quad (13)$$

using as initial value  $x(t_i)$  the endpoint of the solution on the preceding interval (and starting with  $x(t_0) = x_0$ ). The  $\pi$ -trajectory of (12) starting from  $x_0$  is the function  $x(\cdot)$  thus obtained. Observe that this solution may fail to be defined on all of  $[0, +\infty)$ , because of possible finite escape times in one of the intervals, in which case we only have a trajectory defined on some maximal interval. In our results, however, the construction provides a feedback for which solutions are globally defined; we say in that case that the trajectory is well-defined.

We next define the meaning of (globally) stabilizing feedback. This is a feedback law which, for fast enough sampling, drives all states asymptotically to the origin and with small overshoot. Of course, since sampling is involved, when near the origin it is impossible to guarantee arbitrarily small displacements unless a faster sampling rate is used, and, for technical reasons (for instance, due to the existence of possible explosion times), one might also need to sample faster for large states. Thus the sampling rate needed may depend on the accuracy desired when controlling to zero as well as on the rough size of the initial states, and this fact is captured in the following definition. The feedback  $k : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to *s-stabilize* the system (1) if for each pair  $0 < r < R$  there exist  $M = M(R) > 0$ ,  $\delta = \delta(r, R) > 0$ , and  $T = T(r, R) > 0$  such that, for every partition  $\pi$  with  $d(\pi) < \delta$  and for any initial state  $x_0$  such

\*this subsection can be skipped without loss of continuity.

that  $|x_0| \leq R$ , the  $\pi$ -trajectory  $x(\cdot)$  of (12) starting from  $x_0$  is well-defined and it holds that: (uniform attractiveness:)  $|x(t)| \leq r \forall t \geq T$ ; (overshoot boundedness:)  $|x(t)| \leq M(R) \forall t \geq 0$ ; and (Lyapunov stability:)  $\lim_{R \downarrow 0} M(R) = 0$ . If a continuous feedback  $k$  stabilizes the system (1) in the usual sense (namely, it makes the origin of (12) globally asymptotically stable), then it also  $s$ -stabilizes. In this sense, the present notion generalizes the classical notion of stabilization.

The system (1) is *asymptotically controllable* if: (attractiveness:) for each  $x_0 \in \mathbb{R}^n$  there exists some control  $u$  such that the trajectory  $x(t) = x(t; x_0, u)$  is defined for all  $t \geq 0$  and  $x(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ; (Lyapunov stability:) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for each  $x_0 \in \mathbb{R}^n$  with  $|x_0| < \delta$  there is a control  $u$  as above such that  $|x(t)| < \varepsilon$  for all  $t \geq 0$ ; and (bounded controls:) there are a neighborhood  $V$  of 0 in  $\mathbb{R}^n$ , and a compact subset  $U$  of  $\mathbb{R}^m$  such that, if the initial state  $x_0$  above satisfies also  $x_0 \in V$ , then the control as stated can be chosen with  $u(t) \in U$  for almost all  $t$ .

Our main result in [13] is as follows: *A system is asymptotically controllable if and only if it admits an  $s$ -stabilizing feedback.* One implication is easy: existence of an  $s$ -stabilizing feedback is easily seen to imply asymptotic controllability. The interesting implication is the converse, namely the construction of the feedback law. The main ingredients in this construction are: (a) the notion of control-Lyapunov function, (b) methods of nonsmooth analysis, and (c) techniques from positional differential games.

Given a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  and a vector  $v \in \mathbb{R}^n$ , the *lower directional derivative of  $V$  in the direction of  $v$*  is  $DV(x; v) := \limsup_{t \downarrow 0, v' \rightarrow v} \frac{1}{t} (V(x + tv') - V(x))$ . The function  $v \mapsto DV(x; v)$  is lower semicontinuous. For a set  $F \subseteq \mathbb{R}^n$ ,  $\text{co}F$  denotes its convex hull. A *control-Lyapunov pair* for the system (1) consists of two continuous functions  $V, W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that the following properties hold: (positive definiteness:)  $V(x) > 0$  and  $W(x) > 0$  for all  $x \neq 0$ , and  $V(0) = 0$ ; (properness:) the set  $\{x \mid V(x) \leq \beta\}$  is bounded for each  $\beta$ ; and (infinitesimal decrease:) for each bounded subset  $G \subseteq \mathbb{R}^n$  there is some compact subset  $U \subseteq \mathbb{R}^m$  such that

$$\min_{v \in \text{co}f(x, U)} DV(x; v) \leq -W(x) \quad (14)$$

for every  $x \in G$ . If  $V$  is part of a control-Lyapunov pair  $(V, W)$ , it is a *control-Lyapunov function (clf)*.

It was shown in a paper by the PI in 1983 that asymptotic controllability is equivalent to the existence of a pair of functions  $(V, W)$  which satisfy the properties given above, except that the infinitesimal decrease property is expressed in an apparently weaker fashion, namely, by means of derivatives along trajectories (corresponding to relaxed controls). In [24] it was observed that in fact one can reformulate the definition in the above terms, so a system is asymptotically controllable if and only if it admits a control-Lyapunov function. Of course, when the function  $V$  is smooth, condition (14) can be written in the more familiar form found in the literature, namely:

$$\min_{u \in U} \langle \nabla V(x), f(x, u) \rangle \leq -W(x). \quad (15)$$

In contrast to the situation with stability of (non-controlled) differential equations, a system may be asymptotically controllable system and yet there may not exist any possible smooth clf  $V$ . In other words, there is no analogue of the classical theorems due to Massera and Kurzweil. This issue is intimately related to that of existence of continuous feedback, via Artstein's Theorem, which asserts that existence of a differentiable  $V$  is equivalent, for systems affine in controls, to there being a stabilizing regular feedback. Nevertheless, it is possible to reinterpret the condition (15) in such a manner that relation (15) does hold in general, namely by using a suitable generalization of the gradient. Specifically, as remarked in [13], we may use the proximal subgradients of  $V$  at  $x$  instead of  $\nabla V(x)$ , replacing (15) by:

$$\min_{u \in U} \langle \zeta, f(x, u) \rangle \leq -W(x) \quad \text{for every } \zeta \in \partial_P V(x), \quad (16)$$

where  $\zeta$  and  $\partial_P V(x)$  are the proximal subgradients and the subdifferential, respectively, of the function  $V$  at the point  $x$ . The use of proximal subgradients as substitutes for the gradient for a nondifferentiable function plays a central role in our construction of feedback.

The relation (16) says that  $V$  is a "proximal supersolution" of the corresponding Hamilton-Jacobi equation, which is known to be equivalent to the statement that locally  $V$  is a viscosity supersolution of

the same equation. On the other hand, the relation (14) says that  $V$  is an upper minimax solution of the same equation. The coincidence of these two solution concepts reflects the deep and intrinsic connection between invariance properties of function  $V$  with respect to trajectories of the control system (1) and the characterization of these properties in terms of proximal subgradients of  $V$ . The proof of the main result in [13] relies upon the smoothing of a clf  $V$  on compacts; this smoothing is not itself a clf, but it allows finding suitable proximal subgradients, which in turn provide the directions for control.

## 2.5 Universal Controls

A long-standing direction of the PI's research concerns the study of transversality-type theorems which assert the existence of controls with particularly useful characteristics. In the case of inputs universal for observability, the study leads to the deep understanding of relationships between different types of observation spaces and the existence of i/o equations, which in turn give rise to identification procedures; in the case of universality for controllability, one finds algorithms for path planning. We briefly describe some of these ideas next.

The PI's presentation and paper at the ICM [17] dealt with observation spaces, and in particular with the applications of these to the study of the equivalence between i/o high order differential equations  $E(w(t), w'(t), w''(t), \dots, w^{(r)}(t)) = 0$ , where  $w(\cdot) = (u(\cdot), y(\cdot))$  represent the i/o pairs of the system, and state-space representations. In the linear case, such representations form the basis of much of adaptive control and identification theory; and for nonlinear systems, (the discrete-time versions of) our results have also been applied to identification algorithms. Central to this work is the analysis of *observation spaces*, in their different variants. These are essentially spaces spanned by derivatives of functions on states induced by inputs, and different algebraic structures can be introduced, thus obtaining algebras and fields which characterize systems properties (for instance, states of the system appear as the prime spectrum, in the Gelfand-Grothendieck style, of rings of observables; see [17]). Most results about observation spaces depend on the existence of inputs universal for distinguishability. We wish to continue research in this area, and in particular we want to write a detailed technical paper showing that there are controls which are universal with respect to the class of all analytic (finite dimensional) systems (only a brief sketch of this fact appeared in a recent conference proceedings).

A dual version of the universality results allowed us to obtain a result insuring the existence and genericity of universal nonsingular controls; this led us in turn, in [8], to the development of a simple numerical technique for the steering of arbitrary analytic systems with no drift. It is based on the generation of "nonsingular loops" which allow linearized controllability along suitable trajectories. Once such loops are available, it is possible to employ standard Newton or steepest descent methods, as classically done in numerical control. During this last year (1996), we worked with a graduate student at Rutgers (Master's thesis by R. Woolley) in the development of efficient codes for implementing the method in [8]. The resulting code, obtainable by e-mail from the PI, is still preliminary, but runs fast on simple examples. The next section briefly discusses technically the method.

As a final subtopic, we remark that the existence of universal controls for nonsingularity of *discrete time* systems is still not completely resolved, but in current work with F. Wirth we have succeeded in proving the main technical results, and are now looking into the estimation of effective bounds for input lengths.

### Nonsingular Loop Approach: Some technical details\*

The problem of interest concerns, for any given initial and target states  $\xi_0$  and  $\xi_F = 0$  in  $\mathbb{R}^n$ , numerically finding a time  $T > 0$  and a control  $u$  which (approximately) steers  $\xi_0$  to  $\xi_F$ , in (1), i.e.  $x(T, \xi_0, u) \approx \xi_F$ . For now, we assume that the right-hand side  $f$  is continuously differentiable (later, analytic). Classical numerical techniques for this problem are based on variations of steepest descent. The basic idea is to start with a guess of a control, say  $\bar{u} : [0, T] \rightarrow \mathbb{R}^m$ , and to improve iteratively on this initial guess. More precisely, let  $\bar{x} = x(\cdot, \xi_0, \bar{u})$ . If the obtained final state  $\bar{x}(T)$  is already zero, or is sufficiently near zero, the problem has been solved. Otherwise, we look for a perturbation  $\Delta\bar{u}$  so that the new control  $\bar{u} + \Delta\bar{u}$  brings us closer to our goal of steering  $\xi_0$  to the origin. The various techniques differ on the choice of the perturbation; in particular, two possibilities are Newton's method and steepest descent.

As an illustration, take Newton's method. Denote, for any fixed initial state  $\xi_0$ ,  $\alpha(u) := x(T, \xi_0, u)$ , thought of as a partially defined map from  $\mathcal{L}_\infty^m(0, T)$  into  $\mathbb{R}^n$ . This is a continuously differentiable map so expanding to first order there results  $\alpha(\bar{u} + v) = \alpha(\bar{u}) + \alpha_*[\bar{u}](v) + o(v)$  for any other control  $v$  so that  $\alpha(\bar{u} + v)$  is defined, where we use "\*" as a subscript to denote differentials. If we can now pick  $v$  so that

$$\alpha_*[\bar{u}](v) = -\alpha(\bar{u}), \quad (17)$$

then for small enough  $h > 0$  real,

$$\alpha(\bar{u} + hv) = (1 - h)\alpha(\bar{u}) + o(h) \quad (18)$$

will be smaller than the state  $\alpha(\bar{u})$  reached with the initial guess control  $\bar{u}$ . In other words, the choice of perturbation is  $\Delta\bar{u} := hv$ ,  $0 < h \ll 1$ . It remains to solve equation (17) for  $v$ . The operator  $L : v \mapsto \alpha_*[\bar{u}](v)$  is the one corresponding to the solution of the variational equation

$$\dot{z} = A(t)z + B(t)v \quad z(0) = 0, \quad (19)$$

where  $A(t) := \frac{\partial f}{\partial x}(\bar{x}(t), \bar{u}(t))$  and  $B(t) := \frac{\partial f}{\partial u}(\bar{x}(t), \bar{u}(t))$  for each  $t$ , that is,  $Lv = \int_0^T \Phi(T, s)B(s)v(s) ds$ , where  $\Phi$  denotes the fundamental solution associated to  $\dot{X} = A(t)X$ . The operator  $L$  maps  $\mathcal{L}_\infty^m(0, T)$  into  $\mathbb{R}^n$ , and it is onto when (19) is a controllable linear system on the interval  $[0, T]$ , that is, when  $\bar{u}$  is a control nonsingular for  $\xi_0$  relative to the system (1). In other words, onto-ness of  $L = \alpha_*[\bar{u}]$  is equivalent to first-order controllability of the original nonlinear system along the trajectory corresponding to the initial state  $\xi_0$  and the control  $\bar{u}$ . The main point of the paper [8] was to show that it is not difficult to generate useful nonsingular controls for systems with no drift.

Assuming nonsingularity, there exist then many solutions to (17). It is reasonable to pick the least squares solution, that is the unique solution of minimum norm,  $v := -L^\# \alpha(\bar{u})$  where  $L^\#$  denotes the pseudoinverse operator. The technique sketched above is well-known in numerical control. For instance, the derivation in pages 222-223 of the classical text by Bryson and Ho, when applied to solving the optimal control problem having the trivial cost criterion  $J(u) = 0$  and subject to the final state constraints  $x = \psi(x) = 0$ , results in the above formula, and is derived in the same manner as here. It is also possible to combine the technique with line searches over the scalar parameter  $h$  or, even more efficiently in practice, with conjugate gradient approaches.

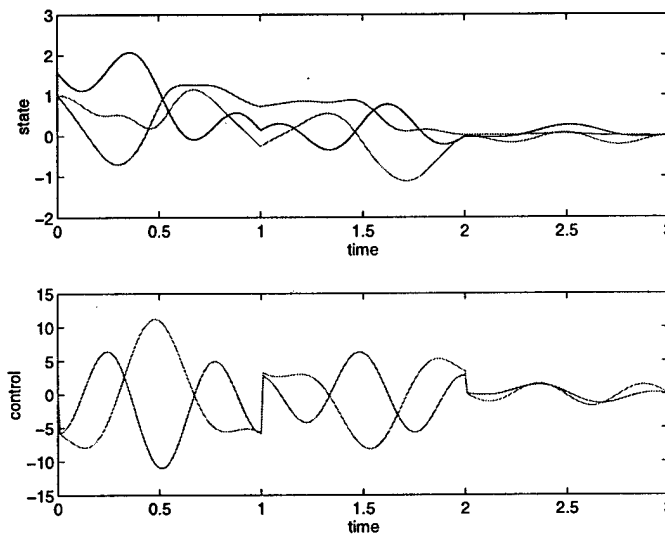
A result in [8] provides a convergence proof of an iterative procedure based on this idea, for analytic systems with no drift  $\dot{x} = G(x)u$ . In order to apply the numerical technique described, one needs to find a control  $\bar{u}$  which leads to nonsingular loops:  $\bar{u}$  is nonsingular for every state  $x$  in a given ball  $\mathcal{B}$ , and  $x(T, x, \bar{u}) = x$  for all such  $x$ . It is shown there that for analytic systems that have the strong accessibility property, controls which are generic—in a Whitney topology sense—are nonsingular for all states. (For analytic systems without drift, Chow's Theorem states that the strong accessibility property is equivalent to complete controllability.) Starting from such a control  $\omega$ , defined on an interval  $[0, T/2]$ , one may now consider the control  $\bar{u}$  on  $[0, T]$  which equals  $\omega$  on  $[0, T/2]$  and is then followed by the antisymmetric extension:  $\bar{u}(t) = -\omega(T - t)$ ,  $t \in (T/2, T]$ . This  $\bar{u}$  is as needed: nonsingularity is due to the fact that if the restriction of a control to an initial subinterval is nonsingular for the initial state, the

\*this subsection can be skipped without loss of continuity.

whole control is, and the loop property is an easy consequence of the fact that there is no drift (so the control appears linearly). In practice, we have implemented instead a finite Fourier series with random coefficients:  $\bar{u}(t) = \sum_{k=1}^l a_k \sin kt$ . which automatically satisfies the antisymmetry property on the time interval  $[0, 2\pi]$ . As an extremely simple example, consider the classical "knife edge" or "unicycle" problem with equations

$$\begin{aligned}\dot{x} &= u_1 \cos \theta \\ \dot{y} &= u_1 \sin \theta \\ \dot{\theta} &= u_2.\end{aligned}$$

We use a two-term Fourier series with  $T = 2\pi$  and randomly generated uniform  $[-1, 1]$  coefficients to obtain (generically) a nonsingular control. The figure shows a typical path and the controls used.



Obviously, this is far from a best possible, or perhaps even desirable, path. But it is striking that the algorithm is able to find a solution (and is guaranteed to do so) without any built-in knowledge whatsoever of the Lie algebraic structure; only the equations of motions are needed (for an internal model) and the form of the variational equation.

### 3 Project Publications 1995/1997

1. (with Y. Lin and Y. Wang) "Input to state stabilizability for parameterized families of systems," *Intern. J. Robust & Nonlinear Control* 5 (1995): 187-205.
2. (with M.A. Dahleh, D.N.C. Tse, and J.N. Tsitsiklis) "Worst-case identification of nonlinear fading memory systems," *Automatica*, 31(1995): 503-508.
3. (With Y. Wang) "Orders of input/output differential equations and state space dimensions," *SIAM J. Control and Optimization* 33 (1995): 1102-1127.
4. (with Y. Wang) "On characterizations of the input-to-state stability property," *Systems and Control Letters* 24 (1995): 351-359.
5. (with Y. Lin and Y. Wang) "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control and Optimization* 34 (1996): 124-160.
6. (with W. Liu and Y. Chitour) "On finite gain stabilizability of linear systems subject to input saturation," *SIAM J. Control and Optimization* 34 (1996): 1190-1219.
7. (with W. Liu and Y. Chitour) "On the continuity and incremental-gain properties of certain saturated linear feedback loops," *Intern. J. Robust & Nonlinear Control* 5(1995): 413-440.
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15. (with F.R. Wirth) "Remarks on universal nonsingular controls for discrete-time systems," to appear.
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31. (with F. Albertini) "Control-Lyapunov functions for time-varying set stabilization," *Proc. European Control Conf.*, Brussels, July 1997, Paper WE-E A5, CD-ROM file ECC515.pdf, 6 pages.