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6. AUTHOR(S)
H. J. Kimble

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California Institute of Technology
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Pasadena, CA 91125

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13. ABSTRACT (Maximum 200 words)

Research underway in the Quantum Optics Group at Caltech explores quantum information processing with continuous quantum variables. The most noteworthy success of this program has been the experimental realization of quantum teleportation for coherent states of the electromagnetic field. Beyond our effort to extend these results to achieve yet higher "quality" of teleportation, we have initiated new research on several fronts, including super-dense quantum coding and teleportation of atomic wavepackets. The research is part of a larger program to lay the foundations for the realization of quantum networks for the distribution and processing of quantum information.

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- "Proposal for teleportation of the wave function of a massive particle," A. S. Parkins and H. J. Kimble, Phys. Rev. A (submitted, 1999; see also quant-ph/9909021).
- "Teleporting an Atomic Wavepacket," A. S. Parkins and H. J. Kimble, *Frontiers of Laser Physics and Quantum Optics*, eds. Z. Xu et al. (Springer, Berlin, 2000), 132.
- "Criteria for continuous-variable quantum teleportation," Samuel L. Braunstein, Christopher A. Fuchs and H. J. Kimble, Journal of Modern Optics **47**, 267 (2000).
- "Dense coding for continuous variables," Samuel L. Braunstein and H. J. Kimble, Phys. Rev. A **61**, 042302.
- "Broadband teleportation," P. van Loock and Samuel L. Braunstein, Phys. Rev. A**62**, 022309 (2000); see also quant-ph/9902030.
- "Experimental Realization of Continuous Variable Teleportation," Akira Furusawa and H. J. Kimble, in *Quantum Information Theory with Continuous Variables*, eds. Samuel L. Braunstein and Arun K. Pati (Kluwer Academic, 2000).
- "Quantum versus Classical Domain for Teleportation with Continuous Variables," Samuel L. Braunstein, Christopher A. Fuchs, H. J. Kimble, and P. van Loock, (accepted, Phys. Rev. A, 2001); available as quant-ph/0012001.

(2) Scientific Personnel

Professor H. J. Kimble, Ph.D., Principal Investigator
Professor Akira Furusawa, Visiting Associate
Dr. Tiancai Zhang, Visiting Scholar
Dr. Peter Lodahl, Postdoctoral Scholar
Graduate Research Assistant: Joseph Buck, Chin-Wen Chou, Kok Win Goh

(3) Report of Inventions -- None.

(4) Scientific Progress and Accomplishments

In 1998, we carried out an experiment that was the first (and remains the only) genuine realization of quantum teleportation. Our protocol exploited squeezed-state entanglement shared between *Alice's sending station* and *Bob's receiving terminal* to teleport an unknown quantum state from *Alice* to *Bob*. The experimentally determined fidelity was greater than that which could be achieved by classical means.

Since this initial effort, we have used the current ARO support in an attempt to extend the results to yet higher values of fidelity (i.e., improved *quality* of teleportation) that would enable the teleportation of manifestly quantum or nonclassical states of light. We are as well attempting to implement improved measurement techniques to characterize the input and output states by way of quantum state tomography. Finally, we have invented and analyzed theoretically other protocols for quantum information processing with continuous quantum variables (including quantum dense coding), and are working toward their experimental implementation.

A principal experimental advance has come from the rebuilding of the entire apparatus from the output of the non-degenerate optical parametric oscillator used to generate entanglement, to the distribution paths to *Alice* and *Bob*. Most importantly, we have improved a critical factor related to beam overlap efficiencies for two squeezed beams from its initial value η^2=80% to η^2=94%. Note that this factor enters directly into the overall fidelity for teleportation. We have also obtained new nonlinear crystals for parametric down conversion to generate the EPR beams as well as for frequency doubling for the pump beam, with some improvement in the efficiency but with blue-light induced absorption still being the major limiting factor. With support from other agencies, we have also been able to replace many other of the critical components of the experiment (which were more more than 10 years old). Most significantly, we have been able to replace the existing Titanium sapphire laser (which was a homemade unit and which was in a constant state of failure) and its associated Argon-ion pump laser.

Theoretical efforts have been directed to a theory of quantum dense coding for continuous quantum variables, of quantum teleportation for broad bandwidth fields, and of the teleportation of the quantum states of matter. As part of an ongoing program to delineate the quantum versus classical boundaries for quantum information processing ,we have carried out an extensive investigation of this question for teleportation of coherent states of light. We have examined the quantum-classical boundary by investigating questions of entanglement and Bell-inequality violations for the EPR states relevant to continuous variable teleportation. The threshold for employing entanglement as a quantum resource in teleportation of coherent states is found to be $F_{classical} = ½$. Likewise, violations of local realism onset at this same threshold $F_{classical} = ½$. These results are in agreement with our previous analysis based upon the utilization of a classical channel without entanglement for teleportation.

(5) Technology Transfer – None.

Proposal for teleportation of the wave function of a massive particle

A.S. Parkins[1] and H.J. Kimble[2]
[1] *Department of Physics, University of Auckland, Auckland, New Zealand*
[2] *Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, CA 91125, U.S.A.*
(September 6, 1999)

We propose a scheme for teleporting an atomic center-of-mass wave function between distant locations. The scheme uses interactions in cavity quantum electrodynamics to facilitate a coupling between the motion of an atom trapped inside a cavity and external propagating light fields. This enables the distribution of quantum entanglement and the realization of the required motional Bell-state analysis.

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In a landmark work of 1993, Bennett *et al.* [1] discovered a procedure for teleporting an unknown quantum state from one location to another. The essential ingredient in their protocol is quantum entanglement of a bipartite system shared by the sender, Alice, and receiver, Bob. This shared entanglement, in unison with suitable measurements performed by Alice and communicated via classical channels to Bob, 'mediates' the state transfer.

Since the work of Bennett *et al.*, a variety of possible experimental schemes for the teleportation of quantum states of *two-state* systems have been proposed, in large part by the quantum optics community (see, e.g., [2–5]). In an exciting recent development, the first experimental investigations of teleportation of such states have been performed [6,7] with, in particular, the polarization state of a photon providing the two-state system of interest.

Complementing this work on two-state systems has been research into the teleportation of states of infinite-dimensional systems [8,9], culminating last year in the experimental demonstration of quantum teleportation of *optical coherent states* [10]. This experiment was based on the specific proposal of [9], utilizing squeezed-state entanglement and balanced homodyne measurements of the light fields. Given that the experiment employed only standard optical elements and measurement techniques, it offers significant promise of further intriguing possibilities for quantum information processing with continuous quantum variables, including quantum dense coding [11], and universal quantum computation [12].

Another burgeoning field of research in quantum information science is the implementation of quantum logic with trapped atoms or ions. Inspired by the proposal of Cirac and Zoller [13] for a quantum computer based on the motional and internal degrees of freedom of a collection of trapped ions, impressive experimental progress has been made towards controlling quantum properties in such systems [14,15]. Particular advantages of trapped atom systems include long coherence times and the exquisite control of transformations between motional and internal states.

In view of this tremendous potential of both light- and motion-based schemes for quantum information processing, it is sensible to investigate possibilities for combining the two approaches and their distinct advantages. One particular application would be to quantum networks for distributed quantum computing and communication, where, e.g., the 'distribution' is accomplished with light fields [16], while local processing is performed on motional states of a collection of trapped atoms. Indeed, with the protocols of [16] in mind, we have recently proposed and analyzed a cavity-QED-based system that enables the transfer of quantum states between the motion of a trapped atom and propagating light fields [17,18], which should lead to new capabilities for the synthesis and control of quantum states for both motion and light.

A particular example from [18] is the possibility of creating an EPR (Einstein-Podolsky-Rosen) state in position-momentum for distantly separated atoms. The creation of such an entangled state between remote particles suggests an avenue for achieving teleportation of an unknown wave function of a trapped particle between the EPR sites. In this letter, we propose and analyze one such protocol that enables the teleportation of an unknown one-dimensional *atomic center-of-mass wave function* and that should be attainable within the context of emerging experimental capabilities for trapping atoms in cavity QED [19,20].

Our proposed teleportation scheme is shown schematically in Fig. 1. Each of Alice ($\mathcal{A}$), Bob ($\mathcal{B}$), and Victor ($\mathcal{V}$) possess an atom trapped inside an optical cavity. The aim is to teleport the (x-dimension) motional state of Victor's atom to Bob's atom. This is achieved via the three usual stages for continuous quantum variables [9,10] – (i) preparation of quantum entanglement between Alice and Bob's atoms, (ii) Bell-state (homodyne) measurement by Alice, and (iii) phase-space displacement $[D(\alpha^*)]$ by Bob (given the classical result α of Alice's measurement). Finally, the state of Bob's atom may be examined by Victor for verification of the quality of the teleportation (or indeed, physically delivered to Victor). Stages (i) and (ii), as illustrated in Fig. 1, employ the cavity-mediated motion-light state transfer scheme of [17], to which we now turn our attention.

Briefly, a single two-level atom (or ion) is tightly confined in a harmonic trap located inside a high-finesse optical cavity. The atomic transition of frequency ω_a is coupled to a single mode of the cavity field of frequency ω_c

and also to an external (classical) laser field of frequency ω_L and strength $\mathcal{E}_L$. The physical setup and excitation scheme are depicted in Fig. 2(a). The cavity is aligned along the x-axis, while the field $\mathcal{E}_L$ is incident from a direction in the y-z plane. Both the cavity field and $\mathcal{E}_L$ are far from resonance with the atomic transition, but their difference frequency is chosen so that they drive Raman transitions between neighboring motional number states (i.e., $\omega_c - \omega_L = \nu_x$, with ν_x the x-axis trap frequency).

A number of assumptions are made in order to achieve the desired motion-light coupling: (i) Atomic spontaneous emission is neglected and the internal atomic dynamics adiabatically eliminated. (ii) The size of the harmonic trap, located at a *node* of the cavity field, is taken to be small compared to the optical wavelength (Lamb-Dicke regime), enabling the approximations $\sin(k\hat{x}) \simeq \eta_x(\hat{b}_x + \hat{b}_x^\dagger)$ and $\mathcal{E}_L(\hat{y}, \hat{z}, t) \simeq \mathcal{E}_L(t)e^{-i\phi_L}$, where η_x ($\ll 1$) is the Lamb-Dicke parameter and $\hat{x} = (\hbar/2m\nu_x)^{1/2}(\hat{b}_x + \hat{b}_x^\dagger)$. (iii) The trap frequency ν_x and cavity field decay rate κ are assumed to satisfy $\nu_x \gg \kappa \gg |(g_0\eta_x/\Delta)\mathcal{E}_L(t)|$, where g_0 is the single-photon atom-cavity mode coupling strength, and $\Delta = \omega_a - \omega_L$. The first inequality allows a rotating-wave approximation to be made with respect to the trap oscillation frequency, while the second inequality enables an adiabatic elimination of the cavity field mode.

Under these conditions, the motional mode dynamics in the x direction is well described by the simple quantum Langevin equation [17]

$$\dot{\tilde{b}}_x \simeq -\Gamma(t)\tilde{b}_x + \sqrt{2\Gamma(t)}\,\tilde{a}_{\rm in}(t)\,, \qquad (1)$$

where $\tilde{b}_x = e^{i\nu_x t}\hat{b}_x$ and $\Gamma(t) = [g_0\eta_x|\mathcal{E}_L(t)|/\Delta]^2/\kappa$. The operator $\tilde{a}_{\rm in}(t)$ obeys the commutation relation $[\tilde{a}_{\rm in}(t), \tilde{a}_{\rm in}^\dagger(t')] = \delta(t-t')$ and describes the quantum noise *input to the cavity field* (in a frame rotating at the cavity frequency). From the linear nature of (1), it follows that the statistics of a (continuous) light field incident upon the cavity can be 'written onto' the state of the atomic motion. This also means that entanglement between separate light fields can be transferred to entanglement between separate motional states, as we discuss below. From a consideration of the input-output theory of optical cavities [21], it also follows that measurements on the cavity output field amount to measurements on the motion of the atom. In particular, one can show that

$$\tilde{a}_{\rm out}(t) \simeq -\tilde{a}_{\rm in}(t) + \sqrt{2\Gamma(t)}\,\tilde{b}_x(t)\,. \qquad (2)$$

So, for a vacuum input field, homodyne measurements on the cavity output field realize position or momentum measurements (or some mixture, depending on the local oscillator phase) on the trapped atom. This enables the necessary Bell-state analysis to be performed.

To begin the teleportation procedure, Victor's atom is prepared in a particular motional state $|\phi\rangle_{\mathcal{V}x}$ in the x dimension (e.g., by the techniques of [15]). With the motion-light coupling switched off in Victor's cavity (i.e., $\mathcal{E}_{L\mathcal{V}} = 0$), this state is assumed to remain unchanged until required by Alice for the Bell-state analysis.

Next, a position-momentum EPR state of Alice and Bob's atoms is prepared by using the motion-light coupling described in (1), with input light fields from a nondegenerate optical parametric amplifier (NOPA). This preparation, described in detail in [18], is depicted in Fig. 2(b). The two quantum-correlated output light fields from a NOPA (operating below threshold) are separated and made to impinge on Alice and Bob's cavities, respectively. Assuming $\Gamma_\mathcal{A} = \Gamma_\mathcal{B} = \Gamma$, after a time $t \gg \Gamma^{-1}$, the following *pure* entangled motional state is prepared,

$$|\psi\rangle_{\mathcal{AB}} = S_{\mathcal{AB}}(r)|0\rangle_{\mathcal{A}x}|0\rangle_{\mathcal{B}x}$$
$$= [\cosh(r)]^{-1}\sum_{m=0}^{\infty}[-\tanh(r)]^m\,|m\rangle_{\mathcal{A}x}|m\rangle_{\mathcal{B}x}\,, \quad (3)$$

where $|m\rangle_{\mathcal{A},\mathcal{B}x}$ are Fock states of the motional modes and $S_{\mathcal{AB}}(r) = \exp[r(\tilde{b}_{\mathcal{A}x}\tilde{b}_{\mathcal{B}x} - \tilde{b}_{\mathcal{A}x}^\dagger\tilde{b}_{\mathcal{B}x}^\dagger)]$, with r the 'entanglement' parameter. Once this state has been prepared, the atom-cavity couplings are turned off ($\Gamma_\mathcal{A}, \Gamma_\mathcal{B} \to 0$), as is the NOPA pump field. Again, we assume that the entangled state (3) remains unchanged until the next step in the procedure.

At this stage in the protocol, the total system state is

$$|\Psi_{\rm I}\rangle = |\phi\rangle_{\mathcal{V}x}\,|\psi\rangle_{\mathcal{AB}}\,. \qquad (4)$$

The Bell-state analysis performed by Alice is depicted in Fig. 3. At a predetermined time, Victor switches on his atom-cavity coupling $\Gamma_\mathcal{V}$ via $\mathcal{E}_{L\mathcal{V}}(t)$, thus converting the state $|\phi\rangle_{\mathcal{V}x}$ to that of a freely propagating field delivered to input beam-splitter BS of Alice's sending station. With due accounting for propagation delay, Alice has likewise switched on the coupling $\Gamma_\mathcal{A}$ from her cavity, where, for simplicity, $\Gamma_\mathcal{V} = \Gamma_\mathcal{A} = \Gamma$. Note that Victor and Alice's cavities both have vacuum inputs at this stage. The two cavity output fields are combined by Alice at the 50/50 beamsplitter BS, the two outputs of which are incident on homodyne detectors $D_\pm$. Through the input-output relation (2), and through the mixing of the cavity output fields at the beamsplitter, these detectors effect homodyne measurements on the modes $\tilde{c}_\pm = 2^{-1/2}(\tilde{b}_{\mathcal{V}x} \pm \tilde{b}_{\mathcal{A}x})$. The effect of these measurements is to project the system state onto quadrature eigenstates of the modes $\tilde{c}_\pm$, given by $|\chi_\pm\rangle_\pm = Q_\pm^\dagger(\chi_\pm)|0\rangle_\pm$, where $Q_\pm^\dagger(\chi_\pm) = (2\pi)^{-1/4}\exp[-(1/2)(\tilde{c}_\pm^\dagger e^{i\theta_\pm} - \chi_\pm)^2 + \chi_\pm^2/4]$, with $\theta_\pm$ the local oscillator (LO) phases [22–24]. In [23,24], this projection is proved with the assumption that the local oscillator photon flux matches the temporal shape of the signal flux [which in our case is set by $\Gamma(t)$], while the variable χ is shown to be equivalent to the integrated homodyne photocurrent.

For the two homodyne measurements we choose LO phases $\theta_+ = 0$ and $\theta_- = \pi/2$. With these choices one can show that, in terms of the original mode operators,

$$Q_+(\chi_+)Q_-(\chi_-) = (2\pi)^{-1/2}\exp\left(-|\alpha|^2/2\right)$$

$$\cdot \exp\left(-\tilde{b}_{\mathcal{V}x}\tilde{b}_{\mathcal{A}x} + \alpha\tilde{b}_{\mathcal{V}x} + \alpha^*\tilde{b}_{\mathcal{A}x}\right), \qquad (5)$$

where $\alpha = (\chi_+ + i\chi_-)/\sqrt{2}$. The motional state of Bob's atom following the homodyne measurements, with results $\chi_\pm$, can thus be written

$$|\varphi\rangle_{\mathcal{B}x} \propto {}_{\mathcal{V}x}\langle 0|_{\mathcal{A}x}\langle 0|Q_+(\chi_+)Q_-(\chi_-)|\Psi_{\rm I}\rangle$$
$$\propto {}_{\mathcal{V}x}\langle\alpha^*|_{\mathcal{A}x}\langle\alpha|\exp\left(-\tilde{b}_{\mathcal{V}x}\tilde{b}_{\mathcal{A}x}\right)|\Psi_{\rm I}\rangle. \qquad (6)$$

Using properties of the squeezing operator $S_{\mathcal{AB}}(r)$ [21], one can further reduce this to

$$|\varphi\rangle_{\mathcal{B}x} \propto {}_{\mathcal{V}x}\langle\alpha^*|\exp\left[\Lambda\left(\tilde{b}_{\mathcal{V}x} - \alpha^*\right)\tilde{b}^\dagger_{\mathcal{B}x}\right]|\phi\rangle_{\mathcal{V}x}|0\rangle_{\mathcal{B}x}, \quad (7)$$

where $\Lambda = \tanh(r)$. Expanding $|\phi\rangle_{\mathcal{V}x}$ in terms of the coherent states, i.e., $|\phi\rangle_{\mathcal{V}x} = \pi^{-1}\int d^2\beta\ {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,|\beta\rangle_{\mathcal{V}x}$, the right-hand-side of (7) becomes

$$\frac{1}{\pi}\int d^2\beta\ {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,\frac{{}_{\mathcal{V}x}\langle\alpha^*|\beta\rangle_{\mathcal{V}x}}{{}_{\mathcal{V}x}\langle\Lambda\alpha^*|\Lambda\beta\rangle_{\mathcal{V}x}}$$
$$\cdot D_{\mathcal{B}}(-\Lambda\alpha^*)D_{\mathcal{B}}(\Lambda\beta)\,|0\rangle_{\mathcal{B}x}, \qquad (8)$$

where $D_{\mathcal{B}}(\beta) = \exp(\beta\tilde{b}^\dagger_{\mathcal{B}x} - \beta^*\tilde{b}_{\mathcal{B}x})$ is the coherent displacement operator for Bob's atom. In the limit of strong squeezing and entanglement ($\Lambda \to 1$), (8) approaches

$$D_{\mathcal{B}}(-\alpha^*)\,\frac{1}{\pi}\int d^2\beta\ {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,|\beta\rangle_{\mathcal{B}x}. \qquad (9)$$

That is, $|\varphi\rangle_{\mathcal{B}x}$ approaches a state which, apart from a coherent displacement by $-\alpha^*$, is identical to the initial motional state (in the x dimension) of Victor's atom.

Given the measurement results $\chi_\pm$, transmitted to Bob via a classical channel, the final step in the teleportation procedure is for Bob to apply a coherent displacement α^* (assuming $\Lambda \simeq 1$) to the motional state of his atom, i.e., $D_{\mathcal{B}}(\alpha^*)|\varphi\rangle_{\mathcal{B}x} \to |\phi\rangle_{\mathcal{B}x}$. In practice, this might be achieved by applying an electric field (in the case of a trapped ion) along the x-axis which oscillates at the trap frequency ν_x, or, alternatively, by applying off-resonant laser fields which drive stimulated Raman transitions between neighboring trap levels [15]. After this, control of Bob's atom can be passed to Victor, who is free to confirm the overall quality of the teleportation protocol, e.g., along the lines analyzed in [25].

Issues of practicality associated with the motion-light state transfer procedure central to our teleportation scheme have been discussed elsewhere [17,18]. In brief, desired conditions are of (i) strong coupling optical cavity QED, such that $g_0^2/(\kappa\gamma) \gg 1$, where γ is the atomic spontaneous decay rate, and (ii) strong confinement of the atoms with minimal motional-state decoherence. Both of these conditions have been achieved separately [14,15,19,20], and we expect that future experiments trapping single atoms inside optical cavities will be able to meet these criteria simultaneously. Note that timescales for motional-state decoherence of trapped ions can be of the order of milliseconds [15]; the typical timescale involved in our teleportation scheme, Γ^{-1}, would likely be of the order of microseconds [17,18]. Finally, calculations in [9] suggest reasonable (nonclassical) teleportation fidelities to be possible with values of the squeezing parameter $r > 1$.

To conclude, we note that the scheme given here is just one of a number of possibilities that we have analyzed. One could, e.g., eliminate Victor's atom and cavity and, as the state to be teleported, choose the motional state of Alice's atom along an axis *orthogonal* to the x-axis. After preparing the entangled (x-dimension) motional state $|\psi\rangle_{\mathcal{AB}}$ of Alice and Bob's atoms, the orthogonal motional modes of Alice's atom could be linearly mixed within the trap itself, in the fashion of a beamsplitter, using suitable interactions with auxiliary laser fields [26,27], after which coupling to the cavity field and homodyne measurement of the output light field would again provide the Bell-state analysis. In addition, as we will discuss elsewhere [27], it is possible to eliminate the NOPA from the scheme and use only trapped atoms interacting with cavity and laser fields both to produce and distribute the quantum entanglement required for the teleportation protocol.

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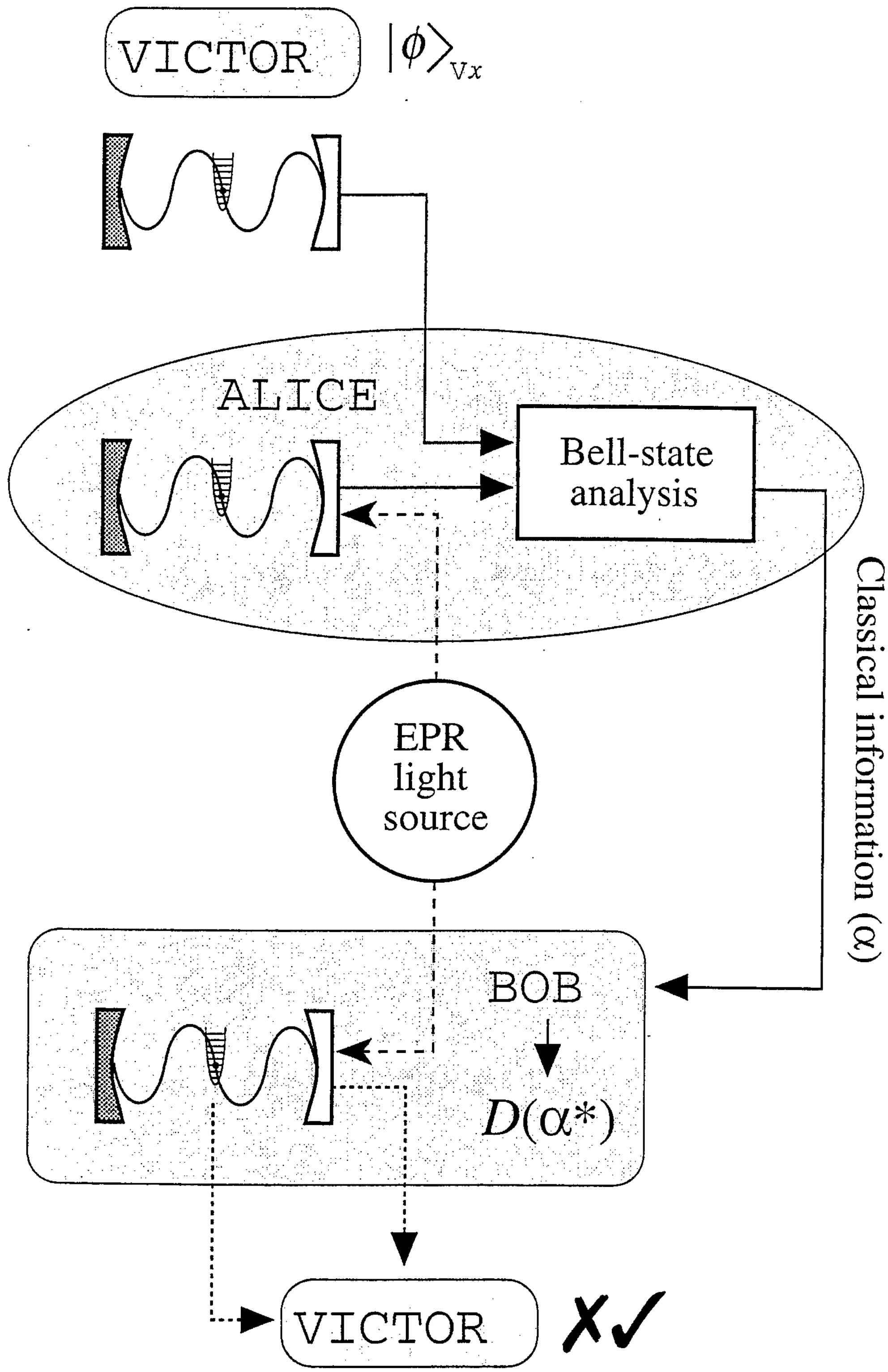
FIG. 1. Schematic of proposed teleportation scheme for atomic wavepackets. Preparation of motional-state entanglement between Alice and Bob and Bell-state analysis by Alice are facilitated by cavity-mediated motion-light couplings.

FIG. 2. (a) Proposed setup and excitation scheme for coupling between the motion of a trapped atom and a quantized optical cavity mode, and thence to a freely propagating external field. The cavity is assumed to be *one-sided*, i.e., one mirror is taken to be perfectly reflecting. (b) Preparation of a position-momentum EPR state of Alice and Bob's atoms. The two output fields from a nondegenerate parametric amplifier (NOPA) impinge on Alice and Bob's cavities, respectively. Faraday isolators (F) facilitate a unidirectional coupling between the entangled light source and the atom-cavity systems.

FIG. 3. Schematic of Alice's Bell-state analysis. The output field representing Victor's unknown state is combined by Alice at a 50/50 beamsplitter (BS) with the output field from her cavity. The resulting output fields from the BS are incident on homodyne detectors $D_{\pm}$. The cavity output fields follow the motional modes, which decay on a timescale Γ^{-1}. The local oscillator fields ($LO_{\pm}$) are pulsed, with temporal profiles chosen to match that of the cavity output fields.


VICTOR
$|\phi\rangle_{Vx}$
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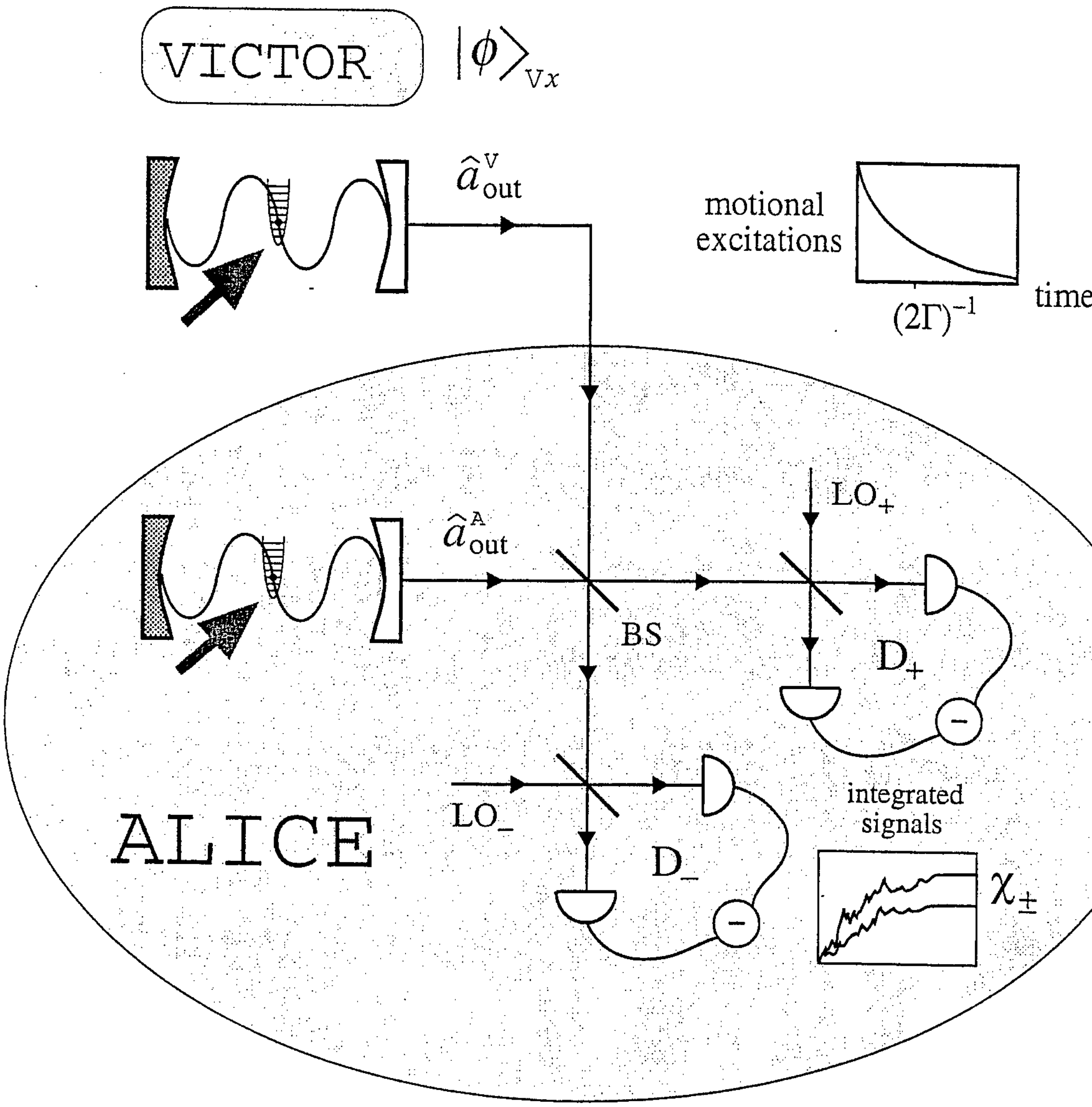
Bell-state analysis
Classical information (α)
EPR light source
BOB
$D(\alpha^*)$
VICTOR

Figure 1



VICTOR
$|\phi\rangle_{Vx}$
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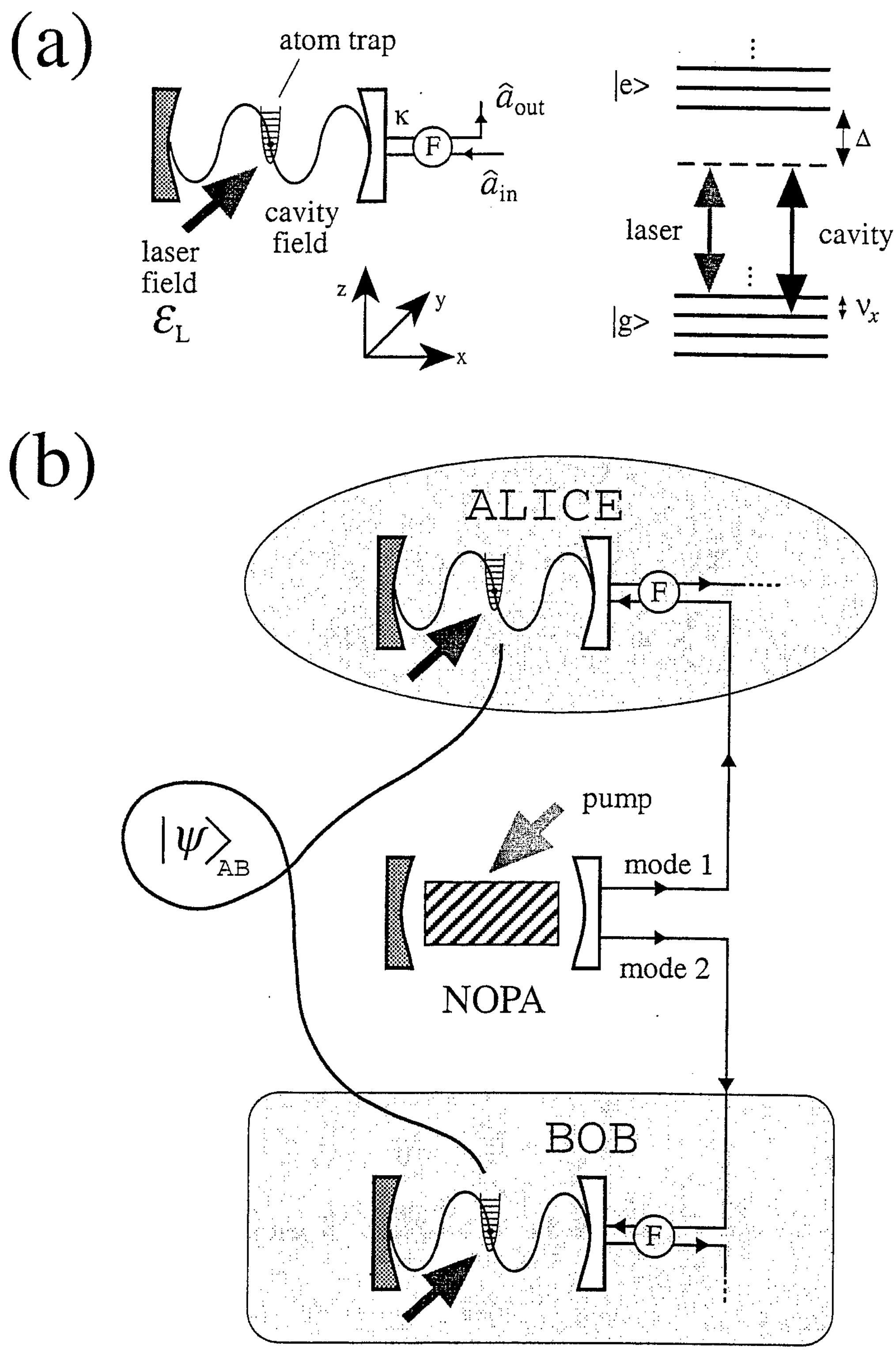
Figure 3


(a)
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mode 1
mode 2
NOPA
BOB
F

Figure 2

Teleporting an Atomic Wavepacket

Scott Parkins[1] and Jeff Kimble[2]

[1] University of Auckland, Physics Department, Private Bag 92019, Auckland, New Zealand

[2] California Institute of Technology, Norman Bridge Laboratory of Physics 12-33, Pasadena, CA 91125, USA

Abstract. We outline a scheme for teleporting an unknown atomic center-of-mass wave function between distant locations. This scheme employs interactions in cavity quantum electrodynamics to facilitate a coupling between the motional degrees of freedom of an atom trapped inside a cavity and external propagating light fields. This enables the distribution of quantum entanglement and the realization of the required motional Bell-state analysis.

1 Introduction

Teleportation of unknown quantum states from one location to another is now an experimental science, with initial attempts directed to the teleportation of the polarization state of a photon [1,2] and then the first complete realization performed with optical coherent states [3]. The former experiments were based on the original procedure for finite-dimensional quantum systems (in particular, two-state systems) proposed by Bennett and coworkers [4], and the latter on a continuous variable (i.e., infinite dimensional) adaptation of this procedure put forward by Vaidman [5] (see also [6,7]). In both instances, though, the essential ingredient in the teleportation procedure is quantum entanglement of a bipartite system shared by the sender, Alice, and the receiver, Bob. For experiments with light, this entanglement can be generated via the process of parametric amplification (see, e.g., [8]).

While the experiments performed thus far have dealt exclusively with light, the procedure of Vaidman offers, in principle, the possibility of teleporting the quantum states of massive particles. The required quantum entanglement in this procedure takes the form of the original Einstein-Podolsky-Rosen (EPR) state of two particles; that is, a state in which the positions and momenta of the particles (labelled 2 and 3) are perfectly anticorrelated and correlated, respectively:

$$x_2 + x_3 = 0\,, \quad p_2 - p_3 = 0\,. \tag{1}$$

Joint measurements performed on particle 1, whose state is to be teleported, and particle 2 then establish a correlation between particles 1 and 3. In particular, measurements of the quantities $x_1 + x_2$ and $p_1 - p_2$, with outcomes

$$x_1 + x_2 = a\,, \quad p_1 - p_2 = b\,, \tag{2}$$

establish the relationship

$$x_3 = x_1 - a\,, \quad p_3 = p_1 - b\,, \tag{3}$$

so that, if the initial state of particle 1 is $\Psi(x_1)$, then the state of particle 3 after the measurements is $e^{ibx_3}\Psi(x_3 + a)$. Appropriate shifts of the state in x_3 and p_3 complete the teleportation.

For the teleportation of optical coherent states using this procedure the relevant variables are the quadrature amplitudes of the electromagnetic fields, the entangled state (1) is generated via parametric amplification, and the joint measurements are performed by combining fields at a beamsplitter and using homodyne detection. In the field of quantum optics, these techniques are now quite standard. In contrast, for the teleportation of the wave function of a massive particle one would seem to be faced with serious difficulties in producing the required states and operations.

However, in recent work we have proposed and analyzed a cavity-QED-based system that enables the transfer of quantum states between the external, or motional, degrees of freedom of a trapped atom and propagating light fields [9,10]. This system should lead to new capabilities for the synthesis, control, and measurement of quantum states for both motion and light. A particular example is the possibility of creating an EPR state of the form (1) where (x_2, x_3) and (p_2, p_3) are indeed the positions and momenta of *distantly separated atoms*. This in turn offers a route to achieving the teleportation of an atomic wavepacket [11], which we summarize in this article.

An overview of our proposed teleportation scheme is given in Fig. 1. Each of Alice ($\mathcal{A}$), Bob ($\mathcal{B}$), and Victor ($\mathcal{V}$) possess an atom trapped inside an optical cavity, with the aim being to teleport the (x-dimension) motional state of Victor's atom to Bob's atom. Again, the three basic steps involved are: (i) preparation of quantum entanglement between Alice and Bob's atoms, (ii) joint (or "Bell-state") measurement by Alice on her atom and Victor's atom, and (iii) phase-space displacement $[D(\alpha^*)]$ by Bob (given the classical result α of Alice's measurement). Finally, the state of Bob's atom may be examined by Victor for verification of the quality of the teleportation. Stages (i) and (ii), as illustrated in Fig. 1, employ the cavity-mediated motion-light state transfer scheme of [9], to which we now turn our attention.

2 Motion-light couplng

Briefly, a single two-level atom (or ion) is tightly confined in a harmonic trap located inside a high-finesse optical cavity. The atomic transition of frequency ω_a is coupled to a single mode of the cavity field of frequency ω_c and also to an external (classical) laser field of frequency ω_L and strength $\mathcal{E}_L$. The physical setup and excitation scheme are depicted in Fig. 2(a). The cavity is aligned along the x-axis, while the field $\mathcal{E}_L$ is incident from a direction in the y-z plane. Both the cavity field and $\mathcal{E}_L$ are far from resonance with the atomic

Fig. 1. Schematic of proposed teleportation scheme for atomic wavepackets

transition, but their difference frequency is chosen so that they drive Raman transitions between neighboring motional number states (i.e., $\omega_c - \omega_L = \nu_x$, with ν_x the x-axis trap frequency).

A number of assumptions are made in order to achieve the desired motion-light coupling:

- Atomic spontaneous emission is neglected and the internal atomic dynamics adiabatically eliminated.
- The size of the harmonic trap, located at a *node* of the cavity field, is taken to be small compared to the optical wavelength (Lamb-Dicke regime), enabling the approximations

$$\sin(k\hat{x}) \simeq \eta_x(\hat{b}_x + \hat{b}_x^\dagger), \qquad \mathcal{E}_L(\hat{y}, \hat{z}, t) \simeq \mathcal{E}_L(t)e^{-i\phi_L}, \tag{4}$$

  where η_x ($\ll 1$) is the Lamb-Dicke parameter and

$$\hat{x} = (\hbar/2m\nu_x)^{1/2}(\hat{b}_x + \hat{b}_x^\dagger). \tag{5}$$

- The trap frequency ν_x and cavity field decay rate κ are assumed to satisfy $\nu_x \gg \kappa \gg |(g_0\eta_x/\Delta)\mathcal{E}_L(t)|$, where g_0 is the single-photon atom-cavity mode coupling strength, and $\Delta = \omega_a - \omega_L$. The first inequality allows a rotating-wave approximation to be made with respect to the trap oscillation frequency, while the second inequality enables an adiabatic elimination of the cavity field mode.

Under these conditions, the motional mode dynamics in the x direction is well described by the simple quantum Langevin equation [9]

$$\dot{\tilde{b}}_x \simeq -\Gamma(t)\tilde{b}_x + \sqrt{2\Gamma(t)}\,\tilde{a}_{in}(t)\ , \tag{6}$$

where $\tilde{b}_x = e^{i\nu_x t}\hat{b}_x$ and $\Gamma(t) = [g_0\eta_x|\mathcal{E}_L(t)|/\Delta]^2/\kappa$. The operator $\tilde{a}_{in}(t)$ obeys the commutation relation $[\tilde{a}_{in}(t), \tilde{a}_{in}^\dagger(t')] = \delta(t-t')$ and describes the quantum noise *input to the cavity field* (in a frame rotating at the cavity frequency).

From the linear nature of (6), it follows that the statistics of a (continuous) light field incident upon the cavity can be "written onto" the state of the atomic motion. For example, in [9] it is shown how this effect can be used to prepare a (pure) squeezed state of the motion of a trapped atom by driving the cavity with broadband squeezed light from a degenerate parametric amplifier. This also means that entanglement between separate light fields can be transferred to entanglement between the motional states of separate atoms, as we discuss below. From a consideration of the input-output theory of optical cavities (see, for example, [12]), it also follows that measurements on the cavity output field amount to measurements on the motion of the atom. In particular, one can show that

$$\tilde{a}_{\rm out}(t) \simeq -\tilde{a}_{\rm in}(t) + \sqrt{2\Gamma(t)}\,\tilde{b}_x(t)\,. \tag{7}$$

So, for a vacuum input field, homodyne measurements on the cavity output field realize position or momentum measurements (or some mixture, depending on the local oscillator phase) on the trapped atom. This enables the necessary Bell-state analysis to be performed.

Fig. 2. (a) Proposed setup and excitation scheme for coupling between the motion of a trapped atom and a quantized optical cavity mode, and thence to a freely propagating external field. The cavity is assumed to be *one-sided*, i.e., one mirror is perfectly reflecting. (b) Preparation of a position-momentum EPR state of Alice and Bob's atoms. Faraday isolators (F) facilitate a unidirectinal coupling between the entangled light source and the atom-cavity systems

3 Teleportation Procedure

3.1 State to be teleported

To begin the teleportation procedure, Victor's atom is prepared in a particular motional state $|\phi\rangle_{Vx}$ in the x dimension (e.g., by the techniques of [13]).

With the motion-light coupling switched off in Victor's cavity (i.e., $\mathcal{E}_{L\mathcal{V}} = 0$), this state is assumed to remain unchanged until required by Alice for the Bell-state analysis.

3.2 Preparation of position-momentum EPR state

Next, a position-momentum EPR state of Alice and Bob's atoms is prepared by using the motion-light coupling described in (6), with input light fields from a nondegenerate optical parametric amplifier (NOPA). This preparation, described in detail in [10], is depicted in Fig. 2(b). The two quantum-correlated output light fields from a NOPA (operating below threshold) are separated and made to impinge on Alice and Bob's cavities, respectively. Assuming $\Gamma_{\mathcal{A}} = \Gamma_{\mathcal{B}} = \Gamma$, after a time $t \gg \Gamma^{-1}$, the following *pure* entangled motional state is prepared,

$$|\psi\rangle_{\mathcal{AB}} = [\cosh(r)]^{-1} \sum_{m=0}^{\infty} [-\tanh(r)]^m \; |m\rangle_{\mathcal{A}x}|m\rangle_{\mathcal{B}x} \,, \tag{8}$$

where $|m\rangle_{\mathcal{A},\mathcal{B}x}$ are Fock states of the motional modes and r is the 'entanglement' parameter. The nature of the correlations inherent in the joint state (8) is most clearly expressed through the Wigner function for this state:

$$\begin{aligned} W(x_{\mathcal{A}}, p_{\mathcal{A}}; x_{\mathcal{B}}, p_{\mathcal{B}}) \\ &= \frac{4}{\pi^2} \exp\left\{- \left[(x_{\mathcal{A}} + x_{\mathcal{B}})^2 + (p_{\mathcal{A}} - p_{\mathcal{B}})^2\right] e^{+2r}\right\} \\ &\quad \times \exp\left\{- \left[(x_{\mathcal{A}} - x_{\mathcal{B}})^2 + (p_{\mathcal{A}} + p_{\mathcal{B}})^2\right] e^{-2r}\right\} && (9) \\ &\to C\, \delta(x_{\mathcal{A}} + x_{\mathcal{B}})\, \delta(p_{\mathcal{A}} - p_{\mathcal{B}}) \quad \text{as } r \to \infty \,, && (10) \end{aligned}$$

with C a constant. Once this state has been prepared, the atom-cavity couplings are turned off ($\Gamma_{\mathcal{A}}, \Gamma_{\mathcal{B}} \to 0$), as is the NOPA pump field. Again, we assume that the entangled state (8) remains unchanged until the next step in the procedure.

3.3 Bell-state analysis

At this stage in the procedure, the total system state is

$$|\Psi_{\mathrm{I}}\rangle = |\phi\rangle_{\mathcal{V}x}\, |\psi\rangle_{\mathcal{AB}} \,. \tag{11}$$

The Bell-state analysis performed by Alice is depicted in Fig. 3. At a predetermined time, Victor switches on his atom-cavity coupling $\Gamma_{\mathcal{V}}$ via $\mathcal{E}_{L\mathcal{V}}(t)$, thus converting the state $|\phi\rangle_{\mathcal{V}x}$ to that of a freely propagating field delivered to input beam-splitter BS of Alice's sending station. With due accounting for propagation delay, Alice has likewise switched on the coupling $\Gamma_{\mathcal{A}}$ from her cavity, where, for simplicity, $\Gamma_{\mathcal{V}} = \Gamma_{\mathcal{A}} = \Gamma$. Note that Victor and Alice's cavities both have vacuum inputs at this stage. The two cavity output fields are combined by Alice at the 50/50 beamsplitter BS, the two outputs of which are incident on homodyne detectors $D_\pm$. Through the input-output relation (7), and through the mixing of the cavity output fields at the beamsplitter, these detectors effect homodyne measurements on the modes $\tilde{c}_\pm = 2^{-1/2}(\tilde{b}_{\mathcal{V}x} \pm \tilde{b}_{\mathcal{A}x})$. In particular, with suitable choices of the local oscillator phases, the setup realizes measurements of the variables $x_\mathcal{A} + x_\mathcal{V}$ and $p_\mathcal{A} - p_\mathcal{V}$, as required by the teleportation protocol.

Mathematically, the effect of these measurements is to project the system state onto quadrature eigenstates of the modes $\tilde{c}_\pm$, and the relevant measurement results, which we denote by $\chi_\pm$, are given by the integrated homodyne photocurrents. Note that this projection requires that the local oscillator photon flux matches the temporal shape of the signal flux [which in our case is set by $\Gamma(t)$] [14,15].

Expanding the motional state of Victor's atom, $|\phi\rangle_{\mathcal{V}x}$, in terms of the coherent states, i.e.,

$$|\phi\rangle_{\mathcal{V}x} = \frac{1}{\pi}\int d^2\beta \; {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,|\beta\rangle_{\mathcal{V}x}, \tag{12}$$

one can show that the motional state of Bob's atom following the Bell-state measurements, $|\varphi\rangle_{\mathcal{B}x}$, is proportional to

$$\frac{1}{\pi}\int d^2\beta \; {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,\frac{{}_{\mathcal{V}x}\langle\alpha^*|\beta\rangle_{\mathcal{V}x}}{{}_{\mathcal{V}x}\langle\Lambda\alpha^*|\Lambda\beta\rangle_{\mathcal{V}x}}\;D_\mathcal{B}(-\Lambda\alpha^*)D_\mathcal{B}(\Lambda\beta)\,|0\rangle_{\mathcal{B}x}\,, \tag{13}$$

where $\alpha = (\chi_+ + i\chi_-)/\sqrt{2}$, $\Lambda = \tanh(r)$, and

$$D_\mathcal{B}(\beta) = \exp(\beta\tilde{b}^\dagger_{\mathcal{B}x} - \beta^*\tilde{b}_{\mathcal{B}x}) \tag{14}$$

is the coherent displacement operator for Bob's atom. In the limit of strong squeezing and entanglement ($\Lambda \to 1$), (13) approaches

$$D_\mathcal{B}(-\alpha^*)\,\frac{1}{\pi}\int d^2\beta \; {}_{\mathcal{V}x}\langle\beta|\phi\rangle_{\mathcal{V}x}\,|\beta\rangle_{\mathcal{B}x}\,. \tag{15}$$

That is, $|\varphi\rangle_{\mathcal{B}x}$ approaches a state which, apart from a coherent displacement by $-\alpha^*$, is identical to the initial motional state (in the x dimension) of Victor's atom.

3.4 Phase-space displacement

Given the measurement results $\chi_\pm$, transmitted to Bob via a *classical* channel, the final step in the teleportation procedure is for Bob to apply a coherent displacement α^* (assuming $\Lambda \simeq 1$) to the motional state of his atom, i.e.,

$$D_\mathcal{B}(\alpha^*)|\varphi\rangle_{\mathcal{B}x} \to |\phi\rangle_{\mathcal{B}x}. \tag{16}$$

In practice, this might be achieved by applying an electric field (in the case of a trapped ion) along the x-axis which oscillates at the trap frequency ν_x,

Fig. 3. Schematic of Alice's Bell-state analysis. The output field representing Victor's unknown state is combined by Alice at a 50/50 beamsplitter (BS) with the output field from her cavity. The resulting output fields from the BS are incident on homodyne detectors $D_{\pm}$. The cavity output fields follow the motional modes, which decay on a timescale Γ^{-1}. The local oscillator fields ($LO_{\pm}$) are pulsed, with temporal profiles chosen to match that of the cavity output fields

or, alternatively, by applying off-resonant laser fields which drive stimulated Raman transitions between neighboring trap levels [13]. After this, control of Bob's atom can be passed to Victor, who is free to confirm the overall quality of the teleportation protocol, e.g., along the lines analyzed in [16].

4 Discussion

To realize our teleportation scheme, required conditions are of (i) strong coupling optical cavity QED, such that $g_0^2/(\kappa\gamma) \gg 1$, where γ is the atomic spontaneous decay rate, (ii) strong confinement of the atoms with minimal motional-state decoherence, and (iii) strongly entangled light fields [9–11]. Each of these conditions have been achieved separately (for cavity QED see, e.g., [17–19]; for trapped ions see, e.g., [13,20]; for entangled light fields, see, e.g., [3]), while the first generation of experiments actually trapping atoms in cavity QED have been performed [21].

Much of the interest in quantum teleportation is rooted in its intimate connection to issues in the fields of quantum information and quantum computation. There, indeed, nonclassical light fields and trapped atoms and ions are leading candidates for the implementation of, for example, elementary quantum computers [22,13,23,24], and quantum dense coding [25] and universal quantum computation [26] with continuous quantum variables. The present work points to the exciting possibility of combining these candidate technologies in distributed quantum networks, where, for example, the "distribution" is achieved with light fields, while "storage" and local processing is achieved with trapped atoms.

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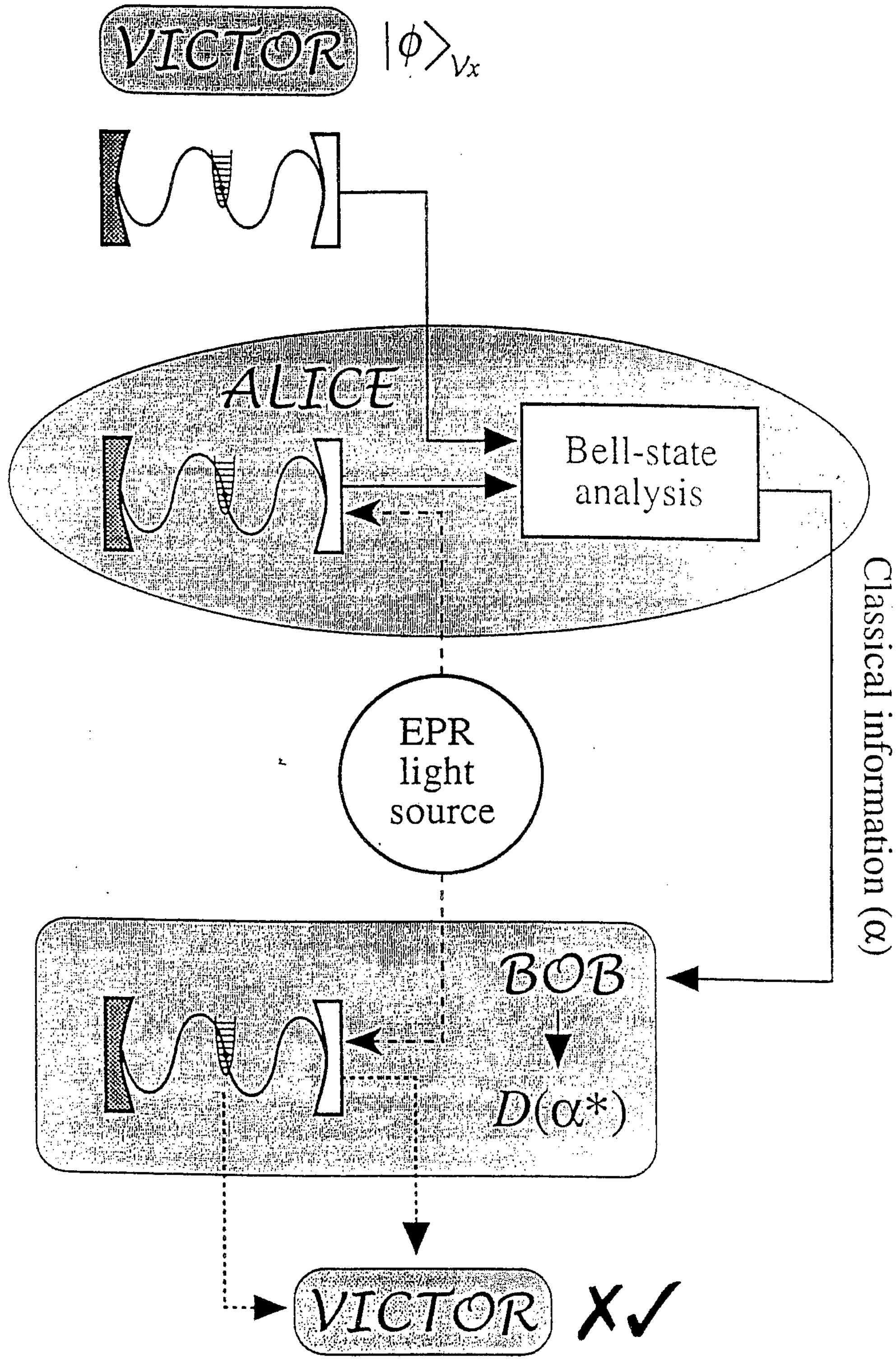

Figure 1, Parkins & Kimble

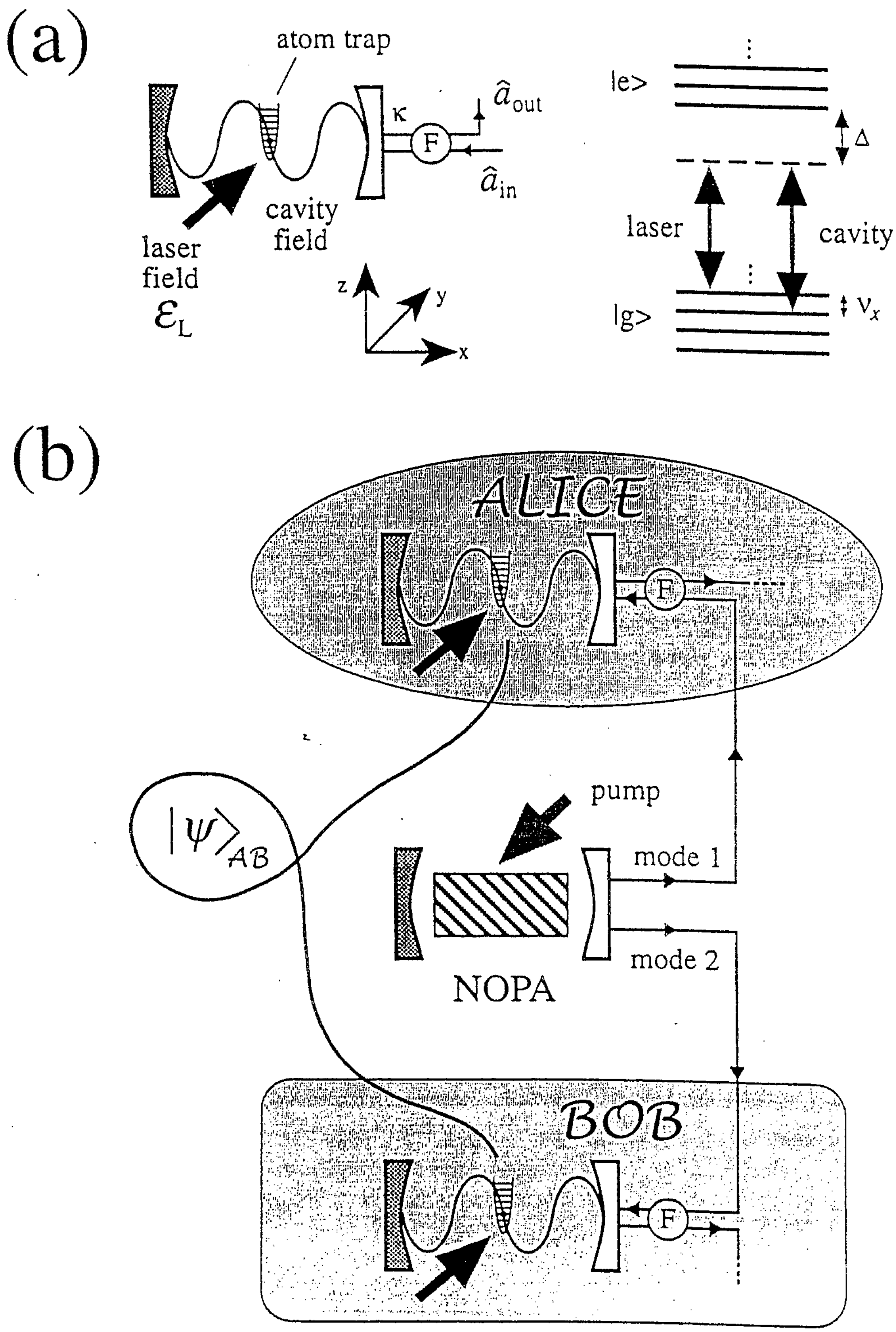

Figure 2, Parkins & Kimble

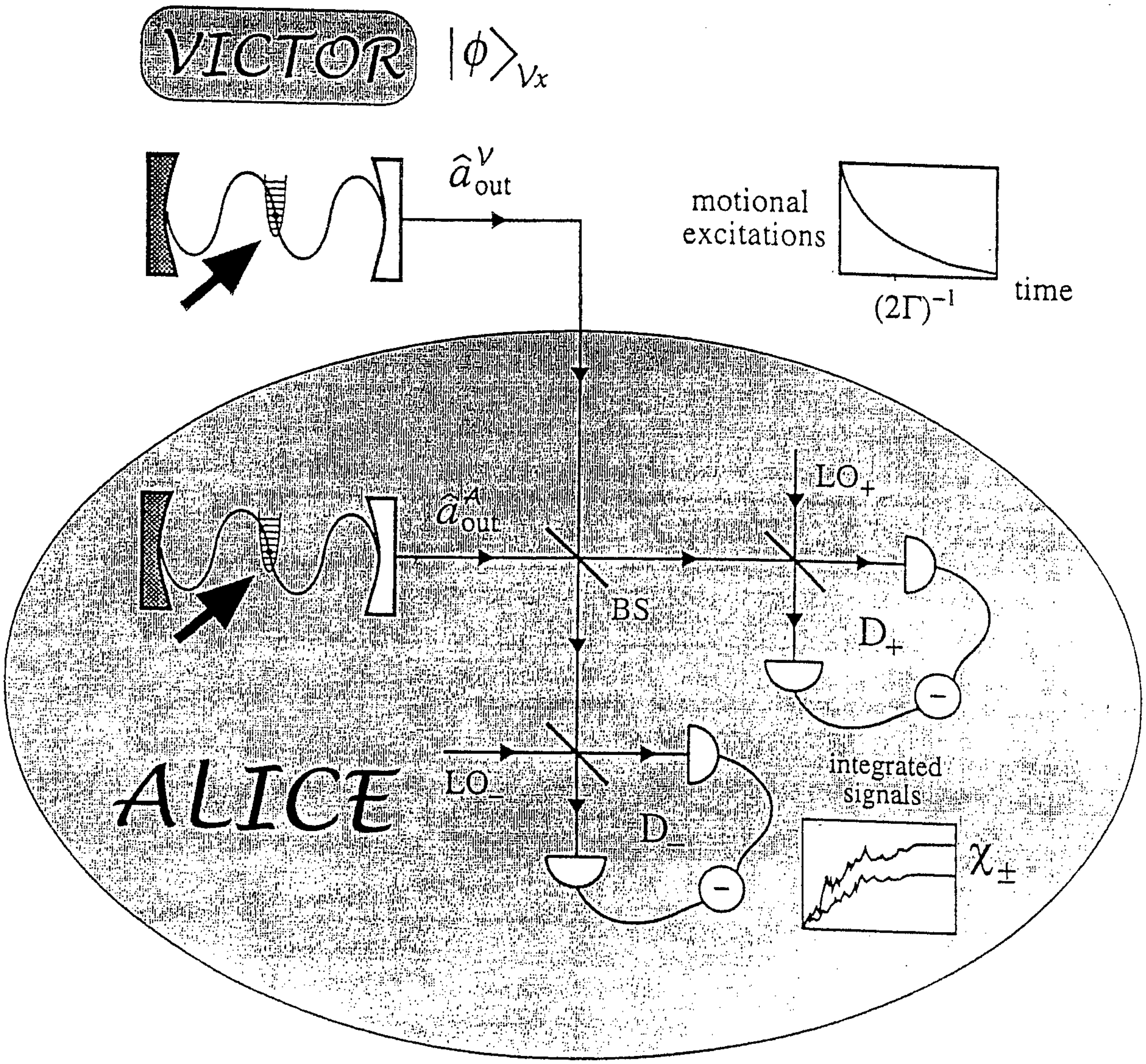

Figure 3, Parkins & Kimble

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Criteria for continuous-variable quantum teleportation

SAMUEL L. BRAUNSTEIN‡, CHRISTOPHER A. FUCHS†, and H. J. KIMBLE†

†Norman Bridge Laboratory of Physics, 12–33 California Institute of Technology, Pasadena, CA 91125, USA
‡SEECS, University of Wales, Bangor LL57 1UT, UK
‡Hewlett-Packard Labs, Math Group M48, Bristol BS34 8QZ, UK

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Abstract. We derive an experimentally testable criterion for the teleportation of quantum states of continuous variables. This criterion is especially relevant to the recent experiment of Furusawa *et al.* where an input–output fidelity of 0.58 ± 0.02 was achieved for optical coherent stages. Our derivation demonstrates that fidelities greater than 1/2 could not have been achieved through the use of a classical channel alone; quantum entanglement was a crucial ingredient in the experiment.

1. Introduction

What is quantum teleportation? The original protocol of Bennett *et al.* [1] specifies the idea with succinct clarity. The task set before Alice and Bob is to transfer the quantum state of a system in one player's hands onto a system in the other's. The agreed upon resources for carrying out this task are some previously shared quantum entanglement and a channel capable of broadcasting classical information. It is not allowed physically to carry the system from one player to the other, and indeed the two players need not even know each other's locations. One of the most important features of the protocol is that it must be able to work even when the state—though perfectly well known to its supplier, a third party Victor—is completely *unknown* to both Alice and Bob. Because the classical information broadcast over the classical channel can be minuscule in comparison to the infinite amount of information required to specify the unknown state, it is fair to say that the state's transport is a disembodied transport [2]. Teleportation has occurred when an unknown state $|\psi\rangle$ goes in and the same state $|\psi\rangle$ comes out.

But that is perfect teleportation. Recent experimental efforts [3–6] show there is huge interest in demonstrating the phenomenon in the laboratory—a venue where perfection is *unattainable* as a matter of principle. The laboratory brings with it a new host of issues: if perfect teleportation is unattainable, when can one say that laboratory teleportation has been achieved? What appropriate criteria define the right to proclaim success in an experimental setting? Searching through the description above, there are several heuristic breaking points, each asking for quantitative treatment. The most important among these are:

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(1) The states should be unknown to Alice and Bob and supplied by an actual third party Victor.
(2) Entanglement should be a verifiably used resource, with the possibility of physical transportation of the unknown states blocked at the outset. There should be a sense in which the output is 'close' to the input—close enough that it could not have been made from information sent through a classical channel alone.
(3) Each and every trial, as defined by Victor's supplying a state, should achieve an output sufficiently close to the input. When this situation pertains, the teleportation is called *unconditional.* (If that is impractical, *conditional* teleportation—where Alice and Bob are the arbiters of success—may still be of interest; but then, at the end of all conditioning, there must be a state at the output sufficiently close to the unknown input.)

To date only the Furusawa *et al.* experiment [3] has achieved unconditional experimental teleportation as defined by these three criteria. The Boschi *et al.* experiment [4] fails to meet Criteria 1 and 2 because their Victor must hand off a (macroscopic) state-preparing device to Alice instead of an unknown state and because of a variety of low system efficiencies [3]. The Bouwmeester *et al.* experiment [5] fails to meet Criteria 2 and 3 because their output states—just before they are destroyed by an extra 'verification' step—can be produced via communication through a classical channel alone [7]. In a similar vein, the Nielsen *et al.* experiment [6] fails to meet these criteria because there is no quantum entanglement shared between Alice and Bob at any stage of the process [8, 9].

But the story cannot stop here. Besides striving for simply better input–output fidelities or higher efficiencies, there are still further relevant experimental hurdles to be drawn from [1]:

(4) The number of bits broadcast over the classical channel should be 'minuscule' in comparison to the information required to specify the 'unknown' states in the class from which the demonstration actually draws.
(5) The teleportation quality should be good enough to transfer quantum entanglement itself instead of a small subset of 'unknown' quantum states.
(6) The sender and receiver should not have to know each other's locations to carry the process through to completion.

And there are likely still more criteria that would seem reasonable to one or another reader of the original protocol (depending perhaps upon the particular application called upon). The point is, these two lists together make it clear that the experimental demonstration of quantum teleportation cannot be a cut and dried affair. On the road toward ideal teleportation, there are significant milestones to be met and passed. Important steps have been taken, but the end of the road is still far from sight.

The work of the theorist in this effort is, among other things, to help turn the heuristic criteria above into pristine theoretical protocols within the context of actual experiments. To this end, we focus on Criterion 2 in the context of the Furusawa *et al.* experiment [3] where the quantum states of a set of continuous variables are teleported (as proposed in [10, 11]). The question is, by what means can one verify that Alice and Bob—assumed to be at fixed positions—actually use some quantum entanglement in their purported teleportation? How can it be known that they did not use the resource of a classical channel alone for the quantum state's transport? What milestone must be met in order to see this? Answering these questions fulfils a result already advertised in [3] and reported in the abstract of the present paper.

Our line of attack is to elaborate on an idea first suggested in [4]. A cheating Alice and Bob who attempt to make do with a classical channel alone, must gather information about the unknown quantum state if they are to have any hope of hiding their cheat. But then the limitations of quantum mechanics strike in a useful way. As long as the allowed set of inputs contains some nonorthogonal states, there is no measurement procedure that can reveal the state's identity with complete reliability. Any attempt to reconstruct the unknown quantum state will be necessarily flawed: information gathering about the identity of a state in a nonorthogonal set disturbs the state in the process [12, 13]. The issue is only to quantify how much disturbance must take place and to implement the actual comparison between input and output in an objective, operationally significant way. If the experimental match (or 'fidelity') between the input and output exceeds the bound set by a classical channel, then some entanglement had to have been used in the teleportation process.

The remainder of the paper is structured as follows. In the following section, we discuss the motivation behind choosing the given measure of fidelity that we do. We stress in particular the need for a break with traditional quantum optical measures of signal transmission, such as signal-to-noise ratio, etc., used in the area of quantum nondemolition (QND) research [14, 15]. In section 3, we derive the optimal fidelity that can be achieved by a cheating Alice and Bob whose teleportation measurements are based on optical heterodyning as in the experiment of Furusawa *et al.* [3]. This confirms that a fidelity of 1/2 or greater is sufficient to assure the satisfaction of Criterion 2 in that experiment. We close in section 4 with a few remarks about some open problems and future directions.

2. Why fidelity?

Ideal teleportation occurs when an unknown state $|\psi\rangle$ goes into Alice's possession and the same state $|\psi\rangle$ emerges in Bob's. What can this really mean? A quantum *state* is not an objective state of affairs existing completely independently of what one knows. Instead it captures the best information available about how a quantum system will react in this or that experimental situation [16, 17]†. This forces one to think carefully about what it is that is transported in the quantum teleportation process. The only option is that the teleported $|\psi\rangle$ must

† On this bit of foundational theory, it seems most experimentalists can agree. See in particular p. S291 of Zeilinger [16] were it is stated that: 'The quantum state is exactly that representation of our knowledge of the complete situation which enables the maximal set of (probabilistic) predictions for any possible future observation.... If we accept that the quantum state is no more than a representation of the information we have, then the spontaneous change of the state upon observation, the so-called collapse or reduction of the wave packet, is just a very natural consequence of the fact that, upon observation, our information changes and therefore we have to change our representation of the information, that is, the quantum state. From that position, the so-called measurement problem is not a problem but a consequence of the more fundamental role information plays in quantum physics as compared to classical physics.'

always ultimately refer to someone lurking in the background—a third party we label Victor, the keeper of knowledge about the system's preparation. The task of teleportation is to transfer what he can say about the system he placed in Alice's possession onto a system in Bob's possession: it is 'information' in its purest form that is teleported, nothing more.

The resources specified for carrying out this task are the previously shared entanglement between Alice and Bob and a classical channel with which they communicate. Alice performs a measurement of a specific character and communicates her result to Bob. Bob then performs a unitary operation on his system based upon that information. When Alice and Bob declare that the process is complete, Victor should know with assurance that whatever his description of the original system was—his $|\psi\rangle$—it now holds for the system in Bob's possession. Knowing with assurance means that there really is a system that Victor will describe with $|\psi\rangle$, not that there *was* a system that he *would have* described with $|\psi\rangle$ just before Alice and Bob declared completion (i.e. as a retrodiction based upon their pronouncement) [7].

In any real-world implementation of teleportation, a state $|\psi_{\rm in}\rangle$ enters Alice and Bob's dominion and a different state (possibly a mixed-state density operator) $\hat{\rho}_{\rm out}$ comes out. As before, one must always keep in mind that these states refer to what Victor can say about the given system (see †). The question that must be addressed is when $|\psi_{\rm in}\rangle$ and $\hat{\rho}_{\rm out}$ are similar enough to each other that Criterion 2 must have been fulfilled.

We choose to gauge the similarity between $|\psi_{\rm in}\rangle$ and $\hat{\rho}_{\rm out}$ by the 'fidelity' between the two states. This is defined in the following way‡:

$$F(|\psi_{\rm in}\rangle,\ \hat{\rho}_{\rm out}) \equiv \langle\psi_{\rm in}|\hat{\rho}_{\rm out}|\psi_{\rm in}\rangle. \tag{1}$$

This measure has the nice property that it equals 1 if and only if $\hat{\rho}_{\rm out} = |\psi_{\rm in}\rangle\langle\psi_{\rm in}|$. Moreover it equals 0 if and only if the input and output states can be distinguished with certainty by *some* quantum measurement. The thing that is really important about this particular measure of similarity is hinted at by these last two properties. It captures in a simple and convenient package the extent to which *all possible* measurement statistics producible by the output state match the corresponding statistics producible by the input state.

To see what this means, take *any* observable (generally a positive operator-valued measure or POVM [17]) $\{\hat{E}_\alpha\}$ with measurement outcomes α. If that observable were performed on the input system, it would give a probability density for the outcomes α given by

‡In order to form this quantity, we must of course assume a canonical mapping or identification between the input and output Hilbert spaces. Any unitary offset between input and output should be considered a systematic error, and ultimately taken into account by readjusting the canonical mapping. See [14, 15] for a misunderstanding of this point. The authors there state, '... fidelity does not necessarily recognize the similarity of states which differ only by reversible transformations....[This suggests] that additional measures are required... based specifically on the similarity of measurement results obtained from the input and output of the teleporter, rather than the inferred similarity of the input and output states'. As shown presently, the fidelity measure we propose does precisely that for all possible measurements, not just the few that have become the focus of present-day QND research.

$$P_{\text{in}}(\alpha) = \langle\psi_{\text{in}}|\hat{E}_\alpha|\psi_{\text{in}}\rangle. \tag{2}$$

On the other hand, if the same observable were performed on the output system, it would give instead a probability density

$$P_{\text{out}}(\alpha) = \text{tr}\,(\hat{\rho}_{\text{out}}\hat{E}_\alpha). \tag{3}$$

A natural way to gauge the similarity of these two probability densities is by their overlap:

$$\text{overlap} = \int [P_{\text{in}}(\alpha)\,P_{\text{out}}(\alpha)]^{1/2}\,\mathrm{d}\alpha. \tag{4}$$

It turns out that regardless of which observable is being considered [18, 19],

$$\text{overlap}^2 \geq \langle\psi_{\text{in}}|\hat{\rho}_{\text{out}}|\psi_{\text{in}}\rangle. \tag{5}$$

Moreover there exists an observable that gives precise equality in this expression [18, 19]. In this sense, the fidelity captures an operationally defined fact about all possible measurements on the states in question.

Let us take a moment to stress the importance of a criterion such as this. It is not sufficient to attempt to quantify the similarity of the states with respect to a few observables. Quantum teleportation is a much more serious task than classical communication. Indeed it is a much more serious task than the simplest forms of quantum communication, as in quantum key distribution. In the former case, one is usually concerned with replicating the statistics of only one observable across a transmission line. In the latter case, one is concerned with reproducing the statistics of a small number of *fixed* noncommuting observables (the specific ones required of the protocol) for a small number of *fixed* quantum states (the specific ones required of the protocol). A full quantum state is so much more than the quantum measurements in these cases would reveal: it is a catalogue for the outcome statistics of an infinite number of observables. Good quality teleportation must take that into account.

A concrete example can be drawn from the traditional concerns of quantum nondemolition measurement (QND) research. There a typical problem is how well a communication channel replicates the statistics of one of two quadratures of a given electromagnetic field mode [14, 15], and most often then only for assumed Gaussian statistics. Thinking that quantum teleportation is a simple generalization of the preservation of signal-to-noise ratio, burdened only in checking that both quadratures are transmitted faithfully, is to miss much of the point of teleportation. Specifying the statistics of two noncommuting observables only goes an infinitesimal way toward specifying the full quantum state when the Hilbert space is an infinite dimensional one [20, 21].

This situation is made acute by noticing that two state vectors can be almost completely orthogonal—and therefore almost as different as they can possibly be—while still giving rise to the same x statistics *and* the same p statistics. To see an easy example of this, consider the two state vectors $|\psi_+\rangle$ and $|\psi_-\rangle$ whose representations in x-space are

$$\psi_\pm(x) = \left(\frac{2a}{\pi}\right)^{1/4} \exp\left((-a \pm ib)x^2\right), \tag{6}$$

for $a, b \geq 0$. In k-space representation, these state vectors look like

$$\tilde{\psi}_{\pm}(k) = \left(\frac{a}{2\pi}\right)^{1/4}\left(\frac{a \pm ib}{a^2+b^2}\right)^{1/2} \exp\left(\frac{-a \mp ib}{4(a^2+b^2)}k^2\right). \tag{7}$$

Clearly neither x measurements nor p measurements can distinguish these two states. For, with respect to both representations, both wave functions differ only by a local phase function. However, if we look at the overlap between the two states we find:

$$\langle\psi_-|\psi_+\rangle = \left(\frac{a(a+ib)}{a^2+b^2}\right)^{1/2}. \tag{8}$$

Taking $b \to \infty$, we can make these two states just as orthogonal as we please.

Suppose now that $|\psi_+\rangle$ were Victor's input into the teleportation process, and—by whatever means—$|\psi_-\rangle$ turned out to be the output. By a criterion that only gauged the faithfulness of the transmissions of x and p [14], this would be perfect teleportation. But it certainly is not so!

Thus the justification of the fidelity measure in Equation (1) as a measure of teleportation quality should be abundantly clear. But this is only the first step in finding a way to test criterion 2. For this, we must invent a quantity that incorporates information about the teleportation quality of many possible quantum states. The reason for this is evident: in general it is possible to achieve a nonzero fidelity between input and output even when a cheating Alice and Bob use no entanglement whatsoever in their purported teleportation. This can come about whenever Alice and Bob can make use of some prior knowledge about Victor's actions.

As an example, consider the case where Alice and Bob are privy to the fact that Victor wishes only to teleport states drawn from a given *orthogonal* set. At any shot, they know they will be given one of these states, just not which one. Then, clearly, they need use no entanglement to 'transmit' the quantum states from one position to the other. A cheating Alice need only perform a measurement $\mathcal{O}$ whose eigenstates coincide with the orthogonal set and send the outcome she obtains to Bob. Bob can use that information to resynthesize the appropriate state at his end. No entanglement has been used, and yet with respect to these states perfect teleportation has occurred.

This example helps define the issue much more sharply. The issue turns on having a general statement of what it means to say that Alice and Bob are given an *unknown* quantum state? In the most general setting it means that Alice and Bob know that Victor draws his states $|\psi_{\rm in}\rangle$ from a fixed set S; they just know not which one he will draw at any shot. This lack of knowledge is taken into account by a probability ascription $P(|\psi_{\rm in}\rangle)$. That is:

All useful criteria for the achievement of teleportation must be anchored in whatever S and $P(|\psi_{\rm in}\rangle)$ are given. A criterion is senseless if the states to which it is to be applied are not mentioned explicitly.

This makes it sensible to consider the average fidelity between input and output

$$F_{\rm av} = \int_S P(|\psi_{\rm in}\rangle) F(|\psi_{\rm in}\rangle, \hat{\rho}_{\rm out})\, {\rm d}|\psi_{\rm in}\rangle, \tag{9}$$

as a benchmark capable of eliciting the degree to which Criterion 2 is satisfied. If S consists of orthogonal states, then no criterion whatsoever (short of watching Alice and Bob's every move) will ever be able to draw a distinction between true teleportation and the sole use of the classical side channel. Things only become interesting when the set $\mathcal{S}$ consists of two or more *nonorthogonal* quantum states [4]: for only then will $F_{\rm av} = 1$ *never* be achievable by a cheating Alice and Bob.

By making the set $\mathcal{S}$ more and more complicated, we can define ever more stringent tests connected to Criterion 2. For instance, consider the simplest non-trivial case: take $\mathcal{S} = \mathcal{S}_0 = \{|\psi_0\rangle, |\psi_1\rangle\}$, a set of just two nonorthogonal states (with a real inner product $x = \cos\theta$). Suppose the two states occur with equal probability. Then it can be shown [12] that the best thing for a cheating Alice and Bob to do is this. Alice measures an operator whose orthogonal eigenvectors symmetrically bestride $|\psi_0\rangle$ and $|\psi_1\rangle$. Using that information, Bob synthesizes one of two states $|\tilde{\psi}_0\rangle$ and $|\tilde{\psi}_1\rangle$ each lying in the same plane as the original two states, but each tweaked slightly toward the other by an angle [22]

$$\phi = \tfrac{1}{2}\arctan\left[\left(\frac{1+\sin\theta}{1-\sin\theta} + \cos 2\theta\right)^{-1} \sin 2\theta\right]. \tag{10}$$

This (optimal) strategy gives a fidelity

$$F_{\rm av} = \tfrac{1}{2}\left(1 + (1 - x^2 + x^4)^{1/2}\right). \tag{11}$$

Even in the worst case (when $x = 1/2^{1/2}$), this fidelity is always relatively high—it is always above 0.933 [23].

This shows that choosing $\mathcal{S}_0$ to check for the fulfilment of Criterion 2 is a very weak test. For an example of the opposite extreme, consider the case where $\mathcal{S}$ consists of every normalized vector in a Hilbert space of dimension d and assume that $\mathcal{S}$ is equipped with the uniform probability distribution (i.e. the unique distribution that is invariant with respect to all unitary operations). Then it turns out that the maximum value $F_{\rm av}$ can take is [24]

$$F_{\rm av} = \frac{2}{d+1}. \tag{12}$$

For the case of a single qubit, i.e. $d = 2$, Alice and Bob would *only* have to achieve a fidelity of 2/3 before they could claim that they verifiably used some entanglement for their claimed teleportation. But, again, this is only if Victor can be sure that Alice and Bob know absolutely nothing about which state he inputs other than the dimension of the Hilbert space it lives in.

This last example finally prepares us to build a useful criterion for the verification of continuous quantum-variable teleportation in the experiment of Furusawa *et al.* [3]. For a completely unknown quantum state in that experiment would correspond to taking the limit $d \to \infty$ above. If Victor can be sure that Alice and Bob know nothing whatsoever about the quantum states he intends to teleport, then on average the best fidelity they can achieve in cheating is strictly zero! In this case, seeing any nonzero fidelity whatsoever in the laboratory would signify that unconditional quantum teleportation had been achieved.

But making such a drastic assumption for the confirmation set $\mathcal{S}$ would be going too far. This would be the case if for no other reason because any present-day Victor lacks the experimental ability to make good his threat. Any Alice and Bob that had wanted to cheat in the Furusawa *et al.* experiment would know that the Victor using their services is technically restricted by the fact that only a handful of manifestly quantum or nonclassical states have ever been generated in quantum optics laboratories [25]. By far the most realistic and readily available laboratory source available to Victor is one that creates optical coherent states of a single field mode for his test of teleportation. Therefore in all that follows we will explicitly make the assumption that S contains the coherent states $|\alpha\rangle$ with a Gaussian distribution centred over the vacuum state describing the probability density on that set. As we shall see presently, it turns out that in the limit that the variance of the Gaussian distribution approaches infinity—i.e. the distribution of states becomes ever more uniform—the upper bound for the average fidelity achievable by a cheating Alice and Bob using optical heterodyne measurements is

$$F_{\text{av}} = \tfrac{1}{2}. \tag{13}$$

Any average fidelity that exceeds this bound must have come about through the use of some entanglement.

3. Optimal heterodyne cheating

We now verify equation (13) within the context of the Furusawa *et al.* experiment. There, the object is to teleport an arbitrary coherent state of a finite bandwidth electromagnetic field. (The extension of the single mode theory of [11] to the multimode case is given in [26].) We focus for simplicity on the single mode case. The quantum resource used for the process is one that entangles the number states $|n\rangle$ of two modes of the field. Explicitly the entangled state is given by [27]

$$|E\rangle_{\text{AB}} = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(\tanh r)^n |n\rangle_{\text{A}}|n\rangle_{\text{B}}, \tag{14}$$

where r measures the amount of squeezing required to produce the entangled state.

In order to verify that entanglement was actually used in the experiment, as discussed in the previous section, we shall assume that the test set S is the full set of coherent states $|\beta\rangle$,

$$|\beta\rangle = \exp(-|\beta|^2/2)\sum_{0}^{\infty}\frac{\beta^n}{n!^{1/2}}|n\rangle, \tag{15}$$

where the complex parameter β is distributed according to a Gaussian distribution,

$$p(\beta) = \frac{\lambda}{\pi}\exp(-\lambda|\beta|^2). \tag{16}$$

Ultimately, of course, we would like to consider the case where Alice and Bob are completely ignorant of which coherent state is drawn. This is described by taking the limit $\lambda \to 0$ in what follows.

It is well known that the measurement optimal for estimating the unknown parameter β when it is distributed according to a Gaussian distribution [28] is the POVM $\{\hat{E}_\alpha\}$ constructed from the coherent state projectors according to

$$\hat{E}_\alpha = \frac{1}{\pi}|\alpha\rangle\langle\alpha|, \tag{17}$$

first suggested by Arthurs and Kelly [29]. This measurement is equivalent to optical heterodyning [30]. These points make this measurement immediately attractive for the present considerations. On the one hand, maximizing the average fidelity (as is being considered here) is almost identical in spirit to the state-estimation problem of [28]. On the other, in the Furusawa *et al.* experiment a cheating Alice who uses no entanglement actually performs precisely this measurement.

We therefore consider an Alice who performs the measurement $\{\hat{E}_\alpha\}$ and forwards on the outcome—i.e. the complex number α—to Bob.* The only thing Bob can do with this information is to generate a new quantum state according to some rule, $\alpha \to |f_\alpha\rangle$. Let us make no *a priori* restrictions on the states $|f_\alpha\rangle$. The task is first to find the maximum average fidelity $F_{\max}(\lambda)$ Bob can achieve for a given λ.

For a given strategy $\alpha \to |f_\alpha\rangle$, the achievable average fidelity is

$$F(\lambda) = \int p(\beta)\left(\int p(\alpha|\beta)|\langle f_\alpha|\beta\rangle|^2 d^2\alpha\right) \mathrm{d}^2\beta \tag{18}$$

$$= \int p(\beta)\left(\int \frac{1}{\pi}|\langle \alpha|\beta\rangle|^2|\langle f_\alpha|\beta\rangle|^2 \,\mathrm{d}^2\alpha\right) \mathrm{d}^2\beta \tag{19}$$

$$= \frac{\lambda}{\pi^2}\int\int \exp(-\lambda|\beta|^2)\exp(-|\alpha-\beta|^2)|\langle f_\alpha|\beta\rangle|^2 \,\mathrm{d}^2\beta\,\mathrm{d}^2\alpha \tag{20}$$

$$= \frac{\lambda}{\pi^2}\int \exp(-|\alpha|^2)\langle f_\alpha|\left(\int \exp(-(1+\lambda)|\beta|^2 + 2\,\mathrm{Re}\,\alpha^*\beta)|\beta\rangle\langle\beta|\,\mathrm{d}^2\beta\right)|f_\alpha\rangle\,\mathrm{d}^2\alpha. \tag{21}$$

Notice that the operator enclosed within the brackets in equation (21), i.e.

$$\hat{\mathcal{O}}_\alpha = \int \exp\left(-(1+\lambda)|\beta|^2 + 2\,\mathrm{Re}\,\alpha^*\beta\right)|\beta\rangle\langle\beta|\,\mathrm{d}^2\beta, \tag{22}$$

is a positive semi-definite Hermitian operator that depends only on the real parameter λ and the complex parameter α. It follows that

$$\langle f_\alpha|\hat{\mathcal{O}}_\alpha|f_\alpha\rangle \le \mu_1(\hat{\mathcal{O}}_\alpha), \tag{23}$$

where $\mu_1(\hat{X})$ denotes the largest eigenvalue of the operator $\hat{X}$.

With this, Bob's best strategy is apparent. For each α, he simply adjusts the state $|f_\alpha\rangle$ to be the eigenvector of $\hat{\mathcal{O}}_\alpha$ with the largest eigenvalue. Then equality is achieved in equation (23), and it is just a question of being able to perform the integral in equation (21).

The first step in carrying this out is to find the eigenvector and eigenvalue achieving equality in equation (23). This is most easily evaluated by unitarily transforming $\hat{\mathcal{O}}_\alpha$ into something that is diagonal in the number basis, picking off the largest eigenvalue, and transforming back to get the optical $|f_\alpha\rangle$. (Recall that eigenvalues are invariant under unitary transformations.)

* We caution, however, that the present considerations do not *prove* the optimality of heterodyne measurement for an arbitrarily adversarial Alice and Bob—they simply make it fairly plausible. Complete optimization requires the consideration of all POVMs that Alice can conceivably perform along with explicit consideration of the structure of the fidelity function considered here, not simply the variance of an estimator as in the state-estimation problem. More on this issue can be found in [32].

The upshot of this procedure is best illustrated by working backward toward the answer. Consider the positive operator

$$\hat{P} = \int \exp\left[-(1+\lambda)|\beta|^2\right]|\beta\rangle\langle\beta|\, d^2\beta. \tag{24}$$

Expanding this operator in the number basis, we find

$$\hat{P} = \pi \sum_{n=0}^{\infty} (2+\lambda)^{-(n+1)} |n\rangle\langle n|. \tag{25}$$

So clearly,

$$\mu_1(\hat{P}) = \frac{\pi}{2+\lambda}. \tag{26}$$

Now consider the displaced operator

$$\hat{Q}_\alpha = \hat{D}\left(\frac{\alpha}{1+\lambda}\right)\hat{P}\hat{D}^\dagger\left(\frac{\alpha}{1+\lambda}\right), \tag{27}$$

where $\hat{D}(\nu)$ is the standard displacement operator [32]. Working this out in the coherent-state basis, one finds

$$\hat{Q}_\alpha = \int \exp\left[-(1+\lambda)|\beta|^2\right]\left|\beta + \frac{\alpha}{1+\lambda}\right\rangle\left\langle\beta + \frac{\alpha}{1+\lambda}\right| d^2\beta \tag{28}$$

$$= \int \exp\left(-(1+\lambda)\left|\gamma - \frac{\alpha}{1+\lambda}\right|^2\right)|\gamma\rangle\langle\gamma|\, d^2\gamma \tag{29}$$

$$= \exp\left(\frac{-|\alpha|^2}{1+\lambda}\right)\int \exp\left(-(1+\lambda)|\gamma|^2 + 2\,\mathrm{Re}\,\alpha^*\gamma\right)|\gamma\rangle\langle\gamma|\, d^2\gamma \tag{30}$$

$$= \exp\left(\frac{-|\alpha|^2}{1+\lambda}\right)\hat{\mathcal{O}}_\alpha. \tag{31}$$

Using this in the expression for $F(\lambda)$ we find,

$$F(\lambda) = \frac{\lambda}{\pi^2}\int \exp\left(-\left(1-\frac{1}{1+\lambda}\right)|\alpha|^2\right)\langle f_\alpha|\left(\hat{D}\left(\frac{\alpha}{1+\lambda}\right)\hat{P}\hat{D}^\dagger\left(\frac{\alpha}{1+\lambda}\right)\right)|f_\alpha\rangle\, d^2\alpha \tag{32}$$

$$\leq \frac{1}{\pi}\frac{\lambda}{2+\lambda}\int \exp\left(-\frac{\lambda}{1+\lambda}|\alpha|^2\right) d^2\alpha \tag{33}$$

$$= \frac{1+\lambda}{2+\lambda}. \tag{34}$$

Equality is achieved in this chain by taking

$$|f_\alpha\rangle = D\left(\frac{\alpha}{1+\lambda}\right)|0\rangle = \left|\frac{\alpha}{1+\lambda}\right\rangle. \tag{35}$$

Therefore the maximum average fidelity is given by

$$F_{\max}(\lambda) = \frac{1+\lambda}{2+\lambda}. \tag{36}$$

In the limit that $\lambda \to 0$, i.e. when Victor draws his states from a uniform distribution, we have

$$F_{\max}(\lambda) \to \tfrac{1}{2}, \quad (37)$$

as advertised in [3].

It should be noted that nothing in this argument depended upon the mean of the Gaussian distribution being $\beta = 0$. Both would need to minimally modify his strategy to take into account Gaussians with a non-vacuum state mean, but the optimal fidelity would remain the same.

4. Conclusion

Where do we stand? What remains? Clearly one would like to develop a toolbox of ever more stringent and significant tests of quantum teleportation—ones devoted not only to Criterion 2, but to all the others mentioned in the section 1 as well. Significant among these are delineations of the fidelities that must be achieved to insure the honest teleportation of nonclassical states of light, such as squeezed states. Some work in this direction appears in [11], but one would like to find something more in line with the framework presented here. Luckily, a more general setting for this problem can be formulated [32] as it will ultimately be necessary to explore any number of natural verification sets $\mathcal{S}$ and their resilience with respect to arbitrarily adversarial Alice and Bob teams.

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Dense coding for continuous variables

Samuel L. Braunstein
Informatics, University of Wales, Bangor LL57 1UT, United Kingdom

H. J. Kimble
Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125
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A scheme to achieve dense quantum coding for the quadrature amplitudes of the electromagnetic field is presented. The protocol utilizes shared entanglement provided by nondegenerate parametric down-conversion in the limit of large gain to attain high efficiency. For a constraint in the mean number of photons $\bar{n}$ associated with modulation in the signal channel, the channel capacity for dense coding is found to be $\ln(1+\bar{n}+\bar{n}^2)$, which always beats coherent-state communication and surpasses squeezed-state communication for $\bar{n}>1$. For $\bar{n}\gg 1$, the dense coding capacity approaches twice that of either scheme.

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An important component of contemporary quantum information theory is the investigation of the classical information capacities of noisy quantum communication channels. Here, classical information is encoded by the choice of one particular quantum state from among a predefined ensemble of quantum states by the sender Alice for transmission over a quantum channel to the receiver Bob. If Alice and Bob are allowed to communicate only via a one-way exchange along such a noisy quantum channel, then the optimal amount of classical information that can be reliably transmitted over the channel has recently been established [1,2].

Stated more explicitly, if a classical signal α taken from the ensemble P_α is to be transmitted as a quantum state $\hat{\rho}_\alpha$, then Holevo's bound for a bosonic quantum channel says that the mutual information $H(A{:}B)$ between the sender A (Alice) and receiver B (Bob) is bounded by [1]

$$H(A{:}B)\leqslant S(\hat{\rho})-\int d^2\alpha P_\alpha S(\hat{\rho}_\alpha)\leqslant S(\hat{\rho}), \qquad (1)$$

where $S(\hat{\rho})$ is the von Neumann entropy associated with the density operator $\hat{\rho}=\int d^2\alpha P_\alpha \hat{\rho}_\alpha$ for the mean channel state.

By contrast, if Alice and Bob share a quantum resource in the form if an ensemble of entangled states, then quantum mechanics enables protocols for communication that can circumvent the aforementioned bound on channel capacity. For example, as shown originally by Bennett and Wiesner [3], Alice and Bob can beat the Holevo limit by exploiting their shared entanglement to achieve dense quantum coding. Here, the signal is encoded at Alice's sending station and transmitted via one component of a pair of entangled quantum states, with then the second component of the entangled pair exploited for decoding the signal at Bob's receiving station. In this scheme, the cost of distributing the entangled states to Alice and Bob is not figured into the accounting of constraints on the quantum channel (e.g., the mean energy). Such neglect of the distribution cost of entanglement is sensible in some situations, as for example, if the entanglement were to be sent during off-peak times when the communication channel is otherwise under utilized, or if it had been conveyed by other means to Alice and Bob in advance (e.g., via a pair of *quantum CDs* with stored, entangled quantum states). Note that in general, no signal modulation is applied to the second (i.e., Bob's) component of the entangled state, so that it carries no information by itself.

Although quantum dense coding has most often been discussed within the setting of *discrete* quantum variables (e.g., *qubits*) [3,4], in this paper we show that highly efficient dense coding is possible for *continuous* quantum variables. As in our prior work on quantum teleportation [5–7], our scheme for achieving quantum dense coding exploits squeezed-state entanglement, and therefore should allow *unconditional* signal transmission with high efficiency, in contrast to the *conditional* transmission with extremely low efficiency achieved in Ref. [4]. More specifically, for signal states α associated with the complex amplitude of the electromagnetic field, the channel capacity for dense coding is found to be $\ln(1+\bar{n}+\bar{n}^2)$, where $\bar{n}$ is the mean photon number for modulation in the signal channel. The channel capacity for dense coding in our scheme thus always beats coherent-state communication and surpasses squeezed-state communication for $\bar{n}>1$. For $\bar{n}\gg 1$, the dense coding capacity approaches twice that of either scheme.

As illustrated in Fig. 1, the relevant continuous variables for our protocol are the quadrature amplitudes $(\hat{x},\hat{p})$ of the electromagnetic field, with the classical signal $\alpha=\langle\hat{x}\rangle+i\langle\hat{p}\rangle$ then associated with the quantum state $\hat{\rho}_\alpha$ drawn from the phase space for a single mode of the field. The entangled resource shared by Alice and Bob is a pair of EPR beams with quantum correlations between canonically conjugate variables $(\hat{x},\hat{p})_{(1,2)}$ as were first described by Einstein, Podolsky, and Rosen (EPR [8]), and which can be efficiently generated via the nonlinear optical process of parametric down conversion, resulting in a highly squeezed two-mode state of the electromagnetic field [9,10]. In the ideal case, the correlations between quadrature-phase amplitudes for the two beams (1,2) are such that

$$\langle(\hat{x}_1-\hat{x}_2)^2\rangle\rightarrow 0, \quad \langle(\hat{p}_1+\hat{p}_2)^2\rangle\rightarrow 0, \qquad (2)$$

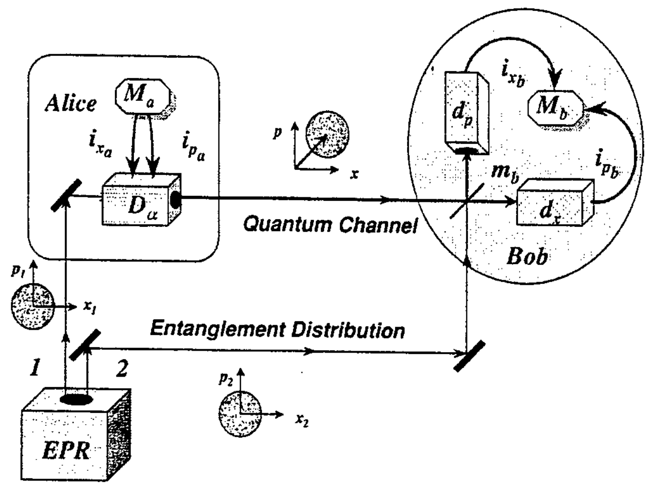

FIG. 1. Illustration of the scheme for achieving super-dense quantum coding for signal states over the complex amplitude $\alpha = x + ip$ of the electromagnetic field. The quantum resource that enables dense coding is the EPR source that generates entangled beams (1,2) shared by Alice and Bob.

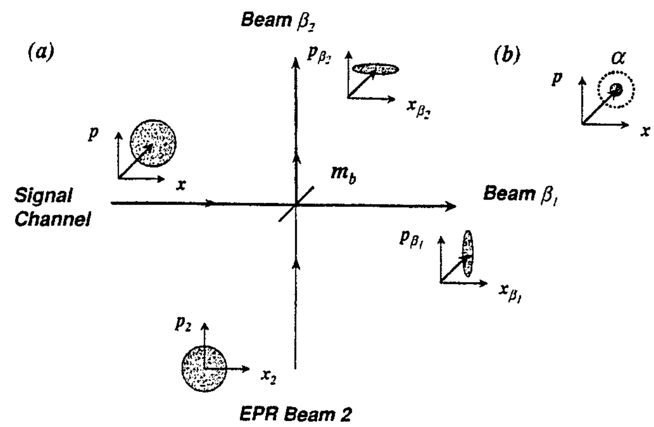

FIG. 2. Depiction of signal decoding at Bob's receiving station. (a) At Bob's 50−50 beam splitter m_b, the displaced EPR beam 1 is combined with the component 2 to yield two independent squeezed beams, with the $\beta_{1,2}$ beams having fluctuations reduced below the vacuum-state limit along $(x_{\beta_1}, p_{\beta_2})$. Homodyne detection at (d_x, d_p) (Fig. 1) with LO phases set to measure $(x_{\beta_1}, p_{\beta_2})$, respectively, then yields the complex signal amplitude α_{out} with variance set by the associated squeezed states. (b) The net effect of the dense coding protocol is the transmission and detection of states of complex amplitude α with an effective uncertainty below the vacuum-state limit (indicated by the dashed circle).

albeit it with an concomitant divergence in the mean photon number $\bar{n}$ in each channel.

Component 1 of this entangled pair of beams is input to Alice's sending station, where the message M_a^α corresponding to the classical signal α_{in} is encoded as the quantum state $\hat{\rho}_{\alpha_{\text{in}}}$ by a simple phase-space offset by way of the displacement operator $\hat{D}(\alpha_{\text{in}})$ applied to 1 [11]. The displacement $\hat{D}(\alpha_{\text{in}})$ can be implemented in a straightforward fashion by amplitude and phase offsets generated by the (suitably normalized) classical currents (i_{x_a}, i_{p_a}) as in Ref. [7]. The state corresponding to Alice's displacement of the EPR beam constitutes the quantum signal and is transmitted along the quantum channel shown in Fig. 1 to Bob's receiving station, (Fig. 2) where it is decoded with the aid of the second component 2 of the original EPR pair of beams and the homodyne detectors (d_x, d_p). The resulting photocurrents (i_{x_b}, i_{p_b}) suitably normalized to produce $\alpha_{\text{out}} = i_{x_b} + i i_{p_b}$ constitute the message M_b^α received by Bob. In the limit $\bar{n} \to \infty$, Eq. (2) ensures $\alpha_{\text{out}} = \alpha_{\text{in}}$, so that the classical message would be perfectly recovered. However, even for finite $\bar{n}$ as is relevant to a channel constrained in mean energy, the finite correlations implicit in the EPR beams enable quantum dense coding with enhanced channel capacity relative to either coherent state or squeezed state communication, as we now show.

Consider the specific case of EPR beams (1,2) approximated by the two-mode squeezed state with Wigner function

$$W_{\text{EPR}}(\alpha_1, \alpha_2) = \frac{4}{\pi^2} \exp[-e^{-2r}(\alpha_1 - \alpha_2)_R^2 - e^{2r}(\alpha_1 - \alpha_2)_I^2 - e^{2r}(\alpha_1 + \alpha_2)_R^2 - e^{-2r}(\alpha_1 + \alpha_2)_I^2], \quad (3)$$

where the subscripts R and I refer to real and imaginary parts of the field amplitude α, respectively (i.e., $\alpha_{R,I} = x, p$). Note that for $r \to \infty$, the field state becomes the ideal EPR state as described in Eq. (2), namely,

$$W_{\text{EPR}}(\alpha_1, \alpha_2) \to C\,\delta(\alpha_{1R} + \alpha_{2R})\,\delta(\alpha_{1I} - \alpha_{2I}). \quad (4)$$

As shown in Fig. 1, signal modulation is performed only on mode 1, with mode 2 treated as an overall shared resource by Alice and Bob (and which could have been generated by Alice herself). The modulation scheme that we choose is simply to displace mode 1 by an amount α_{in}. This leads to a displaced Wigner function given by $W_{\text{EPR}}(\alpha_1 - \alpha_{\text{in}}, \alpha_2)$, corresponding to the field state that is sent via the quantum channel from Alice to Bob.

Upon receiving this transmitted state (consisting of the modulated mode 1), the final step in the dense-coding protocol is for Bob to combine it with the shared resource (mode 2) and retrieve the original classical signal α_{in} with as high a fidelity as possible. As indicated in Fig. 1, this demodulation can be performed with a simple 50−50 beam splitter that superposes the modes (1,2) to yield output fields that are the sum and difference of the input fields and which we label as β_1 and β_2, respectively. The resulting state emerging from Bob's beam splitter has Wigner function

$$W_{\text{sum/diff}}(\beta_1, \beta_2) = W_{\text{EPR}}((\beta_1 + \beta_2)/\sqrt{2} - \alpha, (\beta_1 - \beta_2)/\sqrt{2}). \quad (5)$$

The classical signal that we seek is retrieved by homodyne detection at detectors (d_x, d_p), which measure the analogs of position and momentum for the sum and difference fields (β_1, β_2). For ideal homodyne detection the resulting outcomes are distributed according to

$$P(\beta|\alpha)=\frac{2e^{2r}}{\pi}\exp(-2e^{2r}|\beta-\alpha/\sqrt{2}|^2),$$

where $\beta=\beta_{1R}+i\beta_{2I}$ and represents a highly peaked distribution about the complex displacement $\alpha/\sqrt{2}$. For large squeezing parameter r this allows us to extract the original signal α which we choose to be distributed as

$$P_\alpha=\frac{1}{\pi\sigma^2}\exp(-|\alpha|^2/\sigma^2). \tag{6}$$

Note that mode 1 of this displaced state has a mean number of photons given by

$$\bar{n}=\sigma^2+\sinh^2 r. \tag{7}$$

In order to compute the quantity of information that may be sent through this dense coding channel we note the unconditioned probability for the homodyne statistics is given by

$$P(\beta)=\frac{2}{\pi(\sigma^2+e^{-2r})}\exp\left(\frac{-2|\beta|^2}{\sigma^2+e^{-2r}}\right). \tag{8}$$

The mutual information describing the achievable information throughput of this dense coding channel is then given by

$$H^{\text{dense}}(A:B)=\int d^2\beta d^2\alpha P(\beta|\alpha)P_\alpha \ln\left(\frac{P(\beta|\alpha)}{P(\beta)}\right)$$
$$=\ln(1+\sigma^2e^{2r}). \tag{9}$$

For a fixed $\bar{n}$ in Eq. (7) this information is optimized when $\bar{n}=e^r\sinh r$, i.e., when $\sigma^2=\sinh r\cosh r$ so yielding a dense coding capacity of

$$C^{\text{dense}}=\ln(1+\bar{n}+\bar{n}^2), \tag{10}$$

which for large squeezing r becomes

$$C^{\text{dense}}\sim 4r. \tag{11}$$

How efficient is this dense coding in comparison to single channel coding? Let us place a "common" constraint of having a fixed mean number of photons $\bar{n}$ which can be modulated. For a single bosonic channel Drummond and Caves [12] and Yuen and Ozawa [13] have used Holevo's result to show that the optimal channel capacity is just that given by photon counting from a maximum entropy ensemble of number states. In this case the channel capacity (the maximal mutual information) achieves the ensemble entropy, see Eq. (1), so

$$C=S(\rho)=(1+\bar{n})\ln(1+\bar{n})-\bar{n}\ln\bar{n}. \tag{12}$$

Substituting $\bar{n}=e^r\sinh r$ into this we find

$$C\sim 2r, \tag{13}$$

for large squeezing r. This is just one-half of the asymptotic dense coding mutual information, see Eq. (11). Thus asymptotically, at least, the dense coding scheme allows twice as much information to be encoded within a given state, although it has an extra expense (not included within the simple constraint $\bar{n}$) of requiring shared entanglement.

It is worth noting that this dense coding scheme does *not* always beat the optimal single channel capacity. Indeed, for small squeezing it is worse. The break-even squeezing required for dense coding to equal the capacity of the optimal single channel communication is

$$r_{\text{break-even}}\simeq 0.7809, \tag{14}$$

which corresponds to roughly 6.78 dB of two-mode squeezing or to $\bar{n}\simeq 1.884$. This break-even point takes into account the difficulty of making highly squeezed two-mode squeezed states. No similar difficulty has been factored into making ideal number states used in the benchmark scheme with which our dense coding scheme is compared.

A fairer comparison is against single-mode coherent state communication with heterodyne detection. Here the channel capacity is well known [14–16] for the mean photon number constraint to be

$$C^{\text{coh}}=\ln(1+\bar{n}), \tag{15}$$

which is *always* beaten by the optimal dense coding scheme described by Eq. (10).

An improvement on coherent state communication is squeezed state communication with a single mode. The channel capacity of this channel has been calculated [16] to be

$$C^{\text{sq}}=\ln(1+2\bar{n}), \tag{16}$$

which is beaten by the dense coding scheme of Eq. (10) for $\bar{n}>1$, i.e., the break-even squeezing required is

$$r^{\text{sq}}_{\text{break-even}}\simeq 0.5493, \tag{17}$$

which corresponds to 4.77 dB.

In summary, we have shown how to perform dense quantum coding for continuous quantum variables by utilizing squeezed state entanglement. For a constraint in the mean number of photons that may be modulated $\bar{n}$, the dense coding capacity is found to be $\ln(1+\bar{n}+\bar{n}^2)$. This scheme always beats single-mode coherent-state communication and surpasses single-mode squeezed-state communication for $\bar{n}>1$. Note that in terms of actual implementation, our protocol should allow for high efficiency, *unconditional* transmission with encoded information sent every inverse bandwidth time. This situation is in contrast to implementations that employ weak parametric down conversion, where transmission is achieved *conditionally* and relatively rarely. In fact Mattle *et al.* [4] obtained rates of only 1 in 10^7 per inverse bandwidth time [17]. By going to strong down conversion and using a characteristically different type of entanglement, our scheme should allow information to be sent with much higher efficiency and should simultaneously improve the ability to detect orthogonal Bell states. Indeed, these advantages enabled the first experimental realization of unconditional quantum teleportation within the past year [7]. Beyond the particular setting of quantum communication discussed here, this research is part of a larger program to explore the potential for quantum information processing with continuous quantum variables. Such investigations are quite timely in light of important recent progress concerning the prospects for diverse quantum algorithms with continuous variables, including universal quantum computation [18] and quantum error correction [19–21], with quantum teleportation being a prime example [5,22,23]. Although still in its earliest stages, theoretical protocols have been developed for realistic physical systems that should allow a variety of elementary processing operations for continuous quantum variables, including significantly quantum storage for EPR states. [24,25]

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Broadband teleportation

P. van Loock and Samuel L. Braunstein
Quantum Optics and Information Group, School of Informatics, University of Wales, Bangor LL57 1UT, United Kingdom

H. J. Kimble
Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125

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Quantum teleportation of an unknown broadband electromagnetic field is investigated. The continuous-variable teleportation protocol by Braunstein and Kimble [Phys. Rev. Lett. **80**, 869 (1998)] for teleporting the quantum state of a single mode of the electromagnetic field is generalized for the case of a multimode field with finite bandwith. We discuss criteria for continuous-variable teleportation with various sets of input states and apply them to the teleportation of broadband fields. We first consider as a set of input fields (from which an independent state preparer draws the inputs to be teleported) arbitrary pure Gaussian states with unknown coherent amplitude (squeezed or coherent states). This set of input states, further restricted to an alphabet of coherent states, was used in the experiment by Furusawa *et al.* [Science **282**, 706 (1998)]. It requires unit-gain teleportation for optimizing the teleportation fidelity. In our broadband scheme, the excess noise added through unit-gain teleportation due to the finite degree of the squeezed-state entanglement is just twice the (entanglement) source's squeezing spectrum for its "quiet quadrature." The teleportation of one half of an entangled state (two-mode squeezed vacuum state), i.e., "entanglement swapping," and its verification are optimized under a certain nonunit gain condition. We will also give a broadband description of this continuous-variable entanglement swapping based on the single-mode scheme by van Loock and Braunstein [Phys. Rev. A **61**, 10 302 (2000)].

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I. INTRODUCTION

Teleportation of an unknown quantum state is its disembodied transport through a classical channel, followed by its reconstitution, using the quantum resource of entanglement. Quantum information cannot be transmitted reliably via a classical channel alone, as this would allow us to replicate the classical signal and so produce copies of the initial state, thus violating the no-cloning theorem [1]. More intuitively, any attempted measurement of the initial state only obtains partial information due to the Heisenberg uncertainty principle and the subsequently collapsed wave packet forbids information gain about the original state from further inspection. Attempts to circumvent this disability with more generalized measurements also fail [2].

Quantum teleportation was first proposed to transport an unknown state of any discrete quantum system, e.g., a spin-$\frac{1}{2}$ particle [3]. In order to accomplish the teleportation, classical and quantum methods must go hand in hand. A part of the information encoded in the unknown input state is transmitted via the quantum correlations between two separated subsystems in an entangled state shared by the sender and the receiver. In addition, classical information must be sent via a conventional channel. For the teleportation of a spin-$\frac{1}{2}$-particle state, the entangled state required is a pair of spins in a Bell state [4]. The classical information that has to be transmitted contains two bits in this case.

Important steps toward the experimental implementation of quantum teleportation of single-photon polarization states have already been accomplished [5,6]. However, a complete realization of the original teleportation proposal [3] has not been achieved in these experiments, as either the state to be teleported is not independently coming from the outside [6] or destructive detection of the photons in the teleported state is employed as part of the protocol [5]. In the latter case, a teleported state did not emerge for subsequent examination or exploitation. This situation has been termed "*a posteriori* teleportation," being accomplished via post selection of photoelectric counting events [7]. Without postselection, the fidelity would not have exceeded the value $\frac{2}{3}$ required.

The teleportation of continuous quantum variables such as position and momentum of a particle [8] relies on the entanglement of the states in the original Einstein, Podolsky, and Rosen (EPR) paradox [9]. In quantum optical terms, the observables analogous to the two conjugate variables position and momentum of a particle are the quadrature amplitudes of a single mode of the electromagnetic field [10]. By considering the finite (nonsingular) degree of correlation between these quadratures in a two-mode squeezed state [10], a realistic implementation for the teleportation of continuous quantum variables was proposed [11]. Based on this proposal, in fact, quantum teleportation of arbitrary coherent states has been achieved with a fidelity $F=0.58\pm0.02$ [12]. Without using entanglement, by purely classical communication, an average fidelity of 0.5 is the best that can be achieved if the set of input states contains all coherent states [13]. The scheme with continuous quadrature amplitudes of a single mode enables an *a priori* (or "unconditional") teleportation with high efficiency [11], as reported in Refs. [14,12]. In this experiment, three criteria necessary for quantum teleportation were achieved: (1) An unknown quantum state enters the sending station for teleportation. (2) A teleported state emerges from the receiving station for subsequent evaluation or exploitation. (3) The degree of overlap between the input and the teleported states is higher than that which could be achieved if the sending and the receiving stations were linked only by a classical channel.

In continuous-variable teleportation, the teleportation process acts on an infinite-dimensional Hilbert space instead of the two-dimensional Hilbert space for the discrete spin variables. However, an arbitrary electromagnetic field has an infinite number of modes, or in other words, a finite bandwidth containing a continuum of modes. Thus, the teleportation of the quantum state of a broadband electromagnetic field requires the teleportation of a quantum state which is defined in the tensor product space of an infinite number of infinite-dimensional Hilbert spaces. The aim of this paper is to extend the treatment of Ref. [11] to the case of a broadband field, and thereby to provide the theoretical foundation for laboratory investigations as in Refs. [14,12]. In particular, we demonstrate that the two-mode squeezed state output of a nondegenerate optical parametric amplifier (NOPA) [15] is a suitable EPR ingredient for the efficient teleportation of a broadband electromagnetic field.

In the three above mentioned teleportation experiments, in Innsbruck [5], in Rome [6], and in Pasadena [12], the nonorthogonal input states to be teleported were single-photon polarization states (qubits) [5,6] and coherent states [12]. From a true quantum teleportation device, however, we would also require the capability of teleporting the entanglement source itself. This teleportation of one half of an entangled state (entanglement swapping [16]) means to entangle two quantum systems that have never directly interacted with each other. For discrete variables, a demonstration of entanglement swapping with single photons has been reported by Pan *et al.* [17]. For continuous variables, experimental entanglement swapping has not yet been realized in the laboratory, but there have been several theoretical proposals of such an experiment. Polkinghorne and Ralph [18] suggested teleporting polarization-entangled states of single photons using squeezed-state entanglement where the output correlations are verified via Bell inequalities. Tan [19] and van Loock and Braunstein [20] considered the unconditional teleportation (without postselection of "successful" events by photon detections) of one half of a two-mode squeezed state using different protocols and verification. Based on the single-mode scheme of Ref. [20], we will also present a broadband description of continuous-variable entanglement swapping.

II. TELEPORTATION OF A SINGLE MODE

In the teleportation scheme of a single mode of the electromagnetic field (for example, representing a single pulse or wave packet), the shared entanglement is a two-mode squeezed vacuum state [11]. For infinite squeezing, this state contains exactly analogous quantum correlations as does the state described in the original EPR paradox, where the quadrature amplitudes of the two modes play the roles of position and momentum [11]. The entangled state is sent in two halves: one to "Alice" (the teleporter or sender) and the other one to "Bob" (the receiver), as illustrated in Fig. 1. In order to perform the teleportation, Alice has to couple the

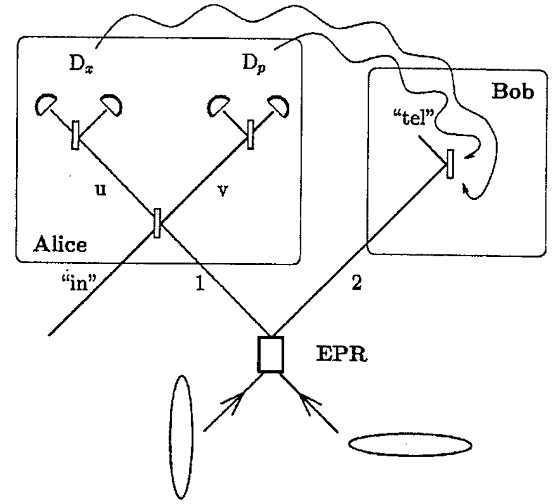

FIG. 1. Teleportation of a single mode of the electromagnetic field as in Ref. [11]. Alice and Bob share the entangled state of modes 1 and 2. Alice combines the mode "in" to be teleported with her half of the EPR state at a beam splitter. The homodyne detectors D_x and D_p yield classical photocurrents for the quadratures x_u and p_v, respectively. Bob performs phase-space displacements of his half of the EPR state depending on Alice's classical results.

input mode she wants to teleport with her "EPR mode" at a beam splitter. The "Bell detection" of the x quadrature at one beam splitter output, and of the p quadrature at the other output, yields the classical results to be sent to Bob via a classical communication channel. In the limit of an infinitely squeezed EPR source, these classical results contain no information about the mode to be teleported. This is analogous to the Bell-state measurement of the spin-$\frac{1}{2}$-particle pair by Alice for the teleportation of a spin-$\frac{1}{2}$-particle state. The measured Bell state of the spin-$\frac{1}{2}$-particle pair determines whether the particles have equal or different spin projections. The spin projection of the individual particles, i.e., Alice's EPR particle and her unknown input particle, remains completely unknown [3]. According to this analogy, we call Alice's quadrature measurements for the teleportation of the state of a single mode (and of a multimode field in the following sections) "Bell detection." Due to this Bell detection, the entanglement between Alice's "EPR mode" and Bob's "EPR mode" means that suitable phase-space displacements of Bob's mode convert it into a replica of Alice's unknown input mode (a perfect replica for infinite squeezing). In order to perform these displacements, Bob needs the classical results of Alice's Bell measurement.

The previous protocol for the quantum teleportation of continuous variables used the Wigner distribution and its convolution formalism [11]. The teleportation of a single mode of the electromagnetic field can also be recast in terms of Heisenberg equations for the quadrature amplitude operators, which is the formalism that we employ in this paper. For that purpose, the Wigner function W_{EPR} describing the entangled state shared by Alice and Bob [11] is replaced by equations for the quadrature amplitude operators of a two-mode squeezed vacuum state. Two independently squeezed vacuum modes can be described by [10]

$$\hat{x}_1 = e^{r}\hat{x}_1^{(0)}, \quad \hat{p}_1 = e^{-r}\hat{p}_1^{(0)},$$
$$\hat{x}_2 = e^{-r}\hat{x}_2^{(0)}, \quad \hat{p}_2 = e^{r}\hat{p}_2^{(0)}, \tag{1}$$

where a superscript (0) denotes initial vacuum modes and r is the squeezing parameter. Superimposing the two squeezed modes at a 50/50 beam splitter yields the two output modes

$$\hat{x}_1 = \frac{1}{\sqrt{2}}e^{r}\hat{x}_1^{(0)} + \frac{1}{\sqrt{2}}e^{-r}\hat{x}_2^{(0)},$$
$$\hat{p}_1 = \frac{1}{\sqrt{2}}e^{-r}\hat{p}_1^{(0)} + \frac{1}{\sqrt{2}}e^{r}\hat{p}_2^{(0)},$$
$$\hat{x}_2 = \frac{1}{\sqrt{2}}e^{r}\hat{x}_1^{(0)} - \frac{1}{\sqrt{2}}e^{-r}\hat{x}_2^{(0)}, \tag{2}$$
$$\hat{p}_2 = \frac{1}{\sqrt{2}}e^{-r}\hat{p}_1^{(0)} - \frac{1}{\sqrt{2}}e^{r}\hat{p}_2^{(0)}.$$

The output modes 1 and 2 are now entangled to a finite degree in a two-mode squeezed vacuum state. In the limit of infinite squeezing, $r \to \infty$, both output modes become infinitely noisy, but also the EPR correlations between them become ideal: $(\hat{x}_1 - \hat{x}_2) \to 0$, $(\hat{p}_1 + \hat{p}_2) \to 0$. Now mode 1 is sent to Alice and mode 2 is sent to Bob. Alice's mode is then superimposed at a 50/50 beam splitter with the input mode "in":

$$\hat{x}_u = \frac{1}{\sqrt{2}}\hat{x}_{\rm in} - \frac{1}{\sqrt{2}}\hat{x}_1, \quad \hat{p}_u = \frac{1}{\sqrt{2}}\hat{p}_{\rm in} - \frac{1}{\sqrt{2}}\hat{p}_1,$$
$$\hat{x}_v = \frac{1}{\sqrt{2}}\hat{x}_{\rm in} + \frac{1}{\sqrt{2}}\hat{x}_1, \quad \hat{p}_v = \frac{1}{\sqrt{2}}\hat{p}_{\rm in} + \frac{1}{\sqrt{2}}\hat{p}_1. \tag{3}$$

Using Eqs. (3) we will find it useful to write Bob's mode 2 as

$$\hat{x}_2 = \hat{x}_{\rm in} - (\hat{x}_1 - \hat{x}_2) - \sqrt{2}\hat{x}_u = \hat{x}_{\rm in} - \sqrt{2}e^{-r}\hat{x}_2^{(0)} - \sqrt{2}\hat{x}_u,$$
$$\hat{p}_2 = \hat{p}_{\rm in} + (\hat{p}_1 + \hat{p}_2) - \sqrt{2}\hat{p}_v = \hat{p}_{\rm in} + \sqrt{2}e^{-r}\hat{p}_1^{(0)} - \sqrt{2}\hat{p}_v. \tag{4}$$

Alice's Bell detection yields certain classical values x_u and p_v for $\hat{x}_u$ and $\hat{p}_v$. The quantum variables $\hat{x}_u$ and $\hat{p}_v$ become classically determined, random variables. We indicate this by turning $\hat{x}_u$ and $\hat{p}_v$ into x_u and p_v. The classical probability distribution of x_u and p_v is associated with the quantum statistics of the previous operators [11]. Now, due to the entanglement, Bob's mode 2 collapses into states that for $r \to \infty$ differ from Alice's input state only in (random) classical phase-space displacements. After receiving Alice's classical results x_u and p_v, Bob displaces his mode

$$\hat{x}_2 \to \hat{x}_{\rm tel} = \hat{x}_2 + \Gamma\sqrt{2}x_u,$$
$$\hat{p}_2 \to \hat{p}_{\rm tel} = \hat{p}_2 + \Gamma\sqrt{2}p_v, \tag{5}$$

thus accomplishing the teleportation [11]. The parameter Γ describes a normalized gain for the transformation from classical photocurrent to complex field amplitude. For $\Gamma = 1$, Bob's displacement eliminates x_u and p_v appearing in Eqs. (4) after the collapse of $\hat{x}_u$ and $\hat{p}_v$ due to the Bell detection. The teleported field then becomes

$$\hat{x}_{\rm tel} = \hat{x}_{\rm in} - \sqrt{2}e^{-r}\hat{x}_2^{(0)},$$
$$\hat{p}_{\rm tel} = \hat{p}_{\rm in} + \sqrt{2}e^{-r}\hat{p}_1^{(0)}. \tag{6}$$

For an arbitrary gain Γ, we obtain

$$\hat{x}_{\rm tel} = \Gamma\hat{x}_{\rm in} - \frac{\Gamma-1}{\sqrt{2}}e^{r}\hat{x}_1^{(0)} - \frac{\Gamma+1}{\sqrt{2}}e^{-r}\hat{x}_2^{(0)},$$
$$\hat{p}_{\rm tel} = \Gamma\hat{p}_{\rm in} + \frac{\Gamma-1}{\sqrt{2}}e^{r}\hat{p}_2^{(0)} + \frac{\Gamma+1}{\sqrt{2}}e^{-r}\hat{p}_1^{(0)}. \tag{7}$$

Note that these equations take no Bell detector inefficiencies into account.

Consider the case $\Gamma = 1$. For infinite squeezing $r \to \infty$, Eqs. (6) describe perfect teleportation of the quantum state of the input mode. On the other hand, for the classical case of $r = 0$, i.e., no squeezing and hence no entanglement, each of the teleported quadratures has *two* additional units of vacuum noise compared to the original input quadratures. These two units are so-called quantum duties or "quduties" which have to be paid when crossing the border between quantum and classical domains [11]. The two quduties represent the minimal tariff for every "classical teleportation" scheme [13]. One quduty, the unit of vacuum noise due to Alice's detection, arises from her attempt to simultaneously measure the two conjugate variables $x_{\rm in}$ and $p_{\rm in}$ [21]. This is the standard quantum limit for the detection of both quadratures [22] when attempting to gain as much information as possible about the quantum state of a light field [23]. The standard quantum limit yields a product of the measurement accuracies which is twice as large as the Heisenberg minimum uncertainty product. This product of the measurement accuracies contains the intrinsic quantum limit (Heisenberg uncertainty of the field to be detected) plus an additional unit of vacuum noise due to the detection [22]. The second quduty arises when Bob uses the information of Alice's detection to generate the state at amplitude $\sqrt{2}x_u + i\sqrt{2}p_v$ [11]. It can be interpreted as the standard quantum limit imposed on state broadcasting.

III. TELEPORTATION CRITERIA

The teleportation scheme with Alice and Bob is complete without any further measurement. The quantum state teleported remains unknown to both Alice and Bob and need not be demolished in a detection by Bob as a final step. How-

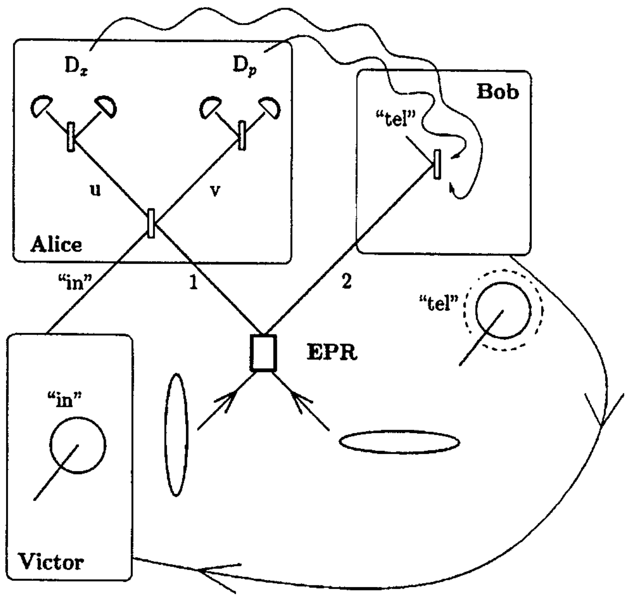

FIG. 2. Verification of quantum teleportation. The verifier "Victor" is independent of Alice and Bob. Victor prepares the input states which are known to him, but unknown to Alice and Bob. After a supposed quantum teleportation from Alice to Bob, the teleported states are given back to Victor. Due to his knowledge of the input states, Victor can compare the teleported states with the input states.

ever, maybe Alice and Bob are cheating. Instead of using an EPR channel, they try to get away without entanglement and use only a classical channel. In particular, for the realistic experimental situation with finite squeezing and inefficient detectors where perfect teleportation is unattainable, how may we verify that successful quantum teleportation has taken place? To make this verification we shall introduce a third party, "Victor" (the verifier), who is independent of Alice and Bob (Fig. 2). We assume that he prepares the initial input state (drawn from a fixed set of states) and passes it on to Alice. After accomplishing the supposed teleportation, Bob sends the teleported state back to Victor. Victor's knowledge about the input state and detection of the teleported state enable Victor to verify if quantum teleportation has really taken place. For that purpose, however, Victor needs some measure that helps him to assess when the similarity between the teleported state and the input state exceeds a boundary that is only exceedable with entanglement.

A. Teleporting Gaussian states with a coherent amplitude

The single-mode teleportation scheme from Ref. [11] works for arbitrary input states, described by any Wigner function W_{in}. Teleporting states with a coherent amplitude as reliably as possible requires unit-gain teleportation (unit gain in Bob's final displacement). Only in this case, the coherent amplitudes of the teleported mode always match those of the input mode when Victor draws states with different amplitudes from the set of input states in a sequence of trials. For this unit-gain teleportation, the teleported state W_{tel} is a convolution of the input W_{in} with a complex Gaussian of variance e^{-2r}. Classical teleportation with $r=0$ then means the teleported mode has an excess noise of two units of vacuum $\frac{1}{2}+\frac{1}{2}$ compared to the input, as also discussed in the previous section. Any $r>0$ beats this classical scheme, i.e., if the input state is always recreated with the right amplitude and less than two units of vacuum excess noise, we may call this already quantum teleportation. Let us derive this result using the least noisy model for classical communication. For the input quadratures of Alice's sending station and the output quadratures at Bob's receiving station, the least noisy (linear) model if Alice and Bob are only classically communicating can be written as

$$\hat{x}_{\text{out},j}=\Gamma_x\hat{x}_{\text{in}}+\Gamma_x s_a^{-1}\hat{x}_a^{(0)}+s_{b,j}^{-1}\hat{x}_{b,j}^{(0)},$$
$$\hat{p}_{\text{out},j}=\Gamma_p\hat{p}_{\text{in}}-\Gamma_p s_a\hat{p}_a^{(0)}+s_{b,j}\hat{p}_{b,j}^{(0)}. \quad (8)$$

This model takes into account that Alice and Bob can only communicate via classical signals, since arbitrarily many copies of the output mode can be made by Bob where the subscript j labels the jth copy. In addition, it ensures that the output quadratures satisfy the commutation relations

$$[\hat{x}_{\text{out},j},\hat{p}_{\text{out},k}]=(i/2)\delta_{jk},$$
$$[\hat{x}_{\text{out},j},\hat{x}_{\text{out},k}]=[\hat{p}_{\text{out},j},\hat{p}_{\text{out},k}]=0. \quad (9)$$

Since we are only interested in one single copy of the output we drop the label j. The parameter s_a is given by Alice's measurement strategy and determines the noise penalty due to her homodyne detections. The gains Γ_x and Γ_p can be manipulated by Bob as well as the parameter s_b determining the noise distribution of Bob's original mode. The set of input states may contain pure Gaussian states with a coherent amplitude, described by $\hat{x}_{\text{in}}=\langle\hat{x}_{\text{in}}\rangle+s_v^{-1}\hat{x}^{(0)}$ and $\hat{p}_{\text{in}}=\langle\hat{p}_{\text{in}}\rangle+s_v\hat{p}^{(0)}$, where Victor can choose in each trial the coherent amplitude and if and to what extent the input is squeezed (parameter s_v). Since Bob always wants to reproduce the input amplitude, he is restricted to unit gain, symmetric in both quadratures $\Gamma_x=\Gamma_p=1$. First, after obtaining the output states from Bob, Victor verifies if their amplitudes match the corresponding input amplitudes. If not, all the following considerations concerning the excess noise are redundant, because Alice and Bob can always manipulate this noise by fiddling the gain (less than unit gain reduces the excess noise). If Victor finds overlapping amplitudes in all trials (at least within some error range), he looks at the excess noise in each trial. For that purpose, let us define the normalized variance

$$V_{\text{out,in}}^{\hat{x}}\equiv\frac{\langle\Delta(\hat{x}_{\text{out}}-\hat{x}_{\text{in}})^2\rangle}{\langle\Delta\hat{x}^2\rangle_{\text{vacuum}}}, \quad (10)$$

and analogously $V_{\text{out,in}}^{\hat{p}}$ with $\hat{x}\to\hat{p}$ throughout $[\langle\Delta\hat{O}^2\rangle\equiv\text{var}(\hat{O})]$. Using Eqs. (8) with unit gain, we obtain the product

$$V_{\text{out,in}}^{\hat{x}}V_{\text{out,in}}^{\hat{p}}=(s_a^{-2}+s_b^{-2})(s_a^2+s_b^2). \quad (11)$$

It is minimized for $s_a=s_b$, yielding $V^{\hat{x}}_{\text{out,in}}V^{\hat{p}}_{\text{out,in}}=4$. The optimum value of 4 is exactly the result we obtain for what we may call classical teleportation $V^{\hat{x}}_{\text{tel,in}}(r=0)\,V^{\hat{p}}_{\text{tel,in}}(r=0)=4$, using Eqs. (6) with subscript out→tel in Eq. (10). Thus, we can write our first "fundamental" limit for teleporting states with a coherent amplitude as

$$V^{\hat{x}}_{\text{out,in}}V^{\hat{p}}_{\text{out,in}}\geq V^{\hat{x}}_{\text{tel,in}}(r=0)\,V^{\hat{p}}_{\text{tel,in}}(r=0)=4. \tag{12}$$

If Victor, comparing the output states with the input states, always finds violations of this inequality, he may already have big confidence in Alice's and Bob's honesty (i.e., that they indeed have used entanglement). Equation (12) may also enable us already to assess if a scheme or protocol is capable of quantum teleportation. Alternatively, instead of looking at the products $V^{\hat{x}}_{\text{out,in}}V^{\hat{p}}_{\text{out,in}}$, we could also use the sums $V^{\hat{x}}_{\text{out,in}}+V^{\hat{p}}_{\text{out,in}}=s_a^{-2}+s_b^{-2}+s_a^2+s_b^2$ that are minimized for $s_a=s_b=1$. Then we find the classical boundary $V^{\hat{x}}_{\text{out,in}}+V^{\hat{p}}_{\text{out,in}}\geq 4$.

However, taking into account all the assumptions made for the derivation of Eq. (12), this boundary appears to be less fundamental. First, we have only assumed a linear model. Secondly, we have only considered the variances of two conjugate observables and a certain kind of measurement of these. An entirely rigorous criterion for quantum teleportation should take into account all possible variables, measurements and strategies that can be used by Alice and Bob. Another "problem" of our boundary Eq. (12) is that the variances $V_{\text{out,in}}$ are not directly measurable, because the input state is destroyed by the teleportation process. However, for Gaussian input states, Victor can combine his knowledge of the input variances V_{in} with the detected variances V_{out} in order to infer $V_{\text{out,in}}$. With a more specific set of Gaussian input states, namely coherent states, the least noisy model for classical communication allows us to determine the directly measurable "fundamental" limit for the normalized variances of the output states

$$V^{\hat{x}}_{\text{out}}V^{\hat{p}}_{\text{out}}\geq 9. \tag{13}$$

But still we need to bear in mind that we did not consider all possible strategies of Alice and Bob. Also for arbitrary s_v (set of input states contains all coherent and squeezed states), Eq. (13) represents a classical boundary, as

$$V^{\hat{x}}_{\text{out}}V^{\hat{p}}_{\text{out}}=(s_v^{-2}+s_a^{-2}+s_b^{-2})(s_v^2+s_a^2+s_b^2) \tag{14}$$

is minimized for $s_v=s_a=s_b$, yielding $V^{\hat{x}}_{\text{out}}V^{\hat{p}}_{\text{out}}=9$. However, since s_v is unknown to Alice and Bob in every trial, they can attain this classical minimum only by accident. For s_v fixed, e.g., $s_v=1$ (set of input states contains "only" coherent states), Alice and Bob knowing this s_v can always satisfy $V^{\hat{x}}_{\text{out}}V^{\hat{p}}_{\text{out}}=9$ in the classical model. Alternatively, the sums $V^{\hat{x}}_{\text{out}}+V^{\hat{p}}_{\text{out}}=s_v^{-2}+s_a^{-2}+s_b^{-2}+s_v^2+s_a^2+s_b^2$ are minimized with $s_a=s_b=1$. In this case, we obtain the s_v-dependent boundary $V^{\hat{x}}_{\text{out}}+V^{\hat{p}}_{\text{out}}\geq s_v^{-2}+s_v^2+4$. Without knowing s_v, Alice and Bob can always attain this minimum in the classical model. In every trial, Victor must combine his knowledge of s_v with the detected output variances in order to find violations of this sum inequality.

Ralph and Lam [24] define the classical boundaries

$$V^{\hat{x}}_{\text{c}}+V^{\hat{p}}_{\text{c}}\geq 2 \tag{15}$$

and

$$T^{\hat{x}}_{\text{out}}+T^{\hat{p}}_{\text{out}}\leq 1, \tag{16}$$

using the conditional variance

$$V^{\hat{x}}_{\text{c}}\equiv\frac{\langle\Delta\hat{x}^2_{\text{out}}\rangle}{\langle\Delta\hat{x}^2\rangle_{\text{vacuum}}}\left(1-\frac{|\langle\Delta\hat{x}_{\text{out}}\Delta\hat{x}_{\text{in}}\rangle|^2}{\langle\Delta\hat{x}^2_{\text{out}}\rangle\langle\Delta\hat{x}^2_{\text{in}}\rangle}\right), \tag{17}$$

and analogously for $V^{\hat{p}}_{\text{c}}$ with $\hat{x}\to\hat{p}$ throughout, and the transfer coefficient

$$T^{\hat{x}}_{\text{out}}\equiv\frac{\mathcal{S}^{\hat{x}}_{\text{out}}}{\mathcal{S}^{\hat{x}}_{\text{in}}}, \tag{18}$$

and analogously $T^{\hat{p}}_{\text{out}}$ with $\hat{x}\to\hat{p}$ throughout. Here, $\mathcal{S}$ denotes the signal to noise ratio for the square of the mean amplitudes, namely $\mathcal{S}^{\hat{x}}_{\text{out}}=\langle\hat{x}_{\text{out}}\rangle^2/\langle\Delta\hat{x}^2_{\text{out}}\rangle$.

Alice and Bob using only classical communication are not able to violate *either* of the two inequalities Eq. (15) and Eq. (16). In fact, these boundaries are two independent limits, each of them unexceedable in a classical scheme. However, Alice and Bob can simultaneously approach $V^{\hat{x}}_{\text{c}}+V^{\hat{p}}_{\text{c}}=2$ and $T^{\hat{x}}_{\text{out}}+T^{\hat{p}}_{\text{out}}=1$ using either an asymmetric classical detection and transmission scheme with coherent-state inputs or a symmetric classical scheme with squeezed-state inputs [24]. For quantum teleportation, Ralph and Lam [24] require their classical limits be simultaneously exceeded, $V^{\hat{x}}_{\text{c}}+V^{\hat{p}}_{\text{c}}<2$ and $T^{\hat{x}}_{\text{out}}+T^{\hat{p}}_{\text{out}}>1$. This is only possible using more than 3 dB squeezing in the entanglement source [24]. Apparently, these criteria determine a classical boundary different from ours in Eq. (12). For example, in unit-gain teleportation, our inequality Eq. (12) is violated for any nonzero squeezing $r>0$. Let us briefly explain why we encounter this discrepancy. We have a priori assumed unit gain in our scheme to achieve outputs and inputs overlapping in their mean values. This assumption is, of course, motivated by the assessment that good teleportation means good similarity between input and output *states* (here, to be honest, we already have something in mind similar to the fidelity, introduced in the next section). First, Victor has to check the match of the amplitudes before looking at the variances. Ralph and Lam permit arbitrary gain, because they are not interested in the similarity of input and output *states*, but in certain correlations that manifest separately in the individual quadratures [25]. This point of view originates from the context of quantum nondemolition (QND) measurements [26], which are focused on a single QND variable while the conjugate variable is not of interest. For arbitrary gain, an inequality as in Eq. (16), containing the input and output mean values, has to be added to an inequality only for variances as in Eq. (15). Ralph and Lam's *best* classical protocol permits output states completely different from the input states, e.g., via asymmetric detection where the lack of information in one quadrature leads on average to output states with amplitudes completely different from the input states. The asymmetric scheme means that Alice is *not* attempting to gain as much information about the *quantum state* as possible, as in an Arthurs-Kelly measurement [21]. The Arthurs-Kelly measurement, however, is exactly what Alice should do in our *best* classical protocol, i.e., classical teleportation. Therefore, our best classical protocol always achieves output states already pretty similar to the input states. Apparently, "the best" that can be classically achieved has a different meaning from Ralph and Lam's point of view and from ours. Then it is no surprise that the classical boundaries differ as well. Apart from these differences, however, Ralph and Lam's criteria do have something in common with our criterion given by Eq. (12): they also do not satisfy the rigor we require from criteria for quantum teleportation taking into account everything Alice and Bob can do. By limiting the set of input states to coherent states, we are able to present such a rigorous criterion in the next section.

B. The fidelity criterion for coherent-state teleportation

The rigorous criterion we are looking for to determine the best classical teleportation and to quantify the distinction between classical and quantum teleportation relies on the fidelity F, for an arbitrary input state $|\psi_{\rm in}\rangle$ defined by [13]

$$F \equiv \langle \psi_{\rm in}|\hat{\rho}_{\rm out}|\psi_{\rm in}\rangle. \tag{19}$$

It is an excellent measure for the similarity between the input and the output state and equals one only if $\hat{\rho}_{\rm out}=|\psi_{\rm in}\rangle\langle\psi_{\rm in}|$. Now Alice and Bob know that Victor draws his states $|\psi_{\rm in}\rangle$ from a fixed set, but they do not know which particular state is drawn in a single trial. Therefore, an average fidelity should be considered [13],

$$F_{\rm av}=\int P(|\psi_{\rm in}\rangle)\langle\psi_{\rm in}|\hat{\rho}_{\rm out}|\psi_{\rm in}\rangle d|\psi_{\rm in}\rangle, \tag{20}$$

where $P(|\psi_{\rm in}\rangle)$ is the probability of drawing a particular state $|\psi_{\rm in}\rangle$, and the integral runs over the entire set of input states. If the set of input states contains simply all possible quantum states in an infinite-dimensional Hilbert space (i.e., the input state is completely unknown apart from the Hilbert-space dimension), the best average fidelity achievable without entanglement is zero. If the set of input states is restricted to coherent states of amplitude $\alpha_{\rm in}=x_{\rm in}+ip_{\rm in}$ and $F=\langle\alpha_{\rm in}|\hat{\rho}_{\rm out}|\alpha_{\rm in}\rangle$, on average, the fidelity achievable in a purely classical scheme (when averaged across the entire complex plane) is bounded by [13]

$$F_{\rm av}\leqslant\frac{1}{2}. \tag{21}$$

Let us illustrate these nontrivial results with our single-mode teleportation equations. Up to a factor π, the fidelity $F=\langle\alpha_{\rm in}|\hat{\rho}_{\rm tel}|\alpha_{\rm in}\rangle$ is the Q function of the teleported mode evaluated for $\alpha_{\rm in}$:

$$F=\pi Q_{\rm tel}(\alpha_{\rm in})=\frac{1}{2\sqrt{\sigma_x\sigma_p}}\exp\left[-(1-\Gamma)^2\left(\frac{x_{\rm in}^2}{2\sigma_x}+\frac{p_{\rm in}^2}{2\sigma_p}\right)\right], \tag{22}$$

where Γ is the gain from the previous sections and σ_x and σ_p are the variances of the Q function of the teleported mode for the corresponding quadratures. These variances are according to Eqs. (7) for a coherent-state input and $\langle\Delta\hat{x}^2\rangle_{\rm vacuum}=\langle\Delta\hat{p}^2\rangle_{\rm vacuum}=\frac{1}{4}$ given by

$$\sigma_x=\sigma_p=\frac{1}{4}(1+\Gamma^2)+\frac{e^{2r}}{8}(\Gamma-1)^2+\frac{e^{-2r}}{8}(\Gamma+1)^2. \tag{23}$$

For classical teleportation $(r=0)$ and $\Gamma=1$, we obtain $\sigma_x=\sigma_p=\frac{1}{2}+\frac{1}{4}V^{\hat{x}}_{\rm tel,in}(r=0)=\frac{1}{2}+\frac{1}{4}V^{\hat{p}}_{\rm tel,in}(r=0)=\frac{1}{2}+\frac{1}{2}=1$ and indeed $F=F_{\rm av}=\frac{1}{2}$. In order to obtain a better fidelity, entanglement is necessary. Then, if $\Gamma=1$, we obtain $F=F_{\rm av}>\frac{1}{2}$ for any $r>0$. For $r=0$, the fidelity drops to zero as $\Gamma\to\infty$ since the mean amplitude of the teleported state does not match that of the input state and the excess noise increases. For $r=0$ and $\Gamma=0$, the fidelity becomes $F=\exp(-|\alpha_{\rm in}|^2)$. Upon averaging over all possible coherent-state inputs, this fidelity also vanishes. Assuming nonunit gain, it is crucial to consider the average fidelity $F_{\rm av}\neq F$. When averaging across the entire complex plane, any nonunit gain yields $F_{\rm av}=0$. This is exactly why Victor should first check the match of the amplitudes for different input states. If Alice and Bob are cheating and fiddle the gain in a classical scheme, a sufficiently large input amplitude reveals the truth. These considerations also apply to the asymmetric classical detection and transmission scheme with a coherent-state input [24] discussed in the previous section. Of course, the asymmetric scheme does not provide an improvement in the fidelity. In fact, the average fidelity drops to zero, if Alice detects only one quadrature (and gains complete information about this quadrature) and Bob obtains the full information about the measured quadrature, but no information about the second quadrature. In an asymmetric classical scheme, Alice and Bob stay far within the classical domain $F_{\rm av}<\frac{1}{2}$. The best classical scheme with respect to the fidelity is the symmetric one ("classical teleportation") with $F_{\rm av}=\frac{1}{2}$.

The supposed limitation of the fidelity criterion that the set of input states contains "only" coherent states is compensated by having an entirely rigorous criterion. Of course, the fidelity criterion does not limit the possible input states for which the presented protocol works. It does not mean we can only teleport coherent states (as we will clearly see in the next section). However, so far, it is the only criterion that enables the experimentalist to rigorously verify quantum teleportation. That is why Furusawa *et al.* [12] were happy to have used coherent-state inputs, because they could rely on a

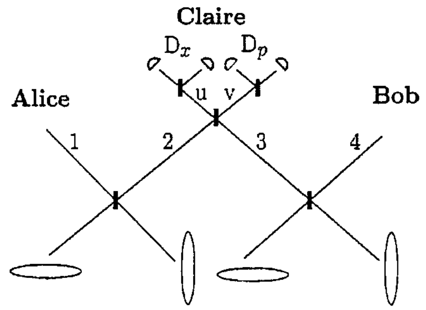

FIG. 3. Entanglement swapping using the two entangled two-mode squeezed vacuum states of modes 1 and 2 (shared by Alice and Claire) and of modes 3 and 4 (shared by Claire and Bob) as in Ref. [20].

strict and rigorous criterion (and not only because coherent states are the most readily available source for the state preparer Victor).

C. Teleporting entangled states: entanglement swapping

From a true quantum teleportation device, we require that it can not only teleport nonorthogonal states very similar to classical states (such as coherent states), but also extremely nonclassical states such as entangled states. When teleporting one half of an entangled state (''entanglement swapping''), we are certainly much more interested in the preservation of the inseparability than in the match of any input and output amplitudes. We can say that entanglement swapping is successful, if the initially unentangled modes become entangled via the teleportation process (even, if this is accompanied by a decrease of the quality of the initial entanglement). In Ref. [20] has been shown, that the single-mode teleportation scheme enables entanglement swapping for any nonzero squeezing ($r>0$) in the two initial entangled states (of which one provides the teleporter's input and the other one the EPR channel or vice versa).

Let us introduce ''Claire'' who performs the Bell detection of modes 2 and 3 (Fig. 3). Before her measurement, mode 1 (Alice's mode) is entangled with mode 2, and mode 3 is entangled with mode 4 (Bob's mode) [20]. Due to Claire's detection, mode 1 and 4 are projected on entangled states. Entanglement is teleported in every single projection (for every measured value of x_u and p_v) without any further local displacement [27]. How can we verify that entanglement swapping was successful? Simply, by verifying that Alice and Bob, who initially did not share any entanglement, are able to perform quantum teleportation using mode 1 and 4 after entanglement swapping [20]. But then we urgently need a rigorous criterion for quantum teleportation that unambigously recognizes when Alice and Bob have used entanglement and when they have not. Now, again, we can rely on the fidelity criterion for coherent-state teleportation. Alice and Bob again have to convince Victor that they are using entanglement and are not cheating. Of course, this is only a reliable verification scheme of entanglement swapping, if one can be sure that Alice and Bob did not share entanglement prior to entanglement swapping and that Claire is not allowed to perform unit-gain displacements (or that Claire is not allowed to receive any classical information). Otherwise, Victor's coherent-state input could be teleported step by step from Alice to Claire (with unit gain) and from Claire to Bob (with unit gain). This protocol, however, requires more than 3 dB squeezing in both entanglement sources (if equally squeezed) to ensure $F_{\rm av}>\frac{1}{2}$ [20]. Using entanglement swapping, Alice and Bob can achieve $F_{\rm av}>\frac{1}{2}$ for any squeezing, but one of them has to perform local displacements based on Claire's measurement results. Any gain is allowed in these displacements, since in entanglement swapping, we are not interested in the transfer of coherent amplitudes (and the two initial two-mode squeezed states are vacuum states anyway). But only the optimum gain $\Gamma_{\rm swap}=\tanh 2r$ ensures $F_{\rm av}>\frac{1}{2}$ for any squeezing and provides the optimum fidelity [20]. Unit gain $\Gamma_{\rm swap}=1$ in entanglement swapping would require more than 3 dB squeezing in both entanglement sources (if equally squeezed) to achieve $F_{\rm av}>\frac{1}{2}$ [20], or to confirm the teleportation of entanglement via detection of the combined entangled modes [19].

We will also give a broadband protocol of entanglement swapping as a ''nonunit-gain teleportation.'' The verification of entanglement swapping via the fidelity criterion for coherent-state teleportation demonstrates how useful this criterion is. Less rigorous criteria, as presented in Sec. III A, cannot reliably tell us if Alice and Bob use entanglement emerging from entanglement swapping. Furthermore, the entanglement swapping scheme demonstrates that a two-mode squeezed state enables *true* quantum teleportation for any nonzero squeezing. Requiring more than 3 dB squeezing, as it is necessary for quantum teleportation according to Ralph and Lam [24], is not necessary for the teleporation of entanglement.

IV. BROADBAND ENTANGLEMENT

In this section, we demonstrate that the EPR state required for broadband teleportation can be generated either directly by nondegenerate parametric down conversion or by combining two independently squeezed fields produced via degenerate down conversion or any other nonlinear interaction.

First, we review the results of Ref. [15] based on the input-output formalism of Collett and Gardiner [28] where a nondegenerate optical parametric amplifier in a cavity (NOPA) is studied. We will see that the upper and lower sidebands of the NOPA output have correlations similar to those of the two-mode squeezed state in Eqs. (2). The optical parametric oscillator is considered polarization nondegenerate but frequency ''degenerate'' (equal center frequency for the orthogonally polarized output modes). The interaction between the two modes is due to the nonlinear $\chi^{(2)}$ medium (in a cavity) and may be described by the interaction Hamiltonian

$$\hat{H}_I=i\hbar\kappa(\hat{a}_1^\dagger\hat{a}_2^\dagger e^{-2i\omega_0 t}-\hat{a}_1\hat{a}_2 e^{2i\omega_0 t}). \tag{24}$$

The undepleted pump field amplitude at frequency $2\omega_0$ is described as a c number and has been absorbed into the coupling κ which also contains the $\chi^{(2)}$ susceptibility. Without loss of generality κ can be taken to be real. The dynam-

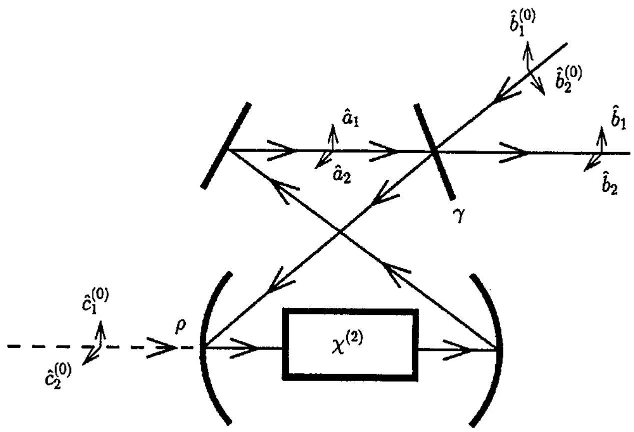

FIG. 4. The NOPA as in Ref. [15]. The two cavity modes $\hat{a}_1$ and $\hat{a}_2$ interact due to the nonlinear $\chi^{(2)}$ medium. The modes $\hat{b}_1^{(0)}$ and $\hat{b}_2^{(0)}$ are the external vacuum input modes, $\hat{b}_1$ and $\hat{b}_2$ are the external output modes, $\hat{c}_1^{(0)}$ and $\hat{c}_2^{(0)}$ are the vacuum modes due to cavity losses, γ is a damping rate and ρ is a loss parameter of the cavity.

ics of the two cavity modes $\hat{a}_1$ and $\hat{a}_2$ are governed by the above interaction Hamiltonian, and input-output relations can be derived relating the cavity modes to the external vacuum input modes $\hat{b}_1^{(0)}$ and $\hat{b}_2^{(0)}$, the external output modes $\hat{b}_1$ and $\hat{b}_2$, and two unwanted vacuum modes $\hat{c}_1^{(0)}$ and $\hat{c}_2^{(0)}$ describing cavity losses (Fig. 4). Recall, the superscript (0) refers to vacuum modes. We define uppercase operators in the rotating frame about the center frequency ω_0,

$$\hat{O}(t)=\hat{o}(t)e^{i\omega_0 t}, \tag{25}$$

with $\hat{O}=[\hat{A}_{1,2};\hat{B}_{1,2};\hat{B}_{1,2}^{(0)};\hat{C}_{1,2}^{(0)}]$ and the full Heisenberg operators $\hat{o}=[\hat{a}_{1,2};\hat{b}_{1,2};\hat{b}_{1,2}^{(0)};\hat{c}_{1,2}^{(0)}]$. By the Fourier transformation

$$\hat{O}(\Omega)=\frac{1}{\sqrt{2\pi}}\int dt\,\hat{O}(t)e^{i\Omega t}, \tag{26}$$

the fields are now described as functions of the modulation frequency Ω with commutation relation $[\hat{O}(\Omega),\hat{O}^\dagger(\Omega')]=\delta(\Omega-\Omega')$ for $\hat{B}_{1,2}$, $\hat{B}_{1,2}^{(0)}$ and $\hat{C}_{1,2}^{(0)}$ since $[\hat{O}(t),\hat{O}^\dagger(t')]=\delta(t-t')$. Expressing the outgoing modes in terms of the incoming vacuum modes, one obtains [15]

$$\hat{B}_j(\Omega)=G(\Omega)\hat{B}_j^{(0)}(\Omega)+g(\Omega)\hat{B}_k^{(0)\dagger}(-\Omega)+\bar{G}(\Omega)\hat{C}_j^{(0)}(\Omega)+\bar{g}(\Omega)\hat{C}_k^{(0)\dagger}(-\Omega), \tag{27}$$

where $k=3-j$, $j=1,2$ (so k refers to the opposite mode to j), and with coefficients to be specified later. The two cavity modes have been assumed to be both on resonance with half the pump frequency at ω_0.

Let us investigate the lossless case where the output fields become

$$\hat{B}_j(\Omega)=G(\Omega)\hat{B}_j^{(0)}(\Omega)+g(\Omega)\hat{B}_k^{(0)\dagger}(-\Omega), \tag{28}$$

with the functions $G(\Omega)$ and $g(\Omega)$ of Eq. (27) simplifying to

$$G(\Omega)=\frac{\kappa^2+\gamma^2/4+\Omega^2}{(\gamma/2-i\Omega)^2-\kappa^2},\qquad g(\Omega)=\frac{\kappa\gamma}{(\gamma/2-i\Omega)^2-\kappa^2}. \tag{29}$$

Here, the parameter γ is a damping rate of the cavity (Fig. 4) and is assumed to be equal for both polarizations. Equation (28) represents the input-output relations for a lossless NOPA.

Following Ref. [29], we introduce frequency resolved quadrature amplitudes given by

$$\begin{aligned}\hat{X}_j(\Omega)&=\frac{1}{2}[\hat{B}_j(\Omega)+\hat{B}_j^\dagger(-\Omega)],\\ \hat{P}_j(\Omega)&=\frac{1}{2i}[\hat{B}_j(\Omega)-\hat{B}_j^\dagger(-\Omega)],\\ \hat{X}_j^{(0)}(\Omega)&=\frac{1}{2}[\hat{B}_j^{(0)}(\Omega)+\hat{B}_j^{(0)\dagger}(-\Omega)],\\ \hat{P}_j^{(0)}(\Omega)&=\frac{1}{2i}[\hat{B}_j^{(0)}(\Omega)-\hat{B}_j^{(0)\dagger}(-\Omega)],\end{aligned} \tag{30}$$

provided $\Omega\ll\omega_0$. Using them Eq. (28) becomes

$$\begin{aligned}\hat{X}_j(\Omega)&=G(\Omega)\hat{X}_j^{(0)}(\Omega)+g(\Omega)\hat{X}_k^{(0)}(\Omega),\\ \hat{P}_j(\Omega)&=G(\Omega)\hat{P}_j^{(0)}(\Omega)-g(\Omega)\hat{P}_k^{(0)}(\Omega).\end{aligned} \tag{31}$$

Here, we have used $G(\Omega)=G^*(-\Omega)$ and $g(\Omega)=g^*(-\Omega)$.

At this juncture, we show that the output quadratures of a lossless NOPA in Eqs. (31) correspond to two independently squeezed modes coupled to a two-mode squeezed state at a beam splitter. The operational significance of this fact is that the EPR state required for broadband teleportation can be created either by nondegenerate parametric down conversion as described by the interaction Hamiltonian in Eq. (24), or by combining at a beam splitter two independently squeezed fields generated via degenerate down conversion [30] (as done in the teleportation experiment of Ref. [12]).

Let us thus define the superpositions of the two output modes (barred quantities)

$$\begin{aligned}\hat{\bar{B}}_1&\equiv\frac{1}{\sqrt{2}}(\hat{B}_1+\hat{B}_2),\\ \hat{\bar{B}}_2&\equiv\frac{1}{\sqrt{2}}(\hat{B}_1-\hat{B}_2),\end{aligned} \tag{32}$$

and of the two vacuum input modes

$$\hat{\tilde{B}}_1^{(0)} \equiv \frac{1}{\sqrt{2}}(\hat{B}_1^{(0)}+\hat{B}_2^{(0)}),$$
$$\hat{\tilde{B}}_2^{(0)} \equiv \frac{1}{\sqrt{2}}(\hat{B}_1^{(0)}-\hat{B}_2^{(0)}). \quad (33)$$

In terms of these superpositions, Eq. (28) becomes

$$\hat{\tilde{B}}_1(\Omega)=G(\Omega)\hat{\tilde{B}}_1^{(0)}(\Omega)+g(\Omega)\hat{\tilde{B}}_1^{(0)\dagger}(-\Omega),$$
$$\hat{\tilde{B}}_2(\Omega)=G(\Omega)\hat{\tilde{B}}_2^{(0)}(\Omega)-g(\Omega)\hat{\tilde{B}}_2^{(0)\dagger}(-\Omega). \quad (34)$$

In Eqs. (34), the initially coupled modes of Eq. (28) are decoupled, corresponding to two independent degenerate parametric amplifiers.

In the limit $\Omega\to 0$, the two modes of Eqs. (34) are each in the same single-mode squeezed state as the two modes in Eqs. (1). More explicitly, by setting $G(0)=\cosh r$ and $g(0)=\sinh r$, the annihilation operators

$$\hat{\tilde{B}}_1=\cosh r\hat{\tilde{B}}_1^{(0)}+\sinh r\hat{\tilde{B}}_1^{(0)\dagger},$$
$$\hat{\tilde{B}}_2=\cosh r\hat{\tilde{B}}_2^{(0)}-\sinh r\hat{\tilde{B}}_2^{(0)\dagger}, \quad (35)$$

have the quadrature operators

$$\hat{\tilde{X}}_1=e^{r}\hat{\tilde{X}}_1^{(0)}, \quad \hat{\tilde{P}}_1=e^{-r}\hat{\tilde{P}}_1^{(0)},$$
$$\hat{\tilde{X}}_2=e^{-r}\hat{\tilde{X}}_2^{(0)}, \quad \hat{\tilde{P}}_2=e^{r}\hat{\tilde{P}}_2^{(0)}. \quad (36)$$

From the alternative perspective of superimposing two independently squeezed modes at a 50/50 beam splitter to obtain the EPR state, we must simply invert the transformation of Eqs. (32) and recouple the two modes

$$\hat{B}_1=\frac{1}{\sqrt{2}}(\hat{\tilde{B}}_1+\hat{\tilde{B}}_2)=\frac{1}{\sqrt{2}}[\cosh r(\hat{\tilde{B}}_1^{(0)}+\hat{\tilde{B}}_2^{(0)})+\sinh r(\hat{\tilde{B}}_1^{(0)\dagger}-\hat{\tilde{B}}_2^{(0)\dagger})]=\cosh r\hat{B}_1^{(0)}+\sinh r\hat{B}_2^{(0)\dagger},$$
$$\hat{B}_2=\frac{1}{\sqrt{2}}(\hat{\tilde{B}}_1-\hat{\tilde{B}}_2)=\frac{1}{\sqrt{2}}[\cosh r(\hat{\tilde{B}}_1^{(0)}-\hat{\tilde{B}}_2^{(0)})+\sinh r(\hat{\tilde{B}}_1^{(0)\dagger}+\hat{\tilde{B}}_2^{(0)\dagger})]=\cosh r\hat{B}_2^{(0)}+\sinh r\hat{B}_1^{(0)\dagger}, \quad (37)$$

and

$$\hat{X}_1=\frac{1}{\sqrt{2}}(\hat{\tilde{X}}_1+\hat{\tilde{X}}_2)=\frac{1}{\sqrt{2}}(e^{r}\hat{\tilde{X}}_1^{(0)}+e^{-r}\hat{\tilde{X}}_2^{(0)}),$$
$$\hat{P}_1=\frac{1}{\sqrt{2}}(\hat{\tilde{P}}_1+\hat{\tilde{P}}_2)=\frac{1}{\sqrt{2}}(e^{-r}\hat{\tilde{P}}_1^{(0)}+e^{r}\hat{\tilde{P}}_2^{(0)}),$$
$$\hat{X}_2=\frac{1}{\sqrt{2}}(\hat{\tilde{X}}_1-\hat{\tilde{X}}_2)=\frac{1}{\sqrt{2}}(e^{r}\hat{\tilde{X}}_1^{(0)}-e^{-r}\hat{\tilde{X}}_2^{(0)}),$$
$$\hat{P}_2=\frac{1}{\sqrt{2}}(\hat{\tilde{P}}_1-\hat{\tilde{P}}_2)=\frac{1}{\sqrt{2}}(e^{-r}\hat{\tilde{P}}_1^{(0)}-e^{r}\hat{\tilde{P}}_2^{(0)}), \quad (38)$$

as the two-mode squeezed state in Eqs. (2). The coupled modes in Eqs. (37) expressed in terms of $\hat{B}_1^{(0)}$ and $\hat{B}_2^{(0)}$ are the two NOPA output modes of Eq. (28), if $\Omega\to 0$ and $G(0)=\cosh r$, $g(0)=\sinh r$.

More generally, for $\Omega\neq 0$, the quadratures corresponding to Eqs. (34),

$$\hat{\tilde{X}}_1(\Omega)=[G(\Omega)+g(\Omega)]\hat{\tilde{X}}_1^{(0)}(\Omega),$$
$$\hat{\tilde{P}}_1(\Omega)=[G(\Omega)-g(\Omega)]\hat{\tilde{P}}_1^{(0)}(\Omega),$$
$$\hat{\tilde{X}}_2(\Omega)=[G(\Omega)-g(\Omega)]\hat{\tilde{X}}_2^{(0)}(\Omega),$$
$$\hat{\tilde{P}}_2(\Omega)=[G(\Omega)+g(\Omega)]\hat{\tilde{P}}_2^{(0)}(\Omega), \quad (39)$$

are coupled to yield

$$\hat{X}_1(\Omega)=\frac{1}{\sqrt{2}}[G(\Omega)+g(\Omega)]\hat{\tilde{X}}_1^{(0)}(\Omega)+\frac{1}{\sqrt{2}}[G(\Omega)-g(\Omega)]\hat{\tilde{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_1(\Omega)=\frac{1}{\sqrt{2}}[G(\Omega)-g(\Omega)]\hat{\tilde{P}}_1^{(0)}(\Omega)+\frac{1}{\sqrt{2}}[G(\Omega)+g(\Omega)]\hat{\tilde{P}}_2^{(0)}(\Omega),$$
$$\hat{X}_2(\Omega)=\frac{1}{\sqrt{2}}[G(\Omega)+g(\Omega)]\hat{\tilde{X}}_1^{(0)}(\Omega)-\frac{1}{\sqrt{2}}[G(\Omega)-g(\Omega)]\hat{\tilde{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_2(\Omega)=\frac{1}{\sqrt{2}}[G(\Omega)-g(\Omega)]\hat{\tilde{P}}_1^{(0)}(\Omega)-\frac{1}{\sqrt{2}}[G(\Omega)+g(\Omega)]\hat{\tilde{P}}_2^{(0)}(\Omega). \quad (40)$$

The quadratures in Eqs. (40) are precisely the NOPA output quadratures of Eqs. (31) as anticipated. With the functions $G(\Omega)$ and $g(\Omega)$ of Eqs. (29), we obtain

$$G(\Omega)-g(\Omega)=\frac{\gamma/2-\kappa+i\Omega}{\gamma/2+\kappa-i\Omega},$$
$$G(\Omega)+g(\Omega)=\frac{(\gamma/2+\kappa)^2+\Omega^2}{(\gamma/2-i\Omega)^2-\kappa^2}. \tag{41}$$

For the limits $\Omega\to 0$, $\kappa\to\gamma/2$ (the limit of infinite squeezing), we obtain $[G(\Omega)-g(\Omega)]\to 0$ and $[G(\Omega)+g(\Omega)]\to\infty$. If $\Omega\to 0$, $\kappa\to 0$ (the classical limit of no squeezing), then $[G(\Omega)-g(\Omega)]\to 1$ and $[G(\Omega)+g(\Omega)]\to 1$. Thus for $\Omega\to 0$, Eqs. (40) in the above-mentioned limits correspond to Eqs. (38) in the analogous limits $r\to\infty$ (infinite squeezing) and $r\to 0$ (no squeezing). For large squeezing, apparently the individual modes of the "broadband two-mode squeezed state" in Eqs. (40) are very noisy. In general, the input vacuum modes are amplified in the NOPA, resulting in output modes with large fluctuations. But the correlations between the two modes increase simultaneously, so that $[\hat{X}_1(\Omega)-\hat{X}_2(\Omega)]\to 0$ and $[\hat{P}_1(\Omega)+\hat{P}_2(\Omega)]\to 0$ for $\Omega\to 0$ and $\kappa\to\gamma/2$.

The squeezing spectra of the independently squeezed modes can be derived from Eqs. (39) and are given by the spectral variances

$$\langle\Delta\hat{\hat{X}}_1^\dagger(\Omega)\Delta\hat{\hat{X}}_1(\Omega')\rangle=\langle\Delta\hat{\hat{P}}_2^\dagger(\Omega)\Delta\hat{\hat{P}}_2(\Omega')\rangle =\delta(\Omega-\Omega')|S_+(\Omega)|^2\langle\Delta\hat{X}^2\rangle_{\text{vacuum}}, \tag{42}$$
$$\langle\Delta\hat{\hat{X}}_2^\dagger(\Omega)\Delta\hat{\hat{X}}_2(\Omega')\rangle=\langle\Delta\hat{\hat{P}}_1^\dagger(\Omega)\Delta\hat{\hat{P}}_1(\Omega')\rangle =\delta(\Omega-\Omega')|S_-(\Omega)|^2\langle\Delta\hat{X}^2\rangle_{\text{vacuum}},$$

here with $|S_+(\Omega)|^2=|G(\Omega)+g(\Omega)|^2$ and $|S_-(\Omega)|^2=|G(\Omega)-g(\Omega)|^2$ ($\langle\Delta\hat{X}^2\rangle_{\text{vacuum}}=\frac{1}{4}$). In general, Eqs. (42) may define arbitrary squeezing spectra of two statistically identical but independent broadband squeezed states. The two corresponding squeezed modes

$$\hat{\hat{X}}_1(\Omega)=S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega),\quad \hat{\hat{P}}_1(\Omega)=S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega),$$
$$\hat{\hat{X}}_2(\Omega)=S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega),\quad \hat{\hat{P}}_2(\Omega)=S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega), \tag{43}$$

where $S_-(\Omega)$ refers to the quiet quadratures and $S_+(\Omega)$ to the noisy ones, can be used as EPR source for the following broadband teleportation scheme when they are combined at a beam splitter:

$$\hat{X}_1(\Omega)=\frac{1}{\sqrt{2}}S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega)+\frac{1}{\sqrt{2}}S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_1(\Omega)=\frac{1}{\sqrt{2}}S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega)+\frac{1}{\sqrt{2}}S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega),$$
$$\hat{X}_2(\Omega)=\frac{1}{\sqrt{2}}S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega)-\frac{1}{\sqrt{2}}S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_2(\Omega)=\frac{1}{\sqrt{2}}S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega)-\frac{1}{\sqrt{2}}S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega). \tag{44}$$

Before obtaining this "broadband two-mode squeezed vacuum state," the squeezing of the two initial modes may be generated by any nonlinear interaction, e.g., apart from the OPA, also by four-wave mixing in a cavity [31].

V. TELEPORTATION OF A BROADBAND FIELD

For the teleportation of an electromagnetic field with finite bandwidth, the EPR state shared by Alice and Bob should be a broadband two-mode squeezed state, as discussed in the previous section. The incoming electromagnetic field to be teleported $\hat{E}_{\text{in}}(z,t)=\hat{E}_{\text{in}}^{(+)}(z,t)+\hat{E}_{\text{in}}^{(-)}(z,t)$, traveling in positive-z direction and having a single polarization, can be described by the positive-frequency part

$$\hat{E}_{\text{in}}^{(+)}(z,t)=[\hat{E}_{\text{in}}^{(-)}(z,t)]^\dagger =\int_W d\omega\frac{1}{\sqrt{2\pi}}\left(\frac{u\hbar\omega}{2cA_{\text{tr}}}\right)^{1/2}\hat{b}_{\text{in}}(\omega)e^{-i\omega(t-z/c)}. \tag{45}$$

The integral runs over a relevant bandwidth W centered on ω_0, A_{tr} represents the transverse structure of the field and u is a units-dependent constant (in Gaussian units $u=4\pi$) [29]. The annihilation and creation operators $\hat{b}_{\text{in}}(\omega)$ and $\hat{b}_{\text{in}}^\dagger(\omega)$ satisfy the commutation relations $[\hat{b}_{\text{in}}(\omega),\hat{b}_{\text{in}}(\omega')]=0$ and $[\hat{b}_{\text{in}}(\omega),\hat{b}_{\text{in}}^\dagger(\omega')]=\delta(\omega-\omega')$. The incoming electromagnetic field may now be described in a rotating frame as

$$\hat{B}_{\text{in}}(t)=\hat{X}_{\text{in}}(t)+i\hat{P}_{\text{in}}(t)=[\hat{x}_{\text{in}}(t)+i\hat{p}_{\text{in}}(t)]e^{i\omega_0 t}=\hat{b}_{\text{in}}(t)e^{i\omega_0 t}, \tag{46}$$

as in Eq. (25) with

$$\hat{B}_{\text{in}}(\Omega)=\frac{1}{\sqrt{2\pi}}\int dt\hat{B}_{\text{in}}(t)e^{i\Omega t}, \tag{47}$$

as in Eq. (26) and commutation relations $[\hat{B}_{\text{in}}(\Omega),\hat{B}_{\text{in}}(\Omega')]=0$, $[\hat{B}_{\text{in}}(\Omega),\hat{B}_{\text{in}}^\dagger(\Omega')]=\delta(\Omega-\Omega')$.

Of course, the unknown input field is not completely arbitrary. In the case of an EPR state from the NOPA, we will see that for successful quantum teleportation, the center of the input field's spectral range W should be around the NOPA center frequency ω_0 (half the pump frequency of the NOPA). Further, as we shall see, its spectral width should be small with respect to the NOPA bandwidth to benefit from the EPR correlations of the NOPA output. As for the transverse structure and the single polarization of the input field, we assume that both are known to all participants.

In spite of these complications, the teleportation protocol is performed in a fashion almost identical to the zero-bandwidth case. The EPR state of modes 1 and 2 is produced either directly as the NOPA output or by the superposition of two independently squeezed beams, as discussed in the preceding section. Mode 1 is sent to Alice and mode 2 is sent to Bob (see Fig. 1) where for the case of the NOPA, these modes correspond to two orthogonal polarizations. Alice arranges to superimpose mode 1 with the unknown input field at a 50/50 beam splitter, yielding for the relevant quadratures

$$\hat{X}_u(\Omega)=\frac{1}{\sqrt{2}}\hat{X}_{\rm in}(\Omega)-\frac{1}{\sqrt{2}}\hat{X}_1(\Omega),$$
$$\hat{P}_v(\Omega)=\frac{1}{\sqrt{2}}\hat{P}_{\rm in}(\Omega)+\frac{1}{\sqrt{2}}\hat{P}_1(\Omega). \tag{48}$$

Using Eqs. (48) we will find it useful to write the quadrature operators of Bob's mode 2 as

$$\hat{X}_2(\Omega)=\hat{X}_{\rm in}(\Omega)-[\hat{X}_1(\Omega)-\hat{X}_2(\Omega)]-\sqrt{2}\hat{X}_u(\Omega)$$
$$=\hat{X}_{\rm in}(\Omega)-\sqrt{2}S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega)-\sqrt{2}\hat{X}_u(\Omega),$$
$$\hat{P}_2(\Omega)=\hat{P}_{\rm in}(\Omega)+[\hat{P}_1(\Omega)+\hat{P}_2(\Omega)]-\sqrt{2}\hat{P}_v(\Omega)$$
$$=\hat{P}_{\rm in}(\Omega)+\sqrt{2}S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega)-\sqrt{2}\hat{P}_v(\Omega). \tag{49}$$

Here we have used Eqs. (44). How is Alice's "Bell detection" which yields classical photocurrents performed? The photocurrent operators for the two homodyne detections, $\hat{i}_u(t)\propto|E_{\rm LO}^X|\hat{X}_u(t)$ and $\hat{i}_v(t)\propto|E_{\rm LO}^P|\hat{P}_v(t)$, can be written (without loss of generality we assume $\Omega>0$) as

$$\hat{i}_u(t)\propto|E_{\rm LO}^X|\int_W d\Omega h_{\rm el}(\Omega)[\hat{X}_u(\Omega)e^{-i\Omega t}+\hat{X}_u^\dagger(\Omega)e^{i\Omega t}],$$
$$\hat{i}_v(t)\propto|E_{\rm LO}^P|\int_W d\Omega h_{\rm el}(\Omega)[\hat{P}_v(\Omega)e^{-i\Omega t}+\hat{P}_v^\dagger(\Omega)e^{i\Omega t}], \tag{50}$$

with a noiseless, classical local oscillator (LO) and $h_{\rm el}(\Omega)$ representing the detectors' responses within their electronic bandwidths $\Delta\Omega_{\rm el}$: $h_{\rm el}(\Omega)=1$ for $\Omega\leqslant\Delta\Omega_{\rm el}$ and zero otherwise. We assume that the relevant bandwidth W ($\sim$MHz) is fully covered by the electronic bandwidth of the detectors ($\sim$GHz). Therefore, $h_{\rm el}(\Omega)\equiv1$ in Eqs. (50). Continuously in time, these photocurrents are measured and fed-forward to Bob via a classical channel with sufficient RF bandwidth. Each of them must be viewed as complex quantities in order to respect the RF phase. The whole feedforward process, continuously performed in the time domain (i.e., performed every inverse-bandwidth time), includes Alice's detections, her classical transmission and corresponding amplitude and phase modulations of Bob's EPR beam. Any *relative* delays between the classical information conveyed by Alice and Bob's EPR beam must be such that $\Delta t\ll1/\Delta\Omega$ with the inverse bandwidth of the EPR source $1/\Delta\Omega$ (for an EPR state from the NOPA: $\Delta t\ll\gamma^{-1}$). Expressed in the frequency domain, the final modulations can be described by the classical "displacements"

$$\hat{X}_2(\Omega)\rightarrow\hat{X}_{\rm tel}(\Omega)=\hat{X}_2(\Omega)+\Gamma(\Omega)\sqrt{2}X_u(\Omega),$$
$$\hat{P}_2(\Omega)\rightarrow\hat{P}_{\rm tel}(\Omega)=\hat{P}_2(\Omega)+\Gamma(\Omega)\sqrt{2}P_v(\Omega). \tag{51}$$

The parameter $\Gamma(\Omega)$ is again a suitably normalized gain (now, in general, depending on Ω).

For $\Gamma(\Omega)=1$, Bob's displacements from Eqs. (51) exactly eliminate $\hat{X}_u(\Omega)$ and $\hat{P}_v(\Omega)$ in Eqs. (49). The same applies to the Hermitian conjugate versions of Eqs. (49) and Eqs. (51). We obtain the teleported field

$$\hat{X}_{\rm tel}(\Omega)=\hat{X}_{\rm in}(\Omega)-\sqrt{2}S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_{\rm tel}(\Omega)=\hat{P}_{\rm in}(\Omega)+\sqrt{2}S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega). \tag{52}$$

For an arbitrary gain $\Gamma(\Omega)$, the teleported field becomes

$$\hat{X}_{\rm tel}(\Omega)=\Gamma(\Omega)\hat{X}_{\rm in}(\Omega)-\frac{\Gamma(\Omega)-1}{\sqrt{2}}S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega)-\frac{\Gamma(\Omega)+1}{\sqrt{2}}S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega),$$
$$\hat{P}_{\rm tel}(\Omega)=\Gamma(\Omega)\hat{P}_{\rm in}(\Omega)+\frac{\Gamma(\Omega)-1}{\sqrt{2}}S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega)+\frac{\Gamma(\Omega)+1}{\sqrt{2}}S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega). \tag{53}$$

In general, these equations contain non-Hermitian operators with nonreal coefficients. Let us assume an EPR state from the NOPA, $S_\pm(\Omega)=G(\Omega)\pm g(\Omega)$. In the zero-bandwidth limit, the quadrature operators are Hermitian and the coefficients in Eqs. (52) and Eqs. (53) are real. For $\Omega\rightarrow0$ and $\Gamma(\Omega)=1$, the teleported quadratures computed from the above equations are, in agreement with the zero-bandwidth results, given by $\hat{X}_{\rm tel}=\hat{X}_{\rm in}$ and $\hat{P}_{\rm tel}=\hat{P}_{\rm in}$, if $\kappa\rightarrow\gamma/2$ and hence $[G(\Omega)-g(\Omega)]\rightarrow0$ (infinite squeezing). Thus, for zero bandwidth and an infinite degree of EPR correlations, Alice's unknown quantum state of mode "in" is exactly reconstituted by Bob after generating the output mode "tel" through unit-gain displacements. However, we are particularly interested in the physical case of finite bandwidth. Apparently, in unit-gain teleportation, the complete disappearance of the two classical quduties for perfect teleportation requires $\Omega=0$ (with an EPR state from the NOPA). Does this mean an increasing bandwidth always leads to deteriorating quantum teleportation? In order to make quantitative statements about this issue, we consider input states with a coherent amplitude (unit-gain teleportation) and calculate the spectral variances of the teleported quadratures for a coherent-state input to obtain a "fidelity spectrum."

A. Teleporting broadband Gaussian fields with a coherent amplitude

Let us employ teleportation equations for the real and imaginary parts of the non-Hermitian quadrature operators. In order to achieve a nonzero average fidelity when teleporting fields with a coherent amplitude, we assume $\Gamma(\Omega)=1$. According to Eqs. (52), the real and imaginary parts of the teleported quadratures are

$$\begin{aligned}
\mathrm{Re}\,\hat{X}_{\mathrm{tel}}(\Omega)&=\mathrm{Re}\,\hat{X}_{\mathrm{in}}(\Omega)-\sqrt{2}\,\mathrm{Re}[S_-(\Omega)]\mathrm{Re}\,\hat{\hat{X}}_2^{(0)}(\Omega)\\
&\quad+\sqrt{2}\,\mathrm{Im}[S_-(\Omega)]\mathrm{Im}\,\hat{\hat{X}}_2^{(0)}(\Omega),\\
\mathrm{Re}\,\hat{P}_{\mathrm{tel}}(\Omega)&=\mathrm{Re}\,\hat{P}_{\mathrm{in}}(\Omega)+\sqrt{2}\,\mathrm{Re}[S_-(\Omega)]\mathrm{Re}\,\hat{\hat{P}}_1^{(0)}(\Omega)\\
&\quad-\sqrt{2}\,\mathrm{Im}[S_-(\Omega)]\mathrm{Im}\,\hat{\hat{P}}_1^{(0)}(\Omega),\\
\mathrm{Im}\,\hat{X}_{\mathrm{tel}}(\Omega)&=\mathrm{Im}\,\hat{X}_{\mathrm{in}}(\Omega)-\sqrt{2}\,\mathrm{Im}[S_-(\Omega)]\mathrm{Re}\,\hat{\hat{X}}_2^{(0)}(\Omega)\\
&\quad-\sqrt{2}\,\mathrm{Re}[S_-(\Omega)]\mathrm{Im}\,\hat{\hat{X}}_2^{(0)}(\Omega),\\
\mathrm{Im}\,\hat{P}_{\mathrm{tel}}(\Omega)&=\mathrm{Im}\,\hat{P}_{\mathrm{in}}(\Omega)+\sqrt{2}\,\mathrm{Im}[S_-(\Omega)]\mathrm{Re}\,\hat{\hat{P}}_1^{(0)}(\Omega)\\
&\quad+\sqrt{2}\,\mathrm{Re}[S_-(\Omega)]\mathrm{Im}\,\hat{\hat{P}}_1^{(0)}(\Omega).
\end{aligned}\tag{54}$$

Their only nontrivial commutators are

$$\begin{aligned}
[\mathrm{Re}\,\hat{X}_j(\Omega),\mathrm{Re}\,\hat{P}_j(\Omega')]&=[\mathrm{Im}\,\hat{X}_j(\Omega),\mathrm{Im}\,\hat{P}_j(\Omega')]\\
&=(i/4)\,\delta(\Omega-\Omega'),
\end{aligned}\tag{55}$$

where we have used Eqs. (30) and $[\hat{B}_j(\Omega),\hat{B}_j^\dagger(\Omega')]=\delta(\Omega-\Omega')$.

We define spectral variances similar to Eq. (10),

$$\frac{\langle\Delta[\mathrm{Re}\,\hat{X}_{\mathrm{tel}}(\Omega)-\mathrm{Re}\,\hat{X}_{\mathrm{in}}(\Omega)]\Delta[\mathrm{Re}\,\hat{X}_{\mathrm{tel}}(\Omega')-\mathrm{Re}\,\hat{X}_{\mathrm{in}}(\Omega')]\rangle}{\langle\Delta\,\mathrm{Re}\,\hat{X}^2\rangle_{\mathrm{vacuum}}}\equiv\delta(\Omega-\Omega')V_{\mathrm{tel,in}}^{\mathrm{Re}\,\hat{X}}(\Omega).\tag{56}$$

We analogously define $V_{\mathrm{tel,in}}^{\mathrm{Re}\,\hat{P}}(\Omega)$, $V_{\mathrm{tel,in}}^{\mathrm{Im}\,\hat{X}}(\Omega)$, and $V_{\mathrm{tel,in}}^{\mathrm{Im}\,\hat{P}}(\Omega)$ with $\mathrm{Re}\,\hat{X}\rightarrow\mathrm{Re}\,\hat{P}$, etc., throughout.

From Eqs. (54), we obtain

$$V_{\mathrm{tel,in}}^{\mathrm{Re}\,\hat{X}}(\Omega)=V_{\mathrm{tel,in}}^{\mathrm{Re}\,\hat{P}}(\Omega)=V_{\mathrm{tel,in}}^{\mathrm{Im}\,\hat{X}}(\Omega)=V_{\mathrm{tel,in}}^{\mathrm{Im}\,\hat{P}}(\Omega)=2|S_-(\Omega)|^2.\tag{57}$$

Here we have used that

$$\begin{aligned}
\langle\Delta\,\mathrm{Re}\,\hat{\hat{X}}_j^{(0)}(\Omega)\Delta\,\mathrm{Re}\,\hat{\hat{X}}_j^{(0)}(\Omega')\rangle&=\delta(\Omega-\Omega')\langle\Delta\,\mathrm{Re}\,\hat{X}^2\rangle_{\mathrm{vacuum}}\\
&=\langle\Delta\,\mathrm{Im}\,\hat{\hat{X}}_j^{(0)}(\Omega)\Delta\,\mathrm{Im}\,\hat{\hat{X}}_j^{(0)}(\Omega')\rangle=\delta(\Omega-\Omega')\\
&\quad\times\langle\Delta\,\mathrm{Im}\,\hat{X}^2\rangle_{\mathrm{vacuum}},
\end{aligned}\tag{58}$$

and analogously for the other quadrature, and

$$\begin{aligned}
&\langle\Delta\,\mathrm{Re}\,\hat{\hat{X}}_j^{(0)}(\Omega)\Delta\,\mathrm{Im}\,\hat{\hat{X}}_j^{(0)}(\Omega')\rangle\\
&\quad=\langle\Delta\,\mathrm{Re}\,\hat{\hat{P}}_j^{(0)}(\Omega)\Delta\,\mathrm{Im}\,\hat{\hat{P}}_j^{(0)}(\Omega')\rangle=0.
\end{aligned}\tag{59}$$

Thus, for unit-gain teleportation at all frequencies, it turns out that the variance of each teleported quadrature is given by the variance of the input quadrature plus twice the squeezing spectrum of the quiet quadrature of a decoupled mode in a "broadband squeezed state" as in Eqs. (43). The excess noise in each teleported quadrature after the teleportation process is, relative to the vacuum noise, *twice* the squeezing spectrum $|S_-(\Omega)|^2$ from Eqs. (42).

We also obtain these results by directly defining

$$\frac{\langle\Delta[\hat{X}_{\mathrm{tel}}^\dagger(\Omega)-\hat{X}_{\mathrm{in}}^\dagger(\Omega)]\Delta[\hat{X}_{\mathrm{tel}}(\Omega')-\hat{X}_{\mathrm{in}}(\Omega')]\rangle}{\langle\Delta\hat{X}^2\rangle_{\mathrm{vacuum}}}\equiv\delta(\Omega-\Omega')V_{\mathrm{tel,in}}^{\hat{X}}(\Omega).\tag{60}$$

We analogously define $V_{\mathrm{tel,in}}^{\hat{P}}(\Omega)$ with $\hat{X}\rightarrow\hat{P}$ throughout. Using Eqs. (52), these variances become for $\Gamma(\Omega)=1$

$$V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=V_{\mathrm{tel,in}}^{\hat{P}}(\Omega)=2|S_-(\Omega)|^2.\tag{61}$$

We calculate some limits for $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)$ of Eq. (61), assuming an EPR state from the NOPA, $S_-(\Omega)=G(\Omega)-g(\Omega)$. Since $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=V_{\mathrm{tel,in}}^{\hat{P}}(\Omega)$ and $\Gamma(\Omega)=1$, we can name the limits according to the criterion of Eq. (12).

Classical teleportation, $\kappa\rightarrow0$. $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=2$, which is independent of the modulation frequency Ω.

Zero-bandwidth quantum teleportation, $\Omega\rightarrow0$, $\kappa>0$. $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=2[1-2\kappa\gamma/(\kappa+\gamma/2)^2]$, and in the ideal case of infinite squeezing $\kappa\rightarrow\gamma/2$: $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=0$.

Broadband quantum teleportation, $\Omega>0$, $\kappa>0$. $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=2\{1-2\kappa\gamma/[(\kappa+\gamma/2)^2+\Omega^2]\}$, and in the ideal case $\kappa\rightarrow\gamma/2$: $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)=2[\Omega^2/(\gamma^2+\Omega^2)]$. So it turns out that also for finite bandwidth ideal quantum teleportation can be approached provided $\Omega\ll\gamma$.

We can express $V_{\mathrm{tel,in}}^{\hat{X}}(\Omega)$ in terms of experimental parameters relevant to the NOPA. For this purpose, we use the dimensionless quantities from Ref. [15],

$$\epsilon=\frac{2\kappa}{\gamma+\rho}=\sqrt{\frac{P_{\mathrm{pump}}}{P_{\mathrm{thres}}}},\quad\omega=\frac{2\Omega}{\gamma+\rho}=\frac{\Omega}{2\pi}\frac{2F_{\mathrm{cav}}}{\nu_{\mathrm{FSR}}}.\tag{62}$$

Here, P_{pump} is the pump power, P_{thres} is the threshold value, F_{cav} is the measured finesse of the cavity, ν_{FSR} is its free spectral range, and the parameter ρ describes cavity losses (see Fig. 4). Note that we now use ω as a normalized modulation frequency in contrast to Eq. (45) and the following commutators where it was the frequency of the field operators in the nonrotating frame.

The spectral variances for the lossless case ($\rho=0$) can be written as a function of ϵ and ω, namely,

$$V_{\text{tel,in}}^{\hat{X}}(\epsilon,\omega)=V_{\text{tel,in}}^{\hat{P}}(\epsilon,\omega)=2\left[1-\frac{4\epsilon}{(\epsilon+1)^2+\omega^2}\right]. \quad (63)$$

Now, the classical limit is $\epsilon\to 0$ ($V_{\text{tel,in}}^{\hat{X}}=2$, independent of ω) and the ideal case is $\epsilon\to 1$ [$V_{\text{tel,in}}^{\hat{X}}(\epsilon,\omega)=2\omega^2/(4+\omega^2)$]. Obviously, perfect quantum teleportation is achieved for $\epsilon\to 1$ and $\omega\to 0$. In fact, this limit can also be approached for finite $\Omega\neq 0$ provided $\omega\ll 1$ or $\Omega\ll\gamma$. Note that this condition is not specific to broadband teleportation, but is simply the condition for broadband squeezing, i.e., for the generation of highly squeezed quadratures at nonzero modulation frequencies Ω.

Let us now assume coherent-state inputs with $\langle\Delta\hat{X}_{\text{in}}^{\dagger}(\Omega)\Delta\hat{X}_{\text{in}}(\Omega')\rangle = \langle\Delta\hat{P}_{\text{in}}^{\dagger}(\Omega)\Delta\hat{P}_{\text{in}}(\Omega')\rangle = \frac{1}{4}\delta(\Omega-\Omega')$ [$\langle\Delta\,\text{Re}\,\hat{X}_{\text{in}}(\Omega)\Delta\,\text{Re}\,\hat{X}_{\text{in}}(\Omega')\rangle=\frac{1}{8}\delta(\Omega-\Omega')$ etc.], at all frequencies Ω in the relevant bandwidth W. In order to obtain a spectrum of the fidelities in Eq. (22) with $\Gamma\to\Gamma(\Omega)=1$, we need the spectrum of the Q functions of the teleported field with the spectral variances $\sigma_x(\Omega)=\sigma_p(\Omega)=\frac{1}{2}+\frac{1}{4}V_{\text{tel,in}}^{\hat{X}}(\Omega)$. We obtain the "fidelity spectrum"

$$F(\Omega)=\frac{1}{1+|S_-(\Omega)|^2}. \quad (64)$$

Finally, with the new quantities ϵ and ω, the fidelity spectrum for quantum teleportation of arbitrary broadband coherent states using broadband entanglement from the NOPA ($\rho=0$) is given by

$$F(\epsilon,\omega)=\left[2-\frac{4\epsilon}{(\epsilon+1)^2+\omega^2}\right]^{-1}. \quad (65)$$

For different ϵ values, the spectrum of fidelities is shown in Fig. 5. From the single-mode protocol (with ideal detectors), we know that any nonzero squeezing enables quantum teleportation and coherent-state inputs can be teleported with $F=F_{\text{av}}>\frac{1}{2}$ for any $r>0$. Correspondingly, the fidelity from Eq. (65) exceeds $\frac{1}{2}$ for any nonzero ϵ at all finite frequencies, as, provided $\epsilon>0$, there is no squeezing at all only when $\omega\to\infty$. However, we had assumed [see after Eqs. (30): $\Omega\ll\omega_0$] modulation frequencies Ω much smaller than the NOPA center frequency ω_0. In fact, for $\Omega\to\omega_0$, squeezing becomes impossible at the frequency Ω [29]. But also within the region $\Omega\ll\omega_0$, effectively, the squeezing bandwith is limited and hence as well the bandwith of quantum teleportation $\Delta\omega\equiv 2\omega_{\max}$ where $F(\omega)\approx\frac{1}{2}$ (<0.51) for all $\omega>\omega_{\max}$ and $F(\omega)>\frac{1}{2}$ (≥ 0.51) for all $\omega\leq\omega_{\max}$. According to Fig. 5, we could say that the "effective teleportation bandwidth" is just about $\Delta\omega\approx 5.8$ ($\epsilon=0.1$), $\Delta\omega\approx 8.6$ ($\epsilon=0.2$), $\Delta\omega\approx 12.4$ ($\epsilon=0.4$), $\Delta\omega\approx 15.2$ ($\epsilon=0.6$), and $\Delta\omega\approx 19.6$ ($\epsilon=1$). The maximum fidelities at frequency $\omega=0$ are $F_{\max}\approx 0.6$ ($\epsilon=0.1$), $F_{\max}\approx 0.69$ ($\epsilon=0.2$), $F_{\max}\approx 0.84$ ($\epsilon=0.4$), $F_{\max}\approx 0.94$ ($\epsilon=0.6$), and, of course, $F_{\max}=1$ ($\epsilon=1$).

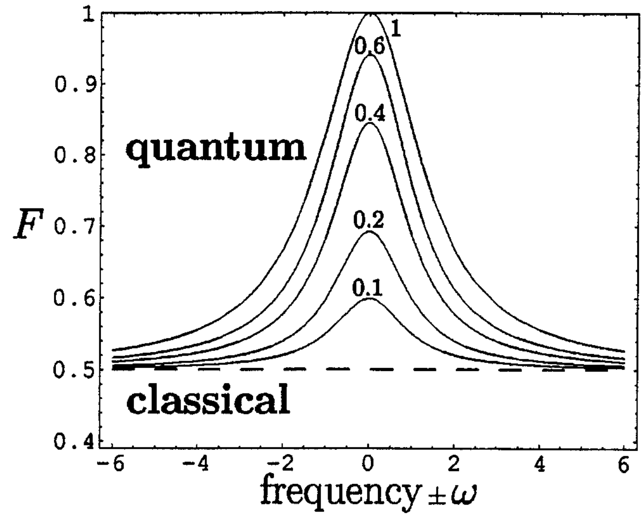

FIG. 5. Fidelity spectrum of coherent-state teleportation using entanglement from the NOPA. The fidelities here are functions of the normalized modulation frequency $\pm\omega$ for different parameter ϵ ($=0.1$, 0.2, 0.4, 0.6, and 1).

B. Broadband entanglement swapping

As discussed in Sec. III, we particularly want our teleportation device to be capable of teleporting entanglement. We will present now the broadband theory of this entanglement swapping for continuous variables, as it was proposed in Ref. [20] for single modes. Before any detections (see Fig. 3), Alice (mode 1) and Claire (mode 2) share the broadband two-mode squeezed state from Eqs. (44), whereas Claire (mode 3) and Bob (mode 4) share the corresponding entangled state of modes 3 and 4 given by

$$\begin{aligned}
\hat{X}_3(\Omega)&=\frac{1}{\sqrt{2}}S_+(\Omega)\hat{X}_3^{(0)}(\Omega)+\frac{1}{\sqrt{2}}S_-(\Omega)\hat{X}_4^{(0)}(\Omega),\\
\hat{P}_3(\Omega)&=\frac{1}{\sqrt{2}}S_-(\Omega)\hat{P}_3^{(0)}(\Omega)+\frac{1}{\sqrt{2}}S_+(\Omega)\hat{P}_4^{(0)}(\Omega),\\
\hat{X}_4(\Omega)&=\frac{1}{\sqrt{2}}S_+(\Omega)\hat{X}_3^{(0)}(\Omega)-\frac{1}{\sqrt{2}}S_-(\Omega)\hat{X}_4^{(0)}(\Omega),\\
\hat{P}_4(\Omega)&=\frac{1}{\sqrt{2}}S_-(\Omega)\hat{P}_3^{(0)}(\Omega)-\frac{1}{\sqrt{2}}S_+(\Omega)\hat{P}_4^{(0)}(\Omega).
\end{aligned} \quad (66)$$

Let us interpret the entanglement swapping here as quantum teleportation of mode 2 to mode 4 using the entanglement of modes 3 and 4. This means we want Bob to perform "displacements" based on the classical results of Claire's Bell detection, i.e., the classical determination of $\hat{X}_u(\Omega) = [\hat{X}_2(\Omega) - \hat{X}_3(\Omega)]/\sqrt{2}, \hat{P}_v(\Omega) = [\hat{P}_2(\Omega) + \hat{P}_3(\Omega)]/\sqrt{2}$. These final "displacements" (amplitude and phase modulations) of mode 4 are crucial in order to reveal the entanglement from entanglement swapping and, for verification, to finally exploit it in a second round of quantum teleportation using the previously unentangled modes 1 and 4 [20]. The entire teleportation process with arbitrary gain $\Gamma(\Omega)$ that led to Eqs. (53), yields now, for the teleportation of mode 2 to mode 4, the teleported mode 4′ [where in Eqs. (53) simply $\hat{X}_{\rm tel}(\Omega) \to \hat{X}'_4(\Omega)$, $\hat{P}_{\rm tel}(\Omega) \to \hat{P}'_4(\Omega)$, $\hat{X}_{\rm in}(\Omega) \to \hat{X}_2(\Omega)$, $\hat{P}_{\rm in}(\Omega) \to \hat{P}_2(\Omega)$, $\hat{\hat{X}}_1^{(0)}(\Omega) \to \hat{\hat{X}}_3^{(0)}(\Omega)$, $\hat{\hat{P}}_1^{(0)}(\Omega) \to \hat{\hat{P}}_3^{(0)}(\Omega)$, $\hat{\hat{X}}_2^{(0)}(\Omega) \to \hat{\hat{X}}_4^{(0)}(\Omega)$, $\hat{\hat{P}}_2^{(0)}(\Omega) \to \hat{\hat{P}}_4^{(0)}(\Omega)$, and $\Gamma(\Omega) \to \Gamma_{\rm swap}(\Omega)$],

$$\begin{aligned}\hat{X}'_4(\Omega) = {} & \frac{\Gamma_{\rm swap}(\Omega)}{\sqrt{2}}[S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega) - S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega)] \\ & - \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{X}}_3^{(0)}(\Omega) \\ & - \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{X}}_4^{(0)}(\Omega),\end{aligned} \tag{67}$$

$$\begin{aligned}\hat{P}'_4(\Omega) = {} & \frac{\Gamma_{\rm swap}(\Omega)}{\sqrt{2}}[S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega) - S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega)] \\ & + \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{P}}_3^{(0)}(\Omega) \\ & + \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{P}}_4^{(0)}(\Omega).\end{aligned}$$

Provided entanglement swapping is successful, Alice and Bob can use their modes 1 and 4′ for a further quantum teleportation. Assuming unit gain in this "second teleportation," where the unknown input state $\hat{X}_{\rm in}(\Omega)$, $\hat{P}_{\rm in}(\Omega)$ is to be teleported, the teleported field becomes

$$\begin{aligned}\hat{X}_{\rm tel}(\Omega) = \hat{X}_{\rm in}(\Omega) & + \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{X}}_1^{(0)}(\Omega) \\ & - \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{X}}_2^{(0)}(\Omega) \\ & - \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{X}}_3^{(0)}(\Omega) \\ & - \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{X}}_4^{(0)}(\Omega),\end{aligned}$$

$$\begin{aligned}\hat{P}_{\rm tel}(\Omega) = \hat{P}_{\rm in}(\Omega) & + \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{P}}_1^{(0)}(\Omega) \\ & - \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{P}}_2^{(0)}(\Omega) \\ & + \frac{\Gamma_{\rm swap}(\Omega)+1}{\sqrt{2}} S_-(\Omega)\hat{\hat{P}}_3^{(0)}(\Omega) \\ & + \frac{\Gamma_{\rm swap}(\Omega)-1}{\sqrt{2}} S_+(\Omega)\hat{\hat{P}}_4^{(0)}(\Omega).\end{aligned} \tag{68}$$

We calculate a fidelity spectrum for coherent-state inputs and obtain

$$\begin{aligned}F(\Omega) = \{ & 1 + [\Gamma_{\rm swap}(\Omega)-1]^2 |S_+(\Omega)|^2/2 \\ & + [\Gamma_{\rm swap}(\Omega)+1]^2 |S_-(\Omega)|^2/2\}^{-1}.\end{aligned} \tag{69}$$

The optimum gain, depending on the amount of squeezing, that maximizes this fidelity [20] at different frequencies turns out to be

$$\Gamma_{\rm swap}(\Omega) = \frac{|S_+(\Omega)|^2 - |S_-(\Omega)|^2}{|S_+(\Omega)|^2 + |S_-(\Omega)|^2}. \tag{70}$$

Let us now assume that the broadband entanglement comes from the NOPA (two NOPA's with equal squeezing spectra), $|S_-(\Omega)|^2 \to |S_-(\epsilon,\omega)|^2 = 1 - 4\epsilon/[(\epsilon+1)^2 + \omega^2]$, $|S_+(\Omega)|^2 \to |S_+(\epsilon,\omega)|^2 = 1 + 4\epsilon/[(\epsilon-1)^2 + \omega^2]$. The optimized fidelity then becomes

$$F_{\rm opt}(\epsilon,\omega) = \left\{ 1 + 2\frac{[(\epsilon+1)^2+\omega^2][(\epsilon-1)^2+\omega^2]}{[(\epsilon+1)^2+\omega^2]^2 + [(\epsilon-1)^2+\omega^2]^2} \right\}^{-1}. \tag{71}$$

The spectrum of these optimized fidelities is shown in Fig. 6 for different ϵ values. Again, we know from the single-mode protocol [20] with ideal detectors that any nonzero squeezing in both initial entanglement sources is sufficient for entanglement swapping to occur. In this case, mode 1 and 4′ enable quantum teleportation and coherent-state inputs can be teleported with $F = F_{\rm av} > \frac{1}{2}$. The fidelity from Eq. (71) is $\frac{1}{2}$ for $\epsilon = 0$ and becomes $F_{\rm opt}(\epsilon,\omega) > \frac{1}{2}$ for any $\epsilon > 0$, provided that ω does not become infinite (however, we had assumed $\Omega \ll \omega_0$). In this sense, the squeezing or entanglement bandwidth is preserved through entanglement swapping. At each frequency where the initial states were squeezed and entangled, also the output state of modes 1 and 4′ is entangled, but with less squeezing and worse quality of entanglement (unless we had infinite squeezing in the initial states so that the entanglement is perfectly teleported) [32]. Correspondingly, at frequencies with initially very small entanglement, the entanglement becomes even smaller after entanglement swapping (but never vanishes completely). Thus, the effective bandwidth of squeezing or entanglement decreases through entanglement swapping. Then, compared to the teleportation bandwidth using broadband two-mode squeezed

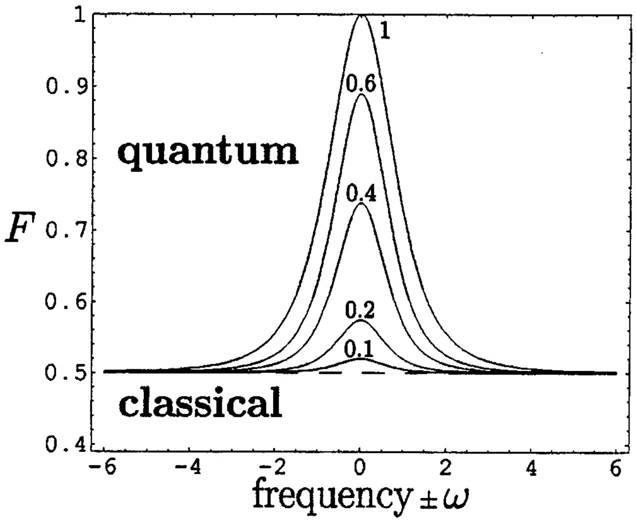

FIG. 6. Fidelity spectrum of coherent-state teleportation using the output of entanglement swapping with two equally squeezed (entangled) NOPA's. The fidelities here are functions of the normalized modulation frequency $\pm\omega$ for different parameter ϵ $(=0.1,\ 0.2,\ 0.4,\ 0.6,\ \text{and}\ 1)$.

states without entanglement swapping, the bandwidth of teleportation using the output of entanglement swapping is effectively smaller. The spectrum of the fidelities from Eq. (71) is narrower and the "effective teleportation bandwidth" is now about $\Delta\omega\approx 1.2$ $(\epsilon=0.1)$, $\Delta\omega\approx 2.6$ $(\epsilon=0.2)$, $\Delta\omega\approx 4.2$ $(\epsilon=0.4)$, $\Delta\omega\approx 5.2$ $(\epsilon=0.6)$, and $\Delta\omega\approx 6.8$ $(\epsilon=1)$. The maximum fidelities at frequency $\omega=0$ are $F_{\max}\approx 0.52$ $(\epsilon=0.1)$, $F_{\max}\approx 0.57$ $(\epsilon=0.2)$, $F_{\max}\approx 0.74$ $(\epsilon=0.4)$, $F_{\max}\approx 0.89$ $(\epsilon=0.6)$, and, still, $F_{\max}=1$ $(\epsilon=1)$.

VI. CAVITY LOSSES AND BELL DETECTOR INEFFICIENCIES

We extend the previous calculations and include losses for the particular case of the NOPA cavity and inefficiencies in Alice's Bell detection. For this purpose, we use Eq. (27) for the outgoing NOPA modes. We consider losses and inefficiencies for unit-gain teleportation (teleportation of Gaussian states with a coherent amplitude). For the case of entanglement swapping (nonunit-gain teleportation), detector inefficiencies have been included in the single-mode treatment of Ref. [20]. By superimposing the unknown input mode with the NOPA mode 1, the relevant quadratures from Eqs. (48) now become

$$\hat{X}_u(\Omega)=\frac{\eta}{\sqrt{2}}\hat{X}_{\rm in}(\Omega)-\frac{\eta}{\sqrt{2}}\hat{X}_1(\Omega)+\sqrt{\frac{1-\eta^2}{2}}\hat{X}_D^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{2}}\hat{X}_E^{(0)}(\Omega),$$

$$\hat{P}_v(\Omega)=\frac{\eta}{\sqrt{2}}\hat{P}_{\rm in}(\Omega)+\frac{\eta}{\sqrt{2}}\hat{P}_1(\Omega)+\sqrt{\frac{1-\eta^2}{2}}\hat{P}_F^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{2}}\hat{P}_G^{(0)}(\Omega). \tag{72}$$

The last two terms in each quadrature in Eqs. (72) represent additional vacua due to homodyne detection inefficiencies (the detector amplitude efficiency η is assumed to be constant over the bandwidth of interest). Using Eqs. (72) it is useful to write the quadratures of NOPA mode 2 corresponding to Eq. (27) as

$$\hat{X}_2(\Omega)=\hat{X}_{\rm in}(\Omega)-[G(\Omega)-g(\Omega)][\hat{X}_1^{(0)}(\Omega)-\hat{X}_2^{(0)}(\Omega)]-[\bar{G}(\Omega)-\bar{g}(\Omega)][\hat{X}_{C,1}^{(0)}(\Omega)-\hat{X}_{C,2}^{(0)}(\Omega)]+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{X}_D^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{X}_E^{(0)}(\Omega)-\frac{\sqrt{2}}{\eta}\hat{X}_u(\Omega),$$

$$\hat{P}_2(\Omega)=\hat{P}_{\rm in}(\Omega)+[G(\Omega)-g(\Omega)][\hat{P}_1^{(0)}(\Omega)+\hat{P}_2^{(0)}(\Omega)]+[\bar{G}(\Omega)-\bar{g}(\Omega)][\hat{P}_{C,1}^{(0)}(\Omega)+\hat{P}_{C,2}^{(0)}(\Omega)]+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{P}_F^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{P}_G^{(0)}(\Omega)-\frac{\sqrt{2}}{\eta}\hat{P}_v(\Omega), \tag{73}$$

where now [15]

$$G(\Omega)=\frac{\kappa^2+\left(\frac{\gamma-\rho}{2}+i\Omega\right)\left(\frac{\gamma+\rho}{2}-i\Omega\right)}{\left(\frac{\gamma+\rho}{2}-i\Omega\right)^2-\kappa^2},$$

$$g(\Omega)=\frac{\kappa\gamma}{\left(\frac{\gamma+\rho}{2}-i\Omega\right)^2-\kappa^2},$$

$$\bar{G}(\Omega)=\frac{\sqrt{\gamma\rho}\left(\frac{\gamma+\rho}{2}-i\Omega\right)}{\left(\frac{\gamma+\rho}{2}-i\Omega\right)^2-\kappa^2},$$

$$\bar{g}(\Omega)=\frac{\kappa\sqrt{\gamma\rho}}{\left(\frac{\gamma+\rho}{2}-i\Omega\right)^2-\kappa^2}, \tag{74}$$

still with $G(\Omega)=G^*(-\Omega)$, $g(\Omega)=g^*(-\Omega)$, and also $\bar{G}(\Omega)=\bar{G}^*(-\Omega)$, $\bar{g}(\Omega)=\bar{g}^*(-\Omega)$. The quadratures $\hat{X}_{C,j}^{(0)}(\Omega)$ and $\hat{P}_{C,j}^{(0)}(\Omega)$ are those of the vacuum modes $\hat{C}_j^{(0)}(\Omega)$ in Eq. (27) according to Eqs. (30).

Again, $\hat{X}_u(\Omega)$ and $\hat{P}_v(\Omega)$ in Eqs. (73) can be considered as classically determined quantities $X_u(\Omega)$ and $P_v(\Omega)$ due to Alice's measurements. The appropriate amplitude and phase modulations of mode 2 by Bob depending on the classical results of Alice's detections are described by

$$\hat{X}_2(\Omega)\rightarrow\hat{X}_{\rm tel}(\Omega)=\hat{X}_2(\Omega)+\Gamma(\Omega)\frac{\sqrt{2}}{\eta}X_u(\Omega),$$
$$\hat{P}_2(\Omega)\rightarrow\hat{P}_{\rm tel}(\Omega)=\hat{P}_2(\Omega)+\Gamma(\Omega)\frac{\sqrt{2}}{\eta}P_v(\Omega). \quad (75)$$

For $\Gamma(\Omega)=1$, the teleported quadratures become

$$\hat{X}_{\rm tel}(\Omega)=\hat{X}_{\rm in}(\Omega)-[G(\Omega)-g(\Omega)][\hat{X}_1^{(0)}(\Omega)-\hat{X}_2^{(0)}(\Omega)] -[\bar{G}(\Omega)-\bar{g}(\Omega)][\hat{X}_{C,1}^{(0)}(\Omega)-\hat{X}_{C,2}^{(0)}(\Omega)] +\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{X}_D^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{X}_E^{(0)}(\Omega),$$

$$\hat{P}_{\rm tel}(\Omega)=\hat{P}_{\rm in}(\Omega)+[G(\Omega)-g(\Omega)][\hat{P}_1^{(0)}(\Omega)+\hat{P}_2^{(0)}(\Omega)] +[\bar{G}(\Omega)-\bar{g}(\Omega)][\hat{P}_{C,1}^{(0)}(\Omega)+\hat{P}_{C,2}^{(0)}(\Omega)] +\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{P}_F^{(0)}(\Omega)+\sqrt{\frac{1-\eta^2}{\eta^2}}\hat{P}_G^{(0)}(\Omega). \quad (76)$$

We calculate again spectral variances and obtain with the dimensionless variables of Eqs. (62)

$$V_{\rm tel,in}^{\hat{X}}(\epsilon,\omega)=V_{\rm tel,in}^{\hat{P}}(\epsilon,\omega)=2\left[1-\frac{4\epsilon\beta}{(\epsilon+1)^2+\omega^2}\right]+2\frac{1-\eta^2}{\eta^2}, \quad (77)$$

where $\beta=\gamma/(\gamma+\rho)$ is a "cavity escape efficiency" which contains losses [15]. With the spectral Q-function variances of the teleported field $\sigma_x(\Omega)=\sigma_p(\Omega)=\frac{1}{2}+\frac{1}{4}V_{\rm tel,in}^{\hat{X}}(\Omega)$, now for coherent-state inputs, we find the fidelity spectrum (unit gain)

$$F(\epsilon,\omega)=\left[2-\frac{4\epsilon\beta}{(\epsilon+1)^2+\omega^2}+\frac{1-\eta^2}{\eta^2}\right]^{-1}. \quad (78)$$

Using the values $\epsilon=0.77$, $\omega=0.56$, and $\beta=0.9$, the measured values in the EPR experiment of Ref. [15] for maximum pump power (but still below threshold), and a Bell detector efficiency $\eta^2=0.97$ (as in the teleportation experiment of Ref. [12]), we obtain $V_{\rm tel,in}^{\hat{X}}=V_{\rm tel,in}^{\hat{P}}=0.453$ and a fidelity $F=0.815$. The measured value for the "normalized analysis frequency" $\omega=0.56$ corresponds to the measured finesse $F_{\rm cav}=180$, the free spectral range $\nu_{\rm FSR}=790$ MHz and the spectrum analyzer frequency $\Omega/2\pi=1.1$ MHz [15].

In the teleportation experiment of Ref. [12], the teleported states described fields at modulation frequency $\Omega/2\pi=2.9$ MHz within a bandwidth $\pm\Delta\Omega/2\pi=30$ kHz. Due to technical noise at low modulation frequencies, the nonclassical fidelity was achieved at these higher frequencies Ω.

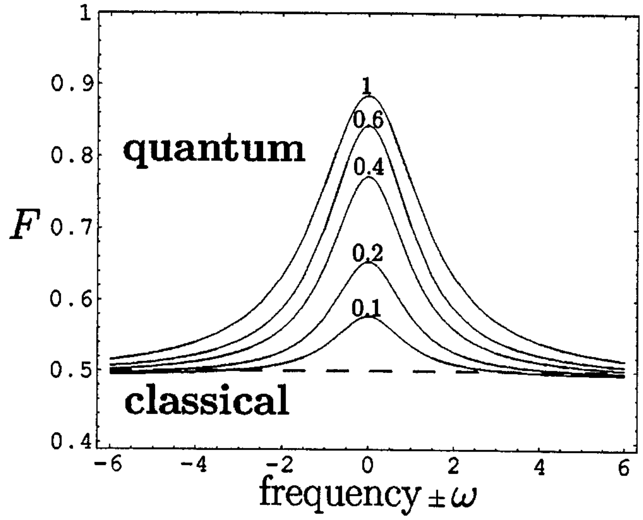

FIG. 7. Fidelity spectrum of coherent-state teleportation using entanglement from the NOPA. The fidelities here are functions of the normalized modulation frequency $\pm\omega$ for different parameter ϵ (=0.1, 0.2, 0.4, 0.6, and 1). Bell detector efficiencies $\eta^2=0.97$ and cavity losses with $\beta=0.9$ have been included here.

The amount of squeezing at these frequencies was about 3 dB. The spectrum of the fidelities from Eq. (78) is shown in Fig. 7 for different ϵ values.

VII. SUMMARY AND CONCLUSIONS

We have presented the broadband theory for quantum teleportation using squeezed-state entanglement. Our scheme allows the broadband transmission of nonorthogonal quantum states. We have discussed various criteria determining the boundary between classical teleportation (i.e., measuring the state to be transmitted as well as quantum theory permits and classically conveying the results) and quantum teleportation (i.e., using entanglement for the state transfer). Depending on the set of input states, different criteria can be applied that are best met with the optimum gain used by Bob for the phase-space displacements of his EPR beam. Given an alphabet of arbitrary Gaussian states with unknown coherent amplitudes, on average, the optimum teleportation fidelity is attained with unit gain at all relevant frequencies. Optimal teleportation of an entangled state (entanglement swapping) requires a squeezing-dependent, and hence frequency-dependent, nonunit gain. Effectively, also with optimum gain, the bandwidth of entanglement becomes smaller after entanglement swapping compared to the bandwidth of entanglement of the initial states, as the quality of the entanglement deteriorates at each frequency for finite squeezing.

In the particular case of the NOPA as the entanglement source, the best quantum teleportation occurs in the frequency regime close to the center frequency (half the NOPA's pump frequency). In general, a suitable EPR source for broadband teleportation can be obtained by combining two independent broadband squeezed states at a beam splitter (actually, even one squeezed state split at a beam splitter is sufficient to create entanglement for quantum teleportation

[33,20]). Provided ideal Bell detection, unit-gain teleportation will then in general produce an excess noise in each teleported quadrature of twice the squeezing spectrum of the quiet quadrature in the corresponding broadband squeezed state (for the NOPA, cavity loss appears in the squeezing spectrum). Thus, good broadband teleportation requires good broadband squeezing. However, the entanglement source's squeezing spectrum for its quiet quadrature need not be a minimum near the center frequency ($\Omega=0$) as for the optical parametric oscillator. In general, it might have large excess noise there and be quiet at $\Omega\neq 0$ as for four-wave mixing in a cavity [31]. The spectral range to be teleported $\Delta\Omega$ always should be in the "quiet region" of the squeezing spectrum.

The scheme presented here allows very efficient teleportation of broadband quantum states: the quantum state at the input (a coherent, a squeezed, an entangled or any other state), describing the input field at modulation frequency Ω within a bandwidth $\Delta\Omega$, is teleported on each and every trial (where the duration of a single trial is given by the inverse-bandwidth time $1/\Delta\Omega$). Every inverse-bandwidth time, a quantum state is teleported with nonclassical fidelity or previously unentangled fields become entangled. Also the output of entanglement swapping can therefore be used for efficient quantum teleportation, succeeding every inverse-bandwidth time.

In contrast, the discrete-variable schemes involving weak down conversion enable only relatively rare transfers of quantum states. For the experiment of Ref. [5], a fourfold coincidence (i.e., "successful" teleportation [7]) at a rate of 1/40 Hz and a UV pulse rate of 80 MHz [34] yield an overall efficiency of 3×10^{-10} (events per pulse). Note that due to filtering and collection difficulties the photodetectors in this experiment operated with an effective efficiency of 10% [34].

The theory presented in this paper applies to the experiment of Ref. [12] where coherent states were teleported using the entanglement built from two squeezed fields generated via degenerate down conversion. The experimentally determined fidelity in this experiment was $F=0.58\pm0.02$ (this fidelity was achieved at higher frequencies $\Omega\neq 0$ due to technical noise at low modulation frequencies) which proved the quantum nature of the teleportation process by exceeding the classical limit $F\leq\frac{1}{2}$. Our analysis was also intended to provide the theoretical foundation for the teleportation of quantum states that are more nonclassical than coherent states, e.g., squeezed states or, in particular, entangled states (two-mode squeezed states). This is yet to be realized in the laboratory.

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Quantum versus Classical Domains for Teleportation with Continuous Variables

Samuel L. Braunstein†, Christopher A. Fuchs‡, H. J. Kimble*, and P. van Loock†
†*Informatics*
Bangor University
Bangor LL57 1UT, UK
‡ *Bell Labs, Lucent Technologies*
600-700 Mountain Ave., Room 2C-420
Murray Hill, NJ 07974, USA
**Norman Bridge Laboratory of Physics 12-33*
California Institute of Technology
Pasadena, California 91125, USA
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Fidelity $F_{classical} = \frac{1}{2}$ has been established as setting the boundary between classical and quantum domains in the teleportation of coherent states of the electromagnetic field (S. L. Braunstein, C. A. Fuchs, and H. J. Kimble, J. Mod. Opt. **47**, 267 (2000)). Two recent papers by P. Grangier and F. Grosshans (quant-ph/0009079 and quant-ph/0010107) introduce alternate criteria for setting this boundary and as a result claim that the appropriate boundary should be $F = \frac{2}{3}$. Although larger fidelities would lead to enhanced teleportation capabilities, we show that the new conditions of Grangier and Grosshans are largely unrelated to the questions of entanglement and Bell-inequality violations that they take to be their primary concern. With regard to the quantum-classical boundary, we demonstrate that fidelity $F_{classical} = \frac{1}{2}$ remains the appropriate point of demarcation. The claims of Grangier and Grosshans to the contrary are simply wrong, as we show by an analysis of the conditions for nonseparability (that complements our earlier treatment) and by explicit examples of Bell-inequality violations.

I. INTRODUCTION

As proposed by Bennett *et al.* [1], the protocol for achieving quantum teleportation is the following. Alice is to transfer an unknown quantum state $|\psi\rangle$ to Bob, using as the sole resources some previously shared *quantum entanglement* and a *classical channel* capable of communicating measurement results. Physical transport of $|\psi\rangle$ from Alice to Bob is excluded at the outset. Ideal teleportation occurs when the state $|\psi\rangle$ enters Alice's sending station and the *same* state $|\psi\rangle$ emerges from Bob's receiving station.

Of course, in actual experiments [2–5], the ideal case is *unattainable* as a matter of principle. The question of operational criteria for gauging success in an experimental setting, therefore, cannot be avoided. We have proposed previously that a minimal set of conditions for claiming success in the laboratory are the following [6].

1. An unknown quantum state (supplied by a third party Victor) is input physically into Alice's station from an outside source.
2. The "recreation" of this quantum state emerges from Bob's receiving terminal available for Victor's independent examination.
3. There should be a quantitative measure for the quality of the teleportation and based upon this measure, it should be clear that shared entanglement enables the output state to be "closer" to the input state than could have been achieved if Alice and Bob had utilized a classical communication channel alone.

In Ref. [6], it was shown that the fidelity F between input and output states is an appropriate measure of the degree of similarity in Criterion 3. For an input state $|\psi_{in}\rangle$ and output state described by the density operator $\hat{\rho}_{out}$, the fidelity is given by [7]

$$F = \langle \psi_{in}|\hat{\rho}_{out}|\psi_{in}\rangle \ . \tag{1}$$

To date only the experiment of Furusawa *et al.* [4] has achieved unconditional experimental teleportation as defined by the three criteria above [6,8,9]. This experiment was carried out in the setting of continuous quantum variables with input states $|\psi_{in}\rangle$ consisting of coherent states of the electromagnetic field, with an observed fidelity $F_{\rm exp} = 0.58 \pm 0.02$ having been attained. This benchmark is significant because it can be demonstrated [4,6] that quantum entanglement is the critical ingredient in achieving an average fidelity greater than $F_{classical} = \frac{1}{2}$ *when* the input is an absolutely random coherent state [10].

Against this backdrop, Grangier and Grosshans [11,12] have recently suggested that the appropriate boundary between the classical and quantum domains in the teleportation of coherent states should be a fidelity $F = \frac{2}{3}$. Their principal concern is the distinction between "entanglement" and "non-separability," where by the latter term, they mean "the physical properties associated with non-locality and the violation of Bell's inequalities

(BI)."* They claim that "due to imperfect transmissions, ... it becomes possible to violate the classical boundary *(i.e.,* $F = \frac{1}{2}$*)* of teleportation without any violation of BI." [11] However, rather than addressing the issue in a direct manner, they then propose the violation of a certain "Heisenberg-type inequality (HI)" as "a more effective – and in some sense 'necessary' – way to characterize shared entanglement." It is this that leads to their condition $F > \frac{2}{3}$ as being necessary for the declaration of successful teleportation. In support of this threshold, they further relate their criterion based on the HI to ones previously introduced in the quantum nondemolition measurement (QND) literature. Finally, in Ref. [12], Grangier and Grosshans find that $F > \frac{2}{3}$ is also required by a criterion they introduce having to do with a certain notion of reliable "information exchange" [12].

The purpose of the present paper is to demonstrate that the conclusions of Grangier and Grosshans concerning the proposed quantum-classical boundary $F = \frac{2}{3}$ are unwarranted and, by explicit counter example, incorrect. Our approach will be to investigate questions of nonseparability and violations of Bell inequalities for the particular entangled state employed in the teleportation protocol of Ref. [13]. Of significant interest will be the case with losses, so that the relevant quantum states will be mixed quantum states. Our analysis supports the following conclusions.

1. Although the argument of Grangier and Grosshans is claimed to be based upon "EPR non-separability of the entanglement resource" [12] [by which they mean a potential violation of a BI], they offer no quantitative connection (by constructive proof or otherwise) between the criteria they introduce (including the threshold $F = \frac{2}{3}$) [11,12] and the actual violation of any Bell inequality. Nothing in their analysis provides a warranty that $F > \frac{2}{3}$ would preclude a description in terms of a local hidden-variables theory. They offer only the suggestion that "$F > \frac{2}{3}$ would be much safer" [11].

2. By application of the work of Duan *et al.* [14], Simon [15], and Tan [16], we investigate the question of entanglement. We show that the states employed in the experiment of Ref. [4] are nonseparable, as was operationally confirmed in the experiment. Moreover, we study the issue of nonseparability for mixed states over a broad range in the degree of squeezing for the initial EPR state, in the overall system loss, and in the presence of thermal noise. This analysis reveals that EPR mixed states that are nonseparable do indeed lead to a fidelity of $F > F_{classical} = \frac{1}{2}$ for the teleportation of coherent states. Hence, in keeping with Criterion 3 above, the threshold fidelity for employing entanglement as a quantum resource is precisely the same as was deduced in the previous analysis of Ref. [6]. Within the setting of Quantum Optics, this threshold coincides with the standard benchmark for manifestly quantum or nonclassical behavior, namely that the Glauber-Sudarshan phase-space function becomes nonpositive-definite, here for any bipartite nonseparable state [17]. By contrast, the value $F = \frac{2}{3}$ championed by Grangier and Grosshans is essentially unrelated to the threshold for entanglement (nonseparability) in the teleportation protocol, as well as to the boundary for the nonclassical character of the EPR state.

3. By application of the work of Banaszek and Wodkiewicz [18,19], we explore the possibility of violations of Bell inequalities for the EPR (mixed) states employed in the teleportation of continuous quantum-variables states. We find direct violations of a CHSH inequality [20] over a large domain. Significant relative to the claims of Grangier and Grosshans is a regime both of entanglement (nonseparability) and of violation of a CHSH inequality for which the teleportation fidelity $F < \frac{2}{3}$ and for which the criterion of the Heisenberg inequalities of Ref. [11] fails. Hence, teleportation with $\frac{1}{2} < F < \frac{2}{3}$ is possible with EPR (mixed) states which do not admit a local hidden variables description. In contradistinction to the claim of Grangier and Grosshans, $F > \frac{2}{3}$ does not provide a relevant criterion for delineating the quantum and classical domains with respect to violations of Bell's inequalities.

4. By adopting a protocol analogous to that employed in *all* previous experimental demonstrations of violations of Bell's inequalities [21–23], scaled correlation functions can be introduced for continuous quantum variables. In terms of these scaled correlations, the EPR mixed state used for teleportation violates a generalized version of the CHSH inequality, though non-ideal detector efficiencies require a "fair sampling" assumption for this. These violations set in for $F > \frac{1}{2}$ and have recently been observed in a setting of low detection efficiency [24]. This experimental verification of a violation of a CHSH inequality (with a fair sampling assumption) again refutes the purported significance of

*Since the terms "entanglement" and "nonseparability" are used interchangeably in the quantum information community, we will treat them as synonyms to eliminate further confusion. We will refer to violations of Bell's inequalities explicitly whenever a distinction must be made between entanglement and local realism *per se*. The only exceptions will be when we quote directly from Grangier and Grosshans [11,12].

the threshold $F = \frac{2}{3}$ promoted by Grangier and Grosshans.

Overall, we find no support for the claims of Grangier and Grosshans giving special significance to the threshold fidelity $F = \frac{2}{3}$ in connection to issues of separability and Bell-inequality violations. Instead, as we will show, it is actually the value $F_{classical} = \frac{1}{2}$ that heralds entrance into the quantum domain with respect to the very same issues. Their claims based upon a Heisenberg-type inequality and a criterion for "information exchange" are essentially unrelated to the issue of a quantum-classical boundary.

All this is not to say that teleportation of coherent states with increasing degrees of fidelity beyond $F_{classical} = \frac{1}{2}$ to $F > \frac{2}{3}$ is not without significance. In fact, as tasks of ever increasing complexity are to be accomplished, there will be corresponding requirements to improve the fidelity of teleportation yet further. Moreover, there are clearly diverse quantum states other than coherent states that one might desire to teleport, including squeezed states, quantum superpositions, entangled states [16], and so on. The connection between the "intricacy" of such states and the requisite resources for achieving high fidelity teleportation has been discussed in Ref. [13], including the example of the superposition of two coherent states,

$$|\alpha\rangle + |-\alpha\rangle \ , \tag{2}$$

which for $|\alpha| \gg 1$ requires an EPR state with an extreme degree of quantum correlation.

Similiarly, Heisenberg-type inequalities are in fact quite important for the inference of the properties of a *system* given the outcomes of measurements made on a *meter* following a *system-meter* interaction. Such quantities are gainfully employed in Quantum Optics in many settings, including realizations of the original EPR *gedanken* experiment [25–27] and of back-action evading measurement and quantum non-demolition detection [28].

Our only point is that the claim of Grangier and Grosshans that $F = \frac{2}{3}$ is required for the "successful quantum teleportation of a coherent state" [12] is incorrect. They simply offer no quantitative analysis directly relevant to either entanglement or Bell-inequality violation issues. In contrast, the prior treatment of Ref. [6] demonstrates that in the absence of shared entanglement between Alice and Bob, there is an upper limit for the fidelity for the teleportation of randomly chosen coherent states given by $F_{classical} = \frac{1}{2}$. Nothing in the work of Grangier and Grosshans calls this analysis into question.

This, however, leads to something we would like to stress apart from the details of any particular teleportation criterion. There appears to be a growing confusion in the community that equates quantum teleportation experiments with fundamental tests of quantum mechanics. The purpose of such tests is generally to compare quantum mechanics to other potential theories, such as locally realistic hidden-variable theories [11,29,30]. In our view, experiments in teleportation have nothing to do with this. They instead represent investigations *within* quantum mechanics, demonstrating only that a particular task can be accomplished with the resource of quantum entanglement and cannot be accomplished without it. This means that violations of Bell's inequalities are largely irrelevant as far as the original proposal of Bennett *et al.* [1] is concerned, as well as for experimental implementations of that protocol. In a theory which allows states to be cloned, there would be no need to discuss teleportation at all – unknown states could be cloned and transmitted with fidelity arbitrarily close to one.

These comments notwithstanding, Grangier and Grosshans did nevertheless attempt to link the idea of Bell-inequality violations with the fidelity of teleportation. It is to the details of that linkage that we now turn. The remainder of the paper is organized as follows. In Section II, we extend the prior work of Ref. [6] to a direct treatment of the consequences of shared entanglement between Alice and Bob, beginning with an explicit model for the mixed EPR states used for teleportation of continuous quantum variables. In Section III we review the criteria Grangier and Grosshans introduced in preparation for showing their inappropriateness as tools for the questions at hand. In Section IV, we demonstrate explicitly the relationship between entanglement and fidelity, and find the same threshold $F_{classical} = \frac{1}{2}$ as in our prior analysis [6]. The value $F = \frac{2}{3}$ is shown to have no particular distinction in this context. In Sections V and VI, we further explore the role of entanglement with regard to violations of a CHSH inequality and provide a quantitative boundary for such violations. Again, $F_{classical} = \frac{1}{2}$ appears as the point of entry into the quantum domain, with the point $F = \frac{2}{3}$ having no particular distinction. Our conclusions are collected in Section VII. Of particular significance, we point out that the teleportation experiment of Ref. [4] did indeed cross from the classical to the quantum domain, just as advertised previously.

II. THE EPR STATE

The teleportation protocol we consider is that of Braunstein and Kimble [13], for which the relevant entangled state is the so-called two-mode squeezed state. This state is given explicitly in terms of a Fock-state expansion for two-modes $(1,2)$ by [31,32]

$$|EPR\rangle_{1,2} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n |n\rangle_1 |n\rangle_2 \ , \tag{3}$$

where r measures the amount of squeezing required to produce the entangled state. Note that for simplicity we consider the case of two single modes for the electromagnetic field; the extension to the multimode case for fields of finite bandwidth can be found in Ref. [33].

The pure state of Eq. (3) can be equivalently described by the corresponding Wigner distribution $W_{\rm EPR}$ over the two modes $(1,2)$,

$$\begin{aligned} & W_{\rm EPR}(x_1,p_1;x_2,p_2) \qquad (4)\\ &= \frac{4}{\pi^2}\frac{1}{\sigma_+^2\sigma_-^2}\exp\Big(-[(x_1+x_2)^2+(p_1-p_2)^2]/\sigma_+^2 \\ &-[(x_1-x_2)^2+(p_1+p_2)^2]/\,\sigma_-^2\Big)\,, \end{aligned}$$

where $\sigma_\pm$ are expressed in terms of the squeezing parameter by

$$\begin{aligned} \sigma_+^2 &= e^{+2r}, \\ \sigma_-^2 &= e^{-2r}, \end{aligned} \qquad (5)$$

with $\sigma_+^2\sigma_-^2 = 1$. Here, the canonical variables (x_j, p_j) are related to the complex field amplitude α_j for mode $j = (1,2)$ by

$$\alpha_j = x_j + ip_j. \qquad (6)$$

In the limit of $r \to \infty$, Eq. (4) becomes

$$C\,\delta(x_1 - x_2)\,\delta(p_1 + p_2) \qquad (7)$$

which makes a connection to the original EPR state of Einstein, Podolsky, and Rosen [25].

Of course, $W_{\rm EPR}$ as given above is for the ideal, lossless case. Of particular interest with respect to experiments is the inclusion of losses, as arise from, for example, finite propagation and detection efficiencies. Rather than deal with any detailed setup (e.g., as treated in explicit detail in Ref. [26]) here we adopt a generic model of the following form. Consider two identical beam splitters each with a transmission coefficient η, one for each of the two EPR modes. We take $0 \le \eta \le 1$, with $\eta = 1$ for the ideal, lossless case. The input modes to the beam splitter 1 are taken to be $(1', a')$, while for beam splitter 2, the modes are labeled by $(2', b')$. Here, the modes $(1', 2')$ are assumed to be in the state specified by the ideal $W_{\rm EPR}$ as given in Eq. (4) above, while the modes (a', b') are taken to be independent thermal (mixed) states each with Wigner distribution

$$W(x,p) = \frac{1}{\pi(\bar n + \frac{1}{2})}\exp\{-(x^2+p^2)/(\bar n + 1/2)\}, \quad (8)$$

where $\bar n$ is the mean thermal photon number for each of the modes (a', b').

The overall Wigner distribution for the initial set of input modes $(1', 2'), (a', b')$ is then just the product

$$W_{\rm EPR}(x_{1'}, p_{1'}; x_{2'}, p_{2'})\,W(x_{a'}, p_{a'})\,W(x_{b'}, p_{b'})\ . \qquad (9)$$

The standard beam-splitter transformations lead in a straightforward fashion to the Wigner distribution for the output set of modes $(1,2), (a,b)$, where, for example,

$$\begin{aligned} x_1 &= \sqrt{\eta}x_{1'} - \sqrt{1-\eta}x_{a'}, \\ x_a &= \sqrt{\eta}x_{a'} + \sqrt{1-\eta}x_{1'}. \end{aligned} \qquad (10)$$

We require $W_{\rm EPR}^{out}$ for the $(1,2)$ modes alone, which is obtained by integrating over the (a,b) modes. A straightforward calculation results in the following distribution for the mixed output state:

$$\begin{aligned} & W_{\rm EPR}^{out}(x_1,p_1;x_2,p_2) \qquad (11)\\ &= \frac{4}{\pi^2}\frac{1}{\bar\sigma_+^2\bar\sigma_-^2}\exp\Big(-[(x_1+x_2)^2+(p_1-p_2)^2]/\bar\sigma_+^2 \\ &-[(x_1-x_2)^2+(p_1+p_2)^2]/\,\bar\sigma_-^2\Big), \end{aligned}$$

where $\bar\sigma_\pm$ are given by

$$\begin{aligned} \bar\sigma_+^2 &= \eta e^{+2r} + (1-\eta)(1+2\bar n), \\ \bar\sigma_-^2 &= \eta e^{-2r} + (1-\eta)(1+2\bar n). \end{aligned} \qquad (12)$$

Note that $W_{\rm EPR}^{out}$ as above follows directly from $W_{\rm EPR}$ in Eq. (4) via the simple replacements $\sigma_\pm \longrightarrow \bar\sigma_\pm$. Relevant to the discussion of Bell inequalities in Sections V and VI is the fact that $\bar\sigma_+^2\bar\sigma_-^2 > 1$ for any $r > 0$ and $\eta < 1$.

III. THE CRITERIA OF GRANGIER AND GROSSHANS

The two recent papers of Grangier and Grosshans argue that "fidelity value larger than $\frac{2}{3}$ is actually required for successful teleportation" [11,12]. In this section, we recapitulate the critical elements of their analysis and state their criteria in the present notation. In subsequent sections we proceed further with our own analysis of entanglement and possible violations of Bell's inequalities for the EPR state of Eq. (11).

Beginning with Ref. [11], Eq. (21), Grangier and Grosshans state the following:

> "As a criteria for non-separability [by which they mean violations of Bell's inequalities], we will use the EPR argument: two different measurements prepare two different states, in such a way that the product of conditional variances (with different conditions) violates the Heisenberg principle."

This statement takes a quantitative form in terms of the following conditional variances expressed in the notation of the preceding section for EPR beams $(1,2)$:

$$\begin{aligned} V_{x_i|x_j} &= \langle \Delta x_i^2\rangle - \frac{\langle x_i x_j\rangle}{\langle \Delta x_j^2\rangle}, \\ V_{p_i|p_j} &= \langle \Delta p_i^2\rangle - \frac{\langle p_i p_j\rangle}{\langle \Delta p_j^2\rangle}. \end{aligned} \qquad (13)$$

with $(i,j) = (1,2)$ and $i \neq j$. Note that, for example, $V_{x_2|x_1}$ gives the error in the knowledge of the canonical variable x_2 based upon an estimate of x_2 from a measurement of x_1, and likewise for the other conditional variances. These variances were introduced in Refs. [26,27] in connection with an optical realization of the original *gedanken* experiment of Einstein, Podolsky, and Rosen [25]. An apparent violation of the uncertainty principle arises if the product of inference errors is below the uncertainty product for one beam alone. For example, $V_{x_2|x_1} V_{p_2|p_1} < \frac{1}{16}$ represents such an apparent violation since $\Delta x_2^2 \Delta p_2^2 \geq \frac{1}{16}$ is demanded by the canonical commutation relation between x_2 and p_2, with here $\Delta x_{1,2}^2 = \frac{1}{4} = \Delta p_{1,2}^2$ for the vacuum state [26,27].

Grangier and Grosshans elevate this concept of inference at a distance from the EPR analysis to "a criteria for non-separability [i.e., violation of Bell's inequalities]." Specifically, they state that "the classical limit of no apparent violation of HI" [and hence the domain of local realism] is determined by the conditions

$$V_{x_2|x_1} V_{p_2|p_1} \geq \frac{1}{16}, \quad \text{and} \quad V_{x_1|x_2} V_{p_1|p_2} \geq \frac{1}{16}. \tag{14}$$

As shown in Refs. [26,27] for the states under consideration, the conditional variances of Eq. (13) are simply related to the following (unconditional) variances

$$\begin{aligned} \Delta x_{\mu_{ij}}^2 &= \langle (x_i - \mu_{ij} x_j)^2 \rangle, \\ \Delta p_{\nu_{ij}}^2 &= \langle (p_i - \nu_{ij} p_j)^2 \rangle. \end{aligned} \tag{15}$$

If we use a measurement of x_j to estimate x_i, then $\Delta x_{\mu_{ij}}^2$ is the variance of the error when the estimator is chosen to be $\mu_{ij} x_j$, and likewise for $\Delta p_{\nu_{ij}}^2$. For an optimal estimate, the parameters (μ_{ij}, ν_{ij}) are given by [26,27]

$$\mu_{ij}^{\text{opt}} = \frac{\langle x_i x_j \rangle}{\langle \Delta x_j^2 \rangle}, \quad \nu_{ij}^{\text{opt}} = \frac{\langle p_i p_j \rangle}{\langle \Delta p_j^2 \rangle}, \tag{16}$$

and in this case,

$$V_{x_i|x_j} = \Delta x^2_{\mu_{ij}^{\text{opt}}}, \text{ and } V_{p_i|p_j} = \Delta p^2_{\nu_{ij}^{\text{opt}}}. \tag{17}$$

The "non-separability" condition of Grangier and Grosshans in Eq. (14) can then be re-expressed as

$$\Delta x_{\mu_{21}}^2 \Delta p_{\nu_{21}}^2 \geq \frac{1}{16}, \quad \text{and} \quad \Delta x_{\mu_{12}}^2 \Delta p_{\nu_{12}}^2 \geq \frac{1}{16}, \tag{18}$$

where we assume the optimized choice and drop the superscript 'opt'. Again, Grangier and Grosshans take this condition of "no apparent violation of HI" as the operational signature of "nonseparability criteria" [violations of Bell inequalities], and hence, by their logic, to delineate the classical boundary for teleportation [11].

To make apparent the critical elements of the discussion, we next assume symmetric fluctuations as appropriate to the EPR state of Eq. (11), $\mu_{ij} = \mu_{ji} \equiv \mu$ and $\nu_{ij} = \nu_{ji} \equiv \nu$, with $\mu = -\nu$. Note that in the absence of losses as in Eq. (4), we have explicitly that $\mu = \tanh 2r$. By contrast, the case considered by Grangier and Grosshans is $\mu = -\nu = 1$, for which their HI becomes

$$\Delta x^2 \Delta p^2 \geq \frac{1}{16}, \tag{19}$$

where

$$\begin{aligned} \Delta x^2 &= \langle (x_1 - x_2)^2 \rangle, \\ \Delta p^2 &= \langle (p_1 + p_2)^2 \rangle. \end{aligned} \tag{20}$$

Note that in general the inequality

$$V_1 V_2 \geq \frac{a^2}{4} \tag{21}$$

implies that

$$V_1 + V_2 \geq V_1 + \frac{a^2}{4V_1} \geq a, \tag{22}$$

so that the criterion of Eq. (19) for *classical* teleportation leads to

$$\Delta x^2 + \Delta p^2 \geq \frac{1}{2}. \tag{23}$$

Hence, the requirement of Grangier and Grosshans for *quantum* teleportation is that

$$\Delta x^2 + \Delta p^2 < \frac{1}{2}, \tag{24}$$

where from Eq. (11), we have that $\Delta x^2 + \Delta p^2 = \bar{\sigma}_-^2$ for the EPR beams $(1,2)$. As we shall see in the next section, it is straightforward to show that this condition corresponds to fidelity $F > \frac{2}{3}$ for the teleportation of coherent states. The claim of Grangier and Grosshans [11] is that the inequality of Eq. (19) serves as "the condition for no useful entanglement between the two beams," where by 'useful' they refer explicitly to "the existence of quantum non-separability [violation of Bell's inequalities]." The inequalities of Eqs. (18) and (19) are also related to criteria developed within the setting of quantum nondemolition detection (QND) [28], as discussed in the next section.

In a second paper [12], Grangier and Grosshans introduce an alternative criteria for the successful teleportation of coherent states, namely that

> "the information content of the teleported quantum state is higher than the information content of any (classical or quantum) copy of the input state, that may be broadcasted classically."

To quantify the concept of "information content" they introduce a "generalized fidelity" describing not the overlap of quantum states as is standard in the quantum information community, but rather the conditional probability $P(\alpha|I)$ that a particular coherent state $|\alpha\rangle$ was actually sent given "the available information I." In effect, Grangier and Grosshans consider the following protocol. Victor sends to Alice some unknown coherent state $|\alpha_0\rangle$, with Alice making her best attempt to determine this state [34], and sending the resulting measurement outcome to Bob as in the standard protocol. Bob then does one of two things. In the first instance, he forwards only this classical message with Alice's measurement outcome to Victor without reconstructing a quantum state. In the second case, he actually generates a quantum state conditioned upon Alice's message and sends this state to Victor, who must then make his own measurement to deduce whether the teleported state corresponds to the one that he initially sent. For successful teleportation, Grangier and Grosshans demand that the information gained by Victor should be greater in the latter case where quantum states are actually generated by Bob than in the former case where only Alice's classical measurement outcome is distributed. It is straightforward to show that Eq. (24) given above is sufficient to ensure that this second criteria is likewise satisfied for the teleportation of a coherent state $|\alpha\rangle$, albeit with the same caveat expressed in [10], namely that neither the set S of initial states $\{|\psi_{in}\rangle\}$ nor the distribution $P(|\psi_{in}\rangle)$ over these states is specified.

We now turn to an evaluation of these criteria of Grangier and Grosshans placing special emphasis on the issues of entanglement and violations of Bell's inequalities, specifically because these are the concepts Grangier and Grosshans emphasize in their work [11,12].

IV. ENTANGLEMENT AND FIDELITY

A. Nonseparability of the EPR beams

To address the question of the nonseparability of the EPR beams, we refer to the papers of Duan *et al.* and of Simon [14,15], as well as related work by Tan [16]. For the definitions of (x_i, p_i) that we have chosen for the EPR beams $(1,2)$, a sufficient condition for nonseparability (without an assumption of Gaussian statistics) is that

$$\Delta x^2 + \Delta p^2 < 1, \tag{25}$$

where Δx^2 and Δp^2 are defined in Eq. (20). This result follows from Eq. (3) of Duan *et al.* with $a = 1$ (and from a similar more general equation in Simon) [35]. Note that Duan *et al.* have $\Delta x_i^2 = \frac{1}{2} = \Delta p_i^2$ for the vacuum state, while our definitions lead to $\Delta x_i^2 = \frac{1}{4} = \Delta p_i^2$ for the vacuum state, where for example, $\Delta x_1^2 = \langle x_1{}^2\rangle$, and that all fields considered have zero mean.

Given the Wigner distribution $W_{\rm EPR}^{out}$ as in Eq. (11), we find immediately that

$$\begin{aligned}\Delta x^2 + \Delta p^2 &= 2\frac{\bar{\sigma}_-^2}{2} \\ &= \eta e^{-2r} + (1-\eta)(1+2\bar{n}).\end{aligned} \tag{26}$$

For the case $\bar{n} = 0$, the resulting state is *always entangled for any $r > 0$ even for $\eta \ll 1$*, in agreement with the discussion in Duan *et al.* [14]. For nonzero $\bar{n}$, the state is entangled so long as

$$\bar{n} < \frac{\eta[1-\exp(-2r)]}{2(1-\eta)}. \tag{27}$$

We emphasize that in the experiment of Furusawa *et al.* [4] for which $\bar{n} = 0$ is the relevant case, the above inequality guarantees that teleportation was carried out with entangled (i.e., nonseparable) states for the EPR beams, independent of any assumption about whether these beams were Gaussian or pure states [36].

By contrast to the condition for entanglement given in Eq. (25), Grangier and Grosshans require instead the more stringent condition of Eq. (24) for successful teleportation. Although they would admit that the EPR beams are indeed entangled whenever Eq. (25) is satisfied,† they would term entanglement in the domain

$$\frac{1}{2} \le \Delta x^2 + \Delta p^2 < 1$$

as not "useful" [11].

With regard to the QND-like conditions introduced by Grangier and Grosshans [11], we note that more general forms for the nonseparability condition of Eq. (25) are given in Refs. [14,15]. Of particular relevance is a condition for the variances of Eq. (15) for the case of symmetric fluctuations as for EPR state in Eq. (11), $\mu_{ij} = \mu_{ji} \equiv \mu$ and $\nu_{ij} = \nu_{ji} \equiv \nu$, with $\mu = -\nu$. Consider for example

$$\Delta x_\mu^2 \equiv \langle (x_2 - \mu x_1)^2\rangle \quad \text{and} \quad \Delta p_\mu^2 \equiv \langle (p_2 + \mu p_1)\rangle^2, \tag{28}$$

as would be appropriate for an inference of (x_2, p_2) from a measurement (at a distance) of (x_1, p_1). In this case, a sufficient condition for entanglement of the EPR beams $(1,2)$ is that

$$\Delta x_\mu^2 + \Delta p_\mu^2 < \frac{(1+\mu^2)}{2}, \tag{29}$$

†Grangier was in fact unaware of Refs. [14,15] when Ref. [11] was originally posted, having had this work pointed out by us.

which reproduces Eq. (25) for $\mu = 1$. The relevance of this condition is that it is in fact possible to achieve an apparent violation of the Heisenberg inequality

$$\Delta x_2^2 \Delta p_2^2 \geq \frac{1}{16} \tag{30}$$

with

$$\Delta x_\mu^2 \Delta p_\mu^2 < \frac{1}{16} \tag{31}$$

for any $r > 0$ and $0 < \eta \leq 1$ for $\bar{n} = 0$. As discussed in Refs. [26,27], μ must be chosen in correspondence to the degree of correlation between the EPR beams, with $0 < \mu \leq 1$.

Hence, an apparent violation of the optimized Heisenberg inequality as in Eq. (31) is sufficient to ensure that the EPR beams are entangled. In applying their QND-like conditions, Grangier and Grosshans demand to the contrary that $\mu = 1$ for all values of r, so that $\Delta x_\mu^2 \Delta p_\mu^2 < \frac{1}{16}$ only for $\bar{\sigma}_-^2 < \frac{1}{2}$ for a violation of their HI (rather than the more general case of $\bar{\sigma}_-^2 < 1$ with optimized μ for any (r, η) and $\bar{n} = 0$). Certainly, $\mu = 1$ is the case relevant to the actual teleportation protocol of Ref. [13]. However, Alice and Bob are surely free to explore the degree of correlation between their EPR beams and to test for entanglement by any means at their disposal, including simple measurements with $\mu \neq 1$.

In their second paper [12], Grangier and Grosshans state that "the intensity of one of the EPR beam can be measured in order to use that information to reduce the noise of the second beam. Then the noise of the corrected beam can be reduced below shot-noise only when the losses on each beam are less than 50%." Again, this statement follows only in the restricted case $\mu = 1$. More generally, for $\bar{n} = 0$, any $r > 0$ with $0 < \eta \leq 1$ is sufficient for reducing the noise of one of the EPR beams below its own shot-noise limit based upon a measurement of the second EPR beam, with the condition for the success of such a procedure being given by the nonseparability condition of Eq. (29) for optimized μ and not by the criteria of Grangier and Grosshans who demand $\mu = 1$.

Although the boundary expressed by the nonseparability conditions of Eqs. (25) and (29) are perhaps not so familiar in Quantum Optics, we stress that these criteria are associated quite directly with the standard condition for nonclassical behavior adopted by this community. Whenever Eqs. (25) and (29) are satisfied, the Glauber-Sudarshan phase-space function becomes nonpositive [17], which for almost forty years has heralded entrance into a manifestly quantum or nonclassical domain. It is difficult to understand how Grangier and Grosshans propose to move from $\Delta x^2 + \Delta p^2 = 1$ to $\Delta x^2 + \Delta p^2 = \frac{1}{2}$ without employing quantum resources in the teleportation protocol (as is required when the Glauber-Sudarshan P-function is not positive definite). Their own work offers no suggestion of how this is to be accomplished.

B. Fidelity

Turning next to the question of the relationship of entanglement of the EPR beams [as quantified in Eq. (25)] to the fidelity attainable for teleportation *with these beams*, we recall from Eq. (2) of Ref. [4] that

$$F = \frac{1}{1 + \bar{\sigma}_-^2}, \tag{32}$$

where this result applies to teleportation of coherent states. [38] When combined with Eq. (26), we find that

$$F = \frac{1}{1 + (\Delta x^2 + \Delta p^2)}, \tag{33}$$

The criterion of Eq. (25) for nonseparability then guarantees that nonseparable EPR states as in Eqs. (4,11)(be they mixed or pure) are sufficient to achieve

$$F > F_{classical} = \frac{1}{2}, \tag{34}$$

whereas separable states must have $F \leq F_{classical} = \frac{1}{2}$, although we emphasize that this bound applies for the average fidelity for coherent states distributed over the entire complex plane [6,38].

We thereby demonstrate that the condition $F > F_{classical} = \frac{1}{2}$ for quantum teleportation as established in Ref. [6] coincides with that for nonseparability (i.e., entanglement) of Refs. [14,15] for the EPR state of Eq. (11). Note that for $\bar{n} = 0$, we have

$$F = \frac{1}{2 - \eta(1 - e^{-2r})}, \tag{35}$$

so that the entangled EPR beams considered here (as well as in Refs. [11,12]) provide a sufficient resource for beating the limit set by a classical channel alone for any $r > 0$, so long as $\eta > 0$. In fact, the quantities $(\Delta x^2, \Delta p^2)$ are readily measured experimentally, so that the entanglement of the EPR beams can be operationally verified, as was first accomplished in Ref. [26], and subsequently in Ref. [4]. We stress that independently of any further assumption, the condition of Eq. (25) is sufficient to ensure entanglement for pure or mixed states [39,40].

The dependence of fidelity F on the degree of squeezing r and efficiency η as expressed in Eq. (35) is illustrated in Figure 1. Here, in correspondence to an experiment with fixed overall losses and variable parametric gain in the generation of the EPR entangled state, we show a family of curves in Figure 1 each of which is drawn for constant η as a function of r. Clearly, $F > F_{classical} = \frac{1}{2}$ and hence nonseparability results in each case. The only apparent significance of $F = \frac{2}{3}$ as championed by Grangier and Grosshans (and which results for $\Delta x^2 + \Delta p^2 = \frac{1}{2}$) is to bound F for $\eta = 0.5$.

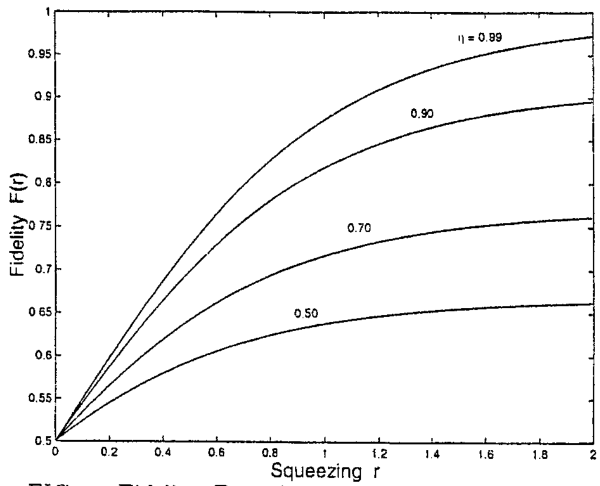

FIG. 1. Fidelity F as given by Eq. (35) versus the degree of squeezing r for fixed efficiency η. From top to bottom, the curves are drawn with $\eta = \{0.99, 0.90, 0.70, 0.50\}$ in correspondence to increasing loss $(1-\eta)$. Note that $F_{classical} = \frac{1}{2}$ provides a demarcation between separable and nonseparable states (mixed or otherwise), while $F = \frac{2}{3}$ is apparently of no particular significance, the contrary claims of Ref. [11,12] notwithstanding. Note that for $\eta = 1$, $r = \frac{\ln 2}{2} = 0.3466$ gives $F = \frac{2}{3}$, corresponding to -3dB of squeezing. In all cases, $\bar{n} = 0$.

As for the criterion of "information content" introduced by Grangier and Grosshans [12], we note it can be easily understood from the current analysis and the original discussion in Ref. [13]. Each of the interventions by Alice and Bob represent one unit of added vacuum noise that will be convolved with the initial input state in the teleportation protocol (the so-called *quduties*). Grangier and Grosshans compare the following two situations: (i) Bob passes directly the classical information that he receives to Victor and (ii) Bob generates a quantum state in the usual fashion that is then passed to Victor. Grangier and Grosshans would demand that Victor should receive the same information in these two cases, which requires that $\bar{\sigma}_{-}^{2} = \Delta x^2 + \Delta p^2 < \frac{1}{2}$, and hence $F > \frac{2}{3}$. That is, as the degree of correlation between the EPR beams is increased, there comes a point for which $\Delta x^2 + \Delta p^2 = \frac{1}{2}$, and for which each of Alice and Bob's excess noise has been reduced from 1 quduty each to $\frac{1}{2}$ quduty each. At this point, Grangier and Grosshans would (arbitrarily) assign the entire resulting noise of $\frac{1}{2} + \frac{1}{2} = 1$ quduties to Alice, with then the perspective that Bob's state recreation adds no noise. Of course one could equally well make the complementary assignment, namely 1 quduty to Bob and none to Alice (again in the case with $\bar{\sigma}_{-}^{2} = \frac{1}{2}$). The point that seems to be missed by Grangier and Grosshans is that key to quantum teleportation is the transport of quantum states. Although they correctly state that "there is *no* extra noise associated to the reconstruction: given a measured β, one can exactly reconstruct the coherent state $|\beta\rangle$, by using a deterministic translation of the vacuum." Bob can certainly make such a state deterministically, but it is an altogether different matter for Victor to receive a classical number from Bob in case (i) as opposed to the actual quantum state in (ii). In this latter case apart from having a physical state instead of a number, Victor must actually make his own measurement with the attendant uncertainties inherent in $|\beta\rangle$ then entering. Analogously, transferring measurement results about a qubit, without recreating a state at the output (i.e., without sending an actual *quantum state* to Victor), is not what is normally considered to constitute quantum teleportation relative to the original protocol of Bennett *et al.* [1].

Turning next to the actual experiment of Ref. [4], we note that a somewhat subtle issue is that the detection efficiency for Alice of the unknown state was not 100%, but rather was $\eta_A^2 = 0.97$. Because of this, the fidelity for classical teleportation (i.e., with vacuum states in place of the EPR beams) did not actually reach $\frac{1}{2}$, but was instead $F_0 = 0.48$. This should not be a surprise, since there is nothing to ensure that a given classical scheme will be optimal and actually reach the bound $F_{classical} = \frac{1}{2}$. Hence, the starting point in the experiment with $r = 0$ had $F_0 < F_{classical}$; the EPR beams with $r > 0$ (which were in any event entangled by the above inequality) then led to increases in fidelity from F_0 upward, exceeding the classical bound $F_{classical} = \frac{1}{2}$ for a small (but not infinitesimal) degree of squeezing. Note that the whole effect of the offset $F_0 = 0.48 < \frac{1}{2}$ can be attributed to the lack of perfect (homodyne) efficiency at Alice's detector for the unknown state. In the current discussion for determining the classical bound in the *optimal* case, we set Alice's detection efficiency instead to $\eta_A^2 = 1$, then as shown above, classical teleportation will achieve $F = \frac{1}{2}$.

Independent of such considerations, we reiterate that the nonseparability condition of Refs. [14,15] applied to the EPR state of Eqs. (4) and (11) leads to the same result $F_{classical} = \frac{1}{2}$ [Eqs. (33) and (34)] as did our previous analysis based upon teleportation with only a classical communication channel linking Alice and Bob [6]. This convergence further supports $F_{classical} = \frac{1}{2}$ as the appropriate quantum-classical boundary for the teleportation of coherent states, the claims of Grangier and Grosshans notwithstanding. Relative to the original work of Bennett *et al.* [1], exceeding the bound $F_{classical} = \frac{1}{2}$ for the teleportation of coherent can be accomplished with a classical channel and entangled (i.e., nonseparable) EPR states, be they mixed or pure, as is made clear by the above analysis and as has been operationally confirmed [4].

We should however emphasize that the above conclusions concerning nonseparability and teleportation fidelity apply to the specific case of the EPR state as in Eq. (11), for which inequality Eq. (25) represents both a necessary and sufficient criterion for nonseparability according to Refs. [14,15]. More generally, for arbitrary entangled states, nonseparability does not necessarily lead to $F > \frac{1}{2}$ in coherent-state teleportation [39,40].

V. BELL'S INEQUALITIES

The papers by Banaszek and Wodkiewicz [18,19] provide our point of reference for a discussion of Bell's inequalities. In these papers, the authors introduce an appropriate set of measurements that lead to a Bell inequality of the CHSH type. More explicitly, Eq.(4) of Ref. [18] gives the operator $\hat{\Pi}(\alpha;\beta)$ whose expectation values are to be measured. Banaszek and Wodkiewicz point out that the expectation value of $\hat{\Pi}(\alpha;\beta)$ is closely related the Wigner function of the field being investigated, namely

$$W(\alpha;\beta) = \frac{4}{\pi^2}\Pi(\alpha;\beta) \ , \tag{36}$$

where $\Pi(\alpha;\beta) = \langle\hat{\Pi}(\alpha;\beta)\rangle$.

For the entangled state shared by Alice and Bob in the teleportation protocol, we identify W^{out}_{EPR} as the relevant Wigner distribution for the modes $(1,2)$ of interest, so that

$$\begin{aligned} &\Pi^{out}_{\mathrm{EPR}}(x_1,p_1;x_2,p_2) \qquad (37)\\ &= \frac{1}{\bar{\sigma}_+^2\bar{\sigma}_-^2}\exp\{-[(x_1+x_2)^2+(p_1-p_2)^2]/\bar{\sigma}_+^2 \\ &\quad -[(x_1-x_2)^2+(p_1+p_2)^2]/\,\bar{\sigma}_-^2\} \ . \end{aligned}$$

Banaszek and Wodkiewicz show that $\Pi^{out}_{\mathrm{EPR}}(x_1,p_1;x_2,p_2)$ gives directly the correlation function that would otherwise be obtained from a particular set of observations over an ensemble representing the field with density operator $\hat{\rho}$, where the actual measurements to be made are as described in Refs. [18,19]. In simple terms, $\hat{\Pi}^{out}_{\mathrm{EPR}}(0,0;0,0)$ is the parity operator for separate measurements of photon number on modes $(1,2)$, with then nonzero (x_i,p_i) corresponding to a "rotation" on the individual mode i that precedes its parity measurement.

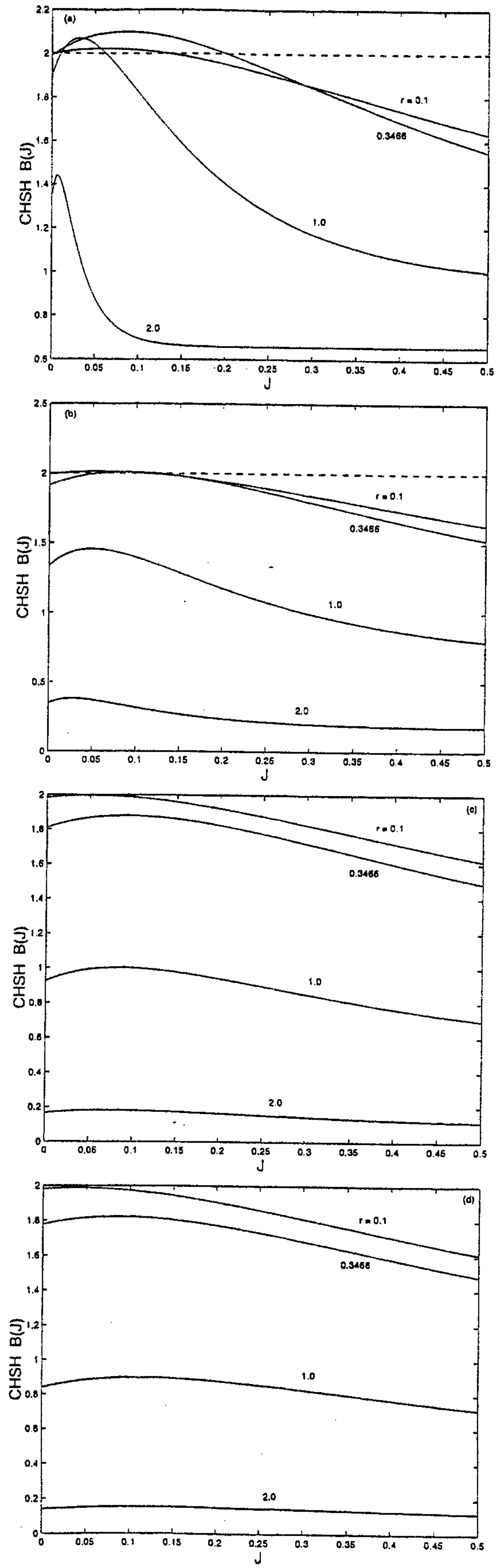

FIG. 2. The function $\mathcal{B}(\mathcal{J})$ from Eq. (38) as a function of $\mathcal{J}$ for various values of (r,η). Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. In each of the plots (a)-(d) a family of curves is drawn for fixed efficiency η and four values of $r = \{0.1, \frac{\ln 2}{2}, 1.0, 2.0\}$. (a) $\eta = 0.99$, (b) $\eta = 0.90$, (c) $\eta = 0.70$, (d) $\eta = 0.50$; in all cases, $\bar{n} = 0$.

The function constructed by Banaszek and Wodkiewicz to test for local hidden variable theories is denoted by $\mathcal{B}$ and is defined by

$$\begin{aligned} &\mathcal{B}(\mathcal{J}) \\ &= \Pi^{out}_{\text{EPR}}(0,0;0,0) + \Pi^{out}_{\text{EPR}}(\sqrt{\mathcal{J}},0;0,0) \\ &\quad + \Pi^{out}_{\text{EPR}}(0,0;-\sqrt{\mathcal{J}},0) - \Pi^{out}_{\text{EPR}}(\sqrt{\mathcal{J}},0;-\sqrt{\mathcal{J}},0)\,, \end{aligned} \tag{38}$$

where $\mathcal{J}$ is a positive (real) constant. As shown in Ref. [18,19], any local theory must satisfy

$$-2 \leq \mathcal{B} \leq 2\,. \tag{39}$$

As emphasized by Banaszek and Wodkiewicz for the lossless case, $\Pi^{out}_{\text{EPR}}(0,0;0,0) = 1$ "describes perfect correlations ... as a manifestation of ... photons always generated in pairs."

There are several important points to be made about this result. In the first place, in the ideal case with no loss ($\eta = 1$), there is a violation of the Bell inequality of Eq. (39) for any $r > 0$. Further, this threshold for the onset of violations of the CHSH inequality coincides with the threshold for entanglement as given in Eq. (25), which likewise is the point for surpassing $F_{classical} = \frac{1}{2}$ as in Eqs. (33,34) and as shown in our prior analysis of Ref. [6] which is notably based upon a quite different approach.

Significantly, there is absolutely nothing special about the point $r = \frac{\ln 2}{2} \approx 0.3466$ (i.e., the point for which $\exp[-2r] = 0.5$ and for which $F = \frac{2}{3}$ for the teleportation of coherent states), in contradistinction to the claims of Grangier and Grosshans to the contrary [11,12]. Instead, any $r > 0$ leads to a nonseparable EPR state, to a violation of a Bell inequality, and to $F > F_{classical} = \frac{1}{2}$ for the teleportation of coherent states. There is certainly no surprise here since we are dealing with pure states for $\eta = 1$ [41].

We next examine the case with $\eta < 1$, which is clearly of interest for any experiment. Figure 2 illustrates the behavior of $\mathcal{B}$ as a function of $\mathcal{J}$ for various values of the squeezing parameter r and of the efficiency η. Note that throughout our analysis in this section, we make no attempt to search for optimal violations, but instead follow dutifully the protocol of Banaszek and Wodkiewicz as expressed in Eq. (38) for the case with losses as well.

From Figure 2 we see that for any particular set of parameters $(r;\eta)$, there is an optimum value $\mathcal{J}_{\max}$ that leads to a maximum value for $\mathcal{B}(\mathcal{J}_{\max})$, which is a situation analogous to that found in the discrete variable case. By determining the corresponding value $\mathcal{J}_{\max}$ at each (r,η), in Figure 3 we construct a plot that displays the dependence of $\mathcal{B}$ on the squeezing parameter r for various values of efficiency η. Note that all cases shown in the figure lead to fidelity $F > F_{classical}$.

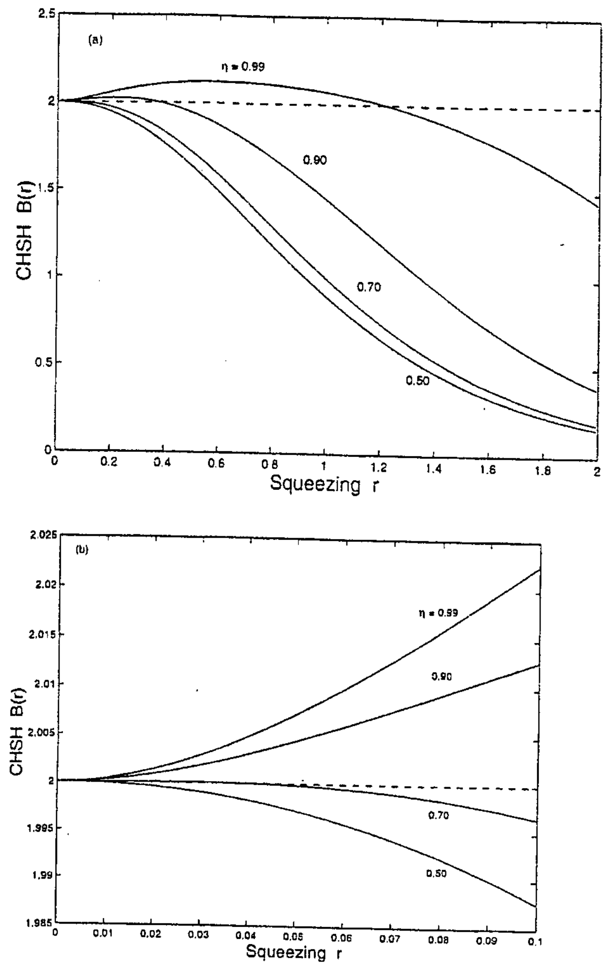

FIG. 3. (a) The quantity $\mathcal{B}$ from Eq. (38) as a function of r for various values of efficiency $\eta = \{0.99, 0.90, 0.70, 0.50\}$ as indicated. At each point in (r,η), the value of $\mathcal{J}$ that maximizes $\mathcal{B}$ has been chosen. Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. Also note that $F > \frac{1}{2}$ for all $r > 0$. (b) An expanded view of $\mathcal{B}$ in the small r region $r \leq 0.1$. Note that in the case $\eta = 0.70$, $\mathcal{B} > 2$ for small r. In all cases, $\bar{n} = 0$.

For $\frac{2}{3} < \eta \leq 1$ there are regions in r that produce direct violations of the Bell inequality considered here, namely $\mathcal{B} > 2$ [42]. In general, these domains with $\mathcal{B} > 2$ contract toward smaller r with increasing loss $(1-\eta)$. In fact as r increases, η must become very close to unity in order to preserve the condition $\mathcal{B} > 2$, where for $r \gg 1$,

$$2(1-\eta)\cosh(2r) \ll 1. \tag{40}$$

This requirement is presumably associated with the EPR state becoming more "nonclassical" with increasing r and hence more sensitive to dissipation [43]. Stated somewhat more quantitatively, recall that the original state $|EPR\rangle_{1,2}$ of Eq. (3) is expressed as a sum over correlated photon numbers for each of the two EPR beams $(1,2)$. The determination of $\mathcal{B}$ derives from (displaced) parity measurements on the beams $(1,2)$ (i.e., projections onto odd and even photon number), so that $\mathcal{B}$ should be sensitive to the loss of a single photon. The mean photon number $\bar{n}_i$ for either EPR beam goes as $\sinh^2 r$, with then the probability of losing no photons after encountering the beam-splitter with transmission η scaling as roughly $p_0 \sim [\eta]^{\bar{n}_i}$. We require that the total probability for the loss of one or more photons to be small, so that

$$(1 - p_0) \ll 1, \tag{41}$$

and hence for $(1-\eta) \ll 1$ and $r \gg 1$ that

$$(1-\eta)\bar{n}_i \sim (1-\eta)\exp(2r) \ll 1, \tag{42}$$

in correspondence to Eq. (40) [44].

On the other hand, note that small values of r in Figure 3 lead to direct violations of the CHSH inequality $\mathcal{B} > 2$ with much more modest efficiencies [43]. In particular, note that for $r = \frac{\ln 2}{2} \approx 0.3466$ and $\eta = 0.90$, $F < \frac{2}{3}$ [from Eq. (35)]. This case and others like it provide examples for which mixed states are nonseparable and yet directly violate a Bell inequality, but for which $F \leq \frac{2}{3}$. Such mixed states do not satisfy the criteria of Grangier and Grosshans (neither with respect to their Heisenberg-type inequality nor with respect to their information exchange), yet they are states for which $\frac{1}{2} < F \leq \frac{2}{3}$ and $\mathcal{B} > 2$, which in and of itself calls the claims of Grangier and Grosshans into question. There remains the possibility that $F > \frac{2}{3}$ might be sufficient to warranty that mixed states in this domain would satisfy that $\mathcal{B} > 2$, and hence to exclude a description of the EPR state in terms of a local hidden variables theory.

To demonstrate that this is emphatically not the case, we examine further the relationship between the quantity $\mathcal{B}$ relevant to the CHSH inequality and the fidelity F. Figure 4 shows a parametric plot of $\mathcal{B}$ versus F for various values of the efficiency η. The curves in this figure are obtained from plots as in Figures 1 and 3 by eliminating the common dependence on r. From Figure 4, we are hard pressed to find any indication that the value $F = \frac{2}{3}$ is in any fashion noteworthy with respect to violations of the CHSH inequality. In particular, for efficiency $\eta \simeq 0.90$ most relevant to current experimental capabilities, the domain $F > \frac{2}{3}$ is one largely devoid of instances with $\mathcal{B} > 2$, in contradistinction to the claim of Grangier and Grosshans that this domain is somehow "safer" [11] with respect to violations of Bell's inequalities. Moreover, contrary to their dismissal of the domain $\frac{1}{2} < F \leq \frac{2}{3}$ as not being manifestly quantum, we see from Figure 4 that there are in fact regions with $\mathcal{B} > 2$. Overall, the conclusions of Grangier and Grosshans [11] related to the issues of violation of a Bell inequality and of teleportation fidelity are simply not supported by an actual quantitative analysis.

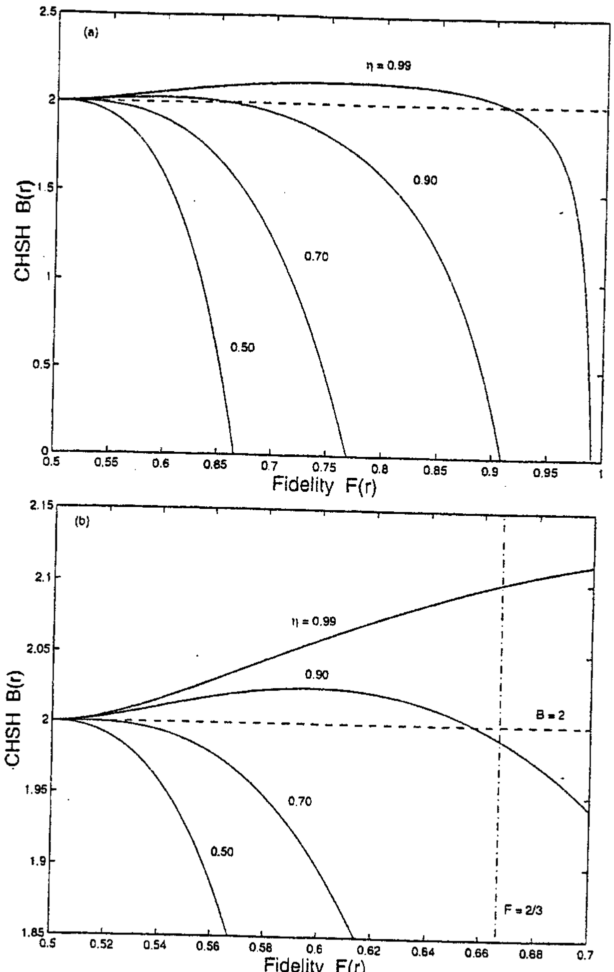

FIG. 4. (a) A parametric plot of the CHSH quantity $\mathcal{B}$ [Eq. (38)] versus fidelity F [Eq. (35)]. The curves are constructed from Figures 1 and 3 by eliminating the r dependence, now over the range $0 \leq r \leq 5$, with r increasing from left to right for each trace. The efficiency η takes on the values $\eta = \{0.99, 0.90, 0.70, 0.50\}$ as indicated; in all cases, $\bar{n} = 0$. Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. (b) An expanded view around $\mathcal{B} = 2$. Note that $\mathcal{B} > 2$ is impossible for $F \leq F_{classical} = \frac{1}{2}$, but that $\mathcal{B} > 2$ for $F > F_{classical}$ in various domains (including for $\eta = 0.70$ at small r). The purported boundary $F = \frac{2}{3}$ proposed by Grangier and Grosshans [11,12] is seen to have no particular significance. Contrary to their claims, $F = \frac{2}{3}$ provides absolutely no warranty that $\mathcal{B} > 2$ for $F > \frac{2}{3}$, nor does it preclude $\mathcal{B} > 2$ for $F < \frac{2}{3}$.

While the above results follow from the particular form of the CHSH inequality introduced by Banaszek and Wodkiewicz [18,19], we should note that another quite different path to a demonstration of the inadequacy of local realism for continous quantum varialbes has recently been proposed by Ralph, Munro, and Polkinghorne [45]. These authors consider a novel scheme involving measurements of quadrature-phase amplitudes for two entangled beams (A, B). These beams are formed by combining *two* EPR states (i.e., a total of four modes, two for each beam). Relevant to our discussion is that maximal violations of a CHSH inequality (i.e., $\mathcal{B} = 2\sqrt{2}$) are predicted for $r \ll 1$, with then a decreasing maximum value of $\mathcal{B}$ for increasing r. Once again, the threshold for onset of the violation of a Bell's inequality coincides with the threshold for entanglement of the relevant fields [i.e., Eq. (25)], with no apparent significance to the boundary set by the Heisenberg-type inequality Eq. (24) of Grangier and Grosshans.

To conclude this section, we would like to inject a note of caution concerning any discussion involving issues of testing Bell's inequalities and performing quantum teleportation. We have placed them in juxtaposition here to refute the claims of Grangier and Grosshans related to a possible connection between the bound $F = \frac{2}{3}$ and violation of Bell's inequalities (here, via the behavior of the CHSH quantity $\mathcal{B}$). However, in our view there is a conflict between these concepts, with an illustration of this point provided by the plot of the CHSH quantity $\mathcal{B}$ [Eq. (39)] versus fidelity F [Eq. (35)] in Figure 4. For example, for $\eta = 0.90$, $\mathcal{B} > 2$ over the range $0.50 < F \lesssim 0.66$, while $\mathcal{B} < 2$ for larger values of F. Hence, local hidden variables theories are excluded for modest values of fidelity $0.50 < F \lesssim 0.66$, but not for larger values $F \gtrsim 0.66$. This leads to the strange conclusion that quantum resources are required for smaller values of fidelity but not for larger ones. The point is that the nonseparable states that can enable quantum teleportation, can *in a different context* also be used to demonstrate a violation of local realism. Again, the juxtaposition of these concepts in this section is in response to the work of Ref. [11], which in any event offers no quantitative evidence in support of their association.

VI. BELL'S INEQUALITIES FOR SCALED CORRELATIONS

The conclusions reached in the preceding section about violations of the CHSH inequality by the EPR (mixed) state for modes $(1, 2)$ follow directly from the analysis of Banaszek and Wodkiewicz [18,19] as extended to account for losses in propagation. Towards the end of making these results more amenable to experimental investigation, recall that the more traditional versions of the Bell inequalities formulated for spin $\frac{1}{2}$ particles or photon polarizations are based upon an analysis of the expectation value

$$E(\vec{a}, \vec{b}) \tag{43}$$

for detection events at locations $(1, 2)$ with analyzer settings along directions $(\vec{a}, \vec{b})$. As emphasized by Clauser and Shimony, actual experiments do not measure directly $E(\vec{a}, \vec{b})$ but rather record a reduced version due to "imperfections in the analyzers, detectors, and state preparation [20]." Even after more than thirty years of experiments, no *direct* violation of the CHSH inequality has been recorded, where by *direct* we mean without the need for post-selection to compensate for propagation and detection efficiencies (also called *strong violations*) [22,23]. Rather, only subsets of events that give rise to coincidences are included for various polarization settings. This "problem" is the so-called detector efficiency loophole that several groups are actively working to close.

Motivated by these considerations, we point out that an observation of violation of a Bell-type inequality has recently been reported [24], based in large measure upon the earlier proposal of Ref. [46], as well as that of Refs. [18,19]. This experiment was carried out in a pulsed mode, and utilized a source that generates an EPR state of the form given by Eq. (11) in the limit $r \ll 1$. Here, the probability $P(\alpha_1, \alpha_2)$ of detecting a coincidence event between detectors (D_1, D_2) for the EPR beams $(1, 2)$ is given by

$$P(\alpha_1, \alpha_2) = M[1 + V\cos(\phi_1 - \phi_2 + \theta)], \tag{44}$$

with then the correlation function E relevant to the construction of a CHSH inequality $-2 \leq S \leq 2$ given by

$$E(\phi_1, \phi_2) = V\cos(\phi_1 - \phi_2 + \theta), \tag{45}$$

where the various quantities are as defined in association with Eqs. (2,3) in Ref. [24]. Note that the quantity M represents an overall scaling that incorporates losses in propagation and detection. Significantly, Kuzmich *et al.* demonstrated a violation of a CHSH inequality ($S_{\exp} = 2.46 \pm 0.06$) in the limit $r \ll 1$ and with inefficient propagation and detection $\eta \ll 1$, albeit with the so-called "detection" or "fair-sampling" loophole.

In terms of our current discussion, this experimental violation of a CHSH inequality is only just within the nonseparability domain $\Delta x^2 + \Delta p^2 < 1$ (by an amount that goes as $\eta r \ll 1$), yet it generates a large violation of a CHSH inequality. If this same EPR state were employed for the teleportation of coherent states, the fidelity obtained would likewise be only slightly beyond the quantum-classical boundary $F_{classical} = \frac{1}{2}$. It would be far from the boundary $F = \frac{2}{3}$ offered by Grangier and Grosshans as the point for "useful entanglement," yet it would nonetheless provide an example of teleportation with fidelity $F > \frac{1}{2}$ and of a violation of a CHSH inequality. Of course, the caveat would be the aforementioned "fair-sampling" loophole, but this same restriction accompanies all previous experimental demonstrations of violations of Bell's inequalities. Once again, we find no support for the purported significance of the criteria offered by Grangier and Grosshans [11,12].

VII. CONCLUSIONS

Beyond the initial analysis of Ref. [6], we have examined further the question of the appropriate point of demarcation between the classical and quantum domains for the teleportation of coherent states. In support of our previous result that fidelity $F_{classical} = \frac{1}{2}$ represents the bound attainable by Alice and Bob if they make use only of a classical channel, we have shown that the nonseparability criteria introduced in Refs. [14,15] are sufficient to ensure fidelity beyond this bound for teleportation with the EPR state of Eq. (11), which is in general a mixed state. Significantly, the threshold for entanglement for the EPR beams as quantified by these nonseparability criteria coincides with the standard boundary between classical and quantum domains employed in Quantum Optics, namely that the Glauber-Sudarshan phase-space function becomes non-positive definite [17].

Furthermore, we have investigated possible violations of Bell's inequalities and have shown that the threshold for the onset of such violations again corresponds to $F_{classical} = \frac{1}{2}$. For thermal photon number $\bar{n} = 0$ as appropriate to current experiments, direct violations of a CHSH inequality are obtained over a large domain in the degree of squeezing r and overall efficiency η. Significant relative to the claims of Grangier and Grosshans [11,12] is that there is a regime for nonseparability and violation of the CHSH inequality for which $F < \frac{2}{3}$ and for which their Heisenberg inequalities are not satisfied. Moreover, the experiment of Ref. [24] has demonstrated a violation of the CHSH inequality in this domain for $(r, \eta) \ll 1$ (i.e., F would be only slightly beyond $\frac{1}{2}$), albeit with the caveat of the "fair-sampling" loophole. We conclude that fidelity $F > \frac{2}{3}$ offers absolutely no warranty or "safety" relative to the issue of violation of a Bell inequality (as might be desirable, for example, in quantum cryptography), in direct disagreement with the assertions by Grangier and Grosshans. Quite the contrary, larger r (and hence larger F) leads to an exponentially decreasing domain in allowed loss $(1 - \eta)$ for violation of the CHSH inequality, as expressed by Eq. (40) [44].

Moreover, beyond the analysis that we have presented here, there are several other results that support $F_{classical} = \frac{1}{2}$ as being the appropriate boundary between quantum and classical domains. In particular, we note that any nonseparable state and hence also our mixed EPR state is always capable of teleporting perfect entanglement, i.e., one half of a pure maximally entangled state. This applies also to those nonseparable states which lead to fidelities $\frac{1}{2} < F \leq \frac{2}{3}$ in coherent-state teleportation. According to Refs. [11,12], this would force the conclusion that there is entanglement that is capable of teleporting truly nonclassical features (i.e., entanglement), but which is not "useful" for teleporting rather more classical states such as coherent states. Further, in Ref. [47] it has been shown that entanglement swapping can be achieved with two pure EPR states for *any nonzero squeezing* in both initial states. Neither of the initial states has to exceed a certain amount of squeezing in order to enable successful entanglement swapping. This is another indication that $F = \frac{2}{3}$, which is exceedable in coherent-state teleportation only with more than 3 dB squeezing, is of no particular significance.

We also point out that Giedke et al. have shown that for all bipartite Gaussian states, nonseparability implies distillability [48,49]. This result applies to those nonseparable states for which $\frac{1}{2} < F \leq \frac{2}{3}$ in coherent state teleportation, which are otherwise dismissed by Grangier and Grosshans as not "useful." To the contrary, entanglement distillation could be applied to the mixed EPR states employed for teleportation in this domain (and in general for $F > \frac{1}{2}$) [50], leading to enhanced teleportation fidelities and to expanded regions for violations of Bell's inequalities for the distilled subensemble.

By contrast, there appears to be no support for the claims of Grangier and Grosshans [11,12] that their so-called Heisenberg inequality and information exchange are somehow "special" with respect to the issues of separability and violations of Bell's inequalities. They have neither found fault in the prior analysis of Ref. [6], nor with the application of the work on nonseparability [14–16] to the current problem. They have likewise provided no analysis that directly supports their assertion that their Heisenberg inequality is in any way significant to the possibility that "the behavior of the *observed* quantities can be mimicked by a *classical* and *local* model." [11] Rather, they attempt to set aside by *fiat* a substantial body of evidence in favor of the boundary $F_{classical} = \frac{1}{2}$ for the teleportation of coherent states with a lack of rigor indicated by their claim that "$F = \frac{2}{3}$ would be much safer." [11]

However, having said this, we emphasize that there is no criterion for quantum teleportation that is sufficient to all tasks. For the special case of teleportation of coherent states, the boundary between classical and quantum teleportation is fidelity $F_{classical} = \frac{1}{2}$, as should by now be firmly established. Fidelity $F > \frac{2}{3}$ will indeed enable certain tasks to be accomplished that could not otherwise be done with $\frac{1}{2} < F \leq \frac{2}{3}$. However, $F = \frac{2}{3}$ is in no sense an important point of demarcation for entrance into the quantum domain. There is instead a hierarchy of fidelity thresholds that enable ever more remarkable tasks to be accomplished via teleportation within the quantum domain, with no one value being sufficient for all possible purposes. For example, if the state to be teleported were some intermediate result from a large-scale quantum computation as for Shor's algorithm, then surely the relevant fidelity threshold would be well beyond any value currently accessible to experiment, $F \sim 1 - \epsilon$, with $\epsilon \lesssim 10^{-4}$ to be compatible with current work in fault tolerant architectures. We have never claimed that $F = \frac{1}{2}$

endows special powers for all tasks such as these, only that it provides an unambiguous point of entry into the quantum realm for the teleportation of coherent states.

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[36] For the experiment of Ref. [4], the measured variances were $\Delta x^2 \approx (0.8 \times \frac{1}{2}) \approx \Delta p^2$, so that $\Delta x^2 + \Delta p^2 \approx 0.8 < 1$.

[37] Eq. (32) is written in the notation of the current paper. It follows from Eq. (2) of Ref. [4] for the case gain $g = 1$, Alice's detection efficiency $\eta_A^2 = 1$ (denoted η^2 in [4]), and with the replacement of efficiencies ($\xi_1^2 = \xi_2^2$) from [4] $\rightarrow \eta$ here.

[38] The expressions of Eqs. (32-35) are strictly applicable only for the case gain $g = 1$ for teleportation of coherent states uniformly distributed over the entire complex plane. More generally, when working with a restricted alphabet of states (e.g., coherent amplitudes selected from a Gaussian distribution), the optimal gain is not unity when referenced to the fidelity averaged over the input alphabet. In fact as shown in Ref. [6], the optimal gain is $g = 1/(1 + \lambda)$ for an input alphabet of coherent states distributed according to $p(\beta) = (\lambda/\pi)exp(-\lambda|\beta|^2)$. When incorporated into the current analysis, we find that nonseparable EPR states are sufficient to achieve $F > (1 + \lambda)/(2 + \lambda)$ (again with optimal gain $g \neq 1$) although F is no longer a monotonic function of r as in Figure 1. This result is in complete correspondence

with our prior result that $F_{classical} = (1+\lambda)/(2+\lambda)$ is the bound for teleportation when only a classical channel is employed. To simplify the discussion, here we set $\lambda = 0$ throughout, with then the optimal gain $g = 1$ and $F_{classical} = \frac{1}{2}$.

[39] Note that for the noisy EPR states considered here with Gaussian statistics, Eqs. (33) and (34) are both necessary and sufficient conditions. However, this is not generally the case for states with Gaussian statistics; there are in fact nonseparable Gaussian states for which Eq. (25) is not true. Close examination of Ref. [14] shows that their results concerning necessary and sufficient conditions for states with Gaussian statistics apply only to Gaussian states in a particular standard form. Explicit counter examples can be constructed from Ref. [40].

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