

## REPORT DOCUMENTATION PAGE

AFRL-SR-BL-TR-01-

0358

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1. REPORT DATE (DD-MM-YYYY) 28/11/2000		2. REPORT TYPE Final Report		3. DATES COVERED (From - To) 01/01/98-30/09/00	
4. TITLE AND SUBTITLE Distributed Systems for Problems in Robust Control and Visual Tracking				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER AF/F49620-98-1-0168	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Allen Robert Tannenbaum				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Georgia Tech School of ECE 777 Atlantic Drive Atlanta, GA 30332-0250				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NM 801 North Randolph Street Arlington, VA 22203-1977				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	

## 12. DISTRIBUTION / AVAILABILITY STATEMENT

Unlimited

Approved for public release,  
distribution unlimited

20010625 115

## 13. SUPPLEMENTARY NOTES

## 14. ABSTRACT

The objectives of this research grant have been development of new theory and algorithms for the analysis and synthesis of robust controllers for nonlinear and possibly distributed feedback systems. A key application is in controlled active vision, including visual tracking, the control of autonomous vehicles, motion planning, and the utilization of visual information in guidance and control. The PI has been employing methods from differential geometry, geometry-driven flows, computational algebraic geometry, as well as interpolation theory and functional analysis. For tracking, the PI has been using a new model of deformable contours based on geodesics and minimal surfaces. He has been combining this local information with more global statistical information (employing Bayesian models) for more powerful region/boundary based models capable of detecting and tracking motion much better. Work was also done for object recognition using differential invariants.

## 15. SUBJECT TERMS

Nonlinear and distributed systems, visual tracking, computer vision, computer-driven flows, deformable active contours.

## 16. SECURITY CLASSIFICATION OF:

a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified
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## 17. LIMITATION OF ABSTRACT

## 18. NUMBER OF PAGES

23

## 19a. NAME OF RESPONSIBLE PERSON

Allen Tannenbaum

19b. TELEPHONE NUMBER (include area code)  
404-894-7582

FINAL REPORT: DISTRIBUTED SYSTEMS FOR PROBLEMS IN  
ROBUST CONTROL AND VISUAL TRACKING  
AFOSR-AF/F49620-98-1-0168

Allen R. Tannenbaum  
Department of Electrical and Computer Engineering  
Georgia Institute of Technology  
Atlanta, GA 30332-0250

## 1 Introduction

The objectives of this project have been the development of a novel theoretical basis and algorithms for the analysis and synthesis of robust and reliable controllers for nonlinear and possibly distributed feedback systems. The applications include our continuing work in controlled active vision, the use of visual information in guidance and control, associated problems in visual tracking, motion planning, and the control of remote autonomous vehicles.

Our work has included the use of curvature-driven partial differential equations to explore problems in controlled active vision. This involves a synthesis of methods from optimal control, image processing, and computer vision. We have devised a noise-resistant skeleton for object recognition as well as a new formulation of optical flow derived from the theory of area-preserving flows. We have explored the combination of statistical methods with our geometric PDE's to develop methods in tracking of dynamic SAR imagery. In particular, we have extended our active contour methods to include more global information making the procedure much more robust. We have included specific estimation theoretic ideas in our geodesic snake framework now as well. We have also continued our work in robust nonlinear control using operator and interpolation theoretic methods as well as considered the use of computational algebraic geometry for problems in optimal control.

In this AFOSR sponsored project, we have considered the use of active contours or snakes for problems in visual tracking. Our general approach is based on the theory of geodesics and minimal surfaces in a conformal geometry. We have introduced some more sophisticated features into our geometric-based cost function. We are incorporating non-local information into our models. We have been exploring the use of adaptive filtering schemes and Bayesian statistics. We also want to couple boundary and region data. One way to do this, is the use of minimum description length (MDL). We are also considering snake algorithms in higher co-dimensions for example for the detection of curves in space. All the usual schemes assume codimension one. Such a program is very important for path-planning. We want to treat detection and tracking together coupling the gradient snakes and estimating region-based features via motion detection term we have defined. Closely, connected to this is the work in the computation of optical flow and in particular weakening the optical flow constraint with one based on the theory of area-preserving mappings. We are now putting in an optical flow term in our active contour functional to better track moving images. We are also considering knowledge-based conformal factors in this context when the number of objects is known. This also allows the inclusion of Bayesian statistics in our tracking.

We are also continuing to explore the use of differential invariants for object recognition. This is essential for some of our visual tracking work. When objects or cameras move, the continued recognition and tracking of the image is a problem of key importance in vision.

Our work on invariant signature manifolds appears particularly well-adapted to this problem, since the signature manifold remains unchanged even while the object and/or camera is in motion, provided the motions belong to the symmetry group in question.

Skeletons are a powerful shape descriptor. The main drawback of classical skeletons is their sensitivity to noise in the object outline. In order to stabilize the skeleton extraction algorithm, we have developed a robust version of skeletons based on using the affine distance. We have shown that this new type of skeleton is quite robust to noise and small deformations of the object. We are using it as one of our key methods of feature extraction for tracking in our continuing research program.

We have been working on the utilization of Groebner bases techniques for certain problems in optimal control. As is well-known, time-optimal problems lead to switching surfaces which typically are defined or may be approximated by polynomial equations. Since the complexity of the switching surfaces can grow to be quite large, this may become quickly a formidable task. Here is where new techniques in computational algebraic geometry may become vital in effectively solving this problem. In our AFOSR sponsored work, we have introduced Groebner bases in this context which will reduce the problem to a combinatorial one. We believe that this methodology may be useful in considering some complex matrix perturbation problems which are connected to antenna array placement.

We have been continuing our project on the generalizations of the linear  $H^\infty$  methodology to nonlinear systems. We have extended our ideas to the general standard problem, and are now working on computer implementations to test on design examples. From a theoretical standpoint this method has been completely justified. We are also continuing to develop a global game theoretic approach to nonlinear  $H^\infty$  optimization.

We now give some details about some of the key results we have developed in our present AFOSR grant.

## 2 Curvature-Driven Flows in Controlled Active Vision

In the past few years, we have become very interested in visual tracking and the general area of the use of visual information in a feedback loop. This is a central area in which the multivariable control methods developed over the past twenty years could have a major impact. In order to work on visual tracking, we have had to develop certain techniques from image processing and computer vision, which has led to several new research directions. Indeed, we have been using geometric invariant flows for various problems in active vision. These flows themselves are very much motivated by ideas in *optimal control*; see [29]. We will now discuss some of the key ideas in curve and surface evolution.

### 2.1 Background on Curve and Surface Evolution

A geometric set or shape can be defined by its boundary. In the case of bounded planar shapes for example, this boundary consists of closed planar curves. We will only deal with closed planar curves, keeping in mind that these curves are boundaries of planar shapes.

A curve may be regarded as a trajectory of a point moving in the plane. Formally, we define a curve  $\mathcal{C}(\cdot)$  as the map  $\mathcal{C}(p) : S^1 \rightarrow \mathbf{R}^2$  (where  $S^1$  denotes the unit circle). We assume that our curves are have no self-intersections, i.e., are embedded.

We now consider plane curves deforming in time. Let  $\mathcal{C}(p, t) : S^1 \times [0, \tau) \rightarrow \mathbf{R}^2$  denote a family of closed embedded curves, where  $t$  parametrizes the family, and  $p$  parametrizes each curve. Assume that this family evolves according to the following equation:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{T} + \beta \vec{N} \\ \mathcal{C}(p, 0) = \mathcal{C}_0(p) \end{cases} \quad (1)$$

where  $\vec{N}$  is the inward Euclidean unit normal,  $\vec{T}$  is the unit tangent, and  $\alpha$  and  $\beta$  are the tangential and normal components of the evolution velocity  $\vec{v}$ , respectively. In fact, it is easy to show that  $\text{Img}[\mathcal{C}(p, t)] = \text{Img}[\hat{\mathcal{C}}(w, t)]$ , where  $\mathcal{C}(p, t)$  and  $\hat{\mathcal{C}}(w, t)$  are the solutions of

$$\mathcal{C}_t = \alpha \vec{T} + \beta \vec{N} \quad \text{and} \quad \hat{\mathcal{C}}_t = \bar{\beta} \vec{N},$$

respectively. (Here  $\text{Img}[\cdot]$  denotes the image of the given parametrized curve in  $\mathbf{R}^2$ .) Thus the tangential component affects only the parametrization, and not  $\text{Img}[\cdot]$  (which is independent of the parametrization by definition). Therefore, assuming that the normal component  $\beta$  of  $\vec{v}$  (the curve evolution velocity) in (1) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \vec{N}, \quad (2)$$

where  $\beta = \vec{v} \cdot \vec{N}$ .

The evolution (2) was studied by different researchers for different functions  $\beta$ . This type of flow was introduced into the theory of shape in [27, 28]. One of the most studied evolution equations is obtained for  $\beta = \kappa$ , where  $\kappa$  is the Euclidean curvature:

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \vec{N}. \quad (3)$$

Equation (3) has its origins in physical phenomena [2, 22]. It is called the *geometric heat equation* or the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (3). Grayson [22] proved that a planar embedded non-convex curve converges to a convex one, and from there to a round point. Note that in spite of the local character of the evolution, global properties are obtained, which is a very interesting feature of this evolution. For other results related to the Euclidean shortening flow, see [2, 3, 22].

Another important example is obtained when one sets  $\beta = 1$  in equation (2):

$$\frac{\partial \mathcal{C}}{\partial t} = \vec{N}. \quad (4)$$

This equation simulates, under certain conditions, the grassfire flow [11]. (More precisely, the unique weak solution of (4) which satisfies the *entropy* condition [44] gives the grassfire flow.) This grassfire flow is also the basis of the morphological scale-space defined by the disk as structuring element. Moreover, one can prove that with different selections of  $\beta$ , other morphological scale-spaces are obtained [29].

In [28], we have studied the following equation in order to develop a hierarchy of shape,

$$\frac{\partial \mathcal{C}}{\partial t} = (1 + \epsilon \kappa) \vec{N}. \quad (5)$$

If  $\epsilon \rightarrow 0$  in (5), the grassfire flow is obtained, and this introduces singularities (*shocks*) in the evolving curve. (The shocks define the well-known skeleton.) On the other hand, if  $\epsilon \rightarrow \infty$ , equation (5) reduces to the classical Euclidean curve shortening flow, which smooths the curve [22, 44]. The combination of these two opposite features gives very interesting properties. When a curve evolves according to (5), the evolution of the curve slope satisfies a reaction-diffusion equation [45]. The reaction term, which tends to create singularities, competes with the diffusion term which tends to smooth the curve. For each different value of  $\epsilon$ , a scale-space is obtained by looking at the solution of (5), and considering the time  $t$  as the scale parameter. We have called the set of all the scale-spaces obtained for all values of  $\epsilon$ , the *reaction-diffusion scale-space*. In particular, we see that the Euclidean shortening flow (equation (3)) defines an Euclidean invariant scale-space (the equation admits Euclidean invariant solutions). In contrast with other scale-spaces, like the one obtained from the classical linear heat equation, this one is a full geometric scale-space. The progressive smoothing given by  $\kappa$  is geometrically intrinsic to the curve.

We now discuss the affine analogue of the Euclidean shortening flow. (The affine group  $SA_2$  is the group generated by unimodular transformations and translations of  $\mathbf{R}^2$ . Under certain natural conditions, it provides a good approximation to the full group of perspective projective transformations.) Then in [36, 40], we show that the simplest non-trivial affine invariant flow in the plane is given by

$$C_t = \kappa^{1/3} \vec{N}. \quad (6)$$

The question now is what happens when a non-convex curve evolves according to (6). The following result answers this question [6]:

**Theorem 1** *Let  $C(\cdot, 0) : S^1 \rightarrow \mathbf{R}^2$  be a smooth embedded curve in the plane. Then there exists a family  $C : S^1 \times [0, T) \rightarrow \mathbf{R}^2$  satisfying*

$$C_t = \kappa^{1/3} \vec{N},$$

*such that  $C(\cdot, t)$  is smooth for all  $t < T$ , and moreover there is a  $t_0 < T$  such that for all  $t > t_0$ ,  $C(\cdot, t)$  is smooth and convex.*

Theorem 1 means that just as in the Euclidean case, a non-convex curve first becomes convex when evolving according to (6). After this, the curve converges to an ellipse from our results in [40]. Because of this, and other related properties (see [41]), we can conclude that equation (6) is the affine analogue of (3) for smooth embedded curves, and thus is called the *affine shortening flow*. (It is also the affine invariant formulation of the geometric heat equation.) One can use it to construct an *affine invariant scale-space* for planar shapes [41]. This is in conjunction with the theory of differential invariants issential for our work in *invariant object recognition*.

## 2.2 Visual Tracking

Much of our recent research in image processing and computer vision has been motivated by problems in controlled active vision, especially visual tracking. We have already described some of the relevant work in control above, and so we would like to consider now some of the

key tools we plan to employ from our work in computer vision and image processing. These include active contours, optical flow and stereo disparity, and certain results from invariant theory for invariant object recognition, as well as the curve and surface evolution methodology sketched in Section 2. We expect these methods to play an integral part in our controlled active vision research program.

### 2.2.1 Geometric Active Contours

In this section, we will describe a paradigm for *snakes* or *active contours* based on principles from curvature driven flows and the calculus of variations. Active contours may be regarded as autonomous processes which employ image coherence in order to track various features of interest over time. Such deformable contours have the ability to conform to various object shapes and motions. Snakes have been utilized for segmentation, edge detection, shape modeling, and visual tracking. Active contours have also been widely applied for various applications in medical imaging. For example, snakes have been employed for the segmentation of myocardial heart boundaries as a prerequisite from which such vital

information such as ejection-fraction ratio, heart output, and ventricular volume ratio can be computed.

In the classical theory of snakes, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain constraints set by the user. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters. Moreover, natural criteria for the splitting and merging of contours (or for the treatment of multiple contours) are not readily available in this framework.

In [26], we propose a deformable contour model to successfully solve such problems, and which will become one of our key techniques for tracking. Our method is based on the Euclidean curve shortening evolution (see Section 2.1) which defines the gradient direction in which a given curve is shrinking as fast as possible relative to Euclidean arc-length, and on the theory of conformal metrics. We multiply the Euclidean arc-length by a conformal factor defined by the features of interest which we want to extract, and then we compute the corresponding gradient evolution equations. The features which we want to capture therefore lie at the bottom of a potential well to which the initial contour will flow. Moreover, our model may be easily extended to extract 3D contours based on motion by mean curvature [26, 32].

The starting point of this work is [14, 33] in which a snake model based on the level set formulation of the Euclidean curve shortening equation is proposed. More precisely, the model is

$$\frac{\partial \Psi}{\partial t} = \phi(x, y) \|\nabla \Psi\| \left( \operatorname{div} \left( \frac{\nabla \Psi}{\|\nabla \Psi\|} \right) + \nu \right). \quad (7)$$

Here the function  $\phi(x, y)$  depends on the given image and is used as a “stopping term.” For example, the term  $\phi(x, y)$  may be chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. One may take [14, 33]

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^2}, \quad (8)$$

where  $I$  is the (grey-scale) image and  $G_\sigma$  is a Gaussian (smoothing filter) filter. The function  $\Psi(x, y, t)$  evolves in (7) according to the associated level set flow for planar curve evolution in the normal direction with speed a function of curvature which was introduced in [38, 44].

It is important to note that the Euclidean curve shortening part of this evolution, namely

$$\frac{\partial \Psi}{\partial t} = \|\nabla \Psi\| \operatorname{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) \quad (9)$$

is derived as a gradient flow for shrinking the perimeter as quickly as possible. The constant *inflation term*  $\nu$  is added in (7) in order to keep the evolution moving in the proper direction. Note that we are taking  $\Psi$  to be negative in the interior and positive in the exterior of the zero level set.

We would like to modify the model (7) in a manner suggested by the curve shortening flow. We change the ordinary arc-length function along a curve  $C = (x(p), y(p))^T$  with parameter  $p$  given by

$$ds = (x_p^2 + y_p^2)^{1/2} dp,$$

to

$$ds_\phi = (x_p^2 + y_p^2)^{1/2} \phi dp,$$

where  $\phi(x, y)$  is a positive differentiable function. Then we want to compute the corresponding gradient flow for shortening length relative to the new metric  $ds_\phi$ .

Accordingly set

$$L_\phi(t) := \int_0^1 \left\| \frac{\partial C}{\partial p} \right\| \phi dp.$$

Then taking the first variation of the modified length function  $L_\phi$ , and using integration by parts (see [26]), we get that

$$L'_\phi(t) = - \int_0^{L_\phi(t)} \left\langle \frac{\partial C}{\partial t}, \phi \kappa \vec{N} - (\nabla \phi \cdot \vec{N}) \vec{N} \right\rangle ds$$

which means that the direction in which the  $L_\phi$  perimeter is shrinking as fast as possible is given by

$$\frac{\partial C}{\partial t} = (\phi \kappa - (\nabla \phi \cdot \vec{N})) \vec{N}. \quad (10)$$

This is precisely the gradient flow corresponding to the minimization of the length functional  $L_\phi$ . The level set version of this is

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \operatorname{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) + \nabla \phi \cdot \nabla \Psi. \quad (11)$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the potential well described by the gradient flow (11). We may also add a constant inflation term, and so derive a modified model of (7) given by

$$\frac{\partial \Psi}{\partial t} = \phi \|\nabla \Psi\| \left( \operatorname{div}\left(\frac{\nabla \Psi}{\|\nabla \Psi\|}\right) + \nu \right) + \nabla \phi \cdot \nabla \Psi. \quad (12)$$

Notice that for  $\phi$  as in (8),  $\nabla\phi$  will look like a doublet near an edge. Of course, one may choose other candidates for  $\phi$  in order to pick out other features.

We now have very fast implementations of these snake algorithms based on level set methods [38, 44]. Clearly, the ability of the snakes to change topology, and quickly capture the desired features will make them an indispensable tool for our visual tracking algorithms. See also [48] for more details about this.

We are also studying an affine invariant snake model for tracking. (The evolution itself works using a level set model of  $\kappa^{1/3}\vec{\mathcal{N}}$  as discussed in Section 2.1.)

Our methods have been extended to 3D images. Indeed, we have developed affine invariant volumetric smoothers in [37]. We also have 3D active contour evolvers for image segmentation, shape modeling, and edge detection based on both snakes (inward deformations) and bubbles (outward deformations) in our work [26, 32]. We have employed affine smoothers in movies as a preprocessing tool for motion estimation. We are working on extensions to dynamic imagery, and the incorporation of more global information for the active contours as well as utilizing Bayesian statistical models.

### 3 Invariant Flows

In this section, we will summarize some of our recent AFOSR supported work on the classification of invariant geometric flows. It is interesting to note how the calculus of variations and thus optimal control type techniques plays such a fundamental role in solving this problem. This is based on our work reported in [37].

#### 3.1 Outline of Invariant PDE's

Consider the evolution of hypersurfaces which are assumed to be represented by the graph of a function. We let the  $p + 1$ -dimensional Euclidean space  $E \simeq \mathbf{R}^p \times \mathbf{R}$ , with coordinates  $x = (x^1, \dots, x^p)$  representing the independent variables, and  $u \in \mathbf{R}$  the dependent variable.

The hypersurface  $\mathcal{S} \subset E$  will be identified with the graph of a function  $u(x)$ , defined on a domain  $x \in D \subset \mathbf{R}^p$ . The symmetry group  $G$  will be a finite-dimensional, connected transformation group acting on  $E$ . Each group transformation  $g \in G$  will map hypersurfaces to hypersurfaces by point-wise transformation.

In Lie's theory of symmetry groups, one replaces the actual group transformations by their infinitesimal generators, which are vector fields on the domain  $E$ , taking the general form

$$\mathbf{v} = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} = \xi^1(x, u) \frac{\partial}{\partial x^1} + \dots + \xi^p(x, u) \frac{\partial}{\partial x^p} + \varphi(x, u) \frac{\partial}{\partial u}. \quad (13)$$

Each vector field generates a local one-parameter group of transformations (or flow) on  $E$ , obtained by integrating the associated system of ordinary differential equations

$$\frac{dx}{d\varepsilon} = \xi(x, u), \quad \frac{du}{d\varepsilon} = \varphi(x, u), \quad (14)$$

where  $\varepsilon$  represents the group parameter. In other words, the group transformations have the Taylor expansion

$$x(\varepsilon) = x + \varepsilon\xi(x, u) + \dots, \quad u(\varepsilon) = u + \varepsilon\varphi(x, u) + \dots. \quad (15)$$

The order  $\varepsilon$  terms in (15) are known as the *infinitesimal group transformations*, and can be identified with the generating vector field (13). The different one-parameter groups combine to generate the entire connected group action of  $G$ .

Fixing the vector field (13), let  $u(x, \varepsilon)$  denote the one-parameter family of hypersurfaces (functions) obtained from a given hypersurface  $u(x, 0) = u(x)$  by applying the group transformation with parameter  $\varepsilon$ . The infinitesimal change in the hypersurface is found by expanding in powers of  $\varepsilon$  using Taylor's Theorem and the chain rule. Thus, the value of the transformed function  $u$  at the new point  $x(\varepsilon)$  is given by

$$u(x(\varepsilon), \varepsilon) = u(x) + \varepsilon\varphi(x, u(x)) + \dots \quad (16)$$

On the other hand, if we are interested in the value of the transformed function at the original point  $x = x(0)$ , we substitute (15) into (16) to deduce the alternative expansion

$$u(x, \varepsilon) = u(x) + \varepsilon Q[u(x)] + \dots \quad (17)$$

The function

$$Q[u] = \varphi(x, u) - \sum_{i=1}^p \xi^i(x, u) \frac{\partial u}{\partial x^i}, \quad (18)$$

is known as the *characteristic* of the vector field (13). The characteristic  $Q$  depends on first order derivatives  $u_i = \partial u / \partial x^i$  because the group transformations are acting on the independent variables  $x$  as well as the dependent variable  $u$ . In particular, a  $G$ -invariant hypersurface is independent of the group parameter  $\varepsilon$ , and hence satisfies the first order partial differential equation  $Q(x, u^{(1)}) = 0$ , indicating its "infinitesimal invariance" under the vector field  $\mathbf{v}$ . Conversely, any infinitesimally invariant function, i.e., any solution to the characteristic equation  $Q = 0$ , is, in fact, invariant under the entire connected transformation group.

Consider the function  $F[u] = F(x, u^{(n)})$  depending on  $x$ ,  $u$ , and the derivatives of  $u$ , denoted by  $u_J = D_J u$ . Here  $D_J = D_{j_1} D_{j_2} \dots D_{j_k}$  are the total derivative operators, which differentiate treating  $u$  as a function of  $x$ . The infinitesimal variation in the function  $F[u]$  under the group generated by the vector field  $\mathbf{v}$  is then given by

$$\left. \frac{d}{d\varepsilon} F[u(x, \varepsilon)] \right|_{\varepsilon=0} = \sum_J \frac{\partial F}{\partial u_J} D_J Q. \quad (19)$$

In (19) we evaluate  $F$  and  $u$  at the original point  $x$ . If we are interested in the value at the transformed point  $x(\varepsilon)$ , we must include an additional term arising from the change of independent variable, as in the passage from (17) to (16). We deduce the expansion

$$F(x(\varepsilon), u^{(n)}(x, \varepsilon)) = F(x, u^{(n)}) + \varepsilon \text{pr } \mathbf{v}(F) + \dots, \quad (20)$$

where

$$\text{pr } \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F \quad (21)$$

defines the "prolongation" of the vector field  $\mathbf{v}$ , denoted  $\text{pr } \mathbf{v}$ , which forms the infinitesimal generator of the prolonged group action on the space of derivatives.

A function  $F(x, u^{(n)})$  is called a *differential invariant* if its value is not affected by the group transformations. Thus we require that the left hand side of (20) be independent of  $\varepsilon$ . The infinitesimal invariance condition is obtained by differentiating with respect to  $\varepsilon$ . This produces

$$0 = \text{pr } \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F. \quad (22)$$

Condition (22), for  $\mathbf{v}$  an arbitrary infinitesimal generator of  $G$ , is necessary and sufficient for  $F$  to be a differential invariant.

A transformation group  $G$  is called a *symmetry group* of a differential equation

$$F(x, u^{(n)}) = 0 \quad (23)$$

if it maps solutions to solutions. The differential equation (23) admits  $G$  as a symmetry group if and only if the infinitesimal invariance condition

$$\text{pr } \mathbf{v}[F] = 0 \quad \text{whenever} \quad F = 0 \quad (24)$$

holds for all infinitesimal generators of  $G$ .

### 3.2 Invariant Hypersurface Flows

The goal is to determine the general form that a  $G$ -invariant evolution equation

$$u_t = K(x, u^{(n)}) \quad (25)$$

must take. Here we have introduced an additional variable  $t$  — the time or scale parameter — which is not affected by our group transformations.

Thus, for  $p = 1$ , we will determine all possible invariant curve evolutions in the plane under a given transformation group, while for  $p = 2$  we find the invariant surface evolutions. According to (21), the infinitesimal change in the  $t$ -derivative of  $u$  at the transformed point is

$$\left. \frac{d}{d\varepsilon} u_t(x, t, \varepsilon) \right|_{\varepsilon=0} = D_t Q + \sum_{i=1}^p \xi^i D_i u_t = Q_u u_t, \quad (26)$$

where

$$Q_u = \frac{\partial Q}{\partial u} = \frac{\partial \varphi}{\partial u} - \sum_{i=1}^p \frac{\partial \xi^i}{\partial u} \frac{\partial u}{\partial x^i}. \quad (27)$$

Therefore, using the infinitesimal condition (24), and substituting for  $u_t$  according to the equation (25), we deduce the basic invariance condition that an evolution equation must satisfy in order to admit a prescribed symmetry group.

**Lemma 1** *A connected transformation group  $G$  is a symmetry group of the evolution equation  $u_t = K[u]$  if and only if the infinitesimal condition*

$$\text{pr } \mathbf{v}(K) = Q_u K \quad (28)$$

*holds for every infinitesimal generator  $\mathbf{v}$  of the group  $G$  with associated characteristic  $Q$ .*

To discover a  $G$ -invariant evolution equation for an arbitrary group, we consider the  $G$ -invariant functionals. An  $n$ -th order *variational problem* consists of finding the extremals (maxima or minima) of a *functional*

$$\mathcal{L}_D[u] = \int_D L(x, u^{(n)}) dx = \int_D L(x, u^{(n)}) dx^1 \wedge \dots \wedge dx^p, \quad (29)$$

subject to certain boundary conditions.

The integrand  $L[u] = L(x, u^{(n)})$ , known as the *Lagrangian*, is a smooth function depending on  $x$ ,  $u$  and the derivatives of  $u$ . A transformation group  $G$  is a symmetry group of a variational problem provided it leaves the functional (29) invariant.

More precisely, given a function  $u(x)$  defined on a domain  $D$ , and a one-parameter subgroup of  $G$ , we let  $u(x, \varepsilon)$  denote the transformed function, which is defined on a transformed domain  $D(\varepsilon)$ . Invariance of the functional requires that  $\mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] = \mathcal{L}_D[u(x)]$ . Using the standard Jacobian change of variables formula for multiple integrals, the infinitesimal invariance condition is then found by differentiating:

$$\begin{aligned} 0 &= \left. \frac{d}{d\varepsilon} \mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] \right|_{\varepsilon=0} \\ &= \left. \frac{d}{d\varepsilon} \int_D L[u(x(\varepsilon), \varepsilon)] \det \left[ \frac{\partial x(\varepsilon)}{\partial x} \right] dx \right|_{\varepsilon=0} \\ &= \int_D [\text{pr } \mathbf{v}(L) + L \text{ div } \xi] dx. \end{aligned} \quad (30)$$

Here  $\text{div } \xi = \sum D_i \xi^i$  is the total divergence arising from the infinitesimal change in the independent variables.

**Lemma 2** *A connected transformation group  $G$  a symmetry group of the variational problem  $\int L dx$  if and only if every infinitesimal generator  $\mathbf{v}$  satisfies the infinitesimal condition*

$$\text{pr } \mathbf{v}(L) + L \text{ div } \xi = 0. \quad (31)$$

The smooth extremals (maxima and minima) of a variational problem are known to satisfy the classical Euler-Lagrange equation,

$$E(L) := \sum_{\#J=0}^n (-D)_J \frac{\partial L}{\partial u_J} = 0, \quad \alpha = 1, \dots, q. \quad (32)$$

where  $(-D)_J = (-D_{j_1})(-D_{j_2}) \dots (-D_{j_k})$  is the signed total derivative. This condition is the infinite-dimensional analog of the vanishing gradient condition for maxima and minima of ordinary functions. The Euler-Lagrange equation is obtained by taking the variational derivative of the functional. For example, if  $\mathcal{L}$  represents the  $G$ -invariant arc-length or surface area functional, the Euler-Lagrange equation will describe the  $G$ -invariant minimal curves or surfaces. In general, the invariance of a variational problem under a given transformation group implies the invariance of its Euler-Lagrange equation. (The converse, however, is not true.) We will be interested in precisely how the Euler-Lagrange equation varies, and this is the result of the following key lemma.

**Lemma 3** Let  $\text{pr } \mathbf{v}$  be the prolonged vector field (21). Let  $L(x, u^{(n)})$  be a Lagrangian. Then

$$E(\text{pr } \mathbf{v}(L) + L \text{div } \xi) = \text{pr } \mathbf{v}(E(L)) + (Q_u + \text{div } \xi)E(L). \quad (33)$$

From this, we can construct invariant evolution equations. Suppose that  $L$  is a  $G$ -invariant Lagrangian, e.g., defining the group invariant arc length or area. Then  $L$  satisfies the infinitesimal invariance condition (31), and hence (33) implies the identity

$$\text{pr } \mathbf{v}[E(L)] + (\text{div } \xi + Q_u)E(L) = 0. \quad (34)$$

Equation (34) means that  $E(L)$  is a relative differential invariant of weight  $-\text{div } \xi - Q_u$ . In particular, the Euler-Lagrange equation  $E(L) = 0$  is invariant under  $G$ , as claimed. On the other hand  $L$  itself is a relative invariant of weight  $-\text{div } \xi$ . Since the prolonged vector field  $\text{pr } \mathbf{v}$  acts as a derivation, the ratio  $E(L)/L$  is a relative differential invariant weight  $-Q_u$ , i.e., it satisfies

$$\text{pr } \mathbf{v} \left[ \frac{E(L)}{L} \right] + Q_u \left[ \frac{E(L)}{L} \right] = 0. \quad (35)$$

Consequently, its reciprocal  $L/E(L)$  (assuming  $E(L) \neq 0$ ) satisfies (28) and defines a  $G$ -invariant evolution equation. We have therefore deduced our fundamental theorem from [37]:

**Theorem 2** Let  $G$  be a transformation group, and let  $L dx$  be a  $G$ -invariant Lagrangian with non-identically zero Euler-Lagrange derivative  $E(L)$ . Then every  $G$ -invariant evolution equation has the form

$$u_t = \frac{L}{E(L)} I, \quad (36)$$

where  $I$  is a arbitrary differential invariant of  $G$ .

Although (36) defines the most general class of invariant evolution equations, the case when the differential invariant  $I$  is constant is not necessarily the simplest one. In the planar Euclidean case,  $L = \sqrt{1 + u_x^2}$  is the Euclidean arc length Lagrangian, so that

$$E(L) = -D_x \frac{\partial L}{\partial u_x} = -\frac{u_{xx}}{(1 + u_x^2)^{3/2}} = -\kappa.$$

Thus the general Euclidean-invariant evolution equation has the form

$$u_t = -\sqrt{1 + u_x^2} \frac{I}{\kappa},$$

where  $I$  is an arbitrary function of  $\kappa$  and its arc length derivatives. Choosing  $I = \kappa$  produces the simplest one (eikonal equation), while  $I = \kappa^2$  produces the Euclidean curve shortening flow.

One can also deduce the following:

**Proposition 1** Suppose  $G$  is a connected transformation group, and  $L dx$  a  $G$ -invariant  $p$ -form such that  $E(L) \neq 0$ . Then  $E(L)$  is a differential invariant if and only if  $G$  is volume-preserving.

**Corollary 1** Let  $G$  be a connected volume preserving transformation group. Then, up to constant multiple, the  $G$ -invariant flow of lowest order has the form

$$u_t = L, \quad (37)$$

where  $\omega = L dx^1 \wedge \dots \wedge dx^p$  is the invariant  $p$ -form of minimal order such that  $E(L) \neq 0$ .

### 3.3 Affine invariant surface flows

We apply the preceding results to describe the simplest possible affine invariant surface evolution. This gives, for convex surfaces, the surface version of the affine shortening flow for curves. The group  $G$  is the (*special*) affine group  $SL(3, \mathbb{R})$ , consisting of all  $3 \times 3$  matrices with determinant 1, combined with the translations. Let  $S$  be a smooth strictly convex surface in  $\mathbb{R}^3$ , which we write locally as a graph  $u = u(x, y)$ .

The simplest affine-invariant area-form is constructed from the affine-invariant metric, which is given by [37]

$$L dx \wedge dy = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2} dx \wedge dy,$$

where

$$\kappa = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2},$$

denotes the usual Gaussian curvature of  $S$ . Corollary 1 allows us to conclude:

**Corollary 2** *Up to constant multiple, the simplest affine-invariant evolution equation has the form*

$$u_t = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2}. \tag{38}$$

## 4 Nonlinear Robust Control

Under AFOSR support, we have worked extensively in nonlinear robust control. Besides the theoretical and practical questions involved in finding an implementable nonlinear design methodology, it is interesting to note that certain associated problems of causality have arisen in this area, which we would like to briefly indicate as well. In fact, as a result of this effort, we have been able to put an explicit causality constraint in commutant lifting theory for the first time [16, 18, 21].

There have been several attempts to extend dilation theoretic techniques to nonlinear input/output operators, especially those which admit a Volterra series expansion. Typically, one is reduced to applying the classical (linear) commutant lifting theorem to an  $H^2$ -space defined on some  $D^n$  (where  $D$  denotes the unit disc). Now when one applies the classical result to  $D^n$  ( $n \geq 2$ ), even though time-invariance is preserved, causality may be lost. Indeed, for analytic functions on the disc  $D$ , time-invariance implies causality. For analytic functions on the  $n$ -disc ( $n > 1$ ), this is not necessarily the case. Consequently, for a dilation result in  $H^2(D^n)$  we need to include the causality constraint explicitly in the set-up of the dilation problem. We will discuss a way of doing this now based on [21, 17, 18].

### 4.1 Nonlinear Interpolation

We now formulate a Causal Commutant Lifting Theorem that is suitable for control applications, in particular the full standard problem. It forms the basis of our research in nonlinear interpolation.

For the standard problem in robust control theory we may extract the following mathematical set-up. We are given complex separable Hilbert spaces  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$  equipped with the unilateral shifts  $S_{\mathcal{E}_1}, S_{\mathcal{E}_2}, S_{\mathcal{F}_1}, S_{\mathcal{F}_2}$ , respectively. Let  $\Theta_1 : \mathcal{E}_1 \rightarrow \mathcal{F}_1$  be a co-isometry intertwining  $S_{\mathcal{E}_1}$  with  $S_{\mathcal{F}_1}$  (i.e.,  $\Theta_1 S_{\mathcal{E}_1} = S_{\mathcal{F}_1} \Theta_1$ ), and let  $\Theta_2 : \mathcal{F}_2 \rightarrow \mathcal{E}_2$  be an isometry intertwining  $S_{\mathcal{E}_2}$  with  $S_{\mathcal{F}_2}$ . We let  $U_{\mathcal{E}_1}$ , be the minimal unitary dilation of  $S_{\mathcal{E}_1}$  on  $\mathcal{K}_{\mathcal{E}_1}$ , and similarly for  $U_{\mathcal{E}_2}$  on  $\mathcal{K}_{\mathcal{E}_2}$ ,  $U_{\mathcal{F}_1}$  on  $\mathcal{K}_{\mathcal{F}_1}$ , and  $U_{\mathcal{F}_2}$  on  $\mathcal{K}_{\mathcal{F}_2}$ .

Now let

$$P_{\mathcal{E}_2}^{(n)} := (I - S_{\mathcal{E}_2}^n S_{\mathcal{E}_2}^{*n}), \quad P_{\mathcal{F}_2}^{(n)} := (I - S_{\mathcal{F}_2}^n S_{\mathcal{F}_2}^{*n}), \quad n \geq 0.$$

We let the sequence  $P_{\mathcal{E}_2}^{(n)}$  define the causal structure on  $\mathcal{E}_2$ , and similarly the causal structure of  $\mathcal{F}_2$  is defined by the sequence  $P_{\mathcal{F}_2}^{(n)}$ . Moreover, the causal structure on  $\mathcal{E}_1$  is defined by a general sequence of operators  $P_1^{(n)}$ ,  $n \geq 0$ , satisfying the standard causal structure conditions [21], and similarly the causal structure on  $\mathcal{F}_1$  is defined by a sequence of operators  $P_2^{(n)}$ ,  $n \geq 0$ , satisfying these conditions as well. We assume that the input/output operators  $\Theta_1, \Theta_2$ , are causal with respect to the above structures. We let  $W : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  denote a causal operator intertwining  $S_{\mathcal{E}_1}$  with  $S_{\mathcal{E}_2}$ , and let  $Q : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  be a causal operator intertwining  $S_{\mathcal{F}_1}$  with  $S_{\mathcal{F}_2}$ .

Define

$$\mathcal{E}_1^{(n)} := (I - P_1^{(n)})\mathcal{E}_1, \quad \forall n \geq 0,$$

and

$$W_n := S_{\mathcal{E}_2}^{*n} W|_{\mathcal{E}_1^{(n)}}.$$

Moreover, let

$$\mathcal{E}_1^{(c)} = \overline{\mathcal{E}_1^{(co)}}$$

where

$$\mathcal{E}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_1}^{*j} \mathcal{E}_1^{(j)} \subset \mathcal{K}_{\mathcal{E}_1}, \quad S_{\mathcal{E}_1}^{(c)} := U_{\mathcal{E}_1}|_{\mathcal{E}_1^{(c)}}.$$

Finally, we define  $W_c : \mathcal{E}_1^{(co)} \rightarrow \mathcal{E}_2$ , by

$$W_c g := W_n g_n,$$

for  $g = U_{\mathcal{E}_1}^{*n} g_n$ ,  $g_n \in \mathcal{E}_1^{(n)}$ ,  $n \geq 0$ .

Note that we can make a similar construction on the spaces  $\mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ . In particular, for a causal  $Q : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ , such that  $Q S_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q$ , we can define  $Q_c : \mathcal{F}_1^{(co)} \rightarrow \mathcal{F}_2$ , where

$$\mathcal{F}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{F}_1}^{*j} \mathcal{F}_1^{(j)}.$$

Next, it is easy to see both  $W_c$  and  $Q_c$  extend by continuity to the closure  $\mathcal{E}_1^{(c)}$ , respectively  $\mathcal{F}_1^{(c)} = \overline{\mathcal{F}_1^{(co)}}$ . Clearly, we also have

$$\|W_c\| = \|W\|, \quad W_c|_{\mathcal{E}_1} = W, \quad W_c S_{\mathcal{E}_1}^{(c)} = S_{\mathcal{E}_2} W_c,$$

and  $\|W - \Theta_2 Q \Theta_1\| = \|(W - \Theta_2 Q \Theta_1)_c\|$ . Now set

$$\mu(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : Q S_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This corresponds to the *classical standard control problem*. We also set

$$\mu_c(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : Q \text{ causal}, Q S_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This is the *causal standard control problem*.

Let  $\hat{\Theta}_1 : \mathcal{K}_{\mathcal{E}_1} \rightarrow \mathcal{K}_{\mathcal{F}_1}$  denote the extension of the co-isometry  $\Theta_1 : \mathcal{E}_1 \rightarrow \mathcal{F}_1$ , that is uniquely defined by

$$\hat{\Theta}_1 U_{\mathcal{E}_1}^{*n} e_1 = U_{\mathcal{F}_1}^{*n} \Theta_1 e_1, \quad \forall e_1 \in \mathcal{E}_1.$$

Note that  $\hat{\Theta}_1$  is also isometric and  $\hat{\Theta}_1 U_{\mathcal{E}_1} = U_{\mathcal{F}_1} \hat{\Theta}_1$ .

We can now state the following key result [19]:

**Theorem 3** *Notation as above.*

1.  $\mu_c(W, \Theta_1, \Theta_2) = \mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2)$ .
2.  $Q_{opt}$  is a causal optimal solution, i.e.,

$$\mu_c(W, \Theta_1, \Theta_2) = \|W - \Theta_1 Q_{opt} \Theta_2\|$$

if and only if  $Q_{opt,c}$  is such that

$$\mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2) = \|W_c - \Theta_2 Q_{opt,c} \hat{\Theta}_1 | \mathcal{E}_1^{(c)}\|.$$

Finally, let us recall how the classical standard problem can be solved using the commutant lifting theorem. Set

$$\begin{aligned} \mathcal{H}_1 &:= \mathcal{E}_1^{(c)} \ominus (\hat{\Theta}_1 | \mathcal{E}_1^{(c)})^* \mathcal{E}_1^{(c)}, \\ \mathcal{H}_2 &:= \mathcal{E}_2 \ominus \Theta_2 \mathcal{F}_2. \end{aligned}$$

Let  $P : \mathcal{E}_2 \rightarrow \mathcal{H}_2$  denote orthogonal projection. Then we define the operator

$$\Lambda = \Lambda(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2) : \mathcal{H}_1 \rightarrow \mathcal{H}_2, \quad (39)$$

by

$$\Lambda h := P W_c h, \quad h \in \mathcal{H}_1. \quad (40)$$

Then using the commutant lifting theorem, one may show that

$$\|\Lambda\| = \mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2).$$

Thus from the above theorem, we have the following result:

**Corollary 3** *Notation as above. Then*

$$\mu_c(W, \Theta_1, \Theta_2) = \|\Lambda(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2)\|.$$

Thus we see that Theorem 3 and Corollary 3, allow one to reduce a causal optimization problem to one involving classical interpolation.

This leads to an explicit computable solution of the nonlinear standard problem based on an iterative interpolation procedure. The computations are based on our previous skew Toeplitz methodology that we developed for distributed  $H^\infty$  control. See [17, 18, 19].

## 4.2 Game Theory and Nonlinear Optimization

We have previously developed an iterative commutant lifting approach to nonlinear system design. The iterative commutant lifting technique is basically a local analytic method for nonlinear system synthesis. We have also been exploring a very different approach applicable to certain systems with saturations (and “hard” noninvertible nonlinearities) based on a game-theoretic interpretation of the classical commutant lifting theorem. This motivates us to formulate a nonlinear commutant lifting result in such a saddle-point, game-theoretic framework.

A related approach to nonlinear design has already been employed by a number of researchers; see [9, 10, 25, 49] and the references therein. As is well known, game theoretic ideas have already been extensively applied in linear  $H^\infty$  theory. In our research, instead of considering general nonlinear systems we have limited ourselves to the concrete (but certainly interesting case) of linear systems with input saturations. Such systems are, of course, essential for many practical applications. We should add that a similar approach is valid for many of the hard, memoryless, noninvertible nonlinearities which appear in control.

In order to motivate our game-theoretic approach to nonlinear  $H^\infty$ , we will first give a “saddle-point” interpretation of the classical Sarason theorem in a special case. We let  $w, m \in H^\infty$  with  $m$  inner. Set  $H(m) := H^2 \ominus mH^2$ , we let  $P_{H(m)} : H^2 \rightarrow H(m)$  denote orthogonal projection, and  $S(m)$  denote the compressed shift. We let  $\|\cdot\|$  denote the 2-norm  $\|\cdot\|_2$  on  $H^2$  as well as the associated induced operator norm.

In our recent AFOSR work, prove that

$$\inf_{q \in H^\infty} \sup_{\|f\| \leq 1} \|(w - mq)f\| = \sup_{\|f\| \leq 1} \inf_{q \in H^\infty} \|(w - mq)f\| = \sup_{\|f\| \leq 1} \inf_{g \in H^2} \|w - mg\|.$$

Now it is easy to show there always an optimal  $q_o$ . We now assume that

$$\|w(S(m))\|_{ess} < \|w(S(m))\|,$$

where  $\|\cdot\|_{ess}$  denotes the essential norm. Then there exists  $f_o \in H^2$ ,  $\|f_o\| = 1$  (a maximal vector), such that

$$\|(w - mq_o)(S(m))f_o\| = \|w(S(m))f_o\| = \|w(S(m))\| = \|(w - mq_o)(S(m))\|.$$

Now

$$\begin{aligned} P_{H(m)}(w - mq_o)f_o &= (w - mq_o)(S(m))f_o = w(S(m))f_o, \\ (w - mq_o)f_o &= w(S(m))f_o. \end{aligned}$$

So

$$\|(w - mq_o)f\| \leq \|(w - mq_o)(S(m))f_o\| = \|(w - mq_o)f_o\|$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ . Moreover,

$$\|w(S(m))f_o\| = \|(w - mq)(S(m))f_o\| \leq \|(w - mq)f_o\|.$$

Hence, we get that

$$\|(w - mq_o)f\| \leq \|(w - mq_o)f_o\| \leq \|(w - mq)f_o\| \tag{41}$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ , and for all  $q \in H^\infty$ . It is a nonlinear analogue of the saddle-point condition (41) that we want to analyze for saturated systems. Indeed, assuming the saddle-point condition (41), we can derive all of the standard consequences of the Sarason theorem. Thus it is precisely the existence of a saddle-point which we have treated in this nonlinear setting.

By virtue of interpretation of the commuting lifting theorem as asserting the existence of a saddle-point, we have derived a global approach to sensitivity minimization for input saturated systems. Thus for  $\sigma_\theta$ , a saturation of magnitude  $\theta < 1$ , and  $m \in H^\infty$  inner, we want to know when there exist  $f_o \in H^2$ ,  $\|f_o\| \leq 1$ ,  $q_o$  continuous, causal, time-invariant, such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \leq \|(w - m\sigma_\theta \circ q_o)f_o\| \leq \|(w - m\sigma_\theta \circ q)f_o\|$$

for all  $f \in H^2$ ,  $\|f\| \leq 1$ ,  $q$  continuous, causal, time-invariant. Such a  $q_o$  (when it exists) will correspond to the optimal compensator, and

$$\mu := \|(w - m\sigma_\theta \circ q_o)f_o\|$$

will be the optimal performance in the weighted sensitivity minimization problem. But this is equivalent to finding  $g_o = q_o(f_o) \in H^2$  such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \leq \|w(f_o) - m\sigma_\theta(g_o)\| \leq \|(w - m\sigma_\theta \circ q)f_o\|. \quad (42)$$

Our approach then has been to follow an analogous line of reasoning which we just outlined in our analysis of the saddle-point condition in the linear case. This leads to nonlinear commutant lifting theorem valid on a *convex space* which can be used to develop a global robust design procedure for nonlinear plants with hard nonlinearities.

## 5 Conclusions

In our work, we we developed a broad research program making use geometric flows, differential invariants, computational geometry, and optimization theory, to study several key problems in systems and control including visual tracking.

We have considered a synthesis of methods from optimal control, image processing, and computer vision for visual tracking. The approach to computer vision and image processing is based on certain curvature dependent evolution equations that may be used for image enhancement, active contours, edge detection, morphology, shape recognition, shape-from-shading, motion planning, and stereo disparity. A number of the resulting flows have certain key invariance properties and so make contact with classical Lie theory. Our efforts can lead to enhanced man-machine interfaces for interactions with computers and more complicated systems such as remote controlled weapons and vehicles. Our work has applications ranging from the airborne laser program, to image-guided surgery, and to automatic target recognition.

Our work in vision is being combined with our longstanding program in distributed and nonlinear control. Computational geometry has begun to play a large part in this effort. Indeed, these methods are having an impact on a number of the algorithms we are developing in controlled active vision as well.

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**Patents:**

"Conformal Geometry and Texture Mappings," (co-inventors Sigurd Angenent, Steven Haker, Allen Tannenbaum, and Ron Kikinis), pending.

## **6 Papers of Allen Tannenbaum and Collaborators under AFOSR-AF/F49620-98-1-0168**

(This list includes papers published after January 1, 1998 the beginning date of AFOSR-AF/F49620-98-1-0168.)

1. "Differential and numerically invariant signature curves applied to object recognition" (with E. Calabi, P. Olver, C. Shakiban), *International Journal of Computer Vision* **26** (1998), pp. 107-135.

2. "Area and length minimizing flows for segmentation" (with Y. Lauziere, K. Siddiqi, and S. Zucker), *IEEE Trans. Image Processing* **7** (1998), pp. 433-444.
3. "Introduction to special issue of *IEEE Trans. Image Processing* on partial differential equation methods in image processing" (with V. Caselles, J. M. Morel, and G. Sapiro), *IEEE Trans. Image Processing* **7** (1998), pp. 269-274.
4. "Shapes, shocks, and wiggles" (with K. Siddiqi, B. Kimia, and S. Zucker), *Journal of Image and Vision Computing* **17** (1999), pp. 365-373.
5. "Curve evolution models for real-time identification with application to plasma etching" (with J. Berg and A. Yezzi), *IEEE Trans. Aut. Control* **44** (1999), pp. 99-104.
6. "Skew Toeplitz solution to the  $H^\infty$  problem for infinite dimensional unstable plants" (with K. Hirata, Y. Yamamoto, and T. Katayama), to appear in *Trans. of the Society of Instrument and Control Engineers*.
7. "On the affine invariant heat equation for nonconvex curves" (with S. Angenent and G. Sapiro), *Journal of the American Mathematical Society* **11** (1998), pp. 601-634.
8. "On skew Toeplitz operators, II" (with H. Bercovici and C. Foias), *Operator Theory: Advances and Applications* **103** (1998).
9. "The shape triangle: parts, protrusions, and bends" (with B. Kimia, K. Siddiqi, and S. Zucker), to appear in *Vision Research*.
10. "On a state space solution to the singular value problem of Toeplitz operators and the computation of the gap" (with K. Hirata and Y. Yamamoto), *Systems and Control Letters*, 1999.
11. "Knowledge-based segmentation of SAR data with learned priors" (with S. Haker and G. Sapiro), *IEEE Trans. Image Processing* **9** (2000), pp. 298-302.
12. "Laplace-Beltrami operator and brain surface flattening" (with S. Angenent, S. Haker, and R. Kikinis), **18** (1999), pp. 700-711. *IEEE Trans. on Medical Imaging*.
13. "On the computation of switching surfaces in optimal control: A Groebner basis approach" (with U. Walther and T. Georgiou), to appear in *IEEE Trans. Aut. Control*.
14. "Conformal surface parametrization for texture mappings" (with S. Angenent, S. Haker, M. Halle, R. Kikinis), *IEEE Trans. on Visualization and Computer Graphics* **6** (2000), pp. 181-190.
15. "Nondistorting flattening maps and the 3D visualization of colon CT images," (with S. Angenent, S. Haker, R. Kikinis), to appear in *IEEE Trans. of Medical Imaging*.
16. "On the computation of affine skeletons of planar cruves and the detection of skew symmetry" (with S. Belelu and G. Sapiro), to appear in *Pattern Recognition*.
17. "Eye tracking: A challenge for robust control," to appear in *Journal of Nonlinear and Robust Control*.

18. "Noise-resistant affine skeletons of planar curves," (with S. Betelu and G. Sapiro), submitted for publication in *IEEE PAMI*.
19. "Hamilton-Jacobi skeletons" (with K. Siddiqi, S. Bouix, and S. Zucker), submitted for publication in *Int. Journal Computer Vision*.
20. "Gradients, curvature, and visual tracking" (with A. Yezzi), in *Computational Methods for Optimal Design and Control* edited by J. Borggaard, J. Burns, E. Cliff, and S. Schreck, Birkhauser-Verlag, 1998.
21. "Multivariable gain margins and spectral interpolation," in *Open Problems in Mathematical Systems and Control Theory*, edited by V. Blondel, E. Sontag, M. Vidyasagar, and J. Willems, Springer, New York, 1998.
22. "Mean curvature flows, edge detection, and medical image segmentation" (with S. Angenent, S. Haker, A. Yezzi), to appear as a book chapter.
23. "Visual tracking, active vision, and gradient flows" (with A. Yezzi), in *The Confluence of Vision and Control*, edited by G. Hager and D. Kriegman, *Lecture Notes in Control and Information Sciences* **237**, Springer-Verlag, New York, 1998.
24. "Switching surfaces and Groebner bases" (with T. Georgiou), in *Learning, Complexity, and Control*, edited by Y. Yamamoto and S. Hara, *Lecture Notes in Control and Information Sciences* **240**, Springer-Verlag, New York, 1998.
25. "On area preserving maps of minimal distortion" (with S. Angenent, S. Haker, and R. Kikinis), in *System Theory: Modeling, Analysis, and Control*, edited by T. Djaferis and I. Schick, Kluwer, Holland, 1999, pages 275-287.
26. "Hyperbolic smoothing of shapes" (with K. Siddiqi, and S. Zucker), *Proceedings of ICCV*, January 1998.
27. "Knowledge based segmentation of SAR images" (with S. Haker and G. Sapiro), *Proceedings of International Conference on Image Processing*, 1998.
28. "On the psychophysics of the shape triangle" (with B. Kimia, K. Siddiqi, and S. Zucker), submitted to the *Vision/Attention Conference*, Providence, RI, 1999.
29. "On the computation of the gap metric" (with K. Hirata and Y. Yamamoto), *Proceedings of MTNS*, 1998.
30. "Harmonic analysis and flattening the brain surface" (with S. Angenent, S. Haker, and R. Kikinis), *Proceedings of MICCAI*, Cambridge, England, 1999.
31. "Categorical features in shape perception" (with B. Kimia, K. Siddiqi, and S. Zucker), *ARVO Conference*, 1999.
32. "On the psychophysics of the shape triangle" (with B. Kimia, K. Siddiqi, and S. Zucker), *Vision/Attention Conference*, Providence, RI, 1999.
33. "A Hamiltonian approach to the eikonal equation" (with K. Siddiqi and S. Zucker), *Proceedings of CVPR'99*.

34. "Conformal geometry and virtual endoscopy" (with S. Angenent, S. Haker, and R. Kikinis), *Proceedings of ISCAS'99*.
35. "Computational algebraic geometry and switching surfaces in optimal control" (with T. Georgiou and U. Walther), to appear in *Proceedings of 1999 IEEE Conference on Decision and Control*, 1999.
36. "The Hamilton-Jacobi skeleton" (with K. Siddiqi and S. Zucker), *Proceedings of ICCV'99*, Corfu, Greece, 1999.
37. "On the evolution of the skeleton" (with J. August and S. Zucker), *Proceedings of ICCV'99*, Corfu, Greece, 1999.
38. "Automated left ventricular measurement during real-time MRI" (with L. Zhao, C. Hardy, S. Warfield, A. Yezzi, L. Panych, R. Kikinis, S. Solomon, S. Maier, and F. Jolesz), submitted to *Proceedings of ISMRM'99*.
39. "High resolution sensing and anisotropic segmentation for SAR imagery" (with T. Georgiou), *Proceedings of IEEE CDC'00*.
40. "Robust control and tracking," *Proceedings of IEEE CDC'00*.
41. "Affine invariant symmetry sets" (with S. Betelu and G. Sapiro), *Proceedings of ECCV'00*, Dublin, Ireland, June 2000.
42. "Nondistorting maps for virtual colonoscopy" (with S. Angenent, S. Haker, and R. Kikinis), *Proceedings of SPIE*, San Diego, February 2000.
43. "New approach for visualization of 3D colon imagery" (with S. Angenent, S. Haker, and R. Kikinis), *MICCAI'00*.
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#### **Books Written Under AFOSR Support**

1. *Feedback Control Theory* (with John Doyle and Bruce Francis), MacMillan Company, New York, 1991.
2. *Robust Control of Distributed Parameter Systems* (with Ciprian Foias and Hitay Özbay), *Lecture Notes in Control and Information Sciences* **209**, Springer-Verlag, New York, 1995.
3. *Feedback Control, Uncertainty, and Complexity*, edited by Bruce Francis and Allen Tannenbaum, *Lecture Notes in Control and Information Sciences* **202**, Springer-Verlag, New York, 1995.
4. *Curvature Flows, Visual Tracking, and Computational Vision*, to be published by SIAM.

#### **Patent Filed Based on AFOSR Projects**

“Conformal Geometry and Texture Mappings,” (co-inventors Sigurd Angenent, Steven Haker, Allen Tannenbaum, and Ron Kikinis), patent pending.

**Students of A. Tannenbaum Supported by AFOSR-AF/F49620-98-1-0168**

1. Anthony Yezzi (Ph. D.)
2. Steven Haker (Ph. D.)

**Awards of A. Tannenbaum During AFOSR-AF/F49620-98-1-0168**

1. George Taylor Research Award (University of Minnesota).
2. Invited Plenary Speaker for IEEE CDC 2000.
3. Plenary Speaker for SIAM Conference on Control 1998.
4. Best Paper Award in FOAM 2000 Conference for the paper: “New algorithms for 3D analysis of open-celled foams,” (M. Montminy, A. Tannenbaum and C. Macosko).