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Abstract

The selection of a lossless 2-port to maximize the wideband power transfer from a generator to a load is a ubiquitous problem in electrical engineering. The mathematical problem is to maximize the wideband transducer power gain over a class of lossless 2-ports. As a numerical optimization problem, wideband impedance matching is difficult because the wideband transducer power gain is a nonlinear, nondifferentiable, badly scaled multivariable function. Therefore, any information on the global solution is valuable to the engineer for assessing the quality of sub-optimal solutions computed by numerical optimizers. In his classic 1950 paper, Fano determined a theoretical upper bound on the transducer power gain [16]. Specifically, the transducer power gain of any lossless 2-port cannot exceed Fano's bound. Development of Fano's approach continued through the 1960s [55], [41]. However, computing these bounds required solving a highly nonlinear system of multivariate inequalities amenable only for simple cases. In the early 1970s, Helton made the amazing connection between operator theory and electrical engineering [28]. Powerful Hardy space techniques were coupled to the electrical engineer's Smith chart computations. In this framework, Nehari's Theorem gave an upper bound on the transducer power gain computable as an (easy) eigenvalue problem. This report shows that continuity conditions make this Nehari bound tight.

Keywords. Impedance matching, transducer power gain, power mismatch, hyperbolic metric, scattering matrix, lossless 2-ports, Darlingtion's Theorem, Fano's bounds, H-Infinity, Hardy spaces, Nehari's Theorem, continuous inner functions, Blaschke-Potapov product, lumped 2-ports, lumped-distributed 2-ports, analytic optimization.

Contents

List of Figures

List of Tables

1

The Matching Problem

Figure 1.1 shows a 2-port with scattering matrix $S(p)$, the generator's reflectance $s_G(p)$, and the load's reflectance $s_L(p)$. The matching problem is to find a lossless 2-port that maximizes the power delivered to the load that is available from the generator. Table 1.1 lists the physical assumptions and mathematical properties of these scattering functions. A complete listing of the H^∞ notation is contained in Section 1.1. If the 2-port is lumped and lossless, then its corresponding scattering

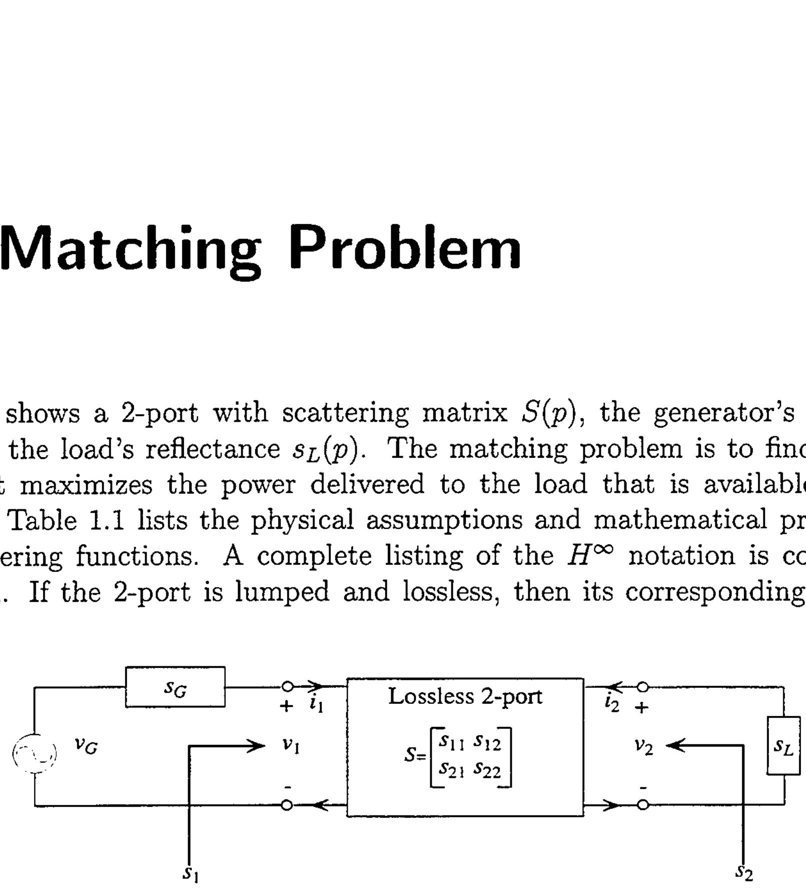

Figure 1.1: Matching circuit and reflectances.

matrix $S(p)$ is a rational, real, inner[1] function. The most general class of functions that model lossless 2-ports are those H^∞ functions that have unitary values:

$$U^+(2) := \{S \in \overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2}) : S \text{ is lossless and real}\}.$$

Implicit in this definition is that a correspondence exists between circuits and their mathematical models. When $S(p)$ is a rational inner function, classical electrical engineering can map $S(p)$ to a lossless 2-port. When $S(p)$ is not rational, $S(p)$

[1] "inner" means $S(j\omega)$ is unitary a.e. [25, page 68], [17, page 186], [53, page 190] rather than the more general notion that $S(j\omega)$ is a partial isometry [48, page 94].

can still map to a lossless 2-port provided $S(p)$ has special functional forms. It is natural to wonder if every scattering matrix in $U^+(2)$ corresponds to a lossless 2-port. Section 3 makes explicit this circuit-scattering question for $U^+(2)$ and several subclasses of scattering matrices.

Table 1.1: Physical and mathematical properties of a 1-port with scattering function $s(p)$ and a 2-port with scattering matrix $S(p)$. The complex frequency p is denoted $p = \sigma + j\omega$. All scattering functions are referenced to a real impedance $z_0 > 0$.

Physical Assumption	Mathematical Model	
	1-port	2-port
Linear, Time-Invariant, and Causal	$s \in H^\infty(\mathbb{C}_+)$	$S \in H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$
Real	$s(p) = \overline{s(\overline{p})}$	$S(p) = \overline{S(\overline{p})}$
Passive	$s \in \overline{B}H^\infty(\mathbb{C}_+)$	$S \in \overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$
Lossless	$\lvert s(j\omega)\rvert = 1$ a.e.	$S(j\omega)^H S(j\omega) = I_2$ a.e.
Continuous on $j\mathbb{R}$	$s \in \mathcal{A}_1(\mathbb{C}_+)$	$S \in \mathcal{A}_1(\mathbb{C}_+, \mathbb{C}^{2\times 2})$

The transducer power gain G_T is the ratio of the power delivered to the load to the maximum power avaliable from the generator [46, pages 606-608]:

$$G_T(s_G, S, s_L; p) := |s_{21}(p)|^2 \frac{1 - |s_G(p)|^2}{|1 - s_1(p) s_G(p)|^2} \frac{1 - |s_L(p)|^2}{|1 - s_{22}(p) s_L(p)|^2}.$$

Here, $s_1(p)$ is the reflectance looking into Port 1 with Port 2 terminated in $s_L(p)$:

$$\begin{aligned} s_1(p) &= \mathcal{F}_1(S, s_L; p) \\ &:= s_{11}(p) + s_{12}(p) s_L(p)(1 - s_{22}(p) s_L(p))^{-1} s_{21}(p). \end{aligned}$$

Likewise, $s_2(p)$ is the reflectance looking into Port 2 with Port 1 terminated in $s_G(p)$:

$$\begin{aligned} s_2(p) &= \mathcal{F}_2(S, s_G; p) \\ &:= s_{22}(p) + s_{21}(p) s_G(p)(1 - s_{11}(p) s_G(p))^{-1} s_{12}(p). \end{aligned}$$

The smallest or worst transducer power gain G_T over a frequency band Ω is

$$\|G_T(s_G, S, s_L)\|_{-\infty,\Omega} := \inf\{G_T(s_G, S, s_L; j\omega) : \omega \in \Omega\}.$$

For a given collection of $\mathcal{U} \subseteq \mathcal{U}^+(2)$ of lossless 2-ports, the matching problem is to maximize the transducer power gain over $\mathcal{U}$:

$$\sup\{\|G_T(s_G, S, s_L)\|_{-\infty,\Omega} : S \in \mathcal{U}\}.$$

In this report, s_G and s_L are fixed and $\Omega = \mathbf{R}$. Therefore, dependence on these elements is suppressed and we will find it handy to simply write

$$g_T(S) := \|G_T(s_G, S, s_L)\|_{-\infty}.$$

With this notation in place, it is worthwhile to consider the mapping $g_T : \mathcal{U} \to \mathbb{R}_+$ as an optimization problem. The standard optimization questions are (1) existence, (2) uniqueness, and (3) computation of optimal and sub-optimal matching circuits. The Weierstrass Theorem guarantees existence of maximizers for those functions continuous on a compact set.

Theorem 1 (Weierstrass [42, page 40], [59, page 152]) *An upper semicontinuous functional on a compact subset $\mathcal{U}$ of topological space achieves a maximum on $\mathcal{U}$.*

The Weierstrass Theorem gives one approach to maximizing the transducer power gain (TPG) by establishing the continuity of g_T and the compactness of $\mathcal{U}$. This straightforward approach suffices when $\mathcal{U}$ is the lumped 2-ports $U^+(2,d)$ of degree d (Section 4). However, $U^+(2)$ is not compact and requires a different approach. The idea is to convert the 2-port matching problem to an equivalent 1-port problem. With an abuse of notation, the transducer power gain is bounded as follows:

$$\begin{aligned}
&\sup\{\|G_T(s_G, S, s_L)\|_{-\infty} : S \in \mathcal{U}\} \\
&= \sup\{\|G_T(\mathcal{F}_2(S, s_G), s_L)\|_{-\infty} : S \in \mathcal{U}\} \\
&= \sup\{\|G_T(s_2, s_L)\|_{-\infty} : s_2 \in \mathcal{F}_2(\mathcal{U}, s_G)\} \\
&\leq \sup\{\|G_T(s_2, s_L)\|_{-\infty} : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

The last supremum is computable by Nehari's Theorem using the "L^∞ Disk Trick". This Nehari bound on the gain holds for any $\mathcal{U} \subseteq U^+(2)$. When $\mathcal{U}$ is the class of all lumped 2-ports $U^+(2,\infty)$, Darlington's Theorem (Section 5) establishes that

$$\begin{aligned}
&\sup\{\|G_T(s_G = 0, S, s_L)\|_{-\infty} : S \in U^+(2,\infty)\} \\
&= \sup\{\|G_T(s_2, s_L)\|_{-\infty} : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\leq \sup\{\|G_T(s_2, s_L)\|_{-\infty} : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

Section 6 eliminates the inequality, provided s_L is sufficiently smooth. The power of this approach is that the global solution of the matching problem is computed by the Nehari bound. The limitations of this approach require that $s_G = 0$. It does not supply the matching circuit. Section 7 increases the magnification to answer when minimizer s_2 exists and is unique. Section 8 concludes by linking the minimizer s_2 to the "dilation theory" of Helton and Douglas [13], [14]. Darlington's Theorem and the "L^∞ Disk Trick" have emerged as the analysis tools in this effort.

1.1 H^∞ Notation

The real numbers are denoted by $\mathbb{R}$. The complex numbers are denoted by $\mathbb{C}$. The set of complex $M \times N$ matrices is denoted by $\mathbb{C}^{M\times N}$. Complex frequency is written $p = \sigma + j\omega$. The open right-half plane is denoted by $\mathbb{C}_+ := \{p \in \mathbb{C} : \Re[p] > 0\}$. The open unit disk is denoted by $\mathbf{D}$ and the unit circle by $\mathbf{T}$. The "end-of-proof" symbol is "///".

$L^\infty(j\mathbb{R})$ denotes the class of Lebesgue-measurable functions defined on $j\mathbb{R}$ with norm determined by the essential bound

$$\|\phi\|_\infty := \text{ess.sup}\{|\phi(j\omega)| : \omega \in \mathbb{R}\}.$$

$C(j\mathbb{R}) \subset L^\infty(j\mathbb{R})$ denotes the class of continuous functions with norm

$$\|\phi\|_\infty := \sup\{|\phi(j\omega)| : \omega \in \mathbf{R}\}.$$

$C_0(j\mathbb{R}) \subset C(j\mathbb{R})$ denotes those continuous functions that vanish at $\pm\infty$. The Hardy space of functions bounded and analytic on $\mathbb{C}_+$ is denoted by $H^\infty(\mathbb{C}_+)$. Its norm is

$$\|h\|_\infty := \sup\{|h(p)| : p \in \mathbb{C}_+\}.$$

$H^\infty(\mathbb{C}_+)$ is also a subspace of $L^\infty(j\mathbf{R})$ obtained by the limit [40, page 153]

$$h(j\omega) = \lim_{\sigma\to 0} h(\sigma + j\omega)$$

that converges pointwise almost everywhere. Convergence in norm occurs if and only if the H^∞ function has continuous boundary values. Those H^∞ functions with continuous boundary values constitute the "disk algebra" denoted by $\mathcal{A}(\mathbb{C}_+)$. Two spaces of the disk algebra that are needed are those that vanish at infinity:

$$\mathcal{A}_0(\mathbb{C}_+) := \mathcal{A}(\mathbb{C}_+) \cap C_0(j\mathbf{R})$$

and those that are constant at infinity:

$$\mathcal{A}_1(\mathbb{C}_+) := 1\dot{+}\mathcal{A}_0(\mathbb{C}_+).$$

These spaces nest as

$$\mathcal{A}_0(\mathbb{C}_+) \subset \mathcal{A}_1(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+) \subset L^\infty(j\mathbb{R}).$$

Tensoring with $\mathbb{C}^{M\times N}$ gives the corresponding matrix-valued functions:

$$L^\infty(j\mathbb{R}, \mathbb{C}^{M\times N}) := L^\infty(j\mathbb{R}) \otimes \mathbb{C}^{M\times N}$$

with norm

$$\|\phi\|_\infty := \text{ess.sup}\{\|\phi(j\omega)\| : \omega \in \mathbf{R}\}.$$

The open unit ball of $H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N})$ is denoted as

$$BH^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) := \left\{h \in H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) : \|h\|_\infty < 1\right\}.$$

The closed unit ball of $H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N})$ is denoted as

$$\overline{B}H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) := \left\{h \in H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) : \|h\|_\infty \leq 1\right\}.$$

An $h \in H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N})$ is called *real*, provided

$$\overline{h(\overline{p})} = h(p)$$

for $p \in \mathbb{C}_+$. The class of real $H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N})$ functions is denoted as

$$\Re H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) = \{h \in H^\infty(j\mathbf{R}, \mathbb{C}^{M\times N}) : \overline{h(\overline{p})} = h(p)\}.$$

1.2 The Caley Transform

Many computations are more conveniently placed in function spaces defined on the open unit disk $\mathbf{D}$ rather than on the open right half-plane $\mathbb{C}_+$. The notation for the relevant function spaces on the disk follows the preceeding nomenclature with the unit disk $\mathbf{D}$ replacing the $\mathbb{C}_+$ and the unit circle $\mathbf{T}$ replacing $j\mathbb{R}$. $H^\infty(\mathbf{D})$ denotes the collection of analytic functions on the open unit disk with essentially bounded boundary values. $C(\mathbf{T})$ denotes the continuous functions on the unit circle, $\mathcal{A}(\mathbf{D})$ denotes the disk algebra

$$\mathcal{A}(\mathbf{D}) := H^\infty(\mathbf{D}) \cap C(\mathbf{T}),$$

and $L^\infty(\mathbf{T})$ denotes the Lebesgue-measurable functions on the unit circle $\mathbf{T}$ with norm determined by the essential bound. A Cayley transform connects the function spaces on the right half plane to their counterparts on the disk.

Lemma 1 ([26, page 99]) *Let* $\mathbf{c} : \mathbb{C}_+ \to \mathbf{D}$ *denote the mapping*

$$\mathbf{c}(p) := \frac{p-1}{p+1};$$

also define the composition operator $\mathbf{c} : H^\infty(\mathbf{D}) \to H^\infty(\mathbb{C}_+)$ *as*

$$h(p) := H \circ \mathbf{c}(p).$$

Then,

(a) $\mathbf{c}$ *is a isometry of* $H^\infty(\mathbf{D})$ *onto* $H^\infty(\mathbb{C}_+)$.

(b) $\mathbf{c}$ *is an isometry mapping* $\mathcal{A}(\mathbf{D})$ *onto* $\mathcal{A}_1(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+)$.

If $\mathbf{c}$ *also denotes the mapping from* $j\mathbf{R}$ *to* $\mathbf{T}$, *then the associated composition operator, also denoted by* $\mathbf{c}$, *maps as* $\mathbf{c} : L^\infty(\mathbf{T}) \to L^\infty(j\mathbf{R})$, *and has the following properties:*

(c) $\mathbf{c}$ *is a isometry of* $L^\infty(\mathbf{D})$ *onto* $L^\infty(\mathbb{C}_+)$.

(d) $\mathbf{c}$ *is an isometry mapping* $C(\mathbf{T})$ *onto* $1\dot{+}C_0(j\mathbf{R}) \subset C(j\mathbf{R})$.

1.3 Nehari's Theorem

Nehari's Theorem is a fundamental tool in H^∞ theory and most conveniently stated on the open unit disk $\mathbf{D}$. Let $\phi \in L^\infty(\mathbf{T})$ admit a Fourier expansion given by

$$\phi(e^{i\theta}) \sim \sum_{n=-\infty}^{\infty} \widehat{\phi}(n) e^{in\theta}.$$

The classic multiplication operator M_ϕ acting on $L^2(\mathbf{T})$ is given by $M_\phi h = \phi h$. Various blocks of M_ϕ have been studied for the last 80 years. Let P denote the orthogonal projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{D})$. The Hankel operator associated with ϕ is the operator H_ϕ mapping $H^2(\mathbf{D})$ into $H^2(\mathbf{D})^\perp$ given by $H_\phi = (I - P)M_\phi$. Its matrix representation with respect to the Fourier basis is

$$H_\phi = \begin{bmatrix} \widehat{\phi}(-1) & \widehat{\phi}(-2) & \widehat{\phi}(-3) & \cdots \\ \widehat{\phi}(-2) & \widehat{\phi}(-3) & \widehat{\phi}(-4) & \cdots \\ \widehat{\phi}(-3) & \widehat{\phi}(-4) & \widehat{\phi}(-5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The essential norm $\|H_\phi\|_e$ is

$$\|H_\phi\|_e := \inf\{\|H_\phi - K\| : K\text{compact}\}.$$

The first result is a simple version of Nehari's Theorem that emphasizes existence and uniqueness of best approximations.

Theorem 2 (Nehari [58], [47]) *Let* $k \in L^\infty(\mathbf{T})$. *Then,*

L-1 $\|k - H^\infty(\mathbf{D})\|_\infty = \|H_k\|$.

L-2 $\|k - \{H^\infty(\mathbf{D}) + C(\mathbf{T})\}\|_\infty = \|H_k\|_e$.

L-3 *If* $\|H_k\|_e < \|H_k\|$ *then* $k \in L^\infty(\mathbf{T})$ *admits a unique best approximation from* $H^\infty(\mathbf{T})$.

1.4 The Weak* Topology

The weak* topology in L^∞ and H^∞ is used in the L^∞ Disk Trick, so a short review follows. A weak* subbasis at $0 \in L^\infty(j\mathbf{R})$ is the collection of weak* open sets

$$O[w, \epsilon] := \{\phi \in L^\infty(j\mathbf{R}) : |\langle w, \phi\rangle| < \epsilon\},$$

where $\epsilon > 0$, $w \in L^1(j\mathbf{R})$, and

$$\langle w, \phi\rangle := \int_{-\infty}^{\infty} w(j\omega)\phi(j\omega)d\omega.$$

Every weak* open set that contains $0 \in L^\infty(j\mathbf{R})$ is a union of finite intersections of these subbasic sets. The following result is one way to handle weak* closure for the L^∞ Disk Trick.

Lemma 2 *If $\phi \in L^\infty(j\mathbf{R})$, then M_ϕ is weak* continuous on $L^\infty(j\mathbf{R})$.*

Proof. To show M_ϕ is weak* continuous, it suffices to shows that M_ϕ pulls subbasic sets back to subbasic sets. Let $\epsilon > 0$, $w \in L^1(j\mathbf{R})$. Then,

$$\begin{aligned}
\psi \in M_\phi^{-1}(O[w, \epsilon]) &\iff M_\phi\psi \in O[w, \epsilon] \\
&\iff |\langle w, \phi\psi\rangle| < \epsilon \\
&\iff |\langle \phi w, \psi\rangle| < \epsilon \\
&\iff \psi \in O[\phi w, \epsilon],
\end{aligned}$$

noting that $\phi w \in L^1(j\mathbf{R})$. ///

A more general way to handle the weak* closure in the proof of the L^∞ Disk Trick is to use weak* sequential compactness. From Simmons [52, page 121]: A metric space is sequentially compact if every sequence in it has a convergent subsequence.

Lemma 3 *Let $K \subseteq \overline{B}L^\infty(j\mathbf{R})$. The following are equivalent:*

(a) *K is weak* compact.*

(b) *K is weak* sequentially closed.*

Proof. A general result that establishes metrizability for subsets of a dual space is the following:

> Rudin [49, Theorem 3.16]: *If X is a separable topological space, if $K \subset X^*$, and if K is weak*-compact, then K is metrizable in the weak*-topology.*

Because $\overline{B}L^\infty(j\mathbf{R})$ weak* compact (Banach-Alaoglu [49, Theorem 3.15]), it follows that $\overline{B}L^\infty(j\mathbf{R})$ is metrizable in the weak* topology. Metrizability links compactness and sequential compactness:

> Simmons [52, page 124] *If K is a metric space, then K is compact if and only K is sequentially compact.*

If $K \subseteq \overline{B}L^\infty(j\mathbf{R})$, then K is also metrizable in the weak* topology. Consequently, weak* compactness of K is equivalent to weak* sequential compactness of K. ///

The Banach-Alaogalu Theorem [49, Theorem 3.15] gives that the unit ball $\overline{B}L^\infty(j\mathbf{R})$ is weak* compact. The preceding results show that same holds for a distorted version of the unit ball. Because this is a critical result, we offer two proofs.

Lemma 4 *Let k, $r \in L^\infty(j\mathbf{R})$ with $r \geq 0$ a.e. Then, the disk*

$$\overline{D}(k,r) := \{\phi \in L^\infty(j\mathbf{R}) : |\phi(j\omega) - k(j\omega)| \leq r(j\omega) \text{ a.e.}\}$$

is weak compact.*

Proof 1. The formal sequence of equivalences

$$\begin{aligned} \phi \in \overline{D}(k,r) &\iff |\phi - k| \leq r \\ &\iff |r^{-1}\phi - r^{-1}k| \leq 1 \\ &\iff r^{-1}\phi \in r^{-1}k + \overline{B}L^\infty(j\mathbf{R}) \\ &\iff \phi \in k + M_r\overline{B}L^\infty(j\mathbf{R}) \end{aligned}$$

leads us to the equality

$$\overline{D}(k,r) = k + M_r(\overline{B}L^\infty(j\mathbf{R})).$$

Observe that $\overline{B}L^\infty(j\mathbf{R})$ is weak* compact. Lemma 2 gives that M_r is weak* continuous. The image of a compact set under a continuous function is compact [52, Theorem 21-B]. Thus, $M_r(\overline{B}L^\infty(j\mathbf{R}))$ is weak* compact and, therefore, implies that $\overline{D}(k,r)$ is weak* compact.

Proof 2. If $\phi \in \overline{D}(k,r)$, then $\|\phi\|_\infty \leq \|k\|_\infty + \|r\|_\infty =: R$ or that $\overline{D}(k,r) \subset R\overline{B}L^\infty(j\mathbf{R})$. A scaled version of Lemma 3 makes weak* compactness equivalent to weak* sequential compactness. Let $\phi_n \in \overline{D}(k,r)$ and assume $\{\phi_n\}$ converges weak* to ϕ. For any $w \in L^1(j\mathbf{R})$, we obtain

$$|\langle w, k - \phi_n\rangle| \leq \langle |w|, r\rangle.$$

Taking the limit gives

$$|\langle w, k - \phi\rangle| \le \langle |w|, r\rangle.$$

Because $w \in L^1(j\mathbf{R})$ is arbitrary, we obtain $|k - \phi| \le r$ or that $\phi \in \overline{D}(k, r)$. Thus, $\overline{D}(k, r)$ is weak* sequentially compact and then must be weak* compact.///

We will need to know that $\mathcal{A}_1(\mathbb{C}_+)$ is weak* dense in $H^\infty(\mathbb{C}_+)$. On the unit circle, $\mathcal{A}(\mathbf{D})$ is weak* dense in $H^\infty(\mathbf{D})$. To obtain this result, let $h \in H^\infty(\mathbf{D})$ and set

$$h_r(z) := h(rz).$$

As $r \to 1$, h_r converges weak* to h [26, page 77], [40, page 13], [12, Exercise 6.43]. Observe that $h_r \in \mathcal{A}(\mathbf{D})$ to get the density claim. The same trick holds on the right half-lane.

Theorem 3 *$\mathcal{A}_1(\mathbb{C}_+)$ is weak* dense in $H^\infty(\mathbb{C}_+)$.*

Proof. Suppose that $w \in L^1(j\mathbf{R})$. Let $h \in H^\infty(\mathbb{C}_+)$. For $\sigma > 0$, define $h_\sigma(j\omega) := h(\sigma + j\omega)$. From Koosis [40, page 153]:

$$\lim_{\sigma \to 0} h(\sigma + j\omega) = h(j\omega) \quad \text{a.e.}$$

Observe that $h_\sigma \in \mathcal{A}_1(\mathbb{C}_+)$. The Lebesgue Dominated Convergence Theorem gives that

$$\lim_{\sigma \to 0} \langle w, h - h_\sigma\rangle = 0,$$

or that h_σ converges weak* to h. This implies $\mathcal{A}_1(\mathbb{C}_+)$ is weak* dense in $H^\infty(\mathbb{C}_+)$. ///

Observe that $U^+(2)$ is closed. Indeed, if $\{S_n\} \subset U^+(2)$ converges to $S \in H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$, then $S_n(j\omega) \to S(j\omega)$ almost everywhere so that

$$I_2 = \lim_{n \to \infty} S_n(j\omega)^H S_n(j\omega) = S(j\omega)^H S(j\omega) \quad \text{a.e.}$$

That is, $S(j\omega)$ is unitary almost everywhere or $S \in U^+(2)$. However, $U^+(2)$ is not weak* closed.

Example 1 *Define $S_n \in U^+(2)$ as*

$$S_n(p) := \left(\frac{1-p}{1+p}\right)^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, S_n converges weak to zero.*

Because zero belongs the weak* closure of $U^+(2)$, we make the extreme conjecture:

Conjecture 1 *The weak* closure of $U^+(2)$ is $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.*

2

The Power Mismatch

The key to analyzing the transducer power gain is the power mismatch. Power mismatch is known in both mathematics and electrical engineering [35]:

> The transformation through a lossless junction [2-port] ... leaves invariant the *hyperbolic distance* ... The hyperbolic distance to the origin of the [Smith] chart is the *mismatch*, that is, the standing-wave ratio expressed in decibels: It may be evaluated by means of the proper graduation on the radial arm of the Smith chart. For two arbitrary points W_1, W_2, the hyperbolic distance between them may be interpreted as the mismatch that results from the load W_2 seen through a lossless network that matches W_1 to the input waveguide.

Hyperbolic metrics have been under mathematical development for the last 200 years. It is fascinating to see how this analysis transcribes to electrical engineering. This section reviews the relevant mathematics, makes explicit the connection to the power mismatch, and concludes by linking the transducer power gain, the lossless 2-port, and the reflectances shown in Figure 1.1.

Mathematically, we start with the *pseudo-hyperbolic*[1] metric on $\mathbf{D}$ [60, page 58]: for all $s_1,\ s_2 \in \mathbf{D}$:

$$\rho(s_1, s_2) := \left|\frac{s_1 - s_2}{1 - \overline{s_1}s_2}\right|.$$

The Möbius group on $\mathbf{D}$ consists of all maps $\mathbf{g} : \mathbf{D} \to \mathbf{D}$ [31, Theorem 5.4c]:

$$\mathbf{g}(s) = e^{j\theta}\frac{s - a}{1 - \overline{a}s},$$

[1] Also known as the Poincaré hyperbolic distance function [54].

where $a \in \mathbf{D}$ and $\theta \in \mathbb{R}$. That ρ is invariant under the Möbius maps $\mathbf{g}$ is fundamental [60, page 58]:

$$\rho(\mathbf{g}(s_1), \mathbf{g}(s_2)) = \rho(s_1, s_2).$$

The *hyperbolic*[2] metric on $\mathbf{D}$ is [60, page 59]:

$$\beta(s_1, s_2) = \frac{1}{2} \log \left(\frac{1 + \rho(s_1, s_2)}{1 - \rho(s_1, s_2)} \right).$$

Because ρ is Möbius-invariant, it follows that β is also Möbius-invariant:

$$\beta(\mathbf{g}(s_1), \mathbf{g}(s_2)) = \beta(s_1, s_2).$$

The series inductor of Figure 2.1 provides an excellent example showing the action of a circuit as Möbius map acting on the unit disk.

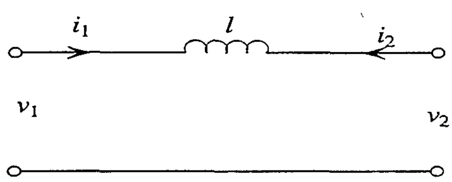

Figure 2.1: Series inductor 2-port.

The series inductor has the scattering matrix [23, Table 6.2]:

$$S(p) = \frac{1}{2 + Lp} \begin{bmatrix} Lp & 2 \\ 2 & Lp \end{bmatrix}$$

and chain scattering matrix:

$$\Theta(p) = \begin{bmatrix} 1 - Lp/2 & Lp/2 \\ -Lp/2 & 1 + Lp/2 \end{bmatrix}.$$

The chain scattering matrix acts on $s \in \mathbf{D}$ as

$$\mathcal{G}(\Theta; s) = \frac{\Theta_{11} s + \Theta_{12}}{\Theta_{21} s + \Theta_{22}} = -\frac{\overline{a}}{a} \frac{s - a}{1 - \overline{a} s}\bigg|_{a=(1+j2/(\omega L))^{-1}}.$$

[2]Also known as the Bergman metric or the Poincaré metic

Figure 2.2 shows the Möbius action of this lossless 2-port on the disk. Frequency is fixed at $p = j$. The upper left panel shows the unit disk partitioned into radial segments. Increasing the inductance, warps the radial pattern to the boundary. The radial segments are geodesics of ρ and β. Because the Möbius maps preserve both metrics, the resulting circles are also geodesics. More generally, the geodesics of ρ and β are either the radial lines or the circles that meet the boundary of the unit disk at right angles.

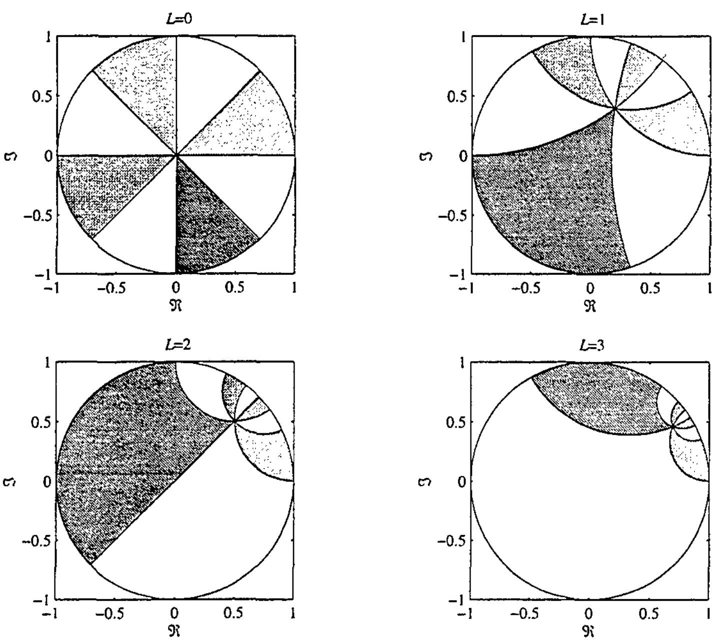

Figure 2.2: Möbius action of the series inductor on the unit disk for increasing inductance values (frequency fixed at $p = j$).

Turning to electrical engineering, basic matching functions are the power mismatch, the VSWR, and the transducer power gain. The power mismatch between two reflectances s_1, $s_2 \in BH^\infty(\mathrm{C}_+)$ is provided by Helton [28]:

$$\Delta P(s_1, s_2; p) := \left| \frac{\overline{s_1(p)} - s_2(p)}{1 - s_1(p)s_2(p)} \right| .$$

This is the pseudo-hyperbolic distance $\rho(\overline{s}_1, s_2)$ between $\overline{s}_1$ and s_2. Referring to Figure 2.2, the power mismatch is the pseudo-hyperbolic distance between $\overline{s}_1$ and s_2 measured along their geodesic. Thus, the geodesics of ρ attach a geometric meaning to the power mismatch and illustrate the quote at the beginning of this section. The voltage standing wave ratio (VSWR) is determined by reflectance s [5, Equation 3.51]:

$$\mathrm{VSWR}(s) = 20\log_{10}\left(\frac{1+|s|}{1-|s|}\right) \quad [\mathrm{dB}].$$

Thus, the VSWR is a scaled version the hyperbolic distance $\beta(s,0)$ from the origin to the reflectance s. Referring to Figure 2.2, the VSWR is the hyperbolic distance from the origin to s measured along its radial line. Thus, the geodesics of β attach a geometric meaning to the VSWR. The transducer power gain G_T is defined in the context of Figure 1.1. As the next result shows, maximizing G_T is equivalent to minimizing the power mismatch at either Port 1 or Port 2—provided the matching 2-port is lossless.

Lemma 5 *Assume the setup of Figure 1.1. Let $S \in U^+(2)$. Let s_G, $s_L \in BH^\infty(\mathbb{C}_+)$. Both $s_1 = \mathcal{F}_1(S, s_L)$ and $s_2 = \mathcal{F}_2(S, s_G)$ are well-defined and belong to $BH^\infty(\mathbb{C}_+)$. Both the LFT law*

$$\Delta P(s_G, \mathcal{F}_1(S, s_L)) = \left|\frac{\overline{s_G} - s_1}{1 - s_G s_1}\right| = \left|\frac{\overline{s_2} - s_L}{1 - s_2 s_L}\right| = \Delta P(\mathcal{F}_2(S, s_G), s_L)$$

and the TPG law

$$G_T(s_G, S, s_L) = 1 - \Delta P(s_G, \mathcal{F}_1(S, s_L))^2 = 1 - \Delta P(\mathcal{F}_2(S, s_G), s_L)^2$$

hold on $j\mathbf{R}$.

Proof. The chain scattering representations are provided by Hasler and Neirynck [23]:

$$\mathcal{G}(\Theta_1; s) := \mathcal{F}_1(S, s), \quad \Theta_1 \sim \frac{1}{s_{21}}\begin{bmatrix} -\det[S] & s_{11} \\ -s_{22} & 1 \end{bmatrix}$$

$$\mathcal{G}(\Theta_2; s) := \mathcal{F}_2(S, s), \quad \Theta_2 \sim \frac{1}{s_{12}}\begin{bmatrix} -\det[S] & s_{22} \\ -s_{11} & 1 \end{bmatrix},$$

where "$\sim$" denotes equality in homogeneous coordinates: $\Theta \sim \Phi$ if and only if $\mathcal{G}(\Theta) = \mathcal{G}(\Phi)$. Because $S(p) \in U^+(2)$, it follows that $S(p)$ is unitary on $j\mathbb{R}$. This forces $\Theta_1(p)$ and $\Theta_2(p)$ to be J-unitary on $j\mathbb{R}$ as provided by Helton [28]:

$$\Theta^H J \Theta = J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Fix $\omega \in \mathbb{R}$. Define the maps $\mathbf{g}_1$ and $\mathbf{g}_2$ on the unit disk $\mathbf{D}$ as

$$\mathbf{g}_1(s) := \mathcal{G}(\Theta_1(j\omega), s)$$

$$\mathbf{g}_2(s) := \mathcal{G}(\Theta_2(j\omega), s).$$

Because $\Theta_1(p)$ and $\Theta_2(p)$ are J-unitary on $j\mathbf{R}$, it follows that $\mathbf{g}_1$ and $\mathbf{g}_2$ are invertible automorphisms of the unit disk onto itself with inverses:

$$\mathbf{g}_1^{-1}(s) = \mathcal{G}(\Theta_1(j\omega)^{-1}, s), \quad \Theta_1(j\omega)^{-1} \sim \begin{bmatrix} -1 & s_{11}(j\omega) \\ -s_{22}(j\omega) & \det[S(j\omega)] \end{bmatrix}$$

$$\mathbf{g}_2^{-1}(s) = \mathcal{G}(\Theta_2(j\omega)^{-1}, s), \quad \Theta_2(j\omega)^{-1} \sim \begin{bmatrix} -1 & s_{22}(j\omega) \\ -s_{11}(j\omega) & \det[S(j\omega)] \end{bmatrix}.$$

Because the $\mathbf{g}_k$'s and their inverses are invertible automorphisms, we obtain from Geobel and Reich [19]:

$$\left| \frac{\mathbf{g}(s_1) - \mathbf{g}(s_2)}{1 - \mathbf{g}(s_1)\overline{\mathbf{g}(s_2)}} \right| = \left| \frac{s_1 - s_2}{1 - s_1 \overline{s_2}} \right|,$$

for s_1, $s_2 \in \mathbf{D}$ and $\mathbf{g}$, denoting either $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_1^{-1}$, or $\mathbf{g}_2^{-1}$. For all $p \in j\mathbf{R}$, we obtain

$$\begin{aligned} \Delta P(s_2, s_L) &= \left| \frac{s_2 - \overline{s_L}}{1 - s_2 s_L} \right| \\ &= \left| \frac{\mathbf{g}_2(s_G) - \overline{s_L}}{1 - \mathbf{g}_2(s_G) s_L} \right| \\ &= \left| \frac{s_G - \mathbf{g}_2^{-1}(\overline{s_L})}{1 - s_G \overline{\mathbf{g}_2^{-1}(\overline{s_L})}} \right| \\ &= \Delta P(s_G, \overline{\mathbf{g}_2^{-1}(\overline{s_L})}). \end{aligned}$$

Then, $\Delta P(s_2, s_L) = \Delta P(s_G, s_1)$, provided we can show $s_1 = \overline{\mathbf{g}_2^{-1}(\overline{s_L})}$. In terms of the chain matrices, this requires us to show

$$s_1 = \mathcal{G}(\Theta_1; s_L) = \overline{\mathcal{G}(\Theta_2^{-1}; \overline{s_L})} = \mathcal{G}(\overline{\Theta_2^{-1}}; s_L).$$

This equality will follow if we can show $\Theta_1 \sim \overline{\Theta_2^{-1}}$ or that

$$\Theta_1 \sim \begin{bmatrix} -1 & s_{11}/\det[S] \\ -s_{22}/\det[S] & 1/\det[S] \end{bmatrix} \sim \begin{bmatrix} -1 & \overline{s_{22}} \\ -\overline{s_{11}} & \overline{\det[S]} \end{bmatrix} \sim \overline{\Theta_2^{-1}}.$$

Because $S(p)$ is inner, $\det[S]$ is inner so that $\overline{\det[S]} = 1/\det[S]$ on $j\mathbf{R}$. Also, on $j\mathbf{R}$, $S(p)$ is unitary so that

$$S^{-1} = \frac{1}{\det[S]} \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix} = \begin{bmatrix} \overline{s_{11}} & \overline{s_{12}} \\ \overline{s_{21}} & \overline{s_{22}} \end{bmatrix}.$$

Then, $\overline{s_{22}} = s_{11}/\det[S]$ and $\overline{s_{11}} = s_{22}/\det[S]$. Thus, $\Theta_1 \sim \overline{\Theta_2^{-1}}$ so that $s_1 = \overline{g_2^{-1}(\overline{s_L})}$ or that the LFT law holds. Although a tedious computation can establish the link between the transducer power gain and the power mismatch, the following classical equality is useful [60, page 58]. For $s_1,\ s_2 \in \mathbf{D}$,

$$1 - \rho(s_1, s_2)^2 = \frac{(1 - |s_1|^2)(1 - |s_2|^2)}{|1 - \overline{s_1} s_2|^2}.$$

///

It is worth remarking that the LFT law is not true if S is strictly passive. For $S \in \overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$, define the gain at Port 1:

$$G_1(s_G, S, s_L) := 1 - \Delta P(s_G, \mathcal{F}_1(S, s_L))^2$$

and the gain at Port 2:

$$G_2(s_G, S, s_L) := 1 - \Delta P(\mathcal{F}_2(S, s_G), s_L)^2.$$

Lemma 5 gives that $G_T = G_1 = G_2$, provided S is lossless. However, when S is passive, there is numerical evidence that $G_T < G_1, G_2$.

Question 1 *Let $S \in BH^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$. Let $s_G,\ s_L \in BH^\infty(\mathbb{C}_+)$. Do the following the TPG inequalities hold?*

$$G_T(s_G, S, s_L) \leq 1 - \Delta P(s_G, \mathcal{F}_1(S, s_L))^2$$

$$G_T(s_G, S, s_L) \leq 1 - \Delta P(\mathcal{F}_2(S, s_G), s_L)^2$$

How can the passivity of S quantify the inequalities?

Finally, a question evaded at the start of this section is the problem of handling reflectances that are not strictly passive.

Question 2 *For which $s_1,\ s_2 \in \overline{B}H^\infty(\mathbb{C}_+)$ is $\Delta P(s_1, s_2)$ well-defined?*

3

Classes of Lossless 2-Ports

The matching problem has been formulated as an optimization problem over $U^+(2)$. Although $U^+(2)$ is a fundamental set, we have only offered a mathematical definition of $U^+(2)$. Because there are many types of lossless matching networks, it is important to put $U^+(2)$ in an electrical engineering context. This will make precise the matching and how it relates to more familiar electrical engineering solutions. We will obtain a nested collection of matching sets:

$$U^+(2,d) \subset U^+(2,\infty) \subset U^+(2) \subset \overline{B}H^\infty(\mathbb{C}_+,\mathbb{C}^{2\times 2}).$$

On the left, $U^+(2,d)$ corresponds to the lumped, lossless 2-ports. Optimization over this set represents an electrical engineering solution. On the right, the H^∞ solution is computable from the measured data. However, it may not correspond to any lossless scattering matrix. This gap between the H^∞ solution and the various electrical engineering solutions is closed by continuity conditions.

There are two important distinctions to observe. First, when we refer to a lossless scattering matrix $S(p)$, the only requirement is that $S(p)$ be a real inner function: $S \in U^+(2)$. Second, when we refer to a 2-port, we assume that an electrical circuit, however impractical, does exist. One of the great topics of electrical engineering is the mapping between N-ports and their corresponding representation by a scattering matrix. Depending on how one defines an N-port, the claim is that every linear, time-invariant, causal, passive N-port admits a scattering matrix. This *circuit-scattering* correspondence is a basic theme of this report.

3.1 $U^+(2,d)$

Any lumped, lossless, 2-port admits a real, rational, lossless scattering matrix $S(p)$. Conversely, a real, rational, lossless scattering matrix $S(p)$ can be mapped to a lumped, lossless 2-port [55, Theorems 3.1, 3.2]. This equivalence permits us to delineate the following class of lossless 2-ports:

$$U^+(2,d) := \{S \in U^+(2) : \deg_{\rm SM}[S(p)] \leq d\},$$

where $\deg_{\rm SM}[S(p)]$ denotes the Smith-McMillan degree of $S(p)$. $S(p)$ is rational if and only it has a finite Smith-McMillan degree. Belevitch's Theorem parameterizes $U^+(2,d)$:

BELEVITCH'S THEOREM [56] $S \in U^+(2,d)$ if and only if

$$S(p) = \begin{bmatrix} s_{11}(p) & s_{12}(p) \\ s_{21}(p) & s_{22}(p) \end{bmatrix} = \frac{1}{g(p)} \begin{bmatrix} h(p) & f(p) \\ \pm f_*(p) & \mp h_*(p) \end{bmatrix},$$

where

B-1 $f(p)$, $g(p)$, and $h(p)$ are real polynomials,

B-2 $g(p)$ is strict Hurwitz of degree not exceeding d,

B-3 $g_*(p)g(p) = f_*(p)f(p) + h_*(p)h(p)$ for all $p \in \mathbb{C}$.

This representation is basic to the Bode-Fano-Youla gain-bandwidth bounds [7, Chapter 4]. Belevitch's Theorem gives the following inclusion:

Lemma 6 *Let* $d \geq 0$. $U^+(2,d) \subset \mathcal{A}_1(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.

Proof. Let

$$S = \frac{1}{g}\begin{bmatrix} h & f \\ \pm f_* & \mp h_* \end{bmatrix},$$

where (f,g,h) is a Belevitch triple. Let M and N denote the degree of $h(p)$ and $g(p)$, respectively. Boundedness forces $M \leq N$. Then,

$$\frac{h(p)}{g(p)} = \frac{h_0 + \ldots + h_M p^M}{g_0 + \ldots + g_N p^N} \overset{p\to\infty}{\to} \begin{cases} 0 & M < N \\ h_N/g_N & M = N \end{cases}.$$

Thus, $h(p)/g(p)$ is continuous across $p = \pm j\infty$. Similar arguments apply to the other entries. Thus, $S(p)$ is continuous at $\pm j\infty$. ///

This inclusion gives us the following compactness of the scattering matrices that represent the lumped, lossless 2-ports of degree not exceeding d.

Theorem 4 *Let* $d \geq 0$. $U^+(2,d)$ *is a compact subset of* $H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.

Proof. Let $C(\mathbf{T}, \mathbb{C}^{2\times 2})$ denote the continuous functions on the unit circle $\mathbf{T}$. Let $\mathcal{R}_M^L$ denote those rational functions $g^{-1}(q)H(q)$ in $C(\mathbf{T}, \mathbb{C}^{2\times 2})$ where $g(q)$ and $H(q)$ are polynomials with degrees $\partial[g] \leq M$ and $\partial[H] \leq L$. The Existence Theorem [10, page 154] shows that $\mathcal{R}_M^L$ is a boundedly compact subset of $C(\mathbf{T}, \mathbb{C}^{2\times 2})$. Lemma 1 shows the Caley transform preserves compactness. Thus, $\mathcal{R}_M^L \circ \mathbf{c}$ is a boundedly compact subset of $1\dot{+}C(j\mathbf{R}, \mathbb{C}^{2\times 2})$. The remarks at the end of Section 1.4 show that $U^+(2)$ is a closed subset of $L^\infty(j\mathbf{R}, \mathbb{C}^{2\times 2})$. The intersection of a closed and bounded set with a boundedly compact set is compact. Thus, $U^+(2) \cap \mathcal{R}_M^L \circ \mathbf{c}$ is a compact subset of $1\dot{+}C(j\mathbf{R}, \mathbb{C}^{2\times 2})$. We claim that $U^+(2,d) = U^+(2) \cap \mathcal{R}_d^d \circ \mathbf{c}$. Observe $\mathcal{R}_d^d \circ \mathbf{c}$ consists of all rational functions with the degree of the numerator and denominator not exceeding d and that are also continuous on $j\mathbf{R}$, including the point at infinity. If $S \in U^+(2) \cap \mathcal{R}_d^d \circ \mathbf{c}$, then $\deg_{\mathrm{SM}}[S] \leq d$. This forces S into $U^+(2,d)$. Consequently, $U^+(2,d) \supseteq U^+(2) \cap \mathcal{R}_d^d \circ \mathbf{c}$. For the converse, suppose $S \in U^+(2,d)$. The Belevitch Theorem and Lemma 6 force S into $\mathcal{R}_d^d \circ \mathbf{c}$. Thus, $U^+(2,d) \subseteq U^+(2) \cap \mathcal{R}_d^d \circ \mathbf{c}$ and equality must hold. Thus, $U^+(N,d)$ is compact. ///

Section 4 will demonstrate that the gain is a continuous function on $U^+(2,d)$. Thus, the matching problem on $U^+(2,d)$ has a solution. The compactness of $U^+(2,d)$ also forces compactness for several classes of 2-ports. For example, the low-pass LC ladders of Figure 3.1 admit the scattering matrix characterization [3, page 121]:

$$s_{21}(p) = \frac{f(p)}{g(p)} = \frac{1}{g(p)}.$$

These scattering matrices ($f(p) = 1$) form a closed and therefore compact subset of $U^+(2,d)$. Consequently, the matching problem on the low-pass ladders of degree not exceeding d admits a solution. Similar closure arguments apply to the high-pass ladders.

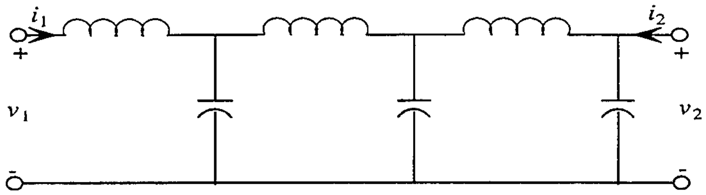

Figure 3.1: A low-pass ladder.

3.2 $U^+(2,\infty)$

A natural generalization drops the constraint on the number of reactive elements in the 2-port and asks: *What is the matching set that is obtained as* $\deg_{\text{SM}}[S(p)] \to \infty$*?* Define

$$U^+(2,\infty) = \overline{\bigcup_{d\geq 0} U^+(2,d)}.$$

The physical meaning of $U^+(2,\infty)$ is that it contains the scattering matrices of all lumped, lossless 2-ports. It is worthwhile to ask if the closure has picked up additional circuits. Belevitch's Theorem not only makes $S \in U^+(2,d)$ continuous on $j\mathbb{R}$, but also forces continuity at infinity. Because $\mathcal{A}_1(\mathbb{C}_+,\mathbb{C}^{2\times 2})$ closed, it follows that $U^+(2,\infty) \subset \mathcal{A}_1(\mathbb{C}_+,\mathbb{C}^{2\times 2})$ is also closed. It is natural to ask: What lossless 2-ports belong to $U^+(2,\infty)$? The transmission line provides an excellent counterexample.

Example 2 (Transmission Line) *A uniform, lossless transmission line of characteristic impedance Z_c and* commensurate *length l is called a unit element (UE) with transmission matrix [3, Equation 8.1]*

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} \cosh(\tau p) & Z_c \sinh(\tau p) \\ Y_c \sinh(\tau p) & \cosh(\tau p) \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix},$$

where τ is the commensurate one-way delay $\tau = l/c$ determined by the speed of propagation c.

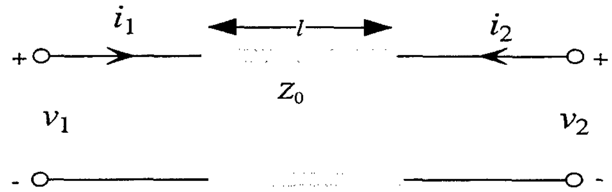

Figure 3.2: The unit element (UE) transmission line.

The impedance matrix is

$$Z_{\text{UE}}(p) = \frac{Z_c}{\sinh(\tau p)} \begin{bmatrix} \cosh(\tau p) & 1 \\ 1 & \cosh(\tau p) \end{bmatrix}.$$

The scattering matrix **normalized to Z_c** *is*

$$S_{\text{UE}}(p) = (Z(p) + Z_c I_2)^{-1}(Z(p) - Z_c I_2) = \begin{bmatrix} 0 & e^{-\tau p} \\ e^{-\tau p} & 0 \end{bmatrix}.$$

The transmission line gives rise to two observations: First, $S_{\mathrm{UE}}(j\omega)$ oscillates out to $\pm\infty$, so $S_{\mathrm{UE}}(j\omega)$ cannot be continuous across $\pm\infty$. Thus, $U^{+}(2,\infty)$ cannot contain such a transmission line. Second, a physical transmission line cannot behave like this near $\pm\infty$. Many electrical engineering books mention only in passing that their models are applicable only for a given frequency band. One rarely sees much discussion that the models for the inductor and capacitor are essentially low-frequency models. This holds true even for the standard model of wire. One cannot shine a light in one end of a 100-foot length of copper wire and expect much out of the other end. These model limitations notwithstanding, the circuit-scattering correspondence will be developed using these standard models. The transmission line on the disk is

$$S_{\mathrm{UE}} \circ \mathbf{c}^{-1}(z) = \begin{bmatrix} 0 & \exp\left(-\tau\frac{1+z}{1-z}\right) \\ \exp\left(-\tau\frac{1+z}{1-z}\right) & 0 \end{bmatrix}$$

and is recognizable as the simplest singular inner function [34, pages 66–67] analytic on $\mathbb{C} \setminus \{1\}$ [34, pages 68–69]. Figure 3.3 shows the essential singularity of the real part of the (1,2) element of $S_{\mathrm{UE}} \circ \mathbf{c}^{-1}(z)$ as z tends toward the boundary of the unit circle.

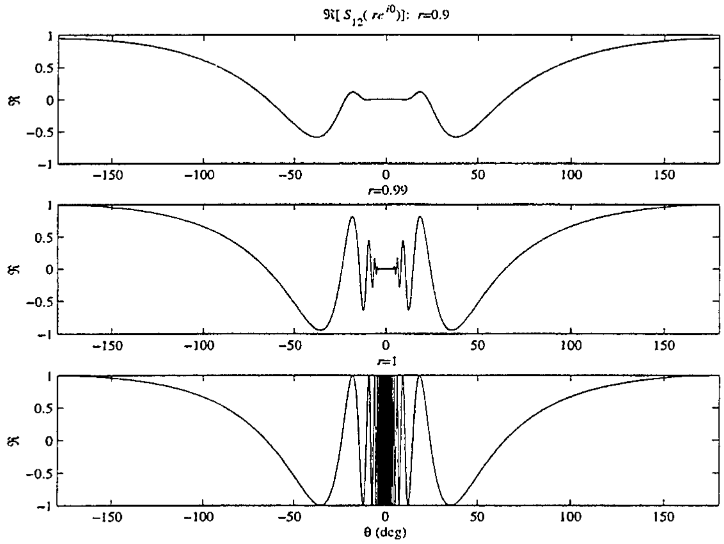

Figure 3.3: Behavior of $\Re[S_{\mathrm{UE},12} \circ \mathbf{c}^{-1}(z)]$ for $z = re^{i\theta}$ as $r \to 1$.

The continuity restriction is substantial. For example, the (infinite) Blaschke products can uniformly approximate any scalar-valued inner function.

FROSTMAN'S THEOREM [40, page 119] Let s be any inner function. For any $\epsilon > 0$, there is a Blaschke product $b(z)$ and $\theta_0 \in [-\pi, \pi]$ such that $\|s - e^{i\theta_0} b\|_\infty \leq \epsilon$.

If all the Blaschke's were continuous, then Frostman's Theorem would force all inner functions to be continuous. Continuity forces any continuous inner function to be rational.

Lemma 7 *Let $h \in H^\infty(\mathbf{D})$ be an inner function. The following are equivalent:*

(a) $h \in \mathcal{A}(\mathbf{D})$.

(b) *h is rational.*

Proof.
(a$\Rightarrow$b) Factor h as $h = cbs$, where $c \in \mathbf{T}$, b is a Blaschke and s is a singular inner function. If $z_a \in \mathbf{T}$ is an accumulation point of the zeros $\{z_n\}$ of b, that is, there is a subsequence $z_{n_k} \to z_a$, then continuity of h on $\overline{\mathbf{D}}$ implies that $0 = h(z_{n_k}) \to h(z_a)$. Continuity of h on $\overline{\mathbf{D}}$ gives a neighborhood $U \subset \mathbf{T}$ of z_a for which $|h(U)| < 1$. Thus, h cannot be inner with b an infinite Blaschke product. Thus, b can only be a finite product and has no accumulation points to cancel the discontinuities of s. More formally, b never vanishes on $\mathbf{T}$ and neither s nor $|s|$ is continously extendable to from the interior of the disk to any point in the support of the singular measure that represents s [34, pages 68–69]. Thus, h cannot have a singular part and we have $h = cb$.
(b$\Rightarrow$a) A rational h also in $H^\infty(\mathbf{D})$ cannot have a pole in $\overline{\mathbf{D}}$. Then h is continuous on $\overline{\mathbf{D}}$ so belongs to the disk algebra. ///

This result lifts to $U^+(2,\infty)$ using the generalizations of Potapov [39]. For $a \in \mathbf{D}$, define the elementary Blaschke factor as provided by Katsnelson and Kirstein [39, Equation 4.2]:

$$b_a(z) := \begin{cases} \frac{|a|}{a}\frac{a-z}{1-\bar{a}z} & a \neq 0 \\ z & a = 0 \end{cases}.$$

To get a matrix-valued version, let $P \in \mathbb{C}^{M\times M}$ be an orthogonal projection: $P^2 = P$ and $P^H = P$. The *Blaschke-Potapov elementary factor* associated with a and P is [39, Equation 4.4]:

$$B_{a,P}(z) := I_M + (b_a(z) - 1)P.$$

There are a couple of ways to see that $B_{a,P}$ is inner. Let U be a unitary matrix that diagonalizes P:

$$U^H P U = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$U^H B_{a,P}(z) U = \begin{bmatrix} b_a(z) I_K & 0 \\ 0 & I_{M-K} \end{bmatrix}.$$

From this, we get the following from by Katsnelson and Kirstein [39, Equation 4.5]:

$$\det[B_{a,P}(z)] = b_a(z)^{\mathrm{rank}[P]}.$$

Definition 1 ([39, pages 320–321]) *The function $B : \mathbf{D} \to \mathbb{C}^{M\times M}$ is called a* left Blaschke-Potapov product *if either B is a constant unitary matrix or there exists a unitary matrix U, a sequence of orthogonal projection matrices $\{P_k : k \in \mathcal{K}\}$, and a sequence $\{z_k : k \in \mathcal{K}\} \subset \mathbf{D}$ such that*

$$\sum_{k\in\mathcal{K}} (1 - |z_k|)\mathrm{trace}[P_k] < \infty$$

and the representation

$$B(z) = \left\{ \overrightarrow{\prod_{k\in\mathcal{K}}} B_{z_k,P_k}(z) \right\} U$$

holds.

Definition 2 ([39, pages 319]) *Let $S \in H^\infty(\mathbf{D}, \mathbb{C}^{M\times M})$ be an inner function. S is called* singular *if and only if* $\det[S(z)] \neq 0$ *for all $z \in \mathbf{D}$.*

Theorem 5 ([39, Theorem 4.1]) *Let $S \in H^\infty(\mathbf{D}, \mathbb{C}^{M\times M})$ be an inner function. There exists a left Blaschke-Potapov product and a $\mathbb{C}^{M\times M}$-valued singular inner function Ξ such that*

$$S = B\Xi.$$

Moreover, the representation is unique up to a unitary matrix U. If

$$S = B_1\Xi_1 = B_2\Xi_2,$$

then $B_2 = B_1 U$ and $\Xi_2 = U^H \Xi_1$.

Critical for our use is that the determinant maps these matrix-valued generalizations of the Blaschke and singular functions to their scalar-valued counterparts.

Theorem 6 ([39, Theorem 4.2]) *Let $S \in \overline{B}H^\infty(\mathbf{D}, \mathbb{C}^{M\times M})$.*

(a) $\det[S] \in \overline{B}H^\infty(\mathbf{D})$.

(b) *S is inner if and only if* $\det[S]$ *is inner.*

(c) *S is singular if and only if* $\det[S]$ *is singular.*

With these results in place, Lemma 7 admits the following matrix-valued generalization.

Corollary 1 *Let $S \in H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$ be an inner function. The following are equivalent:*

(a) $S \in \mathcal{A}_1(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.

(b) *S is rational*

Proof.
(a$\Rightarrow$b) Lemma 1 and Assumption (a) give that $W = S \circ \mathbf{c}^{-1}$ is a continuous inner function in $\mathcal{A}(\mathbf{D}, \mathbb{C}^{2\times 2})$. Theorem 5 gives that $W = B\Xi$ for a left Blaschke-Potapov product B and singular Ξ. Observe that $\det[W] = \det[B]\det[\Xi]$. If W is inner, then $\det[W]$ is inner by Theorem 6(a). Because W is continuous, $\det[W]$ is continuous and Lemma 7 forces $\det[W]$ to be rational. Therefore, $\det[W]$ cannot admit the singular factor $\det[\Xi]$. Consequently, W cannot have a singular factor by Theorem 6(c). Because $\det[W]$ is rational and

$$\det[W] = \det[B] = \prod b_{z_k}^{\mathrm{rank}[P_k]},$$

we see that B must be a *finite* left Blaschke-Potapov product. Consequently, $S = W \circ \mathbf{c}$ is rational. Finally, this gives that S is rational.
(b$\Rightarrow$a) Belevitch's Theorem puts S in $U^+(2,d)$. Lemma 6 puts S in $\mathcal{A}_1(\mathbb{C}_+, \mathbb{C}^{2\times 2})$. ///

Thus, continuity forces $S(p) \in U^+(2,\infty)$ to be rational and the corresponding lossless 2-port to be lumped. Consequently, the scattering matrices of the lumped, lossless 2-ports are exactly all of $U^+(2,\infty)$.

3.3 $U^+(2)$

The definition of $U^+(2)$ is strictly mathematical: $S \in U^+(2)$ if and only if $S(p)$ is an inner function. It is generally accepted that a lossless 2-port admits a scattering matrix $S(p)$. Losslessness gives $S(j\omega)$ unitary values so that $S(p)$ belongs to $U^+(2)$. It is natural to consider the converse.

Question 3 *Does every element in $U^+(2)$ correspond to a lossless 2-port?*

Turning to the inclusion $U^+(2,\infty) \subset U^+(2)$, the preceding sections have established that $U^+(2,\infty)$ is a closed subset of $U^+(2)$ that consists of all rational inner functions parameterized by Belevitch's Theorem. Physically, $U^+(2,\infty)$ models all the lumped 2-ports, but does not model the transmission line. It is natural to wonder what subclass of $U^+(2)$ contains the lumped 2-ports and the transmission line. More precisely, (1) what constitutes a lumped-distributed network? (2) how do we recognize its scattering matrix? Wohlers [55] answers the first question by parameterizing the class of lumped-distributed N-ports, consisting of N_L inductors, N_C capacitors, and N_U uniform transmission lines using the model in Figure 3.4.

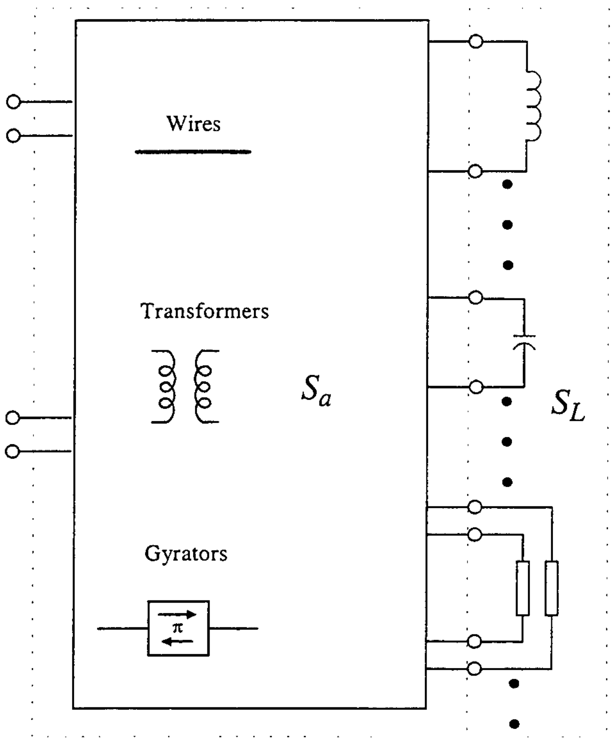

Figure 3.4: State-space representation of a lossless 2-port. The augmented scattering matrix S_a models the non-reactive multiport. The augmented load S_L models the reactive elements.

Let $U^+(N, N_L, N_C, N_U)$ denote this class of scattering matrices. Wohlers [55, pages 168–172] establishes that such scattering matrices $S \in U^+(N, N_L, N_C, N_U)$ exist and have the form,

$$S(p) = \mathcal{F}(S_a, S_L; p) = S_{a,11} + S_{a,12}S_L(p)(I_d - S_{a,22}S_L(p))^{-1}S_{a,21},$$

where the augmented scattering matrix

$$S_a = \begin{bmatrix} S_{a,11} & S_{a,12} \\ S_{a,21} & S_{a,22} \end{bmatrix}$$

models a network of wires, transformers, and gyrators. S_a is a constant, real, orthogonal matrix of size $d = N_L + N_C + 2N_U$. $S_L(p)$ is called the augmented load and models the reactive elements as

$$S_L(p) = \frac{p-1}{p+1} I_{N_L} \oplus -\frac{p-1}{p+1} I_{N_L} \oplus I_{N_U} \otimes \begin{bmatrix} 0 & e^{-\tau p} \\ e^{-\tau p} & 0 \end{bmatrix}.$$

This decomposition assumes: (1) the first $N_L + N_C$ ports are normalized to $z_0 = 1$, and (2) the remaining N_U pairs of ports are normalized to the characteristic impedance Z_{0,n_u} of each transmission line. Although some work has be done charactering these scattering matrices, the reports in Wohlers [55, page 173] are false, as determined by Choi [11].

4

Existence of Matching 2-Ports in $U^+(2,d)$

Proving the existence of a matching 2-port in $U^+(2,d)$ is straightforward: demonstrate that $g_T : U^+(2,d) \to \mathbb{R}_+$ is continuous and use the fact that $U^+(2,d)$ is compact. Because g_T is the composition

$$g_T(S) = 1 - \|\Delta P(s_G, \mathcal{F}_1(S, s_L))\|_\infty^2$$

of linear fractional maps, the bulk of this section establishes continuity of these maps with careful attention to boundary conditions.

Lemma 8 *If $s_G \in BH^\infty(\mathbb{C}_+)$, then*

$$s_1 \mapsto \frac{s_1 - \overline{s_G}}{1 - s_G s_1}$$

is a continuous function on $\overline{B}H^\infty(\mathbb{C}_+)$ into $\overline{B}L^\infty(j\mathbb{R})$.

Proof. Let

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} = \begin{bmatrix} 1 & -\overline{s_G} \\ s_G & -1 \end{bmatrix}.$$

Define

$$\mathcal{G}(\Theta; s_1) := \frac{\theta_{11}s_1 + \theta_{12}}{\theta_{21}s_1 + \theta_{22}}.$$

The problem is to delineate a useful domain and range of $\mathcal{G}(\Theta)$. If $s_G = 0$, $s_1 \mapsto s_1$ and the result is immediate. If $s_G \neq 0$, set $r = \|s_G\|_\infty^{-1} > 1$ and define the open ball

$$B_r := \{s_1 \in H^\infty(\mathbb{C}_+) : \|s_1\|_\infty < r\}.$$

$\mathcal{G}(\Theta)$ is well-defined as a mapping from B_r into $L^\infty(j\mathbb{R})$. Although Θ does not generally have the analytic properties to be a chain scattering matrix ($G(\Theta; 0) = \overline{s_G}$), it will have sufficient algebraic properties. First, we prove that $\mathcal{G}(\Theta)$ is continuous on B_r. Observe that $\Theta \in L^\infty(j\mathbb{R}, \mathbb{C}^{2\times 2})$. Let $s_1 \in B_r$. For all $\Delta s \in H^\infty(\mathbb{C}_+)$ with $\|\Delta s\|_\infty < r - \|s_1\|_\infty$, it follows that $s_1 + \Delta s$ belongs to B_r. Consequently, $\mathcal{G}(\Theta; s_1 + \Delta s)$ is well-defined and the expansion

$$\begin{aligned}
&\mathcal{G}(\Theta; s_1 + \Delta s) - \mathcal{G}(\Theta; s_1) \\
&= \frac{\theta_{11} s_1}{\theta_{21} s_1 + \theta_{21}\Delta s + \theta_{22}} + \frac{\theta_{11}\Delta s_1}{\theta_{21} s_1 + \theta_{21}\Delta s + \theta_{22}} \\
&\quad + \frac{\theta_{12}}{\theta_{21} s_1 + \theta_{21}\Delta s + \theta_{22}} - \frac{\theta_{11} s_1}{\theta_{21} s_1 + \theta_{22}} - \frac{\theta_{12}}{\theta_{21} s_1 + \theta_{22}} \\
&= \mathcal{O}[\Delta s]
\end{aligned}$$

holds in $L^\infty(j\mathbb{R})$. Thus, $\mathcal{G}(\Theta)$ is a continuous function on B_r. Second, we prove that $\mathcal{G}(\theta)$ preserves the unit ball. Observe that

$$\Theta^H J \Theta = \begin{bmatrix} |s_G|^2 & 0 \\ 0 & -|s_G|^2 \end{bmatrix} = |s_G|^2 J.$$

In terms of homogenous coordinates,

$$\frac{b_1}{a_1} = G(\Theta, s_1) \quad \Leftrightarrow \quad \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \Theta \begin{bmatrix} s_1 \\ 1 \end{bmatrix}.$$

Then,

$$\begin{aligned}
[\overline{b_1}\ \overline{a_1}] J \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} &= [\overline{s_1}\ 1] \Theta^H J \Theta \begin{bmatrix} s_1 \\ 1 \end{bmatrix} \\
&= [\overline{s_1}\ 1] |s_G|^2 J \begin{bmatrix} s_1 \\ 1 \end{bmatrix} \\
&= |s_G|^2 (|s_1|^2 - 1) \\
&\leq 0.
\end{aligned}$$

That is, $|b_1|^2 \leq |a_1|^2$ or $\|\mathcal{G}(\Theta, s_1)\|_\infty \leq 1$. Thus, $G(\Theta)$ maps $\overline{B}H^\infty(\mathbb{C}_+)$ into $\overline{B}L^\infty(j\mathbb{R})$. The closure is strict because $\mathcal{G}(\Theta; 1)$ has unit norm. ///

Lemma 9 *If $s_G \in BH^\infty(\mathbb{C}_+)$, then*

$$s_1 \mapsto \|\Delta P(s_G, s_1)\|_\infty$$

is a continuous function on $\overline{B}H^\infty(\mathbb{C}_+)$.

Proof. With power mismatch given as $\Delta P(s_G, s_1) = |\mathcal{G}(\Theta, s_1)|$, Lemma 8 implies that $\Delta P(s_G, \circ)$ is a continuous function on $\overline{B}H^\infty(\mathbb{C}_+)$. Use the fact that the norm is a continuous function on any Banach space to get that the composition $\|\Delta P(s_G, \circ)\|_\infty$ is continuous. ///

Lemma 10 *if s_G, $s_L \in BH^\infty(\mathbb{C}_+)$, then the mappings*

$$S \mapsto \mathcal{F}_1(S, s_L) \quad \text{and} \quad S \mapsto \mathcal{F}_2(S, s_G)$$

are continuous function on $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$ into $\overline{B}H^\infty(\mathbb{C}_+)$.

Proof. It will suffice to demonstrate this result for $\mathcal{F}_1$. Let $S \in \overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$ and set

$$\mathcal{F}(S;p) = \mathcal{F}_1(S, s_L; p) = s_{11}(p) + s_{12}(p)s_L(p)(1 - s_{22}(p)s_L(p))^{-1}s_{21}(p).$$

If $s_L = 0$, then the result is immediate. If $s_L \neq 0$, then set $r = \|s_L\|_\infty^{-1} > 1$ and define the open ball as

$$B_r := \{S \in H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2}) : \|s_{22}\|_\infty < r\}.$$

Then, $\|s_{22}s_L\|_\infty < 1$ shows that

$$\mathcal{F}(S) = s_{11} + s_L s_{12} s_{21} \sum_{k=0}^{\infty} (s_{22}s_L)^k$$

converges in $H^\infty(\mathbb{C}_+)$. Thus, $\mathcal{F}$ is a well-defined as the mapping $\mathcal{F} : B_r \to H^\infty(\mathbb{C}_+)$. Likewise, for all $\Delta S \in H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$, the inequality $\|\Delta S\| < 1 - \|s_{22}s_L\|_\infty$ gives that $\mathcal{F}(S + \Delta S)$ is also well-defined. Then,

$$\begin{aligned}
&\mathcal{F}(S + \Delta S) - \mathcal{F}(S) \\
&= \frac{s_L s_{12} s_{21}}{1 - s_{22}s_L} - \Delta s_{11} - \frac{s_L s_{12} s_{21}}{1 - (s_{22} + \Delta s_{22})s_L} \\
&\quad - \frac{s_L s_{12} \Delta s_{21}}{1 - (s_{22} + \Delta s_{22})s_L} - \frac{s_L \Delta s_{12} \Delta s_{21}}{1 - (s_{22} + \Delta s_{22})s_L} - \frac{s_L \Delta s_{12} \Delta s_{21}}{1 - (s_{22} + \Delta s_{22})s_L}.
\end{aligned}$$

As $\Delta S \to 0$, its elements converge uniformly to zero on $\mathbb{C}_+$. The boundedness of S then implies the second, fourth, fifth, and sixth terms in the expansion converge uniformly to zero. Boundedness also gives that

$$\mathcal{F}(S + \Delta S) - \mathcal{F}(S) = \Delta s_{22} \frac{s_L^2 s_{12} s_{21}}{(1 - s_{22}s_L)^2} + \mathcal{O}[\Delta S],$$

where the expansion holds in $H^\infty(\mathbb{C}_+)$. Thus, $\mathcal{F}$ is a continuous function on $\overline{B}H^\infty(C_+, \mathbb{C}^{2\times 2})$. To verify that $\mathcal{F}$ maps into the unit ball, observe that

$$\mathcal{F}(S) = \mathcal{F}_1(S; s_L) = \mathcal{G}(\Theta; s_L),$$

where, as provided by Hasler and Neirynck [23, page 148]:

$$\Theta \sim \begin{bmatrix} -\det[S] & s_{11} \\ -s_{22} & 1 \end{bmatrix}.$$

(Homogenous coordinates let us drop s_{21} and avoid zero divisions.) The CS decomposition gives the pointwise representation of S on $j\mathbb{R}$ as

$$S = \begin{bmatrix} u_{11} & 0 \\ 0 & u_{22} \end{bmatrix} \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} v_{11} & 0 \\ 0 & v_{22} \end{bmatrix}^H,$$

where $0 \leq t \leq \pi/2$ and u_{11}, u_{22}, v_{11}, and u_{22} have unit norm. Then,

$$\Theta = \begin{bmatrix} u_{11}\overline{v_{11}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \cos(t) \\ -\cos(t) & 1 \end{bmatrix} \begin{bmatrix} v_{22}\overline{u_{22}} & 0 \\ 0 & 1 \end{bmatrix}^H.$$

This decomposition gives that $\Theta^H J \Theta = \{1 - \cos(t)^2\} J$ or that $|\mathcal{F}_1(S; s_L; j\omega)| \leq 1$. Because $\mathcal{F}_1(S; s_L)$ is an element of $H^\infty(\mathbb{C}_+)$, we may invoke the maximum principle to conclude that $\|\mathcal{F}_1(S; s_L)\|_\infty \leq 1$. ///

Lemma 11 *Assume s_G, $s_L \in BH^\infty(\mathbb{C}_+)$. Let $\Omega \subseteq \mathbb{R}$ be non-empty and measurable. Then,*

$$g_1(S) := 1 - \|\Delta P(s_G, \mathcal{F}_1(S, s_L))\|^2_{\infty,\Omega}$$

and

$$g_2(S) := 1 - \|\Delta P(\mathcal{F}_2(S, s_G), s_L)\|^2_{\infty,\Omega}$$

are continuous functions on $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.

Proof. It will suffice to prove this result for g_1. Lemma 10 gives that $\mathcal{F}_1(\circ, s_L)$ is a continuous function on $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$ into $\overline{B}H^\infty(\mathbb{C}_+)$. Lemma 8 gives that $\mathcal{G}$ is a continuous mapping on $BH^\infty(\mathbb{C}_+)$. Then,

$$G_1(s_G, S, s_L) = 1 - |\mathcal{G} \circ \mathcal{F}(S)|^2$$

is also continuous. The pseudo-norm is also continuous, so we get that $g_1 = 1 - \|\mathcal{G} \circ \mathcal{F}\|^2_{\infty,\Omega}$ is continuous. ///

Lemma 12 *Assume s_G, $s_L \in BH^\infty(\mathbb{C}_+)$. Let $\Omega \subseteq \mathbb{R}$ be non-empty and measurable. Then,*

$$g_T(S) := \|G_T(s_G, \mathcal{F}_1(S, s_L))\|^2_{\infty,\Omega}$$

is a continuous function on $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$.

With the gains g_T, g_1, and g_2 continuous on the $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$, existence is settled for compact subsets of 2-ports.

Theorem 7 *Assume s_G, $s_L \in BH^\infty(\mathbb{C}_+)$. Let $\Omega \subseteq \mathbf{R}$ be non-empty and measurable. Let $\mathcal{U}$ be a non-empty compact subset of $\overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{2\times 2})$. Then, g_T, g_1, and g_2 attain their supremum on $\mathcal{U}$.*

In particular, $U^+(2,d)$ is compact so that matching from $U^+(2,d)$ admits a solution. However, this approach does not give any information on "quality" of the solution—how big is the gap $g_T(U^+(2,d)) < g_T(U^+(2,\infty))$? This leads to the solid engineering question: *How complex should the 2-port be to get an acceptable match ?* One approach plots the mismatch as a function of the degree d [1]. As the degree d increases, the mismatch approaches the upper bound computed by Nehari's Theorem. The engineer can graphically make the tradeoff between degree of the matching 2-port and its distance from the Nehari bound. Thus, Nehari's bound provides one benchmark for the matching 2-ports. The remainder of this paper shows when Nehari's bound is tight.

5

Classes of Orbits

The following equalities convert a 2-port problem into a 1-port problem. Let $\mathcal{U}$ be a subset of $U^+(2)$. Then,

$$\begin{aligned}
&\sup\{g_T(s_G, S, s_L) : S \in \mathcal{U}\} \\
&\quad = \; 1 - \inf\{\|\Delta P(s_G, S, s_L)\|_\infty^2 : S \in \mathcal{U}\} \\
&\quad = \; 1 - \inf\{\|\Delta P(s_G, s_1)\|_\infty^2 : s_1 \in \mathcal{F}_1(\mathcal{U}; s_L)\} \\
&\quad = \; 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(\mathcal{U}; s_G)\},
\end{aligned}$$

where

$$\mathcal{F}_1(\mathcal{U}, s_L) := \{\mathcal{F}_1(S, s_L) : S \in \mathcal{U}\},$$
$$\mathcal{F}_2(\mathcal{U}, s_G) := \{\mathcal{F}_2(S, s_G) : S \in \mathcal{U}\}$$

denote the orbit of the load and the orbit of the generator, respectively. Maximizing g_T is equivalent to minimizing ΔP on these orbits. This is why characterizing these orbits is fundamental. Darlington's Theorem makes explicit one class of orbits.

Theorem 8 (Darlington [3]) *The orbits of zero are equal*

$$\mathcal{F}_2(U^+(2,\infty), 0) = \mathcal{F}_1(U^+(2,\infty), 0)$$

and strictly dense in $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$.

Proof. Let $S \in U^+(2,\infty)$. Corollary 1 and Belevitch's Theorem give that

$$S(p) = \frac{1}{g}\begin{bmatrix} h & f \\ \pm f_* & \mp h_* \end{bmatrix},$$

where (f, g, h) is a Belevitch triple. With $s_L = 0$, then

$$s_1 = \mathcal{F}_1(S, 0) = s_{11} = \frac{h}{g}.$$

With $s_G = 0$, then

$$s_2 = \mathcal{F}_2(S, 0) = s_{22} = \frac{\mp h_*}{g}.$$

Lemma 6 gives that $S \in \mathcal{A}_1(\mathbb{C}_+, \mathbb{C}^{2\times 2})$ so that s_1 and s_2 both belong to $\overline{B}\mathcal{A}_1(\mathbb{C}_+)$. The real condition in Belevitch's Theorem (B-1) puts both orbits in $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$. However, Corollary 1 restricts S to be rational. Thus, the orbits cannot be all of $\Re\overline{B}\mathcal{A}_1(C_+)$. Finally, by relabeling S with $1 \leftrightarrow 2$, we get equality between the orbits. To show density, suppose $s \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$. Because the rational functions in $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$ are a dense[1] subset, we may approximate $s(p)$ by a real rational function:

$$s \approx \frac{h}{g} \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+),$$

where $h(p)$ and $g(p)$ may be taken as real polynomials with $g(p)$ strict Hurwitz and for all $\omega \in \mathbf{R}$:

$$g(j\omega)g_*(j\omega) - h(j\omega)h_*(j\omega) \geq 0.$$

By factoring $g(p)g_*(p) - h(p)h_*(p)$ or appealing to the Fejér-Riesz Theorem [48, page 109], we can find a real polynomial $f(p)$ such that

$$f(p)f_*(p) = g(p)g_*(p) - h(p)h_*(p).$$

The conditions of Belevitch's Theorem are met and

$$S(p) = \frac{1}{g(p)} \begin{bmatrix} h(p) & f(p) \\ f_*(p) & -h_*(p) \end{bmatrix}$$

is a lossless scattering matrix that represents a lumped, lossless 2-port. That is, $h(p)/g(p)$ dilates to a lossless scattering matrix $S(p)$ for which $s \approx s_{11}$. Consequentially, both orbits are dense in $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$. ///

The big question is the characterization of an orbit for a general load. For the immediate application of Nehari's Theorem, knowing the following question would be useful.

Question 4 *For what $s_G \in BH^\infty(\mathbb{C}_+)$ is it true that $\mathcal{F}_1(U^+(2,\infty), s_G)$ is dense in $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$?*

[1]Density claims on unbounded regions can be tricky. However, Lemma 1 isometrically maps $\mathcal{A}_1(\mathbb{C}_+) = \mathcal{A}_1(\mathbf{D}) \circ \mathbf{c}$ and preserves the rational functions. Therefore, the dense rational functions in $\mathcal{A}(\mathbf{D})$ map to a set of rational functions in $\mathcal{A}_1(\mathbb{C}_+)$ that must be dense.

The limited knowledge of compatible impedances [57] forces us to take $s_G = 0$. Our optimization set simplifies by Darlington's Theorem:

$$\begin{aligned}
&\max\{g_T(0, S, s_L) : S \in U^+(2, d)\} \\
&\quad = \quad 1 - \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, d); 0)\} \\
&\quad \leq \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, \infty); 0)\} \\
&\quad \overset{\text{Darlington}}{=} \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\quad \leq \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

The next section turns the last "inf" into a "min". This minimum is computable by Nehari's Theorem.

6

Existence of Matching 1-Ports in $\overline{B}H^{\infty}(\mathbb{C}_+)$

The problem is to show that

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}$$

admits minimizers. The trick is to realize the H^∞ is really not relevant. What is relevant is that the mapping

$$s_2 \mapsto \|\Delta P(s_2, s_L)\|_\infty$$

is a lower semicontinuous[1] function on a compact set. The existence of minimizers follows from the Weierstrass Theorem. The key result is the Disk Trick.

6.1 The L^∞ Disk Trick

Lemma 13 (L^∞ Disk Trick) *Let $s_L \in BL^\infty(j\mathbf{R})$. Let $0 \le \rho \le 1$. Define the center function*

$$k := \overline{s}_L \frac{1-\rho^2}{1-\rho^2|s_L|^2} \in BL^\infty(j\mathbf{R}), \tag{6.1}$$

the radius function

$$r := \rho \frac{1-|s_L|^2}{1-\rho^2|s_L|^2} \in \overline{B}L^\infty(j\mathbf{R}), \tag{6.2}$$

[1]Rudin [50, page 38–39] Let h be a real or extended-real function on a topological space X. If $\{x \in X : h(x) > \alpha\}$ is open for every real α, then h is said to be lower semicontinuous. Conversely, h is lower semicontinuous if $\{x \in X : h(x) \le \alpha\}$ is closed for every real α.

and the disk

$$\overline{D}(k,r) := \{\phi \in L^\infty(j\mathbf{R}) : |\phi(j\omega) - k(j\omega)| \leq r(j\omega)\}.$$

Then,

D-1 $\overline{D}(k,r)$ *a closed, convex subset of* $L^\infty(j\mathbf{R})$.

D-2 $\overline{D}(k,r) = \{\phi \in \overline{B}L^\infty(jR) : \rho \geq \|\Delta P(\phi, s_L)\|_\infty\}$.

D-3 $\overline{D}(k,r) = \overline{D}(k,r) \cap \overline{B}L^\infty(j\mathbf{R})$.

D-4 $\overline{D}(k,r)$ *a weak* compact, convex subset of* $L^\infty(j\mathbf{R})$.

Proof. Under the assumption that $\|s_L\|_\infty < 1$, it is straightforward to verify that the center and radius functions are in the open and closed unit balls of $L^\infty(j\mathbf{R})$, respectively.
D-1: Closure and convexity follow from the pointwise closure.
D-2: Basic algebra computes

$$\overline{D}(k,r) = \{\phi \in L^\infty(jR) : \rho \geq \|\Delta P(\phi, s_L)\|_\infty\}.$$

The "free" result is that $\|\overline{D}(k,r)\|_\infty \leq 1$. To see this, let $s := \|s_L\|_\infty$. The norm of any element in $\overline{D}(k,r)$ is bounded by

$$\|k\|_\infty + \|r\|_\infty \leq s\frac{1-\rho^2}{1-\rho^2 s^2} + \rho\frac{1-s^2}{1-\rho^2 s^2} =: u(s,\rho).$$

For $s \in [0,1)$ fixed, we obtain

$$\frac{\partial u}{\partial \rho} = -\frac{-1+s^2}{(\rho s+1)^2}.$$

Thus, $u(s,\circ)$ attains its maximum on the boundary of $[0,1]$: $u(s,1) = 1$. Thus, $\|\overline{D}(k,r)\|_\infty \leq 1$.
D-3: Follows from D-2.
D-4: Convexity follows from D-1. Lemma 4 demonstrates weak* compactness. ///

Let $s_L \in BL^\infty(j\mathbf{R})$. Define the mapping $\Delta\rho(s_2) : \overline{B}L^\infty(j\mathbf{R}) \to \mathbf{R}_+$ as

$$\Delta\rho(s_2) := \|\Delta P(s_2, s_L)\|_\infty.$$

The Disk Trick shows that $\Delta\rho$ is a lower semicontinuous function on $\overline{B}L^\infty(j\mathbf{R})$ in the weak* topology. Consequently, $\Delta\rho$ admits minimizers by the Weierstrass Theorem. This is a trivial result because $\overline{s}_L$ is the minimum. However, $\Delta\rho$ restricted to weak* closed subsets is still lower semicontinuous. In particular, $H^\infty(\mathbb{C}_+)$ is a weak* closed subspace of $L^\infty(j\mathbf{R})$ and we get the following result.

Theorem 9 *Assume* $s_L \in BL^\infty(j\mathbf{R})$. *Then,*

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}$$

admits minimizers.

Because $\Re\overline{B}H^\infty(\mathbb{C}_+)$ is also weak* closed, we get the following result.

Theorem 10 *Assume* $s_L \in BL^\infty(j\mathbf{R})$. *Then,*

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \Re\overline{B}H^\infty(\mathbb{C}_+)\}$$

admits minimizers.

The inequalities now read:

$$\begin{array}{cl} \multicolumn{2}{l}{\max\{g_T(0, S, s_L) : S \in U^+(2, d)\}} \\ = & 1 - \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, d); 0)\} \\ \leq & 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, \infty); 0)\} \\ \overset{\text{Darlington}}{=} & 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\ \leq & 1 - \overset{\text{Theorem 9}}{\min} \{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}. \end{array}$$

The next section starts the process of turning the last "$\leq$" into "$=$". The importance of the equality is that it connects the computable minimum on $\overline{B}H^\infty(\mathbb{C}_+)$ to an optimal matching 2-port—or at least a sequence of 2-ports that converge to the minimal power mismatch.

6.2 The H^∞ Disk Trick

Continuity in the target function projects onto the best approximation. This result lets us squeeze the disk algebra and the H^∞ minimizers. To start the analysis, the next result maps Nehari's Theorem over to the right half-plane and offers a condition for when the disk algebra supports a best approximation.

Theorem 11 (Adapted from Koosis[40, pages 193–195], and Hintzman [32], [33]) *Let* $k \in 1\dot{+}C_0(j\mathbf{R})$. *Then,*

A-1 $\|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty = \|k - \mathcal{A}(\mathbb{C}_+)\|_\infty = \|k - H^\infty(\mathbb{C}_+)\|_\infty$.

A-2 *There is exactly one* $h \in H^\infty(\mathbb{C}_+)$ *such that*

$$|k(j\omega) - h(j\omega)| = \|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty \quad \text{a.e.}$$

A-3 *If* $k \in 1\dot{+}C_0(j\mathbf{R})$ *admits the expansion*

$$K = k \circ \mathbf{c}^{-1} = \sum_{n=-N}^{-1} \widehat{K}(n) z^n,$$

then there is exactly one $h \in \mathcal{R}\mathcal{A}_1(\mathbb{C}_+)$ *such that*

$$|k(j\omega) - h(j\omega)| = \|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty.$$

Proof.
A-1: The inclusions $\mathcal{A}_1(\mathbb{C}_+) \subset \mathcal{A}(\mathbb{C}_+) \subset H^\infty(\mathbb{C}_+)$ force

$$\|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty \geq \|k - \mathcal{A}(\mathbb{C}_+)\|_\infty \geq \|k - H^\infty(\mathbb{C}_+)\|_\infty.$$

Let $K = k \circ \mathbf{c}^{-1} \in C(\mathbf{T})$. Lemma 1 gives

$$\begin{aligned}
\|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty &= \|K \circ \mathbf{c} - \mathcal{A}(\mathbf{D}) \circ \mathbf{c}\|_\infty \\
&= \|K - \mathcal{A}(\mathbf{D})\|_\infty \\
&\overset{\text{Koosis}}{=} \|K - H^\infty(\mathbf{D})\|_\infty \\
&= \|k - H^\infty(\mathbb{C}_+)\|_\infty,
\end{aligned}$$

where the "Koosis" over the equality refers to Koosis [40, page 195–196].
A-2: Existence, uniform modulus, and unicity follow from Koosis [40] or Hintzman [33].
A-3: Use Lemma 1 to map Hintzman's results from the disk to the right half-plane [32]. ///

Example 3 (Hintzman's Counter-Example) *Even when the target function is continuous, best approximations from the disk algebra need not exist [33]. Denote the real and imaginary parts of the following analytic function as*

$$u(z) + iv(z) := -i \sum_{n=2}^{\infty} \frac{z^n}{n \log(n)}.$$

Construct

$$g(e^{i\theta}) = e^{-v(e^{i\theta})}(e^{i\theta} - e^{u(e^{i\theta})}).$$

Then, $g \in C(\mathbf{T})$ *but does* **not** *have a best approximation from* $\mathcal{A}(\mathbf{D})$.

Finally, Carleson and Jacobs [4] have a considerable generalization of Theorem 11 A-3. Their existence and uniqueness results will be used later. For now, we need the H^∞ Disk Trick to convert the power mismatch to a linear problem.

Lemma 14 (H^∞ Disk Trick) *Let $s_L \in BH^\infty(\mathbb{C}_+)$. Let $0 < \rho \leq 1$. Define the center function k and radius function r as in Equations 6.1 and 6.2, respectively. Then,*

$$\{s_2 \in \overline{B}H^\infty(\mathbb{C}_+) : \rho \geq \|\Delta P(s_2, s_L)\|_\infty\} = \overline{D}(k, r) \cap H^\infty(\mathbb{C}_+).$$

Let $a \in H^\infty(\mathbb{C}_+)$ denote a spectral factorization of r. The following are equivalent:

(a) $\overline{D}(k, r) \cap H^\infty(\mathbb{C}_+) \neq \emptyset$.

(b) $\|a^{-1}k - H^\infty(\mathbb{C}_+)\|_\infty \leq 1$.

Proof. The first equality follows from Lemma 13 that gives

$$\overline{D}(k, r) \cap \overline{B}L^\infty(\mathbb{C}_+) = \overline{D}(k, r).$$

Intersect both sides with $H^\infty(\mathbb{C}_+)$ to get

$$\overline{D}(k, r) \cap \overline{B}H^\infty(\mathbb{C}_+) = \overline{D}(k, r) \cap H^\infty(\mathbb{C}_+).$$

To establish is the existence of a spectral factorization, observe that

$$r \geq \rho(1 - \|s_L\|_\infty^2) =: \delta > 0 \quad a.e.$$

Lemma 1 gives that $R = r \circ \mathbf{c}^{-1}$ belongs to $L^\infty(\mathbf{T})$. Because $R \geq \delta > 0$, it follows that $\log(R) \in L^1(\mathbf{T})$ and defines the outer function [15, page 24]:

$$A(z) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log(R(e^{it}))dt\right) \in H^\infty(\mathbf{T}).$$

Lemma 1 gives that $a = A \circ \mathbf{c} \in H^\infty(\mathbb{C}_+)$ and is also an outer function. Thus, a spectral factorization exists.
(a)$\Rightarrow$(b) Suppose $h \in \overline{D}(k, r) \cap H^\infty(\mathbb{C}_+)$. Then, $|h - k| \leq r$ a.e. is equivalent to $|a^{-1}h - a^{-1}k| \leq 1$ a.e. Because $a^{-1}h \in H^\infty(\mathbb{C}_+)$, this last inequality implies *(b)*.
(b)$\Rightarrow$(a) Suppose $\|a^{-1}k - H^\infty(\mathbb{C}_+)\|_\infty \leq 1$. By Theorem 11, there exists an $H \in H^\infty(\mathbb{C}_+)$ such that

$$1 \geq \|a^{-1}k - H^\infty(\mathbb{C}_+)\|_\infty = \|a^{-1}k - H\|_\infty \quad a.e.$$

This implies that $|aH - k| \leq r$ a.e. Because $aH \in H^\infty(\mathbb{C}_+)$, the Disk Trick gives that $aH \in \overline{D}(k, r) \cap H^\infty(\mathbb{C}_+)$ or that (a) holds. ///

Theorem 12 (**H^∞ Mismatch**) *Let $s_L \in BH^\infty(\mathbb{C}_+)$ be non-constant. Then,*

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.$$

Define the center function $k_{\min}$ and radius function $r_{\min}$ as in Equations 6.1 and 6.2. Let $a_{\min} \in H^\infty(\mathbb{C}_+)$ denote a spectral factorization of $r_{\min}$. Then,

Min-1 $\overline{D}(k_{\min}, r_{\min}) \cap H^\infty(\mathbb{C}_+) \neq \emptyset$

Min-2 $\|a_{\min}^{-1} k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty = 1.$

Proof. Because s_L is non-constant, $\rho_{\min} > 0$. The assumption $\|s_L\|_\infty < 1$ and Theorem 9 gives that at least one minimizer exists so that $\rho_{\min}$ is well-defined. Let $s_2 \in \overline{B}H^\infty(\mathbb{C}_+)$ be a minimizer. Lemma 14 implies that Min-1 holds and is equivalent to $\|a_{\min}^{-1} k - a_{\min}^{-1} s_2\|_\infty \leq 1$. If the inequality was strict, there would hold $|k_{\min}(j\omega) - s_2(j\omega)| < r_{\min}(j\omega)$ a.e., or $\rho_{\min} > \|\Delta P(s_2, s_L)\|_\infty$. This inequality contradicts the definition of $\rho_{\min}$, so equality and Min-2 must hold. ///

Theorem 13 (**H^∞ Existence and Uniqueness**) *Let $s_L \in B\mathcal{A}_1(\mathbb{C}_+)$ be a non-constant function. The power mismatch problem*

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}$$

admits a unique solution. Moreover, we obtain the equality

$$\begin{aligned}&\min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}\\ &= \inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\}.\end{aligned}$$

Proof. The assumption $\|s_L\|_\infty < 1$ and Theorem 9 gives that a minimum exists:

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.$$

Because s_L is non-constant, $\rho_{\min} > 0$. Observe that

$$k_{\min} = \overline{s}_L \frac{1 - \rho_{\min}^2}{1 - \rho_{\min}^2 |s_L|^2} \in 1\dot{+}C_0(j\mathbb{R})$$

and

$$r_{\min} = \rho_{\min} \frac{1 - |s_L|^2}{1 - \rho_{\min}^2 |s_L|^2} \in 1\dot{+}C_0(j\mathbb{R}).$$

The assumption that s_L is continuous and $\rho_{\min} > 0$ give that there is a continuous spectral factorization $a_{\min}$ of $r_{\min}$. To see this, observe that Lemma 1 gives that

$R_{\min} = r_{\min} \circ \mathbf{c}^{-1}$ belongs to $C(\mathbf{T})$. Because $R \geq \rho_{\min}(1 - \|s_L\|_\infty^2) > 0$, it follows that $\log(R_{\min}) \in C(\mathbf{T})$ and defines the outer function [15, page 24]:

$$A_{\min}(z) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log(R_{\min}(e^{it}))dt\right) \in \mathcal{A}(\mathbf{T}).$$

Lemma 1 gives that $a_{\min} = A_{\min} \circ \mathbf{c} \in \mathcal{A}_1(\mathbb{C}_+)$ and is also an outer function. Thus, a spectral factorization exists in the disk algebra. Let $s_2 \in \overline{B}H^\infty(\mathbb{C}_+)$ be a minimizer:

$$s_2 := \operatorname{argmin}\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.$$

Theorem 12 gives that

$$1 = \|a_{\min}^{-1}k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty = \|a_{\min}^{-1}k_{\min} - a_{\min}^{-1}s_2\|_\infty.$$

Observe that $a_{\min}^{-1}k_{\min} \in 1\dot{+}C_0(j\mathbf{R})$. Theorem 11(A-2) implies that $a_{\min}^{-1}s_2$ is unique. Because $a_{\min}^{-1}$ is outer, it follows that the minimizer[2] s_2 is also unique. To prove the equality for the disk algebra, set

$$\rho_{\min,\mathcal{A}} := \inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbf{D})\}$$

and observe that $\rho_{\min} \leq \rho_{\min,\mathcal{A}}$. The goal is to prove equality. There are at least two approaches. First, the Disk Trick and Theorem 11(A-1)

$$1 = \|a_{\min}^{-1}k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty = \|a_{\min}^{-1}k_{\min} - \mathcal{A}_1(\mathbb{C}_+)\|_\infty$$

should couple to get equality. Instead, the second approach that we use starts from the lower semicontinuity of the power mismatch

$$\Delta\rho(s_2) := \|\Delta P(s_2, s_L)\|_\infty$$

on $\overline{B}H^\infty(\mathbf{D})$ in the weak* topology. Let $\rho_n > \rho_{\min}$ and $\rho_n \downarrow \rho_{\min}$. There is an $s_n \in \overline{B}H^\infty(\mathbb{C}_+)$ such that $\Delta\rho(s_n) \leq \rho_n$. Let $\epsilon > 0$. Lower semicontinuity implies that

$$\{s \in \overline{B}H^\infty(\mathbb{C}_+) : \Delta\rho(s) < \rho_n + \epsilon\}$$

[2] It is worth noting that Theorem 11(A-2) gives

$$1 = |a_{\min}^{-1}k_{\min} - a_{\min}^{-1}s_2| \quad \text{a.e.},$$

or Helton's Flatness Condition [30]:

$$r_{\min} = |k_{\min} - s_2| \quad \text{a.e.}$$

is a weak* open neighborhood of s_n. Theorem 3 shows that $\mathcal{A}_1(\mathbb{C}_+)$ is weak* dense in $H^\infty(\mathbb{C}_+)$. This weak* density implies that there is an $a_n \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)$ that belongs to this neighborhood. Then,

$$\rho_{\min,\mathcal{A}} \leq \Delta\rho(a_n) \leq \rho_n + \epsilon \downarrow \rho_{\min} + \epsilon.$$

Because $\epsilon > 0$ is arbitrary, it follows that $\rho_{\min,\mathcal{A}} \leq \rho_{\min}$. ///

Thus, for $s_L \in B\mathcal{A}_1(\mathbb{C}_+)$ and a non-constant function, the inequalities now read:

$$\begin{aligned}
&\max\{g_T(0, S, s_L) : S \in U^+(2,d)\} \\
&\quad = \quad 1 - \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2,d); 0)\} \\
&\quad \leq \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2,\infty); 0)\} \\
&\quad \overset{\text{Darlington}}{=} \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\quad \leq \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\quad \overset{\text{Theorem 13}}{=} \quad 1 - \overset{\text{Theorem 9}}{\min}\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

The next section drops the real constraint. Removing real constraint lets us assert that there is a lumped lossless 2-port that can get arbitrarily close to the H^∞ minimum.

6.3 Dropping the Real Constraint

Symmetries in the target function project onto the minimizer. This result permits us to drop the real condition.

Corollary 2 *Let $s_L \in \Re BH^\infty(\mathbb{C}_+)$. Then,*

$$\begin{aligned}
&\min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \Re\overline{B}H^\infty(\mathbb{C}_+)\} \\
&\quad = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

Proof. Let

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.$$

Theorems 9 and 10, with the assumption $\|s_L\|_\infty < 1$, justify the use of the minimums. The real constraint gives the following:

$$\min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}H^\infty(\mathbb{C}_+)\} \geq \rho_{\min}.$$

If we can show that there is a real $s_2 \in \overline{B}H^\infty(\mathbb{C}_+)$ such that $\rho_{\min} = \|\Delta P(s_2, s_L)\|_\infty$, then the first equality is established. Lemma 13 with $\rho_{\min}$ computes the set of minimizers:

$$\emptyset \neq \overline{D}(k, r) \cap \overline{B}H^\infty(\mathbb{C}_+).$$

A real load translates to $s_L(p) = \overline{s_L(\overline{p})}$ for $p \in \mathbb{C}_+$. Define

$$\widetilde{s_L}(p) := \overline{s_L(\overline{p})}$$

and observe that $\widetilde{s_L} \in H^\infty(\mathbb{C}_+)$. Let $s_2 \in \overline{D}(k,r) \cap \overline{B}H^\infty(\mathbb{C}_+)$. Then,

$$\begin{aligned}
\|\Delta P(\widetilde{s_2}, s_L)\|_\infty &= \sup\left\{\left|\frac{\widetilde{s_2}(p) - \overline{s_L(p)}}{1 - \widetilde{s_2}(p) s_L(p)}\right| : p \in \mathbb{C}_+\right\} \\
&= \sup\left\{\left|\frac{\overline{s_2(\overline{p})} - \overline{s_L(p)}}{1 - \overline{s_2(\overline{p})} s_L(p)}\right| : p \in \mathbb{C}_+\right\} \\
&= \sup\left\{\left|\frac{s_2(p) - \overline{s_L(\overline{p})}}{1 - s_2(p) \overline{\overline{s_L(\overline{p})}}}\right| : p \in \mathbb{C}_+\right\} \\
&= \sup\left\{\left|\frac{s_2(p) - \overline{s_L(p)}}{1 - s_2(p) s_L(p)}\right| : p \in \mathbb{C}_+\right\} \\
&= \|\Delta P(s_2, s_L)\|_\infty.
\end{aligned}$$

These equalities show that $\widetilde{s_2}$ is also a minimizer: $\widetilde{s_2} \in \overline{D}(k,r) \cap \overline{B}H^\infty(\mathbb{C}_+)$. Because the set of minimizers is convex, it follows that $\{s_2 + \widetilde{s_2}\}/2 \in \overline{D}(k,r) \cap \overline{B}H^\infty(\mathbb{C}_+)$. That is, the set of minimizers contains a real minimizer whenever the load is real. ///

It turns out that the same result occurs when we only have a minimizing sequence.

Corollary 3 *Let $s_L \in \Re BH^\infty(\mathbb{C}_+)$. Then,*

$$\begin{aligned}
&\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&= \inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\}.
\end{aligned}$$

Proof. Let

$$\rho_\infty := \inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\}.$$

Let $1 > \rho_n \geq 0$ be a sequence converging down to $\rho_\infty \geq 0$. Let k_n and r_n be the associated center and radius functions from Equations 6.1 and 6.2, respectively. The Disk Trick (Lemma 13) computes the level sets:

$$\{s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+) : \|\Delta P(s_2, s_L)\|_\infty \leq \rho_n\} = D(k_n, r_n) \cap \mathcal{A}_1(\mathbb{C}_+).$$

By construction, the intersection is non-empty. Let

$$s_n \in D(k_n, r_n) \cap \mathcal{A}_1(\mathbb{C}_+).$$

Observe that $\widetilde{s_n} \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)$. From the proof in Corollary 2, observe that

$$\|\Delta P(\widetilde{s_n}, s_L)\|_\infty = \|\Delta P(s_n, s_L)\|_\infty.$$

Thus, the level sets are invariant under the "real" operator or

$$\widetilde{s_n} \in D(k_n, r_n) \cap \mathcal{A}_1(\mathbb{C}_+).$$

Because the level sets are convex, it follows that the following real function is also in the level set:

$$\frac{1}{2}\{s_2 + \widetilde{s_2}\} \in \overline{D}(k_n, r_n) \cap \mathcal{A}_1(\mathbb{C}_+).$$

As a element of the level set, $(s_2 + \widetilde{s_2})/2$ satisfies

$$\begin{aligned} \rho_n &\geq \Delta P(\{s_2 + \widetilde{s_2}\}/2, s_L) \\ &\geq \inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\ &=: \rho_{\infty,\Re}. \end{aligned}$$

The set inclusion forces $\rho_{\infty,\Re} \geq \rho_\infty$. Thus, equality holds with a real load. ///

Finally, when a minimizer exists in the disk algebra and the load is real, the same convexity arguments show that a real minimizer also exists.

Corollary 4 *Let $s_L \in \Re BH^\infty(\mathbb{C}_+)$. If the power mismatch admits a solution on $\overline{B}\mathcal{A}_1(\mathbb{C}_+)$:*

$$\rho_{\min} := \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\},$$

then it also admits a solution on $\Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)$ that satisfies

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\}.$$

6.4 Best Match via Darlington+Nehari

For $s_L \in \Re B\mathcal{A}_1(\mathbb{C}_+)$ and a non-constant function, the inequalities now read:

$$\begin{aligned} &\max\{g_T(0, S, s_L) : S \in U^+(2, d)\} \\ &\quad= \quad 1 - \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, d); 0)\} \\ &\quad\leq \quad 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2, \infty); 0)\} \\ &\overset{\text{Darlington}}{=} 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\ &\overset{\text{Corollary 3}}{=} 1 - \inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\ &\overset{\text{Theorem 13}}{=} 1 - \overset{\text{Theorem 9}}{\min}\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\} \\ &\overset{\text{Corollary 2}}{=} 1 - \overset{\text{Theorem 10}}{\min}\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \Re\overline{B}H^\infty(\mathbb{C}_+)\}. \end{aligned}$$

This result is important because it takes the minimum on the unit ball of H^∞—a computable minimum—and shows that this minimum may be approximated to arbitrary precision by a real reflectance in the disk algebra. Darlington's Theorem says this reflectance can be approximated to arbitrary precision by a rational reflectance. This rational reflectance dilates to a lumped, reciprocal, matching circuit. That is, the lumped, reciprocal 2-ports give arbitrarily close performance to the H^∞ minimum. Thus, any upper bounds on the transducer power gain, such as Fano's bounds, are subsumed by the H^∞ computation.

7

Existence of Matching 1-Ports in $\overline{B}\mathcal{A}_1(\mathbb{C}_+)$

The preceding equalities still had some infimums and left unanswered the following question:

Question 5 *What additional constraints on the load will give the existence of a matching 2-port in $U^+(2,\infty)$?*

An answer begins by getting conditions for the existence of the s_{22} element or replacing the "inf" in the preceeding inequalities with a "min" when optimizing over the disk algebra.

7.1 Invariance of the H^∞ Approximant

The basic definitions are set on the unit circle. From Duren [15, page 71]: Let $\phi : \mathbf{R} \to \mathbb{C}$ be periodic with period 2π. The *modulus of continuity* of ϕ is the function

$$\omega(\phi;t) := \sup\{|\phi(t_1) - \phi(t_2)| : t_1, t_2 \in \mathbf{R}, |t_1 - t_2| \le t\}.$$

Λ_α denotes those functions that satisfy a *Lipschitz condition of order* $\alpha \in (0,1]$:

$$|\phi(t_1) - \phi(t_2)| \le A|t_1 - t_2|^\alpha.$$

$C^{n+\alpha}$ denotes those functions with $\phi^{(n)} \in \Lambda_\alpha$ [4]. C_ω denotes those functions that are Dini-continuous:

$$\int_0^\epsilon \omega(\phi;t)t^{-1}dt < \infty,$$

for some $\epsilon > 0$. A sufficient condition for a function $\phi(t)$ to be Dini-continuous is that $|\phi'(t)|$ be bounded [18, section IV.2]. Carleson and Jacobs have an amazing paper that addresses best approximation from the disk algebra [4]:

Theorem 14 (Carleson and Jacobs [4]) *If $k \in L^\infty(\mathbf{T})$, then there always exists a best approximation $h \in H^\infty(\mathbf{D})$:*

$$\|k - h\|_\infty = \|k + H^\infty(\mathbf{D})\|_\infty.$$

If $k \in C(\mathbf{T})$, then the best approximation is unique. Moreover,

(a) *If $k \in C_\omega$ then $h \in C_\omega$.*

(b) *If $k^{(n)} \in C_\omega$ then $h^{(n)} \in C_\omega$.*

(c) *If $0 < \alpha < 1$ and $k \in \Lambda_\alpha$ then $h \in \Lambda_\alpha$.*

(d) *If $0 < \alpha < 1$, $n \in N$, and $k \in C^{n+\alpha}$ then $h \in C^{n+\alpha}$.*

As noted by Carleson and Jacobs [4]: "the function-theoretic proofs ... are all of a local character, and so all the results can easily be carried over to any region which has in each case a sufficiently regular boundary." Provided we can guarantee smoothness across ∞, Theorem 14 carries over to the right half-plane.

Corollary 5 *If $k \in 1\dot{+}C_0(j\mathbf{R})$, then the best approximation*

$$\|k - h\|_\infty = \|k - H^\infty(\mathbb{C}_+)\|_\infty$$

exists and is unique. Moreover, if $k \circ \mathbf{c}^{-1} \in C_\omega$, then $h \circ \mathbf{c}^{-1} \in C_\omega$ so that

$$\|k - h\|_\infty = \|k - H^\infty(\mathbb{C}_+)\|_\infty = \|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty.$$

Proof. The Caley transform $\mathbf{c}$ of Lemma 1 coupled with Theorem 14 gives the existence and uniqueness of $h \in H^\infty(\mathbb{C}_+)$ and the membership of $h \circ \mathbf{c}^{-1}$ in C_ω. Thus, $h \in \mathcal{A}_1(\mathbb{C}_+)$. For the equalities, set inclusions give the following:

$$\begin{aligned} \|k - H^\infty(\mathbb{C}_+)\|_\infty &= \|k \circ \mathbf{c}^{-1} - H^\infty(\mathbf{D})\|_\infty \\ &\leq \|k \circ \mathbf{c}^{-1} - \mathcal{A}(\mathbf{T})\|_\infty \\ &\leq \|k \circ \mathbf{c}^{-1} - \mathcal{A}(\mathbf{T}) \cap C_\omega\|_\infty \end{aligned}$$

Theorem 14(a) gives that

$$\|k \circ \mathbf{c}^{-1} - H^\infty(\mathbf{D})\|_\infty = \|k \circ \mathbf{c}^{-1} - \mathcal{A}(\mathbf{T}) \cap C_\omega\|_\infty.$$

This squeezes the inequalities and we get

$$\|k \circ \mathbf{c}^{-1} - H^\infty(\mathbf{D})\|_\infty = \|k \circ \mathbf{c}^{-1} - \mathcal{A}(\mathbf{T})\|_\infty.$$

In terms of the right half-plane, $h \in \mathcal{A}_1(\mathbb{C}_+)$ and

$$\|k - h\|_\infty = \|k - H^\infty(\mathbb{C}_+)\|_\infty = \|k - \mathcal{A}_1(\mathbb{C}_+)\|_\infty.$$

///

The key result of Corollary 5 is that both inf's in the coset norms are replaced by min's. The trick is to map this result from the norms into the power mismatch.

Corollary 6 ($\mathcal{A}_1(\mathbb{C}_+)$ Existence and Uniqueness) *Let $s_L \in B\mathcal{A}_1(\mathbb{C}_+)$ be a non-constant function. Then,*

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}.$$

Define the center function $k_{\min}$ and radius function $r_{\min}$ as in Equations 6.1 and 6.2. The radius function $r_{\min}$ admits a spectral factorization $a_{\min} \in \mathcal{A}_1(\mathbb{C}_+)$. If $a_{\min}^{-1} k_{\min} \circ \mathbf{c}^{-1} \in C_\omega$, then the power mismatch problem in the disk algebra admits a unique solution that attains the H^∞ minimum also:

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\}.$$

Proof. The assumptions on the load and Theorem 13 give the minimum $\rho_{\min}$. The proof of Theorem 13 and the assumptions on the load give that $a_{\min}$ belongs to $\mathcal{A}_1(\mathbb{C}_+)$. Theorem 12 (Min-2) gives that

$$\|a_{\min}^{-1} k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty = 1.$$

Lemma 5 and the C_ω assumption give that there is a unique $h \in \mathcal{A}_1(\mathbb{C}_+)$ such that

$$\|a_{\min}^{-1} k_{\min} - h\|_\infty = \|a_{\min}^{-1} k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty.$$

That is, $|k_{\min} - a_{\min} h| \le r_{\min}$. Noting that $a_{\min} h \in \mathcal{A}_1(\mathbb{C}_+)$ gives

$$\{a_{\min} h\} \subseteq \overline{D}(k_{\min}, r_{\min}) \cap \mathcal{A}_1(\mathbb{C}_+).$$

Conversely, any element of the intersection is also a minimizer of $\|a_{\min}^{-1} k_{\min} - H^\infty(\mathbb{C}_+)\|_\infty$. Therefore, the inclusion is really an equality. ///

7.2 Best Match via Darlington+Nehari+Carleson

If $s_L \in \Re B\mathcal{A}_1(\mathbb{C}_+)$, the minimizers can be found in the disk algebra—provided $a_{\min}^{-1} k_{\min} \circ \mathbf{c}^{-1} \in C_\omega$. Simple conditions that guarantee this Dini-continuity are as follows.

Lemma 15 *Let $s_L \in \Re B\mathcal{A}_1(\mathbb{C}_+)$ be non-constant. Let*

$$\rho_{\min} = \min\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\},$$

where the "min" is given by Corollary 6. Define the center function $k_{\min}$ and radius function $r_{\min}$ as in Equations 6.1 and 6.2. Let $r_{\min}$ admit a spectral factorization $a_{\min} \in \mathcal{A}_1(\mathbb{C}_+)$, where Corollary 6 supplies the existence of $a_{\min}$. Assume

(a) $s_L' \in 1\dot{+}C_0(j\mathbf{R})$.

(b) $s_L'(p) = \mathcal{O}[p^{-2}]$.

(c) $0 < \|s_L\|_{-\infty}$.

These assumptions force $a_{\min}^{-1} k_{\min} \circ \mathbf{c}^{-1} \in C_\omega$.

Proof. The goal is to show that $k_{\min} \circ \mathbf{c}^{-1}$ and $a_{\min}^{-1} \circ \mathbf{c}^{-1}$ are differentiable with bounded derivative. Their product is Dini-continuous and we can invoke Lemma 6. We start by clarifying the derivative on the circle. Duren [15, page 42] points out the expression $h'(e^{j\theta})$ can have two possible meanings. It may indicate the radial limit of $h'(z)$ or it may indicate the derivative with respect to θ. Integrability forces the two notions to coincide:

> THEOREM [15, page 42] Let $h \in H(\mathbf{D})$. The following are equivalent:
>
> **(a)** $h \in \mathcal{A}(\mathbf{D})$ and is absolutely continuous on $\mathbf{T}$.
>
> **(b)** $h' \in H^1(\mathbf{D})$.
>
> If $h' \in H^1(\mathbf{D})$, then
>
> $$\frac{d}{d\theta} h(e^{j\theta}) = j e^{j\theta} \lim_{r \to 1} h'(re^{j\theta}).$$

We must verify that differentiability in $\mathbb{C}_+$ maps to $\mathbf{D}$. Observe that if $h \in H^\infty(\mathbb{C}_+)$, then $H := h \circ \mathbf{c}^{-1} \in H^\infty(\mathbf{D})$ with

$$\begin{aligned} H'(z) &= h'(\mathbf{c}^{-1}(z)) \frac{d}{dz} \frac{1+z}{1-z} \\ &= h'(\mathbf{c}^{-1}(z)) \frac{2}{(1-z)^2}. \end{aligned}$$

Thus, $H' \in H^\infty(\mathbf{D})$, provided $h' \in H^\infty(\mathbb{C}_+)$ and

$$h'(p) = \mathcal{O}[p^{-2}] \quad p \to \infty.$$

Similar results hold for $K = k \circ \mathbf{c}^{-1} \in L^\infty(\mathbf{T})$. Our goal is to show $K' \in L^\infty(\mathbf{T})$, under the assumptions that $k \in L^\infty(j\mathbf{R})$ and

$$k'(j\omega) = \mathcal{O}[w^2] \quad \omega \to \infty.$$

To keep the notation under control, let

$$k'(j\omega) := \frac{d}{d\omega} k(j\omega).$$

The equation for the center function

$$k_{\min} = \overline{s}_L \, \frac{1 - \rho_{\min}^2}{1 - \rho_{\min}^2 |s_L|^2}$$

gives

$$k'_{\min} = (1 - \rho_{\min}^2) \, \frac{\overline{s}'_L + \rho_{\min}^2 \overline{s}_L^2 s'_L}{(1 - \rho_{\min}^2 |s_L|^2)^2}.$$

Assumptions (a), (b), and $\|s_L\|_\infty < 1$ give that $K_{\min} := k_{\min} \circ \mathbf{c}$ has $K'_{\min} \in C(\mathbf{T})$. Likewise, the equation for the radius function

$$r_{\min} = \rho_{\min} \, \frac{1 - |s_L|^2}{1 - \rho_{\min}^2 |s_L|^2}$$

gives

$$r'_{\min} = \rho_{\min} \, (1 - \rho_{\min}^2) \, \frac{s'_L \overline{s}_L + s_L \overline{s}'_L}{(1 - \rho_{\min}^2 |s_L|^2)^2}.$$

Assumptions (a), (b), and $\|s_L\|_\infty < 1$ give that $R_{\min} := r_{\min} \circ \mathbf{c}$ has $R'_{\min} \in C(\mathbf{T})$. The spectral factorization of $r_{\min}$ has the spectral factorization on the unit disk as

$$A_{\min}(z) := a_{\min} \circ \mathbf{c}^{-1}(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{jt} + z}{e^{jt} - z} \log(R_{\min}(e^{jt}))dt \right).$$

Assumption (c) gives that

$$h(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{jt} + z}{e^{jt} - z} \log(R_{\min}(e^{jt}))dt$$

belongs to $H(\mathbf{D})$. Differentiate with respect to θ and use integration by parts:

$$\begin{aligned}
jzh'(z) &= \frac{d}{d\theta} h(re^{\theta}) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{d\theta} \left\{ \frac{e^{j(t-\theta)} + r}{e^{j(t-\theta)} - r} \right\} \log(R_{\min}(e^{jt}))dt \\
&= -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{jt} + z}{e^{jt} - z} \frac{1}{R_{\min}(e^{jt})} R'_{\min}(e^{jt}) j e^{jt} dt.
\end{aligned}$$

Denote the Poisson kernel as

$$P(r,t) = \Re\left(\frac{e^{jt}+r}{e^{jt}-r}\right).$$

Then,

$$\begin{aligned}
|jzA'_{\min}(z)| &\leq |A_{\min}(z)|\exp\left(-\frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)\frac{R'_{\min}(e^{jt})}{R_{\min}(e^{jt})}\Re(je^{jt})dt\right) \\
&\leq |A_{\min}(z)|\exp\left(+\frac{1}{2\pi}\int_0^{2\pi} P(r,t-\theta)\frac{R'_{\min}(e^{jt})}{R_{\min}(e^{jt})}\sin(t)dt\right) \\
&\leq \infty.
\end{aligned}$$

Thus, $A'_{\min} \in H^\infty(\mathbf{D})$. The continuity of $R'_{\min}/R_{\min}$ gives that $A'_{\min} \in \mathcal{A}(\mathbf{D})$. The exponential form gives that $(1/A_{\min})'$ is also in $\mathcal{A}(\mathbf{D})$. We see that both $K_{\min}$ and $(1/A_{\min})$ have derivatives that are bounded in magnitude. Thus, $K_{\min}/A_{\min} \in C_\omega$. ///

If Lemma 15 holds, then the inequalities now read:

$$\begin{aligned}
&\max\{g_T(0,S,s_L) : S \in U^+(2,d)\} \\
&= \quad 1 - \min\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2,d);0)\} \\
&\leq \quad 1 - \inf\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2,\infty);0)\} \\
&\overset{\text{Darlington}}{=} \quad 1 - \overset{\text{Corollary 4}}{\min}\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \Re\overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\overset{\text{Corollary 3}}{=} \quad 1 - \overset{\text{Corollary 6}}{\min}\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \overline{B}\mathcal{A}_1(\mathbb{C}_+)\} \\
&\overset{\text{Theorem 13}}{=} \quad 1 - \overset{\text{Theorem 9}}{\min}\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\} \\
&\overset{\text{Corollary 2}}{=} \quad 1 - \overset{\text{Theorem 10}}{\min}\{\|\Delta P(s_2,s_L)\|_\infty^2 : s_2 \in \Re\overline{B}H^\infty(\mathbb{C}_+)\}.
\end{aligned}$$

The remaining "inf" and the research problems that arise when tying this 1-port approach back to the matching 2-port conclude this report.

8

Research Topics, Applications, and Transitions

This section organizes the questions raised by this report into several research topics and makes explicit wideband applications of the H^∞ techniques. For concreteness, we conclude with two applications that demonstrate how the electrical engineer can use the information provided by the H^∞ computations to benchmark matching circuits. The key to applying the H^∞ techniques is the conversion of the matching problem—a optimization problem over classes of lossless 2-ports—into an equivalent 1-port problem. It is this 1-port problem that is computable and bounded by the H^∞ techniques. Various continuity conditions make these bounds tight and assert the existence of solutions for the 1-port problem. However, this report did not emphasize that the reflectance s_2 of an optimal matching 1-port is also produced. This matching reflectance arises whenever the Hankel matrix H_ϕ of Nehari's Theorem admits a maximizing vector [58]. Thus, is it is natural to consider how this 1-port matching information lifts to a matching 2-port.

Thus, our first research topic focuses on completing the analysis of the preceeding section regarding the existence of s_2 (Question 5) Computing

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \mathcal{F}_2(U^+(2,\infty); 0)\}$$

over the orbit of $s_G = 0$ forces

$$s_2 = s_{22} + s_{21}s_G(1 - s_{11}s_G)^{-1}s_{12}|_{s_G=0} = s_{22}$$

to be rational (Darlington's Theorem or Corollary 1). A solution for a matching 2-port in $U^+(2,\infty)$ can be obtained—provided one puts conditions on $a_{\min}k_{\min}$ that guarantee Theorem 11(A-3) has a rational minimizer. These conditions and the extension of Theorem 11(A-3) are the first research topic.

The second research topic enlarges the set of lossless 2-ports. The scattering matrix of a lossless 2-port is continuous if and only if the 2-port is lumped. At high frequencies, lumped elements give way to distributed elements, such as the transmission line. The scattering matrix of the transmission line is not continuous (Example 2). The scattering matrices for both the lumped and distributed 2-ports are contained in $U^+(2)$. The converse asks if every scattering matrix in $U^+(2)$ corresponds to a physical 2-port (Question 3). If $U^+(2)$ has a physical meaning, the corresponding matching problem

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \mathcal{F}_2(U^+(2); 0)\},$$

whether or not a solution s_2 exists, and how a solution s_2 dilates to a matching 2-port have a physical meaning. However, not all such s_2's can dilate to a lossless 2-port. Wohlers [55, page 100-101] shows that the 1-port with impedance $z(p) = \arctan(p)$ cannot dilate to an $S \in U^+(2)$. The Douglas-Helton result precisely characterizes those elements in the unit ball of H^∞ that came from a lossless N-port.

Theorem 15 ([13], [14]) *Let $S(p) \in \overline{B}H^\infty(\mathbb{C}_+, \mathbb{C}^{N\times N})$ be a real matrix function. The following are equivalent:*

(a) *$S(p)$ admits an real inner dilation $\mathbf{S}(p)$ such that*

$$\mathbf{S}(p) = \begin{bmatrix} S(p) & S_{12}(p) \\ S_{21}(p) & S_{22}(p) \end{bmatrix}.$$

(b) *$S(p)$ has a meromorphic pseudo-continuation of bounded type to the open left half-plane $\mathbb{C}_-$. That is, there exists a $\phi \in H^\infty(\mathbb{C}_-)$ and an $H \in H^\infty(\mathbb{C}_-, \mathbb{C}^{N\times N})$ such that*

$$\lim_{\substack{\sigma>0 \\ \sigma\to 0}} S(\sigma + j\omega) = \lim_{\substack{\sigma>0 \\ \sigma\to 0}} \frac{H}{\phi}(-\sigma + j\omega) \quad \text{a.e.}$$

(c) *There is an inner function $\phi \in H^\infty(\mathbb{C}_+)$ such that $\phi S^H \in H^\infty(\mathbb{C}_+, \mathbb{C}^{N\times N})$.*

Let $\mathcal{M}$ denote the subset of $\overline{B}H^\infty(\mathbb{C}_+)$ of functions that have meromorphic pseudo-continuations of bounded type. A general Carleson-Jacob line of inquiry opens up to explore when the inequality

$$\inf\{\|\Delta P(s_2, s_L)\|_\infty : s_2 \in \mathcal{M}\} \geq \min\{\|\Delta P(s_2, s_L)\|_\infty^2 : s_2 \in \overline{B}H^\infty(\mathbb{C}_+)\}$$

holds with equality. Thus, this second research topic contains a host of issues all related to the physical meaning of the Darlington dilation.

The third research topic addresses a current limitation of the H^∞ approach: *How does Darlington's Theorem generalize?* Mathematically, we are asking for a "unit-ball" characterization of an orbit so Nehari's Theorem can be applied. This characterization is the General Darlington Problem posed in Question 4: *What is the orbit of a general reflectance $\mathcal{F}_1(\mathcal{U}, s_L)$?* We can also generalize $U^+(2,\infty)$ and ask about the orbit of s_L over all lumped 2-ports. The problem is to characterize all reflectances that belong to

$$\bigcup_{d\geq 0} \mathcal{F}_1(U^+(2,d), s_L)$$

and its closure. The question of when a reflectance s_L belongs to the orbit of another reflectance s'_L means that there is an $S' \in U^+(2)$ such that $s_L = \mathcal{F}_1(S', s'_L)$. The transmission zeros of S' force the inequality

$$\begin{aligned} &\inf\{\|\Delta P(s_G, \mathcal{F}_1(S, s_L))\|_\infty : S \in U^+(2)\} \\ &\geq \inf\{\|\Delta P(s_G, \mathcal{F}_1(S, s'_L))\|_\infty : S \in U^+(2)\}. \end{aligned}$$

This theory of *compatible impedances* is an active research topic in electrical engineering [57] and has links to the Buerling-Lax Theorem [28].

The final research topic revisits the general matching problem and the preceeding topics by restricting the frequency band Ω to be a finite subset of $\mathbf{R}$. The problem is to maximize the transducer power gain

$$\sup\{\|G_T(s_G, S, s_L)\|_{-\infty,\Omega} : S \in \mathcal{U}\},$$

where the frequency band $\Omega = [\omega_{\min}\ ,\ \omega_{\max}]$.

Although these topics are mathematical, the application to commercial venues and the Fleet is direct. Table 8.1 lists several active and passive devices that explicitly use wideband matching to improve performance. The H^∞ applications to the transducers,

Table 8.1: Selected applications that use wideband impedance matching to optimize performance.

Device	Frequency	Reference
Acoustic transducers	VLF, LF	
Antenna: Navy ship	HF	[51]
Antenna: microstrip	SHF	[2], [8]
Circulator: strip-line	SHF	[37]
Radar links: photodetector	RF	[24], [21]
Fiber-Optic links	SHF	[6]
Satellite links	3.6-4.2 GHz	[43]
Amplifiers	2-20 GHz	[9], [20]
Amplifiers	SHF	[38]

antenna, and communication links are immediate. The amplifier is an active 2-port that requires a more general approach. The matching problem for the amplifier is to find input and output matching 2-ports that simultaneously maximize transducer power gain, minimize the noise figure, and maintain stability. Although a more general problem, this amplifier-matching problem fits squarely in the H^∞ framework [27], [28], [29] and is a current topic in ONR's H^∞ research initiative [44].

We close with two demonstrations of the H^∞ techniques. The examples show that the H^∞ techniques are not tied to an operating band. The examples also show how an engineer can assess design requirements and benchmark matching circuits. The first example is the impedance matching of an antenna. Recent measurements were acquired on the forward-mast integrated HF antenna of the LPD 17, an amphibious transport dock. The H^∞ bound for this antenna over 9 to 30 MHz sets the best possible VSWR at approximately 2.37. Figure 8.1 uses this VSWR bound to benchmark several classes of matching circuits. Each circuit's VSWR is plotted as a function of the degree d (the total number of inductors and capacitors). The dashed lines are the VSWR from the low- and high-pass ladders containing inductors and capacitors constrained to practical design values. The solid line is the matching estimated from $U^+(2, d)$. A transformer performs as well as any matching circuit of degree 0 and as well as the low-pass ladders out to degree 6. The high-pass ladders get closer to the VSWR bound at degree 4. A perfectly coupled transformer (coefficient of coupling $k = 1$) offers only a slight improvement over the transformer. Thus, the circuit designer can graphically assess trade-offs between various circuits in the context of knowing the best match possible for any lossless 2-port.

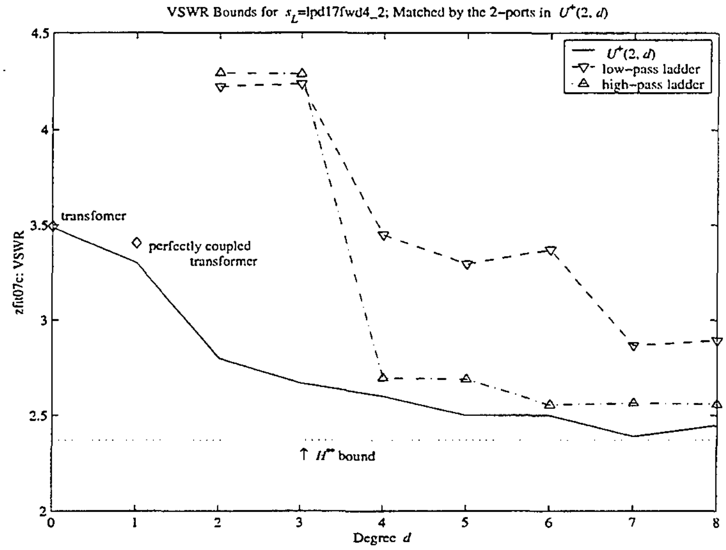

Figure 8.1: Comparing the matching performance of several classes of 2-ports for the LPD17FWD4_2 antenna operating over 9 to 30 MHz with the H^∞ bound [1].

For our final application, we turn to the optical region. A typical fiber-optic link is shown in Figure 8.2 and described by Chapelle [6]: "An optical fiber couples the light emitted from a semiconductor laser diode and guides it to the junction of a photodiode. An input circuit is required to match the very low impedance of the forward biased laser diode to the 50-ohm source impedance. An output circuit is necessary for matching the very high impedance of the reverse biased photodiode the 50-ohm load impedance."

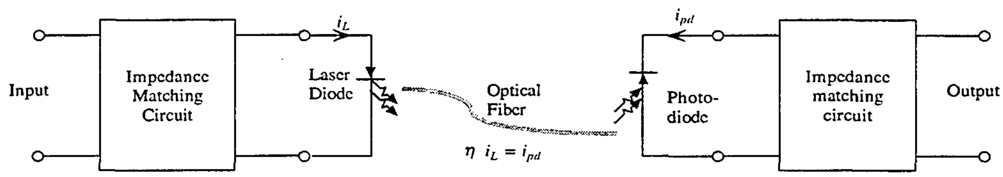

Figure 8.2: A fiber-optic link with impedance-matching circuits [6].

Figure 8.3 presents the equivalent circuits for a laser diode and photodiode [6]. The circuits were used to make the matching computations reported in Table 8.2. Table 8.2 compares the H^∞ bound to the matching performance of an ideal transformer and an ideal transmission line. Knowing the best possible transducer power gain permits the engineer to either select a matching circuit, or to consider other types of matching circuits. For example, neither the transformer or the transmission line reveal the matching performance of the photodiode. The engineer relying on these circuits classes to match the photodiode would be mislead. In contrast, the H^∞ bound provides a benchmark to assess the quality of any matching circuit. In the case of the photodiode, the gap between the H^∞ bound and the matching in Table 8.2 encourages exploration of other circuits. For example, a coupled coil matches the photodiode with $G_{T,\mathrm{cc}} = 0.2348$. From the H^∞ bound, the engineer knows the absolute improvement provided by a coupled coil. The engineer can use this information to assess the benefit of matching with a coupled coil, or continue the search for a better matching circuit.

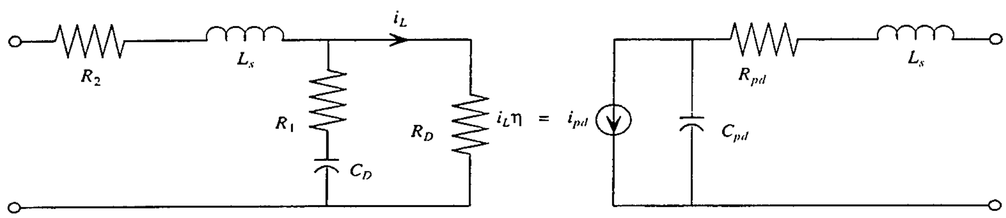

Figure 8.3: Equivalent circuits of a laser diode and photodiode operating over 6 to 12 GHz [6]. Laser diode parameters: $R_D = 4.2\Omega$, $R_1 = 0.5\Omega$, $R_2 = 4.4\Omega$, $C_D = 4.0$ pF, $L_s = 0.2$ nH. Photodiode parameters: $R_{pd} = 2.7\Omega$, $C_{pd} = 0.55$ pF, $L_s = 0.20$ nH.

Table 8.2: Matching results for the laser and photodiode. The transducer power gain G_T estimated for a transformer and a transmission line is compared to the H^∞ bound.

Data	Method	G_T
Chapelle LD	H^∞	0.9840
	Transformer	0.5822
	Transmission Line	0.6913
Chapelle PD	H^∞	0.3620
	Transformer	0.1229
	Transmission Line	0.1555

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14. ABSTRACT
The selection of a lossless 2-port to maximize the wideband power transfer from a generator to a load is a ubiquitous problem in electrical engineering. The mathematical problem is to maximize the wideband transducer power gain over a class of lossless 2-ports. As a numerical optimization problem, wideband impedance matching is difficult because the wideband transducer power gain is a nonlinear, nondifferentialable, badly scaled multivariable function. Therefore, any information on the global solution is valuable to the engineer for assessing the quality of sub-optical solutions computed by numerical optimizers. In his classic 1950 paper, Fano determined a theoretical upper bound on the transducer power gain. Specifically, the transducer power gain of any lossless 2-port cannot exceed Fano's bound. Development of Fano's approach continued through the 1960s. However, computing these bounds required solving a highly nonlinear system of multivariate inequalities amenable only for simple cases. In the early 1970s, Helton made the amazing connection between operator theory and electrical engineering. Powerful Hardy space techniques were coupled to the electrical engineer's Smith chart computations. In this framework, Nehari's Theorem gave an upper bound on the transducer power gain computable as an (easy) eigenvalue problem. This report shows that continuity conditions make this Nehari bound tight.

15. SUBJECT TERMS
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