THE CENTURY OF TURBULENCE THEORY: THE MAIN ACHIEVEMENTS AND UNSOLVED PROBLEMS

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1. Introduction

The flows of fluids actually met both in nature and engineering practice are turbulent in the overwhelming majority of cases. Therefore, in fact the humanity began to observe the turbulence phenomena at the very beginning of their existence. However only much later some naturalists began to think about specific features of these phenomena. And not less than 500 years ago the first attempts of qualitative analysis of turbulence appeared - about 1500 Leonardo da Vinci again and again observed, described and sketched diverse vortical formations ('coherent structures' according to the terminology of the second half of the 20th century) in various natural water streams. In his descriptions this remarkable man apparently for the first time used the word 'turbulence' (in Italian 'la turbolenza', originating from Latin 'turba' meaning turmoil) in its modern sense and also outlined the earliest version of the procedure similar to that now called the 'Reynolds decomposition' of the flow fields into regular and random parts (see, e.g., [1,2]). However, original Leonardo's studies did not form a 'theory' in the modern meaning of this word. Moreover, he published nothing during all his life and even used in most of his writings a special type which could be read only in a mirror. Therefore his ideas became known only in the second half of the 20th century and had no influence on the subsequent investigations of fluid flows.

During the first half of the 19th century a number of interesting and important observation of turbulence phenomena were carried out (such as, e.g., the early pipe-flow observation by G. Hagen [3]) but all of them were only the precursors of the future theory of turbulence. Apparently, the first theoretical works having relation to turbulence were the brilliant papers on hydrodynamic stability published by Kelvin and Rayleigh at the end of the 19th century (apparently just Kelvin who know nothing about Leonardo's secret writings, independently introduced the term "turbulence" into fluid mechanics). However, these papers only 'had relation to turbulence', but did not concern the developed turbulence at all. First scientific description of turbulence was in fact given by Reynolds [4]. In his paper of 1883 he described the results of his careful observations of water flows in pipes, divided all pipe flows into the classes of "direct" and "sinuous" (laminar and turbulent in the modern terminology) flows and introduced the most important dimensionless flow characteristic (now called 'the Reynolds number') $\mathrm{Re} = UL/\nu$, where U and L are characteristic velocity and length scales, and ν is the kinematic viscosity of the fluid. Then Reynolds proposed the famous 'Reynolds-number criterion', according to which the turbulence can exist only if $\mathrm{Re} > \mathrm{Re}_{cr}$ where the critical value Re_{cr} takes different values for different flows and different levels of initial disturbances. And the first serious purely theoretical investigation of the developed turbulence was due again to Reynolds [4]. In his classical paper of 1894 he strictly determined the procedure of 'Reynolds decomposition', derived the 'Reynolds equations' for the mean velocities of turbulent flows and made the first attempt to estimate theoretically the value of Re_{cr} with the help of Navier-Stokes (briefly NS) equations of fluid dynamics. (These equations assume that the fluid is incompressible and have constant density and kinematic viscosity; below only such fluids will be considered.) Therefore the year 1894 may with good reason be considered as the birth year of the modern turbulence theory. After this year turbulence theory was developing energetically during the whole 20th century but up to now it is very far from the completion. Thus, we have quite weighty reasons to call the 20th century *the century of the turbulence theory.*

Of course the 20th century deserves also to be called by more high-grade title of the century of science. In fact during this century the enormous advances were achieved in all sciences and many highly important new scientific domains emerged; Theory of Relativity, Quantum Physics, Nuclear Physics, Physical Cosmology, Molecular Biology are only a few examples. However, in spite of this the modern status of the turbulence theory is quite exceptional and differing from that of all other new sciences.

The reason of such exceptional status is that the other new sciences deal with some very special and complicated objects and processes relating to some extreme conditions which are very far from realities of the ordinary life. These objects and processes are connected, for example, with the movements having enormously high velocities, or manifestations of unprecedentedly high (or, on the contrary, low) energy changes, with extremally small (or large) sizes of involving objects, enormously large or imperceptibly small length- or/and time-scales, and so on. However turbulence theory deals with the most ordinary and simple realities of the everyday life such as, e.g., the jet of water spurting from the kitchen tap. Therefore, the turbulence is well-deservedly often called "the last great unsolved problem of the classical physics".

Such statement was, in particular, often repeated by the famous physicist R. Feynman who even include it, in a slightly different wording, in his textbook [5] intended for high-school and undergraduate university students (the names of three other great physicists to whom this remark is sometimes attributed were indicated by Gad-el-Hak [6]). One of these physicists, A. Sommerfeld, in the late 1940s once noted that he understood long ago the enormous difficulty of the turbulence problem and therefore proposed it in the 1920s to his most talented student Werner Heisenberg; however Heisenberg did not solve this problem which remains unsolved up to now. Finally, the extraordinary status of the problem of turbulence is reflected in the popular funny story about a famous scientist; several versions of this story are met in the available literature. According to S. Goldstein [7] the story reflects the statement made by H. Lamb in 1932 at some meeting in London where Goldstein was present. Goldstein's memory was that Lamb remarked then: "I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other the turbulent motion of fluids. And about the former I am really optimistic." (In other versions of the story H. Lamb was replaced by L. Prandtl, W. Heisenberg, or A. Einstein, and the time and place of the event and sometimes also the first of the mentioned matters were changed; see, e.g., [6].) Let us consider, however, just the above version where turbulence was compared with quantum electrodynamics. It seems that 1932 was too early date for considering the quantum electrodynamics as the most important unsolved physical problem, however somewhat later, say in the late 1940s and early 1950s, it was exactly so - at that time all experts in theoretical physics were tormented with this problem. However, the solution of it was found not much later. The solution made three physicists (R. Feynman, J. Schwinger and S. Tomonaga) the recipients of the 1965 Nobel Prize in physics; then this problem was closed for ever. When recently a group of prominent physicists formulated 10 most important unsolved problems of modern physics (so-called "Physics Problems for the Next Millennium"; see http://feynman.physics.lsa.umich.edu), these problems showed very clearly how far away went the physics of today from the primitive science of 1930-50s when noncontradictory development of quantum electrodynamics seemed to be an unsolvable problem. However, up to now no cardinal changes occurred in the studies of turbulence. Of course, a lot of new particular interesting results relating to turbulence were found in the 20th century and many technical problems of high practical importance were solved, but there were no Nobel Prizes for turbulence studies and most of the riddles of turbulent motion remain mysterious. In fact, even the precise content of the 'problem of turbulence' is still far from being clear at present (a few remarks about this topic will be made at the end of this text).

Let us now return back to Reynolds' classical papers [4]. In the first of them it was stated that the exceeding by the Reynolds number Re of a laminar flow of the critical value Re_{cr} leads to flow turbulization but the mechanism of this transition to a new flow regime was not considered in any detail. In the second paper of 1894 the turbulization was connected with the growth of flow disturbances but only a very crude method of estimation of Re_{cr} was proposed there. In fact, the accurate determination of turbulization conditions was then, and is up to now, complicated by the obvious incompleteness of the mathematical theory of NS equations. Even the conditions guaranteeing the existence and uniqueness of the solutions of the most natural initial-value problems for NS equations were completely unknown at the end of the 19th century (and are far from being perfectly clear even today). Note in this respect, that the famous French mathematician J. Leray, who in the paper [8] and some other works of the 1930s and 1940s made very important contributions to the mathematical theory of NS equations, sometimes was inclined to assume that the transition to turbulence may be produced by the termination of the existence of the solution of NS equations corresponding to the laminar regime of fluid flow. However, this assumption was not confirmed afterwards and therefore the dominating position again became occupied by the old idea of Kelvin, Reynolds and Rayleigh who assumed that flow tubulization is caused not by the nonexistence of the laminar-flow solution of NS equations but by the instability of this solution to small exterior disturbances.

2. Flow Instability and Transition to Turbulence

The early studies of flow instabilities to small disturbances used the simplest approach based on the linearization of the NS equations with respect to the disturbance velocities and pressure. Studies of the solutions of linearized dynamic equations for the disturbance variables which grow in time (or, in the case of a spatially formulated parallel-flow stability problem, in streamwise direction) form the so-called *linear theory of hydrodynamic stability.*

The initial approach to the study of linear stability of steady parallel laminar flows, which was proposed by Stokes, Kelvin and Rayleigh in the second half of the 19th century, is the *normal-mode method.* Here the eigensolutions of the system of linearized NS equations are studied. These solutions are proportional to $\exp(-i\omega t)$, where ω is an eigenvalue which may be real or complex. The considered laminar flow is called unstable with respect to small disturbances if the eigenvalue ω with $\Im m\omega \geq 0$ (where $\Im m$ denotes the imaginary part) does exist, while otherwise the flow is stable. In the spatial approach to the same problem the eigenfunctions proportional to $\exp(ikx)$ are studied where x is the streamwise coordinate of a parallel flow and k is a complex eigenvalue. Here the flow is called unstable if there exists an eigenvalue k with $\Im m k \leq 0$. Spatial approach was first sketched by Orr [9] but in 1907 the spatial eigenvalue problem seemed to be unsolvable and therefore such approach became popular only in the late 1970s. Note, however, that this approach generates some new mathematical problems (relating, e.g., to the validity of the spatial version of the Squire theorem and to the completeness of the corresponding systems of eigenfunctions) and apparently not all of these problems are already solved.

Let us now revert to the classical temporal approach. According to Reynolds' conjecture at values of Re smaller than Re_{cr} all eigenvalues ω_j, j = 1, 2, 3, ... , have negative imaginary parts. Orr [9] and Sommerfeld [10] independently proposed in 1907-1908 to determine Re_{cr} as the smallest value of Re at which there exist at least one real eigenvalue ω_j. The linear equation determining in the case of a plane-parallel flow the eigenvalues ω_j is called therefore the Orr - Sommerfeld (OS) equation.[1] The papers by Orr and Sommerfeld led to numerous computations of the OS-eigenvalues, the values of Re_{cr}, and the "neutral curves" in (Re,k) and (Re,ω)-planes for various parallel and nearly parallel flows. These computations played central role in the development of the theory of hydrodynamic stability during the main part of the 20th century. However, the values of Re_{cr} given by the OS equation often exceeded very much values of Re at which real flow instability and transition to turbulence were observed. Moreover, in the cases of plane Couette and circular Poiseuille flows the OS-method led to conclusion that $\mathrm{Re}_{cr}=\infty$ which contradict to experimental data showing that both these flows become turbulent at moderate values of Re. (By the way, although the validity of the relation $\mathrm{Re}_{cr}=\infty$ for the Poiseuille pipe flow was confirmed by numerous computations with the 100% reliability, the rigorous mathematical proof of this result was not found yet and still represents an unsolved problem.)

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[1] Note that both these authors considered only the simplest case of two-dimensional wave disturbances assuming that the eigenvalue ω with the smallest imaginary part must always correspond to a plane-wave disturbance (this assumption was rigorously proved only by Squire [11] in 1933). Moreover, they both in fact did not use the OS equation since only the case of a plane Couette flow was considered by them and in this case OS differential equation of the fourth order is reducing to a system of two second-order equations. General form of the OS equation (at once for general three-dimensional wave disturbances) was given by Kelvin [12] in 1887, who however made from this equation an incorrect conclusion.

The often observed disagreement between the OS estimates of Re_{cr} and the observed values of Re corresponding to transition to turbulence may have several reasons. It is clear, in particular, that the consideration of only the eigenfunctions of linearized NS equations in fact represents some oversimplification. Proportional to $e^{-i\omega t}$ eigenfunctions are only special solutions of the linarized NS equations which have amplitudes monotonically (more precisely, exponentially) growing or decaying with t. Moreover, already in 1887 the future Lord Kelvin [13] (at that time he was still called William Thomson) found a solution of the linearized NS equations for a plane Couette flow which "at first rises gradually from initial small value and only asymptotically tends to zero". The paper [13] contained some errors indicated by Rayleigh and Orr; apparently therefore all its results (including the correct ones) were long neglected. As to the nonmonotone Kelvin's solution, Orr [9] generalized it finding a whole family of such nonmonotone solutions (again for the case of plane Couette flow) some of which grew up (proportional to some positive powers of t) to quite large values before they began to decay. Orr even stated the assumption that such transient growth of small disturbances may explain the real instability of plane Couette flow. However this important remark also did not attract then any attention. As a results, the interesting results by Kelvin and Orr were long forgotten and some of them were independently rediscovered by other authors only in 1960s and 1970s.

Strong revival of interest to transient (algebraic in t) growth of disturbances arose at the end of the 20th century. During the last twenty years many dozens of papers about such growth were published (papers [14-20] represent only a few examples of them), while much attention to this topic was also given in books [21,22] and a survey [23]. It was shown, in particular, that transient growth of nonmodal disturbances may exceed very much the growth of the linearly unstable wave modes. This circumstance gave rise to keen interest to 'optimal disturbances' undergoing most intensive transient growth in a given laminar flow; see, e.g., papers [15,24,25] devoted to this subject. Note also that in the case of 'subcritical fluid flow' with $Re < Re_{cr}$ all solutions of linearized disturbance equations tend to zero as $t \to \infty$. Therefore here transient growth of any disturbance determined by the linearized NS equations must be replaced by decay at some value t_0 of t. However, even before t_0 an initially small disturbance may grow so much that the linearized NS equations will be inapplicable to it and its further development will be governed by the nonlinear NS equations. Then it is possible that the nonlinear theory will show that the considered disturbance will continue to grow also at some times exceeding t_0. Moreover, it may also happen that growing nonlinearly disturbance will produce by nonlinear interactions some new small transiently growing formations maintaining the process of disturbance-energy growth (at the expense of the mean-flow energy) which finally will lead to transition to turbulence. This reason may sometimes explain the transition of a subcritical flow to turbulence. Some specific nonlinear models of such 'subscritical transitions' (dealing usually not with NS partial differential equations but with more simple finite-dimensional nonlinear systems of ordinary differential equations) were considered, in particular, in papers [26-28] (however in [28] where the onset of turbulence in subcritical plane Poiseuille flow was discussed at length, results found for model equations were confirmed also by references to the results found by DNS of a disturbance development in a channel flow, i.e., by solution of the corresponding initial-value problem for nonlinear NS equations). A similar scheme of transition to turbulence of the Poiseuille flow in a pipe, where only subcritical disturbances exist, was earlier outlined in [29] and compared with the results of simplified numerical analysis of disturbance development described by nonlinear NS equations. There were also some other numerical simulations of temporal or spatial development of flows in plane channels containing initial disturbances of various forms. These simulations showed, in particular, that the flow development may be rather different in the cases where initial disturbances had different forms; see, e.g., typical papers [30-32] and discussion of this topic in the books [21,22].

Let us now say a few words about the present state of the studies of the final stage of the flow transition to turbulence. Recent computations of transient disturbance growths followed by flow turbulization confirm the conclusion obtained earlier from the experimental data which showed that for any laminar flow there are several ways to turbulent regime which realizations depend on a number of often hardly controlled external factors. In the first half of the 20th century almost all performed theoretical studies of flow instability dealt only with linear and (rarely) weakly nonlinear development of disturbances and therefore the real mechanisms of transition to turbulence were then not considered at all. The first physical model of laminar-flow-transition was developed by Landau [33,34] in the early 1940s when he began working on the volume of his fundamental *Course of Theoretical Physics* devoted to continuum mechanics. According to Landau's model transition is produced by a series of subsequent bifurcations of flow regime, where each bifurcation increases by one the number of periodic components of the quasi-periodic fluid motion arising at the preceding bifurcation. This simple model (which was in 1948 supplemented by Hopf [35] by a mathematical example of such instability development) was then almost unanimously accepted by turbulence community as the universal mechanism of flow turbulization. However, the further development of the mathematical theory of dynamic systems showed that Landau's model of flow development not only is nonuniversal but is exceptional in some important respects and thus is rarely observed.

Basing on the available in the early 1970s new results of the dynamic-system theory, Ruelle and Takens ([36]) proposed a new model of transition to turbulence cardinally differing from Landau's model. According to these authors, transition to turbulence is realized by a succession of a few (usually three) "normal" flow bifurcations of Landau-Hopf type, followed by a sudden appearance of a very intricate attracting set (called a "strange attractor") in the phase space of a flow. The flow states corresponding to phase points within the attractor are very irregular and can be characterized as being "chaotic" or "turbulent". Ruelle and Takens' model at first caused some doubts but later it was found that this model agrees quite satisfactorily with some (but not all) experimental data relating to transitions to turbulence and can also explain seemingly paradoxical data of the old numerical experiment by E. Lorenz [37] who considered a low-dimensional numerical model of a convective fluid flow. After this discovery the Ruelle-Takens model gained high popularity and stimulated a great number of further studies of temporal and spatial developments of nonlinear dynamic systems. As a result there appeared enormous (and rather sophisticated) literature on both the general theory of dynamic systems and its applications to flow developments; in this literature the words "chaos", "strange attractor" and some other new terms play the main part.

Results of this very extensive and diverse literature relating to transition to turbulence are, nevertheless, not fully satisfactory up to now. It was, in particular, discovered that there are several different "scenarios" for transition of a dynamic system to chaotic behavior as "parameter of nonlinearity" (e.g., the Reynolds number of a fluid flow) increases. In addition to the scenario by Ruelle and Takens, the Feigenbaum scenario of a cascade of period-doubling bifurcations ([38,39], cf. also the related model described in [40]), and the so-called intermittent-transition scenario by Pomeau and Manneville [41,42] may be mentioned as examples. Note also studied in [28,29] subcritical-flow transition scenarios which don't include any cascade of successive instabilities. During the last 15 years many hundreds of papers and many dozens of books and lengthy surveys appeared were these and some other scenarios of transition of dynamic systems to chaotic regimes are discussed (the books [43-47] and the papers [48-50] discussing the applicability of the concept of chaos to turbulence are only a few examples). Let us mention in this respect also a few laboratory and numerical studies [51-54] where there were described some flow-transition phenomena having features close to those of some of the proposed transition scenarios. However, all the results obtained up to now do not form a complete physical theory of the transition of fluid flows to turbulence. Note that up to now there are no strict conditions of realization of various transition scenarios although it is known that sometimes different scenarios may take place in the same flow depending on some poorly known circumstances. And all the proposed scenarios were mostly compared with computations relating to some finite-dimensional models much simpler than the very intricate infinite-dimensional dynamic system evolving in the space of vector-functions of four variables in accordance with the NS equations (cf., e.g., paper [55] where a scenario for the onset of space-time chaos in a flow was studied on the model example of relatively simple nonlinear partial differential equation and it was shown that even in this case the transition to chaos proves to be quite complicated). Up to now even the question about the existence and properties of strange attractors in the infinite-dimensional phase spaces of real fluid flows is not answered satisfactorily enough (reach in content book [56] in fact covers only the attractor problem of two-dimensional fluid dynamics; see in this respect also the books listed in [134]). Thus, the completely new approaches to the transition-to-turbulence problem developed at the end of the 20th century generated, together with a number of interesting new results, also a great number of new unsolved problems which only confirm the popular assumption about the "insolvability of the problem of turbulence".

3. Development of the Theory of Turbulence in the 20th Century: Exemplary Achievements

Calling the 20th century 'the century of the turbulence theory' we stressed that during this century very great progress was achieved in the studies of turbulence phenomena. And there are two main trends (often overlapping each other) of the turbulence-theory development in the 20th century - elaboration of the methods allowing to determine the practical effects of turbulence, and the investigation of fundamental physical laws controlling the turbulent flows. Below only a few results relating to the second group will be discussed; these results were long assumed to be the most important achievements of the theory of turbulence but at the end of the century it became clear that there are some quite reasonable doubts concerning the classical results discussed below.

3.1. *Similarity Laws of Near-Wall Turbulent Flows*

The class of near-wall parallel (or nearly parallel) laminar flows includes such important examples as flows in broad plane channels (which may be modeled with a good accuracy by plane Poiseuille flows produced by a pressure gradient in a layer between two infinite parallel walls), flows bounded by parallel walls one of which is stationary and the other is moving with constant velocity (plane Couette flows), flows in long circular pipes (circular Poiseuille flows), and boundary-layer flows over flat plates in the absence of the longitudinal pressure gradient (Blasiius boundary-layer flows). Plane Poiseuille, plane Couette, and Blasius boundary-layer flows are bounded by flat walls which for simplicity will be assumed to be smooth. Pipe flows are bounded by a cylindrical wall (also assumed to be smooth) but if pipe radius R is much larger than the 'wall length-scale' $l_w = \nu/u_*$, where $u_* = (\tau_w/\rho)^{1/2}$ is the friction velocity, τ_w - the wall shear stress, and ρ - fluid density (only this case will be considered below), then it is usually possible to neglect, in a reasonable first approximation, the influence of wall curvature, i.e. to consider again the wall as flat one. For fully turbulent near-wall flows the mean-velocity profiles $U(z)$ (where z is the normal-to-wall coordinate) and the skin-friction laws (giving the value of the friction, or drag, coefficient) were carefully studied in the late 1920s and early 1930s by L. Prandl and T. von Kármán who combined a few simple semi-empirical hypotheses with the methods of dimensional analysis (based on definite assumptions about the list of physical parameters which are essential here). Apparently the most important discovery of the mentioned authors was the discovery of the *logarithmic mean-velocity law* for the values of z large with respect to l_w and small with respect to the vertical length scale L (equal to the half-distance between parallel walls, pipe radius, or the boundary-layer thickness). According to this law

$$U(z) = u_*[\mathrm{A}\ln(zu_*/\nu) + \mathrm{B}] \qquad \text{for } l_w << z << L, \tag{1}$$

where A and B are universal constants (and $\kappa = 1/\mathrm{A}$ is called the *von Kármán's constant*).

Logarithmic law (1) was first announced by von Kármán in 1930 at the International Congress of Applied Mechanics at Stockholm. He derived it from a seemingly natural "similarity principle" while Prandtl in 1933 gave another more simple derivation of this law (see, e.g., [7]). Still simpler purely dimensional derivation of this law was proposed in 1944 by Landau [34] This derivation was based on 'rational arguments' stating that at $z \ll L$ the 'external length scale' L cannot affect the flow structure, while at $z >> l_w$ the velocity shear (but not the velocity itself) of a developed turbulent flow cannot be affected by ν (since at such z the velocity gradients are quite small and also the 'eddy viscosity' is much greater than the molecular viscosity). Therefore, at $l_w \ll z \ll L$ the shear dU/dz can depend only on u_* (determining the vertical flux of momentum) and z. Thus $dU/dz = Au_*/z$ there and this implies Eq. (1). Similar arguments were applied by Landau [34] to the first derivation of the logarithmic law of the form

$$T(z) - T(0) = T_*[\mathrm{A_T}\ln(zu_*/\nu) + \mathrm{B_T}(\mathrm{Pr})] \tag{1a}$$

for the profile of mean temperature (or mean concentration of some passive admixture) $T(z)$ in a wall flow with a heat (or mass) transfer from the wall. Here $T_* = j_w/u_*$ is the heat-flux scale of temperature (for definiteness only the case of heat transfer will be mentioned in this paper), j_w is the temperature flux at the wall, while $\mathrm{A_T}$ is a new constant, and $\mathrm{B_T}(\mathrm{Pr})$ is a function of the Prandtl number $\mathrm{Pr} = \nu/\chi$, where χ is the coefficient of thermal diffusivity.

One more elegant derivation of the law (1) was proposed in 1937 by Izakson [57] who recalled that the rational arguments of dimensional analysis led Prandtl to the formulation of the general *wall law* of the form

$$U(z) = u_*\mathrm{f}^{(1)}(u_*z/\nu) \tag{2}$$

(where $\mathrm{f}^{(1)}$ is an universal function) for velocity $U(z)$ at $z << L$. Similar dimensional arguments imply also the validity at $z \gg l_w$ of the *velocity defect law*

$$U_0 - U(z) = u_* f^{(2)}(z/L), \quad \text{where } U_0 = U(L) \tag{3}$$

(for a pipe flow the law (3) was first empirically detected in 1911 by T. Stanton and in 1930 it was justified by dimensional arguments by von Kármán). Then Izakson noted that if an *overlap layer* of not too small and not too large values of z exists where both laws (2) and (3) are simultaneously valid, then it is easy to show that in this layer the wall law (2) must have logarithmic form (1) while the velocity defect law (3) must be also logarithmic and have the form

$$U_0 - U(z) = u_*[-A\ln(z/L) + B^{(1)}]. \tag{4}$$

Here again $A=1/\kappa$, and $B^{(1)}$ is a new constant taking different values for flows in channels, pipes, boundary layers, and for plane Couette flow.

Izakson derivation of two logarithmic laws quickly gained popularity. In particular, in 1938 C. Millikan [58] noted that Izakson's arguments may be applied to flows along both smooth and rough walls (in the latter case the coefficient B will depend on characteristics of wall roughness) and that adding together Eqs. (1) and (4) one may easily derive the famous Prandtl-Nikuradse logarithmic skin-friction law for turbulent flows in smooth-wall and rough-wall pipes and plane channels. (Millikan also remarked that the same method can be applied to turbulent boundary layers. However he did not consider boundary-layer flows and the first derivation of Kármán's skin-friction law for boundary layers by the sketched here method was apparently due to Clauser [59].) Such derivation allows to determine the dependencies of the coefficients of skin-friction laws on logarithmic-law coefficients A, B and $B^{(1)}$; obtained results were found to be in agreement with the available data of velocity and skin-friction measurements. Some further developments of Izakson's method will be indicated slightly later.

The logarithmic velocity-profile and skin-friction laws for wall turbulent flows were conventionally considered as some of the most fundamental (and most valuable for the practice) achievements of the 20th-century turbulence theory. These theoretical results were many times compared with data of direct measurements of turbulent-flow characteristics in pipes, boundary layers and plane channels. As a rule, obtained results agreed more or less satisfactorily with logarithmic laws (see, e.g., the recent survey [60]) but measured values of 'universal coefficients' A, B, and $B^{(1)}$ of these laws proved (and prove up to now) to be rather scattered. During long time the most popular estimates of $\kappa = A^{-1}$ and B were these ones: κ=0.40 (or 0.41, but the values in the range from 0.36 to 0.46 were also sometimes obtained), B = 5.2 (but all values in the range from 4.8 to 5.7, and also some values outside of this range, were met in the literature). The values of $B^{(1)}$ were measured not so often; according to majority of estimates $B^{(1)} \approx 0.6$ for circular pipes and plane channels, and $B^{(1)} \approx 2.4$ for flat-plate boundary layers (see, e.g., the surveys [61,62]). The range of z-values belonging to the so-called *logarithmic layer* of a wall flow, where Eq. (1) is valid, was also subjected to great scatter; most often it was suggested that this layer is extended from the lower limit at $z \approx 50l_w$ (coefficient 50 was sometimes replaced by 30 or by 70) up to the upper limit at $z \approx 0.15L$ (instead of 0.15 the coefficients 0.2 and 0.3 were sometimes used). And at the present time the uncertainty relating to the coefficients $\kappa = A^{-1}$ and B and limits of the 'logarithmic layer' did not become smaller; see below about this matter.

There were also many works extending and generalizing the theory of the logarithmic layer and Izakson's method of derivation of results relating to this layer. Von Mises [63] considered the cases of non-circular pipes while the applications of the same method to near-wall turbulent flows with heat (or mass) transfer were considered in [61,62,64]. Numerous applications to flows along rough walls and wall-flows with non-zero pressure gradients were discussed in surveys [61,65]. Comprehensive generalization of the 'logarithmic-layer theory' to the case of the near-wall layers of turbulent flows in stratified fluids with mean density $\rho(z)$ depending on the vertical coordinate z (first of all to atmospheric and oceanic surface layers) was developed by A.S. Monin and A.M. Obukhov; see, e.g., Chap. 4 of the book [66]. Townsend in the book [67] published in 1956 formulated the general 'Reynolds-number similarity principle' used then for the derivation of the similarity laws (2) and (3) and logarithmic law (1). Simultaneously he also sketched applications of the general similarity arguments to the second moments of velocity fluctuations and, in particular, investigated indicated below wall laws (5) for the second-order moments where k+l+m = 2. More detailed exposition of the applications of Izakson's arguments to moments of velocity-component fluctuations $(u_1,u_2,u_3) = (u,v,w)$ (and temperature fluctuations θ) was presented in the paper [61]. In this paper it was postulated that in the near-wall flow region, where $z \ll L$, the wall similarity law of the form

$$M_{klm}(z) \equiv \langle u^k v^l w^m \rangle = (u_*)^{k+l+m} f_{klm}(zu_*/\nu), \tag{5}$$

is valid, while in the outer flow region, where $z \gg l_w = \nu/u_*$, the outer similarity law of the form

$$M_{klm}(z) = (u_*)^{k+l+m} g_{klm}(z/L). \tag{6}$$

takes place. Here angular brackets denote the probabilistic (ensemble) averaging, while f_{klm} and g_{klm} are two families of universal functions of one variable. If an overlap layer, where $l_w \ll z \ll L$, exists in the considered flow, then both Eqs. (5) and (6) must be valid there and this implied that in this layer the moments of velocity fluctuations take constant values, i.e.

$$M_{klm}(z) \equiv \langle u^k v^l w^m \rangle = a_{klm}(u_*)^{k+l+m} \tag{7}$$

where a_{klm} are universal constants. Related similarity laws can be formulated for many other statistical characteristics of the velocity-component and temperature fluctuations in fully turbulent wall flows (e.g., for correlation functions, spectra and multipoint higher moments of these fluctuations).

Beginning from the 1930s logarithmic velocity and skin-friction laws were used in the engineering practice much more widely than any other scientific results relating t0 turbulence, and very long they were universally treated as indisputable certainty. Such opinion was strongly supported by the unquestionable authority of famous scientists who independently proposed different derivations of these laws and then actively popularized them; the list of such scientists includes the names of L. Prandtl, T. von Kármán, G.I. Taylor, L. Landau, and C.B. Millikan. (By the way, A.N. Kolmogorov also highly estimated logarithmic velocity laws and their derivation from the overlap-layer arguments. He even elucidated these results and their application to the determination of skin-friction laws in two short notes of 1946 and 1952 published in "Doklady of USSR Acad. Sci." and intended for engineers; see the list of his works on turbulence in [68].) However, at present the study of turbulence advanced very much in comparison to its state in the middle of the 20th century and this development produced some doubts in the universal validity of these classical results.

Prandtl's wall law (2) follows from the assumption that at $z \ll L$ the length L cannot affect the flow characteristics. This assumption seemed to be obvious not only in 1925, when the wall law was proposed by Prandtl, but also long after this year, but now it causes doubt by reasons which will be explained below. However the inclusion in the velocity-defect law (3) of the friction velocity u_* determined by flow condition at the wall did not always seem fully motivated and some scientists were long ago inclined to think that the law (3) is in fact of empirical origin. (For this reason some authors even proposed to replace the near-wall velocity scale u_* in (3) by a scale more appropriate to outer-flow conditions; one such example will be mentioned below.) Reverting now to the independence of flow characteristics near the wall (at $z \ll L$) of the length L, let us note that such independence became to be non-obvious after the discovery of the important part playing in turbulent flows by the large-scale organized vortical structures (so called "coherent structures") which affect all regions of the flow. The study of these structures and of their role in turbulence was developed rapidly after the end of the World War II (at great degree under the influence of clear presentation of this topic in Townsend's important book [67] of 1956).

Slightly later Townsend's experimental studies of turbulent boundary layers [69] showed that the intensities $\langle u^2 \rangle$ and $\langle v^2 \rangle$ of the horizontal velocity fluctuations in the 'logarithmic layer' (where $l_w \ll z \ll L$) sometimes take different values in two boundary layers with the same value of u_*. This result clearly contradicted to the wall laws (7) corresponding to mean squares $\langle u^2 \rangle$ and $\langle v^2 \rangle$. Townsend explained this disagreement with the wall laws assuming that turbulent motion in the wall regions of turbulent boundary layers consist of the "active" component [which produced the shear stress $\tau = -\rho\langle uw \rangle$ and satisfies the usual wall laws (1), (2) and (5)], and "inactive" practically irrotational component which is produced by large-scale fluctuations in the outer region of boundary layer and depends on L (i.e. on the boundary-layer thickness, since in [69] only boundary-layer characteristics were discussed). Later Bradshaw [70] (see also [71]) confirmed Townsend's hypothesis by new experimental data and showed that it explains also some other experimental results inexplicable by the traditional theory. Moreover, Bradshaw also repeated Townsend's statement that "inactive motions" contribute nothing to the mean-velocity profiles [and hence do not violate the logarithmic velocity laws] and to vertical (normal-to-wall) velocity fluctuations w. And in the second edition of 1976 of the book [66] Townsend [72] connected the inactive motions with the contributions to the fluid motions made by a definite family of similar to each other vortical structures differing by their length scales. Basing on this idea he derived new equations for quantities $\langle u^2 \rangle$ and $\langle v^2 \rangle$ within the logarithmic layer; these equations included in addition to constant right-hand sides a_{200} and a_{020} of Eqs. (7), also terms proportional to $\ln(z/L)$ (together with small terms which depended on z/l_w and became negligible at very high Re). According to data by Perrry and Li [73], Townsend's equations agree more or less satisfactorily with the results of measurements of mean squares of horizontal velocity fluctuations. (Note, however, that all experimental data relating to higher moments of velocity fluctuations are much more scattered and controversial than results of mean-velocity-profile measurements; cf., e.g., [60].) Arguments similar to those of Townsend [72] were later applied by the present author [74] to theoretical evaluation of intensities of the horizontal wind fluctuations in the unstably stratified atmospheric surface layer. This approach allowed to explain seemingly paradoxical dependence of the intensity of wind fluctuations at few-meter heights above the Earth's surface on the thickness of the planetary boundary layer having the order of 1-2 km.

Townsend's results show that the influence of large-scale coherent structures made incorrect at least some of the classical similarity laws postulating the negligible effect of the external length scale L on the flow characteristics within the flow region where $z \ll L$. Of course, in [72-74] only some particular violations of traditional wall laws were noted. Since, however, at present it is known that large-scale structures of many different types and length scales exist in developed turbulent wall flows, it may be expected that all the similarity laws which neglect the possible influence of the length L are of limited accuracy. And the other fundamental assumption used in the formulations of classical similarity laws of near-wall turbulent flows, according to which the molecular-viscosity effects must be negligibly small at $z \gg l_w = \nu/u_*$, also becomes questionable in the light of recent experimental findings.

Experiments (and numerical simulations) of 1990s definitely show that the developed turbulent flows at large values of Re always include a tangle of intense and very thin vortex filaments which diameters sometimes are of the order of the Kolmogorov length scale η. (This length scale characterizes the spatial extent of viscous influences; for its definition see Eq. (11) below, while more detailed discussion of the role of the filaments may be found, e.g., in [2], Sec. 8.9, and [114], Sec. 5.) In other words, according to modern views the range of scales of organized vortical structures existing in fully-developed turbulent flows extends from the external length scale L up to Kolmogorov's internal length scale η. Since the topology and general structure of the tangle of filaments must depend on Re and the filaments are found in all regions of turbulent flows, the characteristics of the high-Reynolds-number turbulence also may everywhere depend on ν and Re=UL/ν. (Moreover, Barenblatt [83] noted in the paper of 1999 that L. Prandtl in his remark made at the Intern. Congr. of Appl. Mech. of 1930 after the talk by von Kármán where the logarithmic form of the velocity profile first appeared, indicated that at moderate values of Re the influence of near-wall streaks on the flow at greater heights may generate mechanism of possible influence of viscosity ν on turbulence characteristics at $z \gg l_{aw}$. However, later Prandtl apparently never mentioned this effect.)

The arguments presented above imply that the classical logarithmic mean-velocity (and mean-temperature) laws of wall turbulent flows possibly represent only some reasonable approximation which accuracy must be thoroughly checked. Barenblatt, Chorin and Prostokishin, who are apparently the most energetic modern opponents of logarithmic laws, reasonably noted (in [75] and a number of other publications) that the description of the mean velocity $U(z)$ of wall turbulent flows by power laws $U(z) \propto z^k$ was widely used long enough by scientists and engineers and, if the power k was properly chosen for all values of Re of interest, usually led to satisfactory agreement with the data over a wide range of z-values (in this respect usually Schlichting's book [76] is referred). Barenblatt et al. indicated also a great number of more recent publications containing the data illustrating the dependence on Re of the mean flow characteristics of turbulent near-wall flows. (A number of appropriate references may be found in the survey paper [77]; in a short subsequent remark [6] Gad-el-Hak also noted quite reasonably that since any doubt concerning logarithmic laws where long considered as a heresy, most of the papers containing such heresies were apparently rejected by editors of scientific journals.)

In [75] and the other related papers Barenblatt et al. suggested that logarithmic law should be replaced by laws of quite different form. This proposition was directly connected with some general ides introduced in 1972 by Barenblatt and Zeldovich [78]. It was noted in this paper that self-similar solutions of the form $V(x,t) = \mathrm{A}(t)\mathrm{F}[x/l(t)]$ (where x and t are some independent variables) are very often encountered in fluid dynamics and other branches of physics as 'intermediate asymptotics' describing the behavior of the dependent variable V in regions where direct influence on it of peculiar features of the initial or/and boundary conditions is already lost but the system is still far from being in a state of equilibrium. It was then remarked that only a small part of such self-similar solutions may be determined by simple arguments of dimensional analysis. For this part of self-similar solutions the term 'self-similar solutions of the first type' was proposed in [78], while all the other self-similar solutions were called 'self-similar solutions of the second type'. (Later also the terms 'complete similarity' and 'incomplete similarity' were sometimes used by Barenblatt for these two types of self-similar solutions.) In [78] and the subsequent publications of the same authors (in particular, in Barenblatt's book [79]) a great number of self-similar solutions of both types was indicated. The general form of a solution of the first type may be uniquely determined with the help of dimensional arguments; hence it can be easily found and usually includes some factors raised to definite integer (or simple fractional) powers. For a solution of the second type the situation is much more complicated; here only some supplementary physical arguments and experimental data may suggest the general form of the sought for solution which usually includes some factors raised to powers which may take arbitrary values. The corresponding exponents may be determined in some cases from solutions of some supplementary eigenvalue problems of physical origin (see examples in [79]) but very often they must be determined from results of data processing. And the conditions guaranteeing the existence of a self-similar solution of the second type and allowing to determine its form most often are unknown; here the physical intuition and the good luck of the explorer may play the decisive part.

The problem concerning self-similar solutions of the second type in turbulence theory is especially complicated. Recall that the evolution of a fluid flow is governed by system of Navier-Stokes equations. These partial differential equations are very complicated, they cannot be easily analyzed and are insufficiently investigated up to now while their solutions corresponding to turbulent flow regimes are enormously intricate and completely nonexplored. Therefore, it seems that the dynamic equations could not help here in search of needed self-similar solutions. On the other hand, the abundance of self-similar solutions of the second type reliably established in other branches of continuum mechanics gives some reasons to expect that such solutions may play definite part in the turbulence theory too. To verify this expectation, it was only possible to perform the careful examination of the available experimental data of high enough quality.

Such examination of the pipe-flow turbulent data by Nikuradse

[80] (which were indisputably the best ones available in the 1930s and are sometimes referred now too) was carried out by Barenblatt in the early 1990s (see, e.g., [81]) and then presented at greater length in a number of papers (in particular, in the joint paper with Chorin and Prostokishin [75]). According to these data the velocity profile $U(z)$ of a turbulent flow in a pipe satisfied the simple equation of the form

$$U(z)/u_* = C(u_* z/\nu)^\alpha \tag{8}$$

over almost the whole pipe cross-section (except the thin 'viscous sublayer' where $u_* z/\nu$ does not exceed some threshold value of the order of a few tens). In Eq. (8) parameters C and α do not depend on z but vary (rather slowly) with the flow Reynolds number Re = $U_m D/\nu$ (where U_m is the flow velocity averaged over the pipe cross-section and D =2R is the pipe diameter). Careful examination of the Nikuradse data led Barenblatt to proposition of the following expressions for the functions C(Re) and α(Re)

$$C(\mathrm{Re}) = \frac{\ln \mathrm{Re}}{\sqrt{3}} + \frac{5}{2}, \quad \alpha(\mathrm{Re}) = \frac{3}{2 \ln \mathrm{Re}}. \tag{9}$$

Eqs. (8) and (9) were first obtained by treatment of the old data by Nikuradse. However in [75] these equations were compared with a number of more recent pipe-flow and boundary-layer turbulence data and according to results of this paper (which were not unanimously supported) all the considered data agree well with Eqs. (8) and (9). Later results of more detailed comparison by Barenblatt et al. of Eqs. (8) and (9) with velocity profiles $U(z)$ measured in various fully turbulent zero-pressure-gradient boundary layers on flat plates were presented in [82]. (In the case of boundary layers the pipe-flow Eqs. (8)-(9) were used without any modification but now the value of Re was determined as that leading to the best fit of Eqs. (8)-(9) with the available velocity data. This means that here a new 'boundary-layer thickness' Λ was introduced by the condition that the substitution of Re = $\Lambda U_0/\nu$, where U_0 is the free-stream velocity, into Eq. (9) leads to good agreement of Eq. (8) with the measured mean-velocity profile $U(z)$.] In [82] it was found that the velocity profiles of turbulent boundary layers agree well with the power law (8)-(9) in the range of z-values extending from the upper edge of the viscous sublayer (located at $u_* z/\nu$ = 70) to the upper edge of the whole boundary layer above which the homogeneous 'free stream' begins. However, if the 'free stream' is nonturbulent, then in the 'upper sublayer' adjoining the 'free stream' another power law is valid which differs from the law (8)-(9) valid in the 'intermediate layer' located between the viscous and upper sublayers. According to data analyzed in [82], the upper-sublayer velocity profile have the form:

$$U(z)/u_* = \mathrm{B}(u_* z/\nu)^{\beta} \tag{10}$$

where β is an universal constant which is close to 1/5, while B takes different values in different experiments.

Let us now consider at greater length the power law of Eqs. (8) and (9) proposed for the intermediate layers (where z takes not too high and not to low values) of flows in round tubes, plane channels and flat-plate boundary layers. The important questions about the declared universality of these equations and the ranges of z and Re values where they are applicable were widely discussed and up to now produce hot-spirited controversies.

Barenblatt et al. repeatedly stated that they regard the power law (8) as having the theoretical foundation of the same rigor as the foundation of logarithmic law (1). I think that this statement is both correct and incorrect (even if the possibility to measure 'the degree of rigor' will be accepted). It is true that both laws have no fully rigorous proofs. The results given by dimensional analysis which provided the humanity with so many physical laws of the first-rate importance, are always not completely rigorous for a captious mathematician, since they are based on unproved assumptions about the list of physical parameters really affecting the studied process. It is also true that very often the development of science leads to discovery of new factors which were fully neglected in the past and violate the correctness of laws which earlier seemed to be established forever. Nevertheless, the hypotheses used in the dimensional analysis are of physical character and as a rule are based on clear physical intuition without which a physicist cannot be a good scientist. Just physical base of dimensional arguments implying the logarithmic law (1) made this law long undisputed for listed above great scientists.

Of course, physical intuition may sometimes deceive great scientists too and may be questioned by new discoveries. In particular, the discovery in the second half of the 20th century of great part playing in turbulence phenomena by organized vortical structures of various kinds and sizes changed noticeably the situation. It is clear that such structures may depend on some dimensional physical parameters neglected in the traditional derivation of the logarithmic law and this circumstance can restrict the validity of the laws (1), (1a) and (4) or even make them incorrect. Unfortunately, at present there is not too much information about the organized structures which may affect the mean velocity profiles of steady near-wall turbulent flows. (Recall that Townsend [69] and Bradshaw [70] stressed that studied by them 'inactive motions' do not affect the mean velocity; the same statement was also repeated in [71-73] in regard to 'attached eddies' of various sizes.) Nevertheless, since not all coherent structures are known well enough and in principle some of them may affect $U(z)$, it is impossible to exclude the possibility that 'classical similarity laws' represent only a reasonable first approximations valid when possible influences on the mean velocity $U(z)$ of the length L at $z \ll L$ and of the viscosity ν at $z \gg l_w$ are fully neglected. Therefore, found in experiments precise validity of the logarithmic law (1) with universal values of coefficients A and B may be considered as a proof of the negligibility of these influences, while discovered violations of this law or nonuniversality of its coefficients show that there exist some nonnegligible such influences. However, the power law (8) has only much more general grounds: it is supported by the very wide prevalence of 'power laws' and 'incomplite similarities' not only in physics but also in many other scientific fields. Many examples of such 'incomplete similarities' which include power laws with 'anomalous exponents' (which can take arbitrary values), are impressively demonstrated in [78,79]. It was also correctly stressed there that in many cases where incomplete similarities were reliably detected, they could not be derived rigorously from some mathematical equations since such equations were lacking. Nevertheless, this circumstance does not mean that 'incomplite similarities' represent an universal form of the laws of nature which take place everywhere and everywhen. Moreover, while the forms of 'self-similar solutions of the first type' are usually determined by the dimensionality arguments with rather high degree of definiteness, 'similarity solutions of the second type' as a rule may have many different forms. Therefore, if even one is sure that such solution exists, this did not determine automatically its precise form which choice requires the use of some supplementary assumptions. At the same time, in many important cases the existence of a 'self-similar solution of the second type' is not enough for determination of definite verifiable physical conditions and limits of its validity.

Reverting to the logarithmic velocity-profile law, one must say that at present it seems quite possible that the influence of organized structures of various types, which was neglected in conventional derivations of this law, will require to replace it by some more general incomplete-similarity law. On the other hand, such a possibility don't prove that the laws (1) - (4) in all cases must be considered as being incorrect and inappropriate for any practical use. Of course, the law (8) which contains two unknown functions which may be arbitrarily chosen, allows to get rather good agreement with the experimental data. Therefore this law is a good candidate for a new version of the velocity-profile equation which will describe the observed profiles $U(z)$ more accurately than the logarithmic law where only two constants may be varied. In [83] and some other papers by Barenblatt and Chorin the overlap-layer approach to derivation of Eq. (8) was considered; however here it proved to be necessary to add the argument Re to the arguments of functions $f^{(1)}$ and $f^{(2)}$ on the right-hand sides of Eqs. (2) and (3). This addition implies that now in the overlap layer Eq. (1) will be valid where however A and B will be not constants but arbitrary functions of Re. The presence of two arbitrary functions gives too many possibilities to fit these equations to the available experimental data. In [83] it was shown that Eqs. (8)-(9) may be make consistent with the logarithmic law with coefficients A and B dependent on Re, if some consequences of the vanishing-velocity approach of Chorin [83,114] will be additionally taken into account and some very small (and asymptotically negligible) terms will be omitted. Therefore, the derivation of Eqs. (8)-(9) with the help of the overlap-layer method is possible but such derivation is somewhat artificial and therefore less convincing than very elementary Izakson's derivation of the law (1) which however is based on the use of much more special assumptions. Note in this respect, that George and Castillo [91] (this paper will be considered below) also tried to apply the overlap-layer method to a situation where both functions $f^{(1)}$ and $f^{(2)}$ depend additionally on Re, but using some other supplementary assumptions they got quite different form of the overlap-layer velocity profile.

Of course, the derivation of Eqs. (1) and (4) uses some empirical facts too; moreover, the value of coefficients A and B also must be determined here from experimental data. Therefore, it may be said that the logarithmic laws are to some degree of empirical origin. However, it is clear that Eqs. (8)-(9) are empirical to greater degree than the logarithmic law (1). (The possibility to measure 'the degree of empiricism' is somewhat vague but the general sense of this expression is rather clear.) In spite of the connection of Eqs. (8)-(9)

with Chorin's vanishing-velocity method of physical origin, the empirical part of the arguments leading to these equations remains to be quite considerable. Of course, the empirical laws are often of a great importance and there is always a hope that a purely physical base for such a law will be determined later. In the case of Eq. (8) subsequent physical arguments maybe will help to determine the strict conditions of its validity. Moreover, if some necessary, or sufficient, conditions of validity will be found for the law (8), they probably will also help to estimate quantitatively its accuracy.

The accuracy estimate is important for Eqs. (8) and (9) since the degree of their agreement with the available experimental data is up to now a point of controversy. Experimental studies of near-wall turbulent flows continue to be popular and recently several such investigations claiming to be quite accurate were carried out but this did not clarify the situation. Here we will only mention often cited recent papers by Zagarola and Smits [84] and Österlund et al. [85] which both stated that their data confirm the validity of the logarithmic law (1) and both gave rise to a controversy.

Zagarola and Smits' measurements were made in the "superpipe" at Princeton University where strongly compressed air was used as a working fluid. The compression decreases the kinematic viscosity ν of air and thus make possible the study of pipe flows in a wide range of very high Reynolds numbers (data used in [84] covered the range $31\times10^3 \leq \mathrm{Re} \leq 35\times10^6$ where Re is based on the average flow velocity U_{av} and pipe diameter 2R). The authors found that at $\mathrm{Re} > 4\times10^5$ logarithmic law (1) (with coefficients $\kappa = 1/A = 0.436$, $B=6.15$) was valid for values of z in the range $600 l_w < z < 0.07R$. Note that found by Zagarola and Smits limits of the logarithmic layer and the values of 'universal coefficients' A and B differ considerably from 'traditional estimates' of previous investigators (who usually observed log-law at smaller values of Re). And for the range $60 l_w < z < 500 l_w$ (or $60 l_w < z < 0.15R$ if Re is not great enough), which was earlier always considered as a part of (or even the whole) logarithmic layer, it was found that there at all values of Re the velocity profile $U(z)$ has the power form $U(z)/u_* = 8.7(z/l_w)^{0.137}$ - this result clearly disagrees with all previous pipe-flow data. As to the velocity defect law (3), the authors recommended to replace in it the near-wall velocity scale u_* by the outer velocity scale $U_0 = U_{max} - U_{av}$ where U_{max} is the mean velocity at the pipe axis. According to [84], this replacement makes the function $f^{(2)}$ really independent on Re while at large values of Re it changes nothing since then the ratio U_0/u_* takes constant value.

Österlund et al. [85] summarized results of independent experimental studies of flat-plate boundary layers in two wind tunnels: one at the Royal Institute of Technology in Stockholm and the other at the Illinois Institute of Technology in Chicago. These studies covered the range 2500< Re< 27000 of Reynolds numbers Re $= U_0\delta^{**}/\nu$ (where U_0 is the free-stream velocity and δ^{**} is the momentum thickness of boundary layer). According to [85], results of both experiments excellently agree with each other and show that in the studied range of Re-values there exists an 'overlap layer' where logarithmic laws (1) and (4) are both valid with independent on Re constant coefficients: $\kappa= A^{-1} =0.38$, B=4.1, $B^{(1)} = 3.6$ (as the external length-scale L now the thickness $\delta = \delta_{95}$ of the layer where $U(z) \leq 0.95U_0$ was used). In the experiments by Österlund et al. the overlap layer corresponded to the conditions: $200l_w < z < 0.15\delta$. Note that values of coefficients of logarithmic laws and of the overlap-layer limits coincide here neither with values found by Zagarola and Smits nor with conventional values of previous authors.

Barenblatt et al. in [75] and some other papers asserted that Princeton data for $Re > 10^6$ contain a systematic error due to neglect of the wall-roughness influence which becomes important at high Re-values, while all the other data of Princeton group agree very well with Eqs. (8)-(9). However Smits and Zagarola rejected in [86] the accusation that the wall roughness affected substantially their data relating to high values of Re and in [87] they disagreed with the assertion that the low-Re Princeton data confirm the validity of Eq. (8). (According to [87] their data agree with logarithmic law (1) better than with power law (8) even in the case where optimal values of functions C(Re) and α(Re) were determined anew by processing of the Princeton, and not Nikuradse's, data.) Answering to [87], Barenblatt and Chorin published comments [88] repudiating the arguments in this paper, and just then Smits and Zagaropa declared in [86] their disagreement with statements presented in [88]. As to the paper [85], Barenblatt et al. [82] presented some diagrams obtained by processing of the original data used in [85] and showing that these data agree very well with Eqs. (8)-(9). Later, in the note [89] they tried to show that data processing used in [85] had serious defects while correct processing leads to results supporting conclusions formulated in [82] and [75]. However, the note [89] again did not close the polemic: it caused the comments [90] rejecting the made accusations and presenting a diagram showing that the data used in [85] agree with the logarithmic law (1) not worse (maybe even slightly better) than with the power law (8).

The prolonged controversy on the true form of the turbulent-wall-flow velocity profiles was continued at the 53d Annual Meeting of the APS Division of Fluid Dynamic in Washington, D.C. (November 19-21, 2000). The Invited Lecture by A.J. Chorin there was devoted again to his and Barenblatt's theory of the mean-velocity profiles in turbulent boundary layers. The critical estimation of this theory was reflected in three short talks by Buschmann and Gad-el-Hak, Panton, and Nagib et al. [91]. Buschmann and Gad-el-Hak analyzed the experimental and DNS data of mean-velocity measurements or calculations in fully turbulent zero-pressure-gradient boundary layers (with $300 \leq Re \leq 6200$, where again $Re = U_0\delta^{**}/\nu$) obtained by six independent research groups. These data were compared with the results following from both traditional logarithmic laws and recently proposed power laws. The authors found that the log law and power law both agree well with the data within considerable but somewhat different ranges of z values. The log law becomes to be applicable at lower distances from the wall while the power law continues to have a good accuracy in some part of the boundary layer placed above the 'logarithmic layer' where the log law is valid. However, there is a quite considerable flow zone where both laws agree well with all the available data and have there practically the same accuracy.

In Panton's talk [91] (at greater length its contents is described in the informal document [92]) was devoted to studies of the velocity profile of a turbulent pipe flow. Here the traditional overlap-layer arguments were supplemented by corrections taking into account the influence of finite (but high) values of Re. To compute such corrections Panton used the method of matched asymptotic expansion which has many applications to fluid mechanics (see, e.g., [93,94] and short discussions of its applications to high-Reynolds-number turbulent flows in the books [95,96] and surveys [61,62]). Panton considered only the first approximation of this method which he presented in a special form (corresponding to the uniformly valid so-called Poincaré expansion), while the initial profile equation included in his analysis both the log low in the overlap layer and the wake law in a zone adjacent to this layer. Then he showed that the considered by him approximation leads to results describing with a good accuracy numerous experimental and DNA data [including, in particular, the data of papers [84,85]) on the mean velocity and Reynolds-stress profiles $U(z)$ and $\tau(z) = -\langle uw \rangle(z)$]. Obtained composite velocity profiles $U(z)$ in a wide range of Re values agreed rather well with the available data and also with the logarithmic law within the traditional 'logarithmic layer' of z values (where the use of the conventional coefficients $\kappa = 0.41$ and $B = 5.25$ did not lead in most of the cases to disagreement with the data). Moreover, in the case of pipe flows this profile $U(z)$ agrees also well enough with Barenblatt's Eqs. (8), (9) but in another range of z values which includes the outer part of the 'logarithmic layer' and the inner part of the 'wake layer'. Since Panton found that these two laws are valid in different regions, he concluded that it is not appropriate to ask which of these two laws is correct. As to the boundary-layer flows, Panton came to conclusion that used by Barenblatt et al. method for determination of the most appropriate value of $Re = U_0\Lambda/\nu$ don't lead to values of C(Re) and α(Re) which make Eq. (8) to agree well with Österlund's experimental data (at greater length this conclusion is considered in the second document [92]).

Finally, in the talk by Nagib et al. [91] it was stated that the experiments described in [85] were continued by the present authors in the range of very high values of $Re = U_0\delta^{**}/\nu$ exceeding 50 000. The new measurements showed that the mean velocity distribution in the overlap layer of the flat-plate boundary layer for these Reynolds numbers continues to be accurately described by the Reynolds-number-independent log law with the same as in [85] unconventional values of the coefficients $\kappa = 0.38$ and $B = 4.1$.

What may be said in conclusion of this lengthy many-sided discussion? It shows clearly that advocates of two different similarity models cannot convince each other in the correctness of their point of view. Both side refer to (often the same) experimental data trying to prove to opponents that these data confirm their model. This makes an impression that at present the reached accuracy of the available data on near-wall turbulent profiles is simply insufficient for the obtaining of a convincing unique conclusion about the real form of the mean-velocity profile in the intermediate layer of not-too-small and not-too-large values of z. However it seems also that great (and continued to grow) scatter of experimental values for the coefficients A, B and $B^{(1)}$ and for the limits of the logarithmic layer (cf., e.g., the strongly differing results of [84] and [85] which both asserted that their data are precise), contradicts to the idea of an universal overlap layer with logarithmic velocity profile having always the same constant coefficients. Barenblatt et al. [89] remarked in this respect that found in [85] too low value $\kappa = 0.38$ of von Kármán constant contradicts the logarithmic-law universality. Österlund et al. [90] in their answer noted that used by them inner (i.e. lower) limit of the logarithmic layer corresponded best to their data but was much greater than its 'traditional' value; moreover, their data also covered a wider range of high Re values than that used in earlier studies. According to [90], using only the part of their data which corresponded to 'traditional' low range of Re values and 'traditional' overlap-layer limits, the authors got the usual estimate $\kappa = 0.41$. Does this mean that just the further increase of the used values of Re and of the lower limit of the overlap layer implies still greater value $\kappa \approx 0.44$ found in [84]? In fact, the dependence of the value of κ [and of other coefficients of laws (1) and (4)] on the range of Re-values and limits of the considered 'overlap layer' means that either these laws are not universal or the corresponding experimental data are inexact. If the first explanation is true, then the velocity shear dU/dz in the 'intermediate layer' of a wall flow depends not only on u_* and z but also on some other physical quantities which must be directly indicated. Note also that the conventional 'overlap-layer arguments' don't imply conclusions agreeing satisfactorily with the available data when these arguments are applied not to mean-velocity profiles but to more complicated statistical characteristics of wall turbulence (for more details see text printed in small type below). This remark also decreases the confidence in the universal validity of the uniquely determined logarithmic law for the overlap-layer velocity.

Let us now mention one more group of researchers who independently studied the mean-velocity profiles $U(z)$ in near-wall turbulent flows. This group, headed by W.K. George, also modified the traditional 'overlap-layer similarity assumptions' and used a more complicated method for analysis of the nonclosed Reynolds equation for the mean velocity $U(z)$ of a turbulent wall flow. Obtained by them results relating to zero-pressure-gradient boundary layers and to pipe (or channel) flows were summarized in papers [97] and [98], respectively. According to the indicated here new theory, Reynolds number strongly affects all flow regions; therefore the argument Re must be again included in the list of arguments of functions $f^{(1)}$ and $f^{(2)}$ on the right-hand sides of the wall and defect laws (2) and (3). This makes impossible the direct determination of the form of functions $f^{(1)}$ and $f^{(2)}$ in the 'overlap layer' and requires to use here some supplementary hypotheses. Proposed in [97,98] hypotheses implied that the velocity profile $U(z)$ takes in the intermediate 'overlap layer' quite different forms in the cases of boundary-layer flows and flows in pipes and channels: in the first case $U(z)$ satisfies the power-law with respect to the variable $z + a$, and in the second case - the logarithmic law again with respect to $z + a$. (Here a is an auxiliary parameter describing the vertical shift of the coordinate origin and taking different values in different wall flows.) We have no possibility to consider here these rather unexpected results at greater length; note only that physical intuition (which may be incorrect) makes one to be surprised by cardinal difference between the near-wall flow structures in boundary-layer and pipe (or channel) high-Reynolds-number flows. It was also stated in [97,98] that found there results agree satisfactorily with the available experimental data. (This statement was confirmed also by Had-el-Hak [6] who found that results of [97] 'are elegant'.) The found agreement of quite different velocity-profile equations with the same data shows once more that at present the accuracy of the existing data does not permit to determine reliably the true forms of wall-flow velocity profiles.

Completing the discussion of the present situation concerning the choice of the most appropriate theoretical equation for the velocity profiles $U(z)$ of steady turbulent wall-bounded flows one must say that at present there is no equation which will satisfy everybody and will be unanimously recognized as the best one. From this point of view, the situation now is even worse than it was up to the 1980s when the discovery of the logarithmic velocity-profile equation was unanimously considered as one of the most fundamental scientific achievements of the 20th century which solved forever the problem about the form of velocity profile in turbulent wall flows. Now it seems clear that the accuracy of the available experimental and numerical data is insufficient for the determination of the unique 'correct solution' of the problem. At the same time, the great scatter of the found values of logarithmic-law parameters and limits of its validity makes one to suppose that this law represents only a reasonable first approximation which may be useful for engineering practice but cannot be considered as a rigorously established physical law. Therefore, the old velocity-profile problem which tortured L. Prandtl, G.I. Taylor and T. von Kármán in the first quarter of the 20th century, now again became actual and apparently requires supplementary studies of physical mechanisms leading to possible violations of the logarithmic law and to reliably detected violations of related similarity laws for higher-order statistical characteristics of wall-bounded turbulent flows.

Before the appearance of much more accurate experimental (and/or DNS) data (and even after it too), better understanding of the main features of the velocity profiles in various turbulent flows undoubtedly requires (and will require) more direct use of the physical arguments concerning the mechanisms of turbulent mixing.

This is an arduous task: physics of turbulence phenomena is very complicated and even mysterious up to now, dynamic equations are nonclosed and requiring additional hypotheses. Therefore it is not surprising that all approaches discussed above did not use the Navier-Stokes equations of fluid dynamics at all. For this reason the attempt by Nazarenko with coworkers [99] to consider some simplified physical mechanisms producing the near-wall turbulence with logarithmic (or power-law) mean-velocity profiles is worth to be mentioned. These authors studied near-wall turbulence produced by a weak small-scale external forcing. They found that the mean velocity profile of such forced turbulence is very sensitive to the properties of the initial near-wall vorticity penetrating into the outer flow regions. For the case of a simplified dynamic model derived from NS equations the authors found specific conditions guaranteeing the existence of an exact analytic solution of model equations corresponding to the logarithmic (or to power-law) velocity profile. Thus here for the first time it was shown that sometimes these two types of velocity profiles may be obtained under definite conditions from dynamic equations derived from the NS equations. Results of this work in fact stressed again that classical derivations of logarithmic law by Prandtl, Kármán, Izakson, and Millikan in no way can be considered as the conclusive solution of the problem of the velocity profile of near-wall turbulent flows. Such derivations must be also supported by careful physical analysis based on dynamic equations which maybe will explain the interrelation between the power-law and logarithmic velocity profiles.

Above only the mean-velocity profiles $U(z)$ of the near-wall turbulent flows were considered. However any turbulent flow in addition to mean-velocity profile has also a lot of 'statistical characteristics of higher orders' such as higher moments, correlation and structure functions, spectra of fluid-dynamic fields, probability density functions (pdf) of turbulent fluctuations and so on. All these characteristics are peculiar just to given flow and knowledge of many of them may be necessary for solution of some important practical problems. However up to now the higher-order statistical characteristics of wall turbulent flows are poorly known since relating to them experimental data are either missing or are very scattered and unreliable. Moreover, the applications of the 'standard dimensional arguments' of wall-turbulence theory to the higher-order flow characteristics usually lead to results which agree with the available data much worse than results relating to mean-velocity profiles. Recall that the first violations of the 'classical similarity laws' for the 'overlap layer' of near-wall turbulence which were detected by Townsend [69] and Bradshaw [70] (and confirmed by Perry and Li [73]) concerned not the mean-velocity profile but profiles of the second-order moments $\langle u^2\rangle$ and $\langle v^2\rangle$. Since the mentioned here similarity laws were based on the same seemingly obvious dimensional arguments which imply the logarithmic velocity-profile law, the discovery of their violations is very important for future studies of real properties of near-wall turbulence.

It has been already mentioned above that Fernholz and Finley noted in the review [60] that the available mean-velocity data for zero-pressure-gradient boundary layers agree quite satisfactorily with the logarithmic laws (1), (4) [and more general laws (2) and (3)] but the data relating to higher moments of velocity fluctuations are very scattered and disorderly. Note that, nevertheless, in early reviews [61,62] an attempt was made to collect some preliminary (not too reliable) data relating to functions $f_{klm}(zu_*/\nu)$, $g_{klm}(z/L)$ and constants a_{klm} for the cases where $k+l+m = 2$. In particular, it was stated there that apparently $a_{200} \approx 5.5$, $a_{020} \approx 3$, $a_{200} \approx 1$, while $a_{101} = -1$, $a_{110} = a_{011} = 0$. However, later it was stressed in [100] that in fact much data disagree with these estimates [and with general equations (5)-(7) too]. As an example the atmospheric data by Högström [101] and Smedman [102] were presented in [100] which show that in the near-earth logarithmic layer of the atmosphere $\langle u^2\rangle^{1/2}/u_*$ often decreases and $\langle w^2\rangle^{1/2}/u_*$ increases with height in direct contradiction to Eq. (7). Many more recent data relating to various higher-order statistical characteristics of near-wall laboratory or atmospheric turbulence may be found, e.g., in the papers [103]-[107]. These data show that similarity laws (5)-(7) (and similarity laws of the same type corresponding to other characteristics of near-wall turbulence) often disagree with the experimental data or, in the best case, may be considered only as some rough approximations. (In particular, the dependence of statistical characteristics of turbulence on the value of Re was often observed in both the inner, near-wall, and the outer flow regions.) Therefore the search for similarity laws adequately describing higher-order statistical characteristics of wall turbulent flows represents a very difficult problem requiring much further work.

3.2. *Kolmogorov's Theory of Locally Isotropic Turbulence*

Kolmogorov's theory of 1941 (so-called K41 theory, or briefly K41) was first stated in two short notes (of 4 and 3 pages) in "Doklady Akad. Nauk SSSR" ('Reports of USSR Acad. Sci.'). These notes undoubtedly represented one of the highest achievements of the theory of turbulence which, luckily, became very early known in the West. (Up to 1946 Russian "Doklady" were simultaneously published under the title "C. R. Acad. Sci. URSS" in translations to one of three main Western languages. One day in the early 1940s young Cambridge student G.K. Batchelor by chance found these "C. R." in the London library, read Kolmogorov's notes, at once understood their enormous importance and became an urgent popularizer of this work.) So, seven printed pages glorified A.N. Kolmogorov as the brilliant physicists and mechanicians, while earlier he was known only as a famous mathematician. (In fact K41 was the unique achievement in the field of turbulence which was seriously discussed as a work worth the Nobel prize in physics, and probably Kolmogorov would get the Nobel prize if he did not die too early.)

Kolmogorov's theory was based on very clear and convincing physical ideas represented in the form of two hypotheses concerning the mechanisms producing the small-scale turbulent fluctuations. When this theory was developed by Kolmogorov, there were no experimental data to compare with conclusions following from his theory; all of them have the character of pure predictions. Only later numerous experiments confirmed the perfect validity (with the attainable then accuracy) of the main results of Kolmogorov's theory (see, e.g., the books [2,66,95]). Let us stress, however, that the K41 theory did not use at all the dynamic NS equations. In fact, here only intuitive physical reasons were used where the principal part was again played by dimensional arguments. Physical intuition prompted Kolmogorov the idea that the small-scale turbulence fluctuations are produced by a cascade process of energy transfer from the mean flow and the large flow structures to more and more smaller such structures. If so, then it was natural to assume that in the case of very high Reynolds numbers, where cascade process includes many steps, this process must make the small-scale turbulence (corresponding to distances r much smaller than the typical length L of the large-scale flow nonhomogeneities) locally homogeneous, isotropic and depending, in the case of incompressible fluid, only on two dimensional physical parameters. These two parameters are the mean rate ε of the energy transfer over the cascade of eddies (which must be equal to the mean rate of viscous dissipation of the kinetic energy of velocity fluctuations) and the kinematic viscosity of fluid ν. And dependence of only two parameters allows to use dimensional analysis very effectively. In particular, dimensional considerations imply the following result

$$E_{11}(k) = A\varepsilon^{2/3}k^{-5/3}\phi(k\eta), \quad \text{where } \eta = (\nu^3/\varepsilon)^{1/4},\ \phi(0) = 1, \tag{11}$$

$E_{11}(k)$ is the one-dimensional spatial *spectrum* of the streamwise velocity fluctuations, k - the streamwise wave number, η - Kolmogorov's length scale (which has been already met above when the range of length scales of vortical structures was discussed), and A and ϕ are some universal constant and function. Eq. (11) is valid in flows with large values of Re for $k >> 1/L$ (since only such values of k correspond to small-scale turbulence) and it follows from this equation that in the *inertial range* $1/L<<k<<1/\eta$ of wave numbers k spectrum $E_{11}(k)$ has the following simple form:

$$E_{11}(k) = A\varepsilon^{2/3}k^{-5/3} = Bk^{-5/3}, \text{ where } \quad B = A\varepsilon^{-5/3}. \tag{11a}$$

Eq.(11a) represents the famous *five-thirds law* determining the form of the velocity spectrum in the inertial range of wave numbers; this law is one of the most important conclusions following from K41 theory.

First attempts of experimental checking of K41 theory led to confirmation of theoretical predictions; in particular, it was found that velocity spectra of atmospheric turbulence (where Re always takes very high value) are almost always proportional to $k^{-5/3}$ in a wide range of wave numbers. However in the late 1950s the researchers working at Moscow Institute of Atmospheric Physics noted that nevertheless some results of their measurements disagree with original Kolmogorov predictions. The first found disagreement concerned the coefficient B of Eq. (11a). According to K41, at a fixed point of a steady turbulent flow coefficient B must have a constant value. However, real measurements at fixed points of the Earth's atmosphere showed that B fluctuates very strongly - a new spectral measurement made slightly later (say, after 15-20 minutes) gave again a spectrum of the form (11a) but coefficient B often took then quite different value.

This observation led to formulation by Obukhov and Kolmogorov in 1962 of a new, modified, theory of small-scale turbulence, which is now often called the K62 theory (for more details see [2] or [66], Sec. 25). The main idea of it consists in the replacement of the mean dissipation rate ε by the spatially averaged local dissipation rate ε_r. Here $r = 2\pi/k$ is the wave length corresponding to wave number k, and ε_r is obtained by averaging of the local energy dissipation rate $\varepsilon(\mathbf{x},t)$ over a spherical volume of points $\mathbf{x}$ having the radius r/2 and the center at the point to which the considered spectrum $E_{11}(k)$ corresponds.

Let us consider not the one-dimensional spectrum $E_{11}(k)$ but more simple velocity *structure function* of the second order:

$$D_2(r) \equiv \langle[u_1(\mathbf{x} + \mathbf{r}) - u_1(\mathbf{x})]^2\rangle, \quad r = |\mathbf{r}| \tag{12}$$

(here u_1 is velocity component in the direction of vector $\mathbf{r}$ and, as usual, angular brackets denote ensemble averaging). Then, according to K41 for $r << L$

$$D_2(r) = C\varepsilon^{2/3}r^{2/3}f_2(r/\eta), \tag{13}$$

where f_2 is an universal function, $f_2(\infty) = 1$, and $C \approx 4A$ is an universal constant. From (13) it follows that in the inertial range $L \gg r >> \eta$ of distances r Kolmogorov's *two thirds law* of the form

$$D_2(r) = C\varepsilon^{2/3}r^{2/3} \tag{13a}$$

is valid. On the other hand, according to K62 theory for $r << L$

$$D_2(r) = C\langle(\varepsilon_r)^{2/3}\rangle r^{2/3}f_2(r/\eta_r), \quad \text{where } \eta_r = \nu^{3/4}(\varepsilon_r)^{-1/4}. \tag{14}$$

In the inertial range $L >> r >> \eta_r$ (the length η_r fluctuates but usually it is of the same order as η) $f_2(r/\eta_r) = 1$, and hence

$$D_2(r) = C\langle(\varepsilon_r)^{2/3}\rangle r^{2/3}, \quad \text{where } \langle(\varepsilon_r)^{2/3}\rangle \neq \langle\varepsilon_r\rangle^{2/3} = \varepsilon^{2/3}. \tag{15}$$

According to Eq. (15) dimensional coefficient $D = C\langle(\varepsilon_r)^{2/3}\rangle$ of the two-thirds law may fluctuate producing variations of the value of the dimensionless coefficient $C_0 = D/\varepsilon^{2/3}$ (where ε is strictly constant 'mean dissipation rate'). The same arguments may explain the observed variability of the coefficient B of the law (11a).

In his work of 1962 Obukhov assumed that ε_r has lognormal probability distribution with variance depending on r and used this model for a crude estimation of $\langle(\varepsilon_r)^{2/3}\rangle$. Kolmogorov in his version of K62 theory, sketched some general similarity hypotheses which generalized the hypotheses used in K41 (namely, instead of the assumed in K41 local isotropy of the velocity field $\mathbf{v}(\mathbf{x},t)$ he suggested to assume that the probability distributions of the ratios of velocity differences in two pairs of points are invariant with respect to all motions and mirror reflections of this group of points). However, this last hypothesis was never developed to a state of a completed theory. Moreover, Kolmogorov also proposed to use Obukhov's lognormal assumption not only in Eq. (15) but also in the more general equation for the structure function $D_n(r)$ of the arbitrary order n (defined by presented below Eq. (16)). This proposition implied the following approximate estimate of the form of the velocity structure functions of arbitrary orders in the inertial range:

$$D_n(r) \equiv \langle[u_1(\mathbf{x} + \mathbf{r}) - u_1(\mathbf{x})]^n\rangle = C_n(\mathbf{x})(\varepsilon r)^{n/3}(L/r)^{\mu n(n-3)/18}. \tag{16}$$

Here $\varepsilon = \langle\varepsilon_r\rangle$ is the mean rate of the energy dissipation, μ is an universal constant, and $C_n(\mathbf{x})$ depends on the flow macrostructure (and is practically constant in regions of a size much smaller than L). Old K41 theory corresponds to the case where C_n are universal constants and $\mu = 0$; note also for n = 3 both theories imply the same result.

At present it is clear that the lognormal assumption accepted in 1962 by both Kolmogorov and Obukhov was only a crude approximation. [In fact both authors also considered it as only an example allowing to illustrate the possible influence of the dissipation-rate intermittency on the inertial-range spectra and structure functions]. After 1962 a number of attempts were made by different authors to replace this assumption by some more general model of the self-similar cascade process of sequential breakdown of smaller and smaller eddies (the early stage of this development was summarized in Sec. 25 of the book [66]; see also [2]). From all this material only the result due to Novikov [108] will be presented here. Novikov considered three similar to each other spatial volumes (let us say spherical for definiteness) of radii $\rho < r < R$ contained within each other and corresponding to them three averaged dissipation rates ε_ρ, ε_r and ε_R (which are fluctuating random variables). He postulated that self-similarity of the cascade breakdown process is represented by the fact that if all three radii ρ, r, and R belong to the inertial range of lengths, then the random ratios $\varepsilon_\rho/\varepsilon_r$ and $\varepsilon_r/\varepsilon_R$ are statistically independent from each other and have probability distributions depending only on ratios ρ/r and r/R, respectively. Then he showed that from such self-similarity it follows that in the inertial range of distances r

$$D_n(r) = C_n(\varepsilon r)^{n/3} \left(\frac{r}{L}\right)^{\xi_n} \propto r^{\zeta_n}, \qquad \zeta_n = n/3 + \xi_n. \tag{17}$$

A number of measured in various turbulent flows or determined from numerical simulations values of *scaling exponents* ζ_n corresponding to different values of *n* was found during the 1980s and 1990s, in particular, by F. Anselmet et al. (*J. Fluid Mech.*, **140**, 60-89, 1984), R. Benzi et al. (*Phys. Rev.*, **E48**, R29-R32, 1993), G. Stolovitzky et al. (*Phys. Rev.*, **E48**, R3217-R3220, 1993), and J.A. Herweijer and W. van de Water, *Phys. Rev. Lett.*, **14**, 4651-4654, 1995). The first analytical models of the scaling-exponent function $\zeta_n = \zeta(n)$ was proposed by Kolmogorov in 1962 [see Eq. (16)]; its agreement with the subsequently found values of the exponents ζ_n proved to be quite poor. Note that according to Eq. (16) ξ_2 is positive and apparently small (μ is positive by definition but hardly large), $\xi_3 = 0$, and ξ_n are negative for n >3 and $|\xi_n|$ grow very quickly with n. The available data shows that the signs of corrections ξ_n were predicted by Eq. (16) correctly (but ξ_2 is so small, that it is sometimes assumed to be zero), but for higher-order corrections with n >3 values of $|\xi_n|$ are always much smaller than they must be according to Eq. (16)

Later many other 'theoretical models' of scaling exponents corresponding to various particular self-similar models of the cascade process of eddy breakdowns were given by a number of authors; the papers by U. Frisch et al. (*J. Fluid Mech.*, **87**, 719-736, 1978), R. Benzi et al. (see G. Paladin and A. Vulpiani, *Phys. Rev.*, **A35**, 1971-1973, 1987), S. Kida (*J. Phys. Soc. Japan*, **60**, 5-8, 1990), Z.-S. She and E. Lévêque (*Phys. Rev. Lett.*, **72**, 336-39, 1994), B. Dubrulle (*Phys. Rev. Lett.*, **73**, 959-962, 1994), Z.-S. She (*Progr. Theor. Phys. Suppl.*, **130**, 87-102, 1998), J.Jiménez (*J. Fluid Mech.*, **409**, 99-120, 2000), the book [2] and short survey by O.N. Bortav (*Phys. Fluids*, **9**, 1206-1208, 1997) represent only a small part of the material relating to this topic. Many of the proposed quite different analytic models led to results which agreed more or less satisfactorily with available experimental and numerical estimates of the exponents ζ_n, if the model parameters were appropriately chosen. This agreement shows again that up to now available data on high-Reynolds-number turbulence very often do not allow to select uniquely the best of the various proposed theoretical models.

Let us now made some general comments. The K41 theory was based on definite hypotheses which were not (and apparently cannot be) proved rigorously (i.e., derived directly from equations of fluid mechanics). However, these hypotheses seemed, at least, to be quite natural and consistent with physical intuition. In contrast, the reformulation by Obukhov and Kolmogorov of K41 theory as a new K62 theory is far less evident and physically convincing. Of course, the Kolmogorov-Obukhov's attempt of crude estimation of the intermittency effect with the help of replacement of the constant dissipation rate ε by depending on the length r fluctuating characteristic ε_r was a brilliant piece of work, but it was based on a plausible guess only and could not be considered as an adequate physical theory. Therefore it was only natural that at the end of his paper of 1962 Kolmogorov set up a problem of elimination of the quantity ε_r from K62 theory and proposed to use for this purpose two new similarity hypotheses remarking simultaneously that apparently they must be also supplemented by something else. However, the realization of this program is clearly a difficult task and this was not done yet. A partial progress was connected with the appearance of the multifractal formalism of Parisi and Frisch (see [2] about it) where ε_r was not mentioned explicitly. However this formalism represents some idealization of the real situation and it requires the introduction of some supplementary hypotheses.

Differing from K62 modification of the old K41 theory was proposed by Barenblatt and his co-authors (see, e.g., [109-111]). In the paper [109] with Goldenfeld based on some general arguments and the analogy with the problems concerning the near-wall velocity profile and some physical problems of quite different origin the authors assumed that maybe more appropriate correction of the classical two-thirds law (15) of K41 than that of Eq. (17) with n = 2, will be given by an equation of the form

$$D_2(r) = C(\ln Re)(\varepsilon r)^{2/3}\left(\frac{r}{L}\right)^{\alpha(\ln Re)} \tag{18}$$

where L has the same meaning as in Eq. (17) but now coefficient C_2 = C and exponent $\xi_2 = \alpha$ are not constants but functions of ln Re (i.e., slowly changing functions of Re). Expanding these functions in powers of a small parameter $(\ln Re)^{-1}$, the authors assumed that $\alpha(Re) = \alpha_1/\ln Re + O[(\ln Re)^{-2}]$, $C(\ln Re) = C_0 + C_1/\ln Re + O[(\ln Re)^{-2}]$ (constant term was omitted in the series for $\alpha(Re)$ to guarantee the validity of the K41 scaling when $Re \to \infty$). For crude estimate of the function $C(\ln Re)$ the data by Praskovsky and Onsley [112] were used. These authors combined results of spectral measurements of velocity fluctuations in the atmospheric surface layer and in two high-Reynolds-number wind-tunnel flows to verify the possibility of dependence of the Kolmogorov constant $C = C_2$ on the value of the Reynolds number $Re_\lambda = u'\lambda/\nu$ (where $u' = \langle u^2\rangle^{1/2}$ is the root-mean-square value of the streamwise, corresponding to Ox direction, velocity fluctuation and $\lambda = [\langle u^2\rangle/(\partial u/\partial x)^2]^{1/2}$ is the so-called Taylor length microscale). According to [112] values of the coefficient C in eight flows with $2\times10^3 \le Re_\lambda \le 12.7\times10^3$ are weakly decreasing with Re_λ [approximately as $(Re_\lambda)^{-0.1}$]. This dependence on Re differs from that assumed by Barenblatt and Goldenfeld. However, since the results of [112] had low precision (note that the summary tables of the measured C-values collected in [113,114] showed that these values are very scattered but gave no indications of their dependence on Re), it was concluded in [109] that these results may be also crudely approximated by the proposed in this paper equation for C(ln Re). As such approximations even two version of proposed in [109] equation were considered: one with $C_0 = 0$ and the other with $C_0 \neq 0$. Note that if $C_0 \neq 0$, then equation (18) implies that at $Re \to \infty$ limiting regime of 'fully developed turbulence' is realized where Kolmogorov's 'two-thirds law' is valid, while if $C_0 = 0$, then such regime don't exist.

Later Barenblatt and Chorin [83,110,111] generalized Eq. (18) and given above approximate models of the functions $\alpha(Re)$ and $C(Re)$ to the case of the velocity structure functions $D_n(r)$ of orders n

≥ 4, suggesting the following approximate equation for values of these functions in the inertial range $\eta \ll r \ll L$:

$$D_n(r) = (C_n + C_n^1/\ln\mathrm{Re})(\varepsilon r)^{n/3}(\frac{r}{L})^{\alpha_n/\ln\mathrm{Re}}, \quad n = 4, 5, \dots, \tag{19}$$

where C_n, C_n^1 and α_n are some constants. (For n = 2 proposed in [109] equation of the same form as (19) was used; as to the case where n = 3, here the known Kolmogorov's equation $D_3(r) = -(4/5)\varepsilon r$ was used in the inertial range of lengths r.)

Eqs. (18) and (19) correspond to definite concept of the passage to the zero-viscosity limit in fluid mechanics (see, e.g., [110,111]). Recall that according to the K62 small-scale spatial intermittency of the field $\varepsilon(\mathbf{x},t)$ leads to the appearance of small (but finite) changes of 'classical' spectral and structure-function exponents -5/3 and 2/3. (These changes have the same absolute value but opposite signs: they diminish the spectral exponent but increase the structure-function exponent.) At the same time intermittency also produces changes of the form (17) of exponents describing the forms of structure functions of higher orders in the inertial range of lengths r. This prediction of K62 was widely discussed during the last two decades [see, e.g., papers cited after Eq. (17)]. However it was also sometimes contested (e.g., in [115,116]), and Eqs. (19) (and similar equation for n = 2) also corresponds to the assumption that 'intermittency corrections' of the inertial-range exponents tend to zero as Re $\to\infty$. (Just the acceptance of this assumption forced the authors to require that $\alpha(\mathrm{Re}) \to 0$ as Re $\to\infty$.) Since the available experimental and numerical estimates of 'intermittency corrections' are scattered and small, the reliable verification of Eqs. (18), (19) is apparently impossible at present. Let us consider, for example, the situation relating to the 'intermittency correction' ξ_2 corresponding to the second-order structure function $D_2(r)$. The first experimental estimate of ξ_2 given in [117] was close to 0.04, while at present the available non-zero estimates cover the range from 0.05 to 0.02, but zero value is also sometimes accepted. (In particular, Praskovsky and Onsley [112] found that ξ_2 is close to zero at all inspected by them values of Re_λ, and there are also other authors who supposed that the available data are insufficient for proving that $\xi_2 \neq 0$.) Barenblatt et al. [118] tried to use for the verification of their assumption about the dependence of $\alpha = \xi_2$ on Re the data by Benzi et al. [119] who measured the values of functions $D_2(r)$ and $D_3(r)$ in four different flows with Re = 5000, 6000, 18 000, and 300 000 (where different definitions of Re were used for different flows). In [119] it was found that to the summary collection of all obtained data corresponded the practically constant correction $\xi_2 \approx 0.03$. Barenblatt et al. separated data points corresponding to individual experiments and their processing of four separate (rather small) groups of points led to conclusion that the corrections ξ_2 differ in the cases of different experiments decreasing with the growth of Re and possibly tending to zero as Re $\to\infty$. However, Benzi et al. in their reply [120] to the note [118] disagreed with such interpretation of their data. At the beginning they rightly noted that from a theoretical point of view, the dependence of the exponent ξ_2 on Re and its convergence to zero as Re $\to\infty$ does not seem impossible. However then they stated that their experimental data, and also analyzed by them additional data of some other authors covering a larger range of high Re values, show that $\xi_2 \approx 0.03$ in all studied flows and it does not change with the increase of Re. Moreover, it was also noted in [120] that according to data presented in [121] the higher-order scaling exponents ξ_n with n $\leq$ 7 also don't depend on Re. (In [121] an attempt was made to collect results of approximate evaluations of values of ξ_n, n $\leq$ 7, based on data of seven experiments corresponding to quite different turbulent flows and values of Re_λ between 300 and 5000.)

Of course, the experimental results presented in [120,121] cannot be considered as a strict proof of the independence of scaling exponents ξ_n and $\zeta_n = \xi_n + n/3$ on Re. All the measurements of these exponents are rather crude and their results may depend on the choice of the 'inertial range' where the structure functions satisfy the power laws. Note also that in [119-121] the scaling exponents were determined indirectly basing on the 'extended self-similarity' (ESS) hypothesis by Benzi et al. [119] generalizing the concept of the inertial range where structure functions $D_n(r)$ satisfy power laws (17). Eq. (17) implies that in the inertial range any function $D_n(r)$ is proportional to the function $D_m(r)$ raised to the power ζ_n/ζ_m. ESS stated that the proportionality of $D_n(r)$ to $[D_m(r)]^{\zeta_n/\zeta_m}$ is often valid over an unexpectedly wide range of scales r extending far beyond the small-scale limit of the inertial range. [In practical applications it is usually assumed that m = 3; then ζ_m=1 and within the inertial range the ESS representation is equivalent to that of Eq. (17).] The use of the ESS method allows to simplify and make more easy the determination of exponents ζ_n from the experimental data, but in principle found by this method values of ζ_n may be somewhat affected by the extension of the considered range of r values. However, even more important is the absence of any explanation of the ESS phenomenon. ESS clearly represents a surprising similarity property which must be somehow connected to similarity of organized structures determining the shapes of structure functions in the covered by ESS range of lengths r. This generalization of the following from K62 Eq. (17) may be compared with proposed by Barenblatt, Chorin and Goldenfeld Eqs. (18) and (19) which validity also must reflect some unknown symmetry features of flow structures determining the velocity differences. Moreover, Eq. (17) by itself is also a similarity relation which derivation in the framework of K62 is based on the use of some unproved and physically somewhat vague hypotheses. Therefore it is not surprising that Sreenivasan and Dhruva [122] even tried to investigate whether the scaling (17) really exists in high-Reynolds-number turbulence or not. Their measurements in the atmospheric surface layer at $10^4 \leq Re_\lambda \leq 2\times10^4$ led them to the conclusion that apparently in atmospheric turbulence there exists an inertial range where Eq. (17) is valid but its validity is often disturbed by velocity shear and finiteness of Re (see also the discussion of the results of the paper [127] below). However, the paper [122] did not clarify the origin of the similarity law (17).

One more generalization of the K62 scaling (17) for the case of $n = 2$ was proposed by Gamard and George [123]. According to their theory the scaling exponent ξ_2 and Kolmogorov's coefficient $C = C_2$ depend on the Reynolds number Re and ξ_2 tends to zero while C tends to a non-zero constant C_0 as $Re \to \infty$. Thus, this theory stated that the 'classical' turbulent regime of K41 theory is valid in the limiting case of very high Reynolds numbers. The authors applied to the considered by them problem hypotheses of the same type as those used in the papers [97,98] for the evaluation of velocity profiles in turbulent pipe, channel and boundary-layer flows. Obtained in [123] results proved to be in good agreement with the experimental results by Mydlarski and Warhaft [124] relating to spectral measurements in the isotropic turbulent flow produced in a relatively small wind tunnel by an 'active grid' generating intensive turbulent fluctuations. The data by Mydlarski and Warhaft corresponded to a limited range of not too large Reynolds numbers; therefore even the existence here of the intermediate range of wave numbers k where $E_{11}(k) \propto k^{-\alpha}$, $\alpha > 0$, was somewhat unexpected. Note also that in this case the found corrections which must be added to the 'Kolmogorov exponent' -5/3 prove to be positive while according to K62 the intermittency corrections of the spectral exponent are always negative (equal to $-\xi_2$). For this reason the results of this work cannot be compared with the results discussed above which were relating to flows with much higher values of Re.

The present state of the considered above investigations of the K41 theory and of the similarity laws for near-wall turbulent flows, produces an impression that at the end of the 20th century the fundamental achievements of Prandtl, Kármán, Kolmogorov and other giants laying, seemingly for ever, the foundations of the modern theory of turbulence, began to stagger producing doubts and the feeling of uncertainty. Thus, at present the theory of turbulence seems to be more neglected than it was in the middle of the 20th century when the great discoveries of the 1930s, 1940s and 1950s produced universal enthusiasm. Let us nevertheless hope that arising difficulties will be get over and will lead to great progress in understanding of turbulence phenomena in the initial part of the 21st century.

4. Concluding Remarks; Possible Role of Navier-Stokes Equations

It has been already stressed above that both the theory of logarithmic layer of wall-bounded fully turbulent flows developed by Kármán, Prandtl, Izakson, and Millikan in the 1930s and Kolmogorov's K41 theory of locally-isotropic turbulence were based on some seemingly plausible physical hypotheses and dimensionality consideration, while the exact NS equations of fluid dynamics were not used there at all. Both these theories were shortly after their appearance confirmed by seemingly faultless experimental data, became very popular and were unanimously accepted as a final truth by scientific community. It is worth noting that physical basis of the K41 theory at first stimulated enthusiasm only within the community of physicists, while many fluid mecanicians were much in doubt. The closeness of this theory to physical manner of thinking was reflected in a remarkable fact that this theory was later independently developed also by two famous physicists, both the Nobel-prize winners, namely by L. Onzager (in 1945) and W. Heisenberg (in 1947). Moreover, Kolmogorov's theory was first included in textbooks also by famous physicists - in courses of the continuum mechanics written by L. Landau in Russia (then USSR) and by A. Sommerfeld in Germany as parts of the general courses of theoretical physics in many volumes. However later the K41 theory was accepted by everybody and became an important part of modern fluid mechanics.

When it was found in the late 1950s that some of the results of K41 disagree with the data of spectral measurements in the lower atmosphere, Obukhov and Kolmogorov developed a modified K62 theory. As it was told above, this new theory included some description of the influence of the external length scale L (equal to the typical length of large-scale flow nonhomogeneities) on the small-scale turbulence but preserved the assumption about the spatial homogeneity and isotropy of turbulence within small spatial regions of diameters $l \ll L$. This assumption was also left inviolable in the subsequent modifications of K62 by Barenblatt et al. and some others and in numerous studies of cascade models of small-scale intermittency and of scaling exponents (the careful studies [125] of anisotropic contributions to structure functions of various orders and to their scaling laws were rare exceptions in this respect). However, now there is a lot of data showing that the fundamental Kolmogorov's assumption about the isotropy of turbulent fluctuations of scales $l \ll L$ in any high-Reynolds-number flow is quite often violated.

Let us note, for example, that the local isotropy implies that the cospectra $E_{ij}(k)$ of velocity components u_i and u_j, where $i \neq j$, must vanish in the inertial range of wave numbers, i.e., at $|k| \gg 2\pi/L$. However in the lower atmosphere, where Re takes very high value, the cospectrum $E_{13}(k)$ of the horizontal (in the mean-wind direction) and vertical wind components always takes non-zero values in the range of values of k where spectra $E_{11}(k)$ and $E_{33}(k)$ are proportional to $k^{-5/3}$. (Cospectrum $E_{13}(k)$ decreases in this range approximately as $k^{-7/3}$, i.e. faster than spectra $E_{11}(k)$ and $E_{33}(k)$ but not fast enough to become negligibly small; see, e.g., [126]). The simultaneous validity of K41 theory for $E_{11}(k)$ and $E_{33}(k)$ and non-validity for $E_{13}(k)$ requires special explanation which is lacking up to now.

In addition to this, Shen and Warhaft [127] measured recently a number of small-scale characteristics of velocity fluctuations in a homogeneous shear flow (with constant shear dU/dz where U is the mean velocity) behind an active grid. These measurements covered the range $100 \leq Re_\lambda \leq 1100$ of high enough Reynolds numbers Re_λ. For the normalized moments of streamwise-velocity derivative $\partial u/\partial z$

$$S_{2m+1} = \langle(\partial u/\partial z)^{2m+1}\rangle[\langle(\partial u/\partial z)^2\rangle^{(2m+1)/2}]^{-1} \qquad (20)$$

they found that S_3 is decreasing with Re_λ (and possibly tends to zero as $Re_\lambda \to \infty$), while S_5 does not decrease with Re_λ (and is close to 10 at

$Re_\lambda \approx 1000$), while S_7 increases with Re_λ. These results clearly show that studied turbulence is not locally isotropic in the dissipation range of lengths (since at local isotropy all moments S_n of odd orders n must vanish). At the same time it was found that lateral structure functions $D_n(r) = \langle[u(x,y,z+r) - u(x,y,z)]^n\rangle$ of odd orders n = 3, 5, and 7 take non-zero values in the inertial range of lengths r (i.e., for $\eta \ll r \ll L$); hence the studied homogeneous-shear-flow turbulence is anisotropic also in the inertial range of lengths. Thus, results of [127] show that the shear-flow turbulence is locally non-isotropic, at least to $Re_\lambda \approx 1000$, and demonstrates no tendency to become isotropic at higher values of Re_λ. Here again the question arises how the discovered local anisotropy can be combined with the validity of the ordinary laws of two and five thirds which was confirmed by data relating to very different high-Reynolds-number shear flows.

Strong deviations from the predictions of K41 theory were in fact first detected in studies of small-scale fluctuations of temperature (or other passive scalars) in high-Reynolds-number turbulent flows.[2] In particular, at the end of the 1960s it was discovered that the skewness of temperature derivative $S_T = \langle(dT/dx)^3\rangle/[\langle(dT/dx)^2\rangle]^{3/2}$ is different from zero (being of order 1) in the atmospheric flows with very high values of Re, although for locally-isotropic temperature fluctuations $S_T = 0$; see, e.g., [128]. (This excellent survey of the modern studies of passive-scalar fluctuations in turbulent flows contains a long list of references. This fact allows us to omit here all references to papers on this subject, with the exception of very recent papers [129] appearing after the publication of [128].) Later it was found that S_T practically does not depend on Re, i.e. it takes rather high values in all flows. Moreover, also the structure functions of temperature

$$D_{T,n}(r) = \langle[T(\mathbf{x} + \mathbf{r}) - T(\mathbf{x})]^n\rangle, \quad r = |\mathbf{r}|, \tag{21}$$

of odd orders n=2m+1 were found to be different from zero, though the local isotropy implies that all these functions must vanish. There were many attempts to explain these violations of the local isotropy of temperature fluctuations by the influence of 'temperature ramps'

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[22] Generalization of the K41 theory to temperature and other scalar fields (for simplicity, only temperature field will be mentioned here) was carried out independently by A.M.Obukhov and S. Corrsin in 1949-51; see, e.g., [66], Chap. 8. It was found, in particular, that the temperature structure functions and one-dimensional spectra in the inertial ranges of lengths and wave numbers satisfy the same two-thirds and five-thirds laws as structure functions and spectra of velocity.

(where slow temperature growth is suddenly replaced by very rapid decrease or vice versa) and some other strongly asymmetric large-scale temperature structures. However, these attempts were not fully successful and also the origin of the asymmetric temperature structures in scalar turbulence remains enigmatic up to now. Let us note in this respect described in [128] results of the numerical simulation by M. Holzer and E.D. Siggia of the development of temperature fluctuations in a homogeneous Gaussian velocity field without any appreciable structures accompanied with a constant gradient of the mean temperature. It was found that in this case the temperature 'ramp structures' of unknown origin also appeared regularly. In any case, the available at present data relating to small-scale temperature fluctuations show that Kolmogorov's assumption about the isotropy of small-scale turbulent fluctuations in all flows with high enough Reynolds (and Peclét) numbers is usually invalid in the real flow turbulence.

Detected at the end of the 20th century strong deviations of the results of careful measurements of turbulent-flow characteristics from the previous predictions of great scientists are very disturbing for all modern fluid mechanicians. These deviations make highly desirable the comparison of the old theoretical results, based on physically convincing but unproved hypotheses, with conclusions following directly from rigorous dynamic equations of fluid motions. Unfortunately, this natural desire cannot be satisfied easily since the derivation of the specific results relating to high-Reynolds-number fluid flows from the dynamic equations met with unexpected resistance. Below, as everywhere above, only the incompressible fluid flows satisfying the Navier-Stokes equations will be considered. Very complicated properties of these equations have been already noted earlier, and now this complexity becomes especially evident in view of some recently appearing new curious developments relating to this subject.

In the Introduction to these lectures the so-called "Physics Problems for the Next Millennium" have been already mentioned. Let us now explain that the appearance of these problems was stimulated by publication slightly earlier by the Clay Mathematics Institute of a list of seven "Mathematics Millennium Prize Problems" (first announced during the "Millennium Meeting" of mathematicians at the Collège de France in Paris in May 2000). It was announced there that the solution of any of these problem will be rewarded by a prize of $1 million (see [130] and http://www.claymath.org/prize_problems). Clay Institute Problems were considered by their authors as the continuation of the famous "Hilbert's Problems" - a list of 23 then unsolved problems set up by the famous German mathematician D. Hilbert at the International Mathematical Congress of 1900 in Paris for solution in the 20th century. For the subject discussed here it is only of importance that seven Clay Institute Prize Problems include a problem called "Navier-Stokes Equations". A short explanation accompanying the problem title at the internet announcement states that "Our understanding of the Navier-Stokes equations remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in these equations." This is somewhat vague formulation for a problem whose solution is estimated in one million dollars, but it is clear that it is supposed here that the solution must explain the inexplicable features of fluid flows, both laminar and, especially, much more mysterious turbulent. Brief summary of the same prize problem in [130] was expressed as follows: "Prove or disprove the existence and smoothness of solutions to the three-dimensional Navier-Stokes equations (under reasonable boundary and initial conditions)". A little more detailed discussion of this problem by Prof. C.L. Fefferman accompanying the internet notice paid again most attention to unsolved problems relating to the existence, smoothness, and possible singularities of the solutions of three-dimensional NS equations. Moreover, in even more detailed discussion of this problem by P. Constantin [131] much attention was again paid to existence problems for smooth solutions of the NS (and Euler's, where $\nu = 0$) equations, but at the same time some problems on the asymptotic behavior of solutions at large times (closely connected with the secrets of flow instability) and on mysteries of turbulence were also briefly described there. All this is told here to pay attention of the readers to the remarkable fact that mathematical problems of fluid motions were included in a short list of major unsolved mathematical problems which the science of the 20th century left for solution to the 21st century.

Let us now revert to possible applications of the NS equations to studies of turbulence phenomena. A number of difficulties met on this way was discussed by L'vov and Procaccia in 1997 (see [132]). These two scientists were long trying to develop the hydrodynamic theory of turbulence and, in particular, to apply the NS equations to the proof of the existence of a range of the power-law behavior of the velocity structure functions and to the estimation of the corresponding scaling exponents ζ_n (see, e.g., the second paper in [132] and the cited there papers on this subject). Their work showed clearly how complicated this problem is and how difficult it is to obtain here even a modest success. Another very interesting discussion of the problems arising in the hydrodynamic theory of turbulence was published by C. Foias [133] also in 1997. In the title of the paper [133] it was asked: "What do the Navier-Stokes equations tell us about turbulence?", and in the first sentence of it the following answer was proposed: "Until the early eighties, very little; since then, quite a lot." It seems, however, that this answer is a little too optimistic, though it is impossible to neglect serious successes in this field reached during the last twenty years.

The main purpose of Foias in [133] was to make an attempt to find rigorous proofs based on the NS equations of some remarkable results of the turbulence theory which were earlier derived from some combination of the physical intuition with the purely empirical evidence. As the appropriate examples of such theoretical results Kolmogorov's (relating to K41) and Kraichnan's [134] inertial-range laws for three-dimensional (3D) and two-dimensional (2D) turbulence were chosen. (Kraichnan's 2D results were also included since the 2D NS equations are much simpler than the 3D ones.) Some elementary model of the cascade process of energy transfer from larger to smaller eddies was included in Foias' analysis but all intermittency effects were fully neglected. Under this condition the author was able to give practically rigorous proofs of the K41 and Kraichnan's $k^{-5/3}$ and k^{-3} laws for the energy spectrum E(k) in the inertial ranges of wave numbers and of the equations determining the dissipation length scales in three and two dimensions. However, these proofs proved to be rather complicated and they nevertheless included some purely heuristic arguments.

Quite impressive successes were achieved in the studies of the asymptotic behavior of the solutions of the NS equations and of the structure of the corresponding 'attractors' in the infinite-dimensional phase spaces of fluid flows; see, e.g., the books [56,135] where some of the results relating to this topic were considered. (Here again advances were most impressive in the case of 2D turbulence.) However, the development of the rigorous mathematical theory of the high-Reynolds-number turbulence is apparently up to now only in its initial stage.

In the case of developed turbulence most interesting are not individual solutions describing the time evolution of separated flow fields but 'statistical solutions' corresponding to time evolution of the probability measure in the space of all possible fluid-dynamics fields when the initial measure at the time $t = 0$ is given. Instead of the difficult for mathematical treatment probability measure in the infinite-dimensional space of turbulent fields, it is much more convenient to consider corresponding to this measure *characteristic functional* (first introduced, for the case of a random function of one variable, long ago by Kolmogorov [136]). Spatial characteristic functional of the velocity field $\mathbf{u}(\mathbf{x},t) = \{u_1(\mathbf{x},t),u_2(\mathbf{x},t),u_3(\mathbf{x},t)\}$ of a turbulent flow is given by the equation

$$\Phi[\Theta(\mathbf{x}),t] \equiv \Phi[\theta_1(\mathbf{x}),\theta_2(\mathbf{x}),\theta_3(\mathbf{x}),t] = \langle \exp\{i\iiint \sum_{k=1}^{3} \theta_k(\mathbf{x})u_k(\mathbf{x},t)dx_1dx_2dx_3\}\rangle \qquad (22)$$

[here $\mathbf{x} = (x_1,x_2,x_3)$ and integration is extended over the whole space of points $\mathbf{x}$ while the functions $\theta_k(\mathbf{x})$, k = 1,2,3, are chosen to provide convergence of the integral on the right in (22)]. Angular brackets, as usual, denote in (22) the ensemble averaging, i.e., the integration with respect to probability measure. Note that the moments of all orders (both one-point and multipoint) of the velocity field $\mathbf{u}(\mathbf{x},t)$ (where t is fixed) may be easily expressed in terms of the partial *functional derivatives* of various orders of the functional $\Phi[\Theta(\mathbf{x}),t]$ (see, e.g., [66], Sec.4.4, or any of cited below other books where Hopf equation is considered). For determination of the multitime velocity moments relating to velocity values at various space and time points, the spatial-temporal characteristic functional $\Phi[\Theta(\mathbf{x},t)]$ may be used. This functional is given by similar to (22) equation where the functions $\theta_k(\mathbf{x})$ are replaced by functions $\theta_k(\mathbf{x},t)$ and integration is taken over the four-dimensional space of points $(\mathbf{x},t)$. However, such functionals (introduced in the paper [137]) will be not considered below.

Characteristic functional determines uniquely the probability measure of the turbulent velocity field and its time evolution is governed by linear functional derivative equation derived in 1952 by Hopf [138]. *Hopf equation* may be written in the form

$$\frac{\partial\Phi[\Theta(\mathbf{x}),t]}{\partial t} = i(\hat{\theta}_k \frac{\partial D_k D_m \Phi}{\partial x_m}) + \nu(\hat{\theta}_k \Delta D_k \Phi) \qquad (23)$$

where $D_k = D_k(\mathbf{x}) = \delta/\delta\theta_k(\mathbf{x})d\mathbf{x}$ is the functional derivative with respect to the component $\theta_k(\mathbf{x})$ of the vector $\Theta(\mathbf{x})$, Δ is the Laplace operator, the summation is performed over the three values of the twice appearing indices k and m, and $\hat{\theta}_k(\mathrm{x})$ are the components of the vectorial function $\hat{\Theta}(\mathbf{x})$ which may be obtained from the vectorial function $\Theta(\mathbf{x})$ by means of some simple linear operation. Eq. (23) seems to be very attractive, since it is linear, not very clumsy, and determined the whole probability distribution of the velocity field. Unfortunately, the mathematical theory of functional derivative equations was quite undeveloped in the fifties (e.g., nothing was known then about the solvability of such equations and the conditions for the uniqueness of their solutions, and there were no methods for solution computation). Therefore, at first the practical usefulness of the Hopf equation seemed rather questionable. However, during almost a half century separating our time from the early fifties the mathematical theory of the linear functional derivative equations advanced considerably (to a considerable degree just in the connection with induced by Hopf's paper active development of mathematically-oriented studies of statistical fluid mechanics) and this made the situation much less hopeless. A number of results of these studies may be found, in particular, in the papers [139], fundamental monograph by Vishik and Fursikov [140] and the recent books [141] on mathematical fluid mechanics and turbulence theory which include analysis of the Hopf equation.

Let us now say a few words about the paper by Foias, Manley and Temam of 1987 (see [139]), which did not used Hopf's equation directly, but referred to it repeatedly and was ideologically connected with the functional approach to statistical fluid mechanics. Here for the case of isotropic turbulence an attempt was made to connect the derivation of Kolmogorov's 'five-thirds law' for the energy spectrum with the study of statistical solutions of Navier-Stokes equations and even to use the found connection for the determination of lower bound of the range of Reynolds numbers at which the inertial range of wave numbers exists. However, apparently there were no attempts to explain with the help of Navier-Stokes dynamic equations the observed anomalous scaling of the velocity structure functions (i.e., the appearance of the non-zero scaling corrections ξ_n to Kolmogorov's exponents n/3).

Let us now made a small remark of general character at the end of this long text. It is clear that characteristic functional of a random function is a natural generalization of the *characteristic function* of a random variable (or random vector). Method of characteristic functions was introduced into probability theory by the famous Russian scientist A.M. Lyapunov almost exactly one hundred years ago (about 1900) when he applied this method to the first rigorous proof of the Central Limit Theorem of this theory under very general conditions. Later it was found that this method represents an universal tool (of very high efficiency) for the study of the asymptotic behavior of the families of random variables and random functions depending on a parameter tending to infinity. During the 20th century many hundreds of papers (and probably a few dozens of books) were published where characteristic functions were widely used for this purpose. Of course, characteristic functionals are analytically much more complicated than characteristic functions, but the power of analytic methods today also exceeds very much their possibilities in the Lyapunov's time. Let us therefore hope that the method of characteristic functionals will have in the new century a development comparable to that of the method of characteristic functions in the previous century. (Note that in the turbulence theory the investigation of the asymptotic behavior of fluid-dynamical fields as $t \to \infty$ or/and $Re \to \infty$ always plays a very important part.) Since NS equations are very complicated, it is reasonable to elaborate at first the new analytical methods in application to simpler models; from this point of view the numerous recent studies of "nonphysical" Burgers turbulence (to this subject, in particular, the lectures by Uriel Frisch at this summer school were devoted) may be very useful.

It seems natural to expect now that the 21st century will be a century of an astonishingly large growth of turbulent investigations. However, crude dimensional arguments, playing such important part in most fundamental achievements of the previous century, apparently will be of secondary importance for the future development of our science but much more important part will play the deep physical insight and very artful analytical technique.

Acknowledgments. Many colleagues helped me in preparation of this text discussing with me some topics touched in here and/or sending me written materials related to the contents of my talks. I wish to thank here Peter Bradshaw, Peter Constantin, Siegfried Grossmann, John Lumley, Ron Panton, Peter Schmid, Mark Vishik, Zellman Warhaft, and Victor Yudovich whose help was especially important. I am very grateful also to Grisha Barenblatt who regularly sent me his interesting papers on turbulence, but I understand that he will disagree with a number of my opinions. The Department of Aeronautics and Astronautics of MIT and the Department Head Prof. E.F. Crawley do everything to facilitate my systematic work at MIT ant thus help substantially to my work. Financial support for this work was provided by U.S Office of Naval Research through ONR Award No. N00014-01-1-0226, and by Dr. John William Poduska, Sr. and his family through the Poduska Family Foundation giving a special grant to MIT; without this support the preparation of this text would be impossible.

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AQU02-04-0606

THE CENTURY OF TURBULENCE THEORY: THE MAIN ACHIEVEMENTS AND UNSOLVED PROBLEMS

Akiva Yaglom

1. Introduction

The flows of fluids actually met both in nature and engineering practice are turbulent in the overwhelming majority of cases. Therefore, in fact the humanity began to observe the turbulence phenomena at the very beginning of their existence. However only much later some naturalists began to think about specific features of these phenomena. And not less than 500 years ago the first attempts of qualitative analysis of turbulence appeared - about 1500 Leonardo da Vinci again and again observed, described and sketched diverse vortical formations ('coherent structures' according to the terminology of the second half of the 20th century) in various natural water streams. In his descriptions this remarkable man apparently for the first time used the word 'turbulence' (in Italian 'la turbolenza', originating from Latin 'turba' meaning turmoil) in its modern sense and also outlined the earliest version of the procedure similar to that now called the 'Reynolds decomposition' of the flow fields into regular and random parts (see, e.g., [1,2]). However, original Leonardo's studies did not form a 'theory' in the modern meaning of this word. Moreover, he published nothing during all his life and even used in most of his writings a special type which could be read only in a mirror. Therefore his ideas became known only in the second half of the 20th century and had no influence on the subsequent investigations of fluid flows.

During the first half of the 19th century a number of interesting and important observation of turbulence phenomena were carried out (such as, e.g., the early pipe-flow observation by G. Hagen [3]) but all of them were only the precursors of the future theory of turbulence. Apparently, the first theoretical works having relation to turbulence were the brilliant papers on hydrodynamic stability published by Kelvin and Rayleigh at the end of the 19th century (apparently just Kelvin who know nothing about Leonardo's secret writings, independently introduced the term "turbulence" into fluid mechanics). However, these papers only 'had relation to turbulence', but did not concern the developed turbulence at all. First scientific description of turbulence was in fact given by Reynolds [4]. In his paper of 1883 he described the results of his careful observations of water flows in pipes, divided all pipe flows into the

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