A.S. Monin and A.M. Yaglom

Statistical Fluid Mechanics:
The Mechanics of Turbulence

Volume 1, Chapter 5

New English Edition,
revised, updated and augmented by A.M. YAGLOM

CTR Monograph

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ERRATA for Chaps. 2, 3, and 4 of the present book (CTR Monographs 1, May1997, 2, May1998, and 3, September 1999)

(see also Errata in Chap. 3, pp.iv-v, and Chap. 4, p.v)

CHAPTER 2

p.74, line 7 fb: "Eq. (2.41)" should read "Eqs. (2.40) and (2.41)".

p.78, line 13: The point at the line beginning must be deleted.

p.132, line 7: "[$-\delta p/\partial x$]" should read "[$-\partial p/\partial x$]".

p.144, line 10 fb: "Matsudera" should read "Mitsudera".

p.144, line 7 fb: "**248**" should read "**276**".

p.147, line 13: "149" shold read "148".

p.152, line 6 fb: "22" should read "222".

p.157, line 5 fb: "278" should read "288".

p.158, line 9 fb: "(in Russian)." should read:

"(Engl. transl. in the Appendix to the book: Markus, A.S., *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Amer. Math. Soc., Providence, R.I., 1988).".

CHAPTER 3

p.iii, lines 2 and 9 fb: "ii" should read "i"; "75" should read "76".

p.25, line 8: "it was it was" should read "it was".

p.30, line 3 fb: "$\bar{v}$" shold read "$\overline{w}$".

p.33, line 17: ".." should read ".".

p.52, Eq. (3.41): "$w(\mathbf{x},t)$" should read "$w_j(\mathbf{x},t)$".

p.54, line 17 fb: "Ri" should read "Re".

p.57, line 14 fb: "Gustavsson and Hultgren" should read "Gustavsson, and Hultgren".

p.59, Eq. (3.52): "$+\omega t$ " shold read "$-\omega t$ ".

p.67, line 5 fb: "Sec. 3.2)" should read "Sec. 3.22)".

p.92, line 8 fb: "p. 64" should read "p. 65".

P.111, lines 2 fb and 1 fb: the words "which will be considered at greater length in Chap.4" should be deleted.

p.118, line 23 fb: "Kato (1976)" should read "Kato (1976, 1982)".

p.122, lines 16 and 18: "**4**" and "**5**" should read "**A 4**" and "**A 5**".

p.123, line 15: ""natinstability" should read "nature of instability".

CHAPTER 4

p.ii, line 14: "Dordschner" should read "Dorschner".

p.iii, lines 17 and 18: "AND AND" should read "AND".

p.1, line 11: "inadmissible" should read "physically unjustified".

p.32, line 10 fb, and p.48, line 12 fb: "Galdy" should read "Galdi".

p.47, line 7 fb: "(1974)" should read "(1976)".

p.48, line 12 fb: "Galdy" should read 'Galdi".

p.48, lines 12-f fb. Here the papers by Galdi and Straughan (1985a), Mulone and Rionero (1989), Galdi and Padula (1990), and the book by Straughan (1992) are cited in which the method of a generalized Lyapunov functional was applied to the Bénard convection in the rotating fluid layer bounded by two free surfaces. However, recently in the paper by R. Kaiser and L.X. Xu "Nonlinear stability of the rotating Bénard problem, the case Pr = 1", *Nonlinear Diff. Eqs. and Appl.*, **5**, 283-307 (1998) it was shown that just in the case of this rotating Bénard problem Lyapunov's generalized energy functionals used in the mentioned papers and book are unapplicable at Pr = 1 since one nonzero term was missing in all used energy-balance equations. Therefore Kaiser and Xu proposed another form of the energy functional which led to results differing from the previous ones.

p.69, line 9 fb: "Stewart" should read "Stuart".

p.72, line 15 fb: "Eherenstein" should read "Ehrenstein".

p.83, line 19: "Davies" should read "Davis".

p.91, line 23: "Cherhabily" should read 'Cherhabili".

p.173, line 10: "T173-T174." should read: "T173-T174; (1991) Numerical analysis of secondary and tertiary states of fluid flow and their stability properties, *Appl. Sci. Res.*, **48**, 341-351."

p.176, line 13 fb: "AN' should read "Akad. Nauk".

p.179, lines 4 and 10 fb: "George" and "Threshhold" should read: "Georg" and "Threshold".

p.180, line 13 fb: "structure" should read "structures".

p.184, line 2 fb: "Wiederstandes" should read "Widerstandes".

p.186, line 2: "*Tngng.*" should read "*Engng.*"

p.187, line 5 fb: "Ormihres" should read "Ormières".

p.188, line 17: "der Einfluss" should read "den Einfluss".

p. 204, line 4: "dimesional" should read "dimensional".

p.204, line 19: "dimemsional" should read "dimensional".

p.218, line 8 fb: "(1994)" should read "(1993)".

CHAPTER 5. FURTHER WEAKLY-NONLINEAR APPROACHES TO LAMINAR-FLOW STABILITY: BLASIUS BOUNDARY-LAYER FLOW AS A PARADIGM

Landau's equation and its generalizations considered in Sec. 4.2 represent a particular weakly-nonlinear approach to the study of flow stability, based on the assumption that the disturbance amplitude A is small enough to justify the expansion of solutions of fluid-dynamic equations in powers of A. However this approach has a severe limitation: here the evolution of only one isolated mode of disturbance is traced while its interaction with all other modes is only roughly characterized by the values of real or complex Landau's constants of various orders.

A comprehensive nonlinear theory of hydrodynamic stability must include a more direct description of interdependencies between disturbance modes. The complexity of the problem does not permit a universal analytical treatment. However, there is a vast number of approximate methods applicable to one or another particular case. Recall in this respect that some such approximate methods have already been briefly mentioned in Sec. 4.2 when the papers by Benney and Lin (1960), Benney (1961, 1964), Stuart (1962a,b), Ito (1980), and Danaila *et al.* (1998) were cited. In these papers the simultaneous development of two or more modes of disturbance was considered and therefore, instead of one Landau's equation, more general systems of differential equations for amplitudes of these modes were used [a typical example of such a system is given by Eqs. (4.43)]. However, in Sec. 4.2 no details and/or applications of these multimode analyses were presented.

In contrast to this, in the present chapter and the next some approximate methods for the study of multimode weakly-nonlinear flow instabilities will be considered at greater length, together with a number of applications of these stability theories to development of disturbances in some steady laminar flows of great practical importance. However, since the amount of material relating to this subject accumulated up to now is really enormous, a rather strict selection of topics was necessary here, and even then it has been impossible to include in the present chapter an adequate description of results of weakly-nonlinear instability theory for a wide range of laminar flows. Therefore, at first only results relating to one such flow will be considered at full length, but this will allow us to shorten considerably the presentation of similar results for other flows. As to the choice of the primary example, it was made easy by the quite exceptional place occupied in fluid mechanics by the Blasius boundary layer growing on a flat plate aligned with a parallel flow with constant velocity U_0 (and hence with zero pressure gradient). This model laminar flow is quite a good approximation to many flows met in nature and in engineering facilities, which makes it one of the most important laminar flows. Moreover, this flow has a rather simple structure, and it has been carefully studied by a number of outstanding scientists who obtained many interesting results about it, often directly relating to weakly-nonlinear stability. Note also that these results show very convincingly that in the case of the Blasius boundary layer the multimode type of instability plays an especially important part, and in fact determines the development of disturbances leading to transition of this flow to turbulence. Therefore it seems natural to devote the present chapter entirely to the study of weakly-nonlinear multimode instability of the Blasius boundary layer flow and only after this to consider some other laminar flows.

5.1. RESONANCE MECHANISMS OF WAVE-DISTURBANCE GROWTH; TWO-WAVE AND THREE-WAVE RESONANCES

In physics the term 'resonance' is most often used to describe the rapid growth of the amplitude of a steady-state periodic oscillation of a physical system affected by an external oscillating force when the frequency of the force oscillations approaches the fundamental frequency of the system considered (or one of these frequencies if there are many of them). The same term was also met in Sec. 3.32 of this book where, however, it had a slightly different meaning - there, the growth of flow disturbances produced by the degeneracy of the system of eigenfrequencies of the linear stability problem was called the 'resonance growth'. It was explained in Sec. 3.32 why the word 'resonance' is appropriate here - if there are two eigenfrequencies taking the same value ω_0 and both the corresponding flow oscillations are excited, then either of them may be considered as a force affecting the other and producing resonance growth of the oscillation amplitude. Since in Sec. 3.32 only the linear stability theory, dealing with disturbances of exceptionally small amplitude, was considered, resonance conditions were there formulated in terms of eigenvalues of linearized wave equations and no attempts to evaluate the resonance growth of amplitudes were made. However, a more general formulation states that the 'resonance mechanism of disturbance growth' means that there are several modes of disturbance such that their interactions efficiently excite some (or all) of them leading to rapid increase of the corresponding amplitudes. According to this formulation, a resonance mechanism includes reciprocal interactions among two or even more modes, and hence it cannot be studied in the framework of the one-mode Landau weakly-nonlinear theory. However the general weakly-nonlinear approach, based on the assumption that initial amplitudes of all disturbances considered are small (but not infinitesimal) can be used here too. This section will be devoted entirely to weakly-nonlinear theory of resonances and other intermode interactions appearing in fluid flows.

In Sec. 3.32 we considered only the particular *two-wave resonances* which are due to the coincidence of the frequencies of two wave modes of infinitesimal disturbance and lead to power-law growth of amplitudes of these modes. It was explained there that such resonances are rather numerous, and can occur for both two-dimensional (2D) and three-dimensional (3D) wave disturbances of steady plane-parallel or axisymmetric-parallel flows. The use of the adjective 'infinitesimal' means that in Sec. 3.32 only linearized equations of motion were considered, and hence all the resonances studied were of the elementary linear type. [The possible importance of nonlinear resonance effects was mentioned only once, on p. 70 of Chap. 3, with reference to the paper by Benney and Gustavsson (1981) but was not discussed there.] At the end of Sec. 3.32 it was also stressed that the resonance growth rates of amplitudes found in all the papers discussed were much smaller than the growth rates of disturbances observed in the available laboratory experiments and numerical simulations. This discrepancy clearly shows that there are some other growth mechanisms, more efficient than the linear resonance mechanism.

Let us now consider a more complicated situation relating to the manifestation of resonances in nonlinear physical systems (exemplified by a viscous fluid flow consisting of a steady primary flow with a finite disturbance superimposed on it). Note, first of all, that the nonlinear resonance is much more versatile than the linear one. In the simplest case of a one-dimensional oscillation $u(t)$ the quadratic term ($\propto u^2$) of the oscillation equation leads to the appearance, in addition to the harmonic oscillation of fundamental angular frequency ω_0, of the second harmonic proportional to $\exp(2i\omega_0 t)$; therefore the system may resonate here also if the external force has a frequency close to $2\omega_0$. Higher harmonics $\exp(ki\omega_0 t)$, $k = 3, 4, \ldots$, also appear in many nonlinear systems together with the primary oscillation. In general, the response of a nonlinear system to a sinusoidal external disturbance may be highly nontrivial and lead to exceedingly complicated behavior; see, e.g., Chirikov's survey (1979), Secs. 3 and 4, and the book by Rabinovich and Trubetskov (1989), Chap. 13. In particular, the phase space of a nonlinear system, even a one-dimensional one, can include a number of different resonance bands which can overlap, complicating the situation considerably. However this topic will not be considered in this book where the main attention will be paid to other aspects of the nonlinear resonance.

The possibility of nonlinear resonance produced by the interaction of a primary oscillation of a frequency ω_0 with an external force of double frequency $2\omega_0$ (or of frequency $k\omega_0$, $k > 2$) means that in a nonlinear system the simultaneous appearance of two oscillations with frequencies ω_0 and $k\omega_0$, where k is an integer, may also sometimes produce rapid amplitude growth. From this one may deduce that, for example, in a two-dimensional steady fluid flow the interaction of a pair of two-dimensional Tollmien-Schlichting (T-S) waves of finite amplitude can lead to resonance not only in the case considered in Sec. 3.32, where both waves have the same frequency ω_0 (i.e., ω_0 is a degeneracy point of the corresponding eigenvalue spectrum), but also in cases where these two T-S waves have frequencies ω_0 and $k\omega_0$ where k is an integer. A similar increase in the number of possible resonance effects is produced by the nonlinearity of the equations of motion when three-dimensional (3D) wave disturbances are considered instead of simple two-dimensional T-S waves. (Such 3D resonances were also analyzed in Sec. 3.32, in the framework of the linear stability theory.) We see that in the case of wave disturbances of finite amplitude there are many more possibilities for two-wave resonances than in the case of waves of infinitesimal amplitude. Moreover, since the product of harmonic oscillations of frequencies ω_1 and ω_2 may be represented by a linear combination of two harmonic oscillations of frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$, the interaction of two oscillations of frequencies ω_1 and ω_2 in a nonlinear system may cause a 'resonance growth' of a third oscillation of frequency $\omega_1 + \omega_2$ (or $\omega_1 - \omega_2$), if such an oscillation is also present. In other words, the nonlinear resonance may be produced in a nonlinear system with quadratic nonlinearities by a triad of small (but finite) harmonic oscillations with frequencies ω_1, ω_2, and ω_3 (which can be incommensurable with each other) such that

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{5.1}$$

for some choice of the signs of the frequencies considered (the sign of the frequency of a sinusoidal oscillation may be equally correctly considered as being positive or negative). Similarly, nonlinear resonances may also be produced by n-tuples of harmonic disturbances, where $n > 3$, with frequencies $\omega_1, \omega_2, \dots, \omega_n$ of any signs whose sum is equal to zero. Note however that condition (5.1) and the other frequency relations indicated above imply only the possibility of resonance but are not sufficient for its occurrence. In practice the emergence of a resonance and the rate of the corresponding resonance growth of amplitude depend on the general structure of the nonlinear system considered, and on numerical values of its characteristics. Note also that in the cases of exponentially growing or decaying harmonic oscillations the variables ω in Eq. (5.1) designate the real physical frequencies - as in the many other relations dealing with exponentially growing or decaying oscillations which we will meet below. [Thus, for T-S waves corresponding to points of the (k,Re)-plane away from the neutral curve the symbol ω will as a rule designate the real part, $\omega^{(r)} = \Re e\omega$, of the complex eigenvalue ω of the Orr-Sommerfeld eigenvalue problem. As to the imaginary parts $\omega^{(i)} = \Im m\omega$, they determine exponential factors $\exp(-\omega^{(i)}t)$ which will usually be included in the corresponding amplitudes $A(t)$.] Moreover, if one takes it that the frequencies are positive by definition, then the + signs in Eq. (5.1) and the similar relations must, of course, be replaced by $\pm$ signs.

Above, only the case of a one-dimensional oscillation $u(t)$ satisfying some nonlinear ordinary differential equation was considered (although wave disturbances depending on spatial coordinates were sometimes mentioned as examples). Let us now discuss the case of oscillations relating to fluid mechanics at slightly greater length. Here the oscillating disturbances always have the form of vector fields $\mathbf{b}(\mathbf{x},t) = \{u(\mathbf{x},t), v(\mathbf{x},t), w(\mathbf{x},t), p(\mathbf{x},t)\}$ (where u, v, w, and p are the three velocity components and pressure) depending on time t and coordinates $\{x, y, z\} = \mathbf{x}$ and satisfying the Navier-Stokes (N-S) partial differential equations. In such a case the study of resonance conditions and of possible types of resonance effect represents a complicated problem, where it is difficult to expect that practically useful results can be found for general disturbances. A very important particular class of disturbances, which played the central part in Chaps. 2 and 3 of this book (and has already been mentioned occasionally in the present section too), is the class of *wave disturbances* of the form $\mathbf{b}(\mathbf{x},t) = A(t)\mathbf{B}e^{i(kx - \omega t)}$ (where $\mathbf{B} = \{U, V, W, P\}$ is a constant vector, and $A(t)$ is the amplitude, which is slowly changing with t) or some other related wave form. Therefore, it is natural to suppose that investigation of the nonlinear wave resonance must represent an important part of the nonlinear stability theory. The remark above about 'other related wave forms' stressed that in some cases it is convenient to consider only one- or two-dimensional waves, where the three-dimensional 'spatial wave factor' e^{ikx} is replaced by e^{ikx} or $e^{i(k_1x+k_2y)}$, and the vector $\mathbf{B}$ may depend on spatial coordinates not entering the given exponents. [In the spatial formulation of the problem of hydrodynamic stability the amplitude A is assumed to be dependent, not on t but on the spatial coordinate or coordinates (most often, on the streamwise coordinate x). In some cases it is also reasonable to assume that $A = A(\mathbf{x},t)$ is a slowly varying function of both the time and spatial coordinates; see, e.g., the discussion of the corresponding one-mode stability problems in Sec. 4.22 and 4.24(b) and of the three-wave resonances of waves with amplitudes $A(\mathbf{x},t)$ in Craik (1985), Chap. 5]. As to the four-dimensional vector $\mathbf{b} = \{u, v, w, p\}$, in the case of a plane-parallel flow of incompressible fluid it may often be replaced by the two-dimensional vector $\mathbf{w} = \{w, \zeta\}$ (where w is the vertical velocity and $\zeta = \zeta_3$ is the vertical vorticity; see., e.g., Sec. 3.33), while in the case where only two-dimensional wave disturbances are studied it is enough to consider only the scalar stream-function field $\psi = \psi(x,z,t)$ (a similar change of arguments must then be applied also to the vector $\mathbf{B}$). For the sake of simplicity, we will first consider scalar waves of the form $u(\mathbf{x},t) = A(t)Ue^{i(kx - \omega t)}$ (or of the related one- and two-dimensional forms) representing one component of the vector $\mathbf{b}(\mathbf{x},t)$, and only later will pass to more general vector waves. Let $u(\mathbf{x},t) = A(t)Ue^{i(kx - \omega t)}$ be a wave of small enough amplitude which satisfies some nonlinear wave-propagation equation including a nonlinear quadratic term. A very important particular case is that in which $(\mathbf{k}, \omega)$ are the eigenvalues of the corresponding linearized equation (i.e., where ω is the eigenfrequency of the eigenvalue problem corresponding to a given value $\mathbf{k}$ of the wave vector or, if spatial disturbance development in a plane-parallel flow is studied, the streamwise component k_1 of the vector $\mathbf{k} = \{k_1, k_2\}$ is the eigenvalue corresponding to given values of ω and of k_2 or k_2/k_1). In this case the nonlinear equation may be used for approximate evaluation of the effect of nonlinearity on the evolution of an initially very small (practically infinitesimal) wave disturbance. Quadratic nonlinearity entering the equation will produce a term proportional to $exp[2i(\boldsymbol{kx} - \omega t)]$, representing a wave with doubled frequency and wave number. As a rule the values ($2\mathbf{k}$, 2ω) will not be the eigenvalues of the linearized problem; then the nonlinear equation for the amplitude $A(t)$ will be reducible to an equation of Landau's type considered in Sec. 4.2. However, in some exceptional cases both ($\mathbf{k}$, ω) and ($2\mathbf{k}$, 2ω) will be eigenvalues of a linearized problem and here the resonance effect may take place. In fact, in this case a very small wave proportional to $e^{2i(kx - \omega t)}$ may be generated in the flow by background "noise" (including turbulence and not only acoustic waves) of environmental origin and then its interaction with the square of the first wave disturbance produced by quadratic nonlinearity of the wave equation will lead to the resonance growth of disturbance component with doubled frequency and wave number. Thus, in the case of finite disturbances *a two-wave resonance may be possible in a fluid flow if there is an eigenvalue ($\mathbf{k}$, ω) of the linearized equation of motion such that ($2\mathbf{k}$, 2ω) is also an eigenvalue.* [Of course, resonance may also be possible if ($\mathbf{k}$, ω) and ($l\mathbf{k}$, $l\omega$), where l is an integer exceeding 2, are linear eigenvalues. However, this is a higher-order resonance where growth terms are proportional to higher powers of small initial amplitudes and it will be not considered in this book.]

The condition printed above in italics is valid quite infrequently. However, *three-wave resonances* may also occur in fluid flows, and the conditions making such a resonance possible can often be satisfied more easily than conditions for the two-wave resonance. Suppose that ($\mathbf{k}_1$, ω_1) and ($\mathbf{k}_2$, ω_2) are both eigenvalues of the linearized equation, determining the infinitesimal wave modes of a disturbance. Then the waves with wave-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and angular frequencies ω_1 and ω_2 may be simultaneously excited and their interaction (described by the part of the nonlinear term proportional to the product of two waves) will generate waves with wave vectors and frequencies $(\mathbf{k}_3, \omega_3) = (\mathbf{k}_1+\mathbf{k}_2, \omega_1+\omega_2)$ and $(\mathbf{k}_4, \omega_4) = (\mathbf{k}_1-\mathbf{k}_2, \omega_1-\omega_2)$. The arguments which were summarized above for the case of harmonic oscillations in a nonlinear system [and led to Eq. (5.1)] now show that in the case of waves of small (but not infinitesimal) amplitudes satisfying a quadratically nonlinear wave equation, the three-wave nonlinear resonance may occur if together with the waves with wave number and frequency values $(\mathbf{k}_1,\omega_1)$ and $(\mathbf{k}_2,\omega_2)$, a third wave is present which has the same $(\mathbf{x},t)$-periodicity as one of the waves produced by nonlinear interaction of the above-mentioned waves, i.e., if $(\mathbf{k}_3, \omega_3) = (\mathbf{k}_1+\mathbf{k}_2, \omega_1+\omega_2)$ [or $(\mathbf{k}_1-\mathbf{k}_2, \omega_1-\omega_2)$]. Hence here the conditions making the resonance possible may be written in the form

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \tag{5.2}$$

where it is assumed that signs of the frequencies and wave vectors can be chosen arbitrarily [if this assumption is not accepted, then Eq. (4.2) must be written in the form

$$\omega_1 \pm \omega_2 = \omega_3, \quad \mathbf{k}_1 \pm \mathbf{k}_2 = \mathrm{k}_3; \tag{5.2a}$$

see, e.g., Phillips (1960, 1974a,b)]. Phillips assumed that the three waves considered have small amplitudes $A_i(t)$, $i = 1, 2, 3$, of the same order of magnitude, and, substituting the sum of three waves into the nonlinear wave-propagation equation (whose form depends upon the nature of the waves considered), he obtained for the case of three real (sinusoidal) steady waves satisfying the conditions

$$\omega_1 + \omega_2 = \omega_3, \qquad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \tag{5.2b}$$

the following approximate equations determining, to the order of the squares of the initial amplitudes, the rates of change of the three amplitudes:

$$\frac{dA_1}{dt} = C_1 A_2 A_3, \quad \frac{dA_2}{dt} = C_2 A_3 A_1, \quad \frac{dA_3}{dt} = C_3 A_1 A_2, \tag{5.3}$$

where C_1, C_2 and C_3 are three interaction coefficients dependent on the particular wave motion considered, and on the wavenumbers involved and their geometrical configuration. [For more details see the paper by Bretherton (1964), the book by Craik (1985) specially devoted to wave interactions in fluid flows, and the papers and books cited below in this section, many of which contain rigorous derivations of these equations for a number of particular cases.] Craik considered the complex wave disturbances in a plane-parallel fluid flow where the amplitudes A_1, A_2 and A_3, and also the frequencies ω_1, ω_2 and ω_3, may be complex (as they are, e. g., in the case of unsteady T-S waves). Here the first condition (5.2b) takes the form $\Re e(\omega_1 + \omega_2) = \Re e\,\omega_3$ (while the second does not change), and Eqs. (5.3) take the form:

$$\frac{dA_1}{dt} = \omega_1^{(i)} A_1 + C_1 A_2^* A_3, \quad \frac{dA_2}{dt} = \omega_2^{(i)} A_2 + C_2 A_1^* A_3, \quad \frac{dA_3}{dt} = \omega_3^{(i)} A_3 + C_3 A_1 A_2, \tag{5.4}$$

where $\omega_n^{(i)} = \Im\mathrm{m}\,\omega_n$, $n = 1,2,3$, and the asterisk denotes a complex conjugate. In the case of a spatial formulation of the parallel-flow stability problem the frequencies ω_1, ω_2 and ω_3 take real values but the streamwise components, k_{11}, k_{21} and k_{31}, of the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ may be complex. Therefore in this case the first condition (5.2b) remains unchanged, while the second condition must be replaced by equation $\Re\mathrm{e}(\mathbf{k}_1 + \mathbf{k}_2) = \Re\mathrm{e}\,\mathbf{k}_3$ (where the symbol $\Re\mathrm{e}$ relates only to the streamwise components of the wave vectors), and Eqs. (5.4) must be written as

$$\frac{dA_1}{dx} = -k_{11}^{(i)} A_1 + B_1 A_2^* A_3, \quad \frac{dA_2}{dx} = -k_{21}^{(i)} A_2 + B_2 A_1^* A_3, \quad \frac{dA_3}{dx} = -k_{31}^{(i)} A_3 + B_3 A_1 A_2, \tag{5.4a}$$

where $k_{n1}^{(i)} = \Im\mathrm{m}\,k_{n1}$, $n = 1,2,3$, and the B_n are new interaction coefficients. Note also that, as in the case of complex frequencies, the superscript (r) in the symbols for the real parts of streamwise wave vectors representing exponentially growing or decaying waves will often be omitted, and the real parts will be denoted by symbol k_1 (or k, if the wave is two-dimensional).

Eqs. (5.4) and (5.4a) represent the nonlinear approximation of lowest order in the weakly-nonlinear stability theory. In the approximation of the next order with respect to wave amplitudes, additional third-order terms appear on the right-hand sides of Eqs. (5.4) and (5.4a); see e.g. Eqs. (5.11) below, the papers by Usher and Craik (1975) and Goncharov (1981), and the book by Craik (1985), Secs. 16.3 and 25-26. The quadratic terms of amplitude equations characterize only resonant modes, while for the more usual nonresonant modes cubic terms follow the linear ones; see, e.g., Landau-Stuart's Eq. (4.40) and Stuart's Eqs. (4.43). Presence of quadratic terms clearly means that for resonant modes nonlinearity begins to be important 'sooner' (i.e., at smaller amplitudes) than for nonresonant modes.

The computation of the interaction coefficients C_1, C_2 and C_3 is a complicated problem, for which a number of special methods (applicable to one or other particular wave-interaction situation) have been developed (see e.g. the important early paper by Simmons (1969) and the discussion of this topic in Craik's book). The problem is significantly complicated by the fact that real physical media are very often *dispersive*. This means that the wavenumber $\mathbf{k}$ and the frequency ω of waves in the medium cannot be chosen arbitrarily but must satisfy a definite *dispersion relation*

$$\omega = D(\mathbf{k}), \tag{5.5}$$

where the function $D(\mathbf{k})$ (one- or many-valued) may depend on physical parameters affecting wave propagation and on the dimensionless characteristics of the corresponding physical regime [e.g., on the Reynolds number; see e.g. Eq. (2.90) in Chap. 2 and, for more details, Karpman's monograph (1975)]. Therefore in many cases it is not easy to find triads of wave vectors $\mathbf{k}_i$ and frequencies ω_i satisfying both Eqs. (5.2b) (or a related equation differing by the signs of some terms and/or by replacement of ω_i by $\Re e\omega_i$) and (5.5), since such triads (if they exist) represent only some infrequent exceptions. In particular, Phillips (1960, 1961) who was one of the first to look for nonlinear wave resonances in fluid flows, studied inviscid gravity waves in deep water (where $\mathbf{k}$ is a two-dimensional vector and the dispersion relation has the form $\omega^2 = g|\mathbf{k}|$, g is the acceleration due to gravity). He found that here the condition (5.2b) cannot be fulfilled at all (note that such a dispersion relation evidently prevents two-wave resonance also). Hence Phillips was forced to pass to four-wave resonances of quadruples of waves satisfying the conditions $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$, $\omega_1 + \omega_2 = \omega_3 + \omega_4$. He found that such quadruples of gravity waves really exist, and hence the four-wave resonances may occur here and produce unsteadiness of the gravity waves. Then he determined the corresponding amplitude equations which, in the case of four-wave resonance contain, in the lowest-order nontrivial approximation, terms of third order in wave amplitudes on the right-hand sides; see also Craik (1985), Secs. 8 and 23. In some plasma and geophysical wave problems the five-wave resonances produced by a coupled pair of resonant wave triads having one member in common are also of importance; such resonances are discussed in Craik (1985), Sec. 16.2, while for examples of resonances of this type appearing in fluid flows see, e.g., the small-type text in the final part of Sec. 5.3 in this chapter). As to three-wave resonances, they may be important in many physical situations (the case of gravity waves on a deep-water surface may be considered as an exception) and therefore the literature devoted to study of such resonances is quite extensive. In particular, it was found by McGoldrick (1965) that such resonances may occur in the case of capillary-gravity waves and ripples on the water surface (which are affected by both the gravity and the surface tension and have a dispersion relation of the form $\omega^2 = g|\mathbf{k}| + \gamma|\mathbf{k}|^3/\rho$, where γ is the coefficient of surface tension and ρ is the density of water). In the case of gravity waves in a heavy liquid beneath a solid plate, the surface tension plays no role but here the waves are affected by the elastic properties of the plate, and this leads to a dispersion relation which again makes three-wave resonance possible; see e.g. Marchenko (1991, 1999). It was also found that three-wave resonances may occur among effectively-inviscid internal gravity waves in stratified flows with density depending on the vertical coordinate; among various large-scale geophysical waves (e.g. those depending on Earth's angular velocity, such as Rossby waves in the troposphere and plasma waves in the ionosphere at much greater heights); and among many other types of interacting nonlinear waves. Note also that the three-wave resonance may be realized in triads of waves of different types (in particular, in triads consisting of two gravity waves on the surface of a stratified liquid and one internal wave in the same liquid, or of two short capillary-gravity waves and one longer purely-gravity wave unaffected by surface tension). A number of theoretical and experimental studies of three-wave resonances in fluid flows may be found in papers by McGoldrick (1965, 1970a,b, 1972), McGoldrick *et al.* (1966), Phillips (1966, 1967, 1974a,b, 1977, 1981), Longuet-Higgins and Smith (1966), Longuet-Higgins and Gill (1967), Craik (1968), Nayfeh (1971), Brekhovskikh *et al.* (1972), Loesch (1974), Ripa (1981), Bannerjee and Korpel (1982), Yuen and Lake (1982), Hogan (1984), Henderson and Hammack (1987), Perlin *et al.* (1990), Christodoulides and Dias (1994), Trulsen and Mei (1996) and many others. [These publications and the book by Craik (1985) also contain many supplementary references relating to this topic.] Since nonlinear dispersive waves may occur in quite different media and situations, the nonlinear wave-resonance theory has many applications to problems outside classical fluid mechanics; in such cases the theory has often been developed independently of studies of waves in ordinary fluids. As typical examples of publications dealing with three-wave resonances relating to waves of other origins, we may mention the papers by Jurkus and Robson (1960) on nonlinear electronics, by Khokhlov (1961) on electromagnetic wave propagation in dispersive conductors, and by Dimant (2000) on nonlinear interactions among ionospheric waves; the books and papers by Armstrong *et al.* (1962), Bloembergen (1965, 1968) and Akhmanov and Khokhlov (1972) on nonlinear optics, by Davidson (1972), Weiland and Wilhelmsson (1977) and Turner and Boyd (1978) on plasma waves; and the general survey by Kaup *et al.* (1979) [containing an extensive bibliography and supplemented by Kaup's paper (1981)]. However, in the framework of the present purposes only waves in an incompressible Navier-Stokes fluid are of interest, and in this chapter only the case of waves in nearly plane-parallel Blasius boundary-layer-flows will be investigated.

In almost all the papers and books cited above which deal with wave resonances in fluid mechanics, waves in immovable fluids (where there is no basic flow) were considered. In this case the total energy of any group of waves interacting with each other must be conserved. [This means, in particular, that if the wave energy $E \propto |A|^2$ is always positive, the coefficients C_i, $i = 1, 2, 3$, of Eqs. (5.3) cannot all have the same sign. More complicated cases, in which the excitation of waves lowers the total energy of the system so that the wave energy must be considered as negative, are often encountered in plasma physics, and have also been considered in application to fluid mechanics, e.g. by Cairns (1979), Craik and Adam (1979), and Craik (1985), Secs. 2.3 and 14.3; however, they will not be discussed in this book.] Energy conservation implies that the growth of one wave may be achieved only as a result of energy exchange between various waves, leading to energy redistribution and the attenuation of some other wave (or waves). Such energy redistribution changes the wave amplitudes (and also the wave shapes, which become distorted by the growth of supplementary waves extracting energy from the primary one) producing unsteadiness and hence making the waves unstable. Emerging unsteadiness of waves is often of oscillatory type [in which the energy of one of the waves decreases for a time because of transfer to other waves but then begins to grow anew when the energy transfer changes sign; see e.g. Fig. VII.3 in Phillips (1974b)]. Such unsteadiness clearly represents an interesting physical phenomenon which differs strongly from the monotonic growth of disturbance amplitudes leading to transition of laminar flows to turbulence. If the viscosity of the fluid cannot be neglected, then the redistribution of energy between interacting waves will be accompanied by their viscous decay, but here too the growth of one wave amplitude must necessarily be accompanied by simultaneous attenuation of others.

The difference between the wave instabilities observed in immovable fluids and the flow instabilities studied in Chaps. 2-4 above is due first of all to the fact that in these chapters only instabilities of steady laminar *shear flows* [most often of plane-parallel flows with nonuniform velocity distributions $U(z)$] were considered. In such flows the most important mechanism of disturbance growth is connected with the transfer of energy from the primary flow to the disturbance (reverse energy transfer is also possible in principle but it appears much more rarely). This mechanism plays the leading role in the majority of cases of transition to turbulence, and also in all the flow instabilities studied in Chaps. 2-4 (see in this respect Sec. 4.1 on the energetics of instability phenomena). Therefore it is natural to suppose that the same mechanism may have an essential effect on the resonant growth of wave disturbances in steady shear flows, and thus lead to some new interesting and important instability phenomena.

Apparently Raetz (1959) [see also the discussion of this work by Stuart (1962a) and in Raetz's survey (1964)] was the first to suggest that three-wave nonlinear resonances may play an essential part in the transition to turbulence of a laminar boundary layer with, say, the Blasius velocity profile $U(z)$. Shortly after this Benney and Niell (1962) expressed doubts about the possibility of a nonlinear resonance leading to a large growth of some wave amplitudes; however, later their doubt was found to be groundless [and the importance of nonlinear resonance was then stressed, in particular, by Benney and Gustavsson (1981)]. As will be shown below, the main idea of Raetz proved to be correct and very important; for this reason Raetz's unpublished report of 1959 stimulated the appearance of a great number of papers further developing this idea. This matter will be discussed at greater length in the next section; first, however, the results of two relatively old (but quite typical) papers relating to some special cases of nonlinear resonance of waves in shear flows will be briefly considered, as illustrations of the general tendency of nonlinear-resonance studies.

Kelly's paper (1968) was devoted to the search for resonant interactions of waves in two particular plane-parallel inviscid shear flows - in a Bickley jet, where $U(z) = U_0\text{sech}^2(z/H)$ in $-\infty < z < \infty$, and in a stably stratified plane mixing layer with the velocity profile $U(z) = U_0\tanh(z/H)]$ and the density profile $\rho(z) = \rho_0\exp[-\beta\tanh^3(z/H)]$. In this paper only two-wave resonances, involving pairs of neutrally-stable two-dimensional waves, were considered. For waves proportional to $\exp\{i(k_jx-\omega_jt)\}$, $j = 1, 2$ (where k_j and ω_j are real and positive), nonlinear resonance is possible if $k_2 = 2k_1$, $\omega_2 = 2\omega_1$; it was found that to identify waves satisfying these conditions, the data collected by Drazin and Howard in their survey (1966) of stability of parallel flows in inviscid fluids are very useful.

According to Drazin and Howard, a Bickley jet can support a pair of neutral two-dimensional waves with the following wave numbers and frequencies: $k_1 = 1/H$, $\omega_1 = (2/3)U_0/H$, and $k_2 = 2/H$; $\omega_2 = (4/3)U_0/H$ (the stream-function vertical profiles $\psi_1(z)$ and $\psi_2(z)$ of these waves were also given by Drazin and Howard). Thus, these two waves may interact resonantly. As for the stratified mixing layer, Miles (1963) showed that there can exist an infinite number of two-dimensional neutral modes depending on the value of the overall Richardson number Ri*= $g\beta H/(U_o)^2$ (which characterizes the flow stability). Results given by Drazin and Howard show that at Ri* = 4/3 the resonance conditions $k_2 = 2k_1$, $\omega_2 = 2\omega_1$ are satisfied for the first two modes; while for Ri* > 4/3 other resonant cases may occur which also involve higher modes (in particular, at Ri* = 10/3 a three-wave resonance may occur among the first three modes). Kelly studied the interactions of these pairs of two-dimensional waves in the Bickley jet and in the stratified mixing layer, and found that, at least in the stably stratified mixing layer at Ri* = 4/3, two-wave resonance can occur, leading to the rapid temporal growth of a wave disturbance with a fixed spatial periodicity. This continuous growth shows that in this case the nonlinear interaction of the waves with each other and with the primary flow leads to transfer of primary-flow energy to the waves.

Slightly later Craik (1968) examined the possibility of resonant gravity-wave interactions in a horizontal liquid layer with the linear velocity profile $U(z) = -u_1 z$, $0 \geq z > -\infty$. (The condition that $|U(z)| \to \infty$ as $z \to -\infty$ is not essential here, because the gravity-wave motions involve only a thin upper layer of liquid.) It was indicated above that the two-wave and three-wave resonant interactions cannot occur among gravity waves in a liquid at rest, while such interactions among quadruples of waves can occur here but cannot produce continuous growth of any of the waves. Craik found that in a shear flow with a linear velocity profile, two-wave and three-wave resonant interactions are also impossible among two-dimensional gravity waves, but three-wave resonant interactions are now possible among two- and three-dimensional gravity waves. He did not try to examine all possible resonant triads of such gravity waves but limited himself to consideration of triads comprising one two-dimensional wave proportional to $\exp[i(kx-\omega t)]$), and two symmetric oblique waves which are proportional to $\exp[i(k_1 x \pm k_2 y - \omega_1 t)]$ and thus have inclination angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$, with the same absolute value but opposite signs. The frequencies ω and ω_1 and the wave numbers k, k_1 and k_2 were assumed to be real, i.e., all the waves considered were neutrally stable. However, these frequencies and wave vectors could not take arbitrary real values, but had to satisfy a definite dispersion relation. Craik showed that in the homogeneous shear flow with constant shear u_1, the frequency ω and wave vector $\mathbf{k} = (k_1, k_2)$ of a gravity wave must satisfy a dispersion relation of the form

$$\omega^2 - (k_1 u_1/|\mathbf{k}|)\omega = g|\mathbf{k}|, \quad |\mathbf{k}| = (k_1^2 + k_2^2)^{1/2}. \tag{5.6}$$

Relation (5.6) allows us to examine conditions under which a triad of waves with wave numbers and frequencies $(\mathbf{k}_1, \omega_1)$, $(\mathbf{k}_2, \omega_1)$ and $(\mathbf{k}_3, \omega)$, where $\mathbf{k}_1 = (k_1, k_2)$, $\mathbf{k}_2 = (k_1, -k_2)$ and $\mathbf{k}_3 = (k, 0)$, may satisfy conditions (5.2b). Since $\omega(k_1,k_2) = \omega(k_1,-k_2)$ by virtue of Eq. (5.6), these conditions now take the very simple form:

$$k_1 = k/2, \quad \omega_1 = \omega/2. \tag{5.7}$$

Thus the values of k, k_2 and ω must be chosen so that Eq. (5.6) will be valid for the following two wavevector-frequency combinations: i) $(k, 0, \omega)$ and ii) $(k/2, k_2, \omega/2)$. Assuming, without loss of generality, that k and ω are positive, Craik showed that such values of k, k_2 and ω exist only under the condition that u_1 is also positive and so large that

$$u_1/(gk)^{1/2} \geq [7 + (48)^{1/2}]/[8 + (48)^{1/2}]^{1/2} \approx 3.60. \tag{5.8}$$

This means that three-wave resonant interactions, which are completely impossible for gravity waves in a motionless liquid, may be possible for such waves in a homogeneous shear flow for wave triads of special form, but only in the cases where the shear u_1 is positive and large enough. Craik also showed that, under condition (5.8), to every permissible value of $u_1/(gk)^{1/2}$ there correspond two permissible values of $k_2 > 0$ and of the angle $\theta = \tan^{-1}(2k_2/k)$. Moreover, here the two values of θ coincide with each other and are close to 74° when $u_1/(gk)^{1/2}$ takes its minimum permissible value (close to 3.6), while when the value of $u_1/(gk)^{1/2}$ increases one of the two values of θ is continuously growing and the other is continuously decreasing, tending to values 90° and 60° as $u_1/(gk)^{1/2} \to \infty$.

The subsequent part of Craik's paper is devoted to a lengthy approximate evaluation of the growth rates for triads of interacting plane waves satisfying the resonant conditions (5.2b). Assuming that the viscosity ν is very small and that the initial complex amplitudes $A_1(0)$, $A_2(0)$ and $A_3(0)$ of the three surface gravity waves considered have small (but not infinitesimal) absolute values, Craik derived, under rather general conditions, a system of equations for the functions $A_1(t)$, $A_2(t)$ and $A_3(t)$, of the form (5.4). Here the first terms on the right-hand sides (which describe the viscous decay of the waves) can usually be neglected, while for the leading terms of the expressions for the coefficients C_i, $i = 1, 2, 3$, the following order-of-magnitude estimates were obtained: $C_1 = O(\omega^2/k\nu)$, $C_2 = O(\omega^2/k\nu)$, but $C_3 = O(\omega k)$. It is remarkable that at small values of ν (i.e. large Re) the coefficients C_1 and C_2, determining the growth rates of amplitudes of the two oblique waves, turn out to be much greater than the coefficient C_3. Hence here the oblique waves grow very rapidly, while the amplitude of the two-dimensional wave changes much more slowly. This shows that in this case a very strong resonant interaction of the three waves takes place, and the oblique waves effectively extract energy from the primary flow while the amplitude of the two-dimensional wave changes only a little. Of course, since these estimates of the wave growth rates were based on the assumption that all the amplitudes are small, the estimates are valid only during a restricted time interval.

In conclusion Craik briefly considered also the resonant-interaction problem for interfacial gravity waves in a two-layer flow where for $z > 0$ and $z < 0$ the liquid has different densities ρ_1 and $\rho_2 > \rho_1$ and the flow has constant but different velocity gradients $dU/dz = u_1$ and u_2. He found that here the condition for three-wave resonance can often be satisfied by much smaller values of the velocity gradients than those given by Eq. (5.8). However we will not linger on this special problem in this book.

5.2. RESONANCE AND SECONDARY-INSTABILITY MANIFESTATION IN BOUNDARY-LAYER DEVELOPMENT

The title of this chapter and the short introduction indicated that the chapter will be devoted to weakly-nonlinear mechanisms of instability development in a steady laminar boundary layer in zero pressure gradient (a "flat plate" boundary layer). Consideration of this flow alone was justified by the prevalence and great practical importance of boundary-layer flows in nature and in industry. It was also noted that properties of the laminar flat-plate boundary-layer flow (often called 'the Blasius flow' since its velocity profile was computed by Blasius as long ago as 1908; cf. Chap. 2, p. 110) have been intensively studied by both theorists and experimenters during many years. These studies led to many interesting results which, unfortunately, have not solved all the problems relating to boundary-layer flow instability and transition to turbulence, but nevertheless have considerably clarified the situation and had a great effect on the whole weakly-nonlinear theory of hydrodynamic instability.

The main topics of the present chapter are the nonlinear resonance among three wave-like disturbances with small amplitudes of the same order of magnitude appearing in the boundary-layer flow, and the secondary instabilities of two-dimensional waves of small but finite amplitudes with respect to wave disturbances of other types. However, some other multimode weakly-nonlinear theories of hydrodynamic instability will also be briefly considered. Let us stress again that although enormous amounts of research effort were devoted during the whole 20th century to the study of instability and transition to turbulence of flat-plate boundary layers, our understanding of these processes is still far from being complete. This statement repeats the remark made more than thirty years ago by Tani (1969), and the intervening years have not shaken its correctness. One of the first puzzles relating to instability was produced by the discovery in the classical experiments on boundary-layer stability by Schubauer and Klebanoff (1956) [see also Schubauer (1958)], Klebanoff and Tidstrom (1959) and Klebanoff *et al.* (1962) of the fact that the development in a boundary layer of an initially small two-dimensional disturbance always leads to the appearance slightly later of some fast-growing, spanwise-periodic three-dimensional structures. [This fact was later confirmed by many other authors; see, for example, the papers by Tani (1967) and Komoda (1967) which preceded a great number of more detailed experiments and numerical simulations, some of which will be discussed below.] The streamwise development of these structures was thoroughly studied by Klebanoff *et al.* (and then also by Tani, Komoda, and many others). All the above-mentioned authors followed Schubauer and Skramstad (1947) in using a vibrating ribbon to excite waves in the boundary layer. However

Klebanoff *et al.*, and then Tani, Komoda, and some others, modified this method by inserting spacers (typically small pieces of adhesive tape) at regular intervals beneath the ribbon, thus producing a weak spanwise periodicity of the disturbance. The spanwise wavelength then depended on the distance between adjacent pieces of tape, and hence could be varied; in the above-mentioned experiments it was always chosen to be equal to that appearing in experiments without any spanwise forcing. Therefore here the usual 3D periodicity was present at the beginning of the studied flow.

The appearance of flow three-dimensionality evidently contradicted the known results of linear stability theory (see, e.g., Sec. 2.81) according to which the most unstable small disturbances in any plane-parallel flow of viscous fluid have the form of two-dimensional waves independent of the spanwise coordinate y. This contradiction could evidently be explained only by some nonlinear effects which were neglected in the linear theory. However, to develop such an explanation it was necessary to go beyond the Landau approach where only the evolution of disturbance amplitude, but not the change of its shape, was considered.

Benney and Lin (1960) and Benney (1961, 1964) were among the first who attempted to explain theoretically the growth of three-dimensionalities in disturbed plane-parallel flows. For this purpose they applied second-order weakly-nonlinear theory (which preserves only terms of the first two orders in disturbance amplitudes) to the simultaneous development, in a plane-parallel shear flow, of two rather small disturbances: a two-dimensional (2D) wave proportional to $\exp(ikx-i\omega t)$ and a three-dimensional (3D) wave proportional to $\exp(ikx-i\omega_1 t)\cos(k_1 y)$ where ω and ω_1 are complex parameters having the same real parts, $\omega_0 = \Re e\omega = \Re e\omega_1$. (In the first two papers the case of a flow in an unbounded space with a hyperbolic-tangent velocity profile was considered, while in the third paper the simplified model of a plane-parallel boundary-layer flow having the piecewise-linear velocity profile shown in Fig. 3.1a was used as the primary flow.) Although the velocity profiles studied differed from the real boundary-layer profile, some of the features of the predicted wave developments recalled phenomena observed in the boundary-layer stability experiments. However, the agreement with experimental data was only qualitative and quite incomplete, and the subsequent attempt by Nakaya (1980) to repeat the calculations using the Blasius boundary-layer velocity profile instead of some simplified model of it did not lead to more satisfactory results. Moreover, Stuart (1962a) noted that the assumption used in the above-mentioned papers, that 2D and 3D waves have the same (real) frequency, contradicts the available experimental data, and Craik (1971) stressed that in these papers the spanwise wavelength was chosen quite arbitrarily while experiments show that it has a definite preferred value. [In fact, Klebanoff *et al.* and then also Anders and Blackwelder (1980) and some other early experimenters found that this wavelength always takes the same value; however later it was shown that this statements is incorrect.] Antar and Collins (1975) relaxed Benney and Lin's, and Benney's, assumptions and accepted that $\Delta\omega_0 = \Re e\,\omega - \Re e\,\omega_1$ may differ from zero (and then used in their computations, relating to both Blasius and Falkner-Skan boundary-layer velocity profiles, values of $\Delta\omega_0$ given by numerical-simulation results). Then Nelson and Craik (1977) considered another relaxation of Benney and Lin's and Benney's models, assuming that $\Delta\omega_0 = 0$ but accepting that the streamwise wave numbers of 2D and 3D waves may take different values, while Herbert and Morkovin (1980) studied the superposition of a 2D wave, with the wave vector $\mathbf{k}_1 = (k_1, 0)$, and a spanwise wave with wave vector $\mathbf{k}_2 = (0, k_2)$. These generalizations of the previous models, which will not be considered at length here, yielded a somewhat better (but not too good) agreement with the experimental data available in the 1970s [see Craik's paper (1980) especially devoted to comparison of various theoretical models with the experimental data of Klebanoff *et al.* (1962) and Kovasznay *et al.* (1962)].

Stuart (1962a,b) supplemented his criticism of the Benney and Lin model by a sketch of a somewhat different approach to the study of development of three-dimensionality in plane-parallel flows. Namely, he applied a weakly-nonlinear analysis of Landau's type to the time evolution of a pair of interacting small wave disturbances (one 2D and the other 3D) having finite real amplitudes, arbitrary complex frequencies ω_1, ω_2 and real wave vectors $\mathbf{k}_1 = (k_1, 0)$ and $\mathbf{k}_2 = (k_1, k_2)$. Neglecting terms of higher than third order with respect to wave amplitudes, Stuart showed that here Landau's equation (4.34) is replaced by the system (4.43) of two coupled nonlinear equations for the amplitudes $A_1(t)$ and $A_2(t)$ of the two waves. However he did not try to compute the coefficients of these equations in preparation for solving these equations for any particular plane-parallel flow. Instead, he confined himself to a description of the equilibrium finite-amplitude solutions of these equations, and discussion of the stability of the resulting equilibrium states (see Sec. 4.21 above). Later Itoh (1980) carried out an approximate evaluation of the coefficients of Eqs. (4.43) for a plane Poiseuille flow with Re = HU_0/ν varying in the range from 4000 to 8000, which covers both subcritical and supercritical conditions (here H and U_0 are the half-distance between the walls and the maximum velocity of the undisturbed flow). It was also assumed by Itoh that $k_1H = 1$ while k_2H takes a number of values not exceeding 1. Moreover, the popular assumption that the contribution of the least stable eigenmode of the linearized equations of motion must dominate the eigenfunction expansions of both 2D and 3D nonlinear waves was also accepted and used to simplify the computations. The coefficients of Eqs. (4.43) were found to be dependent on the phase difference of the two waves considered, but this dependence could be eliminated by averaging the solutions over the period of 'fast oscillations' produced by the difference of the primary frequencies of 2D and 3D waves. Using such averaging Itoh found some conditions for stability of a two-dimensional wave, superimposed on a plane Poiseuille flow, to three-dimensional wave disturbances, and he estimated the threshold value of the 2D-wave amplitude A_1 above which the three-dimensional waves are continuously growing. Using numerical values for the coefficients in the equation, the equilibrium solutions (4.44) for Poiseuille flows could be found and their stability characteristics determined. Even earlier, these stability characteristics were studied by Volodin and Zel'man (1977) for the simpler case of two interacting two-dimensional T-S waves in the Blasius boundary-layer flow at supercritical values of Re.

Let us recall that in the 1960s, when Benney and Lin began to study the time evolution of two-mode disturbances in plane-parallel flows, they assumed that the streamwise wave numbers of a two-dimensional and a three-dimensional wave have the same value of k. This assumption [which was later rejected by Nelson and Craik (1977)] evidently excluded the possibility of including the 2D and 3D waves in a resonant triad of wave disturbances. However, even before the appearance of these studies, Raetz (1959) stated that according to his computations for the plane-parallel model of a Blasius boundary layer, there exist some triads of three-dimensional wave disturbances which satisfy the conditions (5.2b). These conditions imply that the corresponding waves may interact resonantly, producing rapid growth of wave amplitudes, and Raetz suggested that such resonant instability of wave triads may play an important part in the transition of boundary layers to turbulence.

Raetz considered only neutral 3D waves corresponding to real values of the eigenfrequency ω. Because his report of 1959 did not contain a complete description of the computations, Stuart, preparing his survey lecture (1962a), computed anew one more resonant triad for a Blasius boundary layer at a supercritical value of Re (this triad also consisted of three-dimensional neutrally-stable waves; according to Raetz resonant triads do not exist among purely two-dimensional waves). However, neither Raetz nor Stuart computed the corresponding growth rates [determined by the values of three coefficients C_i in equations (5.3)]. This was, of course, a necessary step; recall that in general conditions (5.2b) are necessary for the resonant character of three-wave interactions, but do not guarantee that resonance will actually occur in all cases where these conditions are valid.

A much more detailed study of resonant three-wave interactions in boundary-layer flows was carried out by Craik (1971). He noted that Raetz's and Stuart's limitation to only neutrally-stable waves strongly restricts the class of resonant wave triads to be studied, since such triads can in principle also exist among both subcritical and supercritical waves. [In the case of non-neutral waves the real frequencies ω_i must of course be replaced, in the first condition (5.2b), by the real parts $\Re e\,\omega_i$ of the corresponding complex eigenvalues $\omega_i = c_i/k_I$ of the Orr-Sommerfeld equation (2.41), but no limitations are imposed by these conditions on their imaginary parts.] However Craik did not consider the general case of arbitrary triads of three-dimensional Tollmien-Schlichting (T-S) waves with any wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. As in his paper of 1968 on gravity waves, discussed at the end of the previous subsection, he examined only special triads comprising one 2D wave propagating in the streamwise direction (proportional to $\exp[i(kx - \omega t)]$) and two 3D waves proportional to $\exp[i(k_1 x \pm k_2 y - \omega_1 t)]$ (i.e., inclined at equal but opposite angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ to the flow direction). Hence he assumed that $\mathbf{k}_1 = (k, 0)$, $\mathbf{k}_2 = (k_1, k_2)$ and $\mathbf{k}_3 = (k_1, -k_2)$ and thus the resonance conditions (5.2b) took the very simple form $k_1 = k/2$, $\Re e\,\omega_1 = \Re e\,\omega/2$ [cf. (5.7)]. The reason for giving much attention to such special triads was connected with the fact that here the three waves have the same phase velocity $c = \Re e\,\omega/k = \Re e\,\omega_1/k_1$ and hence there is only one critical layer, at the height z_0 where $U(z_0) = c$. Since it is known that the most intensive interaction of a small wave with the primary steady flow takes place in the vicinity of the critical layer [see, e.g., the discussion of the role of a critical layer in nonlinear resonant interactions by Mankbadi (1990, 1991, 1994), Mankbadi *et al.* (1993), and Goldstein (1995)], it is natural to expect that in the case where three interacting waves have the same critical layer these waves may extract energy from the primary flow in a particularly powerful manner. This expectation was confirmed by Craik (1968) for the case of gravity waves on the surface of a liquid shear layer; the results of his paper of 1971 (see below) also agreed with the stated expectation rather well. (A small correction of the conclusion that in this case the oblique-wave growth must take a maximal value was discovered later, and will be discussed in Sec. 5.4; see, in particular, Figs. 5.13-5.15 there and the text relating to these figures.)

In the case of the T-S waves in a plane-parallel primary flow the search for possible resonant triads may be facilitated considerably by the use of the Squire theorem presented in Sec. 2.81. According to this theorem $\omega_j(k_1,k_2,\mathrm{Re}) = \omega_j(|\mathbf{k}|,0,k_1\mathrm{Re}/|\mathbf{k}|)$, where $\omega_j(k_1,k_2,\mathrm{Re}) = c_j(k_1,k_2,\mathrm{Re})/k_1$ is the jth eigenfrequency of the general O-S equation (2.41) corresponding to the wave vector $\mathbf{k} = (k_1,k_2)$ and Reynolds number Re, while $|\mathbf{k}| = (k_1{}^2+k_2{}^2)^{1/2}$. Thus here we need only the eigenvalues $\omega_j(k, 0, \mathrm{Re})$ of the two-dimensional O-S equation (2.44) for various values of k and Re. One convenient method for determination of such resonant triads in the case of a plane-parallel flow with a given value of Re begins with the plotting of the curves $\Re e\,\omega(k_1,k_2)/k_1 \equiv c_r(k_1,k_2) =$ const. and $\Im m\,\omega(k_1,k_2)/k_1 \equiv c_i(k_1,k_2) =$ const. [where $c(k_1,k_2)$ is the eigenvalue of the O-S eigenvalue problem (2.41)-(2.42) with the greatest imaginary part] in the (k_1,k_2)-plane (the Squire theorem is, of course, very useful here). Then one may select an arbitrary point $(k, 0)$ on the k_1-axis, plot a curve $c_r =$ const. passing through this point, and then determine the two intersections (symmetric with respect to the k_1 axis) of the line $k_1 = k/2$ with the plotted curve. These intersections determine the values k_2 and $-k_2$ such that the oblique waves with wave vectors $\mathbf{k}_2 = (k_1,k_2)$ and $\mathbf{k}_3 = (k_1,-k_2)$ together with the 2D wave with the wave number $\mathbf{k}_1 = (k, 0)$ make up a resonant triad. Therefore, normally for any wave number k the resonant triad may be found which consists of a 2D wave with the wave vector $\mathbf{k}_1 = (k, 0)$ and two symmetric oblique waves with wave vectors $\mathbf{k}_2 = (k/2,k_2)$ and $\mathbf{k}_3 = (k/2,-k_2)$.

Craik (1971) applied this method to find a number of resonant triads for two-dimensional Blasius boundary layers with various values of $\mathrm{Re} \equiv \mathrm{Re}^* = U_0\delta^*/\nu$. (Here U_0 is the free-stream velocity and δ^* is the displacement thickness. Below these velocity and length scales will be used to make dimensionless all physical parameters relating to the Blasius boundary layer; therefore, the symbol k will now denote the dimensionless combination $k\delta^*$, the symbol ω the combination $\omega\delta^*/U_0$ and so on.) Fig. 5.1a shows one of Craik's examples, for Re* = 882. Here two resonant triads are denoted by dotted arrows; the first of them with $k = 0.254$ includes the linearly-most-unstable 2D wave as its first component while its second and third components are linearly stable, and the second triad with $k =$ 0.46 includes the linearly-stable 2D wave but linearly-unstable oblique waves. Later F. Hendriks [see his Appendix at the end of Usher and Craik's paper (1975)] extended Craik's calculations to the six additional resonant triads (with $0.1 \le k \le 0.5$) in the Blasius boundary layer with Re* = 882; some comparisons of the results of Craik's and Hendriks' calculations with experimental data were made by Craik (1980). A number of other examples of resonant triads of the same type in Blasius boundary layers with various values of Re (mostly supercritical, i.e. exceeding Re_{cr}) and k may be found, in particular, in Volodin and Zel'man's paper (1978) and the book by Schmid and Henningson (2000). As an example, two triads computed 1/ by Schmid and Henningson for Re* = 750 are presented in Fig. 5.1b; Подп. к 5.1а here the first triad (where $k \approx 0.37$) comprises three waves which have close to zero negative growth rates according to linear stability theory, while all members of the second triad (where $k \approx 0.18$) are essentially stable (have appreciably negative rates of growth). In Figs. 5.2a,b, also taken from Schmid and Henningson's book, the Подп. к Figs. linear stability properties of all resonant triads of Craik's type with 5.2а,б $k \le 0.5$ are shown for two Reynolds numbers, Re* = 600 and 1000. One may see that at Re* = 600 there is a range of wavenumbers k where the 2D component of a resonant triad is linearly unstable, while the 3D components are linearly stable at any k; on the other hand, at Re* = 1000 both 2D and 3D component may be simultaneously unstable. Note, however, that for linearly-stable oblique T-S waves entering a resonant triad, the rates of their resonant growth usually greatly exceed the rates of their decay given by the linear theory of stability. Therefore the linear stability of such waves in fact plays no part here.

An essential part of Craik's paper of 1971 was devoted to approximate evaluation of the coefficients C_1, C_2 and C_3 of equations (5.4) for the amplitudes of three resonant waves. It was based on the use of nonlinear equations for the velocity components of a steady primary flow disturbed by three T-S waves of small amplitude. Craik's main attention was given to asymptotic results for large values of kRe. He found that if the critical layer is located far from flow boundaries, then, under rather general conditions, in the case of fairly large (but finite) values of the Reynolds number the magnitudes of the coefficients are $C_1 = O(\mathrm{Re})$, $C_2 = O(\mathrm{Re})$, and $C_3 = O(1)$, for an arbitrary velocity profile $U(z)$ of the primary flow (here it is assumed that the 1st and 2nd waves are the oblique ones while the 2D wave has number 3). This shows that, again, the amplitudes of two oblique waves grow very fast at the expense of the energy of the primary flow, while the amplitude of the two-dimensional wave changes much more slowly.

Since Craik's results relating to the Blasius velocity profile $U(z)$ and triads of waves shown in Fig. 5.1a were found to be very complicated and cumbersome, he also considered the simpler model of a piecewise linear velocity profile of the form

$$U(z) = bz\ , b > 0,\ \text{ for } 0 \le z \le H,\quad U(z) = bH = U_0 \quad \text{for } z > H \qquad (5.9)$$

(shown in Fig. 3.1a in Chap. 3). For this profile, Craik was able to find explicit asymptotic equations for the coefficients of equations (5.4) which showed at once that in this case $C_1 = O(\mathrm{Re})$, $C_2 = O(\mathrm{Re})$ and $C_3 = O(1)$ at large values of $\mathrm{Re} = U_0H/\nu$.

Later Reutov (1990) examined resonant wave interactions in the model of a boundary-layer flow with the velocity profile (5.9), assuming that $\nu = 0$ and hence $\mathrm{Re} = \infty$ [note that stability with respect to infinitesimal disturbances of such an inviscid flow was thoroughly investigated by Tietjens (1925) whose results were used extensively by Craik]. Reutov's idea was to show that results similar to those found by Craik may be obtained more easily for a simpler case of inviscid fluid. In the inviscid flow with the velocity profile (5.9) the dispersion relation determining the frequency ω of a three-dimensional wave proportional to $\exp[i(k_1x + k_2y - \omega t]$ has the form

$$\omega = k_1U_0\left[1 - \frac{1}{2|\mathbf{k}|H}(1 - \exp(-2|\mathbf{k}|H)\right], \quad |\mathbf{k}| = (k_1^2 + k_2^2)^{1/2}. \qquad (5.10)$$

[Eq. (5.10) was also obtained by Tietjens (1925); it is, of course, much simpler than the dispersion relations for viscous plane-parallel flows where the possible values of ω at given $\mathbf{k}$ are given by the eigenvalues of the corresponding O-S problem (2.41)-(2.42).] Like Craik, Reutov considered only the wave triads consisting of a two-dimensional wave which is proportional to $\exp[i(kx-\omega t)]$ and two oblique waves proportional to $\exp[i(k_1x \pm k_2y - \omega_1 t)]$. Conditions (5.2b) then take the form (5.7), and by virtue of Eq. (5.10) these conditions will be valid here if, and only if, $k/k_1 = 2$ and $k_2/k_1 = \sqrt{3}$. Hence here possible resonant triads consist of a 2D wave with wave number k (which can take any value) and two oblique waves with streamwise wave number $k/2$ which are inclined at angles $\pm 60°$ to the primary-flow direction. Investigating the nonlinear interaction between three plane waves of small amplitudes, Reutov paid his main attention to the subject studied in his earlier paper [Reutov (1985)], namely to the most important contribution to this interaction, which is produced in the vicinity of the critical layer where $U(z) = c$. He was able to show that at small positive values of t, when it is sufficient to keep only the terms of first and second order in the amplitudes, oblique waves grow exponentially while the amplitude of a two-dimensional wave remains practically constant. Thus here also a strong nonlinear resonance takes place, and leads to very effective transfer of energy from the inviscid steady flow to a pair of oblique waves; the two-dimensional wave plays the role of a catalyst, stimulating this process but preserving practically constant amplitude.

For the case of nonlinear development of resonant wave disturbances in viscous plane-parallel (or nearly plane-parallel) fluid flows, Zel'man (1974) proposed to take an average of the equations of motion over an 'intermediate region' which is much smaller than typical scales of change of the most important 'slow variations' of wave amplitudes but much greater than the wave lengths and periods of unimportant rapid oscillations, which considerably complicate the required solution. This method generalizes Landau's approach (1944) considered at the beginning of Sec. 4.21, and also has many features in common with the popular method of multiple scales, which was mentioned several times in Chap. 4 and will be mentioned again later in this chapter. This method of averaging has many applications to various physical problems relating to nonlinear oscillations and waves [see e.g. Chap. 11 of the book by Rabinovich and Trubetskov (1989)]. In particular, after 1974 this method was often applied to fluid-dynamic equations, where it facilitated the evaluation of interaction coefficients in the equations for amplitudes of resonant wave systems. One of the first examples of its use was due to Volodin and Zel'man (1978) who applied this method to the study of the spatial, rather than temporal, development of resonant wave triads of Craik's type in a Blasius boundary-layer flow. [For further applications of the method of averaging to development of disturbances in boundary layers see Zel'man and Kakotkin (1985), Zel'man (1991) and Zel'man and Maslennikova (1993a).] Volodin and Zel'man based their analysis on the numerical integration of the averaged nonlinear equations for the vertical velocity w and the vertical vorticity $\zeta = \zeta_3$ of the disturbed flow [i.e. Eqs. (3.44) and (3.54) supplemented by nonlinear terms]. These equations are equivalent to the Navier-Stokes equations for velocity components (since components u and v may be determined if w and ζ are known) but the velocity-vorticity equations do not contain the pressure; therefore they are more convenient and are used very frequently [see e.g. the review paper by Gatski (1991)]. The computational procedure used by the above authors allowed them to determine the values of the interaction coefficients $B_{1,2}$ and B_3 in Eqs. (5.4a), relating to wave amplitudes $A_i(x)$, $i = 1, 2, 3$, for a number of values of Re* in the range from 650 to 1300, and of the non-dimensionalized wave number $k_1 = k/2$ in the range from 0.19 to 0.5. (Recall that in the case of a two-dimensional least-stable T-S wave with $k_2 = 0$, values of Re* and k uniquely determine the value of ω.) By virtue of the 3D Orr-Sommerfeld equation (2.41), if the values of Re* and ω are fixed then the value of k_1 determines the value of k_2 for the most unstable wave and therefore also determines the inclination angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ of the oblique components of the wave triad [for more details see Kachanov and Michalke (1994, 1995) and Kachanov (1996)]. Volodin and Zel'man found that in the spatial formulation of the stability problem the ratio $|B_{1,2}|/|B_3|$ also takes fairly large values, which increase appreciably with increasing Re* (the coefficients B_1 and B_2 coincide here for reasons of symmetry). They also applied the approximate method of Bouthier (1973) to incorporate the effect of streamwise variation of the boundary-layer flow into the computation; it was found that this effect does not invalidate the important conclusion formulated above. This conclusion was later confirmed by the analytical investigations of resonant-triad development in a streamwise-growing boundary layer by Smith and Stewart (1987), Nayfeh (1987a,b) [who criticized some of the assumptions of Smith and Stewart which were called in question also by Mankbadi *et al.* (1993) and Wu (1993, 1995); see also Healey's (1995a) critical discussion of various approximations used in derivation of multimode amplitude equations], and the papers by Khokhlov (1993, 1994) (who used a slight modification of Smith and Stewart's assumptions), and Zel'man and Maslennikova (1984, 1993a) (some results of the latter authors will be discussed below).

Craik (1971) considered, at the end of his paper, several exact solutions of some particular amplitude equations of the form (5.4) [for the simplest case of Eqs. (5.3) with constant coefficients C_i, exact solutions, represented in terms of elliptic functions, were found independently by Jurkus and Robson (1960), Armstrong *et al.* (1962) and Bretherton (1964)]. Craik's solutions also include examples where amplitudes of some waves become infinite after a finite time. (These singularities apparently show that the wave energy grows faster than exponentially; of course, the second-order equations (5.4) cease to be valid in such cases before the predicted 'infinite instability burst' occurs.) Later Craik and Adam (1978) and Craik (1978, 1985) also considered three-wave resonances, for waves with amplitudes depending on both spatial coordinates and time. In this case the left-hand sides of Eqs. (5.3a) must be supplemented by the terms $(\mathbf{v}_i\cdot\nabla)A_i$ where $\mathbf{v}_i$ is the appropriately-defined velocity of the ith wave. Wave systems of such types are met in a number of diverse physical problems; therefore the exact solutions of some of the corresponding amplitude equations found by Craik may have many applications. However, this subject will not be discussed here at any length.

Craik (1971) found also that 'direct computation' of the interaction coefficients C_i with the help of the Navier-Stokes equations was quite complicated and labor-consuming. Therefore Usher and Craik (1974) tried to replace the ordinary form of N-S equations in this computation by the variational formulation suggested by Bateman and presented in the textbook by Dryden, Murnaghan and Bateman (1956). This attempt was stimulated by the fact that in the case of a similar problem for capillary-gravity waves a variational analysis by Simmons (1969) proved to be much more simple and elegant than the 'direct evaluation' of the interaction coefficients by McGoldrick (1965) by means of Euler's inviscid equations of motion. According to Usher and Craik the viscous terms and the non-self-adjointness of the N-S equations considerably complicate the derivation of an appropriate variational formulation of these equations. Nevertheless, such a formulation can be found, and it indeed allows computation of the interaction coefficients more simply than by Craik's method of 1971. However, the subsequent rapid increase in the power of digital computers, combined with the development of improved numerical methods, soon made applications of the variational method unnecessary.

A more complete, but still weakly-nonlinear, theory of resonant three-wave interactions, which takes into consideration terms of third order in wave amplitudes, was developed by Usher and Craik (1975). Recall that the Landau and Stuart-Landau equations (4.34) and (4.40) for the amplitude of a single mode, and Stuart's equations (4.43) for amplitudes of a pair of non-resonantly interacting modes, both include terms of the third order (but the second-order terms are absent from these equations). Hence the amplitude equations, which include terms up to the third order, generalize both the one-mode and two-mode equations, by Landau and Stuart, and equations (5.4) for the three-wave resonant interactions. The third-order amplitude equations for three-wave resonant interactions derived by Usher and Craik [and later by Weiland and Wilhelmsson (1977) and Goncharov (1981); see also Craik (1985), Secs. 16.3 and 25-26] have the form

$$\begin{aligned}
\frac{dA_1}{dt} &= \omega_1^{(i)}A_1 + C_1A_2^*A_3 + A_1(c_{11}\,|\,A_1\,|^2 + c_{12}\,|\,A_2\,|^2 + c_{13}\,|\,A_3\,|^2),\\
\frac{dA_2}{dt} &= \omega_2^{(i)}A_2 + C_2A_1^*A_3 + A_2(c_{21}\,|\,A_1\,|^2 + c_{22}\,|\,A_2\,|^2 + c_{23}\,|\,A_3\,|^2),\\
\frac{dA_3}{dt} &= \omega_1^{(i)}A_3 + C_3A_1A_2 + A_3(c_{31}\,|\,A_1\,|^2 + c_{32}\,|\,A_2\,|^2 + c_{33}\,|\,A_3\,|^2).
\end{aligned} \tag{5.11}$$

(In the case of non-resonant three-wave interactions, third-order amplitude equations have the same form but with $C_1 = C_2 = C_3 = 0$; therefore, non-vanishing of the latter coefficients shows that the wave interactions are resonant.)
Compared with Eqs. (5.4), the new equations include nine additional unknown coefficients c_{ij}. Usher and Craik gave their main attention to the case of a resonant triad of Craik's type, consisting of two symmetric oblique waves and one plane 2D wave. As in Craik (1971), they assumed that numbers 1 and 2 correspond to the oblique waves while number 3 corresponds to the 2D wave; then the oblique-wave symmetry implies that $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{12} = c_{21}$, and $c_{31} = c_{32}$. Therefore, in this case only five new coefficients need evaluation. Nonlinear N-S equations for velocity components lead to some lengthy expressions for these coefficients, which show that at large values of Re all coefficients take large values (proportional to some positive powers of Re). These asymptotic estimates force one to conclude that at large values of Re the second-order equations (5.4) may be valid only for waves with rather small amplitudes.

Craik (1975) studied equilibrium solutions of the third-order three-wave amplitude equations (5.11) and the stability of these solutions. Recall that the third-order Landau and Stuart-Landau equations (4.34) and (4.40) imply that if Landau's constant $\delta > 0$, then at small supercritical values of $\mathrm{Re} > \mathrm{Re}_{cr}$ there is an equilibrium periodic solution of Eq. (4.40) which separates from the steady primary flow by a Hopf bifurcation. On the other hand, if $\delta < 0$, then an equilibrium solution exists under slightly subcritical conditions Re $< \mathrm{Re}_{cr}$ where a periodic wave of finite amplitude ~~separ~~ates from the primary flow if its initial amplitude exceeds a small, but finite, critical value [proportional to $(\mathrm{Re}_{cr} - \mathrm{Re})^{1/2}$]. Craik used Eqs. (5.11) for investigation of the stability of equilibrium solutions of Eqs. (4.40) with respect to pairs of symmetric oblique waves of small amplitude, and determination of conditions making possible a second bifurcation, leading to the appearance in the flow of a resonant triad, consisting of the same two-dimensional wave as that entering the primary equilibrium solution together with symmetric oblique waves of finite amplitudes. These results of Craik are relevant to the results by Herbert (1984a, 1985, 1986, 1987, 1988a,b) relating to the secondary-instability mechanism of boundary layer instability which will be considered slightly later in this subsection.

bifurcates

Let us now revert to discussion of Craik's (1971) resonant triads consisting of one two-dimensional T-S wave with the angular frequency ω and wave vector $(k, 0)$ and two fully symmetric oblique waves with the same frequency $\omega/2$ and wave vectors $(k/2, \pm k_2)$. Following Craik we will assume that three waves of a triad have small amplitudes of the same order of magnitude. Assume that the value of ω is determined by the conditions of the experiment and is therefore known. [This condition is fulfilled, in particular, if the 2D plane wave is excited by some device oscillating with a fixed frequency - e.g., by a vibrating ribbon used by Schubauer and Skramstad (1947) and then by many others; or by an acoustic radiator used, among others, by Morkovin and Paranjape (1971), Yan *et al.* (1988), and a number of authors cited by Nishioka and Morkovin (1986); or by a heating element with periodically varying temperature used, in particular, by Liepmann *et al.* (1982); or by localized periodically-alternating blowing and suction of fluid considered by Konzelmann *et al.* (1987)).] Among the flow disturbances produced by a device oscillating with frequency ω, a dominant role is played by the least-stable T-S wave having this frequency; such a wave is always two-dimensional and its wavenumber k can be uniquely determined from the two-dimensional O-S eigenvalue problem (2.44), (2.42). If this wave is linearly unstable, then it will grow, and some time later will excite a pair of oblique waves forming, with the primary T-S wave, Craik's resonant triad (or, maybe, a fast-growing triad close to this - such a possibility will be discussed later).

In the case when Craik's triad alone is excited (here, only this case will be considered) the spanwise wavenumbers $\pm k_2$ of the oblique waves with frequency $\omega/2$ and streamwise wavenumber $k/2$ may be determined from the three-dimensional O-S eigenvalue problem (2.41), (2.42). [A number of results relating to computation of the 3D T-S waves by numerical solution of this eigenvalue problem (where k_1, and not ω, is considered as a complex eigenvalue), supplemented by comparison of the results obtained with the available experimental and numerical-simulation data, were presented by Kachanov and Michalke (1994, 1995) and briefly discussed by Kachanov (1996).] Consider now another case, where the disturbances penetrate into the boundary layer from an unsteady external stream generating background "noise", including an extensive collection of various weak fluctuations. Then among the boundary-layer waves produced by these fluctuations the two-dimensional T-S wave with the greatest linear growth rate will naturally dominate the initial stage of disturbed-flow development. At a given value of Re, such a wave has definite values of ω and k which may be determined with the help of Eqs. (2.44) and (2.42) [see Figs. 2.23 and 2.26]; hence here also the values of ω and k may be considered as known. Knowledge of ω and k (and hence also of $\omega/2$ and $k/2$) again allows the spanwise wavenumbers $\pm k_2$ of the oblique components of the resonant triad to be determined uniquely, from Eqs. (2.41) and (2.42). We see that in the framework of Craik's model the spanwise periodicity of the 3D structure may usually be determined uniquely.

However, the uniqueness of the value of k_2 at given values of Re, ω and k is contradicted by the data of Saric and Thomas (1984), Saric *et al.* (1984), and Kozlov *et al.* (1984) who found that in their experiments (where a 2D wave was excited by a vibrating ribbon) the observed value of k_2 depended not only on ω and k but also on the initial amplitude of the excited 2D wave (for more details see the next Sec. 5.3). On the other hand, the exact symmetry of oblique waves entering Craik's triad clearly requires the initial real amplitudes $|A(0)|$ and phases $\theta(0)$ (where $|A(0)|e^{i\theta(0)} \equiv A(0)$ is the initial complex amplitude) of two oblique waves to coincide with each other. This requirement appreciably restricts the Craik model of development of three-dimensional disturbances in a boundary layer. These two circumstances led Herbert (1983a, 1984a) and Saric and Thomas (1984) to doubt the universal applicability of Craik's model of resonant-triad generation of three-dimensionality in steady plane-parallel (or nearly plane-parallel) shear flows and to attribute some of the observed 3D structures in such flows to the secondary-instability mechanism.

The suggestion that the discrepancy between Craik's theory and experiment, and the apparent restrictions of this theory, required it to be replaced by a more universal secondary-instability approach was not unanimously supported. In particular, Zel'man and Maslennikova (1985, 1989, 1990, 1993a) [see also Maslennikova and Zel'man (1985)] showed that exact symmetry of oblique waves (which implies that the initial amplitudes of two oblique waves must take the same value), and exact equality of the oblique-wave real frequency and streamwise wavenumber to half those of the accompanying plane wave, are not necessary for the rapid resonant growth of two oblique waves entering the wave triad. They considered wave triads where the frequencies and wave vectors of the plane wave and the two oblique waves are $\{\omega, k, 0\}$ and $\{\omega/2, k_1 \approx k/2, \pm k_2\}$, respectively, but with $k/2 - k_1 \neq 0$ [for given values of ω and k_2 the values of k and k_1 may be uniquely determined with the help of the O-S equations (2.44) and (2.41); therefore only the values of ω and k_2 can be chosen arbitrarily]. According to the results of Zel'man and Maslennikova's computations, if the initial amplitudes $|A_2(0)|$ and $|A_3(0)|$ of two oblique waves are not equal and k_1 does not coincide exactly with $k/2$, the growth of oblique-wave amplitudes nevertheless remains much larger than the growth of the plane-wave amplitude. Moreover, the nonlinear interactions usually lead to rapid equalization of amplitudes $|A_2(t)|$ and $|A_3(t)|$ and to recovery of oblique-wave symmetry, and after a short time these two amplitudes become greater than the amplitude of the 2D wave. Still later, fast-growing oblique waves start to influence the plane wave very strongly, and cause its explosive growth which is more rapid than the exponential growth of the oblique waves.

The above formulation is the temporal one, but in fact Zel'man and Maslennikova considered the spatial, not temporal, growth of boundary-layer waves which is more convenient for comparison with the experimental data. Therefore they used amplitude equations of the form (5.4a), not (5.4), and determined the corresponding interaction coefficients B_n, $n = 1, 2, 3$, by the method generalizing that applied by Volodin and Zel'man (1978) to the study of spatial development of strictly symmetric wave triads (see the discussion of this paper on p.24). A typical example of results more general than those found in 1978 is shown in Fig. 5.3, taken from Zel'man and Maslennikova's paper (1993a). In this figure the dependence of wave amplitudes $|A_i(x)|$, $i = 1, 2, 3$, on the streamwise coordinate x is replaced by their dependence on Reynolds number $\mathrm{Re}^+ = U_0\delta^+/\nu = (U_0x/\nu)^{1/2} = (\mathrm{Re}_x)^{1/2}$, where $\delta^+ = (\nu x/U_0)^{1/2} \approx 0.58\delta^*$ is a new scale of the boundary-layer thickness which is often used instead of δ^* (it was used, in particular, in Chap. 2, p. 110). Fig. 5.3 corresponds to some definite values of the dimensionless frequency $F = \omega\nu/U_0^2$ and spanwise wavenumber $K_2 = \nu k_2/U_0$ (the frequency F was already used in Chap. 2 - see Figs. 2.26 and 2.27) and to the case where initially $|A_1| \gg |A_2| \gg |A_3|$ (where $|A_1|$ is the plane-wave amplitude) and the initial phases of the waves are matched. Logarithmic scaling of the amplitudes allows us to see clearly the region of exponential growth of oblique-wave amplitudes and the explosive growth of plane-wave amplitude at $\mathrm{Re} > \mathrm{Re}_n$ (for simplicity the usual notation Re will be used to denote the particular Reynolds number Re^+). Zel'man and Maslennikova (1993a) also presented figures showing examples of the plane-wave and oblique-wave amplitude-growth curves for (a) a fixed value of the initial oblique-wave amplitudes $|A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$ and three different initial plane-wave amplitudes $|A_1|(\mathrm{Re}_0)$, (b) fixed values of both $|A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$ and $|A_1|(\mathrm{Re}_0)$, but with three different values of initial phase mismatch, and (c) a fixed value of $|A_1|(\mathrm{Re}_0)$, and three different values of $|A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$ (it was assumed here that $\mathrm{Re}_0 = 500$, $|A_1|(\mathrm{Re}_0) > |A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$, and that in cases (a) and (c) there is no phase mismatch). The figure corresponding to case (a) illustrated the existence of a threshold value of $|A_1|$ below which the plane wave cannot excite the rapid growth of three-dimensional oblique waves which in turn produces the later explosive growth of the plane wave itself.

Zel'man and Maslennikova (1990, 1993a) stated that a more general and accurate version of the three-wave-resonance theory described in their papers showed that the three-wave resonance could be considered as the universal dominant mechanism of the so-called subharmonic (S-type or, alternatively, N-type - the latter name will be used consistently in Secs. 5.3 and 5.4) instability development in boundary layers (for more detailed discussion of this type of instability development see Sec. 5.3). However, this statement also was not universally accepted. Moreover, it did not imply that the other possible mechanisms are worthless; the authors only insisted on the possibility of interpreting the subharmonic disturbance development in the framework of the appropriately-modified three-wave-resonance theory in all cases. However it will be shown below that the three-wave-resonance approach often leads to results which are very close to those given, e.g., by the secondary-instability theory, which presupposes that the wave modes have amplitudes of two different orders of magnitude (see Fig. 5.16a). Note also that in some cases the secondary-instability computation allows the derivation of the required results to be simplified considerably. Moreover, the secondary-instability mechanism seems to be the most appropriate one in the widely-studied cases where a primary plane wave of finite amplitude is produced by a vibrating ribbon and later excites some secondary waves which are initially very weak. In addition, this mechanism is important in itself since it has many applications to problems unrelated to three-wave resonances. On the other hand, Mankbadi (1990, 1991, 1993a,b), Mankbadi *et al.* (1993), and Wu (1993, 1995) in their approximate evaluation of the resonant growth rates of two symmetric oblique waves (with frequencies and streamwise wavenumbers which are close, but not necessary equal, to half of those corresponding to the 2D wave of the triad) used quite another method [based on the idea that the dominant part of the nonlinear wave interactions is concentrated in the neighborhood of the critical layer; for more details see the end of the present subsection, printed in small type]. Apparently this new method could in some cases replace both the resonant-triad and the secondary-instability methods, but its range of applicability is not clear up to now [cf. Healey (1995a)]. Note also that Jennings *et al.* (1995) considered the most general resonant triads consisting of three oblique waves [recall that just such triads were earlier discussed by Raetz (1959) and Stuart (1962a,b)] and showed that rapid growth of oblique-wave amplitudes is possible in this case also. The paper by Jennings *et al.* supplemented Zel'man and Maslennikova's results, showing that the three-wave resonance mechanism of generation and development of three-dimensional structures in boundary layers has a much wider domain of applicability than was assumed in the 1970s; however, this does not exclude the possibility that other mechanisms may also play important parts in some cases of boundary layer transition to turbulence and are therefore worth studying.

Let us also stress that the available experimental data relating to evolution of Blasius laminar boundary layers disturbed by a two-dimensional 'primary wave' (some of these data will be discussed in Secs. 5.3-5.5 below) definitely show that very different three-dimensional structures may appear in the course of this evolution. Therefore, it seems natural to suppose that there exist many different mechanisms of generation of such structures. Having this in mind, and also recalling remarks above relating to the *secondary-instability mechanism* of generation of flow three-dimensionality, we will now pass to discussion of this mechanism.

The secondary-instability approach to development of flow instabilities is based on a simple two-stage model. The first stage consists of the growth of some relatively simple small disturbance in accordance with the linear hydrodynamic-stability theory considered in Chap. 2. When this 'primary disturbance' becomes strong enough, it becomes unstable with respect to some disturbances of a quite different form, and then the second stage of disturbance development begins. Recall that in Sec. 4.22 the secondary instability of the two-dimensional equilibrium disturbances of a plane Poiseuille flow was briefly discussed on p. 72, where also a number of references touching upon this subject was presented, while on pp. 92-93 of the next subsection 4.23 even some tertiary and quaternary flow instabilities were mentioned, and a few references relating to such instabilities were indicated. Now we will consider a model where the superposition of some two-dimensional T-S wave of a finite amplitude A on the Blasius boundary-layer flow is considered as the 'primary flow', whose stability with respect to three-dimensional background ('environmental') waves of small amplitudes must be investigated. (For the sake of simplicity all amplitudes will now usually be assumed to be real and the possible effect of the 'phase mismatch' will as a rule be ignored.) Thus here the 'primary flow' has the velocity $\mathbf{V}_1(x,z,t) = \mathbf{V}(z) + A\mathbf{v}_1(z)e^{i(kx-\omega t)}$ where $\mathbf{V}(z) = \{U(z),0,0\}$ (and $U(z)$ is the Blasius velocity profile at streamwise distance x, if the locally-plane-parallel model of the boundary-layer flow is used), while $\mathbf{v}_1(z)$ is the velocity profile of the selected T-S wave, normalized in a reasonable way, and A is its amplitude. (Normalization of the vector-function $\mathbf{v}_1(z) = \{u(z),v(z),w(z)\}$ is necessary to give meaning to the amplitude A. In particular, if $\mathbf{v}_1(z)$ is normalized by the condition that $\max_z u(z)/U_0 = 1$ where U_0 is the free-stream velocity, then A measures the maximal streamwise velocity of the T-S wave as a fraction of U_0.) Note also that the representation of $\mathbf{V}_1$ used here involves some other conventional approximations of the linear stability theory, excluding local parallelism; in particular, the velocity-profile distortion by disturbances is here neglected for both the steady Blasius boundary layer and the periodic T-S wave within it [for more details see Herbert's surveys (1988a,b)]. The primary flow with velocity $\mathbf{V}_1$ is disturbed by a 'secondary disturbance' of velocity $\mathbf{v}_3(x,y,z,t)$ where $|\mathbf{v}_3| \ll |\mathbf{V}_1|$. The last condition makes it possible to apply linear stability theory, i.e. to base the stability analysis on the N-S equations for the velocity field $\mathbf{V}_1(x,z,t) + \mathbf{v}_3(x,y,z,t)$ linearized with respect to the velocity and pressure $(\mathbf{v}_3, p_3)$ of the disturbance. Thus, in contrast to the theory of three-wave resonance, where the amplitudes of all three waves are assumed to be of the same order of smallness and the equations of motion are expanded into subsequent powers of all amplitudes, in the secondary-instability theory the amplitude A of the 2D wave is considered as a fixed finite parameter and only the amplitude of the supplementary 3D disturbance is assumed to be small.

The papers of the 1980s on secondary instability of steady shear flows cited in Chap. 4 contain much material directly relating to the present topic [in fact this instability of laminar boundary layers was briefly discussed even earlier, by Görtler and Witting (1958) and Maseev (1968a,b)]. In solving the secondary-instability problem it is convenient to use, instead of a stationary frame, a frame moving in the Ox direction with the phase velocity c of the T-S wave having velocity $A\mathbf{v}_1(z)e^{i(kx-\omega t)}$, i.e., to replace x by the variable $x' = x - ct$. In this frame the primary flow is independent of time and periodic in x, i.e., here $\mathbf{V}_1(x,z,t) = \mathbf{V}_1(x',z)$, where $\mathbf{V}_1(x',z) = \mathbf{V}_1(x'+\lambda_x,z)$, $\lambda_x = 2\pi/k$. Therefore, the frame transformation reduces the secondary-stability problem to the study of the linear stability of a steady but streamwise-periodic, locally plane-parallel flow. Numerical investigation of this linear stability problem for the plane-parallel model of a Blasius boundary-layer flow (and also for some other flows) was carried out, in particular, by Orszag and Patera (1983) who obtained some interesting new results which were later confirmed by other authors. However, a much more explicit study of the secondary instability of the primary flow considered here was accomplished by Herbert (1983b, 1984a, 1985, 1986, 1987, 1988a,b) [see also Herbert and Santos (1987), Herbert *et al.* (1987) and Crouch and Herbert (1993)]. Therefore, here the main attention will be given to this work.

Herbert used the fact that the linear stability analysis of steady periodic flow with respect to a small three-dimensional disturbance may be reduced to study of a Floquet system of linear differential equations with periodic coefficients. The main properties of such systems may be found, e.g., in Coddington and Levinson's textbook (1955); various applications of Floquet theory to hydrodynamic stability were considered, in particular, by Kelly (1967), Clever and Busse (1974), Davis (1976), Barkley and Henderson (1996), and Schulze (1999) [see also Craik (1995) and references therein].

However, Floquet theory was primarily developed in relation to the study of nonlinear periodic oscillations, and therefore in fluid mechanics it was most often applied to investigations of stability of time-periodic primary flows. Since Herbert considered, instead of this, the case of spatially periodic primary flow, it is reasonable to present here some details of his method.

The normal-mode concept, which was widely applied in Chap. 2 to problems relating to the linear stability theory for steady non-periodic flows, may now be used in exactly the same form for description of the dependence of the disturbance on the variables y and t. Here it leads to the representation of the disturbance velocity $\mathbf{v}_3(x',y,z,t)$ in the form of a superposition of modes depending on parameters k_2 and Ω (and admitting separate study) of the form

$$\mathbf{v}_3(x',y,z,t) = e^{i(k_2 z - \Omega t)}\mathbf{v}_4(x',z). \tag{5.12}$$

As in Sec. 2.5, the spanwise wave number k_2 may be assumed real (by virtue of the spanwise homogeneity of the primary flow), while (again exactly as in Chap. 2) the parameter Ω is generally complex: $\Omega = \Omega_r + i\Omega_i$. [Note that here Ω_r characterizes the frequency shift of the 3D disturbance with respect to the frequency ω of the primary T-S wave; modes with $\Omega_r = 0$ travel with the primary flow of velocity $\mathbf{V}_1(x,z,t)$.] As to the dependence of $\mathbf{v}_4(x',z)$ on the streamwise coordinate x', the Floquet theory implies that it may be represented in the form

$$\mathbf{v}_4(x',z) = e^{\gamma x}\mathbf{v}_5(x',z), \tag{5.13}$$

where $\gamma = \gamma_r + i\gamma_i$ is a complex *characteristic exponent* of the problem and $\mathbf{v}_5(x',z)$ is a periodic function of x': $\mathbf{v}_5(x' + \lambda_\xi, z) = \mathbf{v}_5(x',z)$. The periodicity of $\mathbf{v}_5(x',z)$ allows it to be expanded in a Fourier series and thus to obtain the following general form of the three-dimensional disturbance $\mathbf{v}_3(x',y,z,t)$:

$$\mathbf{v}_3(x',y,z,t) = e^{\gamma x' + i(k_2 y - \Omega t)}\sum_{ms}\hat{\mathbf{v}}_m(z)e^{imkx'}, \quad -\infty < m < \infty, \tag{5.14}$$

where wave numbers k and k_2 are real, and constants γ and Ω are complex.

The additional complex parameter γ leads to the appearance here of new possible forms of disturbance. Note first of all that the values γ and $\gamma + ink$ of this parameter, where n is an integer of either sign, lead to the same collections of functions (5.14), differing only in numbering of the Fourier coefficients. Therefore, it is possible to assume that $-k/2 < \gamma_i \leq k/2$. Moreover, it is also reasonable to subdivide the set of all disturbances of the form (5.14) into three classes of more special disturbance modes:

a) *Fundamental modes*, $\gamma_i = 0$. Here

$$\mathbf{v}_3(x',y,z,t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{imkx'}, \quad -\infty < m < \infty. \tag{5.14a}$$

b) *Subharmonic modes*, $\gamma_i = k/2$. Here

$$\mathbf{v}_3(x',y,z,t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{imk_1 x'},\ k_1 = k/2,\ m=2n+1,\ -\infty < n < \infty. \tag{5.14b}$$

c) *Detuned modes*, $0 < |\gamma_i| < k/2$. Here, if $2\gamma_i/k = \varepsilon$, then $0 < \varepsilon < 1$ and

$$\mathbf{v}_3(x',y,z,t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{i(m+\varepsilon)k_1 x'}, k_1 = k/2,\ m=2n,\ -\infty < n < \infty. \tag{5.14c}$$

The word "detuned" simply implies a streamwise wave number somewhere between the fundamental and the subharmonic modes. The terms corresponding to $m = \pm 1$ are the dominant ones on the right side of Eq. (5.14a) describing the 3D fundamental modes. These terms show that the primary 2D mode having the streamwise wavenumber k may excite resonant 3D waves with the same streamwise wavenumber (in the temporal presentation of the theory it means that a 2D wave of frequency ω may excite 3D waves of the same frequency). This process is associated with the so-called *primary resonance* in a Floquet system. In Eq. (5.14b) the main terms are also those with $m = \pm 1$; they correspond to subharmonic 3D modes having streamwise wavenumber $k_1 = k/2$ (or, in temporal presentation, to subharmonic modes of frequency $\omega/2$). The resonant excitation in a Floquet system of 3D waves with the streamwise wavenumber (or frequency) equal to half of the corresponding characteristic of the primary 2D wave represents a phenomenon which is often called the *principal parametric resonance* (the adjective 'parametric' is used because in many real physical systems the primary oscillation of frequency ω represents oscillatory variations of some physical parameter affecting the system). Real detuned modes must include on the right-hand side of Eq. (5.14c) two complex-conjugate summands with opposite detuning parameters $\pm\varepsilon$. Herbert [in (1988a,b) and some other papers] called real detuned modes the *combination modes*; and said that they participate in the *combination resonances* [see Santos and Herbert (1986), Herbert and Santos (1987) and Herbert *et al.* (1987); cf. also the surveys by Nayfeh (1987a,b)].

As will be shown later, all the above-mentioned types of secondary-instability resonances can participate in the development of fluid-flow instability. However here only the evident similarity of the principal parametric resonance to Craik's three-wave resonance will be emphasized. This similarity makes the principal parametric resonance especially interesting for the analysis of boundary-layer instabilities. Note in this respect that resonances of such type occur also in many other physical systems. Apparently the first description of such phenomenon in scientific literature is due to Faraday (1831), who discovered that when a vessel containing liquid is made to vibrate vertically, some vibrations of the free surface of the liquid have a frequency equal to only half of that of the vessel. This seemingly unusual *Faraday resonance* (or *Faraday waves, Faraday instability*) attracted much attention and was later studied by many authors both theoretically and experimentally [in particular, Rayleigh (1883a,b) participated in both kinds of studies]. Nevertheless, a satisfactory theory of this resonance was developed only in the second half of the 20th century and its study is not yet complete; see, e.g., the papers by Benjamin and Ursell (1954), Miles (1984, 1993), Guthart and Wu (1994), Friedel *et al.* (1995), Wright *et al.* (2000), and the survey by Miles and Henderson (1990) containing many supplementary references [cf. also the paper by Schulze (1999) indicating some conditions under which the principal parametric resonance cannot occur].

Let us now return to a description of Herbert's work. The imaginary part Ω_r of the parameter Ω determines the time growth of the 3D-disturbance amplitude, which is proportional to $\exp(\Omega_i t)$. However, this is correct only for amplitudes at fixed points of the frame of reference moving with velocity c, while amplitudes at fixed points x of the stationary frame will be proportional to $|\exp[-i\Omega t + \gamma(x-ct)]| = \exp(\gamma_r x)\exp[(\Omega_i - \gamma_r c)t]$. Models of purely temporal growth of disturbances (which were the main objects of investigation in all early theoretical studies and continue to be widely studied up to now; see, e.g., Sec. 2.92) correspond to assumption that $\gamma_r = 0$, while purely spatial growth in the laboratory frame corresponds to condition $\Omega_i = \gamma_r c$.

Substitution of the above expressions for the disturbance modes (5.14a,b,c) (assumed to be real) into linearized N-S equations for the velocity disturbances leads to infinite systems of coupled linear differential equations for the functions $\hat{\mathbf{v}}_m(z)$. A numerical solution may be obtained if the Fourier series are truncated, making the infinite systems finite. Numerical studies by Herbert (1984a,b, 1985, 1986, 1988b), Herbert and Santos (1987), Herbert *et al.* (1987), and Crouch and Herbert (1993) [see also Herbert's survey (1988a)] showed that reasonable accuracy may be achieved even when truncation is very severe - in the case of subharmonic modes it is often enough to preserve only the terms with $m = -1$ and $m = 1$, while for fundamental modes the truncation of all terms with $|m| > 1$ (i.e., inclusion in the analysis only the terms with $m = -1$, 0 and 1) gives in many cases satisfactory accuracy. [This conclusion, which confirms the above statements about the dominant terms of Eqs. (5.14a,b) agrees also with the results of subsequent numerical investigations of Blasius boundary-layer secondary instability by Wang and Zhao (1992) and Ustinov (1994)]. The resulting systems depend on the boundary-layer and primary-wave velocity profiles $U(z)$ and $\mathbf{v}_1(z)$ and include parameters A, k, ω, k_2, γ_r, γ_i, Ω_r, Ω_i, characterizing the primary T-S wave, and 3D disturbances interacting with this wave. (Strictly speaking, ω takes a complex value if the primary T-S wave is not neutral, while if spatial, and not temporal, development of disturbances is considered, then ω is real but k is complex. However, we will follow Herbert and assume that the T-S wave varies slowly in comparison with the 3D disturbance, and hence the T-S amplitude A may be assumed to be locally constant and both parameters ω and k will be real. More general models where k or ω may be complex were considered by Wang and Zhao (1992) but will not be discussed here.) Parameters A, k and ω characterize the primary T-S wave and may be assumed to be known; as to the other parameters mentioned, the majority can take any values, which may be chosen on the basis of available data or physical arguments. This, however, is not true for all parameters, since, as in the case of the Orr-Sommerfeld equation, the systems of equations for functions $\hat{\mathbf{v}}_m(z)$ with appropriate boundary conditions define eigenvalue problems - their solutions exist only for special values of some of the parameters ('eigenvalues' of the problem, which depend on the chosen values of the other parameters). And, exactly as in the case of the O-S equation (where the eigenvalues are the real and imaginary parts of ω or, if a spatial formulation of the stability problem is used, of k), only two of the above real parameters must be treated here as eigenvalues determined by the requirement of solubility of the system. Note also that in the case of the spanwise wavenumber k_2 it is natural to suppose that the value to which the highest growth rate of the wave amplitude corresponds should be just the wavenumber that is most likely to appear in experiments. This assumption [which is entirely similar to that used in the linear stability theory for determination of the value of k in the O-S equation (2.44)] provides a criterion for determination of the preferred spanwise periodicity. Some of the results obtained in this way will be considered, together with the appropriate experimental and numerically-simulated data, in Sec. 5.4 of this chapter. It will be also shown there that numerical solutions of the amplitude equations for resonant waves in a boundary layer, and for the disturbance modes (5.14a,b,c) of its secondary instability, allows many observable characteristics of the boundary-layer instability to be determined, yielding information about the most appropriate instability models and values of the corresponding parameters. As will be seen, in spite of the essential differences between resonance and secondary-instability mechanisms, the quantitative consequences of the two theories sometimes (though not always) lead to results which are very close to each other. Note in this respect that both theories were independently proposed at a time when almost no reliable data existed for comparison with theoretical predictions. In the case of the secondary-instability theory the early (and nowadays rarely cited) papers by Görtler and Witting (1958) and Maseev (1968a,b) are worth mentioning in this respect. It is curious to note that both the German authors and the Russian one (in the first of his two papers) independently chose practically the same title, which was later used also by Herbert (1988a). Herbert noted in this paper that Maseev's papers (the first being published in Russian in a small-circulation collection of papers written by lecturers from a Moscow engineering college, while the second was translated into English but is very short and not entirely clear) apparently contained some new, nontrivial correct ideas about the role of the secondary instability in boundary-layer transition to turbulence (in fact, these ideas had something in common with the contents of the earlier paper by Görtler and Witting). In particular, Maseev gave, without explicit proof, some reasonable estimates of the threshold amplitudes of the 2D wave needed for the generation of three-dimensionality with a given spanwise wavenumber k_2 (see Fig. 5.4) [the estimates are compatible with the data of Klebanoff *et al.* (1962)]. A similar schematic graph was given by Görtler and Witting who did not indicate scales but stated that their graph agrees with the experimental data of Schubauer (1958).

Let us say now a few words about the papers of Mankbadi (1990, 1991, 1993a) and some related work. In the 1990 and 1991 papers Mankbadi considered fully-resonant triads, where all waves have small amplitudes and the same phase velocity c. For these conditions he analyzed the role of the critical layer, where $U(z) = c$, in triad development. He found that the main contribution to the growth rates of wave amplitudes is due to wave interactions in the neighborhood of the critical layer, and at large values of Re this neighborhood is the only flow region where nonlinearity strongly affects the wave dynamics. Therefore Mankbadi used the linearized N-S equations both below and above the narrow sublayer where $U(z) \approx c$ matching then their solutions with the nonlinear-equation solution for the singled out sublayer. In Mankbadi (1993a) a more general triad was considered, in which frequencies and streamwise wave numbers of oblique waves were close, but not necessarily equal, to half those of the 2D wave. According to Mankbadi, in this case too the oblique- and plane-wave growth rates G_o and G_p at large values of Re are determined with high accuracy by the contributions of the neighborhood of the critical layers (which in this case are clearly close to each other for all three waves). Based on this, Mankbadi carried out an asymptotic evaluation of the growth rates, and found that if the initial amplitude of the plane wave is much greater than the oblique-wave amplitudes and Re is large enough, then $G_o \gg G_p$ and oblique waves with quite different spanwise wavenumbers k_2 can grow rapidly extracting energy very efficiently from the undisturbed flow (i.e. a three-wave resonance of some sort takes place for a wide range of k_2-values, and the plane wave then plays the role of a catalyst stimulating growth of oblique waves). The positive growth rates G_o depend on the plane-wave amplitude and the values of Re* and k_2; hence $k_{2,\text{pr}} = k_{2,\text{pr}'}(A,\text{Re}^*)$ where $k_{2,\text{pr}}$ is the preferred value of k_2 corresponding to the maximal value of G_o. Dependencies of G_o on A, Re* and k_2, and of $k_{2,\text{pr}}$ on A and Re*, computed by Mankbadi were in good agreement with the available experimental and numerical data (see Figs. 5.15a,b in Sec. 5.4). This agreement clearly increases confidence in Mankbadi's results but since the problems solved by him are quite involved, a supplementary check of all his arguments remains desirable.

A more complicated asymptotic theory of the spatial development of resonant triads in a Blasius boundary layer at large values of Re was developed by Mankbadi *et al.* (1993). Here the wave triads considered included one plane wave and a pair of symmetric oblique waves, having frequencies ω and $\omega/2$ and streamwise wave numbers k and $k_1 \approx k/2$, respectively. [Such triads were also analyzed, by a quite different method, by Zel'man and Maslennikova (1993a); the interest of theoreticians in them was stimulated by papers by Corke and Mangano (1988,1889) describing experimental investigations of development of such wave triads in a boundary layer.] Since the value of k_1 could vary, the spanwise wavenumbers $\pm k_2$ and the inclination angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ could also take different values. Mankbadi *et al.* estimated the wave growth rates G_o and G_p by a somewhat refined method of critical-layer analysis which took into account the nonlinear critical-layer effects which lead to the appearance, in the amplitude equations, of nonlinear integral terms which account for the influence of the upstream wave history. Their main attention was paid to the case where the plane wave is linearly unstable while the oblique waves are linearly stable (i.e., decaying according to the linear stability theory), and where amplitudes of all three waves are small but the initial amplitudes of the oblique waves are much smaller than that of the plane wave. It was shown that at first the plane wave causes fast growth of oblique waves while the plane wave itself continues growing for some time at a rate close to that given by the linear stability theory (this growth rate is much smaller than the simultaneous growth rates of the oblique waves). Later, when amplitudes of the oblique waves become considerably greater than the plane-wave amplitude, nonlinearity begins to affect the evolution of the plane wave as well. At this stage the self-interaction of oblique waves becomes important and considerably changes the law of their growth, leading to oscillations of their growth rates, at first around their earlier high growth rate and then around the zero growth rate corresponding to the final saturation stage. These conclusions agree with some experimental results by Corke and Mangano (1988, 1989) (for more information about their work see Sec. 5.3) but in general there are not enough data to confirm the results; moreover, it was noted by Healey (1995) that the assumptions used by Mankbadi *et al.* may be valid only at unrealistically large Reynolds numbers. Some results supplementing those discussed here were presented, in particular, by Goldstein (1994, 1995) and Wu (1995) but we have no space to discuss them here.

5.3. K AND N REGIMES OF INSTABILITY DEVELOPMENT IN BOUNDARY LAYERS; EXPERIMENTAL STUDIES OF THE N REGIME

Experimental data which could be compared with the weakly-nonlinear theories considered above appeared only relatively recently. Therefore, it is no wonder that for some time these theories did not attract much attention. Recall that at the beginning of Sec. 5.2 the classical papers of Schubauer and Klebanoff (1956), Klebanoff and Tidstrom (1959) and Klebanoff *et al.* (1962) were cited as the primary source of experimental information about the nonlinear development of 3D wave disturbances in boundary-layer flows. In particular, the last-named has for many years been referred to very frequently by experts in the flow-stability theory. However, it has already been mentioned that the experimental data contained in these papers agreed only qualitatively with the early theoretical models by Benney and Lin (1960) and Benney (1961, 1964) of the two-mode disturbance development. The point is that in these theoretical papers it was assumed that two-dimensional and three-dimensional modes have the same frequency, while according to the results of Klebanoff and his co-authors this is not the case. However, these results disagree even more strongly with Craik's model of a resonant-triad interaction, where the frequency ω_1 of the two three-dimensional waves is taken equal to one-half of the frequency ω of the two-dimensional wave. Klebanoff and his co-workers studied the development of disturbances produced by a vibrating ribbon in a flat-plate boundary layer and found three-dimensional flow oscillations, but their frequency ω_1 differed only slightly from the fundamental frequency ω_0 of ribbon oscillations and of the 2D wave produced by it. These 3D oscillations appeared at relatively small values of x (i.e., soon after the origin of the 2D wave) and later, at larger values of x, these regular oscillations were transformed into irregular bursts of high-frequency fluctuations (so-called 'spikes'; see Fig. 5.21 in the beginning of Sec. 5.5) which preceded the formation of turbulent spots and final transition to turbulence (cf. the short description of boundary-layer instability in Sec. 2.1; for more detailed characterization of the boundary-layer instability considered here see again Sec. 5.5). However, no subharmonic waves with half the fundamental frequency were found in these experiments.

It is now clear that these experimental results did not prove the incorrectness of Craik's model but only showed that the nonlinear development of boundary-layer disturbances observed by Klebanoff *et al.* was not due to Craik's resonance mechanism. Note in this respect that Morkovin and Reshotko (1990) reasonably remarked that even in the cases of similar flow geometries and initial velocity fields there is no universality in the instability and transition process; because of the wide variety of external-flow disturbances feeding this process, and the large number of permissible nonlinear developments of them there is a great variety of possible behavior. This remark [which in a less definite form was also stated by Herbert and Morkovin (1980) and was often repeated by later authors; Shaikh and Gaster's paper (1994) is just a typical example] describes excellently the conclusion following from numerous experimental results collected during the whole 20th century. So it may also explain quite convincingly the reason for the deviation of Klebanoff's experimental results from the predictions of Craik's theory.

For a number of years after Craik's theory of 1971, no subharmonic disturbances of frequency $\omega_0/2$ were observed in boundary layers where a two-dimensional 'fundamental wave' of frequency ω_0 was generated by some means [although *two-dimensional* subharmonics of the 'fundamental frequency' ω_0 were repeatedly found in mixing layers with antisymmetric velocity profiles, e.g., by Sato (1959), Browand (1966) and Miksad (1972), and also in plane and circular jets - see, e.g., Wehrmann and Wille

(1958)]. Therefore, it was usually assumed during these years that Craik's theory was inapplicable to real boundary-layer instabilities. Apparently the first work in which it was shown that subharmonic waves of frequency $\omega_1 = \omega_0/2$ do indeed sometimes appear in a constant-pressure boundary layer perturbed by a ribbon vibrating with the frequency ω_0 (corresponding, at a given value of Re, to a 2D wave unstable according to the linear stability theory) was that of Kachanov, Kozlov and Levchenko (1977) in Novosibirsk, Russia. These authors made hot-wire anemometer measurements of the streamwise velocity fluctuations $u(x,y,z,t)$ [the deviations of instantaneous streamwise velocities from the undisturbed velocity $U(z)$] in a ribbon-excited boundary layer. Then they determined normalized amplitudes $A = u'/U_0$ of these fluctuations (where, as above, u' is the appropriately defined[1] real amplitude of u-fluctuations and U_0 is the free-stream velocity; note that in the experiments of Kachanov *et al.* the initial values of A were much smaller than in the experiments of Klebanoff *et al.*). Kachanov *et al.* measured the frequency spectra $P_u(f)$ (where $f = \omega/2\pi$ is the frequency measured in Hz) of the streamwise-velocity fluctuations $u(t)$ (describing the spectral composition of these fluctuations) at various points (x, y, z). They found that, together with the main spectral peak at the frequency f_0 of ribbon oscillations and higher harmonics of frequencies $2f_0$ and $3f_0$ (which are typical for any nonlinear wave development and were seen almost everywhere in the flow), velocity fluctuations with frequencies much below f_0 were also observed at large enough values of x. Moreover, at such values of x subharmonic fluctuations of frequency $f_1 = f_0/2$ were also detected at all points of observation (as a typical example see Fig. 5.5a, where peaks at frequencies $2f_0$ and $3f_0/2$ are produced by nonlinear interactions of the primary wave of frequency f_0 with itself and with the subharmonic of frequency $f_0/2$, and where peaks at $3f_0$ and $5f_0/2$ are due to interactions of the same primary wave and its subharmonic with the second harmonic). Another example of the same type is shown in Fig. 5.5b, taken from the paper by Kachanov and Levchenko (1984); here a relatively wide low-frequency range of amplitude fluctuations with a peak at $f = f_0/2$ is seen at both values of x and f_0. The appearance of subharmonic fluctuations in

[1] It is often convenient to define the fluctuation amplitude as the root-mean-square value (i.e., as the square root of the temporal mean value of squared fluctuations). Such definition is widely used, in particular, in studies of turbulent flows.

the experiments by Kachanov *et al.* (1977) coincided with the onset of three-dimensionality, producing appreciable spanwise variations of flow characteristics. These results strongly suggested to the authors that Craik's three-wave resonance took place at the corresponding values of x.

The results found by Kachanov *et al.* in 1977 were later confirmed, supplemented by many details, and expounded in research papers and surveys both by members of the Novosibirsk group [see, e.g., Kachanov *et al.* (1978, 1980, 1982), Kachanov and Levchenko (1982, 1984), Kachanov (1987, 1994a,b), Boiko *et al.* (1999)], and by other scientists, partially in collaboration with those from this group [see, e.g., Thomas and Saric (1981), Saric *et al.* (1981), Saric and Thomas (1984), Santos and Herbert (1986), Thomas (1987), Yan *et al.* (1988), Corke and Mangano (1988, 1989), Corke (1989, 1990, 1995), Saric, Kozlov and Levchenko (1984), Kozlov, Levchenko and Saric (1984), and Bake *et al.* (1996, 2000)]. It was also noted by Saric and Thomas (1984) and Herbert (1988a) that some related results (which will be described later) had been observed in early flow-visualization studies by Knapp and Roache (1968) which did not attract much attention at the time.

Comparison of the results of the above-mentioned papers with those found by Klebanoff and his co-workers clearly shows that there exist at least two different routes of boundary-layer transition to turbulence. The first of these transition regimes, whose study was initiated by Klebanoff's work, usually corresponds to relatively large initial amplitude of a two-dimensional wave disturbance (with values of u'/U_0 of the order of 1% or more, where u' is the amplitude of streamwise-velocity fluctuations at the distance from the wall where this amplitude is a maximum). Herbert and Morkovin (1980) proposed to call this regime the *K-Regime* (for Klebanoff); their proposition was widely accepted and will be used in this book too. As was indicated earlier, the K-regime includes the formation of three-dimensional structures leading to appearance of bursts of high-frequency fluctuations which are later transformed into separate turbulent spots; these spots multiply and grow with time, then start merging with each other, and finally occupy the whole boundary layer. Only the first stage of this regime was studied by Klebanoff *et al.* (1962) and only this regime was briefly considered in Sec. 2.1. The second regime, discovered in experiments of the Novosibirsk group, is often called the *N-Regime* [see, e.g., Kachanov's survey papers (1987, 1994a,b)], and below we will normally use this name. (Other names found in the literature are *Subharmonic Regime* and *S-Regime*; these names stress the importance here of the subharmonic resonance.) The N-regime of disturbance development does not lead to the appearance of 'turbulent spots' (localized regions of very strong fluctuations), and usually occurs only under some special initial conditions (in particular, at initial values of u'/U_0 appreciably smaller than 1%) and is rarely realized in natural and engineering flows (therefore, it was not mentioned in Sec. 2.1). In particular, the emergence of the N-regime requires that the boundary layer contains a two-dimensional T-S wave with rather small initial amplitude u'/U_0 [but not smaller than about 0.3%; this last condition was first mentioned in qualitative form by Görtler and Witting (1958), was independently presented, together with the quantitative (but numerically incorrect) Fig. 5.4 by Maseev (1968a,b) and later was proved by different theoretical methods by Orszag and Patera (1983), Herbert (1984a, 1985, 1988a) and Zel'man and Maslennikova (1984, 1993a)]. According to many authors, the N-regime may begin either with a nonlinear wave resonance of Craik's type or with a secondary-instability phenomenon. Saric and Thomas (1984), who found that the spanwise periodicity and the character of the observed nonlinear wave development can depend on the initial value of u'/U_0, even recommended distinguishing these two origins of boundary-layer three-dimensionality by introducing the attributes 'C-type' (for Craik) and 'H-type' (for Herbert) (the data motivating their proposal will be considered later). However, later it was shown that the nonlinear resonance may have many different forms, and often it cannot easily be distinguished from the secondary-instability development of flow disturbances.

Before the detailed consideration (in this and the next sections) of the results relating to the N-regime of disturbance development in a boundary layer and then (in Sec. 5.5) of the main features of the K-regime, it is worth making some general remarks about this subject. Note that both the regimes were discovered in experiments where a ribbon vibrating with a constant angular frequency ω was used to generate the primary disturbance. Hence, we consider here only the so-called 'normal transition scenarios', which begin with the emergence in the flow of a linearly-unstable (or linearly-stable but transiently growing) Tollmien-Schlichting wave. However, it was noted in Chap. 2 (Sec. 2.92, p. 118) that both in laboratory experiments and in real life external-stream disturbances can be large enough for 'by-pass transition' to occur, with no observable small-amplitude T-S waves at the beginning of the process as in 'normal transition'.

In fact, the N-regime of disturbance development can really occur only in cases of rather low levels of external disturbances (of background or environmental origin). In such cases it is often even unimportant whether only one periodic T-S wave or a more complicated disturbance appears first. It will be explained below that even in the case of a single primary plane wave the N-regime development quickly leads to a disturbance spectrum of rather complicated form. The K-regime corresponds to the cases of boundary layers with a higher level of external disturbances; here also the primary disturbance may not necessarily have the form of a single T-S wave. Usually the K-regime leads to the scenario of transition to turbulence through the stage of 'turbulent spots' (see Sec. 2.1); therefore, the final stages of the K-regime may also be realized in 'by-pass transition'. Note also that when the N-regime of boundary-layer evolution develops without further disturbance for a long enough time, it may gradually acquire some features of the K-regime; see in this respect the discussion of papers by Bake *et al.* (1996, 2000) at the end of this section and in Sec. 5.5.

Let us now consider at greater length the data relating to the first stage of the N-regime. Kachanov *et al.* (1977) in their experiments showed only that an appreciable subharmonic component of velocity fluctuations with frequency $f_1 = f_0/2$ appeared simultaneously with the onset of flow three-dimensionality. This observation gave reason to suggest that Craik's three-wave resonance may have been present but, of course, it could not be considered as a proof of such resonance. Therefore a much more detailed study of the instability phenomenon observed in 1977 was carried out by Kachanov and Levchenko (1982, 1984). Here frequency spectra of streamwise velocity fluctuations in the constant-pressure boundary layer [identical to that studied by Kachanov *et al.* (1977)] were measured at a number of distances x from the plate leading edge and heights z above the plate (one of the results obtained is shown in Fig. 5.5b). Then narrow-band frequency filters were used to isolate (*a*) the 'primary wave' of velocity fluctuations produced by ribbon vibrations of frequency f_0, and (*b*) the subharmonic waves of half that frequency. The phases ϕ and streamwise wavenumbers k_1 of the primary and subharmonic waves were measured, and it was shown that the phase synchronism required for resonance (usually reducing to the condition $\phi_{1,0} = \phi_{2,0} + \phi_{3,0}$, where $\phi_{i,0}$ is the initial phase of the ith wave, and $i = 1$ for the primary wave) actually occurred, and that the resonance condition $k_1 = k/2$ of Eqs. (5.7) was satisfied with high accuracy. It was also found that the amplitude of the subharmonic wave of frequency $f_0/2$ grew rapidly with x (from the viewpoint of a fluid element, with time t measured from the moment of wave excitation by the vibrating ribbon) over a considerable range of x, while the amplitude of the primary wave changed only a little in this range (see Fig. 5.6, and also Fig. 5.3 which shows some subsequent results relating to more general initial conditions). All this supports very convincingly the suggestion by Kachanov *et al.* (1977) that the three-wave resonance predicted by Craik was really observed in their experiments.

Kachanov and Levchenko also measured the spanwise distributions of the amplitude and phase for both the primary wave and the subharmonics; one typical result of such measurements is shown in Fig. 5.7. These measurements confirmed that the primary wave is two-dimensional, while the subharmonic of frequency $f_0/2$ is three-dimensional and the dependence of its amplitude on y is close to that of the function $B\cos(k_2 y)$ (corresponding to a pair of symmetric oblique waves with spanwise wavenumbers $\pm k_2$), where B depends on x and z (and also on the frequency f_0 of the primary wave). Experimental data of the type presented in Fig. 5.7 were used by Kachanov and Levchenko to determine the spanwise wavenumber k_2 and the angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ between the propagation directions of the plane 2D wave and of the two subharmonic oblique waves. According to the results obtained, $|\theta_{1,2}| \approx 63\text{-}64^\circ$ in the main part of the region where strong three-wave resonance was observed. These values differ from the theoretical estimate $|\theta_{1,2}| \approx 50^\circ$ obtained by Volodin and Zel'man in 1978 (when there were no experimental data to compare with predictions) for a version of Craik's three-wave-resonance model of disturbance development. However, Kachanov and Levchenko did not pay too much attention to this discrepancy, which did not shake their confidence in the discovery of Craik's resonant structure. Subsequent theoretical studies, which will be considered later, showed that Kachanov and Levchenko were right, since Volodin and Zel'man's estimate of the angle $|\theta_{1,2}|$ was based on an oversimplification of the problem.

Kachanov and Levchenko's data also included the measured values of vertical (z-wise, normal-to-wall) profiles of the real amplitude $|A|$ and the phase ϕ (where $|A|e^{i\phi} = A$ is the complex amplitude[2]) for both the primary 2D wave and its subharmonics of half the primary frequency. The profile measurements were made in the flow region where strong resonant interactions take place among waves of these two types. Results obtained for amplitude $A(z)$ and phase $\phi(z)$ of the primary-wave streamwise velocity fluctuations $u(x,z,t) = A(z)\exp[i\{kx-\omega t+\phi(z)\}]$ were found to be very close to the corresponding conclusions of linear stability theory relating to the two-dimensional T-S wave considered. As to the measured vertical

[2] Below in this chapter, in cases where complex amplitude is not considered, the real amplitude $|A|$ will usually be denoted as A.

profile of subharmonic-wave real amplitude A, its accuracy was also confirmed by Corke and Mangano's (1989) measurements, which will be considered slightly later. It will be also noted later in this section that, according to the experimental results of Corke and Mangano (1989) and Corke (1995), the vertical profile of subharmonic-wave amplitude found by Kachanov and Levchenko is close to the profiles corresponding to subharmonic waves entering more general resonant triads, which satisfy the resonant conditions (5.7) not exactly but only approximately. Moreover, in the survey (1994a) Kachanov compared vertical profiles of the subharmonic-wave amplitude and phase presented in Kachanov and Levchenko (1982, 1984) with some theoretical and numerically-simulated estimates of these profiles, and showed that the experimental data agree excellently with these estimates. [For more details see Figs. 5.16a,b in Sec. 5.4 and the text there relating to these figures (including that printed in small type).]

Continuing the consideration of experimental data relating to the N-regime of nonlinear disturbance development in boundary-layer flow, we note the visualization studies of boundary-layer instabilities carried out in the early 1980s by Saric and his co-authors [who in fact began with independent repetition of the early observations by Knapp and Roache (1968) which long remained unknown to the majority of scientists]. These studies showed that three-dimensional vortical structures, which appear in the Blasius boundary layer in the course of nonlinear development of an initially two-dimensional Tollmien-Schlichting wave, differ considerably in the cases of the K-regime and the N-regime of laminar-flow breakdown. In both cases nonlinear effects produce some regular process of distortion of the primary 2D wave into three-dimensional vortices reminiscent of the Greek letter Λ, with tips directed downstream (so-called 'Λ-vortices'). In the case of the K-regime, these vortices form an ordered vortical structure of peak-valley splitting in which the successive peaks are spatially in phase and follow regularly behind one another (see a typical flow-visualization picture in Fig 5.8a). On the other hand, in the case of the N-regime the structure consists of spanwise rows of Λ-vortices where successive rows are out of phase and the peaks of one row are aligned with the valleys in the next row (see Fig. 5.8b). Just such a 'staggered vortical structure' was first observed in visualizations of disturbed boundary-layer flow by Knapp and Roache (1968); this structure clearly corresponds to twice the streamwise wave length (i.e., half the wave number) of the ordered K-regime structures in Fig. 5.8a. Later, structures of both types were independently found and described by Thomas and Saric (1981) and Saric *et al.* (1981) who applied to boundary-layer flows the technique of air-flow visualization by smoke developed by Corke *et al.* (1977). More detailed analysis of the data of Saric and his co-workers was presented by Saric and Thomas (1984), Saric *et al.* (1984), Kozlov *et al.* (1984), Craik (1985), Thomas (1987), Herbert *et al.* (1987), Herbert (1988a,b), and Nayfeh (1987a,b); in these publications numerous flow-visualization pictures were presented (Figs. 5.8a,b represent just one such example). The first high-quality pictures were published by Saric and Thomas (1984), who used flow visualization to observe the nonlinear wave development in a zero-pressure-gradient boundary layer disturbed by a vibrating ribbon, at different values of the disturbance level u'/U_0 (where, as above, u', observed not far from the ribbon, is the maximum with respect to z of the amplitude of the streamwise-velocity oscillations in the excited plane T-S wave, and U_0 is the free-stream velocity). At u'/U_0 = 0.7%, Saric and Thomas found the usual K-type nonlinear development which was earlier observed by Schubauer, Klebanoff and his co-authors, and a number of other experimenters. However, for $u'/U_0 < 0.5\%$ the character of the picture changed, and instead of an ordered peak-valley vortical structure corresponding to that in Fig. 5.8a a staggered structure of the type shown in Fig. 5.8b was observed. Moreover, Saric and Thomas also found that some important details of the staggered structure depended critically on the initial value of u'/U_0. At $u'/U_0 = 0.3\%$ they obtained a picture which agreed excellently with Craik's fully-resonant triad: here the angular frequency of the 3D oblique waves was equal to $\omega/2$, with high precision, and the streamwise wavenumber of these waves was practically equal to $k/2$, where k is the wavenumber of the two-dimensional T-S wave excited by the vibrating ribbon. At the same time, the spanwise wavenumber k_2 of oblique 3D waves found at this value of u'/U_0 agreed well with the value given by the general O-S equation (2.41) for a three-dimensional T-S wave with angular frequency $\omega/2$ (the angular frequency of ribbon oscillations is now denoted by ω) and streamwise wavenumber $k/2$, while the vertical profile of the 3D-wave amplitude $u_3'(z)$ measured by a hot-wire anemometer agreed with amplitude calculations based on Craik's (1971) resonant-triad theory. However, at slightly higher disturbance level, $u'/U_0 = 0.4\%$, the value of k_2 given by flow-visualization data was more than twice as large as that corresponding to a three-dimensional T-S wave with frequency $\omega/2$ and streamwise wavenumber $k/2$. The results of Saric *et al.* (1984) also showed that spanwise periodicity of the 3D structures depended very significantly on the disturbance level. These results, which have already been mentioned in Sec. 5.2 (see p. 28), clearly showed that the observed vortical structure could not always be due to the simple Craik mechanism of three-wave resonance, which has the same form at any value of the 2D-wave amplitude.

Important subsequent experimental studies of the N-regime of wave development in Blasius boundary-layer flow were carried out by Corke and Mangano (1988, 1989), Corke (1989, 1990, 1995), and Bake *et al.* (1996, 2000). These authors produced controlled wave disturbances in a boundary layer by means other than the old but still-popular vibrating ribbon. In particular, Corke and his group used the method proposed by Liepmann *et al.* (1982) and then refined by Robey (1987). Instead of the usual vibrating ribbon, Liepmann *et al.* used a heating wire, placed in the initial part of a water boundary layer and excited electrically to give a temperature varying periodically with given frequency $f = \omega/2\pi$. They used a single wire which was stretched spanwise from wall to wall of the test rig; since wire-temperature variations generate local changes of flow viscosity (and local buoyancy forces), the spanwise heating wire excites a 2D wave of frequency f in the flow. Robey noted that this technique lends itself to 3D forcing since the heater geometry can be prescribed arbitrarily. He used a heater array consisting of 32 rectangular surface elements separated by narrow gaps. In Robey's experiments, individual elements were aligned in a single spanwise row, but by varying the distribution of the phase and/or amplitude of the temperature fluctuations across the span of the array he could produce many different 3D disturbances. In the experiments of Corke's group, this method was applied to the air boundary layer in a wind tunnel where a single heating wire, whose temperature fluctuated with frequency f, was supplemented by a spanwise array, at a fixed x-location close to that of the first wire, of individual heating segments of fixed spanwise length s, again separated by narrow gaps. The temperature of the heating segments oscillated with a fixed frequency (most often with the subharmonic frequency $f_1 = f/2$); moreover, these authors also introduced a definite phase shift ϕ between temperature variations at any two adjacent segments. This arrangement generated time-periodic and spanwise-periodic variations of flow velocity which excited a pair of symmetric oblique waves. These waves propagated streamwise, and their spanwise wavenumbers $\pm k_2$ and inclination angles $\theta = \pm\tan^{-1}(k_2/k_1)$

depended on f, ϕ and s and hence could be changed by changing values of some of these parameters. (Usually the values of k_2 and θ were adjusted by changing the phase shift ϕ.) The amplitudes of the plane and oblique waves depended on the amplitudes of heating-wire and heating-array temperature oscillations; hence both wave amplitudes could be arbitrarily varied. Thus, the heating method had an important advantage over the vibrating ribbon, since here all the important parameters of both the 2D plane and 3D oblique waves could be prescribed by experimenters. Results were recorded by smoke-flow visualization by smoke and by hot-wire measurements of all three velocity components.

Recall that at the beginning of Sec. 5.2, it was noted that Klebanoff *et al.* (1962) also artificially generated spanwise periodicity of the boundary-layer disturbances, but the purpose of this procedure was then quite different. In the old work of 1962 and in all repetitions of it by other authors, spanwise forcing was used only to shorten the time needed for the natural appearance of spanwise variations of the nominal 2D disturbance. Therefore, the experiments by Corke's group, where the amplitudes, frequencies, streamwise and spanwise wavelengths of all waves of a triad, and also the degree of phase synchronism between plane and oblique waves could be prescribed beforehand by the investigators, were much more informative than those of Klebanoff *et al.* and their successors.

Corke and Mangano (1988, 1989) began their experiments with boundary-layer observations in the absence of any heating-wire forcing. They found that then the boundary-layer velocity profile preserved the Blasius shape down the whole length of the wind-tunnel test section, and among the observed weak disturbances induced by background noise the least-stable T-S wave played the dominant part. Then the authors switched on the wall-to-wall heating wire, using two different temperature-oscillation frequencies f corresponding to values $F \times 10^6 = 88$ and $F \times 10^6 = 79$ of the dimensionless frequency $F = 2\pi f\nu/U_0^2 = \omega\nu/U_0^2$ (used in Figs. 2.26, 2.27, in Sec. 2.92 and also in Fig. 5.3). At the position of the heating elements exciting the waves, both frequencies corresponded to linearly-unstable T-S waves (in these experiments $x_1 = 45$ cm and $\mathrm{Re}^+ = (U_0 x_1/\nu)^{1/2} = 430$ at the position of the heating wire, where x_1 is measured from the beginning of the test section upon whose wall the boundary layer was developed). At larger values of x_1 (where Re increased because of boundary-layer growth) these T-S waves became stable to infinitesimal disturbances. The hot-wire measurements showed that, even for a very small initial amplitude of the wave excited by the heating wire (much smaller than the initial wave amplitudes used in all studies of the K-regime of boundary-layer breakdown), the T-S wave corresponding to the frequency F of the excitation was easily detected against the background of much weaker external noise. Moreover, the initial exponential growth and later decay of this T-S wave, predicted by the linear stability theory, was also found in the experiments of Corke and Mangano. This agreement with the linear theory showed that the locally-plane-parallel approximation used in the theory was sufficiently accurate. However, conclusions based on the linear stability theory were in fact unimportant here, since the linear-theory rates of wave growth and decay were negligibly small in comparison with the rates of change due to nonlinear interactions, which were the main object of the study.

As to the experiments where both 2D and 3D waves were artificially excited, Corke and Mangano considered only cases where $f_1 = f/2$, and restricted themselves to the study of three special cases. In two of these cases the dimensionless frequency took the value $F\times10^6 = 79$ (and hence $F_1\times10^6 = 2\pi f\nu/U_0^2\times10^6 = 39.5$) and the phase shift ϕ took either the value corresponding to oblique-wave inclination angles $\theta_{1,2} = \pm45^\circ$, or a value such that $\theta_{1,2} = \pm59^\circ$ (cases 1 and 2, respectively), while in the third case the values were $F\times10^6 = 88$, $F_1\times10^6 = 44$ and $\theta_{1,2} = \pm61^\circ$. In all three cases flow visualization showed a 'staggered vortical structure' of the type presented in Fig. 5.8b. For case 3 the spanwise distributions of the amplitude $A = u'_{\max}/U_0$ (where as before $u'_{\max}$ is the value of u at the height z where it is a maximum) and the phase ϕ of the primary 2D wave of frequency $F = 79\times10^{-6}$, and of the sum of 3D oblique waves with half this frequency, are shown in Fig. 5.9. [These distributions were determined from hot-wire measurements at points with different values of y, fixed $x = 150$ cm (measured from the location of the array of heaters) and a value of z corresponding to the critical layer where the mean velocity $U(z)$ is equal to the phase velocity c of the 2D wave.] Fig. 5.9 confirms that the amplitude and phase of the primary wave have uniform spanwise distributions, as must be the case for a plane wave, while for subharmonic oscillations of half the frequency these distributions are consistent with the sum of two symmetric oblique waves with spanwise wavenumbers $\pm k_2$. Similar results were obtained by Corke and Mangano for two other cases; cf. also Fig. 5.7 showing the results of Kachanov and Levchenko (1984).

The measurements of the streamwise velocity fluctuations u at a number of points on the centerline ($y = 0$), with different values of the coordinate x and coordinate z corresponding to the maximum amplitude of these fluctuations, allowed Corke and Mangano to determine the downstream development of the streamwise-velocity amplitude u'_{max} of both the plane wave (having frequency F) and the subharmonic oblique waves (with frequency $F_1 = F/2$); see Fig. 5.10. Fig. 5.10a shows that the rates, $G = d(\ln A)/dA = dA/Adx$, of the downstream growth of the oblique-wave amplitude A differ in the three cases considered, but in all of them these rates considerably exceed those given by the linear stability theory, over a wide range of x-values (i.e., of times t measured from the moment of wave excitation). Corke and Mangano showed also that the rates $G = G(x)$ in all three cases change strongly with x - at first increasing with x to some maximal value G_{max} (different in the three cases and also occuring at different values of x) and then decreasing with x. On the other hand, Fig. 5.10b shows that the amplitude of the 2D plane wave changes much more slowly. Recall that Fig. 5.6 showed similar behavior of the amplitudes of 2D and 3D waves; however, it represented the results of Kachanov and Levchenko's experiments where only the 2D wave was artificially excited, while oblique 3D waves were mainly due to background noise. Hence it was natural to suppose that the observed 3D waves are just those with the highest rate of growth in the presence of the excited plane wave. Thus, it was assumed that the excited 3D waves, together with the primary 2D wave, form Craik's fully-resonant triad which, according to Craik's theory, extracts energy from the undisturbed flow in the most powerful way. As to Fig. 5.10, here all three waves of the considered triads were artificially excited and their frequencies, wave vectors and amplitudes could be chosen by the experimenters; therefore, it was not clear beforehand whether they would or would not satisfy Eqs. (5.7) representing Craik's conditions of strict resonance.

Since the frequencies $F = \omega\nu/U_0^2$ and $F_1 = \omega_1\nu/U_0^2$ were chosen so that $F_1 = F/2$, the second condition (5.7) was valid in all three cases studied by Corke and Mangano. However, the first condition, which concerns the wavenumbers and guarantees that the primary 2D wave and subharmonic 3D waves have exactly the same phase velocity, was not automatically satisfied in their experiments. Note that under the conditions of these experiments k could be determined with the help of the O-S equation (2.44) as the streamwise wave number of the least-stable plane T-S wave in the Blasius boundary layer having the given frequency $\omega = FU_0^2/\nu$. It was found that the agreement of the values of k determined in this way with the directly-measured values of k was usually rather close and this evidently confirmed the accuracy of both methods.

Values of ω and k determine the phase velocity $c = \omega / k$ of the primary T-S wave. As to the phase velocity c_1 of the 3D oblique waves, knowledge of $k_2/k_1 = \tan\theta$ (or of the value of k_2 which could be determined from Fig. 5.9 and similar figures for the two other cases studied) allowed k_1 to be computed from the three-dimensional O-S equation (2.41) [This equation has the same form as the 2D Eq. (2.44) and satisfies the same boundary conditions (2.42), but it determines only the vertical profile of the vertical velocity amplitude $W(z)$. For discussion of the computations of the horizontal velocity components see the papers by Kachanov and Michalke (1994,1995) and Kachanov (1996), and also the earlier papers by Chen and Bradshaw (1984) and Tang and Chen (1985) demonstrating the use of 2D linear stability computations for determination of eigenvalues and eigenfunctions of the 3D linear stability problem]. Calculations of k_1 with the help of the O-S equations led to values of $c_1 = k_1/\omega_1$ according to which the condition $c = c_1$ was satisfied with high accuracy in Corke and Mangano's case 1, while in cases 2 and 3 it was not satisfied although the differences between the two phase velocities were not large. Values of c_1 determined from the experimental data led to much closer agreement with Craik's resonant conditions, in all three cases, than did the values computed from the linear O-S equations. Corke and Mangano therefore concluded that in the presence of the primary 2D mode the 3D subharmonic modes reach phase-velocity synchronization with the primary mode in the course of their development, whatever the initial conditions, and noted that this conclusion agrees with Herbert's secondary-instability theory but disagrees with Craik's theory of fully-resonant triads. This topic will not be further discussed here; note only that, according to Fig. 5.10, considerable growth of 3D waves was observed in all three cases studied (but was different in different cases). Because of this one may suppose that a subharmonic resonance of some form occured in all these cases.

Corke and Mangano carried out a more detailed investigation of the properties of the wave triads studied by them, and found that all properties observed in their experiments agreed well with the predictions by Herbert (1983b, 1988a) and Herbert *et al.* (1987) [and also with subsequent results by Crouch and Herbert (1993)] relating to evolution of secondary-instability waves in boundary layers (see in this respect Fig. 5.15a in Sec. 5.4, which is taken from Corke and

Mangano's paper). However, as was noted above, some of the properties observed in cases 2 and 3 were found to be inconsistent with those of fully-resonant triads. Therefore Corke and Mangano concluded that the C-type and H-type of nonlinear development of subharmonic waves in the N-regime of boundary-layer instability growth may be distinguished in practical situations, and in their experiments case 1 corresponded to C-type development, while cases 2 and 3 corresponded to H-type development. Note, however, that later Zel'man and Maslennikova (1993a) generalized Craik's concept of the fully-resonant triad and stated that their version of the three-wave-resonance theory admitted deviations of wave characteristics from the strict-resonance conditions (5.2b) and led to results which also agreed very well with Corke and Mangano's data. Furthermore, Fig. 5.15b shows that the method proposed by Mankbadi (1991, 1993a) for approximate evaluation of the growth rates of oblique waves entering symmetric wave triads gives results which agree excellently with the experimental data of Corke and Mangano in all three cases studied.

A more thorough analysis of Corke and Mangano's data, supplemented by results of a few additional experiments of the same type, was carried out by Corke (1987, 1989, 1990, 1995). In his papers the main attention was paid to the spectra of the velocity fluctuations and the explanation of their origin. In this respect Corke investigated the spatial development of various harmonics generated by nonlinear interactions of 2D and 3D waves with themselves and with each other, and by higher-order interactions of these 'harmonics of the lowest order' among themselves and with the primary 2D and 3D waves. In the 1990 and 1995 papers the effect of 'mode detuning' (noncoincidence of the frequency f_1 of an artificially-excited 3D-wave with the 'resonant frequency' $f/2$) was specially studied. Corke (1995) used the same combination of heating elements as Corke and Mangano (which allowed the frequencies of 2D and 3D waves to be set to any values) to excite a pair of symmetric oblique waves with dimensionless frequency $F_1 \times 10^6 = 39.5$ (corresponding to $f_1 = 16$ Hz) and $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1) = \pm 59^\circ$ together with a 2D (plane) T-S wave whose dimensionless frequency F took different values in the five successive experiments. The values of $F \times 10^6$ used were: 79 (this is the 'tuned case' where $F_1 = F/2$) and 81, 84, 86 and 88 (they correspond to frequencies $f = 32$, 32.8, 33.5, 34.75 and 36 Hz). In all Corke's wave triads the streamwise wavenumbers k and k_1 of the 2D and 3D primary waves satisfied the 'wavelength resonance condition' $k_1 = k/2$ with high accuracy, but the 'frequency resonance condition'

$\omega_1 = \omega/2$ was satisfied only in the 'tuned case'. Measurements of the spectra of streamwise-velocity fluctuations downstream of the heating elements showed, in all cases, numerous 'higher-order waves', produced by nonlinear interactions among existing waves and having frequencies and wave vectors equal to differences or sums of those of the pre-existing waves. Recall that in the case of simple fully-resonant triads quite similar 'oscillations and waves of higher orders' were also observed by Kachanov *et al.* (1977) and Kachanov and Levchenko (1982,1884) and some of them are shown in Fig. 5.5a.

Corke's results corresponding to the 'tuned case', where $f = 2f_1 = 32$ Hz, agreed excellently with those found by Corke and Mangano (1989), while among the 'detuned cases' (where $f \neq 2f_1$) only some representative results for the 'most-detuned' case where $F \times 10^6 = 88$ (i.e., $f = 36$ Hz) were described at length in his paper of 1995. In this 'most-detuned' case the artificially-excited 2D wave with frequency $f = 36$ Hz and wave number k, together with 3D oblique waves with frequency $f_1 = 16$ Hz and wave vectors $(k/2, \pm k_2)$, generated a number of supplementary 3D wave harmonics with 'combined' frequencies and wave numbers (in particular, with frequencies $20 = 36 - 16$, $4 = 20 - 16$, $32 = 16 + 16$, and $24 = 20 + 4$ Hz). Among these 'higher-order harmonics', the lowest order had 3D waves with frequency $f_2 = 20$ Hz $= f_1 + \Delta f$, $\Delta f = 4$ Hz, and wave vectors $\mathbf{k} = (k/2, \pm k_2)$ produced by nonlinear interactions of primary 2D and 3D waves. These waves are especially interesting since, together with the original 2D and 3D waves, they form a 'five-wave resonant system' consisting of two 'detuned resonant triads' with frequency-wavevector combinations $(f, k, 0)$, $(f_1, k/2, k_2)$, $(f_1+\Delta f, k/2, -k_2)$, and $(f, k, 0)$, $(f_1, k/2, -k_2)$, $(f_1+\Delta f, k/2, k_2)$ [cf. the related 'tuned five-wave resonances' mentioned in Sec. 5.1 and considered by Craik (1985), Sec.16.2]. The corresponding 'detuned resonances' explain well the rapid growth observed by Corke (which began immediately after the appearance of the wave of frequency 20 Hz) of both the primary 3D wave of frequency 16 Hz and the 3D harmonics of frequency 20 Hz (see Fig. 5.11). Note that in the early stages of disturbance development the 'harmonics' had smaller amplitude than the primary 3D wave; this was, of course, natural since 'harmonics' did not exist at the very beginning and had to be generated by interaction of the primary 2D and 3D waves. However, after their appearance the harmonics began to grow faster than the primary 3D wave, and some time later their amplitudes overtook that of the slowly-growing 2D wave. This situation is entirely similar to that predicted by Zel'man and Maslennikova for the modified cases of Craik's fully-resonant triad, where two oblique waves have initially different amplitudes (see Fig. 5.3 above). Using the data of some preliminary experiments of Corke's group, Mankbadi (1993b) proposed some approximate equations describing the dependence of the amplitudes of the 2D wave, and of two pairs of symmetric oblique waves entering 'a pair of detuned resonant triads', on Re (i.e. on the streamwise coordinate x determining the value of Re). The equation given for two oblique-wave amplitudes included cubic terms (more general than those in Eqs. (5.11) for the 'fully-resonant case') which allowed the saturation of the oblique wave to be determined. Mankbadi's amplitude equations were simplified by Corke (1995), who presented them in the form of three equations for the three amplitudes; these equations contained eleven constant coefficients requiring special determination. In this context Corke also discussed some data from his amplitude measurements which will not be considered here.

According to Corke, both oblique waves (with frequencies 16 and 20 Hz) of the 'detuned triad' had practically the same phase velocity (and hence the same 'critical layer'). They also had the same normalized vertical amplitude profile $|A(z/\delta^*)|/A_{max}$, which did not differ much from the amplitude profile of the oblique components of Craik's 'tuned' resonant triad with $f_1 = f/2$, $k_1 = k/2$, which was measured both by Kachanov and Levchenko (1982, 1984) and by Corke and Mangano (1989). (The results found by these two groups were rather close to each other; they will be discussed in Sec. 5.4 and shown there in Fig. 5.16a. At the same time, Corke and Mangano's results showed that in their cases 2 and 3, where k_1 took values close, but not equal, to $k/2$, the normalized profiles of subharmonic-wave amplitudes did not differ much from those observed in the 'fully-resonant' case 1.) On the other hand, the amplitude profile of the 'higher-order harmonic' with low frequency $f = 4$ Hz differed considerably from that in Fig. 5.16a, while as a rule the mean value of the amplitude $|A(z)|$ of this (and other) higher-order harmonic components of velocity fluctuations grew significantly as x increased.

Corke also showed that in the course of disturbance development new wave components were repeatedly generated by numerous nonlinear interactions among existing components. Thus, the detuned-triad resonance studied in his paper led to the appearance of a broad range of streamwise-growing discrete modes at intervals equal to the lowest difference frequency (equal to 4 Hz in the case considered here). An example of Corke's observations of the downstream growth of a number of such higher-order harmonic components is shown in Fig. 5.12. Let us recall that Figs. 5.5a,b show that frequency spectra of the nonlinearly-developing disturbances in a Blasius boundary layer perturbed by a vibrating ribbon are in fact very far from the pair of discrete lines at frequencies ω_0 and $\omega_0/2$ corresponding to a resonance triad of Craik's type. And detuned resonances generated by background noise, with low detuning Δf, may be one of the mechanisms producing the rapid growth of energy of low-frequency fluctuations and thus leading to formation of spectra of the type presented in Fig. 5.5a,b.

Another method of controlled wave excitation, proposed by Gaponenko and Kachanov (1994), was used by Bake, Kachanov and Fernholz (1996) and Bake, Fernholz and Kachanov (2000). These authors carried out their experiments in a wind tunnel at the Technical University of Berlin, having an axisymmetric test section with a diameter of 441 mm and a total length of 6000 mm. The boundary layer studied developed on the wall of the test section. At a free-stream velocity $U = 7.2$ m/sec the boundary-layer thickness δ at the position of wave excitation (corresponding to $x = x_s = 547$ mm if $x = 0$ corresponds to the beginning of the test ection) was close to 6 mm (with $\delta^* \approx 2$ mm). Under this condition the undisturbed normalized velocity profile $U(z/\delta^*)/U_0$ had practically the same Blasius form (which corresponds to flat-plate boundary layers) at all streamwise and spanwise measurement positions. The wave disturbances were introduced into the boundary layer by a 'slit generator' consisting of a long narrow slit (with 0.5 mm width, 5 mm depth and 260 mm length in spanwise - i.e. circumferential - direction) cut into the inner wall, and a set of 32 small tubes (with a spanwise spacing of 8 mm) placed under the slit and connected to three loudspeakers. The loudspeakers were fed by three different time-periodic signals which combined with each other inside the slit generator forming, near the outlet of the slit, a field of flow fluctuations corresponding to a 2D or 3D disturbance of any type of interest to the investigators.

Bake *et al.* used the primary frequency $f = 62.5\text{Hz}$ (corresponding to $F = 2\pi f\nu/U_0^2 = 115.5\times10^{-6}$ and to subharmonic frequencies $f_1 = f/2 = 31.25$ Hz and $F_1 = 57.8\times10^{-6}$) and studied four cases of excited wave disturbances:

I) The primary 2D wave of frequency f and large amplitude A is excited simultaneously with a pair of oblique subharmonics of frequency f_1 and low amplitude $A_1 \ll A$. The spanwise wavenumbers of the oblique waves $\pm k_2$ were determined by the spanwise spacing of the tubes feeding the slit generator, but the phases of primary and subharmonic waves could be prescribed by the experimenters and were chosen to be close to values which, according to previous data, are most favorable for the subharmonic resonance.

II) Only the pair of subharmonic waves with the same characteristics as in case I was excited.

III) Only the primary wave (the same as in the case I) was excited.

IV) The same three waves as in case I were excited, but the phase shift between the fundamental and subharmonic waves was selected to be *least* favorable for the subharmonic resonance.

In cases II-IV no indication of resonance was found; therefore only results for case I will be discussed below. Results relating to the initial stage of the disturbance development (for $\Delta x = x - x_s < 250$ mm) as a rule agreed well with those of the previous investigations. It was found that at the chosen values of f, k_2, A, A_1, and the phase shift between primary and subharmonic waves the resonance conditions (5.7), guaranteeing the equality of phase velocities of three waves, were satisfied with good accuracy. Hence it was only natural that for $\Delta x < 250$ mm the results of Bake *et al.* for spanwise distributions of the amplitudes and phases of the primary and subharmonic waves, for the normalized vertical profiles of the same amplitudes and phases, and for the 'growth curves' representing the dependence of the amplitudes of three waves on the streamwise coordinate x, did not differ much from the values of the same characteristics found for fully-resonant wave triads, e.g., by Kachanov and Levchenko (1984), Saric *et al.* (1984), Kozlov *et al.* (1984), Thomas (1987), and Corke and Mangano (1989), who usually also restricted themselves to not-too-large values of Δx (some results from these papers are shown in Figs. 5.6, 5.7, 5.9, and in Figs. 5.16 and 5.17 to be discussed in Sec. 5.4).

Note, however, that the wind tunnel used by Bake *et al.* had a very long test section and the region $\Delta x < 250$ mm is only a small part of it. In fact, the main purpose of the investigators was to study the N-regime of wave-disturbance development over a downstream range much greater than any explored previously. They found that at large values of x the wave development which began as the N-regime, unexpectedly acquired some features which were previously considered as typical only for the K-regime. However, these results can be discussed only with those relating to the K-regime of disturbance development, and this discussion must be postponed until Sec. 5.5.

5.4. COMPARISON OF THEORETICAL PREDICTIONS FOR THE N-REGIME WITH EXPERIMENTAL AND NUMERICAL DATA

Let us begin this section with a discussion of the remark made in Sec. 5.3 (see p. 46) that the discrepancy between Kachanov and Levchenko's (1982, 1984) experimental value of the inclination angle $\theta = |\theta_{1,2}|$ of observed subharmonic oblique components of a resonant wave triad, and the theoretical estimate of this angle by Volodin and Zel'man (1978), may be explained by some defects of Volodin and Zel'man's theory. The first hint indicating that, contrary to the conclusion of this theory, the value of the angle $\theta = |\theta_{1,2}|$ is apparently not universal but depends on the value of the plane-wave amplitude A_1 was given by Zel'man and Maslennikova (1984). In subsequent more explicit studies (1989, 1990, 1993a) these authors proved that there is a direct link between the values of θ and A_1. This proof confirmed the experimental results of Saric and Thomas (1984) and Saric *et al.* (1984) which have been already mentioned in Sec. 5.3 (on p. 48) and will be considered at greater length slightly later. Moreover, the proof clearly implies that the unique value of θ given by Volodin and Zel'man in 1978 cannot be universal.

The point is that in 1978 Volodin and Zel'man followed Craik's paper of 1971 and considered only 'fully-resonant triads' consisting of one plane and two symmetric oblique T-S waves exactly satisfying Eqs. (5.7), where k and k_1 are real parts of streamwise wavenumbers of plane and oblique waves, and ω and ω_1 are the real parts of the corresponding frequencies. [For the sake of brevity, the words 'real parts' are applied here to both wavenumbers and frequencies. Of course, in the overwhelming majority of actual stability problems only one of these two wave characteristics takes complex values.] Later Zel'man and Maslennikova (1989, 1990, 1993a) generalized Craik's model admitting, in particular, that two oblique waves may not be strictly symmetric (e.g., the amplitudes of these waves may differ from each other) while the two Eqs. (5.7) may be valid not exactly but only approximately. According to the results of these papers (some of which have already been mentioned in Sec.5.2) in the cases of these more general wave triads rapid resonant growth of the oblique waves also occurs quite often; see, e.g., Fig. 5.3 taken from the paper (1993a) and also Figs. 5.10 and 5.11 showing some experimental data confirming this conclusion. Now we will continue the discussion of the corresponding theoretical and experimental results.

Fig. 5.3 is only one example illustrating the general results given by Zel'man and Maslennikova (1993a). According to these results, if the value of $\mathrm{Re}^+ = U_0\delta^+/\nu = (U_0x/\nu)^{1/2}$ (or of $\mathrm{Re}^*=U_0\delta^*/\nu \approx 1.72\mathrm{Re}^+$) is given, then under rather general conditions there exists, for a given plane T-S wave of frequency ω, streamwise wavenumber k, and amplitude A_1, a large set of pairs of oblique 3D-waves of frequency $\omega_1 \approx \omega/2$ inclined at angles $\pm\theta$ to the undisturbed-flow direction; together with the primary plane wave these form 'resonant triads'. (These triads, as a rule, do not satisfy Eqs. (5.7) exactly, but nevertheless they are 'resonant' since the corresponding amplitude equations include resonant quadratic terms. Therefore, here the growth rates $G_o = dA_{2,3}/A_{2,3}dx$ of the oblique-wave amplitudes $A_2 = A_3$ strongly exceed the growth rate G_p of the plane-wave amplitude, which remains close to that given by linear stability theory.) For a number of such generalized resonant triads Zel'man and Maslennikova computed the interaction coefficients B_1, B_2 and B_3 of Eqs. (5.4a) by a method similar to that used by Volodin and Zel'man (1978); a few results of these computations are shown in Fig. 5.3. In the cases where two oblique waves entering the triads had the same amplitude, the same frequency $\omega/2$ and matched phases, these waves had variable streamwise wavenumbers $k_1 \approx k/2$ and values of spanwise wavenumbers $\pm k_2$ filling a rather wide range. For the existence of a collection of pairs of oblique waves resonantly excited by a given plane wave, it is only necessary that Re (according to any suitable definition) be high enough and that A_1 be not too small. Note in this respect that the existence of the threshold value A_{tr} of A_1, below which no growing 3D waves can be excited, was predicted quite early by Görtler and Witting (1958) and Maseev (1968a,b), and that Maseev's Fig. 5.4 implies also that at any $A > A_{tr}$ there is a finite range of k_2 values corresponding to 3D waves growing in the presence of the given plane wave.

The ranges of admissible values of k_2 and k_2/k_1, corresponding to positive growth rates G_o, and also the preferred values of k_2 and k_2/k_1, $(k_2)_{pr}$ and $(k_2/k_1)_{pr}$, (corresponding to the greatest value of G_o), depend on A_1, ω and Re, while the value of G_o itself depends on A_1, ω, Re and k_2/k_1. (Note that for given values of Re, $\omega/2$, and k_2, the streamwise wavenumber k_1 of the corresponding most-unstable oblique wave may be determined uniquely with the help of the O-S equation (2.41). However, the strict equality $k_1 = k/2$ will be valid here only for one special value of k_2.) Fig. 5.13, which is based on the results of Zel'man and Maslennikova's computations, shows a typical example of the dependence of G_o (non-dimensionalized with δ^+ as the unit of length) on $K_2 = k_2\nu/U_0$ (and also on $k_2/k_1 = \tan\theta$) for some definite values of the dimensionless parameters $\mathrm{Re}^+ = U_0\delta^+/\nu$ and $F_1 = \omega_1\nu/U_0^2$ and a number of values of the amplitude A_1 (measured as fractions of U_0). This figure shows that here the preferred value of k_2/k_1, which must be met most often in real boundary-layer flows, is not constant but grows with the value of A_1. According to the results of Zel'man and Maslennikova (1993a) (only partially represented in Fig. 5.13) the value of $(k_2/k_1)_{pr}$ depends very little on Re^+ and F_1, while at values of A_1 only slightly above the threshold value (which makes possible the resonant growth of some oblique waves), $(k_2/k_1)_{pr} \approx 1$ (and $|\Delta k| = |k_1 - k/2|$ is very small, i.e., the resonant triads are here close to Craik's conditions of perfect resonance). With an increase of the amplitude A_1, the range of values of k_2/k_1 corresponding to resonance conditions where $G_0 > 0$ also increases, the value of $(k_2/k_1)_{pr}$ grows and approaches 2 and $|\Delta k|$ also grows (but, nevertheless, $|\Delta k|/k$ remains relatively small). Hence, contrary to the expectation of Craik (1971), among the triads including one 2D and two symmetric 3D waves the growth rate of 3D waves usually attains its greatest value for a triad satisfying only approximately, but not exactly, the resonance conditions (5.2b).

Recall that Kachanov and Levchenko (1982, 1984) stressed that in their experiments the symmetric wave triads appearing in the flow (and hence apparently corresponding to the most rapid growth of oblique waves) were those exactly satisfying Eqs. (5.7). However, this statement apparently shows only that in these experiments $|\Delta k|/k$ was so small that it was difficult to distinguish it from zero. As to the results of Corke and Mangano (1989) shown in Fig. 5.10, according to which three wave triads with different values of the oblique-wave angle $\theta = \tan(k_2/k_1)$ also have different rates of oblique-wave growth, they clearly conform to the results just discussed. Having this in mind, Zel'man and Maslennikova (1993a) compared growth rates G_0 found by Corke and Mangano for three pairs of oblique waves excited in their experiments with the results of their own computations. This comparison showed that the experimental values of G_0 found by Corke and Mangano for three different values of k_2/k_1 (corresponding to two values of F_1 relatively close to each other, and to known values of Re and A_1 which differed very little in the three cases) agree very well with the computed values of G_0. As will be shown below in Figs. 5.15a,b, the closeness of the oblique-wave amplifications measured by Corke and Mangano to theoretical estimates was also confirmed by Corke and Mangano themselves, using a quite different theoretical model, and then by Mankbadi (1993a) who made comparisons with results of computations based on the use of one more theoretical model. The good agreement found by the above-mentioned authors between one set of experimental data and the results of three different theories apparently shows that these three theories in fact differ much less than appears at first sight.

Let us now pass to Fig. 5.14 which is also based on results by Zel'man and Maslennikova (1993a). This figure shows values of k_2/k_1 observed in some recent experiments where A_1 took different values. It is natural to believe that the values of k_2/k_1 observed in experiments are just those which correspond to maximal growth rates of oblique waves (recall that a similar assumption has been widely used in comparisons of the results from linear stability theory with experimental data). Therefore in Fig. 5.14 the values of k_2/k_1 observed in experiments are compared with theoretical estimates of $(k_2/k_1)_{\mathrm{pr}}$. Two types of these theoretical estimates are shown in the figure: the simplest ones derived for the plane-parallel model of Blasius boundary layer, and the improved estimates based on the theory of Zel'man and Kakotkin (1982) which took into account the non-parallelism of the boundary-layer. As can be seen, the latter estimates agree excellently with the available data, confirming the idea that the values of k_2/k_1 observed in experiments where only a plane wave is artificially excited are very close to $(k_2/k_1)_{pr}$.

Similar results were obtained by Herbert (1983b, 1984a, 1988a,b) [and by Herbert and Bertolotti (1985)] in studies of the secondary-instability mechanism of generation of three-dimensional structures in a boundary layer by means of a principal parametric resonance of oblique waves [see Eq. (5.14b) and the text relating to it]. In these studies it was also found that at a given value of Re (Herbert used the Reynolds number Re^+) and for a given plane T-S wave of frequency ω and not-too-small amplitude A_1, there is usually a wide range of values of k_2 (and hence of k_2/k_1) corresponding to pairs of fast-growing 3D waves of frequency $\omega/2$. This range widens, and the growth rates G_0 increase, with an increase of A_1 above some rather small threshold value, while the preferred values, $(k_2)_{pr}$ and $(k_2/k_1)_{pr}$, of k_2 and k_2/k_1 corresponding to the greatest possible value of G_0 increase monotonically (but relatively slowly) with an increase of the amplitude A_1 or of Reynolds number Re*. Herbert's results also agree entirely satisfactorily with some experimental and numerical-simulation data (see e.g. Fig. 5.15a,c). Mankbadi (1993a) also tried to estimate theoretically the growth characteristics of the oblique components of a resonant triad at different values of parameters A_1, Re*, and k_2 [of course instead of the value of k_2 it may be used the value of k_2/k_1 or of $|\theta| = \tan^{-1}(k_2/k_1)$]. He applied for this purpose his 'nonlinear-critical-layer method', which was briefly discussed at the end of Sec. 5.2. Like the other authors mentioned above, Mankbadi found that at given not-too-small values of A and Re*, positive values of the oblique-wave growth rate G_0 correspond to a wide range of values of k_2 and this range widens (and the maximum value of G_0 increases) when A_1 and/or Re* increase. His quantitative results agreed quite satisfactorily with the experimental data of both Kachanov and Levchenko (1984) and Corke and Mangano (1989), and with the results of Spalart and Yang's (1987) numerical simulation of disturbance development in Blasius boundary-layer flow disturbed by a vibrating ribbon (see, in particular, Fig. 5.15b,c). Thus, three different methods of computation of the resonant-triad development in the boundary layer led to results which are close to each other and agree satisfactorily with both experimental and numerical-simulation data. At the same time, all the above-mentioned results clearly contradict the early conclusion of Volodin and Zel'man (1978) based on their use of the original model of Craik (1971).

Let us now pass to comparison of the measurements by Kachanov and Levchenko (1984), of the vertical profiles of the amplitude $A_{1/2}(z)$ and phase $\phi_{1/2}(z)$ of the subharmonic oblique waves entering the resonant triad, with the available theoretical results. [Now notations $A_{1/2}$ and $\phi_{1/2}$ are used instead of the notations A_2, A_3, $A_{2,3}$, and ϕ_2, ϕ_3 used above.] It was mentioned above that Kachanov (1994a) compared these profiles with several theoretical and numerically-simulated estimates. The first theory used by him for this purpose was the well-known three-wave-resonance theory. However, the initial form of this theory proposed by Craik (1971) was too crude to give sufficiently accurate values of the subharmonic-wave amplitude $A_{1/2}(z)$; therefore the refinements of Craik's theory by Zel'man and Maslennikova (1989, 1990, 1993a), briefly described above, were used by Kachanov to determine curve 3 in Fig. 5.16a. Moreover, Kachanov considered also the results of Herbert's (1984a) theory of secondary instability of the primary plane wave, which relate to computation of profiles of $A_{1/2}$; these results led to curve 1 in the same figure. Finally, we can also use the results of numerical solutions of the Navier-Stokes equations describing the downstream propagation of a wave triad in the boundary-layer - such a solution was computed, in particular, by Fasel *et al.* (1987) and led to the results presented as curves 2 in Figs. 5.16a,b.[3] One may see that, once more, two different theoretical approaches and the numerical simulation all give results which agree very well with the experimental data and with each other.

Herbert (1984a, 1986), Herbert and Santos (1987), Herbert *et al.* (1987), Crouch and Herbert (1993), Zel'man and Maslennikova

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[3] Some other attempts at numerical simulation of the N-regime of boundary layer instability developments were carried out by Spalart and Yang (1987) and Laurien and Kleiser (1989) (one result of the former authors is shown in Fig. 5.15c). However, in both these papers the less-accurate temporal, and not spatial, simulation was performed (see the small-type text below for discussion of the difference between these two approaches and the remarks about this topic in the next footnote [4]). Hence, the results found in these papers were less complete than those of Fasel *et al.*, and for this reason, except for Fig. 5.15b, these results will not be considered here. On the other hand, Rist and Fasel (1995) improved somewhat on the numerical method of Fasel *et al.*; however, as to the results relating to N-regime, the paper of 1995 contains only the indication that here "the quantitative agreement between numerical results and experiments was at least as good or even better than that achieved by Fasel *et al.* (1987)".

(1993a), and Kachanov (1994a) showed also that the results of measurements by Kachanov and Levchenko (1982, 1984), Saric *et al.* (1984), and Corke and Mangano (1989) for the dependence of the primary-wave and subharmonic-wave amplitudes on x (or on Re $\propto x^{1/2}$), which are presented, in particular, in Figs. 5.6 and 5.10, agree excellently with the results of available computations of the spatial amplitude growth. The accuracy achieved was found to be practically the same for computations based on Herbert's secondary-instability analysis and on the three-wave-resonance theory of Zel'man and Maslennikova. The same, if not better, accuracy was found in comparisons with numerical solutions by Fasel *et al.* of the initial-value problem for Navier-Stokes equations, describing development of a three-wave disturbance in the Blasius boundary layer. Some results confirming the statements made here are collected in Figs. 5.17a-c. Let us also note that Mankbadi (1991, 1993a) compared amplitude-growth data for the primary and subharmonic waves in a boundary layer, found in experiments by Kachanov and Levchenko (1984) and Corke and Mangano (1989), with results of his theoretical calculations by the nonlinear-critical-layer method and with the appropriate numerical-simulation results; his comparison showed yet again that there is good agreement between the available experimental, theoretical and numerical data (see Figs. 5.15b,c above).

Figs. 5.13-5.17 require some comments. Let us note first of all that the numerical-simulation data presented in Fig. 5.15c were obtained by numerical simulation of temporal (and not spatial) disturbance development. This means that the authors assumed that the disturbance studied was streamwise-periodic, and then used the N-S equations for computation of its evolution in time. This assumption presupposed that the parallel-flow approximation was used, but this corresponds to the real experimental conditions somewhat more poorly than the spatial-growth approximation used in spatial simulations where the disturbance is assumed to be time-periodic (with a prescribed frequency) while its dependence on coordinate x has to be computed with the help of the N-S equations (cf. a similar comparison of temporal and spatial solutions of the Orr-Sommerfeld eigenvalue problem in Chap. 2, pp. 113-114). Moreover, the influence of boundary-layer growth can be, at best, only crudely taken into account in the framework of the temporal approach[4], while in the case of a

[4] The simplest way of doing this is based on the supplementation of the N-S equations by an artificial 'force term' guaranteeing the existence of a solution describing the plane-parallel Blasius boundary layer with time-dependent thickness $\delta(t)$, growing at a rate equal to that registered by an observer who moves streamwise with a reasonably chosen velocity. According to Gaster's (1962) arguments, the group velocity c_g of a packet of T-S waves (which depends only weakly on the vertical coordinate z) may be chosen as such a 'reasonable velocity'. Then the corresponding time-dependent plane-parallel

spatial numerical simulation the dependence of the primary flow on x offers no difficulty. However, for temporal simulations much less computer resources (memory and computation time) are needed and the determination of the appropriate outflow boundary conditions at the downstream end of the computation domain is much easier than in the case of spatial simulation; therefore it is not surprising that the temporal approach to flow simulations has been very popular. In addition to Spalart and Yang (1987), temporal numerical simulations of boundary-layer instability development have been carried out by Wray and Hussaini (1984), Zang and Hussaini (1985, 1987, 1990), Laurien and Kleiser (1989), Zang (1992), and some others (see also the description of some related numerical-simulation results in Sec.5.5). In particular, Zang (1992) showed that results of temporal numerical simulation agree well with the data of Corke (1990) relating to the effect of mode detuning on wave triad development in boundary layers. However Fasel *et al.* (1987), whose data are shown in Figs. 5.16 and 5.17, carried out a spatial numerical simulation of boundary-layer instability, and the spatial approach was also discussed and used by Murdock (1986), Fasel (1990), Fasel and Konzelmann (1990), Konzelmann (1990), Rist (1990, 1996), Kleiser and Zang (1991), Kloker (1993), Rai and Moin (1993) (who studied the case of a compressible boundary layer with a high level of external disturbances), Joslin *et al.* (1993), Reed (1994), Rist and Fasel (1995), and Rist and Kachanov (1995), while the corresponding outflow boundary conditions were discussed by Kloker *et al.* (1993).

Now let us pass to other subjects. In some of the above-mentioned papers by Herbert it was indicated that the secondary instability of a plane T-S wave in a laminar boundary-layer flow may manifest itself in the plane-wave instability with respect to some 3D Squire (Sq) waves, which satisfy Eqs. (2.46) of Chap. 2 and the conditions indicated there. (Recall that all Sq waves are rapidly damped and hence decay as $t \to \infty$; however, as pointed out in Chap. 3, these waves may nevertheless make a large contribution to the transient growth of very small disturbances.) The instability with respect to Sq, and not T-S, wave disturbances was first considered by Herbert (in short, H) in his studies (1983a, 1984b) of the secondary instability of a plane T-S wave in a plane Poiseuille flow, where the midplane symmetry of the undisturbed velocity profile produces serious difficulties for the possibility of the plane-wave secondary instability with respect to oblique T-S waves [more will be said about this in the next chapter of this book; cf. also Wu (1996)]. Based on his experience of plane-channel secondary instability, H stated in the papers (1983b, 1984a, 1988a) on the secondary instability of the Blasius boundary-layer flow that here the instability with respect to 3D Squire waves may also take place, in principle.

Later Zel'man and Maslennikova (in short, Z-M) in the paper (1993a) criticized Herbert's conclusion, stating that for the triad comprising a plane T-S wave and a pair of Sq waves with half the streamwise wavenumber, the resonance frequency condition (5.7) is strongly violated. According to Z-M, this shows that resonance among one T-S plane and two 3D subcritical Sq

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boundary layer may be considered as a temporal model of the real streamwise-growing boundary layer [cf. the remark in Chap. 4, p.103, about a similar method of numerical simulation of the steady plane-parallel model of a Blasius boundary layer]. This method of approximate allowance, in temporal numerical simulations, for the spatial (streamwise) growth of a boundary layer was used, in particular, by Spalart and Yang (1987) and later gained great popularity.

waves is impossible; thus, a T-S mode cannot stimulate fast growth of some Sq modes. Furthermore Z-M indicated that the form of the vertical profile of the subharmonic-wave amplitude computed by Herbert (1984a) (see Fig. 5.16a) clearly showed that here the subharmonic wave was represented by a three-dimensional T-S wave and not by a Sq wave which has a quite different amplitude profile. Therefore, Z-M (1993a) considered only amplitude equations of the forms (5.4) and (5.4a) corresponding to wave triads comprising three T-S waves. According to their results, numerical solutions of such equations agreed well with all available data for the initial stage of the N-regime of boundary layer development. In particular, the results of their computations agreed very well with the data of Kachanov and Levchenko (1982, 1984) for the profile of the subharmonic-wave amplitude (see again Fig. 5.16a) and of Saric *et al.* (1984) relating to the streamwise-growth curves for amplitudes of primary and subharmonic waves (see Fig. 5.17a).

However, Herbert (1983b, 1984a, 1988a) did not assert that the excitation by a plane T-S wave of two Sq waves really plays an important part in the development of three-dimensional structures in the Blasius boundary-layer flow; he only indicated that this mechanism must be also considered. In fact, results presented in his papers (1984a, 1988a) clearly show that interactions among triads of T-S waves play the dominant part in the development of three-dimensionality in boundary layers.[5] On the other hand, the assertion by Z-M (1993a) about the impossibility of strong excitation in a boundary-layer flow of oblique Sq waves by a plane T-S wave was not correct. The point is that, even earlier, Nayfeh (1985) proved that a strong interaction of a T-S wave with a pair of Sq waves is quite possible in the Blasius boundary layer. Slightly later, and independently, this result was confirmed by Zang and Hussaini (1990). These authors computed several solutions of N-S equations describing the downstream propagation, in a plane-parallel flow with Blasius velocity profile, of wave triads consisting of a linearly-unstable plane T-S wave and a pair of symmetric 3D Sq waves with half the streamwise wavenumber. Growth curves for amplitudes of one plane T-S and two oblique Sq waves determined by Zang and Hussaini had the same form as the growth curves in Figs. 5.17a-c, and thus clearly showed that a plane T-S wave may stimulate rapid growth of two symmetric Sq modes. The computations by Zang and Hussaini also showed that a resonance triad consisting of one T-S and two Sq waves produces in a boundary layer a vortical structure, which depends on the value of the plane-wave amplitude in exactly the same way as was found in the experiments of Saric and Thomas (1984). However, Zang and Hussaini did not try to compare their results quantitatively with any real experimental data relating to the N-regime of boundary-layer instability development. Therefore, their work cannot be used for a reliable determination of the physical mechanism which produced the N-regime of boundary-layer development observed in this or that specific experiment.

To identify this mechanism, it is necessary to use the results of comparisons of specific experimental data with the predictions of various

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[5] However, the Squire waves also possibly made some contribution to the secondary disturbances computed by Herbert and his co-authors [such a possibility was explicitly stated by Crouch and Herbert (1993)]. It is also possible that some small Squire-wave contribution was present even in some of the computational results of Zel'man and Maslennikova; as was pointed out by E. Reshotko (personal communication) Squire waves sometimes appear quite unexpectedly in numerical solutions of the Navier-Stokes equations.

in their stu[...] ~~the~~ of nonlinear development of secondary disturbances in boundary layer theoretical models. Let us consider from this point of view the results shown in the above Figs. 5.13-5.17. All these figures illustrate the excellent agreement of the experimental results with the calculations. Among the theoretical models considered, those developed by Z-M were most often used in the figures. These models generalize Craik's model of a resonant triad comprising three T-S waves (one plane and two oblique, but the strict symmetry of the oblique waves and precise fulfillment of the resonance conditions are not now required). Excellent agreement of the model predictions with the observed data allows one to conclude that the general three-wave-resonance model describes one of the instability mechanisms which can produce the N-regime of disturbance development in boundary-layer flows. In other words, the Craik-type resonance among three T-S waves satisfying, exactly or approximately, the resonance conditions (5.7) may quite satisfactorily explain the observed features of the N-regime.

On the other hand, the good agreement of Herbert's (1984a) and Crouch and Herbert's (1993) computational results with experimental data, demonstrated by Figs. 5.15a, 5.16a and 5.17a,c show that secondary instability of a primary plane T-S wave with respect to a pair of symmetric oblique T-S (not Sq - Fig. 5.16a proves this quite definitely) waves may also be a mechanism leading to the development of 3D structures in boundary layers, as observed by several groups of experimenters. The fact that both the three-wave-resonance theory and the secondary-instability theory lead to results which equally well describe the available experimental data does not seem surprising. The point is that both theories relate to practically the same situation of downstream propagation of a triad of T-S waves approximately satisfying the resonance conditions. The only difference is that in the secondary-instability theory the amplitudes A_2 and A_3 of the two oblique waves are assumed to be much smaller than the plane-wave amplitude A_1, while in the three-wave-resonance theory these three amplitudes are assumed to be of the same order of magnitude (but both theories are restricted to cases of three waves which all have sufficiently-small amplitudes). In such situations it seems natural to suppose that there must be an intermediate range of ratios A_2/A_1 and A_3/A_1 within which both theories will be applicable. In principle, the secondary-instability theory must be considered as the more justified in cases where the plane T-S wave has already been growing for some time, so that its amplitude has reached a finite value, while oblique T-S waves have just been produced and hence have very small amplitude; the opposite opinion seems natural in cases where all three waves have already been growing for some time and have more nearly equal amplitudes. However, it is known that in the physical sciences theoretical equations very often turn out to be applicable over a wider range of conditions than those under which the equations were derived. So it is quite possible that the close agreement between the results of the secondary-instability and three-wave-resonance theories over a wide range of amplitude conditions is just one more illustration of this fact.

The numerical-simulation results shown in Figs. 5.16-5.17 also support the above statement that the N-regime of boundary-layer instability development is due to strong interaction among triads of T-S waves. Let us begin with Figs. 5.16a and 5.17a, which show excellent agreement between the results of the numerical simulation of Fasel *et al.* (1987), the experimental data of Kachanov and Levchenko (1982, 1984) and the theoretical work of H (1984a) and of Z-M (1990, 1993a). Recall again that the theory of Z-M is based on the assumption that the main features of the N-regime of boundary-layer development are due to the appearance in the flow of a resonant triad, comprising one plane T-S wave of relatively small amplitude and two symmetric oblique T-S waves of approximately half the frequency. Therefore, Fig. 5.16a apparently implies that both the numerical results of Fasel *et al.* and the theory of H also relate to situations where resonant triads including three T-S waves play the dominant role. In the case of the simulation data, this assumption also agreed well with the description of the computations. In fact, Fasel *et al.* considered the model of a laminar plane-parallel boundary-layer flow disturbed by vertical (normal to the wall) velocity oscillations produced by periodic blowing and suction of fluid through a narrow strip in the upstream part of the plate [see also the description of this disturbance model by Konzelmann *et al.* (1987), which may be compared to the description of five different models of this type by Berlin *et al.* (1999)]. The vertical velocity fluctuations were represented in the simulation of Fasel *et al.* by the sum of a spanwise-independent component proportional to $\sin(\omega_0 t)$, and a spanwise-periodic component proportional to $\sin(\omega_1 t)\cos(k_2 y)$. It was assumed here that $\omega_1 = \omega_0/2$ while the values of ω_0, k_2 and the amplitudes of the two components of the disturbance could be varied. It is natural to expect that such disturbances will generate a plane T-S wave of frequency ω_0 and a pair of oblique T-S waves having frequency $\omega_0/2$ and opposite spanwise wavenumbers $\pm k_2$; moreover, the amplitude of the plane wave could be chosen within the amplitude range corresponding to the N-regime. However, it seems highly improbable that vertical velocity oscillations produced by blowing and suction of fluid could generate Squire waves, which have zero vertical velocity.

Note in conclusion that Ustinov (1994) also tried to compare some results that follow from three different theoretical models of the nonlinear development in a boundary-layer flow of resonant triads comprising three T-S waves. Models considered by him included Craik's three-wave model leading to amplitude equations of the form (5.4), a DNS model based on numerical solutions of the N-S equations describing the downstream propagation of resonant T-S-wave triads [here the approximations applied by Ustinov (1993) to computations of a plane-channel flow were used], and Herbert's secondary-instability model (where Ustinov did not suppose that Sq waves would play any role). According to computations for the cases where $A_1 \gg A_2 = A_3$ (here A_1, A_2 and A_3 have the same meaning as above) Herbert's theory leads to results which agree very well with numerical solutions of N-S equations, while Craik's approach leads, if the initial amplitude of the 2D wave is not small enough, to results differing considerably from those of the other two models. These results apparently show that the question of the accuracy of different proposed theories of the N-regime of the boundary-layer instability development cannot be considered to have been fully answered at present.

Most of the results considered above in this section and almost all the figures (Figs. 5.5a,b and 5.12 being exceptions) are related to the study of development in a boundary layer of plane and oblique Tollmien-Schlichting waves entering a resonant (but not necessary fully-resonant) wave triad. As to Figs. 5.5a,b and 5.12, they clearly show that a disturbed boundary layer usually includes not just one resonant wave triad but a great variety of disturbances of different types. Moreover, if the nonlinear development of disturbances is studied as the initial stage of laminar-flow transition to turbulence, then one has no right to consider only isolated wave triads, since such flow conditions are very far from real pre-transition situations. Therefore it is reasonable to mention here some other scenarios of disturbance development in a boundary layer which may also play a significant role in transition processes. Note, however, that there are many different scenarios which may be realized under one or another combination of flow conditions. Below, only a few typical examples of such scenarios will be briefly considered; some other examples (which are far from exhausting all the possibilities) will be considered in Sec. 5.6.

Let us first cite the study by Zel'man and Smorodsky (1990) of the influence of resonant interactions on the downstream propagation in a boundary layer of a narrow packet of three-dimensional T-S waves. However, this work will not be discussed at this place, since propagation of wave packets will be separately considered in Sec. 5.6, and for now attention will be paid only to disturbances consisting of a finite number of individual T-S waves. Recall in this respect that in Corke's (1990, 1995) experiments the development of a artificially-produced resonant triad was accompanied by the appearance of a great number of secondary waves. [In fact, many such waves were observed by Kachanov and Levchenko (1984) too; see also Kachanov (1994a).] According to Corke, superposition of primary and secondary waves often included, in particular, the 'five-wave resonant systems' consisting of two 'resonant triads' (maybe of a detuned type) which both include the same primary 2D wave. And Z-M (1993a) (here again this abbreviation of 'Zel'man and Maslennikova' is used) independently computed the time evolution of a 'five-wave resonant system' comprising a primary plane wave with frequency ω and wavenumber k, and two pairs of nonsymmetric oblique waves (i.e. having different initial amplitudes) with frequency-wavevector combinations $\{\omega/2,\ k_1,\ \pm k_2\}$ and $\{\omega/2,\ k_1^*,\ \pm k_2^*\}$. (In the above combinations k_2 and k_2^* are arbitrarily chosen parameters while k_1 and k_1^* may then be computed with the help of the 3D O-S equation.) Accurate determination of all interaction coefficients entering the five amplitude equations corresponding to this system, and subsequent numerical integration of these equations, allowed Z-M to determine the streamwise development of all five waves for different initial conditions and different values of the parameters Re, ω, k_2 and k_2^* affecting disturbance development. Fig. 5.18 represents a typical example of the results obtained [the dependence of wave amplitudes on x is replaced here by their dependence on Re^+ =

$(U_0x/\nu)^{1/2}$]. One may see that, as in the results for one asymmetric strictly-resonant wave triad shown in Fig. 5.3, the amplitudes of the four subharmonic 3D waves grow rapidly with x; moreover, their amplitudes quickly become almost equal and their growth curves cross the 2D-wave growth curve together.

Similar results were obtained by Z-M (1993b) for wave systems comprising more than five individual waves. Such systems may be used for modeling, by discrete wave combinations, the process of filling-in of low frequencies of the velocity-fluctuation spectra in a disturbed boundary layer (this process leads to the formation of the low-frequency band clearly seen in Figs. 5.5a,b). Moreover, results for many-wave systems are needed to describe Corke's (1995) observations of a great number of secondary, tertiary and quaternary 3D waves in a boundary-layer flow. Z-M began attempts to explain, by the weakly-nonlinear instability theory, the process of spectrum filling peculiar to the N-regime in their (1990, 1992) papers, and the work was continued in the (1993b) paper. They considered the case of a laminar boundary layer which is disturbed at time $t = 0$ by a plane, linearly-unstable T-S wave having frequency ω_0, wave vector $\mathbf{k}_0 = (k, 0)$, and very small amplitude. This wave will begin to grow in accordance with the results of the linear stability theory. When the wave amplitude becomes large enough, interaction with the permanently-existing background noise will start, resulting in the extraction from the noise of two fast-growing secondary oblique T-S waves with frequency $\omega_1 \approx \omega_0/2$ and wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$, where $k_1 \approx k/2$ (recall that Z-M considered only T-S, but not Sq, waves). During this stage of disturbance development the primary plane wave will continue to grow at a rate close to that given by the linear stability theory (which is much smaller than the growth rate of oblique waves). When the amplitudes of all three waves become approximately equal, the oblique waves will strongly affect the plane wave, leading to its explosive growth (cf. Fig. 5.3 above where the evolution of one non-symmetrical resonant triad was shown). However, even before the beginning of the explosive growth of the primary 2D wave, but at some value of x where the oblique waves of the first order are already rather large, the evolved first-order waves will begin to excite two new pairs of symmetric oblique waves (again at the expense of the energy of the background fluctuations) having frequency $\omega_2 \approx \omega_0/4$ and wave vectors $\mathbf{k}_3 = (k_1', k_2')$, $\mathbf{k}_4 = (k_1', -k_2')$, $\mathbf{k}_5 = (k_1'', k_2'')$, $\mathbf{k}_6 = (k_1'', -k_2')$, where $k_1' \approx k_1'' \approx k_1/2$. These two pairs of the 3D waves of the second order will form, together with two oblique waves of the first order, two new resonant triads. Then the same process may be repeated with respect to waves of frequency ω_2 and so on. As a result a cascade transfer of the energy to more and more low-frequency 3D waves will take place, filling the low-frequency part of the spectrum (of course, direct nonlinear interactions between all the generated waves will also contribute substantially to the filling of this spectral range).

For the locally plane-parallel model of a Blasius boundary layer Z-M (1993b) studied quantitatively the first two steps of the cascade process of spectrum filling. To do this, they determined a system of 7 differential equations for the amplitudes A_i, $i = 0, 1, ..., 6$, of 7 interacting waves: amplitude A_0 of the primary plane wave, amplitudes A_1, A_2 of two secondary oblique waves, and amplitudes $A_3, ..., A_6$ of the four tertiary 3D waves. One typical example of the computed dependencies of the amplitudes of these seven waves on $\mathrm{Re}^* = U_0\delta^*/\nu \approx 1.72(U_0 x/\nu)^{1/2} \propto x^{1/2}$ is shown in Fig. 5.19. Here it has been assumed for simplicity that $\omega_1 = \omega_0/2$ and $\omega_2 = \omega_0/4$, while the initial amplitudes $A_1, ... , A_6$ of the six oblique secondary and tertiary waves (normalized by division into the free-stream velocity U_0) were taken to be equal to 10^{-5} (this value represented, in the model considered, the relative intensity of the background noise of flat frequency spectrum, but it was found that the results were almost the same for a wide range of these values) and the initial value of A_0 was chosen to be much greater than 10^{-5}. The computations were carried out for a number of values of ω_0, k_2'/k_1' and k_2''/k_1'' while k_2/k_1 was chosen to be equal to 2 which, according to Z-M (1990), is the value corresponding to maximal growth-rate of the amplitudes $A_1 = A_2$. Fig. 5.19 shows the results for a specific value of ω_0 and for values of k_2'/k_1' and k_2''/k_1'' which lead to the fastest growth of the amplitudes $A_3 = A_4$ and $A_5 = A_6$ of the tertiary 3D waves. The figure shows that subharmonic waves of frequency $\omega_0/2$ begin to grow from the moment of their appearance (corresponding, in the case considered here, to the value of x for which $\mathrm{Re}^* = 850$), while the primary-wave amplitude is almost unchanged at first (cf. similar results in Figs. 5.3, 5.6 and 5.10, where the results for resonant triads including only waves of frequencies ω_0 and $\omega_0/2$ were presented). As to the amplitudes of the tertiary subharmonics of frequency $\omega_0/4$, they even diminish slightly at first. However, beginning from a value of x corresponding to $\mathrm{Re}^* = 1050$, when $A_1 = A_2$ reaches some threshold level, the amplitudes $A_3 = A_4$ and $A_5 = A_6$ also begin to grow rapidly (all amplitudes $A_1, ..., A_6$ are then growing approximately as exponential functions of the streamwise coordinate x and thus also of the time t) while the amplitude A_0 continues to change very slowly. Only later, where Re^* reaches a value $\mathrm{Re}_N^* \approx 1200$, the amplitude of the primary plane wave begins to grow very rapidly (faster than exponentially) while all the subharmonics continue to grow exponentially with time. Z-M assumed that Re_N^* must be close to the empirical value of the Reynolds number, Re_{tr}, characterizing the transition of the boundary layer to turbulence, and they derived from this assumption some results relating to transition prediction; however in this chapter the later stages of transition to turbulence will not be discussed.

Z-M (1993b) also considered the amplitude equations for cases where a number of detuned (i.e., having frequency-ratios differing from the simple values 2 and 4 considered above) two- and three-dimensional waves of various amplitudes were introduced into the boundary-layer flow at some initial value of the coordinate x. In particular, they studied the case where the 2D T-S wave and two pairs of 3D waves, with frequencies and wave vectors of the form $(\omega_0,k_0,0)$, $(\omega_1,k_1,\pm k_2)$ and $(\omega_0-\omega_1,k_1',\pm k_2')$, were simultaneously introduced into a boundary layer flow, and investigated the dependence of the characteristics of the corresponding instability developments on the 'detuning parameter' $1-2\omega_1/\omega_0$. They also considered the development of a complicated wave system, comprising two 2D and ten 3D detuned waves close to those actually observed by Corke (1990). In this case they found many coincidences between the wave behavior given by their theory and that observed in the laboratory experiment. Some other results of Z-M allowed them to interpret, in a natural way, some observations by Yan *et al.* (1988) who also observed the cascade process of filling in the velocity-fluctuation spectrum in the course of instability development in a boundary layer. The methods used by Z-M can in principle be applied also to interpretation of Corke's (1995) results presented in Fig. 5.12, but the corresponding computations are rather complicated and apparently have not yet been carried out. Nevertheless, the results discussed above definitely show that the multimode weakly-nonlinear stability theory may be very useful for the quantitative theoretical description of many phenomena observed during the initial stage of transition of the boundary-layer flow from laminar to turbulent flow regime.

Let us now consider the investigation by Nayfeh and Bozatli (1979a) of the possibility that a primary plane T-S wave of frequency and wavenumber (ω, k) in a Blasius boundary layer can excite, by means of the principal parametric resonance of secondary-instability theory, a two-dimensional T-S wave with frequency and wavenumber close to half those of the primary wave. Recall that at the beginning of Sec. 5.1 it was indicated that nonlinear resonance may occur among two waves with frequency-wavenumber combinations $(2\omega, 2\mathbf{k})$ and $(\omega, \mathbf{k})$; therefore, in principle, such resonance in a Blasius boundary layer seems to be probable. Moreover, since the two 2D waves considered will have critical layers which are close to each other, it seems natural to expect that their nonlinear interaction will be rather powerful. Nayfeh and Bozatli analyzed the spatial development of disturbances, i.e., they considered the primary wave with real frequency ω and with wavenumber k which may be complex, and assumed that the 2D wave excited by the primary wave has the frequency-wavenumber combination $(\omega/2, k_1)$ where k_1 may also be complex but is such that $\Re e[k/2 - k_1] = \Delta k_1$ is a small detuning parameter. (Frequencies and wavenumbers are assumed here to be made dimensionless by using the displacement thickness δ^* and free-stream velocity U_0 as length and velocity scales.) To compute the interaction between the primary and the secondary waves the authors used the *method of multiple scales* (see the book by Nayfeh (1981) for a description of this method and a number of its applications). The computations were performed for three values of the dimensionless frequency $F \times 10^6 = \omega\nu/U_0^2$, namely 60, 52 and 40, and a wide range of Reynolds numbers. However, the results were rather disappointing: they showed that to trigger the parametric instability in a Blasius boundary layer and achieve rapid growth of the secondary 2D wave, the amplitude (peak value) of the primary plane wave must exceed a critical value close to 29% of the free-stream velocity U_0. Since it is known that in a boundary layer secondary instabilities of many other types become significant at considerably smaller amplitudes of the primary wave, it became clear that this instability mechanism cannot play a significant role.

Later Healey (1994, 1995, 1996) turned anew to the study of a possible two-wave resonance between a pair of two-dimensional waves in a Blasius boundary layer with frequency-wavenumber combinations (ω, k) and (ω_1, k_1), where ω_1 and k_1 have real parts twice as large as those of ω and k. (Note that the subscript 1 now refers to the wave with larger frequency and wavenumber.) Healey somewhat changed Nayfeh and Bozatli's problem formulation by admitting that both parameters ω and k (and naturally ω_1 and k_1 too) may take complex values. Recall that Nayfeh and Bozatli assumed that ω and ω_1 are real, $\omega_1 = 2\omega$, while k and k_1 are complex and such that the real part of k_1 is close to twice the real part of k. The assumptions used allowed Nayfeh and Bozatli to choose values of ω and Re almost arbitrarily; moreover, they spoke only of *closeness* of the real parts of k_1 and $2k$, since in 1979 it was believed that at real values of ω and $\omega_1 = 2\omega$ the condition $\Re e k_1 = 2\Re e k$ could not be satisfied exactly. As will be explained below, it was found recently that this assumption is incorrect, but this discovery does not invalidate Nayfeh and Bozatli's reasoning.

Nayfeh and Bozatli used the traditional spatial formulation of the stability problem inspired by the experiments of Schubauer and Skramstad (1947), and many of their followers, where a plane wave of fixed frequency was artificially produced in the initial part of a laminar boundary layer, and the subsequent development of this wave and any further instability phenomena generated by it were studied. The admittance by Healey of complex values for both the frequency and the wavenumber clearly expanded considerably the class of plane waves considered, and simultaneously forced Healey to change the resonance conditions, giving them the form of two equalities: $\Re e\omega_1 = 2\Re e\omega$ and $\Re e k_1 = 2\Re e k$. Augmenting the set of waves considered of course meant that a new physical situation, which led to a new stability problem, was being studied. Healey's problem formulation corresponded to the case where the primary plane wave had an amplitude which was not constant but was modulated as $A(t) = A_0\exp(-\omega^{(i)}t)$. To illustrate the importance of the instability phenomena produced by such a wave, Healey referred to the remark by Gaster (1980) who pointed out that the amplitude threshold above which a flow disturbance leads to the boundary-layer breakdown and transition to turbulence is often several times lower in the case of a modulated wave-packet disturbance than in the case of a disturbance having the form of a sinusoidal plane wave. He also noted the results of subsequent experiments by Shaikh and Gaster (1994) on randomly-modulated wavetrains, which again showed that modulation enhances the nonlinear effects of a disturbance. These facts stimulated Healey's study of the instability of a boundary layer disturbed by an amplitude-modulated wave.

Healey (1994) investigated whether there exist complex eigenvalues $k = k^{(r)} + ik^{(i)}$ and $k_1 = k_1^{(r)} + ik_1^{(i)}$ of two O-S eigenvalue problems (2.44), (2.42) (where $c = \omega/k$ and $U(z)$ is the Blasius velocity profile) with complex parameters $\omega = \omega^{(r)} + i\omega^{(i)}$ and $\omega = \omega_1 = \omega_1^{(r)} + i\omega_1^{(i)}$ respectively, satisfying the condition $\omega_1^{(r)} = 2\omega^{(r)}$, which are such that $k_1^{(r)} = 2k^{(r)}$. Performing some complicated computations, Healey showed that such pairs (ω, k) and (ω_1, k_1) exist at all high enough values of Re, and that it is also possible to find more special pairs (ω, k) and (ω_1, k_1) of complex frequency-wavenumber combinations where not only and $k_1^{(r)} = 2k^{(r)}$ but even $\omega_1 = 2\omega$ and $k_1 = 2k$. In particular, Healey found that at Re* = 2100 the latter equalities are valid if ω = 0.04318 − 0.01819i and k = 0.1433 − 0.0600i (here as usual δ^* and U_0 are used as length and velocity units). In Healey's (1995, 1996) papers many results supplemented those given in Healey (1994) are presented. In particular, in the paper (1995) the location in the complex plane of the resonant pairs (ω, k) and $(2\omega, 2k)$ is analyzed in its dependence of the Reynolds number and it is also shown that if the condition $k_1 = 2k$ is replaced by less restrictive condition $k_1^{(r)} = 2k^{(r)}$, then it is possible to satisfy this condition, together with the condition $\omega_1 = 2\omega$, by a combination of real ω and complex k and k_1. [For example, at Re* = 2000 these conditions are satisfied for $\omega = 0.0817$, $k = 0.256 - 0.0101i$, $\omega_1 = 2\omega = 0.1634$, $k_1 = 0.512 + 0.225i$.] However, the results relating to resonant wave pairs with a real value of ω (i.e., corresponding to the traditional problem of spatial wave development) do not completely undermine the early belief that such pairs do not exist. The point is that in the wave pairs found by Healey one of the two considered frequency-wavenumber combinations necessarily belongs to the higher-order O-S eigenvalues describing rapidly-damped higher modes, which earlier were never taken into account. Nor do the new results contradict those of Nayfeh and Bozatli (1979a), since Healey showed only that there exist pairs of waves for which resonance interaction is in principle possible, but said nothing about the efficiency of this interaction. At the same time it seems physically doubtful that interactions including higher-order modes may really play an essential part in boundary-layer instability development.

Healey also considered the equations for the complex amplitudes $A_1(x)$ and $A_2(x)$ of two two-dimensional waves with complex values of ω and k satisfying the conditions given above for resonance interaction to be possible. According to his results these equations, accurate to the order of the quadratic nonlinearities, have the form

$$\frac{dA_1}{dx} = -k^{(i)}A_1 + b_1 A_1^* A_2, \quad \frac{dA_2}{dx} = -k_1^{(i0}A_2 + b_2 A_1^2 \tag{5.15}$$

where b_1 and b_2 are the interaction coefficients corresponding to the situation considered. (Here again nonvanishing of these coefficients shows that the interaction is a resonant one.) For details of the derivation of Eqs. (5.15) and evaluation of their coefficients see Healey (1995), where a small wavenumber detuning of two waves is also allowed [cf. also Dangelmayr (1986)]. Some of the results implied by these equations were verified by Healey in some specially-arranged wind-tunnel experiments where development of modulated waves, and also the influence of the phase difference between two waves (which according to equations (5.15) must be rather significant) were measured. The experimental data confirmed, to sufficient accuracy, the theoretical results (including, in particular, the detection of resonances under just the conditions indicated by the theory). However, the full clarification of the role of modulated waves in real boundary-layer breakdown and transition processes evidently requires much further work.

Let us now continue the description of the work by Nayfeh and Bozatli. In their papers (1979b, 1980) these authors used the method of multiple scales to study of the nonlinear interactions between two two-dimensional T-S waves of different frequencies and wavenumbers and also between three such waves of frequencies ω_1, $\omega_2 > \omega_1$, and $\omega_2 - \omega_1$. They found that a 2D wave of moderate amplitude has little influence on its 2D subharmonic and therefore a 2D wave of frequency ω and moderate amplitude cannot generate a fast growing 2D wave of frequency $\omega/2$ [this result clearly confirms the conclusion of the paper (1979a)]. However, a 2D wave has a strong influence on its second harmonic, so a moderate-amplitude wave of frequency ω may generate a secondary wave of frequency 2ω. Moreover, waves of frequencies ω_1 and ω_2 have a strong influence on a wave of frequency $\omega_2-\omega_1$ often making it unstable (i.e., growing in time). Many of Nayfeh and Bozatli's results were verified in experiments by Saric and Reynolds (1980) in which a vibrating ribbon in a boundary layer was used to excite either one plane wave of fixed frequency ω_1 or two plane waves of frequencies ω_1 and ω_2. [This experiment was stimulated by the similar one by Kachanov *et al.* (1980) where oscillations of two frequencies were introduced in a boundary layer by two separate ribbons.] In particular, the experiments showed that a primary plane wave of frequency ω_1 may generate a plane wave of frequency $2\omega_1$ with an amplitude approximately twice that of the primary wave, but no cases of generation of subharmonic waves with frequency $\omega_1/2$ were detected. When waves of two frequencies ω_1 and ω_2 were introduced into the flow, secondary waves of frequencies $\omega_1 - \omega_2$ (and also $2\omega_1 - \omega_2$) were detected, but the streamwise development of their amplitudes did not follow the predictions of Nayfeh and Bozatli. In fact, Saric and Reynolds' experimental data agreed satisfactorily with some of Nayfeh and Bozatli's theoretical results but strongly disagreed with others; hence revision of the theory seems necessary. However, this subject will not be discussed further here, since all the instabilities considered in Nayfeh and Bozatli's papers led to much smaller growth rates than those corresponding to the three-wave resonances, and therefore these instabilities can hardly play an important part in transition of a boundary layer to turbulence.

Still later Nayfeh (1985) showed that if a two-dimensional primary T-S wave in a Blasius boundary layer is disturbed by a single secondary T-S wave which has a frequency and streamwise wavenumber equal to half of those of the primary wave but is three-dimensional, with spanwise wavenumber k_2 larger than some small critical value, then the principal parametric resonance become very effective and leads to fast growth of the secondary wave. This result clearly agrees well with those considered earlier in this subsection.

The secondary-instability problem considered by Nayfeh (1985) [and also those studied by Herbert (1983b, 1984a), Herbert *et al.* (1987) and Bertolotti (1985)] deals with the principal parametric resonance in a boundary-layer flow, which leads to the appearance of subharmonic 3D waves and of the staggered vortical structure shown in Fig. 5.8b. Recall now that in the K-regime of the evolution of a disturbed boundary layer observed by Klebanoff and his co-authors an ordered, and not staggered, vortical structure was observed. Trying to simulate this regime, Nayfeh and Bozatli (1979c) [see also Neyfeh (1987a,b)] introduced a four-wave instability model. In the Nayfeh-Bozatli (N-B) model four different O-S waves interact with each other in a boundary-layer flow: they are the primary plane wave with frequency and wave vector $(\omega, k, 0)$; its second harmonic, a plane wave of frequency 2ω and wave vector $(k_1 \approx 2k, 0)$; and two oblique waves with frequency and wave vectors $(\omega, k, \pm k_2)$. The downstream propagation of the N-B wave system was analyzed by Z-M (1984, 1989, 1993a), who determined the values of all interaction coefficients of the corresponding system of four amplitude equations, and performed numerical integration of this system for a number of initial conditions. Here, in fact, two different resonances are simultaneously realized - resonant growth of the second harmonic stimulated by the first one, and fast growth of two oblique waves produced by Craik's three-wave resonance interaction of the amplified second harmonic with a pair of oblique waves with half the preimary frequency ω. The spatial development of the four-wave system leads to generation of an ordered system of vortices, sketched in Fig. 5.8a and typical of the K-regime of instability development. Since in this section we are discussing only the N-regime, and not the K-regime, the N-B model will not be considered here in any detail (but it will be mentioned in the next Section, Sec. 5.5, devoted to study of the K-regime).

5.5. WEAKLY-NONLINEAR INSTABILITIES IN THE K-REGIME OF BOUNDARY-LAYER DEVELOPMENT

The K-regime of the boundary-layer instability development was discovered and explored in the late 1950s and early 1960s by Klebanoff and his co-authors. In this book, these results were very briefly considered in Sec. 2.1 (where even the name 'K-regime', which marks Klebanoff's contribution, was not mentioned) and, in a little more detail, were presented in the beginning of Sec. 5.2 with a subsequent brief mention in the beginning of Sec. 5.3 where the name 'K-regime' first appeared. Below some recent studies of this regime will be described at greater length; therefore, it is appropriate to make here some additional remarks about its main features.

In Sec. 5.2 it was indicated that Klebanoff *et al.* (1962) studied the downstream evolution of the three-dimensional structures which, according to the results of a number of earlier experimental investigations, regularly appear at some downstream position in a laminar boundary layer containing somewhere in the beginning of it a vibrating spanwise ribbon exciting in the flow a two-dimensional linearly-unstable T-S wave. Since Klebanoff *et al.* were interested first of all in the spatial development of the appeared 3D structures, they generated artificially a weak spanwise periodicity of the amplitude of ribbon vibrations (with the same spanwise wavelength λ_y which was earlier observed in boundary layers excited by ribbon vibrations with y-independent amplitude). Then the amplitude of the y-periodicity of the streamwise disturbance velocity was measured at different values of the streamwise coordinate x. The obtained results (some of them are shown in Fig. 5.20) showed that spanwise modulation of the disturbance velocity grows rapidly with x producing a specific peak-valley wave structure with a constant 'fundamental spanwise wavelength' λ_y. As it was indicated in Sec. 5.3, later it was found by other authors that this structure consists of a strictly ordered collection of streamwise 'Λ-vortices' and that two quite different orderings of vortices are realized in the cases of large and small amplitudes of the initial two-dimensional T-S wave (see Figs. 5.8a,b above and the text relating to them). Klebanoff *et al.* considered only the case of relatively large initial T-S waves and therefore they dealt only with a regularly ordered vortical structure of the type shown in Fig. 5.8a (but the authors did not use flow visualization and therefore could not observe the ordering of vortices). However, hot-wire-anemometer measurements by Klebanoff *et al.* allowed them to discover that in the studied case of large T-S-wave amplitude the bursts of high-frequency velocity oscillations of short duration regularly appear (and then are repeated within each period of the primary T-S wave) at downstream peak positions of the spanwise velocity distribution. These high-frequency bursts were called "spikes" by Klebanoff *et al.* since some spikes are seen in the traces of disturbance velocity against time (see Fig. 5.21).

In the course of their downstream evolution spike structures are doubled (see Fig. 5.21 again), then tripled and so on. Klebanoff *et al.* associated the spikes with the formation in the flow of a family of small hairpin-shaped vortices produced by the inflectional instability of high-shear layers formed around the large-scale vortical structures. The 'legs' of hairpin vortices may be gradually converging in the course of their evolution; this process may explain, in particular, the formation of 'ring vortices' which are also sometimes observed in the later stages of the K-regime. According to Klebanoff *et al.* a breakdown of medium-size vortices into smaller and still smaller vortical structures leads at first to the appearance of spikes and then to transformation of spikes into wholly irregular "turbulent spots" which are the precursors of the final transition to turbulence. Because of the connection with irregular turbulent spots the spikes were long considered as also irregular ("random") embryos of the future spots. Moreover, in the accordance with the point of view of Klebanoff *et al.*, it was also long assumed that spikes arise from local (both in time and space) inflections of the disturbed Blasius velocity profile. In fact, it can be shown that near the inflection points produced by a low-frequency disturbance, a strong flow instability to high-frequency oscillatory disturbances must be developing locally [this statement was due to Betchov (1960) and its support by Klebanoff *et al.* (1962) made it quite popular afterwards; see, e.g., numerous references to subsequent studies of the 'local high-frequency secondary instability' (briefly, LHSI) in reviews by Nayfeh (1987a) and Kachanov (1991a, 1994a) and the paper by Kachanov *et al.* (1993)]. However, as will be explained later in this section, in the late 1980s and early 1990s it was discovered that apparently spikes have quite another origin, are not 'wholly irregular', and their transformation into "turbulent spots" does not occurs at once but only after some specific intermediate stages. Moreover, some more recent experimental and numerical-simulation data give the impression that the origin of ring vortices may differ from that sketched above; this topic will be also discussed later in this section.

Among the first experimental results relating to the K-regime were those of Nishioka *et al.* (1975, 1980) [see also Nishioka (1985), Nishioka and Asai (1985a,b) and Asai and Nishioka (1989)] who performed detailed measurements of the instability development in a plane-channel flow. This flow is usually modeled as plane Poiseuille flow, but it has many features similar to those of Blasius boundary-layer flow. In particular, it was shown in the above-mentioned papers [and also in the similar experimental work of Kozlov and Ramazanov (1981, 1983, 1984a,b)] that plane-channel flow may also undergo K- and N-regimes of instability development. And studying the K-regime of disturbance growth in a channel-flow Nishioka *et al.* obtained the first experimental corroboration of the fact that LHSI may really take place during the K-regime of the flow development. However the channel-flow instabilities will be considered at length only in the next Chap. 6; so now we will go over to discussion of the original research and survey papers by Kachanov *et al.* (1984, 1985, 1989), Borodulin and Kachanov (1988, 1989, 1994, 1995), Kachanov (1987, 1990, 1991a, 1994a,b), Dryganets *et al.* (1990), Bake *et al.* (1996, 2000), Lee (1998,2000), and Lee et al. (2000) [see also the recent books by Boiko *et al.* (1999) and Schmid and Henningson (2000)] where many results of recent experimental studies of K-regime in boundary layers are presented. ital.

The experiments described and discussed in the above-mentioned papers and books often (but not always) were based on the use of the experimental method which was first tested by Klebanoff *et al.* (1962) and then became quite popular. This means that here again the laminar boundary layer on a flat plate placed in a wind tunnel was disturbed by a spanwise-oriented vibrating ribbon, and simultaneously a weak spanwise nonhomogeneity of the resulting disturbance was artificially produced by an array of identical pieces of tape placed beneath the ribbon. However, in contrast to the earlier experiments by Kachanov *et al.* (1977, 1978, 1980), in all the experiments considered here the amplitude of ribbon fluctuations was chosen to be so great that it guaranteed the realization of the K (and not N) regime of disturbance development. Moreover, the authors tried to make the experimental conditions as close as possible to those of the experiment by Klebanoff *et al.*

(1962). But the new experiments differed from the early studies of Klebanoff's group by more sophisticated measurement techniques and by more careful investigation of the frequency (and spanwise-wavenumber) composition of the velocity fluctuations at various points $\mathbf{x} = (x, y, z)$ of the boundary layer.

Passing to the consideration of these more recent experimental studies of boundary-layer instability development, one must note first of all the results of Borodulin and Kachanov (1988). These authors showed that LHSI does in fact occur in boundary layers but leads to some special nonlinear effects, which must be distinguished from the production of spikes. The point is that spikes usually appear at considerably greater 'height' (distance from the wall) than the velocity-profile inflection (which is expected to be the site of any quasi-inviscid instabilities) and have amplitudes exceeding those of LHSI-produced formations. Borodulin and Kachanov (1988) [see also the subsequent discussion of their results in the surveys by Kachanov (1990, 1991a, 1994a,b) and Borodulin and Kachanov (1994, 1995) and the theoretical papers by Zel'man and Smorodsky (1991a,b) and Kachanov, Ryzhov and Smith (1993)] often observed both types of nonlinear formations at the same values of coordinates x and y but quite different values of the vertical coordinate z. The lower formations were always observed just at the heights of velocity-profile inflections, and the measurements agreed very well with theories of local high-frequency secondary instability [see e.g. the discussion of this matter in Kachanov *et al.* (1993) and the papers cited there by Smith, and by Smith and co-authors on this subject]. However the spikes [whose importance for boundary-layer instability development was demonstrated quite early by Klebanoff *et al.* (1962)] certainly have an origin unrelated to LHSI.

Spectral analysis of velocity fluctuations performed by Kachanov and his co-authors showed that in the K-regime of boundary layer development numerous higher harmonics of the primary 2D wave, with frequencies $\omega_n = n\omega$, $n = 2, 3, \ldots$, and values of n up to several tens, always exist in the flow together with the oscillations of frequency $\omega_1=\omega$ equal to that of the ribbon vibrations and of the primary plane wave produced by them. Thus, an amplitude of the primary wave larger than that leading to the N-regime leads to an intensity of high-harmonic generation much greater than in the N-regime.

Klebanoff *et al.* did not observe so many higher harmonics of 2D velocity oscillations and did not pay much attention to them, but according to Kachanov *et al.* (1984, 1985, 1989) these harmonics are highly important in the K-regime. Therefore, the latter authors concluded that Klebanoff *et al.* underestimated the role of higher harmonics of the primary wave. However, Rist and Fasel (1995), who performed careful numerical simulation of the K-regime of boundary-layer instability development as observed by Kachanov *et al.* [using in this simulation the same model of disturbance generator as that used by Fasel *et al.* (1987), whose work was discussed in Sec. 5.4] disagreed with the above-mentioned conclusion. Rist and Fasel indicated that although Kachanov *et al.* tried to repeat experiment by Klebanoff *et al.* very accurately, there were nevertheless some small differences in experimental conditions. These differences led, in particular, to a considerable greater initial value (measured just downstream of the vibrating ribbon) of the ratio $A_{2,3}/A_1$ of the 3D-wave amplitude to that of the primary 2D wave in the experiments of Kachanov *et al.* than in the similar experiments of Klebanoff *et al.* This explains why fewer higher 2D harmonics were significantly excited in the experiments of Klebanoff's group, and there these harmonics really were of somewhat smaller importance. However, almost all the experimental results of Kachanov *et al.* (1984, 1985) were confirmed, with high accuracy, by Rist and Fasel's numerical-simulation data [see also the papers by Rist and Kachanov (1995) and Rist (1996), where some supplementary numerical data are presented].

Kachanov *et al.* measured downstream-growth curves for the amplitudes of various higher harmonics of the primary oscillation, and found that these amplitudes begin to grow rapidly at approximately the same value of x at which the primary-plane-wave amplitude begins to grow faster than predicted by linear stability theory. The streamwise growth of amplitudes of the primary wave and its higher harmonics is arrested (and is sometimes replaced by a decrease) just in the region where spikes appear [see Fig. 5.22 where data about the spatial growth of the first 6 harmonics are presented together with data relating to growth of the total disturbance intensity; amplification curves for 17 harmonics may be found in Borodulin and Kachanov (1988) and Kachanov (1994a)]. According to Kachanov *et al.*, the spike, i.e. the short-term highly-localized outbreak of high-frequency oscillations is produced, not by a sudden rapid increase of intensity of all higher harmonics, but by the local (valid within short ranges of y and z values) phase synchronization of all harmonics shown in Fig. 5.23, leading to strong amplification of the observed oscillations. (As indicated above, the spanwise coordinate y of a spike was found in all cases to be close to a peak position of the spanwise wave shown in Fig. 5.20.) Therefore, the new theory considered spikes not as random formations but as regular structures, naturally produced by deterministic evolution of the Fourier composition of the appearing upstream flow disturbances. The experimental data shown in Figs. 5.22 and 5.23 were later confirmed by observations of the Novosibirsk group, and also agree very well with the results of thorough direct numerical simulations of the K-regime by Kloker (1993), Rist and Fasel (1995) and Rist (1996) [see also the paper by Rist and Kachanov (1995) where new numerical-simulation results were compared with the new measurements from Novosibirsk].[6] The regular character of spike structures was also confirmed in careful experiments (which will be discussed in Sec. 5.52) by Breuer *et al.* (1997) devoted to study of development of some localized disturbances in a boundary layer; see also the survey paper by Bowles (2000). Thus, the data presented in Figs. 5.22 and 5.23, and the new explanation of the origin of spikes following from them, may now be considered as reliable.

Using the data of the Novosibirsk experiments (of which Figs. 5.22 and 5.23 represent only a small part), Kachanov (1987) [see also his papers (1990, 1991a, 1994a,b)] proposed a wave-resonance theory of the K-regime of boundary-layer instability development. This theory assumes that K-regime leads to the emergence of a cascade of successive four-wave resonances, generalizing the four-wave resonance studied by Nayfeh and Bozatli (briefly, N-B) in the paper (1979c). Recall that N-B resonance includes 2D and 3D waves with frequency-wavevector combinations $(2\omega, k, 0)$, $(\omega, k', 0)$, and $(\omega, k'', \pm k_2)$, where $k \approx 2k'$, $k'' \approx k'$, and $k_2 = k_0$ corresponds to the spanwise periodicity of 3D disturbances observed in experiments by various authors (i.e., to the fundamental wavelength λ_y of spanwise waves seen in Fig. 5.20). It was also noted in Sec. 5.4 that this resonance generates the vortical system typical of the K-regime of boundary-layer development. According to Kachanov, there is a cascade of resonances leading to the rapid growth of 3D structures in the K-regime comprising resonances among quadruples of waves

[6] In Sec. 5.4, attempts by Wray and Hussaini (1984), Zang and Hussaini (1985, 1987, 1990), Spalart and Yang (1985), Murdock (1986), Laurien and Kleiser (1989), Kleiser and Zang (1991), Zang (1992) and some others to simulate numerically the boundary-layer instability development were mentioned. These papers contain a number of results relating to the K-regime of such development and almost all of them agree satisfactorily with available experimental data. However, these results are less accurate and less complete than those by Rist and Fasel (1995) and Rist and Kachanov (1995); therefore results of the earlier numerical simulations of the K-regime will not be considered in this book.

with frequencies and spanwise wavenumbers $(n_1\omega, 0)$, $(n\omega, 0)$ and $(n\omega, \pm mk_0)$. Here n_1, n and m are integers, $n_1 = 1, 2, 3, ..., n \approx n_1/2$, $mk_0 \approx k_2$, where k_2 is the spanwise wavenumber corresponding to spanwise periodicity of small-scale disturbances while k_0 is the 'primary' (or 'fundamental') spanwise wavenumber mentioned above, describing the spanwise waves which appear in experiments, either naturally or as the result of artificial disturbances such as pieces of tape under a vibrating ribbon.

These wave quadruples may be produced by nonlinear interactions of waves of the same type but with smaller values of n_1, n and m. The ensuing interactions among the quadruples may be resonant and similar to those taking place in the case of an N-B quadruple where $n_1 = 2$ and $n = m = 1$. Kachanov (1987) showed that the above-mentioned cascade of resonances may lead to the appearance of spikes at the locations where they were actually observed in experimental studies of the K-regime. Slightly later, a numbers of waves which may participate in 'Kachanov's resonances' were identified in observations of the K-regime by Borodulin and Kachanov (1989, 1994) [see also Kachanov *et al.* (1989), Kachanov (1990, 1991a, 1994a,b)]. Borodulin and Kachanov found that for some Kachanov's wave quadruples, the phase velocities of the four waves were quite close to each other, making strong four-wave interaction quite probable. They also stated that at $n = 1$ and 2 the most rapid growth of the oblique waves present in these quadruples is reached for $m \approx 4$ to 7. Kachanov's cascade of resonances clearly fills the high-frequency and high-wavenumber parts of the frequency and spanwise-wavenumber spectra (these parts correspond to small-scale oscillations of 'spike type') but it does not generate 'genuine subharmonics' corresponding to large-scale oscillations.

Kachanov's wave-resonance theory did not seriously contradict the numerical-simulation results of Rist and Fasel (1995) who found that the higher spanwise harmonics of the primary 2D wave, which correspond to the 3D $(n\omega_1,\pm mk_0)$-modes with $n = 1$, $m = 1, 2, ..., 8$, appear in the flow successively and then begin to grow rapidly with downstream distance x, while their initial growth rates increase with m, reaching a maximum for $m = 7$ and 8 (see Fig. 5.24; supplementary data may be found in Rist (1996), where similar growth curves are also given for some (n,m)-modes where $n = 0$ or 2). This figure shows that all the modes considered reach approximately the same saturation level at $x = 420$ mm, which is close to the position where spikes first appear. However, a more thorough treatment of the results of a subsequent, more refined numerical simulation of the same type carried out by Rist led to a conclusion differing from that formulated above. New numerical results, presented in Rist and Kachanov (1995) and Rist (1996), give a clearer picture of the flow than that derived from previous experiments and simulations. As pointed out in these papers, the new results showed large amplification rates of spanwise modes with high values of m [cf. also the related earlier results by Zang (1992)], which cannot be explained by the resonances considered by Kachanov (1987). According to the papers of 1995 and 1996, the modes corresponding to $m = 2, 3, ..., 8$, are apparently just higher harmonics of the (1,1)-mode produced by non-resonant nonlinear interactions. If so, then their amplification with x must be of the same origin as the amplification of higher temporal harmonics of the primary T-S mode and of other products of non-resonant two-wave interactions (cf. the amplification curves in Fig. 5.12). It is clear, however, that a final solution of the numerous problems relating to the origin and subsequent evolution of higher 2D and 3D instability modes in the K-regime of boundary-layer development requires much additional work.

Note in conclusion that Rist and Fasel (1995), Rist and Kachanov (1995), and Rist (1996) also used numerical-simulation results for the preliminary investigation of various 3D vortical structures appearing in the K-regime, and compared the computed structures with experimental data. Particular attention was paid here to the study of the Λ-shaped structures ('Λ-eddies') observed in numerous flow-visualization experiments (see, e.g., the visualization pictures in Fig. 5.8a; two examples of Λ-structure given by numerical-simulation results are shown in Fig. 5.25). Rist and Kachanov also noted that, according to the new simulation data, at a late stage of flow development (which corresponds to the appearance and subsequent multiplication of spikes) ring-like vortices connected to spikes emerge, pinching off from the downstream 'tips' of the pre-existing Λ-vortices. (This mechanism of generation of ring vortices clearly differs from the previous suggestions sketched in the beginning of this section where results of the early experiments of Klebanoff *et al.* were discussed.) The numerical results showed the appearance of 'spikes' at the same points, and with the same amplitudes and durations, as those which were observed in wind-tunnel experiments; one such example is shown in Fig. 5.26.

Recall now that when the results of Bake *et al.* (1996, 2000) relating to the N-regime of boundary-layer instability development were briefly reviewed at the end of Sec. 5.3, it was promised that similarities between some features of the N- and K-regimes found by these authors would be described in Sec. 5.5. It was noted in Sec. 5.3 that the wind tunnel used by Bake *et al.* had a very long test section and only the results relating to the initial part of it agreed well with experimental data from previous investigations of the N-regime. (It was said in Sec. 5.3 that such agreement was observed for $\Delta x < 250$ mm, where Δx is the streamwise distance between the point of measurements and the disturbance generator. In fact the first deviations from the ordinary N-regime were observed by the authors as early at $\Delta x = 220$ mm, but they were weak enough to be ignored for the purposes of Sec. 5.3.) The main study of the structure of developed disturbances far from the disturbance generator was made by Bake *et al.* at $\Delta x = 380$ mm and here the behavior was quite different from that observed earlier in the N-regime; in fact, the measurements represented some mixture of features typical of the N- and K-regimes. At $\Delta x = 380$ mm strong spanwise modulation of streamwise disturbance velocity, of the type shown in Fig. 5.20, was clearly seen (faint signs of such modulation were present at $\Delta x = 220$ mm) and, what is especially important, Klebanoff 'spikes' very similar to those observed repeatedly in the K-regime were also present. The spikes had the same shape as in the case of the K-regime, and again they appeared in the outer part of the boundary layer at spanwise peaks of disturbance velocity and could be doubled and tripled. However, in contrast to the K-regime, they now appeared periodically in time with the subharmonic period $T_1 = 2\pi/\omega_{1/2} = 4\pi/\omega_1$ and not with the primary-wave period $T = 2\pi/\omega_1$ (where ω_1 is the 'fundamental frequency' of the primary 2D wave and $\omega_{1/2} = \omega_1/2$). Moreover, the vortical structure generated by the developing disturbances again consisted of Λ-vortices, but they were now positioned in space in the staggered order shown in Fig. 5.8b, and not regularly as in Fig. 5.8a. However, in spite of these differences, the subsequent development of spikes and vortices was very similar to that observed in late stages of K-regime development. Therefore, there is reason to assume that prolonged N-regime development may lead to transition of a boundary layer to turbulence by the same process that takes place at large amplitudes of the primary wave leading to the K-regime of boundary-layer development.

Let us now consider briefly the results relating to the long-time evolution of spikes appearing in the K-regime of instability development. This topic differs from the subjects considered above, since spikes have some features which invalidate standard methods of weakly-nonlinear stability theory. Observations by Kachanov *et al.* (1984, 1985, 1989), Borodulin and Kachanov (1988, 1989, 1994, 1995) and some others showed that spikes include a great number of phase-synchronized 2D and 3D modes strongly interacting with each other (cf. Figs. 5.22-5.24). Therefore, ordinary systems of equations for mode amplitudes are of little use in this case. Recall now that spikes are localized in small spatial domains (spanwise localization is especially strong; see e.g. Fig. 5.23). Observations also showed that a newly-formed spike at first moves away from the wall but on reaching the upper part of the boundary layer it moves downstream at practically constant z, and with practically constant velocity close to that of the external stream [see Fig. 5.27a,b and their discussion by Kachanov *et al.* (1993), accompanied by some supplementary data; similar results were obtained by Acarlar and Smith (1987) for evolution in a boundary layer of "hairpin vortices", which are similar to spikes in many respects]. During its downstream travel a spike preserves its shape (and also its spatial size and temporal extent), i.e. it does not disperse as do, for example, the ordinary wave packets in which individual waves have different phase velocities determined by the dispersion law (5.5). Not only the spatial form but also the spectral composition and the amplitude of a spike are in the main preserved during its convection downstream. This circumstance was first stressed by Borodulin and Kachanov (1988) and was confirmed by their subsequent experimental studies; see also Kachanov's surveys (1991a, 1994a). It allows spikes, once fully-formed, to be considered as *coherent structures*, i.e. flow formations with a definite degree of ordering which is preserved during long time intervals. The term 'coherent structure' appeared in fluid mechanics only in the second half of the 20th century and was not at once universally recognized [for example, it was not used at all by Monin and Yaglom (1971, 1975)], but now it is clear that such structures play a very important part in the mechanics of turbulence [see, e.g., the book by Holmes *et al.* (1996)]. Note, however, that coherent structures of many different types are met in fluid mechanics, especially in fully-turbulent flows, and spikes represent a very special type of such formations. Spikes appear in laminar flows at a relatively late stage of instability development; they are strongly localized, mobile, and have definite boundaries, and thus may be associated with the notion of *solitons*.

The term 'soliton' was apparently first introduced by Zabusky and Kruskal in their paper (1965) devoted to plasma waves, but in fact it has a long history, being directly connected with the observation by J. Scott Russell in 1834 of a strange *solitary wave* in the Edinburgh to Glasgow canal. The wave was produced by a suddenly stopped boat and had the form of a rounded well-defined heap of water elevated above the mean level and for a long time rapidly moving forward (i.e., in the direction of boat motion before the stop) without any change of form or speed - Russell pursued it on a horse for more than a mile. This observation stimulated subsequent attempts by Russell to generate such waves in the laboratory, and led to publication in 1844 of his report to the British Association for the Advance of Science devoted to this subject. Russell's solitary wave attracted considerable attention, but only in the second half of the 20th century was it discovered that it represents a particular case of a wide class of flow phenomena which are met in many parts of quite different physical sciences, and have numerous important applications. At present soliton studies form a special science to which an enormous and very diverse literature has been devoted [here it will be enough to name only the relatively small introductory books by Lamb (1980), Drazin (1983) and Drazin and Johnson (1989)]. Up to now there is no universally recognized strict definition of the soliton; to follow Drazin's books one may say that this word means usually a solution of a nonlinear equation or system of equations which describes a wave or collection of waves of a conservative form which is spatially localized, mobile, and may strongly interact with other objects of the same type, retaining its identity after the interaction.

Kachanov and his group stressed the similarity of spikes to solitons mainly on the basis of their localization and conservation of form. However, the relation of spikes to coherent structures was also emphasized by this group; therefore Kachanov suggested applying to spikes the new name 'CS-solitons' (CS for 'coherent structure', hence 'CS-soliton' may be deciphered as ''solitonlike coherent structure'). This name indicates the special place of spikes in both collections - of coherent structures and of solitons. As indicated above, solitons usually represent some special solutions of a definite nonlinear equation or equations (in particular, the strict theory of Russell's 'solitary waves' emerged when Korteweg and de Vries (1895) discovered the nonlinear equation for surface waves in a liquid of finite depth and proved that this equation has solitary-wave solutions). Therefore, the identifications of spikes with a special kind of soliton seemed incomplete without a nonlinear equation to describe them.

The first attempts to develop an analytical theory of the soliton-like formations in flat-plate boundary layers were made independently by Zhuk and Ryzhov (1982) and Smith and Burggraf (1985). In both papers the boundary-layer disturbances considered were those which, in the case of small enough amplitude and large streamwise length scale, may be described with good accuracy by the so-called Benjamin-Ono (briefly, B-O) equation, a nonlinear integro-differential equation of the form

$$\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial^2 A/\partial\xi^2}{\xi-x}d\xi+\varphi(t,x) \tag{5.16}$$

where $A=A(x,t)$ is the unknown amplitude of the disturbance [replaced in the integrand by $A(\xi,t)$], the integral, if divergent, is understood as the Cauchy principal value, while $\varphi(t,x)$ is the 'source term', which may be absent in some cases but in others may have different origins and forms. Eq. (5.16) [without the source term] was derived by Benjamin (1967) and Ono (1975) [and used by Davis and Acrivos (1967)[7]] to describe the variation of amplitude of two-dimensional long internal waves of small amplitude in a stratified fluid of great depth, and it was shown by Benjamin and by Ono that this equation has soliton solutions of the same form as those known for the Korteweg-de Vries equation. Later it was discovered that the same equation may also be applied to many other nonlinear waves of large streamwise lengthscale and small amplitude in steady shear flows bounded by a wall [the above-mentioned papers by Zhuk and Ryzhov, and Smith and Burggraf and also those by Goncharov (1984), Romanova (1984), Demekhin and Shkadov (1986), Benjamin (1992), and Matsuno (1996) are just typical examples]. However in these papers neither the K-regime of boundary-layer transition nor the spikes were considered explicitly.

Application of the B-O equation to the development of strongly nonlinear disturbances in a boundary-layer flow was studied, in particular, by

[7] Therefore instead of the name 'Benjamin-Ono (or B-O) equation' the name 'Benjamin-Davis-Acrivos (or BDA) equation' is sometimes used.

Rothmayer and Smith (1987). However here only a rather special one-parameter family of soliton solutions of the B-O equation was considered, and these solutions proved to be inappropriate to describe the spikes observed in the K-regime. Then Zhuk and Popov (1989) found some new soliton solutions of Eq. (5.16) (with non-zero 'source term'), while Ryzhov (1990) investigated a more general three-parameter family of soliton solutions of the homogeneous B-O equation (some important features of this solution are shown in Fig. 5.28). Ryzhov's investigation was continued by Kachanov, Ryzhov and Smith (1993), Ryzhov (1994), and Bogdanova-Ryzhova and Ryzhov (1995) who applied this and some related soliton solutions to a description of real fluid-mechanics instabilities [see also Kachanov's survey papers (1991a, 1994a) and an interesting survey by Ryzhov and Bogdanova-Ryzhova (1998) containing a long list of references]. In particular, Bogdanova-Ryzhova and Ryzhov (1990) studied the soliton solutions of the inhomogeneous B-O equation describing the evolution of disturbances in a boundary layer on a rough wall (where the effect of roughness elements may be described by a definite form of the source term $\varphi(t,x)$). These authors also cited some papers in which the same equation was applied to development of atmospheric and oceanic waves affected by a mountain ridge or by large bottom irregularities, which also generated source terms, of a form different from that applying to the rough wall [for more details see Ryzhov and Bogdanova-Ryzhova's survey (1998)]. Detailed study of the solutions of the inhomogeneous B-O equation and their applications to water-wave problems was carried out also by Matsuno (1996) whose paper contains many supplementary references relating to this topic.

Further, Ryzhov (1990) and Kachanov *et al.* (1993) showed that the determining parameters of Ryzhov's family of soliton solutions may be chosen so that the general form of these solutions, and a number of their numerical characteristics, are very close to those found by Kachanov (1991b), Borodulin and Kachanov (1988, 1994, 1995) and in some other experiments on Klebanoff's spikes in the early stage of their downstream evolution (see, e.g., Fig. 5.29). These results may be considered as the confirmation of the soliton nature of spikes. The deviations in Fig. 5.29 of the experimental results for far-downstream points of observation from the theoretical curves may be explained by the fact that B-O equation deals only with two-dimensional disturbances; therefore it represents a two-dimensional spike model which is inapplicable to the later, essentially three-dimensional, stages of spike evolution. Many details of these later stages of spike development were discussed by Kachanov, Ryzhov and Smith (1993) and studied experimentally by Borodulin and Kachanov (1995). A generalization of the B-O equation to the case of three-dimensional near-wall disturbances was proposed by Shrira (1989) in connection with the study of 3D waves in the upper layer of the ocean. Later Abramyan *et al.* (1992) proved that Shrira's equation has three-dimensional soliton solutions which may possibly be used to describe spikes in the three-dimensional stage of their evolution.

It has already been mentioned that the time evolution of spikes leads finally to their transformation into 'turbulent spots'. Such spots (one of which is shown in Fig. 2.2) represent the spatial regions where the flow becomes truly turbulent, i.e. it becomes irregular, is accompanied by random ('stochastic') fluctuations, and therefore cannot be studied mathematically without the use of probability-theory concepts. Recall that it was long assumed that randomization of the boundary-layer flow takes place when spikes (considered as irregular formations) first appear. However it was found later that spikes themselves are regular structures which may be described by deterministic equations of motion, while random velocity fluctuations emerge only at the later stages of spike development. The process of gradual development of the 'flow randomness' associated with spikes in an initially laminar boundary layer disturbed by a two-dimensional T-S wave was studied experimentally by Dryganets *et al.* (1990) whose results were discussed by Kachanov (1994a); see also the descriptions of experiments by Breuer *et al.* (1997) in Sec. 5.62. An analytical model of the gradual randomization of a spike and subsequent formation of a spot was briefly outlined by Smith in Kachanov *et al.* (1993) and then developed further by Smith (1995). Bogdanova-Ryzhova and Ryzhov (1995) considered the model of randomization of a soliton by a wall hump and then returned to the problem of the possible connection between solitons and the onset of random flow disturbances in Ryzhov and Bogdanova-Ryzhova (1998). Note in this respect that many different mechanisms may be responsible for the appearance of random fluctuations in real boundary-layer flows; a definite part may be played also by 'local high-frequency secondary instability' (LHSI) of a flow disturbed by a T-S wave, and by penetration into this flow of background (environmental) disturbances in the form of random 2D and 3D T-S waves or wave packets, corresponding to the continuous part of the spectrum of the boundary-layer Orr-Sommerfeld eigenvalue problem. However a detailed analysis of the appearance of randomness in a laminar boundary flow lies outside of the contents of this chapter on weakly-nonlinear stability theory.

Note in conclusion that recent experemental studies of the K-regime of boundary-layer instability development by Lee [see Lee (1998, 2000), Lee *et al.* (2000) and references therein] lead to some results differeing from those considered above. Lee studied disturbance development in the boundary layer on a flat plate mounted in the low-turbulence water channel at Peking University. In these experiments a wave disturbance was excited in the flow by periodic pumping of water in and out of the boundary layer through a spanwise oriented narrow slit near the leading edge of the plate. Then the disturbance development was recorded by hot-wire measurements at a number of downstream positions and by numerous photos of the evolution of flow structures visualized by hydrogen bubbles. Lee used the ~~the~~ name 'CS-solitons' to denote some new flow structures which fill out the whole thickness of a boundary layer and have quite different form in the near-wall region (where the long streaks appear at the 'peak positions' of the spanwise velocity modulation), in the middle part of the boundary layer, and in its upper part (where Kachanov's 'CS-solitons' were travelling most of the time). According to Lee, the upper part of CS-solitons is produced by short chains of ringlike vortices appearing periodically (with the same frequency as that of the primary T-S wave) at the tips of Λ-vortices which breakdown generates spikes. Lee's CS-solitons differ from Kachanov's ones, but both these formations are strongly localized spanwise and preserve their main features up to final breakdown (leading to the appearance of turbulent spots). Lee noted that some of his results are similar to those observed earlier by Hama and Nutant (1963) and Williams *et al.* (1984); as to the disagreements of some Lee's conclusions with those by Kachanov, they were partially gotten over in their joint work [see, e.g., Lee *et al.* (2000) and references in Lee (2000)]. Nevertheless, at present it seems that Lee's results require further careful investigation and that his claim on the possible finding of the 'universal transition scenaro' is questionable. However, since the 'transition scenarios' are only indirectly connected with the main content of the present chapter, Lee's results will not be considered here at greater length.

5.6. SOME OTHER SCENARIOS OF INSTABILITY DEVELOPMENT IN BOUNDARY LAYERS

The N- and K-regimes of boundary-layer instability development considered in the previous section of this chapter have a very important common feature - in both of them the instability process starts with the appearance in the flow of a linearly-unstable two-dimensional Tollmien-Schlichting wave. [This T-S wave is often identified with that solution of the O-S eigenvalue problem (2.42), (2.44) corresponding to the eigenvalue ω (or k) which has the maximal (or minimal if k is the O-S eigenvalue) imaginary part. Such identification is then justified by the assumption that any disturbance to excite a T-S wave may enter the boundary layer from the disturbed free-stream flow, and hence the most-unstable T-S wave must play the dominant role in boundary-layer evolution.] The simplest case, the instability regime initiated by a sole plane T-S wave, was investigated in the famous experiments by Schubauer and Skramstad (1947) and in numerous subsequent similar boundary-layer stability studies (including all the experimental studies of the N- and K-regimes considered above) which used a vibrating ribbon (or some other periodically-oscillating device) for the excitation in the flow of a weak 2D wave of fixed frequency ω. Recall however the remark made in Sec. 2.92 (p. 118) which stated that in the majority of boundary-layer transitions to turbulence met in wind- and water-tunnel experiments and in real life the appearance in the flow of an isolated linearly-unstable T-S wave of small amplitude growing in accordance with the laws of linear stability theory, is not observed at all, i.e. this stage of instability development is *by-passed*. Therefore, the scenarios of the boundary-layer instability development which begin with the appearance in the flow of the most-unstable plane T-S wave of small amplitude are inapplicable to the majority of real-life boundary-layer-transition phenomena. Note that the term *by-pass transition* is often used in engineering practice to describe response to such high levels of free-stream turbulence that transition starts at Reynolds numbers far below the critical value predicted by linear instability theory, so that no stage of the route to randomness discussed above, nor the behavior of the simple finite-amplitude subcritical modes discussed in Secs. 5.3 and 5.4 have any relevance.

In this book no attempts will be made to consider all scenarios of boundary-layer instability development and transition to turbulence met in practice. However, at least some of the regimes of instability development differing from the N and K regimes considered above must, clearly, be discussed here.

5.61. *Oblique and Streak-Breakdown Transition Scenarios*

Let us begin with a remark about the paper by Goldstein and Choi (1989). These authors considered the case of a plane-parallel shear layer ("mixing layer") between two parallel streams with uniform velocities U_1 and $U_2 \neq U_1$. Then they studied the evolution in this flow of a pair of linearly-unstable symmetric oblique waves of the same amplitude A and frequency ω, with two-dimensional wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$. The waves were assumed to be harmonic in time (i.e., ω is real) but streamwise-growing (k_1 is complex with a negative imginary part while k_2 is real). It was found that the two waves strongly interact with each other and, as in the case of Craik's resonant triads satisfying conditions (5.7), strong nonlinear wave interaction is concentrated in the neighborhood of the common critical layer of these two waves. Using known methods of approximate asymptotic analysis of the critical-layer contribution to nonlinear wave interactions [see, e.g., the review by Maslowe (1986) and the subsequent related paper by Goldstein (1995)], Goldstein and Choi derived an equation for the amplitude $A = A(x)$. This equation proved to be integro-differential and cubically nonlinear and described the rapid streamwise growth of the amplitude A. A similar method was applied by Wu *et al.* (1993) to the study of disturbance development in a near-wall fluid layer above a horizontal plate oscillating sinusoidally in the x-direction; here again the nonlinear interaction between a pair of symmetric oblique waves leads to rapid growth of flow disturbances. [Note that later Wu and Stewart (1995) showed that rapid growth of three-dimensional disturbances in a plane shear layer may also be produced by the interaction between another pair of T-S waves having the same critical layer - of one two-dimensional T-S wave and one oblique wave having the same phase velocity. However this instability mechanism will not be considered in this chapter.]

The development in a plane-channel (i.e. plane Poiseuille) flow of disturbances initiated by the appearance of a pair of oblique waves having amplitude A, frequency ω and wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$ was apparently first studied by Lu and Henningson (1990) and Schmid and Henningson (1992a,b) who performed temporally-developing direct numerical simulations of this development. The authors considered the oblique-wave development in a steady laminar flow as the first step en route to transition of this flow to turbulence, an alternative to the N- and K-routes. To describe this new route the term *oblique transition* was used by these authors, while in the book by Schmid and Henningson (2000) the name *O-type transition* was also used. As to the boundary-layer flows, the development of a pair of oblique waves was first considered as a possible transition mechanism in the case of compressible flow; see, e.g., Thumm *et al.* (1989, 1990), Chang and Malik (1992, 1994), Fasel *et al.* (1993), and Sandham *et al.* (1994). [Special interest in the compressible case was stimulated by the fact that Squire's theorem of Sec. 2.81 is not valid here and, in contrast to incompressible boundary layers, the most-unstable wave in compressible shear layers is usually an oblique one; see, e.g., Reshotko (1976).] However, since in this book compressible flows are not considered, these papers will not be discussed here.

A spatially-developing numerical simulation of the oblique transition in the incompressible Blasius boundary layer was apparently first carried out by Joslin *et al.* (1993) who solved the exact Navier-Stokes equations together with the approximate 'parabolic stability equations' (see Sec. 2.92, p. 117), and by Berlin *et al.* (1994) whose numerical simulation covered a greater number of flow-development stages than that of Joslin *et al.* The paper by Berlin *et al.* (1994) contains the first outline of Berlin's extensive numerical study of development of oblique waves in a boundary-layer flow, while the final results of this study were summarized in Berlin's doctoral thesis (1998) and in the paper by Berlin *et al.* (1999). The experimental part of the work, which was also included in the latter paper, was based on results from the doctoral dissertation of Wiegel (1996). One more recent doctoral dissertation devoted to experimental study of oblique transitions in plane Poiseuille and Blasius boundary-layer flows was presented by Elofsson (1998b); his results relating to boundary-layer instability are given also in Elofsson (1998a) and Elofsson and Alfredsson (2000). Then Schmid and Henningson (2000) presented results of somewhat different temporally-developing numerical simulations of the plane-Poiseuille and boundary-layer oblique transitions; these results will be also considered a little later.

Numerical simulations of the 'oblique-transition regime' (or the 'oblique-transition scenario' as this regime is often called) may be realized by solving the N-S equations for the disturbance velocity in a given laminar flow under the condition that at some $x = x_0 > 0$ there is a 'disturbance generator' which generates a pair of symmetric oblique waves propagating streamwise in the flow. This means that here the oblique waves are included in the 'inflow boundary condition' stating that the 'inflow velocity' at $x = x_0$ is represented by the Blasius velocity profile plus velocity profiles of two symmetric oblique waves. In the case of spatial simulation the frequency ω, spanwise wavenumbers $\pm k_2$, amplitude A and phase ϕ of oblique waves are real constants which may be chosen by the researcher implementing the simulation. If the plane-parallel model of a boundary layer is used, then the streamwise wavenumber $k = k_1$ may be determined as the complex eigenvalue, having the smallest imaginary part, of the corresponding Orr-Sommerfeld equation (2.41) with given values of ω and k_2 (and $c = \omega/k_1$). In the more general case when a locally plane-parallel approximation is used, k_1 is a slowly-changing complex function of x which is given by the eigenvalue with the smallest imaginary part of the local Orr-Sommerfeld equation (2.41) (corresponding to the Blasius velocity profile $U(z) = U(z,x)$ at the streamwise coordinate x). In temporal simulations the wavenumbers k_1 and k_2, amplitude A and phase ϕ are real and may take arbitrary values; while the frequency ω is the complex eigenvalue of the corresponding O-S equation, with given values of k_1 and k_2, that has the greatest imaginary part. In experimental studies of oblique transition the 'disturbance generator' must be realized, of course, as some device exciting the oblique waves with prescribed values of ω, A and k_2. In the boundary-layer experiments by Wiegel (1996), Elofsson (1998a,b), and Elofsson and Alfredsson (2000) this device was similar to the 'wave generator' proposed by Gaponenko and Kachanov (1994) and then used by Bake *et al.* (1996, 2000), while in channel-flow experiments by Elofsson and Alfredsson (1995, 1998) [see also Elofsson (1998b)] the oblique waves were produced by a pair of 'oblique ribbons' vibrating with the frequency ω and amplitude A and placed at equal and opposite angles to the mean-flow direction. This latter method of oblique wave generation was also used by Elofsson and Lundbladh (1994) who, simultaneously with their experiments, carried out a numerical simulation of this transition where as 'disturbance generator' a pair of vibrating oblique ribbons was simulated.

The early temporally-developing direct numerical simulations of the disturbance development in a plane-channel flow disturbed by a pair of small (but not infinitesimal) symmetric oblique waves performed by Schmid and Henningson (1992a,b) showed that strong nonlinear interaction between two waves arises almost at once, and produces a rapid growth of the disturbance energy and the appearance of a number of new disturbance structures, essentially accelerating transition to turbulence. Subsequent more complete spatial numerical simulations by Berlin *et al.* (1994, 1999) [see also Henningson *et al.* (1995)], and temporal numerical simulations by Schmid and Henningson (200~~0~~), of the analogous development of a 1 Blasius boundary layer disturbed by a pair of oblique waves, revealed many important features of the process, and substantiate Schmid and Henningson's idea of the possible importance of the oblique-wave mechanism in laminar-turbulent transition.

In papers by Berlin *et al.* it was again shown that a strong nonlinear interaction of a pair of symmetric oblique waves develops much more rapidly than in the case of a single 2D Tollmien-Schlichting wave initiating the nonlinear N- and K-routes to transition. This circumstance can be explained by the fact that both the N- and K-routes begin with the exponential growth of an unstable T-S wave according to the laws of the linear stability theory, and only when the amplitude of this T-S wave becomes large enough does the nonlinear resonant-triad interaction begin to play an essential part. However, the linear-theory prediction of the growth rate of a wave corresponding to eigenfunctions of the Orr-Sommerfeld equation is very small in comparison, not only with the growth of wave disturbances produced by their nonlinear interactions but even with the non-modal transient disturbance growth due to non-normality of the linearized Navier-Stokes equations [see in this respect Chap. 3 above where the meaning of the term 'non-normality' was explained on pp. 115-116, and also the expressive Fig. 5.30 taken from the paper by Reddy and Henningson (1993)]. Recall that in order to eliminate the stage of very slow growth of disturbances following linear-theory laws, Klebanoff *et al.* (1962) and many of their followers artificially excited three-dimensional disturbances in the vicinity of a spanwise vibrating ribbon. This was necessary since otherwise the test section of a low-turbulence wind tunnel would usually be too short for the most interesting stages of flow development to be reached

Berlin *et al.* (1994) carried out a spatial numerical simulation of a Blasius boundary-layer flow with a pair of oblique waves in it, and used the simulation results to study the appearance and subsequent growth in the flow of a number of new wave structures produced by nonlinear interactions between the primary oblique waves. The inflow conditions specified at $x = x_0$ corresponded to a Blasius boundary layer with $\mathrm{Re}^* = U_0\delta^*/\nu = 400$ (where U_0 is the free-stream velocity and δ^* is the displacement thickness at the inflow; recall that at such a low Re^* unstable T-S waves do not exist in a boundary-layer flow) plus a pair of oblique waves with frequency $\omega_0 = 0.08$ (this and all other quantitative characteristics of the simulation discussed below are non-dimensionalized by the scales U_0 and δ^*), spanwise wavenumber $k_{2,0} = 0.192$ (the value of k_1 was then determined from the O-S eigenvalue problem) and amplitude $A = 0.01$. A more extensive and careful numerical simulation of the same type (where five different models of inflowing oblique waves were considered) was carried out by Berlin *et al.* (1999). Here somewhat different values of Re*, ω_o, $k_{2,0}$ and A, and of the investigated range of x-values, were chosen to achieve a satisfactory match with the conditions of Wiegel's experiments, whose data were then compared with the numerical simulations. In Fig. 5.31a,b results from the two papers by Berlin *et al.* are presented, showing the dependence on $(x - x_0)/\delta^*$ (in Fig. 5.31a) or on x in mm (in Fig. 5.31b) of the energies E of a number of (n,m)-Fourier components with frequencies and spanwise wavenumbers $(\omega, k_2) = (n\omega_0, mk_{2,0})$. (The numbers n and m may be always assumed to be nonnegative since the symmetry of the (ω, k_2) and $(\omega, -k_2)$ modes means that modes with negative values of k_2 need not be considered explicitly.) In Fig. 5.31a the energies are divided by the inflow energy of the primary (1,1) mode, and hence here $E(0) = 1$ for (1,1) mode and $= 0$ for all other modes; in Fig. 5.31b energies E are measured in some conventional dimensional units, and the coordinate x_0 of the 'wave generator' was here close to 186 mm.

It is easy to see that the quantitative results of the 1994 and 1999 simulations do not coincide; the differences are apparently due to the use of different numerical methods, models of inflowing waves, and outflow conditions (Fig. 5.31a clearly corresponds to conditions annihilating waves at the outflow end of the computational domain). Qualitatively however, the two collections of results are sufficiently close to each other. Both simulations show that the energy of the primary oblique waves does not change much with the streamwise coordinate x [in Fig. 5.31a it grows slightly at first and then remains almost constant, while in Fig. 5.31b it begins to decrease slowly immediately after the peak at the wave-generation point, but in both cases the energy changes for this mode are small in comparison with those for the other modes]. Fig. 5.31a shows the generation of a rather energetic (0,0) mode describing the distortion of the mean velocity profile by nonlinear waves, but this effect is not considered at all in Fig. 5.31b. However, according to both figures the most important feature of the oblique-wave interaction is the rapid growth of the (0,2) mode, greatly exceeding the growth of all other modes and quickly making this mode the most important disturbance structure. The (0,2) mode does not oscillate and has half the spanwise wavelength of the primary oblique waves oscillating with frequency ω_0. Thorough analysis of the results of the numerical simulations by Berlin *et al.* (1999) and the flow visualizations performed by Wiegel (1996), Elofsson (1998a), and Elofsson and Alfredsson (2000) showed that in the case of oblique transition the (0,2) mode represents a periodic array of pairs of counter-rotating streamwise vortices with spanwise wavelength half that of the primary oblique waves.

Recall now that, according to results presented in Chap. 3, pp. 92-93,[8] arrays of streamwise vortices are just the structures which are subjected to the greatest transient growth produced by the so-called lift-up effect studied, in particular, by Landahl (1975, 1980, 1990) and discussed in Secs. 3.22, 3.32 and 3.33. Therefore, after the generation of the (0,2) mode by the direct nonlinear interaction between primary modes (1,1) and (1,−1) its subsequent growth is due to two different factors: the quadratically-nonlinear interactions among existing oblique waves and the linear lift-up effect. The combined action of two growth mechanisms explains naturally the excess of the growth rate of the (0,2) mode over those of the (2,0) and (2.2) modes, which are also produced by direct nonlinear interactions of primary waves. As indicated by Landahl, the lift-up effect leads to the transformation of the streamwise vortices into a spanwise-periodic collection of horizontal streaks of fluid with alternating low and high streamwise velocity. Such streaky structures are in fact clearly seen in flow-visualization photos of boundary-layer flows by Wiegel, Elofsson, and Elofsson and Alfredsson, and on contour plots of disturbance velocities and vorticities determined from the data given either by the appropriate numerical simulations or by detailed hot-wire-anemometer measurements [see, e.g., Berlin *et al.* (1994, 1999), Berlin (1998), Elofsson (1998a,b), and Elofsson and Alfredsson (2000)].

In a range of x-values where distinct horizontal streaks are seen, the streak amplitude A_s at a fixed value of x depends on the amplitude A of the primary oblique waves and, for not-too-large values of A, it is proportional to A^2 as it must be in the case of streak

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[8] Note that the results of Butler and Farrell (1992) presented in Chap. 3 were obtained for a simplified, strictly plane-parallel model of the Blasius boundary layer. The optimally-growing disturbance structures for the more accurate model of a streamwise-thickening boundary layer were studied by Andersson *et al.* (1999) and Luchini (2000) but will not be considered here.

generation by quadratic interaction of two oblique waves. With increasing x, the amplitude A_s grows approximately linearly at first and then saturates but, if the primary forcing amplitude A is too low, then after the initial growth A_s begins to decrease and the streaks gradually disappear [these facts were first discovered by Joslin *et al.* (1993) and then confirmed in other papers mentioned above]. According to results of both the numerical simulations and the experiments, if A is not too low then streaks of saturated large amplitude A_s become unstable with respect to high-frequency oscillations, and this instability leads at first to oscillations of streaks and then to their breakdown and transformation into collections of irregular small-scale vortices forming the turbulent flow regime. Such a scenario of transition to turbulence was studied for both plane-channel and boundary-layer flows, in particular, by Henningson *et al.* (1995), Schmid *et al.* (1996), Alfredsson and Matsubara (1996), Reddy *et al.* (1998), Berlin *et al.* (1999), Brandt *et al.* (2000), and Elofsson and Alfredsson (2000); see also Schmid and Henningson's book (200~~0~~). 1 Most of these studies were based on the analysis of the appropriate numerical-simulation data (which was supplemented by Brandt *et al.* also by some stability-theory computations), but Alfredsson and Matsubara, and Elofsson and Alfredsson performed in laboratory flat-plate boundary layers direct experimental studies of the streak-breakdown process. However, in this chapter the transition to turbulence is not discussed; therefore here only a few remarks about the streak-breakdown transition scenario will be made.

Let us begin with a short consideration of results given by numerical simulations of the oblique transition in a boundary-layer flow carried out by Henningson, Schmid and their coworkers and described in the papers by Henningson *et al.* (1995) and Shmid *et al.* (1996), and the book by Schmid and Henningson (200~~0~~). 1 (These papers and the book contain also results of similar simulation of the oblique transition in a plane Poiseuille flow which will be briefly discussed in the next chapter.) As was indicated earlier in this section, Schmid and Henningson performed temporal, not spatial, numerical simulations which differed in some respects from the spatial simulations by Berlin *et al.* (1994, 1999). In their temporal simulations the authors used the same model of a strictly plane-parallel Blasius boundary layer, with thickness $\delta(t)$ growing with time, which was used by Spalart and Yang (1987) and was briefly described in footnote [4] on p. 64. As to the above-mentioned difference from the approach by Berlin *et al.*, it is connected with the inclusion of weak random background disturbances (supplementing a pair of oblique waves of much greater amplitude) in the numerical model of disturbed boundary-layer flow used by Schmid and Henningson. To model an oblique boundary-layer transition these authors disturbed the Blasius boundary layer not only by a pair of primary symmetric oblique waves of finite amplitude with wave vectors $(k_1,\pm k_2)$ but also by random 'noise waves' of much smaller amplitudes having the following 'neighboring multiple wave vectors' (0,0) (random 'mean-velocity correction'), $(k_1,0)$, $(2k_1,0)$, $(0,\pm k_2)$, $(2k_1, \pm k_2)$, $(0, \pm 2k_2)$, $(k_1,\pm 2k_2)$, and $(2k_1,\pm 2k_2)$. Hence, contrary to the previous numerical models where all 'higher modes' were produced entirely by nonlinear interactions of the two primary oblique waves among themselves and with their higher harmonics, here weak random higher modes were assumed to exist right from the start, and could grow by extracting energy from the much more energetic primary waves. In Fig. 5.32 an example, computed by Schmid and Henningson, of dependencies on the time t of the energies E of the primary mode (1,1) and of three selected 'higher modes' is shown. (Here t and E are measured in some conventional units, and the numbers in parentheses indicate the ratios $(K_1/k_1,\ K_2/k_2)$ of the streamwise and spanwise wavenumbers of the mode to those of primary waves.) The modes represented in Fig. 5.32 are not the same as in Figs. 5.31a,b (where moreover spatial, and not temporal, wave amplification was simulated), but nevertheless the qualitative results of the initial part of Fig. 5.32 (say, until $t \approx 500$) are reminiscent of those given in Figs. 5.31a,b. However, at larger values of t the horizontal-streak array [generated by streamwise vortices which are also (0,2)-mode structures] becomes unstable with respect to local high-frequency fluctuations, begins to oscillate in disorderly fashion, and then breaks down. As a result, the flow becomes turbulent, containing a large collection of various finite-amplitude structures. [This process is partially reflected in the right part of Fig. 5.32; cf. also Waleffe (1995) and Hamilton *et al.* (1995) where the streak breakdown and the following stages of disturbance development were also considered, and it was shown that, at $Re > Re_{cr}$, streak breakdown leads to regeneration of roll structures and may be a part of a self-sustaining process forming a steady near-wall turbulent regime]. Numerical results presented in Figs. 5.31a,b and 5.32 may be supplemented also by figures in the paper by Elofsson and Alfredsson (2000) showing the dependence on x of the amplitudes of various (n,m)-modes in the Blasius boundary-layer flow studied experimentally by these authors. However, space limitations give us no possibility to linger here on these experimental results.

Note that the N- and K-regimes of disturbance development in a boundary layer both begin with the growth of a primary linearly-unstable two-dimensional Tollmien-Schlichting wave. Then this wave stimulates the appearance in the flow of some three-dimensional T-S waves (not the same for the two regimes) forming, together with the primary wave, an unstable wave system (this instability is clearly the secondary one). Thus, both regimes may be interpreted as initial stages of the *TS-wave-secondary-instability transition scenario;* as explained earlier, whether the N-regime or the K-regime will be realized depends only on the primary-wave amplitude. Quite another route to boundary-layer transition is represented by the *oblique-transition scenario* (or 'O-regime') considered above, where the first stage consists of the development in the flow of a pair of symmetric oblique T-S waves. In parallel with these two scenarios Schmid *et al.* (1996), Reddy *et al.* (1998) and Schmid and Henningson (200~~0~~) 1 considered also a third *streak-breakdown transition scenario* which does not include the stage of growing T-S waves [i.e., represents some particular type of the *by-pass transitions* considered by Morkovin (1969); cf. Chap. 2, p. 118]. This third scenario has many features in common with the oblique-transition scenario but it completely disregards the first stage of the latter regime, connected with TS-wave development, and begins with a collection of streamwise vortices which is a (0,2)-mode structure, while in the oblique transitions (0,2)-structures are produced by nonlinear development of a pair of symmetreic oblique waves.

At the very beginning of this section it was noted that if one assumes that any T-S wave may penetrate the boundary-layer from the disturbed free-stream flow, then it seems natural to suppose that the most unstable of such waves must dominate the initial stage of the development of flow instability. However, if the free-stream flow is so disturbed that all possible T-S waves are present there and can penetrate the near-wall flow region, then similar penetration must be possible also for many non-modal disturbances (i.e. those differing from T-S waves) existing in the boundary-layer environment. It seems equally natural to assume that the initial stage of boundary-layer instability development will be dominated, not by the most-unstable T-S wave but by the optimally-growing disturbance of non-modal type which, during the initial stage of disturbance development, grows much faster than any T-S wave (again see Fig. 5.30). As explained in Chap. 3, if the transient, rather than the asymptotic (relating to $t \to \infty$) disturbance growth, is considered and the disturbances are assumed to be so small that their development may be described by linear instability theory, then the optimally-growing disturbance will be non-modal, and in the case of a boundary-layer flow will have the form of a spanwise-periodic array of streamwise vortices. Therefore, it seems reasonable to suppose that a disturbance development starting with the appearance in the boundary layer of streamwise vortices of small amplitude may also be a quite important mechanism of real boundary-layer transition to turbulence. Exactly this mechanism was called the ‘streak-breakdown transition scenario’ in the papers mentioned in the previous paragraph.

Is it possible to estimate quantitatively, if only roughly, the relative likelihood of various transition regimes for different flows met in practice? It is clear that for this it is necessary, first of all, to estimate somehow the probabilities of the appearance of disturbances of various types, with various amplitudes, frequencies and wave vectors. However, such an estimate is impossible without detailed knowledge of the qualitative and quantitative characteristics of free-stream turbulence and other environmental `noise' while in practice these characteristics can seldom be determined with satisfactory accuracy. Thus, the problem of likelihood estimation cannot have a general solution and may be solved, even partially, only in some exceptional cases. Hence it is only natural that Schmid *et al.* (1996), Reddy *et al.* (1998), and Schmid and Henningson (2000) did not try to study the problem in its general form but considered only two more special subproblems, having definite relevance to a rough assessment of the likelihoods of different routes to transition. 1/

It has already been indicated earlier in this section that a pair of symmetric oblique waves may lead to `oblique transition' of the boundary-layer flow only if the initial amplitude A of these waves is not too small. Otherwise the waves will at best only begin to grow and later they (and also the streamwise vortices, if they were generated by primary waves) will begin to decay and finally disappear. This means that there exists some threshold amplitude A_{tr} of the oblique waves, oblique transition being possible only if $A > A_{tr}$. [Of course, the threshold amplitude may take different values for oblique waves with different values of (k_1,k_2) or (ω,k_2); below, the symbol A_{tr} will always be applied to ‘optimal waves’ corresponding to the greatest threshold amplitude.] Recall now that in Sec. 5.2 it was indicated that a threshold amplitude exists also in the case of resonant-triad interactions: at too small an amplitude of the primary plane T-S wave, resonant growth of the oblique wave becomes impossible. In Sec. 5.2 only the stage of resonant growth of oblique waves was considered; it is clear, however, that for the full realization of the TS-wave-secondary-instability transition scenario the initial amplitude A of the linearly unstable plane T-S wave must exceed a definite threshold value A_{tr}, which is apparently greater than the threshold value determining the possibility of a transient growth of oblique-wave amplitude. Finally, a definite threshold value A_{tr} of the initial amplitude of the streamwise vortices must also exist, and determine whether or not an array of such vortices can be transformed into a periodic array of streamwise streaks and then disintegrate into a collection of disordered (‘turbulent’) vortical structures. Hence, for all transition scenarios considered above, there exists an initial threshold amplitude A_{tr} determining whether the corresponding initial (oblique) disturbances may or may not lead to transition. The value of A_{tr} does not determine the likelihood of this transition scenario but it is clear that a decrease in this value increases the chances that the scenario will be realized in practice. Therefore the evaluations of amplitudes A_{tr} may be quite useful for the assessment of the likelihood of various transition regimes.

Another problem, also having relation to attempts to determine which of the scenarios is the most likely, is the problem of estimation of the 'transition time' T_0 (or streamwise distance) which is necessary for completion of the transition to turbulence (if it may be achieved) by the route considered. The point is that if the time T_0 is large, then there is a real chance that during this time some extraneous disturbances will begin to interfere with the normal flow development and will disrupt the transition process. Hence, an increase in T_0 diminishes the likelihood of the scenario.

For the case of a Blasius boundary-layer flow, an approximate estimate of the values of A_{tr} and T_0 corresponding to the three transition scenarios listed above was made by Schmid *et al.* (1996) [see also Schmid and Henningson's book (200~~0~~)]. This estimate was based on the results of the 1/ temporal numerical simulations of the three transition scenarios described above, performed by the authors. All the simulations were of the same type as the simulation of the oblique transition which was briefly described earlier in this section, and led to the results shown in Fig. 5.32. (Thus, in these simulations the boundary layer was also assumed to be plane parallel with thickness $\delta(t)$ growing with time, and the initial disturbances included 'random noise' whose energy was about 1% of the energy of primary disturbance.) The primary disturbances - a plane T-S wave, or a pair of symmetric oblique waves, or a spanwise periodic array of streamwise vortices - were always chosen to be close to the optimal ones (which grow with time most rapidly), but the initial amplitudes A of these disturbances were varied, and, in all cases in which transition to turbulence was found to be possible, the simulations were continued up to transition. These computations yielded, for the three scenarios, the dependence of T_0 on the value of the initial amplitude A, and thence the value of A_{tr}, being the greatest value of A at which the transition could not be reached (i.e. it corresponded to $T_0 = \infty$). Fig. 5.33 shows results obtained by Schmid *et al.* for the Blasius boundary layer with the initial Reynolds number Re* = 500. Here the initial amplitude A is replaced by the initial energy of the primary disturbance $E = \frac{1}{2V}\int_W (u^2 + v^2 + w^2)d\mathbf{x}$, where W is the periodic box domain of the computations, V is its volume, and E and the other dimensional quantities are non-dimensionalized using δ^* and U_0 as length and velocity scales. It is seen that the threshold energy E_{tr} takes its lowest value for the oblique transition, and the highest for the streak-breakdown regime which begins with the appearance of an array of streamwise vortices. The TS-wave-secondary-instability regime (which may be either of N- or of K-type) takes an intermediate place, but at high values of the initial energy E it develops more slowly (leading to a greater value of T_0) than the streak-breakdown regime and this increases the competitiveness of the latter regime.

5.62. *Linear and Nonlinear Development of Localized Disturbances*

The three transition scenarios considered above all begin with the appearance of some spatially-unbounded disturbance in a laminar Blasius boundary layer. However, it was noted in Chap. 3, p. 99, that real disturbances appearing in various natural, engineering and laboratory flows are as a rule initially localized in some finite fluid volumes. In this respect several papers were cited in Chap. 3 which were devoted to studies of the temporal evolution of localized disturbances in wall-bounded shear flows. Most of these papers dealt with inviscid flows, which are not considered in this chapter [an important exception is the paper by Henningson *et al.* (1993), some results of which will be discussed below]. Moreover, in Chap. 3 only results relating to initial disturbances of very small amplitudes, whose evolution may be described by linearized Navier-Stokes equations, were studied. On the other hand, Gaster and Grant (1975) and Breuer and Haritonidis (1990) (these papers were considered in Chap. 3), who tried to describe the results of their wind-tunnel observations of the evolution of localized disturbances in a boundary-layer flow in the framework of linear stability theory, both found that the deductions from this theory agree with observations only during some initial time interval and become invalid at later times. Hence it is clear that the linear theory is insufficient for a satisfactory description of the development of localized disturbances.

As well as the above-mentioned work by Gaster and Grant, and by Breuer and Haritonidis, other attempts to study evolution of artificially produced localized disturbances in laboratory flat-plate boundary layers have been made; the papers by Gaster (1984, 1990), Tso *et al.* (1990), Cohen *et al.* (1991) and Breuer *at al.* (1997) are just typical examples of such work. Gaster, and Tso *et al.* paid their main attention to a late stage of the disturbance evolution directly connected with formation of turbulent spots and transition to turbulence; since this chapter is devoted mainly to the weakly-nonlinear effects, their papers will be mentioned only occasionally below. Results of the other two papers mentioned will be described below at greater length; first, however, data of quite a different origin will be considered.

The purely theoretical results available at present cannot satisfactorily describe the weakly-nonlinear stage of localized-disturbance development, but results of numerical simulations are more informative. Apparently one of the first attempts to apply nonlinear numerical simulation (i.e., the numerical solution of the appropriate initial-value problem for the nonlinear Navier-Stokes equations) to the study of the evolution of a localized disturbance in a boundary-layer flow was made by Breuer (1988). His numerical-simulation results were then carefully analyzed by Breuer and Landahl (1990). The numerical solution of the nonlinear initial-value problem considered in Breuer's dissertation (1988) and his paper with Landahl related to the evolution in the Blasius flow [assumed to be plane-parallel but with thickness $\delta(t)$ growing with time] of a localized disturbance initially having the form shown schematically in Chap. 3, Fig. 3.2; see also Eqs. (5.19), (5.19a) and Fig. 5.38a below, also relating to this initial disturbance. [As noted in Chap. 3, the same model of the initial disturbance was also used in stability computations by Russell and Landahl (1984), Henningson (1988), Breuer and Haritonidis (1990) and Henningson *et al.* (1993); later it was also accepted as one of the three initial conditions considered by Bech *et al.* (1998).] Breuer and Landahl's paper represented a continuation of the work of Breuer and Haritonidis (1990), where the same initial-value problem was solved for the inviscid linearized N-S equations; some of the results obtained there were shown in Fig. 3.3. These results agreed satisfactorily with Breuer and Haritonidis' wind-tunnel experimental data (relating to a flat-plate boundary layer where localized disturbances of a shape close to that shown in Fig. 3.2 were artificially produced) but only for small and moderate values of dimensionless time $\tau = tU_0/\delta^*$. However, for larger values of τ the numerical results of Breuer and Landahl's nonlinear computations agreed better with the available experimental data than those of Breuer and Haritonidis' solution of linearized N-S equations.

Breuer and Landahl (1990) [and also Landahl, Breuer and Haritonidis (1987)] stressed that both Breuer's computational results and the experimental data of Breuer and Haritonidis showed that the disturbance evolving from a strongly-localized initial disturbance in a boundary layer consists of two very different parts. Recall that in Chap. 3 (pp. 24-27) it was pointed out that in the case of small disturbances in a plane-parallel steady inviscid flow, the general solution of the corresponding linear initial value problem includes terms of two different types. [In Chap. 3 this result was attributed to Gustavsson (1978) but in fact it was already mentioned by Case (1960) for the case of two-dimensional disturbances.] The first type is formed by the so-called 'convective components' (the adjective 'convective' is sometimes replaced here by 'transient'); these components are convected streamwise with the local flow velocity $U(z)$ and they often undergo considerable transient growth followed by a rapid decline. (In connection with the phenomenon of 'transient growth', much attention was paid in Chap. 3 to these components.) The disturbance components of the second type are 'dispersive waves', i.e., waves with phase velocities depending on their frequencies and wave numbers. In Chap. 3 it was stressed that in the case of an 'ideal' (inviscid) fluid the phase velocities c of the wave component do not coincide with the discrete eigenvalues of Rayleigh's eigenvalue problem. However, in the case of fluids with non-zero viscosity the phase velocities of the 'dispersive waves' are just the eigenvalues of the Orr-Sommerfeld eigenvalue problem and the 'dispersive component' of any evolving disturbance is represented by some collection of T-S waves.

Discussion of the 'convective' (i.e. 'transient') and 'dispersive' flow disturbances in Chap. 3 related only to inviscid fluids and to very small (regarded as 'infinitesimal') disturbances. However, the closing sentence of the last paragraph implies that the same notions may also be applied to disturbances in viscous flows. (Recall that small transiently-growing disturbances in viscous fluid flows were in fact considered at length in Sec. 3.3 of Chap. 3.) Also, the results of the above-mentioned papers by Breuer, Landahl, and Haritonidis (and the experimental results of Tso *et al.*) confirmed that the division of the set of all disturbances in steady plane-parallel (or nearly plane-parallel) flows into 'convective' and 'dispersive' parts is fully appropriate in the case of finite-amplitude disturbances in viscous flows, where the two types can often be easily distinguished. These results also showed that convective disturbances are really transient ones - they grow considerably during a short initial time (or streamwise) intervals but then begin to decay rapidly and as a rule entirely disappear shortly afterwards. Therefore, in studying the long-time evolution of localized disturbances leading to transition to turbulence, it is reasonable to pay the main attention to dispersive wave disturbances.

One of the results found by Breuer and Landahl is shown in Fig. 5.34; it is similar in many respects (although not identical) to that presented in Fig. 3.3b. Note, in particular, that both figures show the appearance of a strong tilted shear-layer between low-speed and high-speed regions produced by the lift-up effect; this result was confirmed later by the results of numerical simulations, both linear and nonlinear, of the development of a localized disturbance in plane Poiseuille and boundary-layer flows by Henningson *et al.* (1993) [one of the linear results of these authors was shown in Fig. 3.17]. Note that the linear and nonlinear instability of the boundary layer to two-dimensional waves ($k_2 = 0$) with high values of k_1, found by

Breuer and Landahl, strongly contradicted the results of Henningson *et al.* but, as the latter authors showed, this was due to the insufficient resolution of Breuer and Landahl's computations in the wall-normal direction. However, many other results of these two groups of authors agree quite well with each other (and were confirmed also by the results of careful experiments by Cohen, Breuer and Haritonidis (1991) and Breuer, Cohen and Haritonidis (1997) which will be discussed later).

So, Breuer and Landahl found that nonlinear effects strongly influence the temporal evolution of disturbance structures and the behavior of disturbances at large values of dimensionless time τ. Two-dimensional spatial spectra of the normal-to-wall disturbance velocity $w(x,y,z,t)$ at $z/\delta^* = 1.05$, computed by them for a number of values of τ, show that at $\tau = 43$ the spectrum contours have smooth oval shapes and there is a unique spectral peak at the common center of these ovals. This simple spectral shape is close to that corresponding to the initial conditions in Fig. 3.2. However at larger values of τ the shape becomes much more complicated, and a number of new spectral peaks emerge at points $(k_1\delta^*,k_2\delta^*)$ corresponding to larger values of spanwise wavenumber k_2. In particular, at $\tau = 136$ peaks were found at $(k_1\delta^*,k_2\delta^*) \approx (0,0.7)$, $(0.1,1.3)$, $(0.2,2.0)$ which recall the series of harmonics of increasing order produced by nonlinear interactions. Henningson *et al.* showed analytically that, after the appearance of the peaks of the energy distribution at wave vectors $(\pm k_1,\pm k_2)$, the nonlinear interactions will give rise to new peaks at $(\pm 2k_1,0)$ and $(0,\pm 2k_2)$ (the latter will be the most rapidly growing), and also at $(\pm k_1,\pm 3k_2)$, $(0,\pm 4k_2)$, etc., corresponding to propagation of energy up the spanwise wavenumber axis. Moreover, both groups of investigators found that solutions of the nonlinear initial-value problem imply the generation, at later stages of the instability-development process, of a system of long spanwise-alternating streaks of high- and low-speed fluid (see for example Fig. 5.35 by Hennington *et al.*; similar figures were also presented by Breuer and Landahl, and Bech *et al.*). These streaks then form streamwise-elongated vortical structures, recalling the streamwise Λ-vortices observed in other regimes of boundary-layer transition, and still later produce turbulent spots which are the precursors of full transition to turbulence (these stages of instability development were more explicitly described by Cohen *et al.* and Breuer *et al.*). Therefore, the results support Morkovin's idea of the ordinariness of so-called 'by-pass boundary-layer transitions' whose late stages do not differ much from those for transitions initiated by primary T-S waves. Moreover, they allow the *localized-disturbance scenario* of boundary-layer transition to be added to the other three transition scenarios considered above.

The nonlinear interactions play an important part in the temporal evolution in laminar shear flows of high- and moderate-amplitude wave packets consisting of a collection of two- and three-dimensional T-S waves. Let us recall that in Sec. 3.31 it was mentioned that such wave packets were used by a number of researchers as natural models of localized disturbances in a boundary layer. In particular, Gaster (1975) used a wave-packet model to describe quantitatively the results of Gaster and Grant's (1975) experiments on the development of a localized disturbance, produced by a short acoustic pulse, in the boundary layer on a flat plate. The streamwise evolution of such disturbances was investigated by a hot-wire anemometer measuring the values of the streamwise disturbance velocity u at points with $z = 3.2\delta^*$ (i.e., placed slightly above the boundary layer) and various values of coordinates x and y. As stated in Sec. 3.31, Gaster modeled this evolution by representing the values of the streamwise-velocity disturbances $u(x,y,z,t)$ at positive values of x in the form:

$$u(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(k_2,\omega;z)\exp[i\{k_1(k_2,\omega)x + k_2 y - \omega t\}]dk_2 d\omega \qquad (5.17)$$

[this equation appeared in Sec. 3.31 as Eq. (3.52)]. Here $u(k_2,\omega;z)$ is the Fourier transform, with respect to y and t, of the initial value of the streamwise velocity disturbance at $x = 0$ and a fixed value of z, and $k_1(k_2,\omega)$ is the complex eigenvalue with the smallest imaginary part appearing in the spatial 3D Orr-Sommerfeld eigenvalue problem (2.41), (2.42) (corresponding to given values of k_2 and ω). Gaster and Grant dealt with the supercritical ($\mathrm{Re} > \mathrm{Re}_{cr}$) boundary layer to which a small disturbance was introduced at $x = 0$. Hence there existed that plane wave which grows most rapidly with x, having frequency $\omega = \omega_0$ and the streamwise wavenumber $k_1 = k_1(0,\omega)$ given by the O-S eigenvalue with numerically-greatest negative imaginary part. Moreover, the waves with (k_2,ω)-values close to $(0,\omega_0)$ and $k_1 = k_1(k_2,\omega)$ are also spatially growing in this case and their rate of growth is only slightly smaller that that of the most unstable 2D wave. There is also a larger region of the (k_2,ω)-plane which corresponds to the collection of all spatially-growing waves. Gaster described the evolution of the localized disturbance by an approximate numerical value of the integral in Eq. (5.17) in which only spatially-growing waves were taken into account. Thus, the approximate solution of the initial-value problem he considered has the form of a superposition of spatially-growing two- and three-dimensional T-S waves each of which is the least stable of the waves with the same values of k_2 and ω, grows in accordance with the prerdictions of the linear stability theory and does not interact with any of the other waves.

In Gaster and Grant's experiments the amplitude of a wave packet took rather low values and they found that in this case the theoretical model (5.17) led to results which agreed well with their observations at the majority of the measuring stations. However, they noted that the data obtained at the largest value of x disagreed with the predictions of Eq. (5.17). The authors explained this discrepancy by the influence of nonlinear effects at large x. This explanation is evidently confirmed by the results presented above, relating to transition scenarios starting with the growth of T-S waves. In fact, these results show that even when there is only one such wave whose amplitude exceeds a relatively small threshold value, it necessarily begins to interact at once with the background disturbances ("noise") that always exist in practice. Moreover, in the case of a group of growing T-S waves, their nonlinear interactions must necessarily become apparent after quite a short period of independent development. Therefore model (5.17) of wave-packet development can represent only an approximation applicable to packets of small initial amplitude during some limited initial period of time.

Cohen, Breuer and Haritonidis (1991) and Breuer, Cohen and Haritinidis (1997) repeated the experiments by Gaster and Grant (1975) using a low-turbulence wind tunnel with a test section about 6 m long. Cohen *et al.* made hot-wire measurements of the mean-velocity profile $U(z)$ (depending only on the local value of δ^*)[9] and of

[9] In both papers it was assumed that the boundary layer is plane-parallel but in the treatment of data relating to a given value of x, values of δ^* and Re* corresponding to this x were used. [A more precise analysis of some data of Cohen *et al.*, which took into account the streamwise growth of the boundary layer, was developed by Cohen (1994).] Measurements by Cohen *et al.* and Breuer *et al.* showed that in their studies the pressure gradient in the boundary layer was slightly negative, and therefore the function $U(z)$ was slightly closer to a Falkner-Skan profile for $\beta \approx 0.01$ (see Chap. 2, p.119) than to

the three disturbance-velocity components u, v and w at a great number of points $\mathbf{x} = (x,y,z)$ inside the boundary layer, while Breuer *et al.* measured only the streamwise disturbance velocity u but with a rake of hot-wire probes to make simultaneous measurements of u at eight values of z. The long wind-tunnel test section made possible the observation of boundary-layer development over a large range of x. Moreover, the disturbance generator (which produced short sinusoidal air pulses of acoustic origin) allowed the amplitude A of the initial localized disturbance to be varied easily. The Reynolds number $\mathrm{Re}^* = U_o\delta^*/\nu$ at the location of this generator was close to 1000 (well above the critical value) in these experiments.

Cohen *et al.* and, later, Breuer *at al.* found that at small enough values of A three different stages of streamwise development of wave packets may be observed. The first stage (called the *linear stage* by these authors) corresponded very well to Gaster and Grant's observations; here Gaster's Eq. (5.17) (based on the linear stability theory) described the disturbance evolution with high accuracy. Spectral analysis of the velocity fluctuations showed that during this stage the disturbance included both two- and three-dimensional T-S waves corresponding to ranges of dimensionless frequencies $\omega = 2\pi f\delta^*/U_0$ (where f is the dimensional frequency measured in Hz) and spanwise wavenumbers $K_2 = k_2\delta^*$ centered around the values $\omega = \omega_0$ and $K_2 = 0$ corresponding to the most rapidly growing T-S wave (which is two-dimensional, i.e. with $K_2 = 0$, by virtue of Squire's theorem - see Chap. 2, p. 75). [Symbols ω, K_2 and K_1 $(= k_1\delta^*)$ will now denote dimensionless frequencies and wavenumbers.] One example of the (K_2,ω)-spectrum found in the linear stage of localized-disturbance development is shown in Fig. 5.36a. In full accordance with Gaster's model (5.17), during the linear stage the values of $K_1 = K_1(K_2,\omega)$ could be determined for all waves considered by means of the O-S equation, as those corresponding to the most-unstable wave with given values of ω and K_2. It was also found that as the wave packet moved downstream all wave components evolved according to the O-S equations (and hence independently from each other). Therefore, in the linear stage of disturbance development the most rapidly growing T-S wave, and a group of T-S waves with frequency and wave number close to the most rapidly growing wave [and hence with values of K_2 and ω close

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the Blasius profile corresponding to $\beta = 0$. However, the Blasius approximation was found to be accurate enough to be usable in the analysis of the experimental data.

to $K_2 = 0$ and $\omega = \omega_0$ corresponding to the most unstable T-S wave], gained energy most effectively. As a result, a relatively narrow band of two- and three-dimensional T-S waves centered at the most-amplified wave quickly began to play the dominant role in the evolution of the wave packet. In the initial series of experiments by Cohen *et al.* the first (linear) stage was observed from $x = 160$ cm to $x = 220$ cm (the disturbance generator being placed at $x = x_0 = 81$cm from the plate leading edge). The amplitude A of the wave packet was close to 0.3% of U_0 at $x = 160$ cm and grew to 0.46% of U_0 at $x = 220$ cm (i.e., during the linear stage it continued to be quite small).

At $x = 220$ cm the second stage of wave-packet development began. Here, in addition to the spectral peak at $(0,\omega_0)$ two additional spectral peaks of the two-dimensional (K_2,ω)-spectrum appeared at the points $(K_{2,1},\omega_1)$ and $(-K_{2,1},\omega_1)$ which corresponded to a definite pair of symmetric oblique waves (see Fig. 5.36b). Cohen *et al.* discovered that the peak frequency ω_1 was equal to $\omega_0/2$, i.e. half the frequency of the most-amplified 2D wave. [This fact agrees with Gaster's (1990) discovery of spectral peaks at frequencies $\omega/2$ and $3\omega/2$ (the latter was clearly due to secondary nonlinear interactions) in the wave packet produced in the boundary layer on a flat plate by a sinusoidal acoustic signal of frequency ω.] Cohen *et al.* also found that to the peak frequency ω_1 and spanwise wavenumber $K_{2,1}$ corresponded to the T-S wave with complex streamwise wavenumber $K_1 = K_1(K_{2,1},\omega_1)$ having the real part $\Re e K_1(K_{2,1},\omega_1)$ close to half the real part $\Re e K_1(0,\omega_0)$ of the complex streamwise wavenumber $K_1(0,\omega_0)$ of the most unstable 2D wave. Hence the two new spectral peaks appearing in the second stage of the localized-disturbance development together with the already existing spectral peak at the point $(0,\omega_0)$ corresponded to a Craik's resonant triad of T-S waves. And to the spectral regions surrounding two peaks at points $(K_{2,1},\omega_1)$ and $(-K_{2,1},\omega_1)$ there corresponded two symmetric bands of subharmonic oblique waves with frequencies close to $\omega_0/2$, recalling the band of subharmonic oblique waves appearing during the N-regime of instability development initiated by the primary unstable plane T-S wave (this band is clearly seen in Figs. 5.5a,b). During the second stage of wave-packet development the two bands of subharmonic oblique waves gain energy very effectively, so that the oblique waves experienced rapid growth, exceeding considerably the growth of waves corresponding to the primary peak centered at the point $(0,\omega_0)$. Cohen *et al.* suggested that this gain was due to a number of three-wave resonances. They also found that the primary band of waves, with frequencies close to ω_0 and small values of $|K_2|$, began to lose its energy somewhere in the initial part of the second stage (where the growth of its waves turns into decay) and disappeared entirely in the third stage (see Figs. 5.36c,d). Cohen *et al.* and Breuer *et al.* called the second stage of the wave-packet development the *subharmonic stage*. In the first series of experiments considered above, made in 1991, this stage was observed between $x = 220$ cm and $x = 300$ cm and within this range the amplitude A of the wave packet increased from 0.46% of U_0 to 5.2% of U_0 (this growth evidently considerably exceeds that observed within the first stage). Cohen (1994) tried to apply to description of this stage of disturbance development the weakly-nonlinear stability theory, which used the amplitude-power expansions where only a few terms of lowest orders are taken into account. [The attempt by Zel'man and Smorodsky (1990) to describe a wave-packet evolution by a system of amplitude equations of the lowest nontrivial order also relates to just this stage.]

The third and final stage of wave-packet development is strongly nonlinear. Here a number of new spectral peaks, representing sums and differences of spectral characteristics of primary and secondary waves and due to the nonlinear interactions of these waves, appear in the disturbance spectra. In particular, the (0,0)-mode corresponding to velocity-profile distortion also emerges from such interactions, and may lead to the appearance of local profile inflections producing quasi-inviscid flow instabilities and high-frequency small-amplitude velocity oscillations. These oscillations have random phases and may later contribute to the formation of turbulent spots, indicating the imminence of transition to turbulence [see, however, the closing part of Sec. 5.5 where the appearance of turbulent spots is connected with the evolution of Klebanoff's spikes, which have an origin other than inflection-generated oscillations]. In the series of experiments by Cohen *et al.* considered above, the third stage was observed between $x = 320$ cm. and $x = 350$ cm and was accompanied by rapid growth of disturbance energy leading at $x = 350$ cm to a very high value of amplitude A, close to 27% of U_0. To study the second and third stages of the wave-packet development, Cohen *et al.* performed a number of experiments with larger initial values of A to shift the second and third stages of wave-packet development upstream and thus make observations easier. A more detailed experimental study of the late stage of wave-packet development was carried out by Breuer *et al.*

(1997) while Cohen (1994) published some theoretical considerations relating to the initial stage of wave-packet development, and compared his theoretical results with the experimental data of Cohen *et al.* (1991).

The theoretical results by Cohen (1994) were based on an improved linear model of the evolution of waves in a laminar boundary-layer flow. This model took into account the nonparallelism of the flow (leading to weak dependence of δ^* on x) by an approximate method developed by Saric and Nayfeh (1975) and Nayfeh and Padhye (1979). Cohen extended Gaster's (1975) model to the case of a slightly nonparallel boundary layer and then calculated anew the time evolution of amplitudes for a large number of two- and three-dimensional T-S components of the wave packet studied by Cohen *et al.* (1991). Data obtained in the latter work for the evolution of amplitudes of individual waves were then compared with the evolution predicted by the extended linear stability theory. Cohen found that the results of the linear stability theory relating to the most rapidly growing two-dimensional T-S wave of frequency ω_0, or to any T-S waves with values of (K_2,ω) close to $(0,\omega_0)$ and high rates of the 'linear' spatial growth, agreed very well with the experimental data within the whole first stage of wave-packet development and a considerable part of the second stage. However, the subharmonic oblique modes with frequencies close to $\omega_0/2$ begin to grow much faster than predicted by linear stability theory, before the end of the first stage of wave-packet development (this was not observed in experiments since at corresponding values of x the subharmonic modes were still rather weak). Thus Cohen (1994) concluded that in the case of wave propagation in a laminar boundary layer, nonlinear effects often become significant at appreciably smaller values of x (measured from the leading edge of the plate) than was assumed earlier, and these effects make the linear stability theory inapplicable to subharmonic wave modes for all but rather small values of x.

Measurements carried out by Breuer *et al.* (1997) dealt only with streamwise velocities $U(z)$ and $u(x,y,z,t)$ but they were made in a very dense grid of spatial points and times, and provided the authors with a vast amount of numerical data. (In particular, a great number of repeated observations yielded large ensembles of data, guaranteeing the accuracy of statistical characteristics.) The results found by Breuer *et al.* supported, and made more precise, the conclusions of the paper by Cohen *et al.* As an example of the new results, Fig. 5.36 shows the constant-velocity contours in the (τ,y)-

plane and the corresponding two-dimensional (K_2,ω)-spectra for velocities $u(x,y,z,t)$ at $z/\delta^* = 0.5$ and for four values of x relating to the first, second, and third (two x-values) stages of wave-packet development.[10] Here $\tau = (t-T_0)U_0/(x-x_0)$ is non-dimensionalized time, t is dimensional time of the measurement (counted from the moment of air-pulse ejection by disturbance generator), x is the x-coordinate of the measurements counted from the leading edge of the plate, x_0 = 81 cm is the x-coordinate of the disturbance generator, while T_0 is the delay time, proportional to $(x-x_0)/U_0$ with a proportionality coefficient chosen to make the origin of the time τ close enough to the time when the leading edge of the wave packet reaches the measurement coordinate x.

The two upper diagrams in Fig. 5.36, labeled as Fig. 5.36a, are for x = 170 cm, within the first (linear) stage of disturbance development (the wave-packet amplitude A was here close to 0.6% of U_0). At this value of x the u-velocity contours had the form of smooth swept-back crescents, which was also the form of the wave-packet observed by Gaster and Grant at points far from the disturbance generator (closer to the generator, Gaster and Grant's wave packet had an oval shape). The (K_2,ω)-spectrum shows that, at this x, most of the wave-packet energy is concentrated in the band of 2D modes (and 'almost 2D' modes with $|K_2| \ll 1$), centered at the mode with $(K_2,\omega) = (0,0.09)$ which is just the most-unstable T-S wave at the Re* corresponding to x = 170 cm. There are also two much smaller spectral peaks at points $(K_2,\omega) \approx (\pm 0.25,\ 0.085)$, which apparently represent weak 'oblique-wave contributions' to the disturbance energy at x-values corresponding to the first stage of wave-packet development as noted by Cohen *et al.* (1991)

At x = 250 cm, in the second (subharmonic) stage of disturbance development, the nonlinear effects were much more influential and this is clearly seen in Fig. 5.36b. In particular, two 'side spectral peaks' appeared here, at the frequency $\omega_1 \approx \omega_0/2$ and spanwise wavenumbers $\pm K_{2,1} \approx \pm\ 0.25$. These peaks acquired their energy from preexisting 'background noise' and the values of ω_1 and $K_{2,1}$ implies that 3D waves corresponding to them have a phase velocity close to that of the most-unstable 2D wave. This means that

[10] The measurements by Breuer *et al.* discussed here related to waves excited by an acoustic pulse with a different amplitude from that used in the experiments by Cohen *et al.* (1991). Therefore the streamwise locations of the three stages of wave-packet development mentioned in our discussion of results by Cohen *et al.* are not the same as those in the series of experiments considered here.

these 3D waves, together with the most-unstable 2D wave, form a resonant wave triad (but not necassary of Craik's 'fully-resonant type' where Eqs. (5.7) are valid precisely). Thus the growth of subharmonic modes corresponding to the side peaks and to spectral regions adjacent to them may be due to three-wave resonance or to secondary instability of the primary waves to subharmonic disturbances - cf. the discussion of the N-regime of boundary-layer development in Secs. 5.3 and 5.4. The velocity contours at $x = 250$ cm show that some streamwise elongated structures appeared, with some similarity to streamwise Λ-vortices. Note also the appearance of a group of waves, apparently produced by nonlinear wave interactions, with (K_2,ω)-values belonging to the 'low-K_2, low-ω' spectral region near the mean-flow distortion mode with $(K_2,\omega) = (0,0)$.

The two lower pairs of diagrams in Fig. 5.36 (Figs. 5.36c and 5.36d) correspond to streamwise coordinates $x = 270$ cm and $x = 280$ cm, in the third, strongly-nonlinear stage of wave-packet development. The corresponding velocity contours show the formation at $x = 270$ cm of a system of elongated structures including spanwise-alternating streaks of fluid having in turn higher and lower streamwise velocity than the mean $U(z)$. At $x = 280$ cm this system is more compact and gives the impression of approaching the 'turbulent spot' stage [other experimental results given in the paper by Breuer *et al.* (1997) allowed the authors to suggest that the formation of turbulent spots actually began near $x = 282.5$ cm]. The velocity spectrum at $x = 270$ cm shows that the primary band of waves centered at the most-unstable $(0,\omega_0)$-wave has practically disappeared here, but the subharmonic bands with frequencies close to $\omega_0/2$ became considerably more pronounced. The spectral peak at the coordinate origin, and the adjacent region of 'low-K_2, low-ω' points corresponding to mean-flow distortions and nearly-2D low-frequency waves also grew considerably in comparison to those at $x = 250$ cm. In addition two small spectral bands appeared near the peaks at points $(\pm 2K_{2,1},0)$, produced by nonlinear interactions of $(\pm K_{2,1},\omega_1)$ and $(0,\omega_0) = (0,2\omega_1)$ modes. At $x = 280$ cm the primary spectral band adjacent to the point $(0,\omega_0)$ completely disappeared, bands around the subharmonic peaks at $(\pm K_{2,1},\omega_1)$ became less energetic than at $x = 270$ cm, while bands near points (0,0) and $(\pm 2K_{2,1},0)$, became much more pronounced and other peaks appeared near the points $(\pm 3K_{2,1},\omega_1)$. [According to Breuer *et al.*, diagrams more detailed than Fig. 5.36d show in the case where $x = 280$ cm, additional spectral peaks at points $(0, \omega_1)$, $(0, 3\omega_1)$, and near the points $(3K_{2,1}, 2\omega_1)$ and $(3K_{2,1}, 3\omega_1)$.] These results clearly agree with observations by Breuer and Landahl (1990) and Henningson *et al.* (1993) of the 'propagation of the disturbance energy along the K_2-axis'.

Further results of Breuer *et al.* (1997) describe in more detail the spatial and spectral structures accompanying the wave-packet development. In Fig. 5.37, contours in the (ω,z)-plane of the frequency spectra $P(\omega; \mathbf{x}) = P(\omega;x,y,z)$ of velocity oscillations $u(\mathbf{x},t)$ at points $\mathbf{x} = (x,y,z)$ are shown for $y/\delta^* = 4.7$ and four values of x corresponding to the different development stages. These contours again illustrate that at $x = 170$ cm (i.e., in the linear stage) the disturbance energy is concentrated near the frequency $\omega_0 = 0.09$ corresponding to the most-unstable T-S wave, and that near $x = 250$ cm (in the subharmonic stage) an additional band of oscillations, with frequencies close to $\omega_0/2 = 0.045$, appears. These figures also show vertical profiles of different spectral components, which agree well with the results of linear stability theory at $x = 170$ cm, while by $x = 250$ cm they have become more complicated. However the data for $x = 270$ cm and 280 cm, relating to the strongly-nonlinear third stage, show rather energetic high-frequency components of u-fluctuations which are absent from Fig. 5.36. The reason for this discrepancy is apparently that Fig. 5.36 shows spectra of the ensemble-averaged velocity fields, and if the high-frequency oscillations have random phases they will be canceled by ensemble averaging. However, spectral contours in Fig. 5.37 were obtained from spectra computed for individual observations by subsequent ensemble averaging. It is clear that here contributions of oscillations with the same frequency but different phases to various individual spectra will be added to each other in the sum of individual spectra and will be represented by the ensemble-averaged contributions in the averaged spectra of Fig. 5.37. Therefore, the high-frequency velocity oscillations shown in Figs. 5.37c,d (but absent from Figs. 5.36c,d) must be real. They may be connected, e.g., with local velocity-profile inflections due to distortions of the local mean-velocity profiles by evolved disturbances; such local profile inflections were also observed by Breuer *et al.*

The appearance of high-frequency velocity fluctuations with random phases clearly means that the flow has acquired disorderly features typical of turbulence. Hence the observations summarized in Figs. 5.37a-d have a direct bearing on studies of the onset of randomness in boundary-layer flows. Note that Breuer *et al.* also consistently observed, during the late stages of instability development, the appearance of 'spike disturbances' of the same type as found by Klebanoff *et al.* (1962), and later by many others, in boundary layers excited by a vibrating ribbon. Therefore, the experiments of Breuer *et al.* proved very convincingly that spikes are a rather general phenomenon, unrelated to any special mechnism of disturbance generation. Moreover, since the authors repeated their observations many times, collecting an ensemble of observations under identical conditions, they were able to show that spikes are quite repeatable - they regularly appear at practically the same points and preserve the same main features in all repetitions. Hence the observations by Breuer *et al.* confirmed the earlier statement of Borodulin and Kachanov about the regular, non-random nature of spikes. On the other hand, LHSI-produced small-amplitude high-frequency oscillations have random phases and amplitudes, and thus these disturbances may generate the early flow randomness.

The final part of the paper of Breuer *et al.* is devoted entirely to discussion of the late-stage transformations of the wave packet studied. These transformations lead at first to the appearance of 'turbulent spots' (as noted above, the authors found that their formation begins near $x = 282.5$ cm; recall that it is connected also with the latest stages of spike development discussed at the end of Sec. 5.5) and then to the onset of the laminar-flow breakdown to a chaotic ('turbulent') state. [Quite another approach to the study of these transformations was sketched by Waleffe (1995) in the paper cited above; see also the recent survey by Bowles (2000).]. Additional information about the 'breakdown-stage' of wave-packet development is contained in Gaster's (1990) description of the results of his experiments; less detailed observations relating to this stage were described by Tso *et al.* (1990). However, this final stage of instability development is beyond the scope of the present chapter.

In the above discussion of the (temporal or spatial) development of localized disturbances in a laminar boundary layer it was usually assumed that the initial disturbance had a form close to that sketched in Fig. 3.2 of Chap. 3. This rather special assumption was accepted here, since it was widely used in numerical simulations of this development performed by various researchers. Therefore, even in the analysis of experiments where the initial disturbance was produced by some 'disturbance generator' and clearly did not coincide with that in Fig. 3.2, the data were often compared with numerical results for this special initial form of disturbance.

One of the purposes of the recent numerical-simulation work by Bech, Henningson and Henkes (1998) was just the verification of the influence of the initial form of a localized disturbance on its subsequent development. The authors also made an attempt to verify the accuracy of the approximate method of temporal numerical simulation of disturbance development, used in this and almost all previous simulation studies. Finally, apparently the main object pursued by the authors was the determination of the influence of non-zero pressure gradient dp/dx on the development of localized disturbances in a laminar boundary layer.

To determine the influence of the initial form of a disturbance, Bech *et al.* chose three different forms and solved the corresponding initial-value problems numerically for the full Navier-Stokes equations. All chosen forms of the initial velocity field $\mathbf{u}(\mathbf{x}) = \{u(\mathbf{x}),v(\mathbf{x}),w(\mathbf{x})\}$, where $\mathbf{x} = (x,y,z)$, corresponded to 'localized disturbances', with values of $\mathbf{u}(\mathbf{x})$ differing noticeably from zero only in a bounded region surrounding the coordinate origin. Moreover, all these disturbance forms could be represented in terms of a scalar streamfunction $\psi(x,y,z)$ which for the three cases considered had the forms:

$$\psi = Axyz^3\exp(-[x^2+y^2+z^2]), \tag{5.18}$$

$$\psi = Axz^3\exp(-[x^2+y^2+z^2]), \tag{5.19}$$

and

$$\psi = 0.5Ar^2z^3\exp(-[r^2+z^2]), \quad r^2 = x^2+y^2. \tag{5.20}$$

Here the coordinates x, y, z are assumed to be non-dimensionalized by some length scales l_x, l_y and l_z (defined in the paper of Bech *et al.* separately for three models), A is a disturbance amplitude, small in comparison with the free-stream velocity U_0, and the velocity fields $\{u, v, w\}$ in the three cases are expressed in terms of the function ψ by the following three different equations:

$$\{u, v, w\} = \{0, -\partial\psi/\partial z, \partial\psi/\partial y\}, \tag{5.18a}$$

$$\{u, v, w\} = \{\partial\psi/\partial z, 0, -\partial\psi/\partial x\}, \tag{5.19a}$$

and

$$\{u, v, w\} = \{-(\partial\psi/\partial z)xr^{-2}, -(\partial\psi/\partial z)yr^{-2}, (\partial\psi/\partial r)r^{-1}\}. \tag{5.20a}$$

The simulations were carried out with Re* = 950 at $x = 0$, and all the lengths, velocities and times relating to process of disturbance development were made dimensionless by the length δ^* at $x = 0$ and the free-stream velocity U_0.

The first model of Eqs. (5.18) and (5.18a) just corresponds to the form sketched in Fig. 3.2 of Chap. 3; the equations given here for the initial velocity field agree exactly with those used by Henningson *et al.* (1993), and are almost identical to those used by Breuer and Haritonidis (1990) and Breuer and Landahl (1990). (Recall the remark in Chap. 3, p. 28, that in this model disturbance the initial streamwise velocity $u(\mathbf{x})$ is everywhere equal to zero, but it undergoes rapid transient growth and soon becomes greater than the other two velocity components.) A schematic form of the initial vertical-velocity contours for this model is shown in Fig. 5.38a. In the second model (5.19), (5.19a) the initial spanwise velocity $v(\mathbf{x})$ is equal to zero; this model describes a wave packet where the energy is mainly concentrated in plane 2D waves. The initial velocity contours for this model are shown in Fig. 5.38b. The third model (5.20), (5.20a) has already been used in numerical simulations by Henningson *et al.* (1993); in the case of this model it is assumed that $l_x = l_y$ and hence here the initial disturbance is axisymmetric with respect to the vertical z-axis (see Fig. 5.38c).

The numerical simulations of the disturbance development in a boundary layer presented in the main part of the paper by Bech *et al.* were temporal ones, i.e., they were based on the assumption that the flow is plane-parallel and the spatial Fourier-components of disturbance velocities evolve in time. (As is now usual, the parallel-flow assumption was supplemented by the assumption that the boundary-layer thickness δ is not constant but grows with time; cf. footnote [4] on page 64.) To verify the accuracy of the somewhat simplified temporal approach, one of the temporal simulations was repeated, using a spatial approximation which assumes that the flow is steady but may be nonparallel, and that disturbances are time-periodic and spatially evolving. The latter approximation is evidently more accurate than the temporal one but it is also more complicated and more expensive in computer time. Comparison of the results of the two simulations revealed some small inaccuracies of the temporal-simulation results, but also showed that these inaccuracies appear only at rather late stages of wave-packet development, while the overwhelming majority of predictions of the temporal simulations agreed quite satisfactorily, qualitatively and quantitatively, with those of the approximate spatial simulation. Thus, it was concluded that the results of temporal simulations were sufficiently reliable to be investigated in detail.

At the beginning of the paper of Bech *et al.* some simpler small-amplitude results were considered. Here the authors analyzed numerical solutions of the linear initial-value problem (corresponding to linearized N-S equations) for three chosen forms of the initial velocity field, where the streamfunction amplitude A was chosen to make the maximum $|w_0|$ of the initial vertical velocity equal to $10^{-5}U_0$. (Solutions were computed for two values of the pressure gradient, but for now only the case of a Blasius boundary layer, with $dp/dx = 0$ will be discussed.) 'Linear' (given by the linear stability theory) temporal growths of the disturbance energy and of the maximal values of streamwise and vertical velocities were computed for various values of the time t. It was found that results for the three initial conditions described differ considerably from each other, as must be the case since both the partition of the developing disturbance into convective and dispersive components, and also the TS-wave composition of the dispersive component, were different in the three cases. Then the flow patterns arising from the three chosen initial conditions were reconstructed, for the latter stages of the 'linear development', from simulation results. Figs. 5.39a-f show velocity contours of u and w for the dimensionless time $t = 300$. Figs. 5.39a and 5.39c show that in cases 1 and 3 [corresponding to initial streamfunctions (5.18) and (5.20)] the elongated streaky structures, composed of alternating streaks of low and high streamwise velocities, emerged in the flow before $t = 300$. It seems evident that these streaky structures were produced here by the transiently-growing part of the disturbance, subjected to Landahl's (1980) mechanism of streamwise elongation. This mechanism affects only the velocity u; therefore contours of vertical velocity w in Figs. 5.39d-f, which again are quite similar to each other in cases 1 and 3 but have somewhat different forms in case 2, represent typical wave-packet structures corresponding to the dispersive part of the developing disturbance. Recall that in case 2 [initial streamfunction (5.19)] the initial disturbance had vanishing spanwise velocity and contained no $(0,k_2)$-modes, so that the generation of spanwise inhomogeneity played an important part in the formation of streaky structures. For this reason the initial disturbance (5.19), (5.19a) produced no streaky structures by $t = 300$ and Fig. 5.39b is quite different from Figs. 5.39a and 5.39c.

Bech *et al.* also analyzed the appearance of weakly-nonlinear effects on disturbance development. They first of all supplemented the computations with initial amplitude A corresponding to the condition $|w_0|/U_o = 10^{-5}$ with computations for larger values of A corresponding to $|w_0|/U_o = 5\times10^{-5}$ and 10^{-4}. Then the authors studied the expansions of their solutions in powers of the amplitude A, and extracted from these expansions the linear terms (describing the results of the linear stability theory) and the weakly-nonlinear quadratic (proportional to A^2) and cubic (proportional to A^3) terms. This procedure allowed them to isolate contributions of some nonlinear interactions to the developed disturbances; in particular, the component corresponding to wave vector $(0,2k_{2,0})$ (and marking the beginning of the energy transport to higher spanwise wavenumbers) was detected in the quadratic part of the disturbance, accompanied by the most energetic T-S wave of the linear theory, with wave vector $\mathbf{k} = (k_{1,0},k_{2,0})$.

To study strongly-nonlinear effects on wave-packet propagation, the authors further extended the range of values of the initial disturbance amplitude, and in addition to the above-mentioned values they carried out numerical simulations for cases where $|w_0|/U_o = 10^{-3}$, 5×10^{-3}, and 10^{-2}. A preliminary study of numerical-simulation results for disturbances with $|w_0|/U_o = 5\times10^{-3}$ showed that in the case of the third model [Eqs. (5.20), (5.20a)] strongly nonlinear effects develop more slowly than in the cases of the other two models of the initial disturbance. Therefore, it was found that for complete analysis of the nonlinear development of the disturbance (5.20), (5.20a), the range of investigated values of t should be considerably extended. For this reason the authors decided to study strongly nonlinear effects only for the first and second models of the initial disturbance.

For these two models, the authors were able to cover, in their numerical simulations, all the stages of nonlinear development of a localized disturbance in the Blasius boundary layer found in the experimental and numerical-simulation studies of earlier authors. In particular, the subharmonic disturbance growth produced by secondary subharmonic instability of primary waves was detected in data relating to case 2, with the initial conditions (5.19), (5.19a), and appeared here much earlier than in the experiments of Cohen *et al.* (1991) with a considerably smaller initial disturbance amplitude. Also in case 2, when the subharmonic growth of oblique waves began, the generation of the streaky structures, absent from Fig. 5.39b, also began and took practically the same form as in the case of the other two initial conditions and in the experiments by Cohen *et al.* (1991) and Breuer *et al.* (1997). This means, in particular, that exactly as in the experiments, nonlinear effects led to cascade transfer of energy up the spanwise-wavenumber axis. The numerical simulations of Bech *et al.* also show that the streaky structures sometimes reach breakdown only with a very high amplitude of disturbance velocity - e.g., in the case of the initial conditions (5.18), (5.18a) with $|w_0| = 5\text{x}10^{-3}$ the streaks continue to exist at an amplitude of streamwise-velocity oscillations close to 30% of U_0. This demonstration of the high value of velocity amplitude needed for breakdown of streaky structures agrees, in particular, with results by Reddy *et al.* (1998) relating to streaks in a plane-channel flow. Nevertheless, breakdown of the streaky structures, and emergence of the disorderly high-frequency fluctuations accompanied by rapid increase of the disturbance kinetic energy and of the maximal values of velocity fluctuations[11], were also detected in the numerical simulations for both the high-amplitude initial conditions, if the value of $|w_0|$ (and hence also of A) was large enough. Bech *et al.*'s large-amplitude simulations of disturbance development also revealed many other details of streaky-structure breakdown, transition to the unordered flow regime, and the accompanying flow phenomena. However, again these results are outside the scope of this chapter.

As noted above, a considerable part of the paper by Bech *et al.* is devoted to the study of the development of localized disturbances in boundary layer with an adverse pressure gradient $dp/dx > 0$ decelerating the fluid motion. Boundary layers with non-zero pressure gradients are met very often in practical applications of fluid mechanics, and have therefore attracted much attention by investigators. Therefore, it is only natural that the nonlinear instability of pressure-gradient boundary layers is considered in a great number of publications; the papers and dissertations by Bertolotti (1985), Herbert and Bertolotti (1985), Wubben *et al.* (1990), Goldstein and Lee (1992), Kloker (1993), Zel'man and Maslennikova (1993a), Kosorygin (1994), Kloker and Fasel (1995), van Hest (1996), van Hest *et al* (1996), Corke and Gruber (1996), Liu (1997), Liu and Maslowe (1999), and Borodulin *et al.* (2000) represent only a small part of this work. Bech *et al.* were interested in

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[11] Note that the growth of the disturbance kinetic energy does not necessary imply the growth of disturbance velocities. For example, in the case of transient growth of localized disturbances in plane shear flows studied by Landahl (1980), the growth of disturbance energy due to the "lift-up effect" described by him is produced by the disturbance elongation increasing its volume, and not by the growth of velocities of individual fluid particles.

boundary layers with adverse pressure gradient since here, at a not-too-small absolute value of the Falkner-Skan parameter β (see Chap. 2, p. 119) the profile $U(z)$ has a pronounced inflection point where $U''(z) = 0$, and is inviscidly unstable with respect to small-amplitude disturbances according to the classical results of Rayleigh (see Chap. 2. Sec. 2.82). This increased linear instability of a laminar boundary layer in adverse pressure gradient (in comparison to the case of a boundary-layer with zero pressure gradient) must also strongly influence the nonlinear instability effects and produce some additional phenomena worth special study. Bech *et al.*, who performed simulations for $\beta = 0$ (i.e. $dp/dx = 0$) and $\beta = -0.155$, in fact detected a number of interesting differences between disturbance developments in these two flows. However, volume limitations give no possibility for discussion in this book of results for boundary layers with non-zero pressure gradients.

Figures for Chapter 5

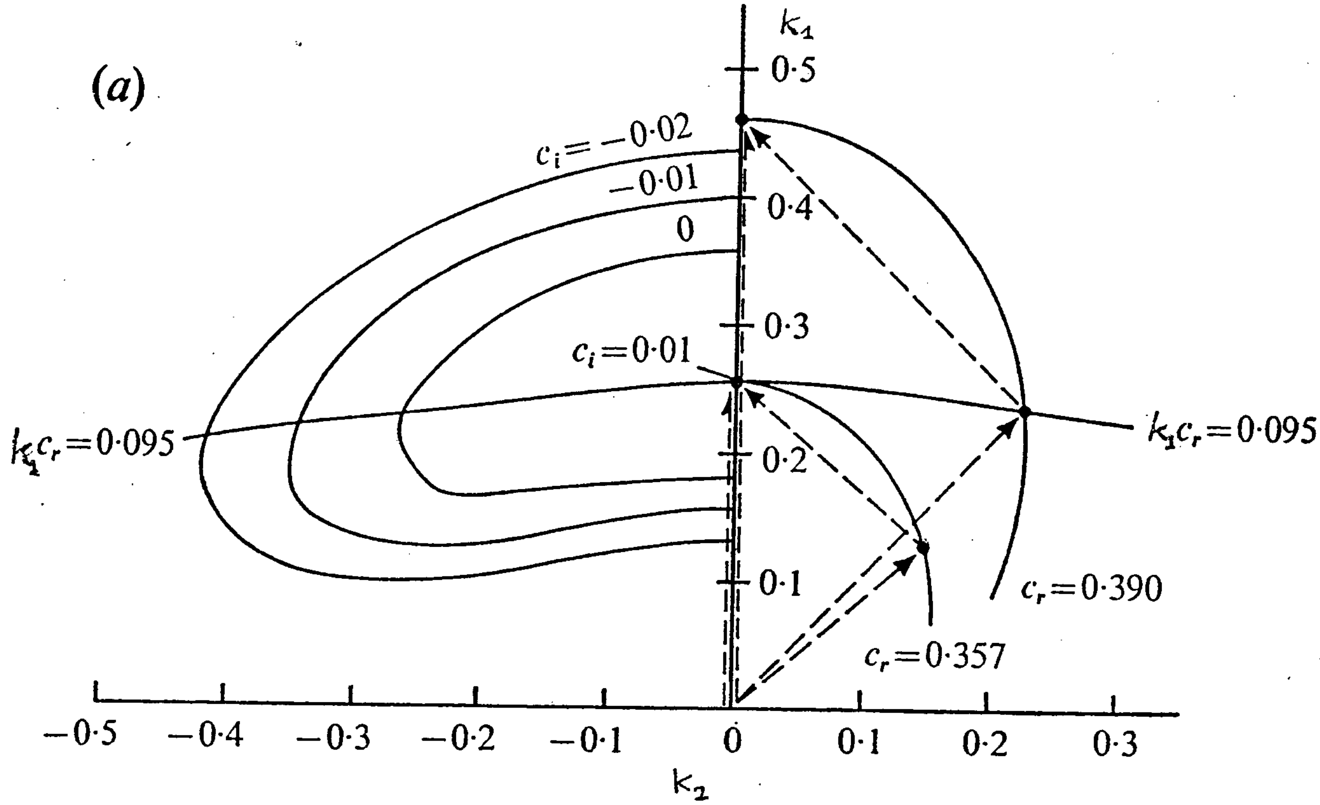


Figure 5.1, (a) and (b). (For caption see the next page.)

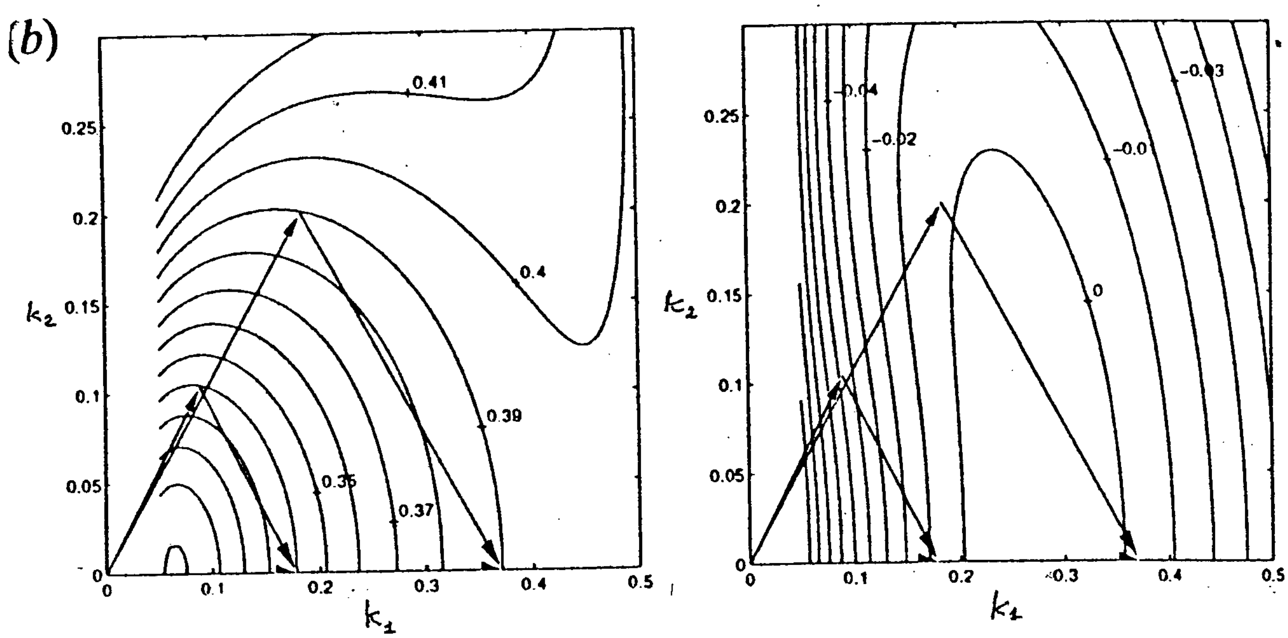


Figure 5.1. Examples of calculated resonant wave triads in the Blasius boundary layers with Re* = 882 and Re* = 750.

(a) Contours in the (k_1,k_2)-plane of phase-velocity values $c_r = \Re e c = \Re e(\omega/k_1)$ and values of $c_i = \Im m c = \Im m(\omega/k_1)$ (determining the growth, or dacay, rate $k_1 c_i$) for temporally evolving T-S waves in a boundary layer with Re* = 882. Since $c_r(k_1,k_2) = c_r(k_1,-k_2)$ and $c_i(k_1,k_2) = c_i(k_1,-k_2)$, contours for c_i are shown only for $k_2 \leq 0$, and those for c_r only for $k_2 \geq 0$. Two resonant triads with wave vectors $\mathbf{k}_1 = (k, 0)$, $\mathbf{k}_{2,3} = (k/2, \pm k_2)$ satisfying the condition $c_r(k, 0) = c_r(k/2, \pm k_2)$ are shown by arrows [after Craik (1971)].

(b) Contours in the (k_1,k_2)-plane of phase velocity $c_r = \Re e c = \Re e(\omega/k_1)$ (the left diagram) and of $c_i = \Im m c = \Im m(\omega/k_1)$ (the right one) for temporally evolving T-S waves in a boundary layer with Re* = 750. Two examples of resonant triads are shown by arrows [after Schmid and Henningson (2000)].

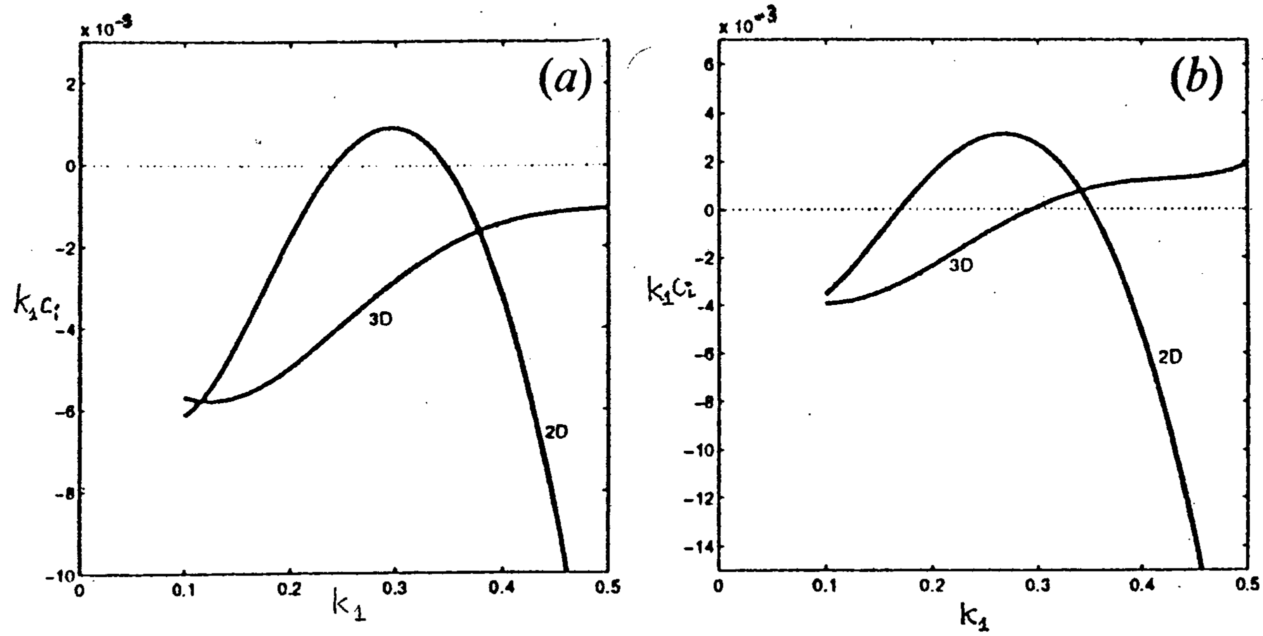


Figure 5.2. Temporal growth (or decay) rates $k_1 c_i = \Im m\,\omega$ for 2D and 3D components of resonant triads of Craik's type with various values of streamwise wavenumber k_1 of 2D wave in Blasius boundary layers with (a) Re* = 000, and (b) Re* = 000 [after Schmid and Henningson (2000)].

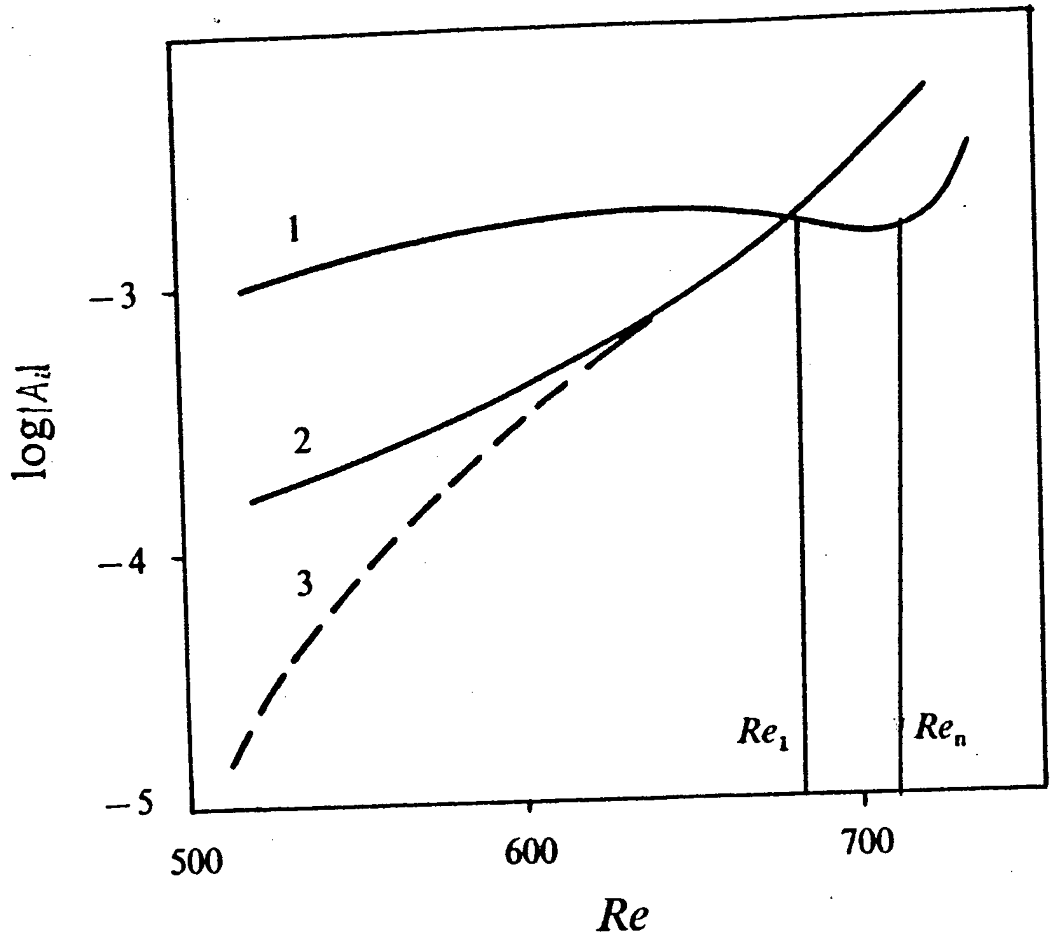


Figure 5.3. Calculated dependence of the wave amplitudes $|A_i(x)|$, i = 1, 2, 3, on Re = $(U_0x/\nu)^{1/2} \propto x^{1/2}$ for the wave triad consisting of a plane wave 1 of frequency ω and wave vector $\mathbf{k}_1 = (k, 0)$ and oblique waves 2 and 3 of frequency $\omega/2$ and wave vectors $\mathbf{k}_{2,3} = (k_1, \pm k_2)$ in the case when initial (at Re = 525) amplitudes $|A_{i,0}|$ and phases $\phi_{i,0}$ satisfy the conditions: $|A_{1,0}| \gg |A_{2,0}| \gg |A_{3,0}|$, $\phi_{1,0} = \phi_{2,0} + \phi_{3,0}$. It was assumed here that $F = \omega\nu/U_0^2 = 115\times10^{-6}$ and $K_2 = \nu k_2/U_0 = 0.18\times10^{-3}$; the values of k and k_1 were then determined from the Orr-Sommerfeld equations (2.44) and (2.41) which showed that $k_1 \approx k/2$ [after Zel'man and Maslennikova (1993a)].

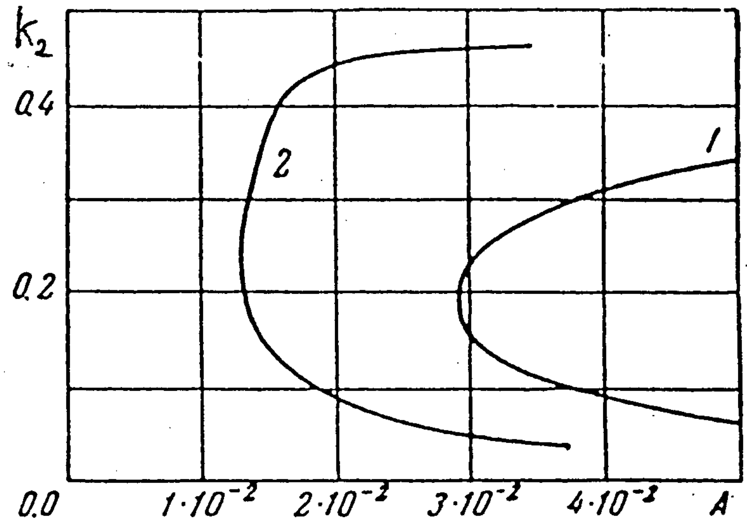


Figure 5.4. Threshold amplitude A of the plane T-S wave with streamwise wavenumber k_1 in a Blasius boundary layer for the onset of three-dimensionality with spanwise wavenumber k_2 [after Massev (1968b)]. Curve 1: Re* = 1203, k_1 = 0.43; curve 2: Re* = 519, k_1 = 0.27. All the quantities are non-dimensionalized by scales δ^* and U_0.

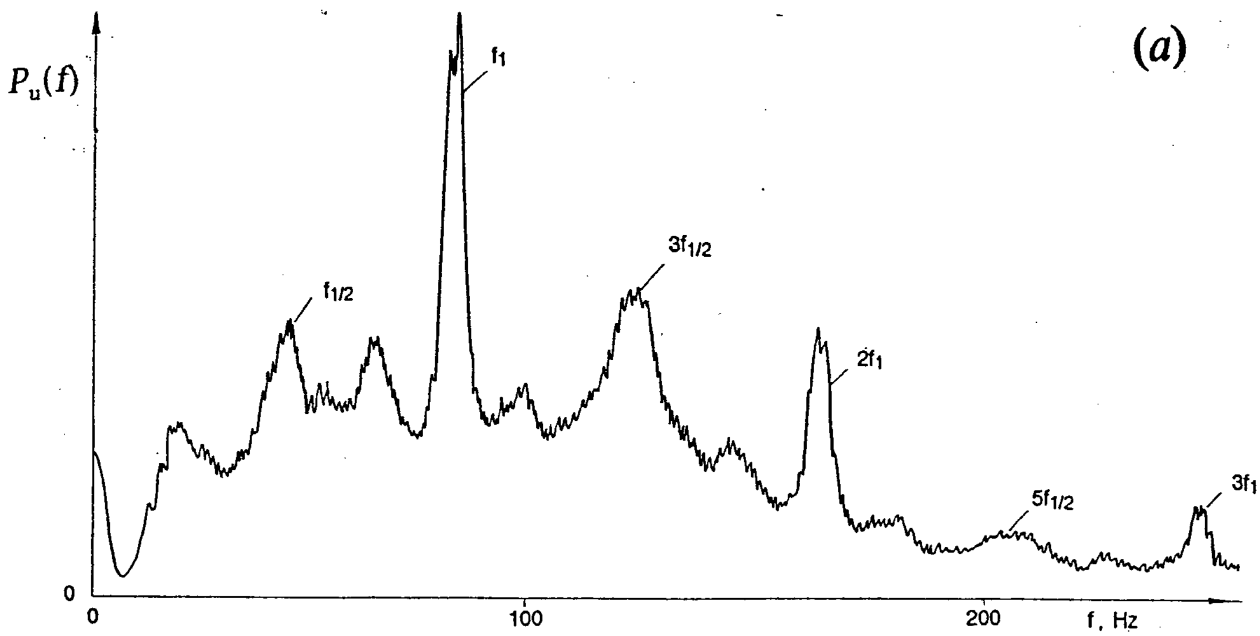


Figure 5.5, (a) and (b). (For caption see the next page.)

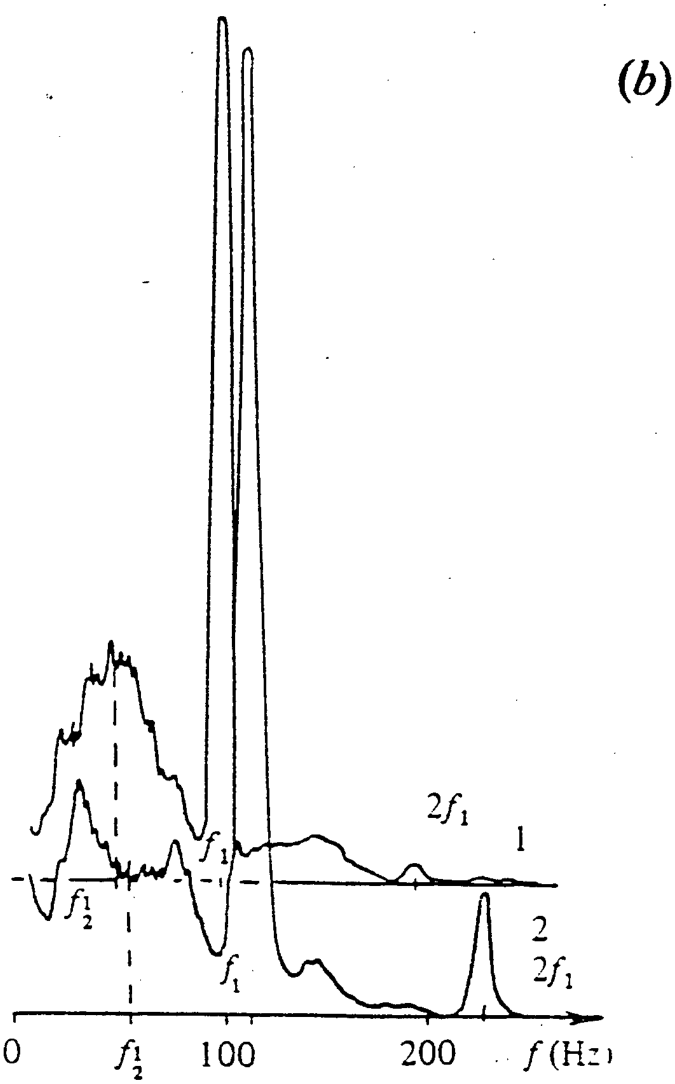


Figure 5.5. Examples of the amplitude spectra $P_u(f)$ of streamwise-velocity fluctuations $u(t)$ in a laboratory flat-plate boundary layer disturbed by a ribbon vibrating with frequency f_0.

(a) Typical form of spectrum $P_u(f)$ measured by Kachanov, Kozlov and Levchenko (1977) [after Kachanov (1994a)]. Peaks denoted as $f_1, f_{1/2}, 3f_{1/2}, 2f_1, 5f_{1/2}$ and $3f_1$ correspond to frequencies f_0, $f_0/2$, $3f_0/2$, $2f_0$, $5f_0/2$ and $3f_0$.

(b) Spectra $P_u(f)$ measured inside a boundary layer at two values of frequency f_0 and coordinate x (measured from plate leading edge) but fixed values of y and z : 1. $f_0 = 96.4$ Hz ($F_0 = 2\pi f_0 \nu/U_0^2 = 109\times10^{-6}$), $x = 600$ mm (Re $= (U_0 x/\nu)^{1/2} = 608$); 2. $f_0 = 111.4$ Hz ($F_0 = 124\times10^{-6}$), $x = 640$ mm (Re $= 633$) [after Kachanov and Levchenko (1984)].

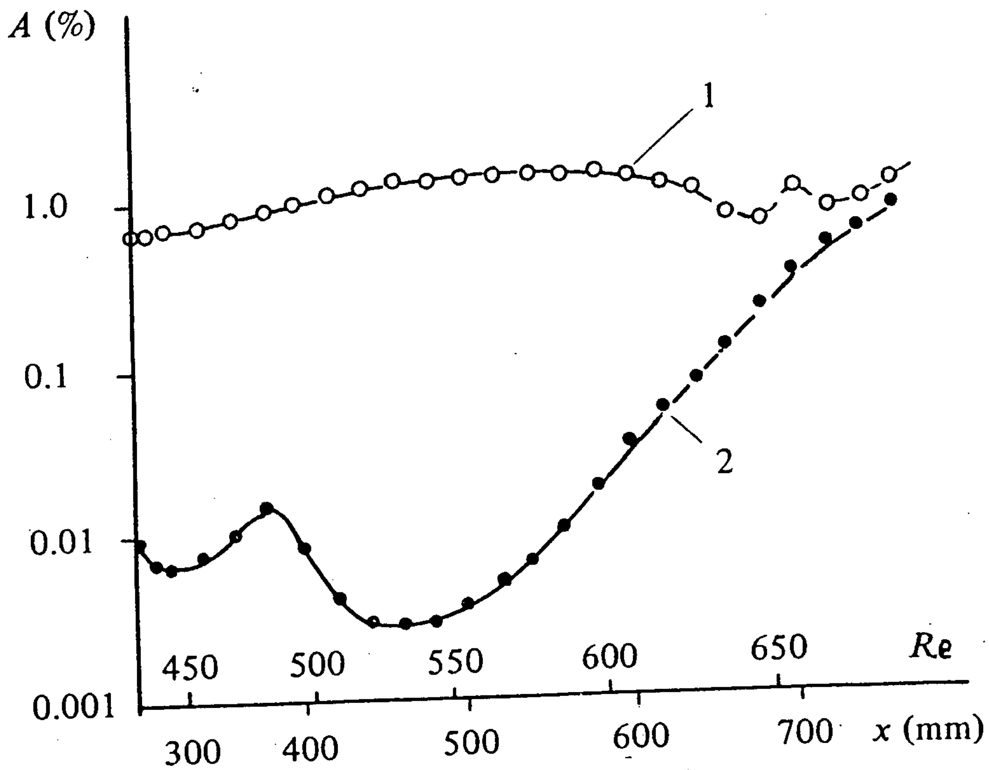


Figure 5.6. Dependence of the dimensionless amplitudes $A = u'/U_0$ of the primary plane wave (1) and subharmonic oblique waves (2) on the coordinate x [and Re = $(U_0x/\nu)^{1/2}$] at y = -2.5 mm, z/δ = 0.26 (δ is the boundary-layer thickness), according to measurements by Kachanov and Levchenko (1984).

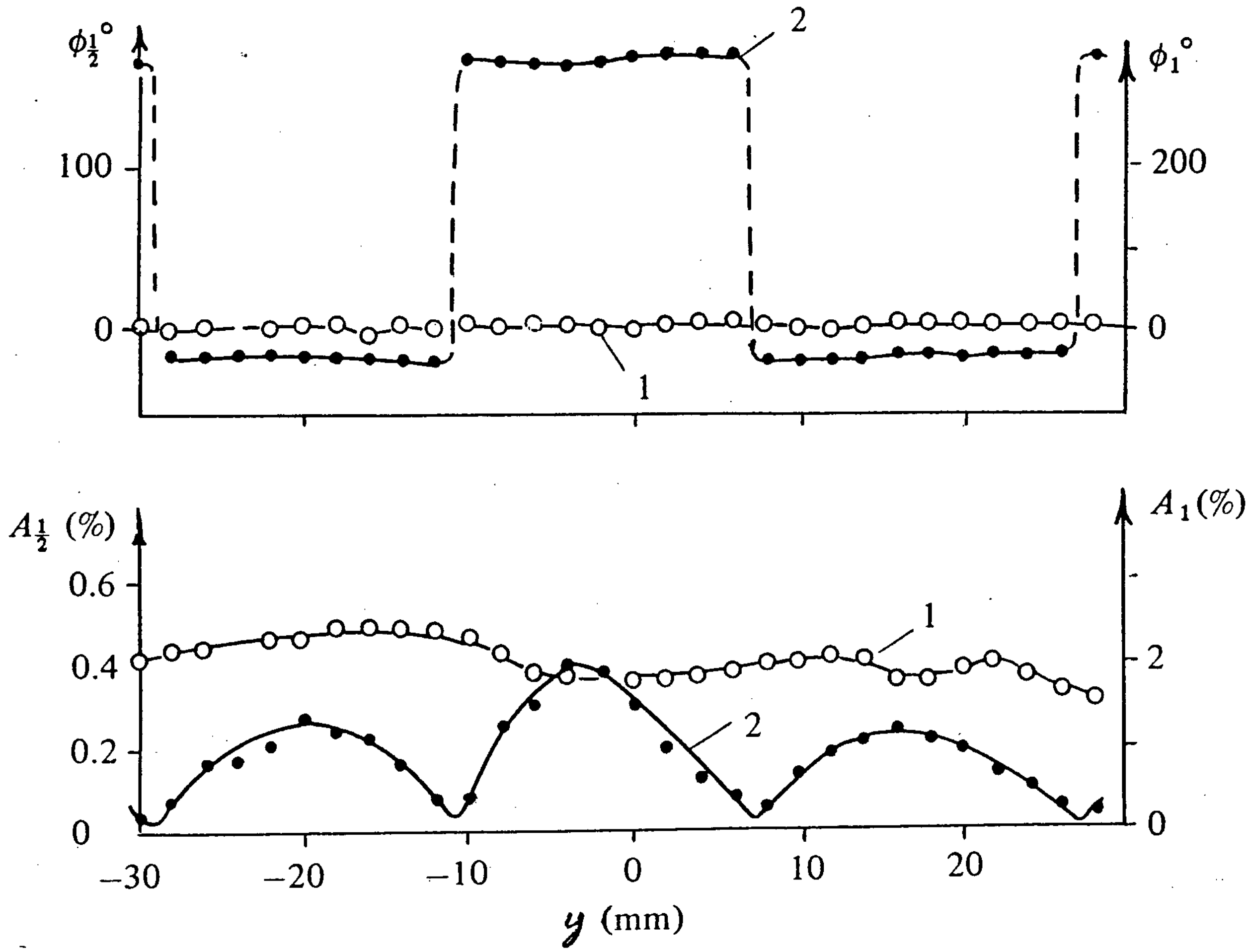
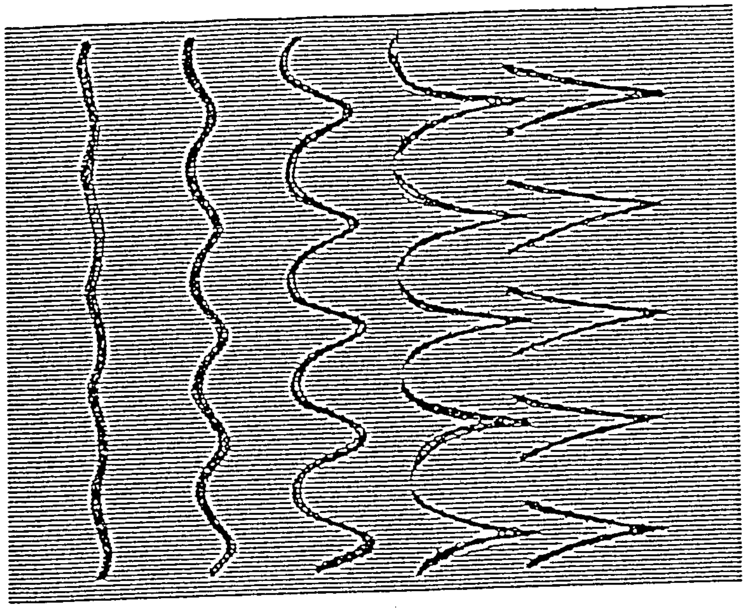
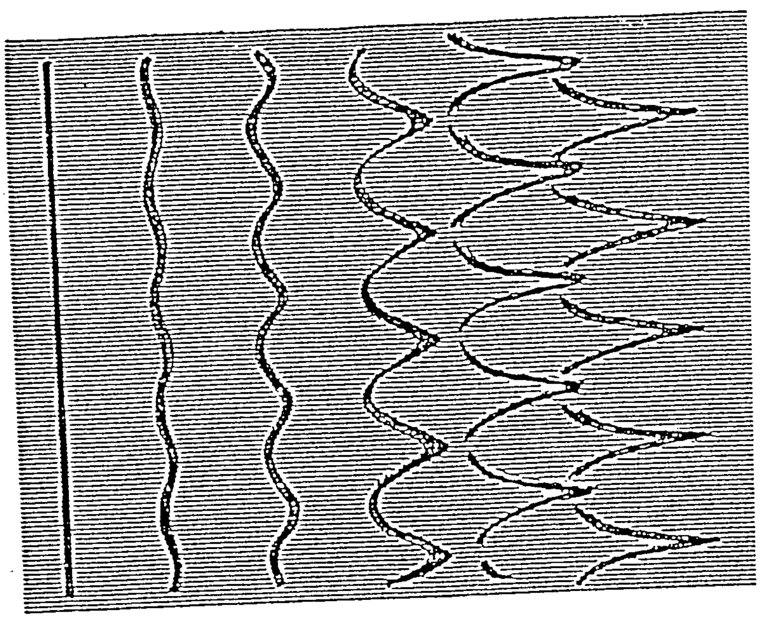


Figure 5.7. Measured dependence of phases ϕ_1 and $\phi_{1/2}$, and amplitudes A_1 and $A_{1/2}$ of the primary plane wave (1) and subharmonic waves (2) on the spanwise coordinate y [after Kachanov and Levchenko (1984)].



(a)



(b)

Figure 5.8. (a) Regular system of Λ-vortices typical for the K-regime of disturbance development in a boundary layer.
(b) Staggered system of Λ-vortices typical for the N-regime of disturbance development. The figures show flow streaklines appearing when the disturbed flow is visualized by smoke [after Herbert *et al.* (1987)].

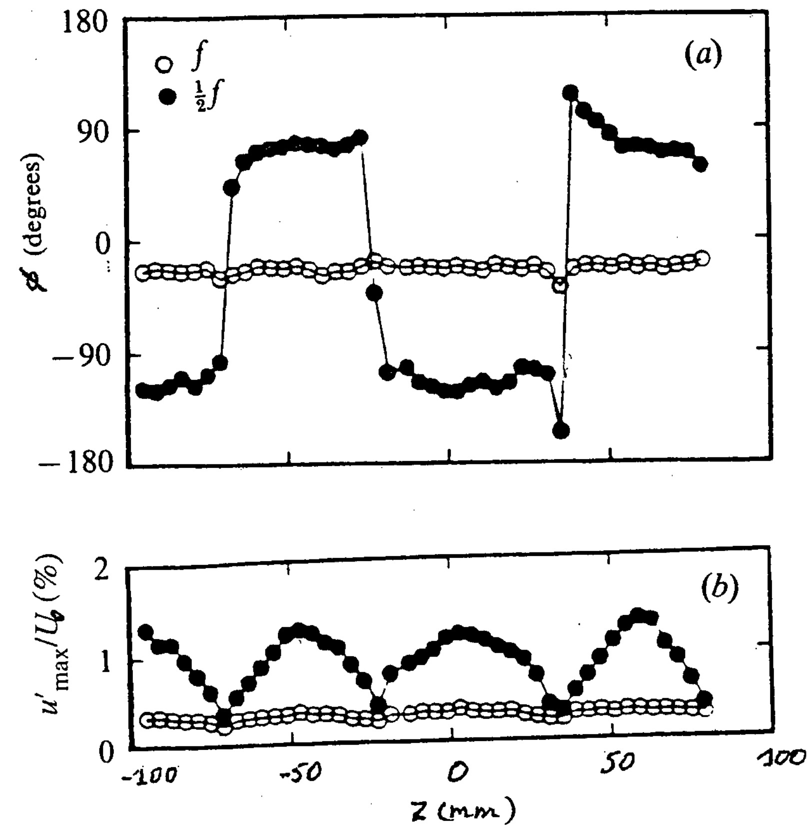


Figure 5.9. Spanwise distributions of phases ϕ (a) and maximum (with respect to z) amplitudes A (b) for the primary plane wave of frequency f (o) and subharmonic oblique waves of frequency $f/2$ (•) for the case 3 of Corke and Mangano's measurements [after Corke and Mangano (1989)].

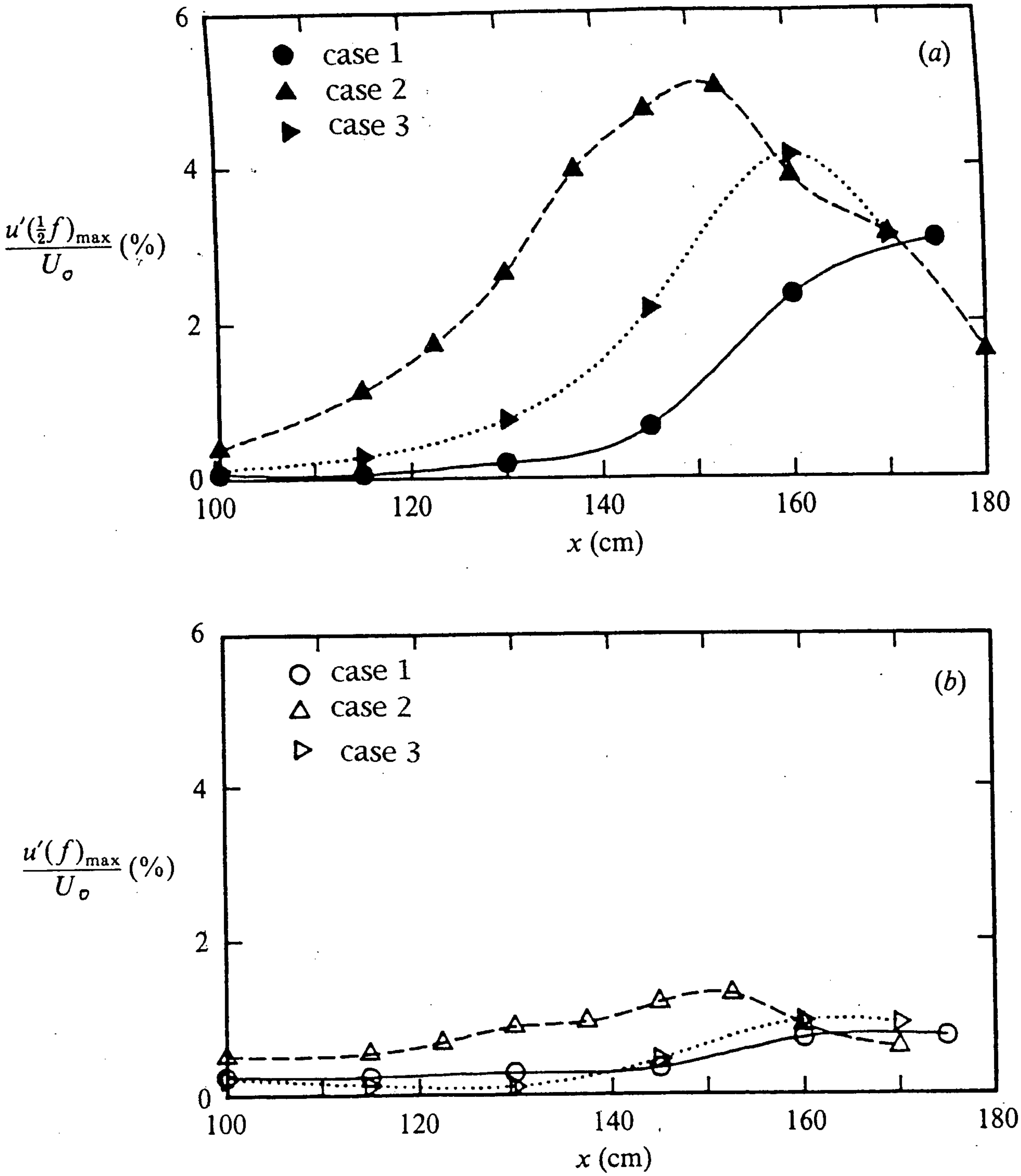


Figure 5.10. Streamwise developments of maximum amplitudes of streamwise velocity fluctuations for (a) subharmonic waves of frequency $f/2$, and (b) primary waves of frequency f in the cases 1, 2, and 3 [after Corke and Mangano (1989)].

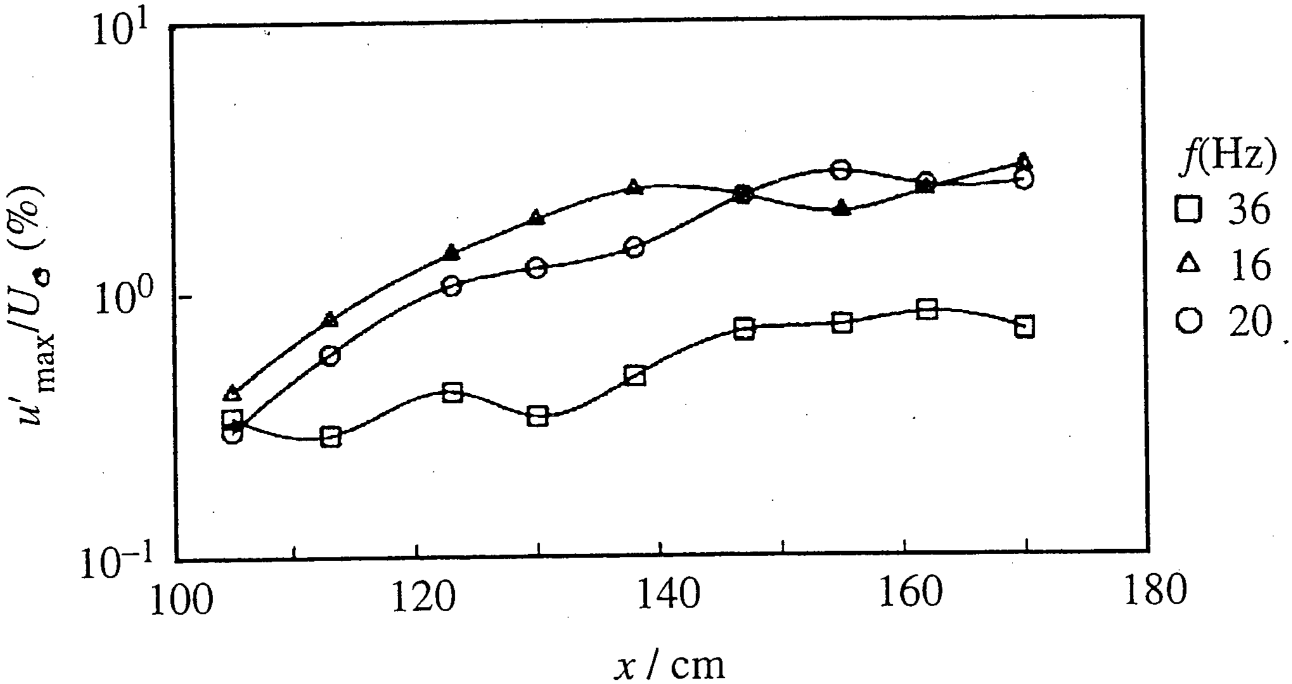


Figure 5.11. Streamwise development of maximum amplitudes of artificially exited plane wave of frequency 36 Hz and oblique wave of frequency 16 Hz, together with development of the produced by their nonlinear interaction 3D wave of frequency 20 Hz [after Corke (1995)].

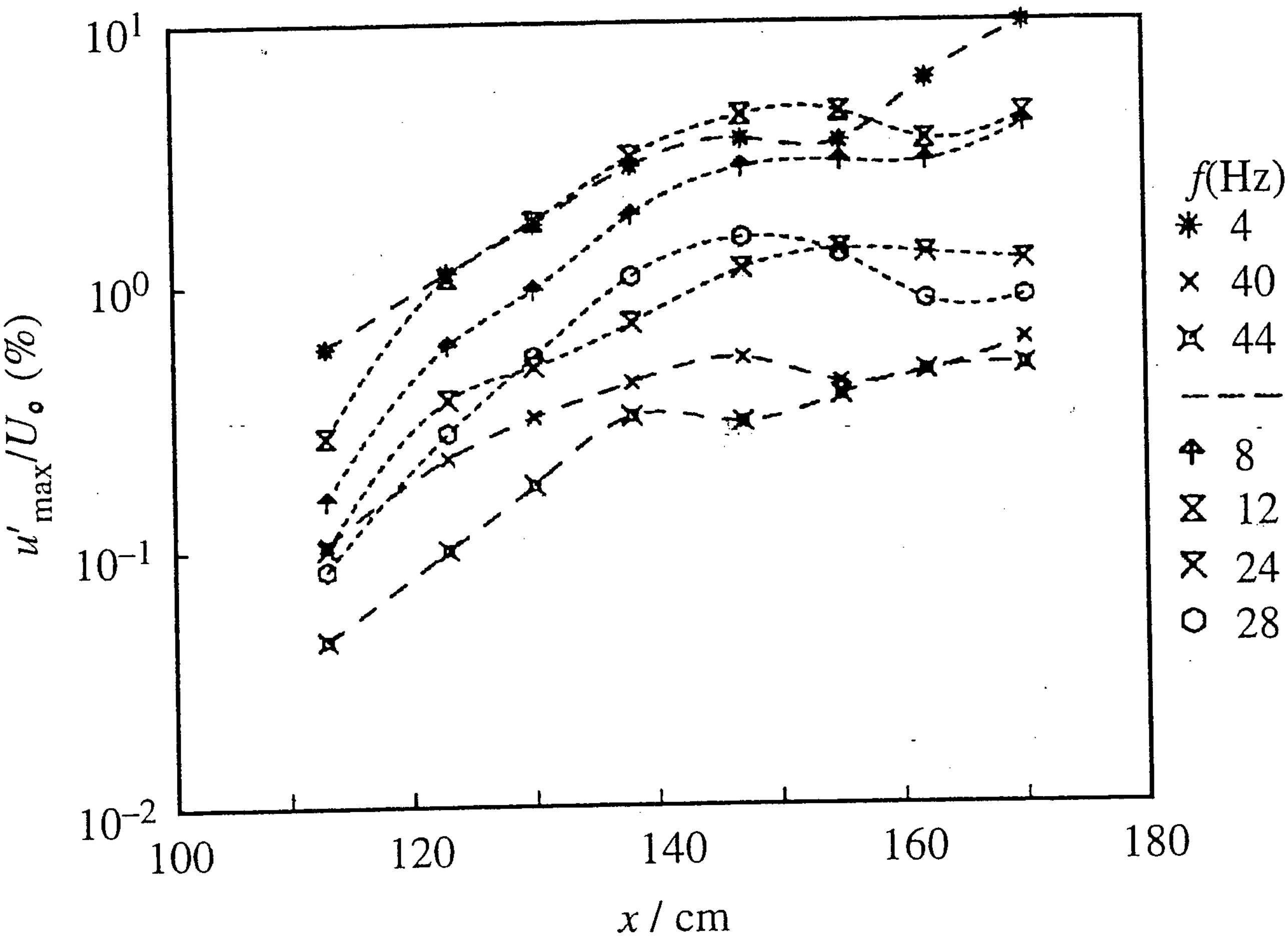


Figure 5.12. Streamwise development of a number of waves produced by nonlinear interactions of waves from 'detuned resonance triad' artificially excited by Corke [after Corke (1995)].

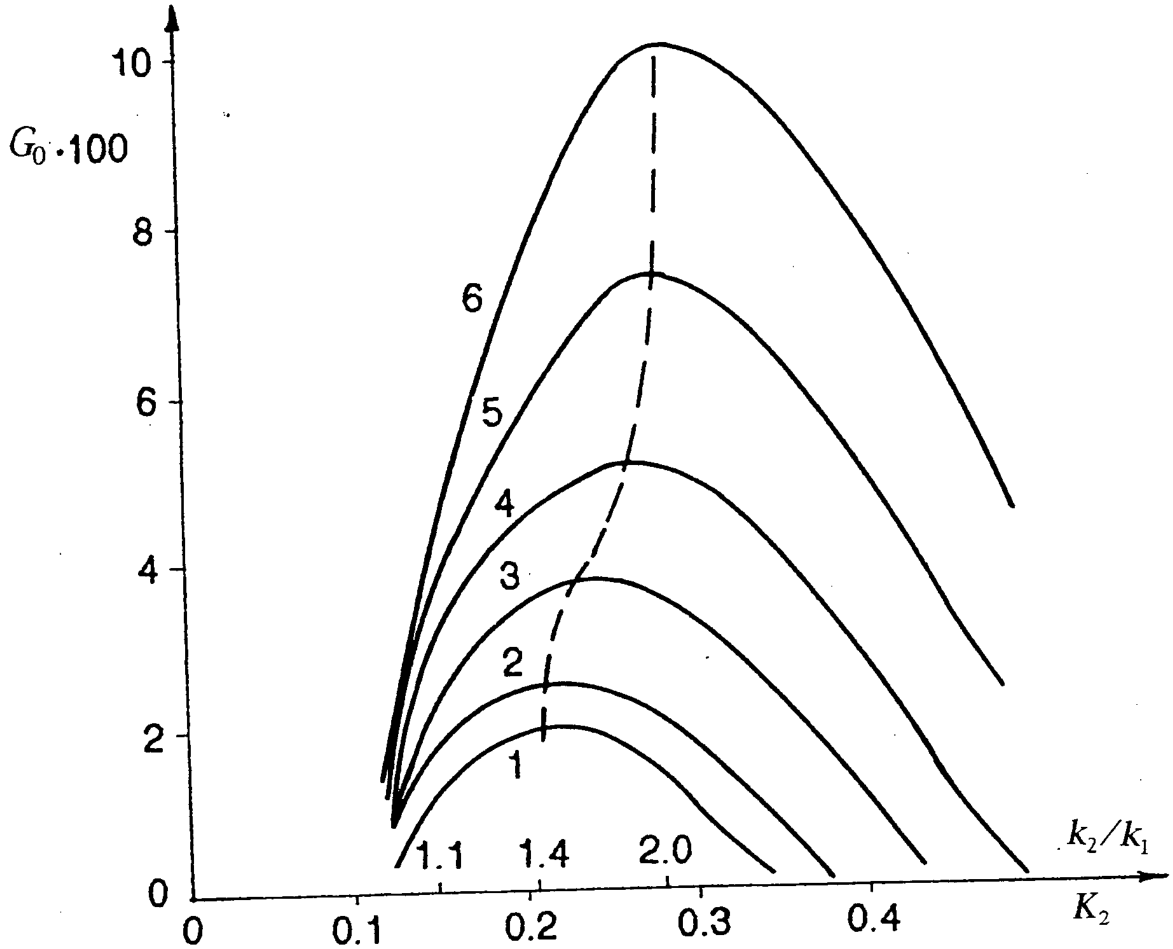


Figure 5.13. Dependence of the amplification rate $G_0 = dA(x)/Adx$ of the oblique-wave amplitude $A \equiv A_2 = A_3$ of a resonant wave triad on $K_2 = k_2\nu/U_0$ (and k_2/k_1) for different values of the plane wave amplitude A_1 [and $F_1 \equiv \omega_1\nu/U_0^2 = 115\times10^{-6}$, $\mathrm{Re}^+ = (U_0x/\nu)^{1/2} = 640$]. All dimensional quantities are non-dimensionalized by scales $\delta^+ = (\nu x/U_0)^{1/2}$ and U_0. Curves 1, 2, ..., 6 correspond to A_1 = 0.14, 0.21, 0.28, 0.40, 0.53, 0.72%, dotted line shows the dependence of optimal values $(k_2/k_1)_{pr}$ and $(K_2)_{pr}$ on A_1 [after Zel'man and Maslennikova (1953a) and Kachanov (1994a)].

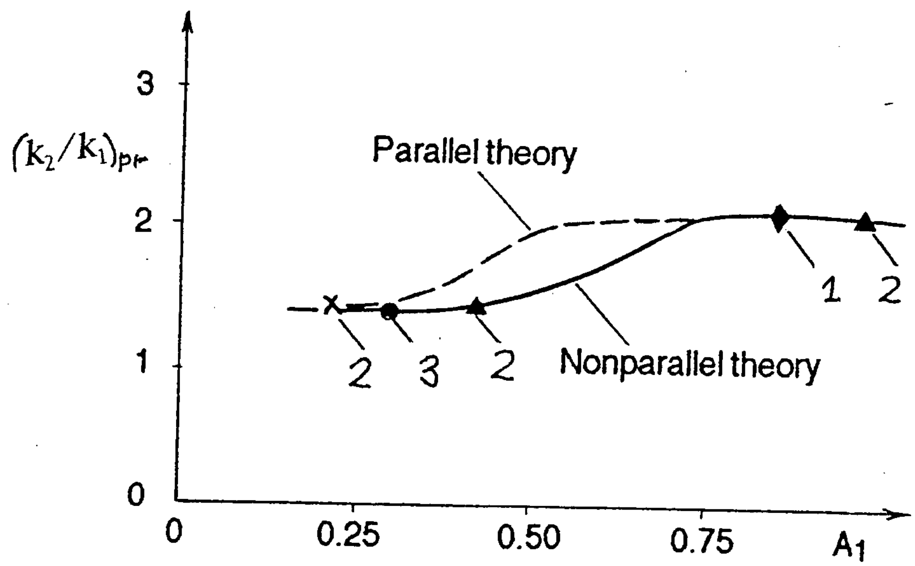
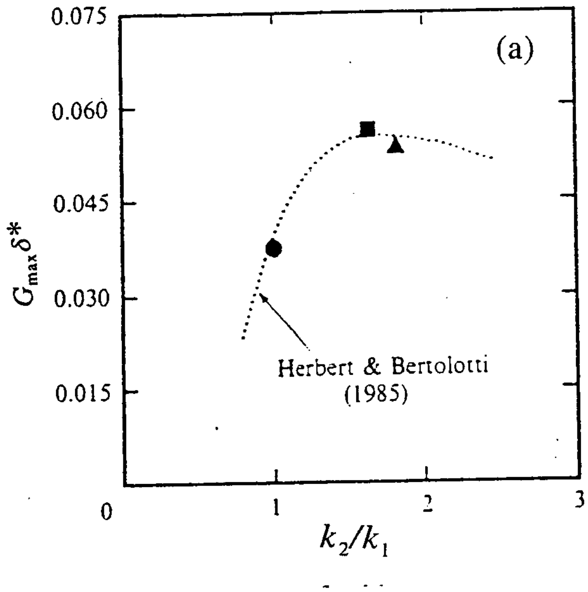


Figure 5.14. Dependence of the the value of $(k_2/k_1)_{pr}$ corresponding to the most amplified oblique subharmonics of the plane wave on the plane-wave amplitude A_1. Experimental points correspond to laboratory observations: 1 - of Kachanov and Levchenko (1982,1984); 2, 3 - of Saric *et al.* (1984); 4 - of Saric and Thomas (1984) [after Zel'man and Maslennikova (1993a) and Kachanov (1994a)].



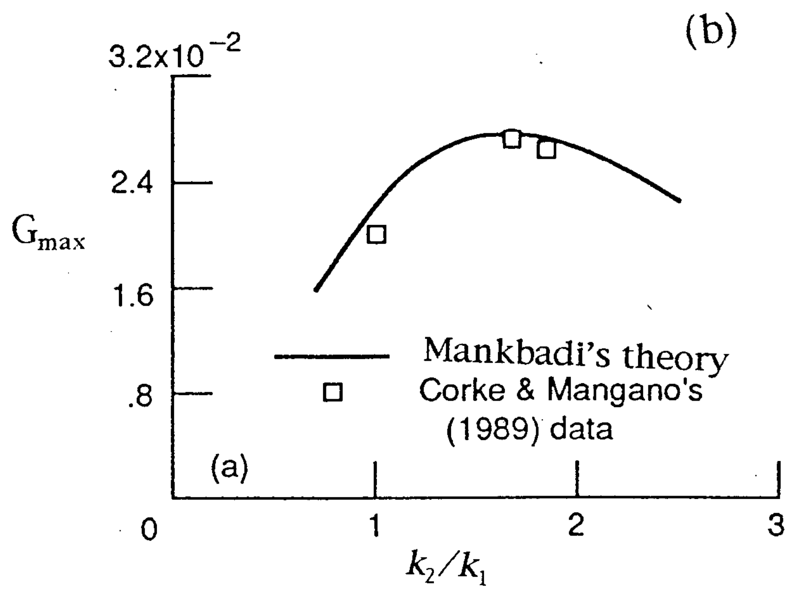


Figure 5.15, (a), (b) and (c). (For caption see the next page.)

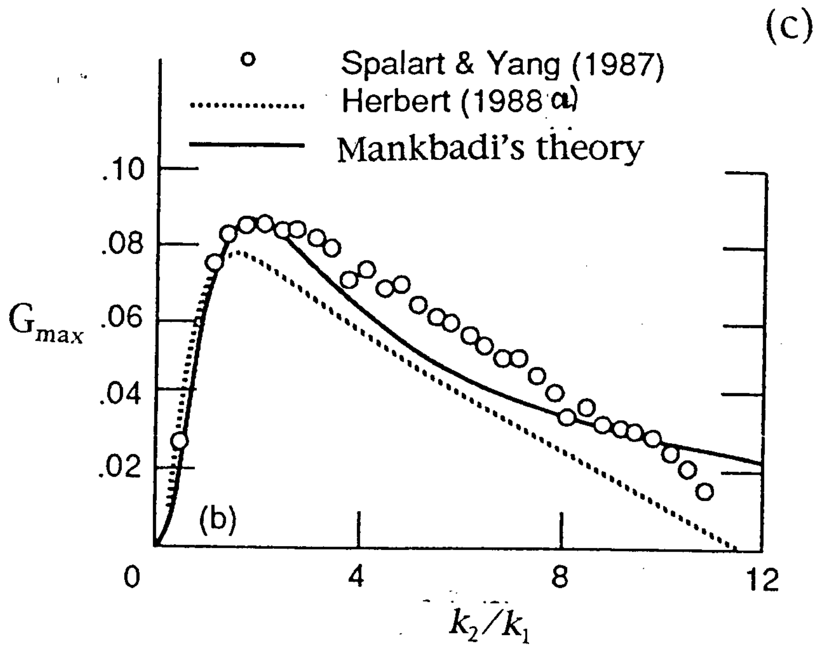


Figure 5.15. (a) Comparison of the measured by Corke and Mangano (1989) maximal values A_{max} of the oblique-wave amplitudes $A_2 = A_3$ in three studied cases with the theoretical dependence of A_{max} on k_2/k_1 which follows from the application to the their experiments results of the secondary-instability theory developed by Herbert (1983b,1988a) and Herbert and Bertolotti (1985) [after Corke and Mangano (1989)].

(b) Comparison of the maximal oblique-wave growth rate G_{max} observed by Corke and Mangano in three cases studied in their experiments with theoretical estimate of the dependence of G_{max} on k_2/k_1 following from Mankbadi's theory of critical-layer nonlinearity [after Mankbadi (1993a)].

(c) Comparison of Mankbadi's theoretical estimate of the dependence of G_{max} on k_2/k_1 with the corresponding theoretical estimate by Herbert (1988a) and results of numerical simulation by Spalart and Yang (1987) of disturbance development in a boundary layer with a vibtrating ribbon in it; for $F \equiv \omega\nu/U_0^2 = 58.8\times10^{-5}$ and initial conditions $A_1(0) = 1.4\%$ and $\mathrm{Re}^{+}(0) = 950$ [after Mankbadi (1993a)].

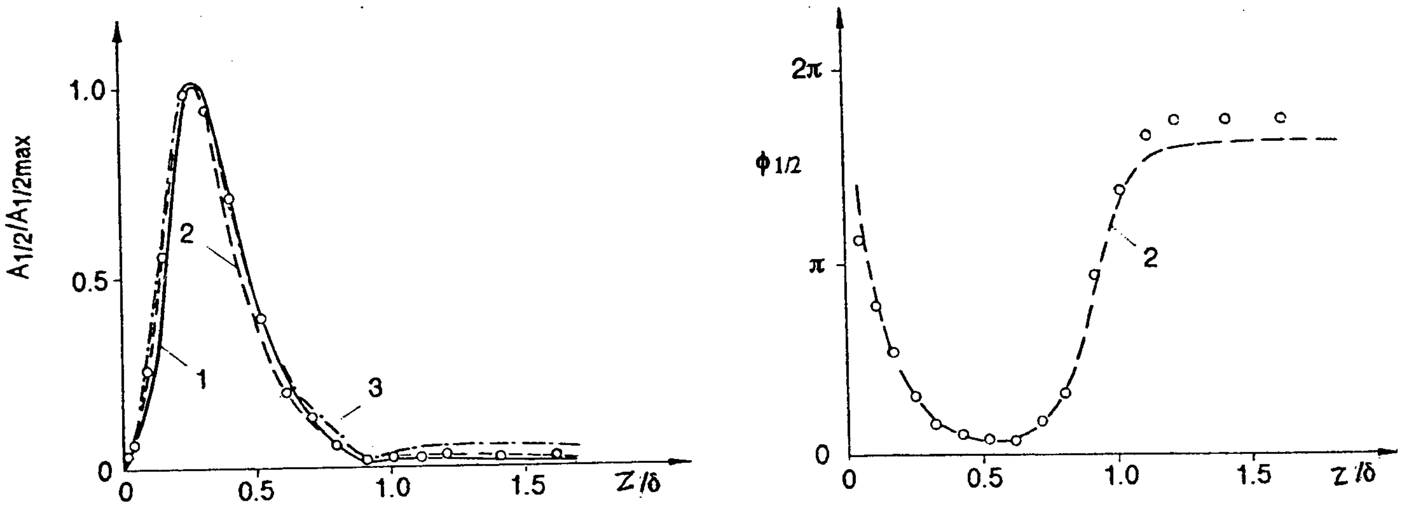


Figure 5.16. Measured (*points*) and calculated (*curves*) vertical profiles of the amplitude $A_{1/2}(z)$ (*left*) and the phase $\phi_{1/2}(z)$ (*right*) of the subharmonic 3D wave of frequency $\omega/2$ resonantly amplified in the N-regime of instability development in a Blasius boundary layer. Experimental data by Kachanov and Levchenko (1982). Calculations: 1 - secondary-instability theory of Herbert (1984a); 2 - numerical simulation of Fasel *et al.* (1989); 3 - resonant-triad theory of Zel'man and Maslennikova (1989,1990, 1993a) [after Kachanov (1994a)].

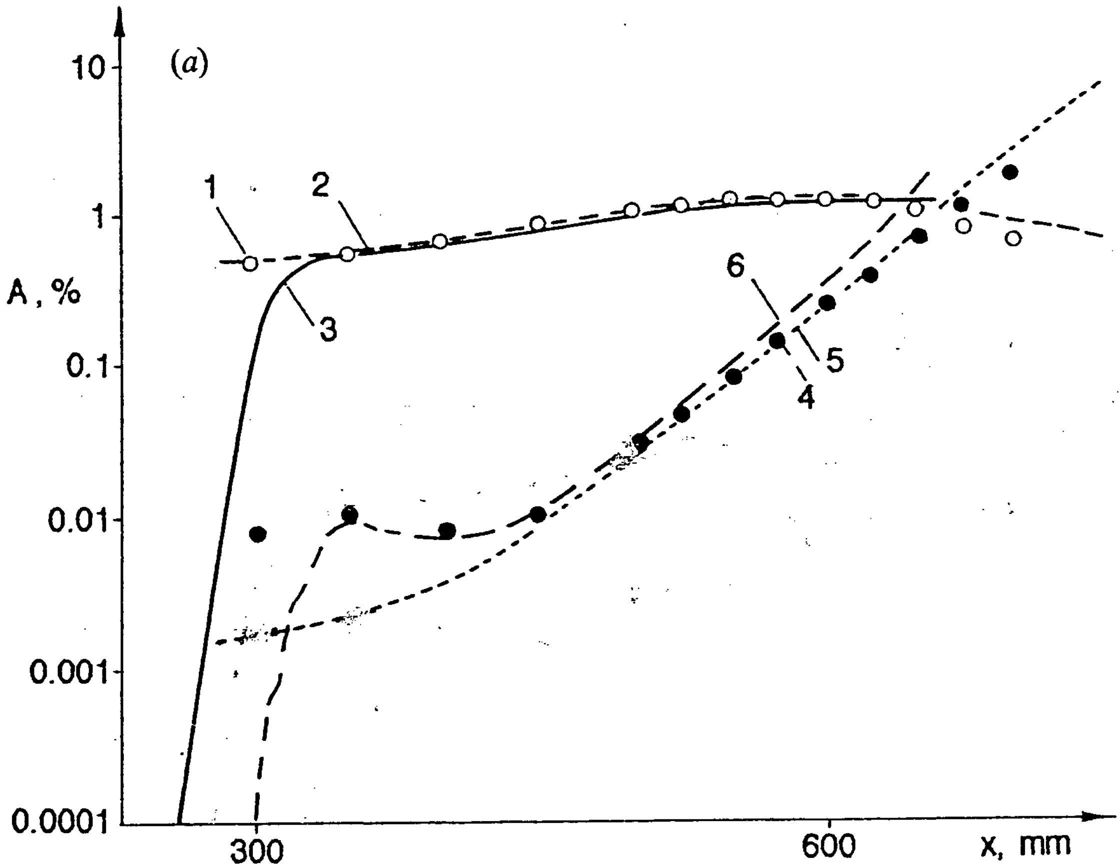


Figure 5.17, (a), (b) and (c). (For caption see the next page.)

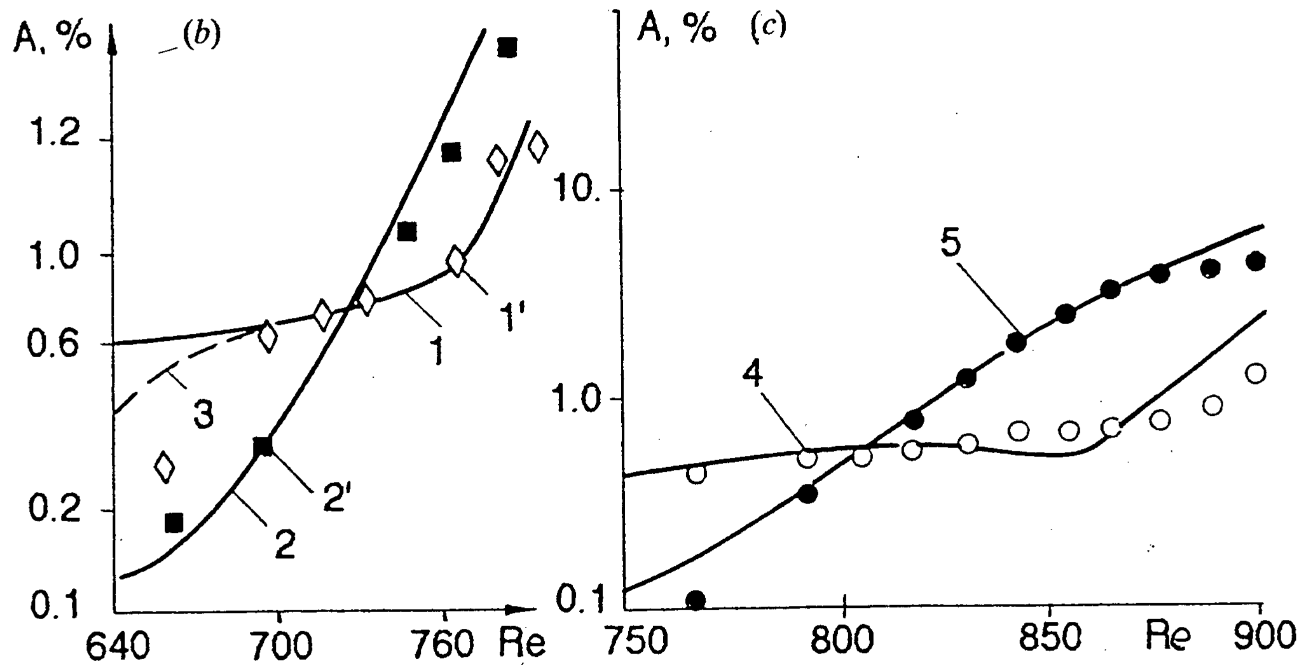


Figure 5.17. (a) Resonant streamwise amplifications of the plane wave amplitude $A_1(x)$ (results 1, 2, 3) and of the amplitude $A_{1/2}(x)$ of the two subharmonic waves of twice smaller frequency (results 4, 5, 6) during the initial stage of the N-regime of instability development. Experimental data (points 1 and 4) by Kachanov and Levchenko (1982); calculations (curves): 2 and 5 - Herbert's (1884a) theory; 3 and 6 - numerical simulations by Fasel *et al.* (1989).

(b) and (c) The same resonant amplifications at more late stages of the N-regime when values of $A_{1/2}(x)$ (2, 2' and 5) overtake those of $A_1(x)$ (1, 1', 3 and 4); $Re=(xU_0/\nu)^{1/2} \propto x^{1/2}$. (b): expereimental data (points 1' and 2') by Saric *et al.* (1984), calculated curves 1 and 2 - theory by Maslennikova and Zel'man (1985) and Zel'man and Maslennikova (1993a); dotted curve 3 - theory taking non-parallelism into account. (c): experimental data (points 4 and 5) by Corke and Mangano (1989); theoretical calculations (curves) by Crouch and Herbert (1993) [all figures after Kachanov (1994a)].

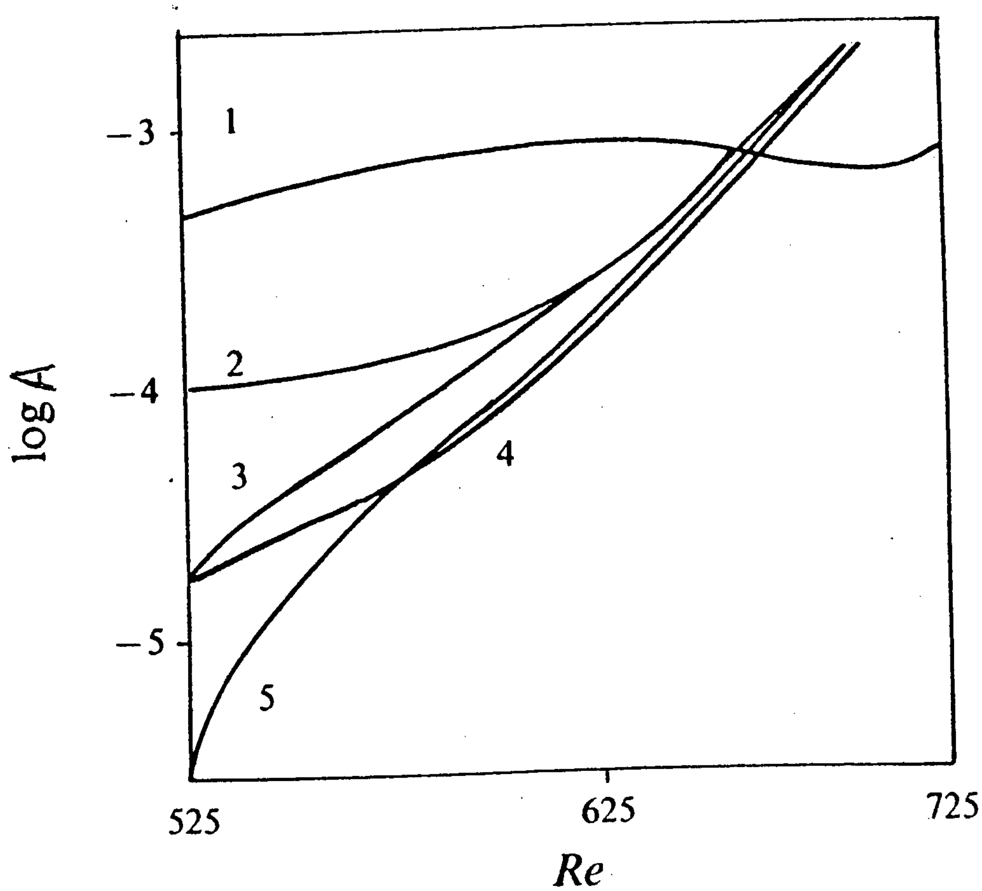


Figure 5.18. Downstream-growth curves for amplitudes of five-wave disturbance system in a Blasius boundary layer. The system includes the plane wave 1 of frequency ω and wave vector $\{k, 0\}$ and oblique-wave pairs 2-3 and 4-5 with frequency-wave vector values $\{\omega/2, k_1, \pm k_2\}$ and $\{\omega/2, k_1^*, \pm k_2^*\}$ where $F = \omega\nu/U_0^2 = 230\times10^{-6}$, $K_2 = k_2\nu/U_0 = 0.171\times10^{-3}$, $K_2^* = k_2^*\nu/U_0 = 0.15\times10^{-3}$; Re $= (U_0x/\nu)^{1/2}$ [after Zel'man and Maslennikova (1993a)].

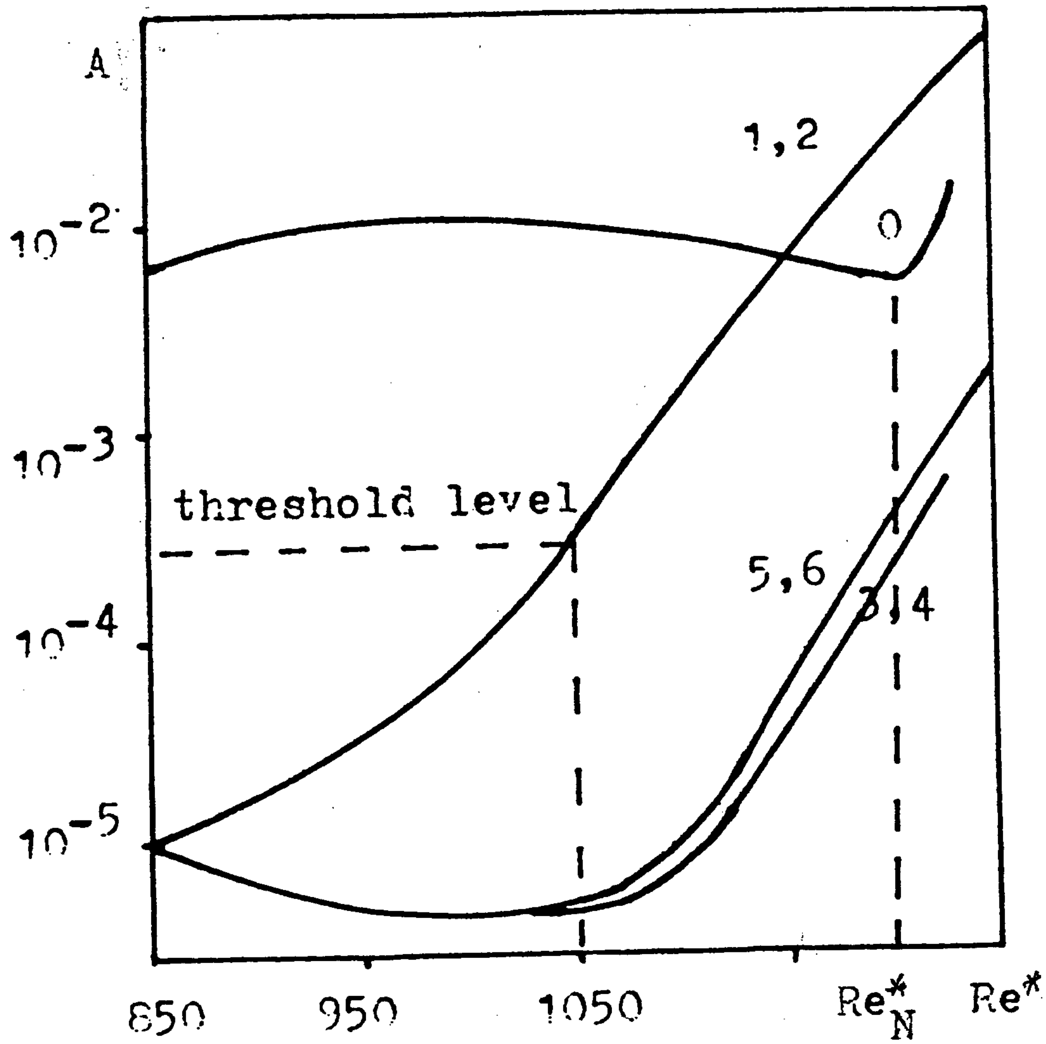


Figure 5.19. Amplification curves for seven-wave system including the primary plane wave 0 with frequency-wave vector (f-w) combination $\{\omega_0, k, 0\}$, a pair of secondary oblique waves 1-2 with f-w combination $\{\omega_0/2, k_1, \pm k_2\}$ and two pairs of tertiary oblique waves 3-4 and 5-6 with f-w combinations $\{\omega_0/4, k_1', \pm k_2'\}$ and $\{\omega_0/4, k_1'', \pm k_2''\}$. Here $F_0 = \omega_0 \nu / U_0^2 = 122\times10^{-6}$, $k_2/k_1 = 2$, $k_2'/k_1' = 2.8$, $k_2''/k_1'' = 3.44$, $Re^* = (U_0\delta^*/\nu)^{1/2}$ [after Zel'man and Maslennikova (1993b)].

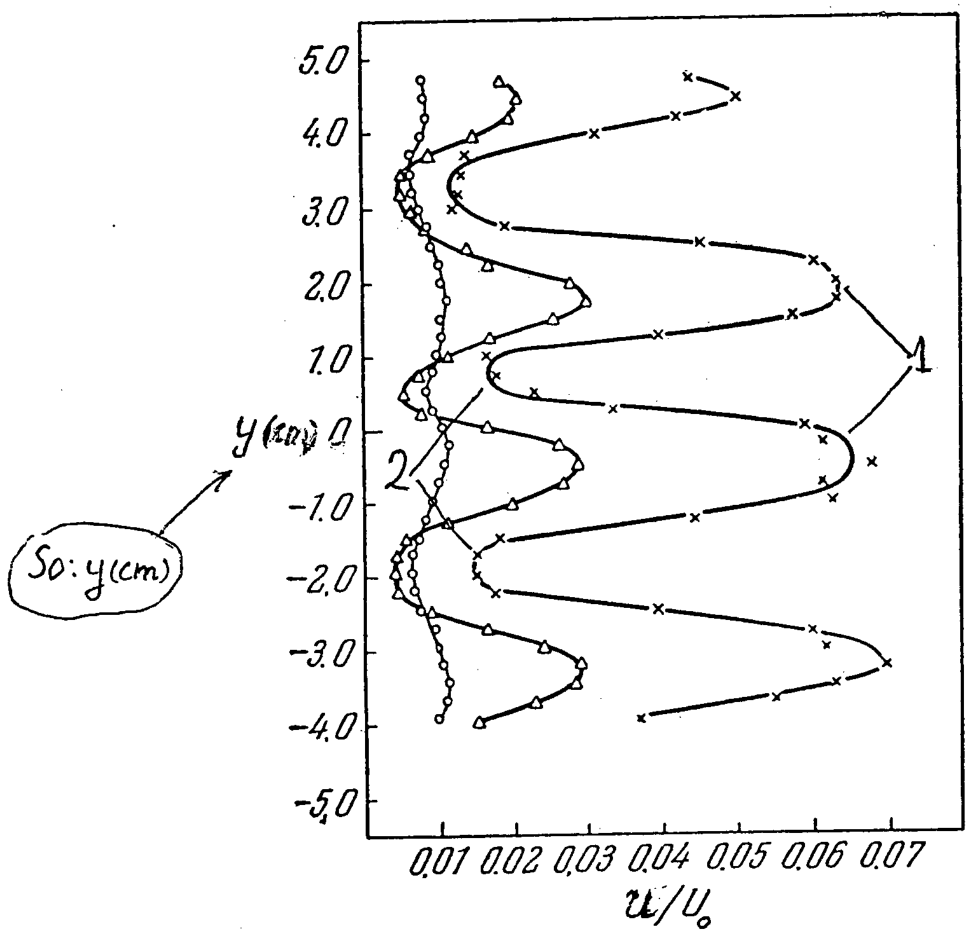


Figure 5.20. Downstream growth of spanwise modulation of the amplitude *u'* of streamwise disturbance velocity in a boundary layer disturbed by vibrating ribbon.

o - data for x = 7.6 cm; Δ - for x = 15.2 cm; $\times$ - for x = 19cm, where x is measured from the trailing edge of the ribbon; 1 - modulation 'peaks', 2 - 'valleys' [after Klebanoff *et al.* (1962)].

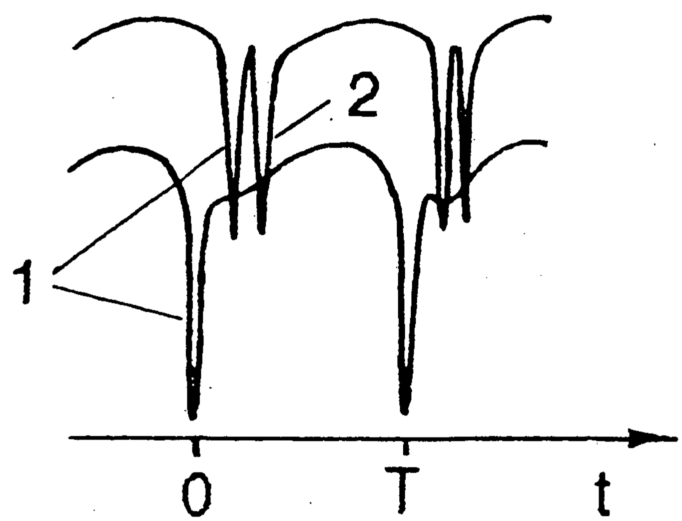


Figure 5.21. Typical single and double spikes in a boundary layer flows [after Klebanoff *et al.* (1962)]. 1 - 1st spike; 2 - 2nd spike; T - fundamental period of spike repetitions.

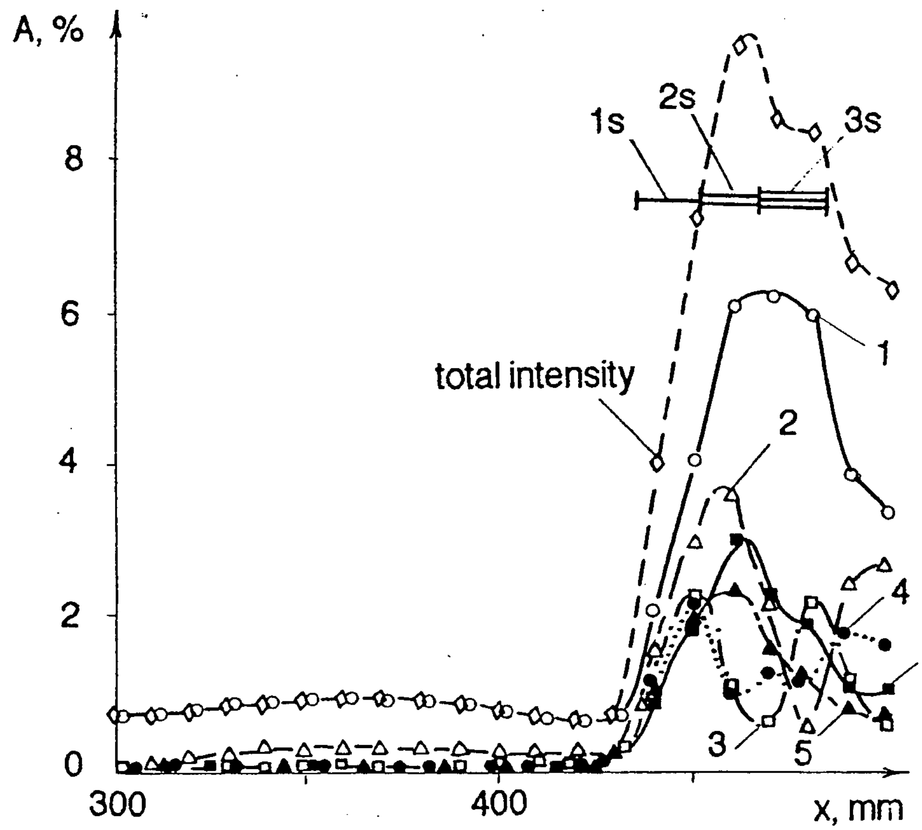


Figure 5.22. Amplification of total intensity of streamwise disturbance velocity $u(\mathbf{x},t)$ and amplitudes of its harmonics with frequencies ω_1, ω_2, ... , ω_6 (symbols and curves1, 2, ... ,6; ω_1 - fundamental frequency of primary T-S wave, $\omega_n = n\omega_1$) observed at y corresponding to peak position of spanwise modulation, fixed value of z and variable x-coordinate. Streamwise intervals 1s, 2s, 3s - places of formation of the 1st, 2nd, and 3rd spike [after Kachanov *et al.* (1984) and Kachanov (1994a)].

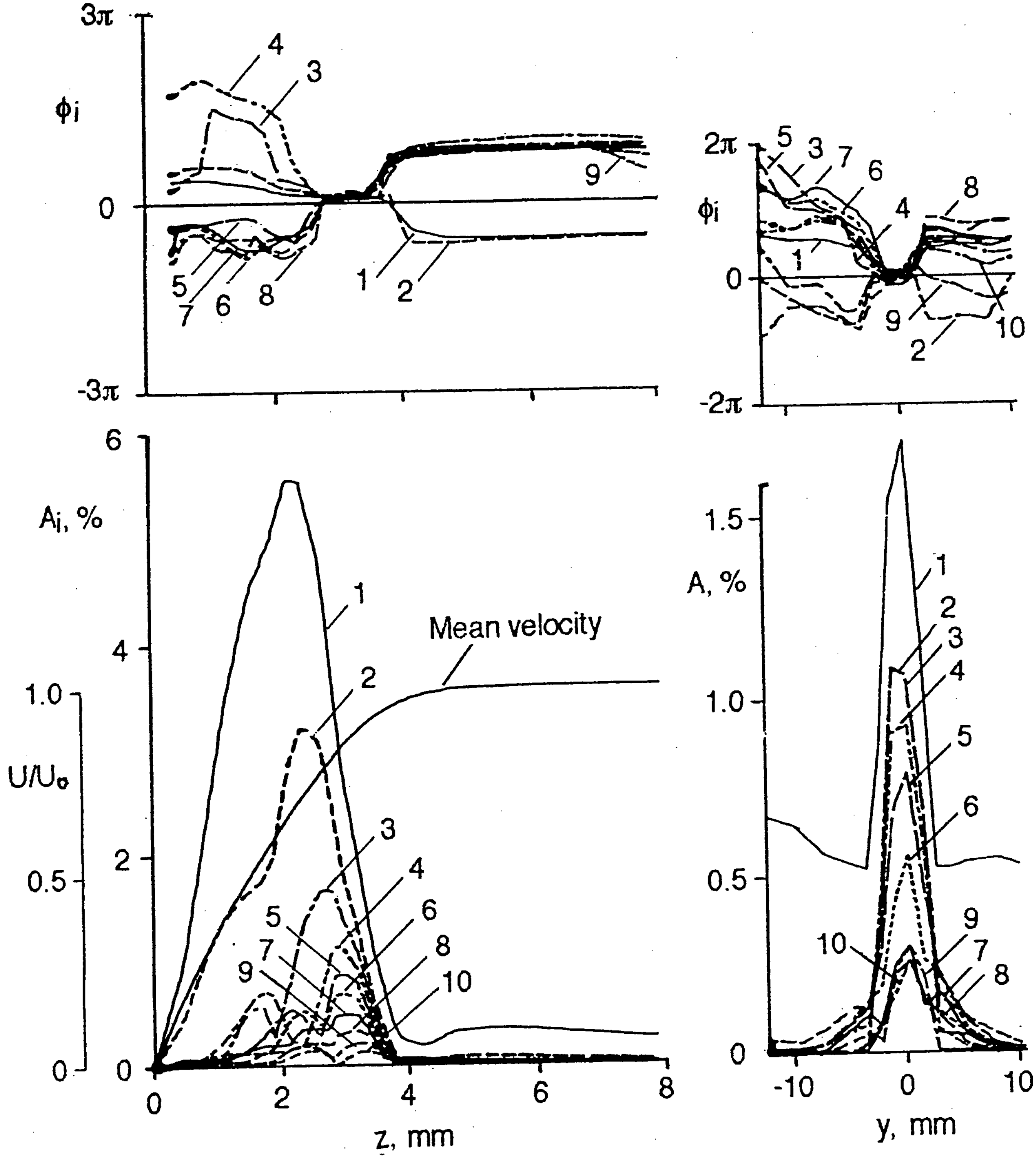


Figure 5.23. Vertical (*left*) and spanwise (*right*) profiles of the amplitudes A_i (*bottom*) and phases ϕ_i (*top*) of streamwise-velocity harmonics with frequencies $\omega_1, \omega_2, \ldots, \omega_{10}$ measured at the stage of developed spikes (curves 1, 2, ... , 10) [after Borodulin and Kachanov (1992) and Kachanov (1994a)]. Mean-velocity profile is added to vertical amplitude profiles to show the boundary-layer thickness.

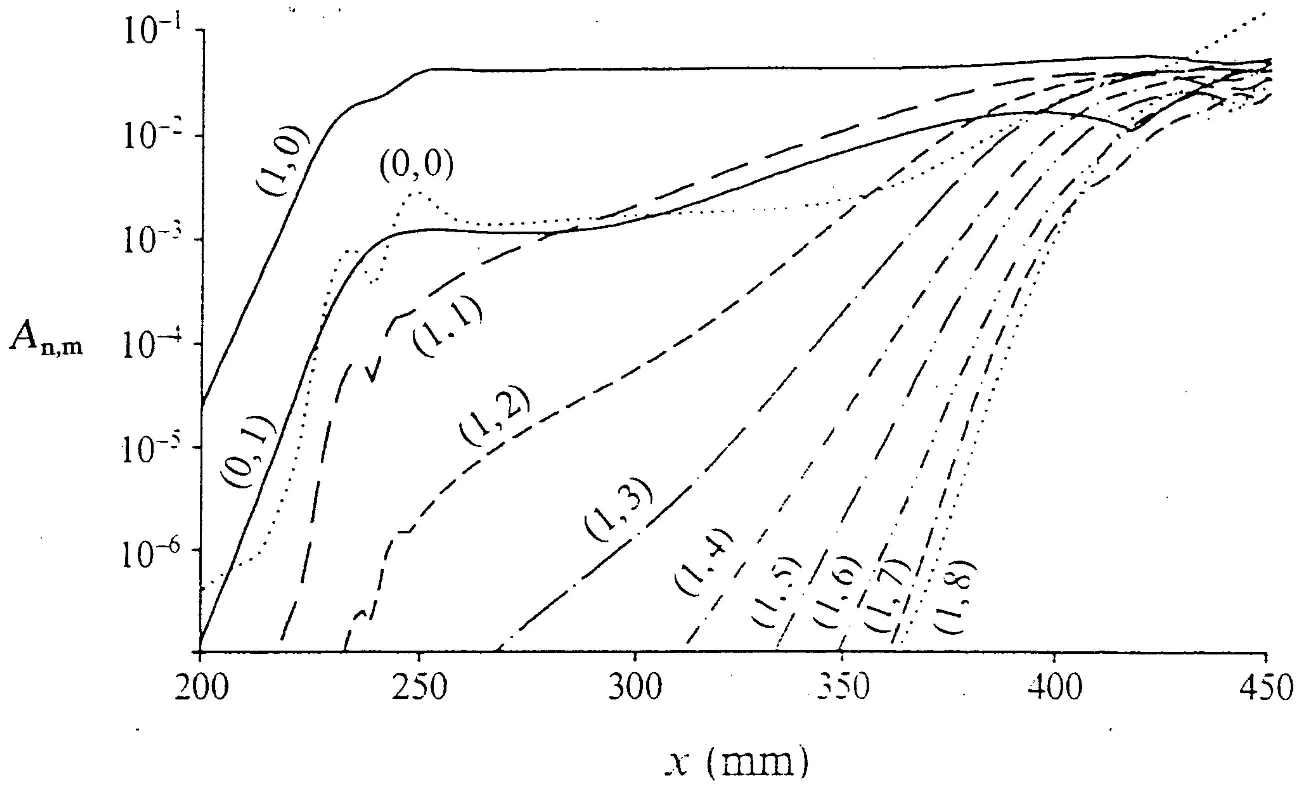


Figure 5.24. Streamwise development of values of amplitudes $A_{n,,m}$ of (n,m)-Fourier components of velocity $u(\mathbf{x},t)$ at the heights z where these amplitudes takes maximal values [after Rist and Fasel (1995)].

(n,m)-Fourier component corresponds to frequency $\omega_n = n\omega_1$ and spanwise wavenumber $k_{2,m} = mk_0$ (where ω_1 - fundamental frequency of primary wave, k_0 - wavenumber of the fundamental spanwise periodicity (0,1) shown in Fig. 5.21). Symbols (0,0) and (1,0) correspond to amplitudes of the 'mean flow correction' and 'primary 2D wave'.

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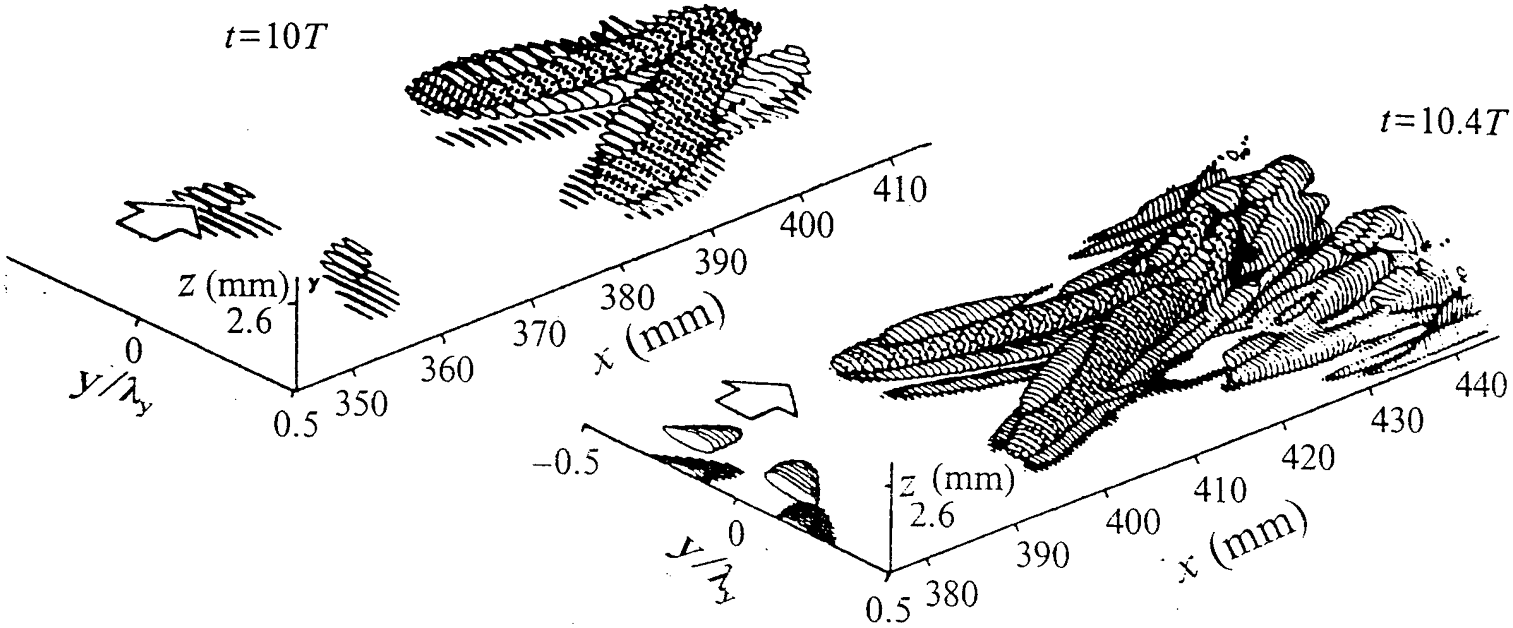

Figure 5.25. Two examples of Λ-vortices appearing at two time instants in the numerical simulation of unstability developments in a flat-plate boundary layer by Rist and Fasel.

Shown here three-dimensional Λ-shaped structures are bounded by surfaces $|\eta_x|$ = constant where η_x is the x-component of flow vorticity; λ_y is the primary spanwise wavelength clearly seen in Fig. 5.20 [after Rist and Fasel (1995)].

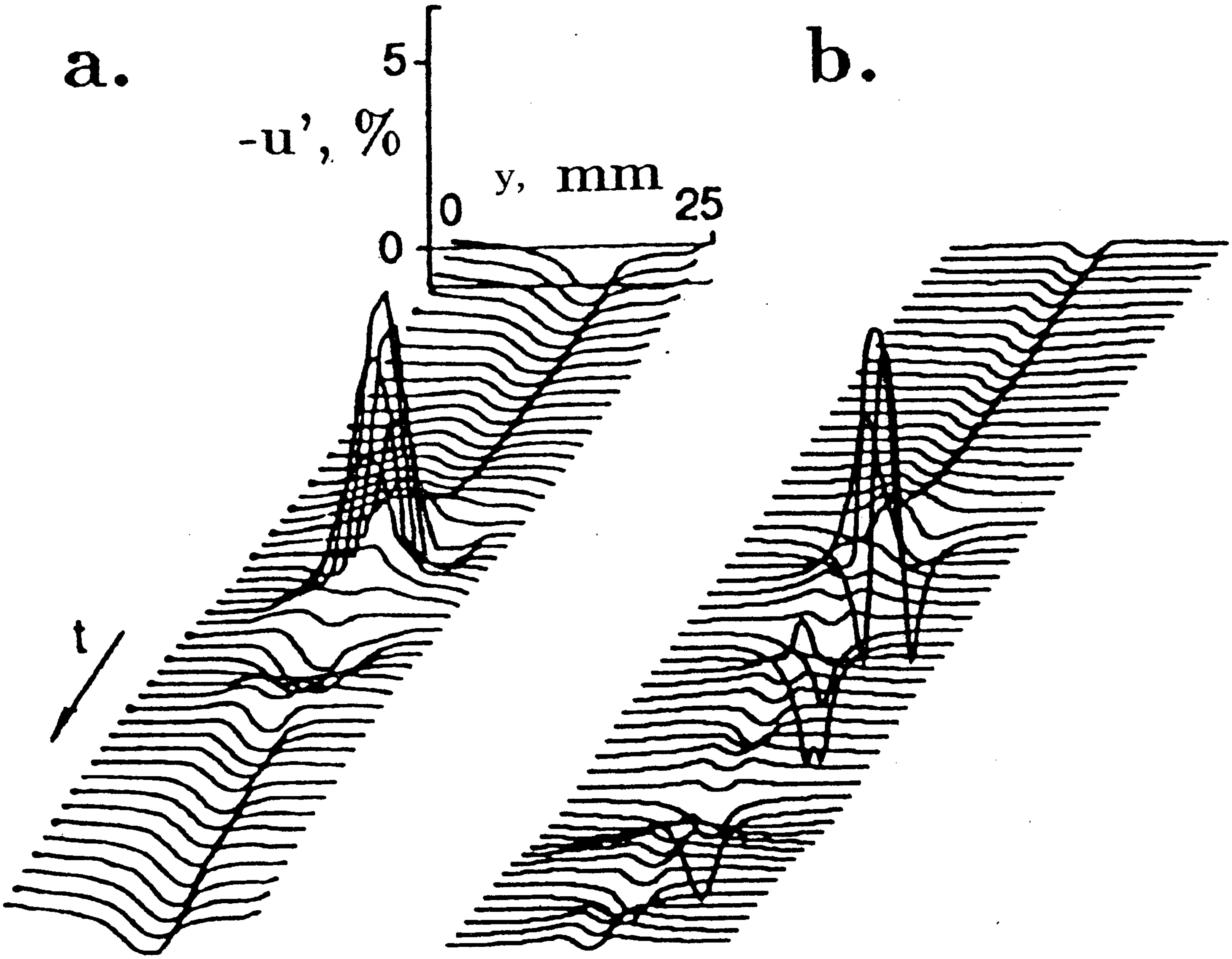

Figure 5.26. Comparison of 'spike signal' in the (u', t, y)-space (u' is the stremwise disturbance velocity measured in parts of U_0) appearing in numerical simulation of boundary-layer instability development by Rist and Kloker at x = 500 mm, z = 8 mm (a) with the 'spike signal' observed at the same values of x and z in the corresponding laboratory experiment of Kachanov and Borodulin (b) [after Rist and Kachanov (1995)].

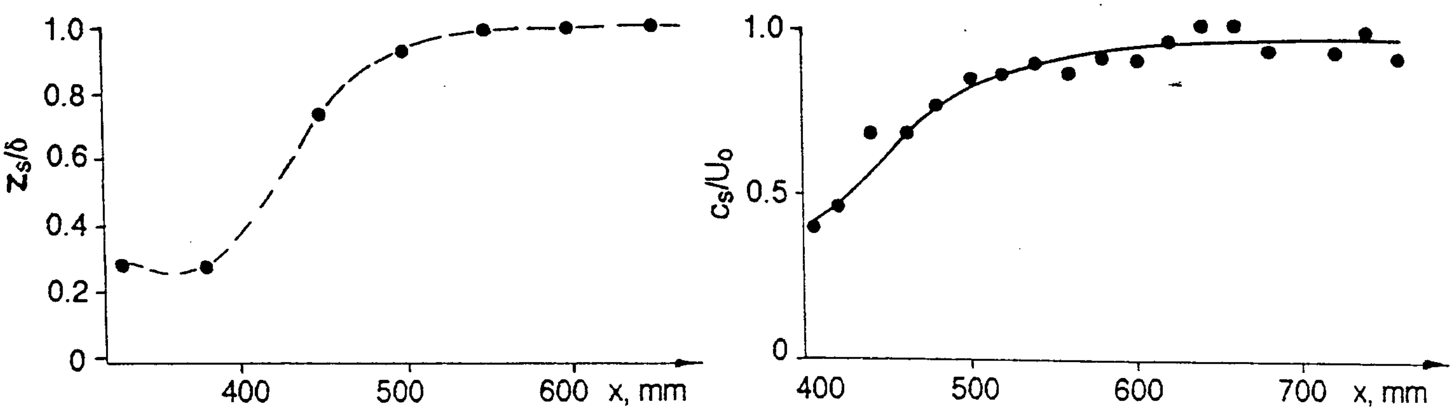

Figure 5.27. (a) Dependence of the dimensionless height z_s/δ (where δ is the boundary-layer thickness) of the center of an appearing spike on streamwise coordinate x during spike downstream evolution. (b) Dependence on x of the streamwise velocity c_s of a spike [after Borodulin and Kachanov (1994)].

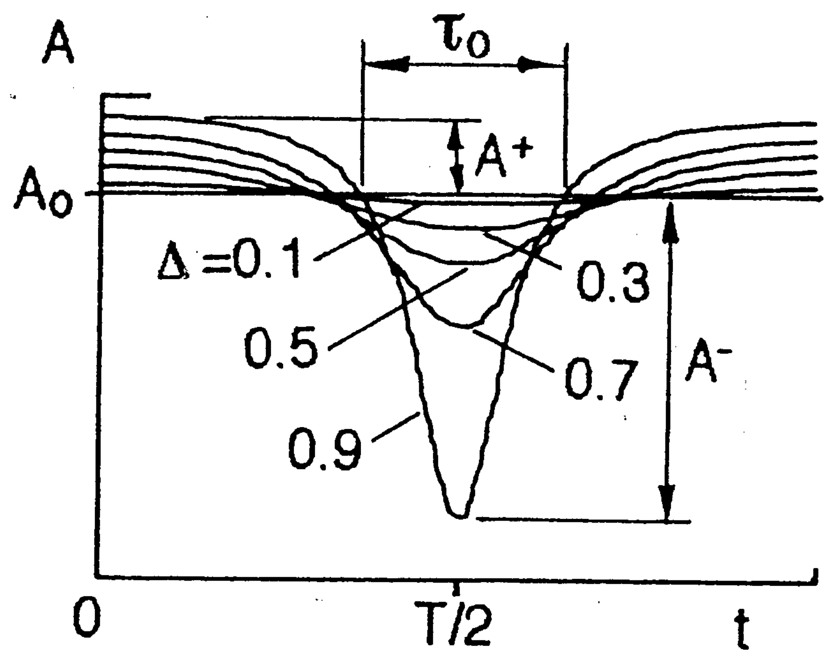

Figure 5.28. Schematic form of a family of Ryzhov's three-parameter soliton solution of the Benjamin-Ono equation (5.16) corresponding to various values of the amplitude parameter Δ and fixed values of other two parameters (determining scales of the dependence of A on t and x). Here $A_0 = T^{-1}\int_0^T A(t)dt$ is the mean amplitude, A^-, A^+ and τ_0 are some numerical form characteristics and T is fundamental spike period [after Kachanov, Ryzhov and Smith (1993)].

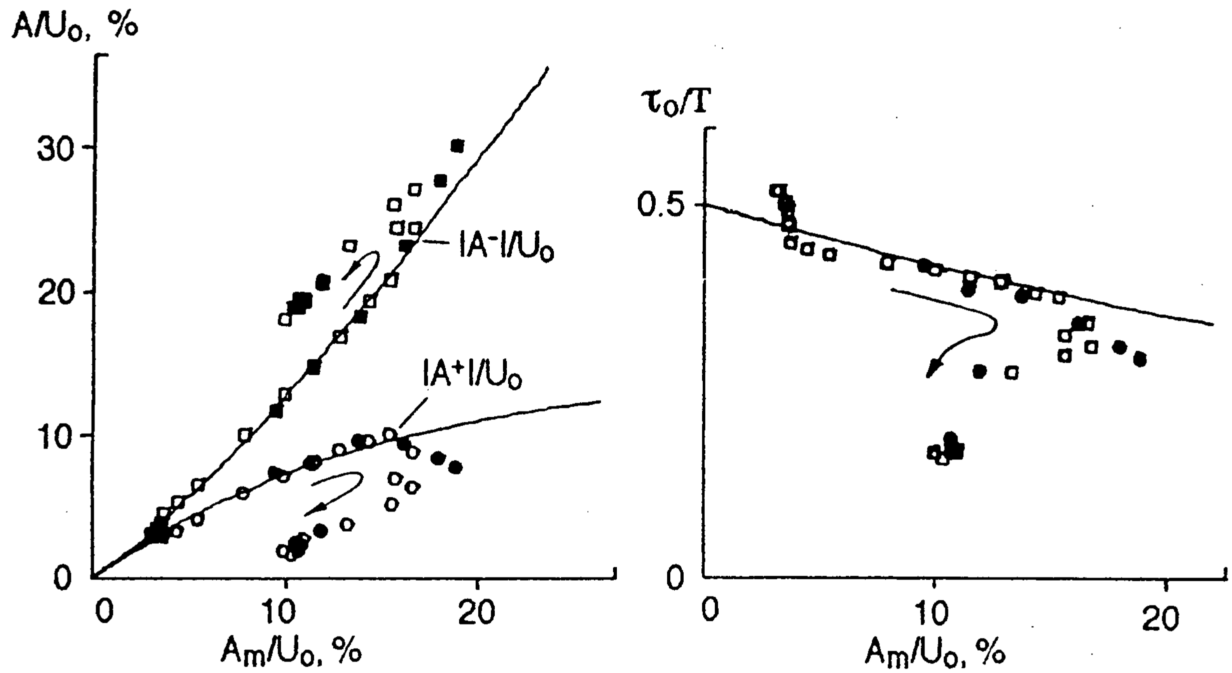

Figure 5.29. Experimental (*points*) and theoretical (*curves*) dependencies of soliton form characteristics A^-, A^+ and τ_0 on the soliton magnitude $A_m = (A^+ + A^-)/2$. Theoretical curves correspond to shown in Fig. 5.28 solution of B-O equation with appropriately chosen parameter values; points - data of Borodulin and Kachanov (1988) [after Kachanov, Ryzhov and Smith (1993)].

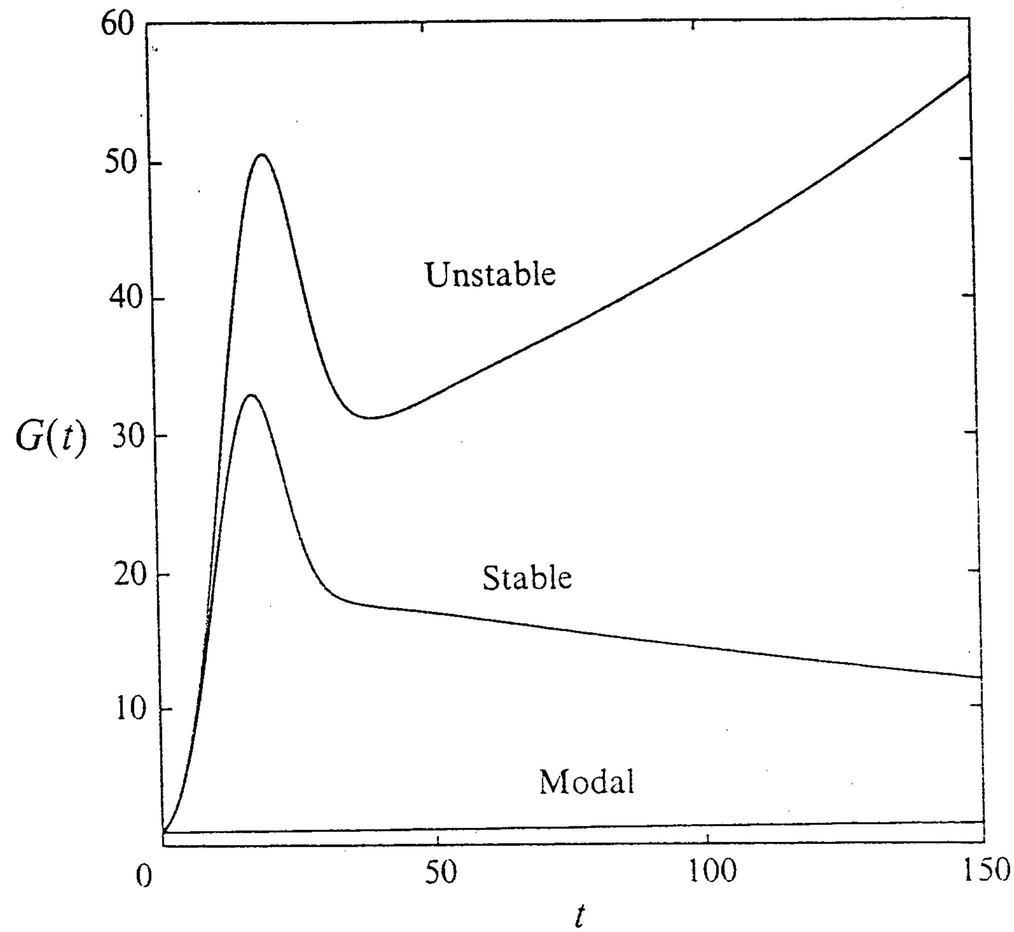

Figure 5.30. The dependence on t of the growth curve $G(t) = E(t)/E(0)$ for the energy E of plane-wave disturbances with $k_1 = 1$ [all physical quantities are non-dimensionalized by scales $H_1 = H/2$ and $U_0 = U(H/2)$] in a plane Poiseuille flow between walls at $z = 0$ and $z = H$. The curves labelled 'Unstable' and "Stable"correspond to the 'optimal' 2D wave having the greatest transient growth in the linearly unstable Poiseuille flow with $\mathrm{Re} = U_0 H_1/\nu = 8000$ and linearly stable flow with Re = 5000, respectively, while 'Modal' curve shows the growth of the unstable solution of the Orr-Sommerfeld eigenvalue problem with $k_1 = 1$ and Re = 8000 [after Reddy and Henningson (1993)].

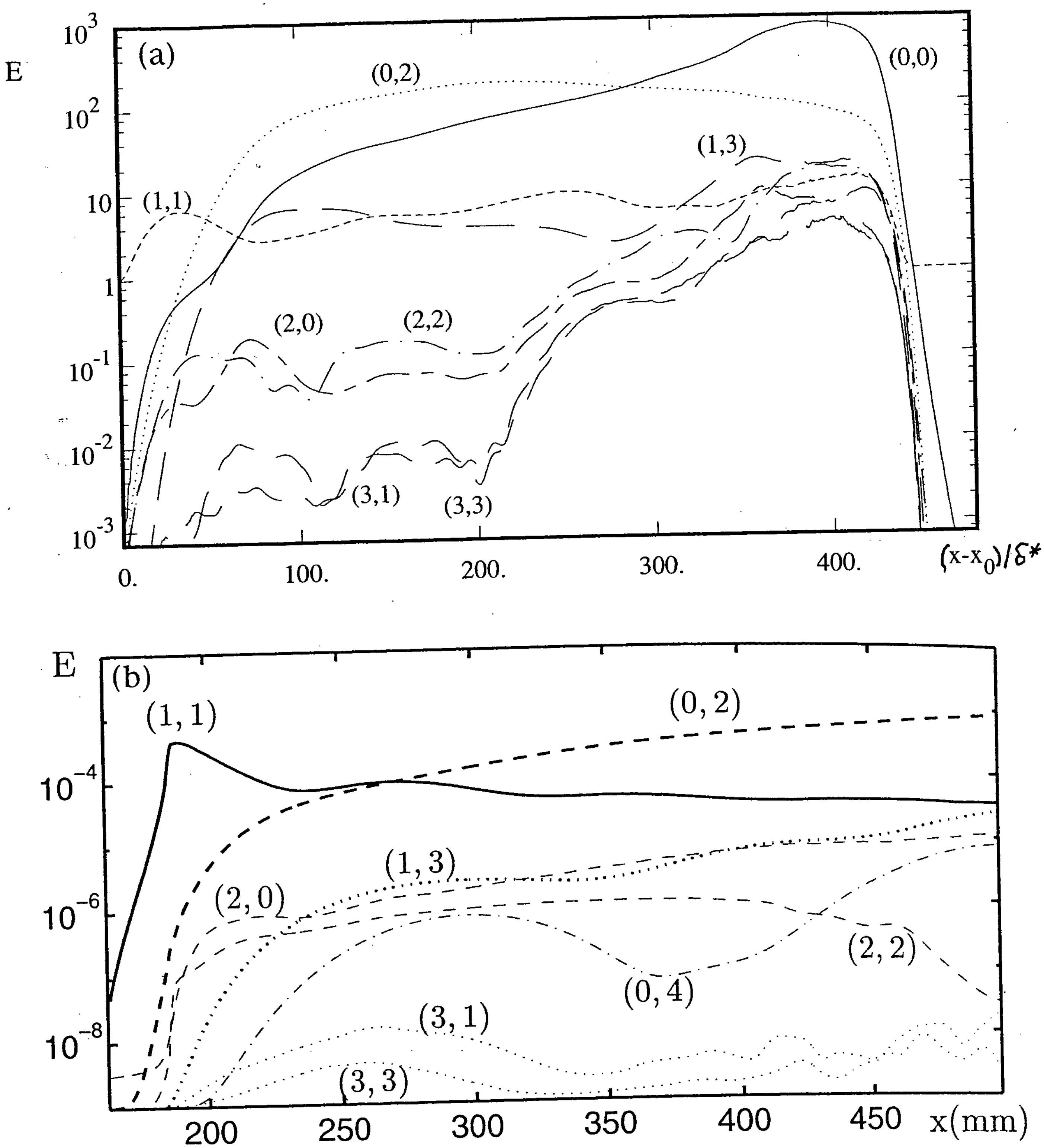

Figure 5.31. (a) The dependence of the energies E of a number of (n,m)-Fourier components [i.e., waves with frequencies and spanwise wavenumbers $(n\omega_0, mk_{2,0})$] on x in a Blasius boundary layer with a pair of oblique waves with frequencies and spanwise wavenumbers $(\omega_0, \pm k_{2,0})$. (a) Dependence of E on $(x - x_0)/\delta*$ in the case when the initial energy of (1,1)-mode is equal to 1 [after Berlin *et al.* (1994)]. (b) Dependence of E (measured in some conventional units) on x (in mm); coordinate x_0 of the 'wave generator' is here close to 186 mm [after Berlin *et al.* (1999)].

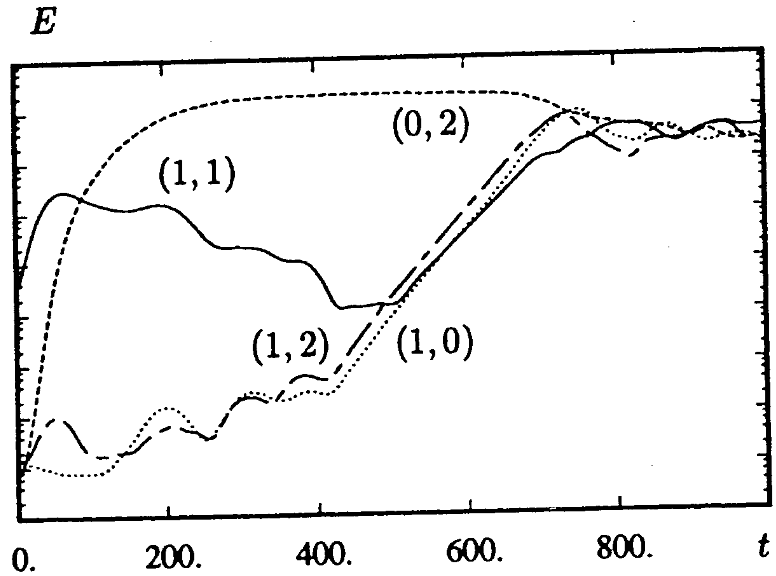

Figure 5.32. Computed energy-growth curves for a number of (n,m)-waves with streamwise and spanwise wavenumbers $(nk_{1,0}, mk_{2,0})$ in a boundary layer disturbed by 'primary oblique waves' with wavenumbers $(k_{1,0}, \pm k_{2,0})$ and a 'weak noise' consisting of supplementary small-amplitude (n,m)-waves with n = 0, 1, 2, and m = 0, 1, 2 [after Schmid and Henningson (2000)].

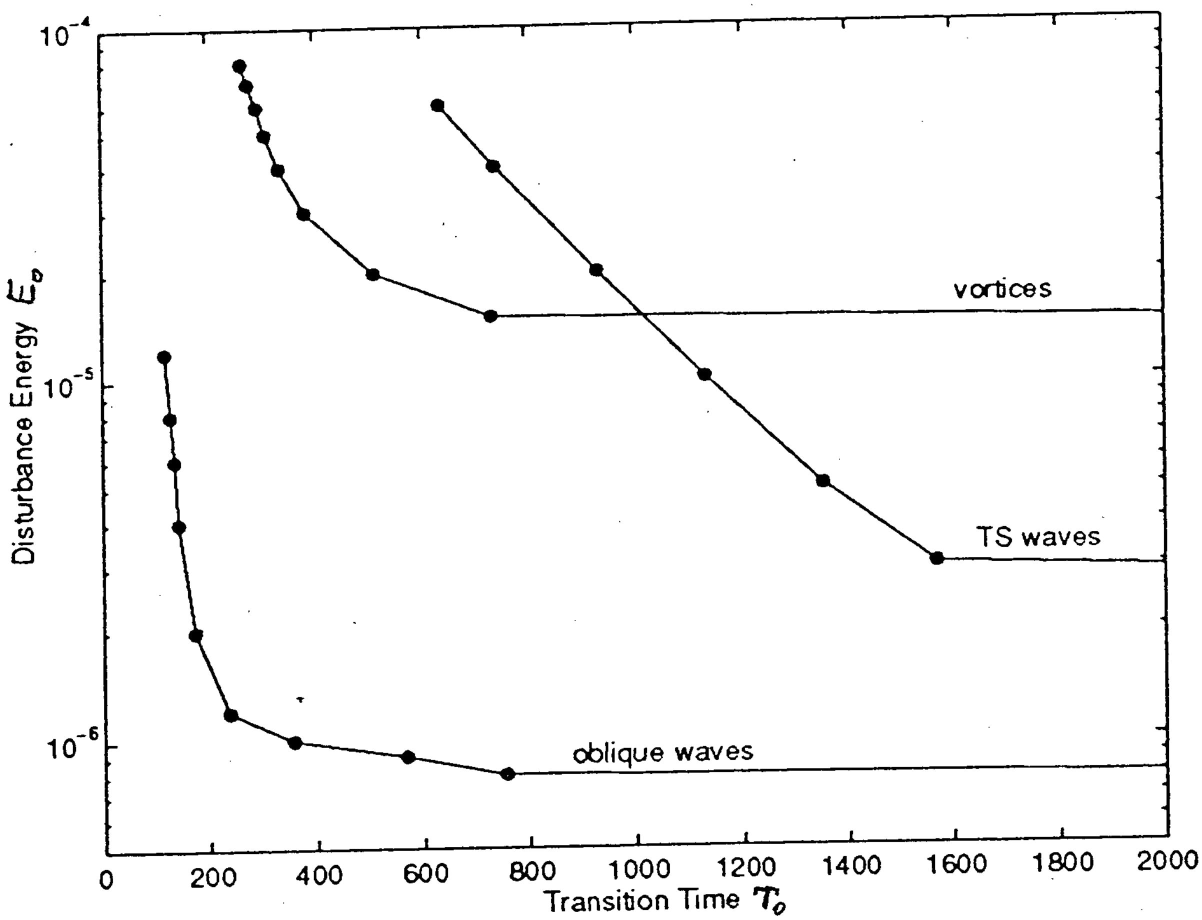

Figure 5.33. Dependence of the transition time T_0 on the disturbance initial energy E_0 (which is proportional to A_0^2) for three transiton scenarios: i) 'oblique transition' (labelled 'oblique waves'), ii) TS-wave-secondary-instability scenario' (K-regime, labelled 'TS waves') and iii) 'streak-breakdown scenario' (labelled 'vortices' since streaks are produced by streamwise vortices which are (0,2)-structures). The results are for temporally growing Blasius boundary layer with initial Re* = 500 [after Schmid *et al.* (1996)].

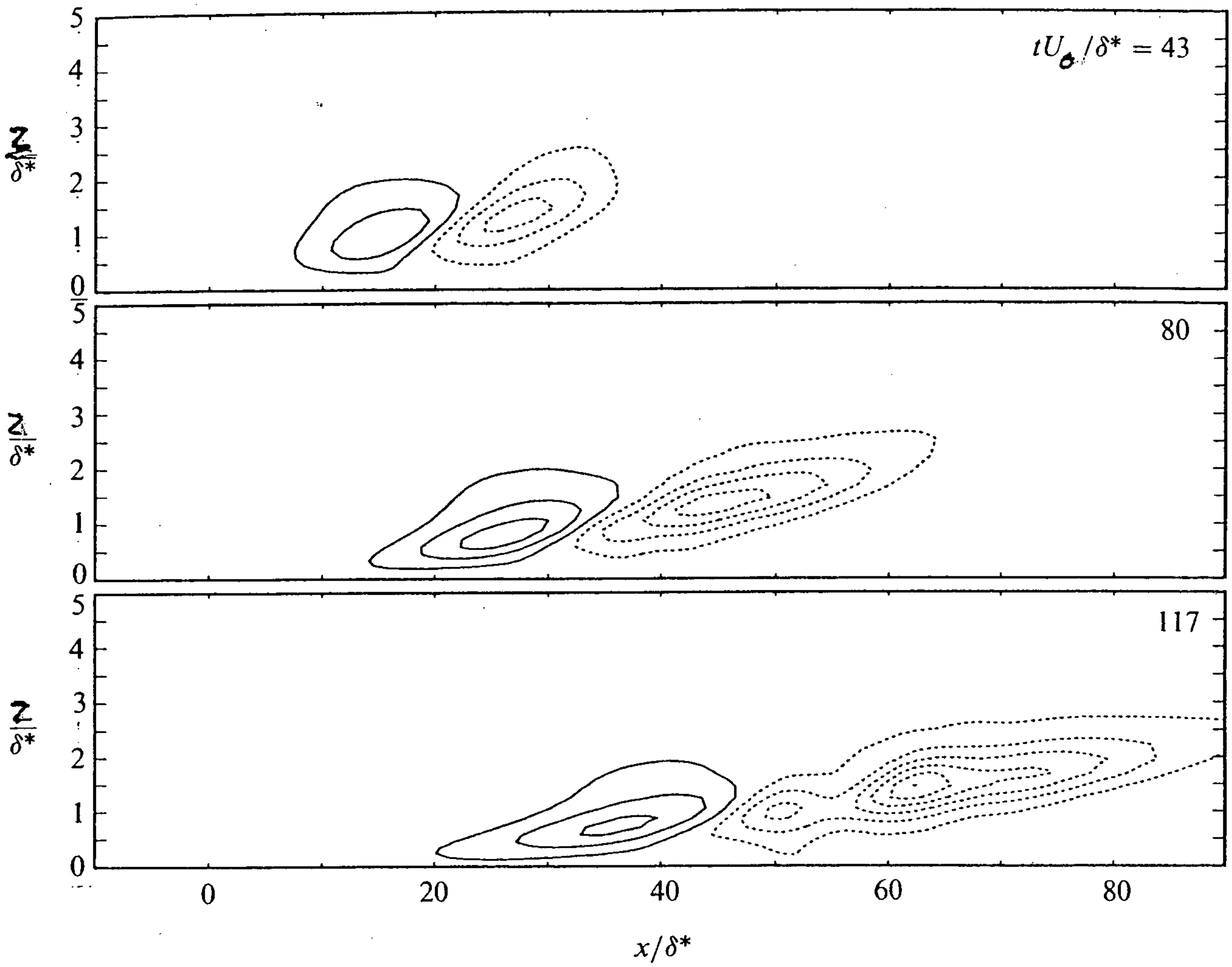

Figure 5.34. Computed contours in the (x,z)-plane of the streamwise disturbance velocity $u(x,y,z,t)$ at $y = 0$ and several values of t for a localized disturbance of finite amplitude with given value at $t = 0$. Solid and dotted lines represent positive and negative velocity values; contour spacing is 2% of U_0 [after Breuer and Landahl (1990)].

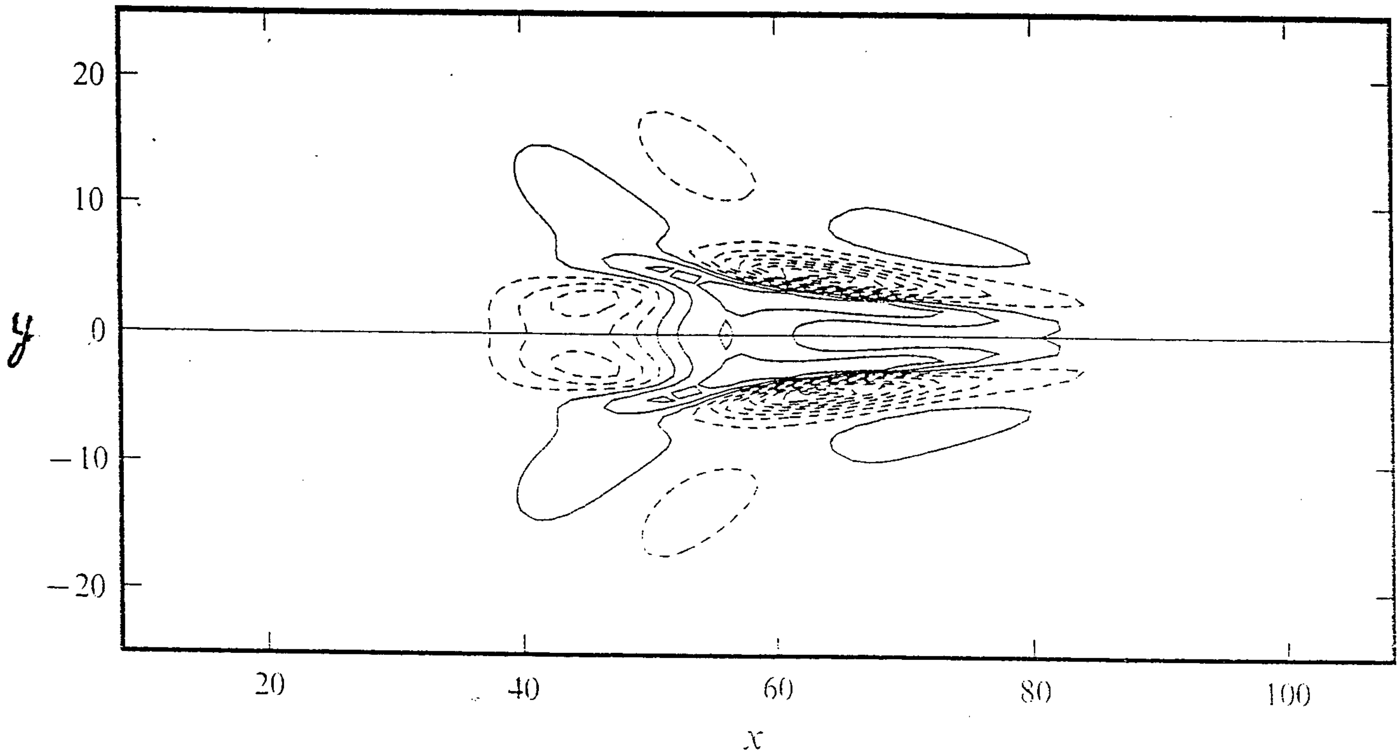

Figure 5.35. Contours in the horizontal (x,y)-plane of the vertical velocity $w(x,y,z,t)$ at $z = 0.99$, $t = 117$ (all dimensional quantities are non-dimensionalized by scales δ^* and U_0). Solid and dotted lines represent positive and negative velocity values; contour spacing is 0.001 [after Henningson *et al.* (1993)].

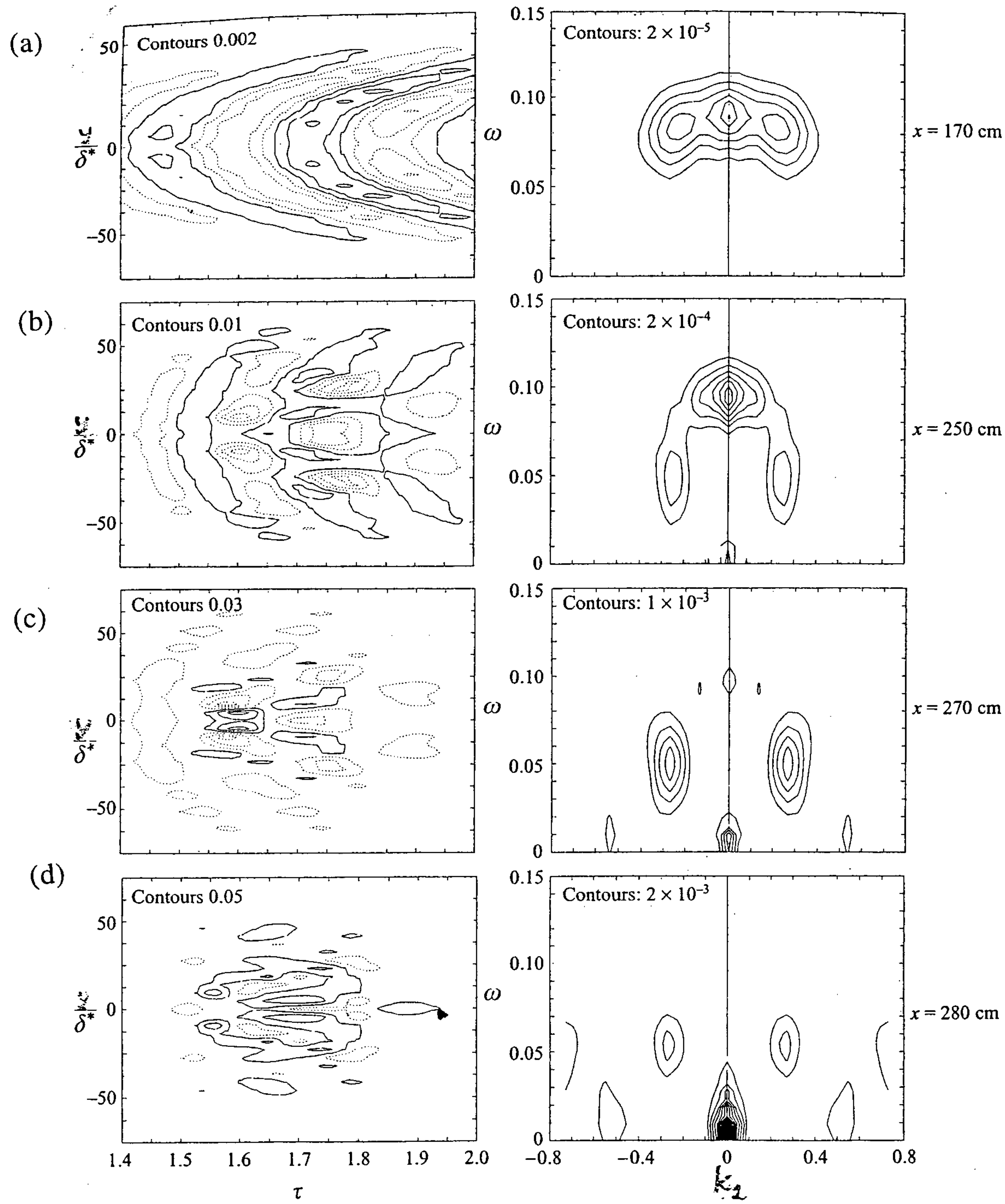

Figure 5.36. Overall view of the dependence of velocity $u(x,y,z,t)$ on y and $\tau = (t - T_0)U_0//(x - x_0)$ (*left-hand column*) and of the dependence of corresponding wavenumber-frequency spectra on ω and k_2 (*right-hand column*) at $z/\delta^* = 0.5$ and four different values of x [data for separate x-values are noted by marks (a), (b), (c) and (d)]. Solid and dotted lines show positive and negative values, respectively [after Breuer *et al.* (1997)].

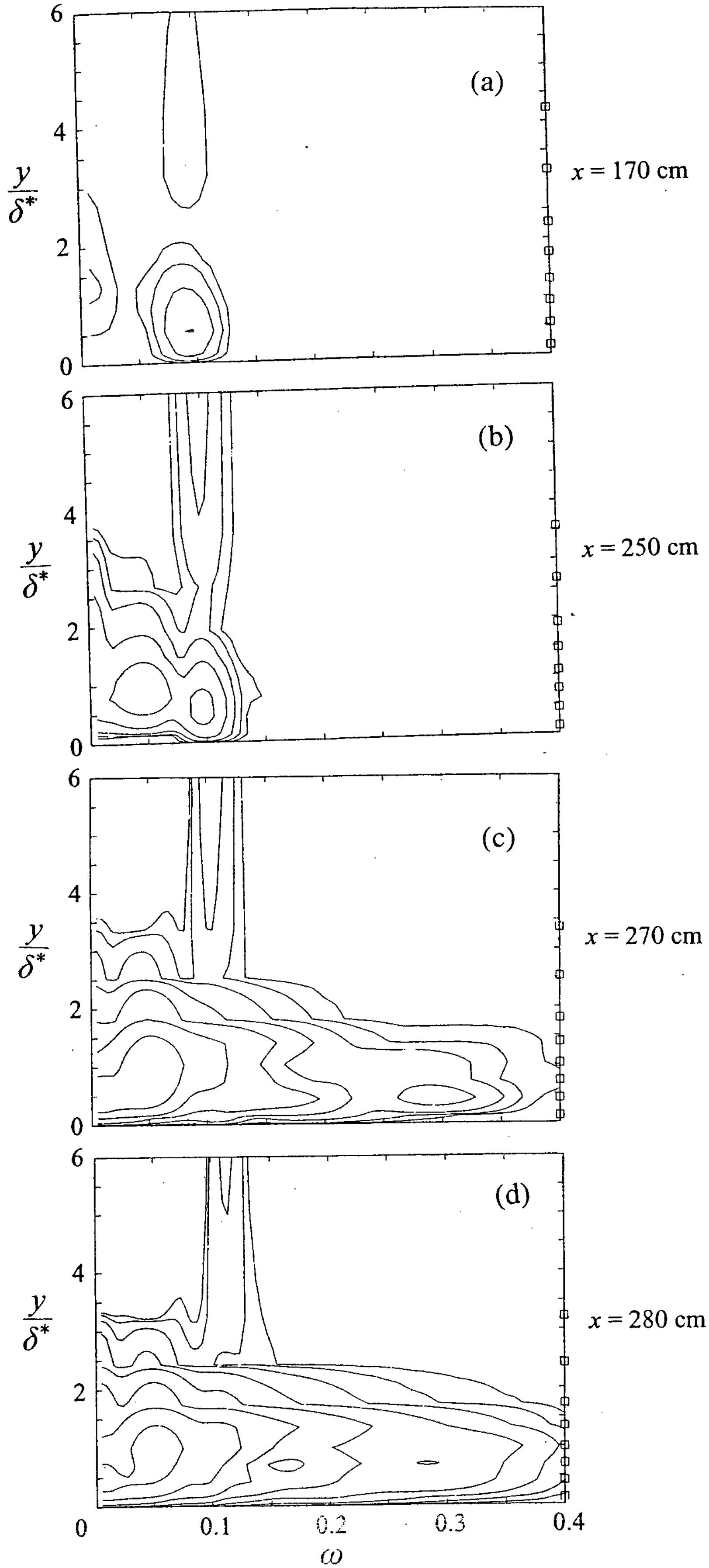

Figure 5.37. Contours in the (ω, z)-plane of averaged frequency spectra $P_z(\omega)$ of streamwise velocity fluctuations $u(x,y,z,t)$ at four different values of x and $y/\delta^* = 4.7$. Spectra $P_z(\omega)$ were computed for a number of independently observed velocity fields and then were averaged over the ensemble of made observations [after Breuer *et al.* (1997)].

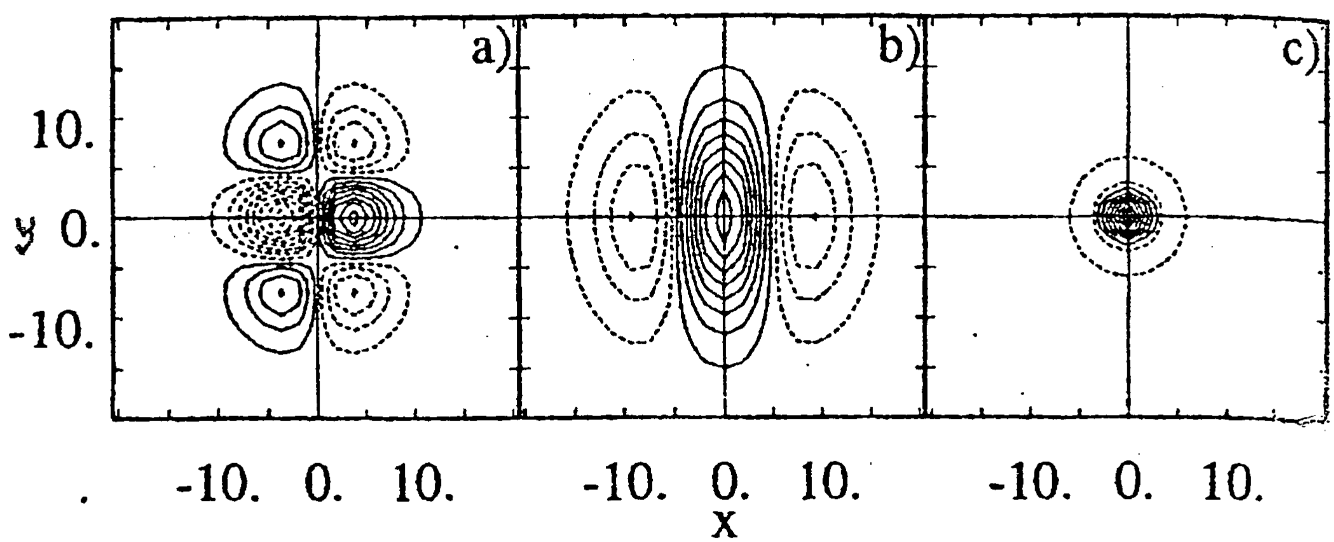

Figure 5.38. Contours in the (x,y)-plane of the initial vertical velocity $w(x,y,z)$ at $z/\delta^* = 1.5$ for three selected models of the initial velocity field. The marks (a), (b) and (c) correspond to the first, second and third models [after Bech *et al.* (1998)].

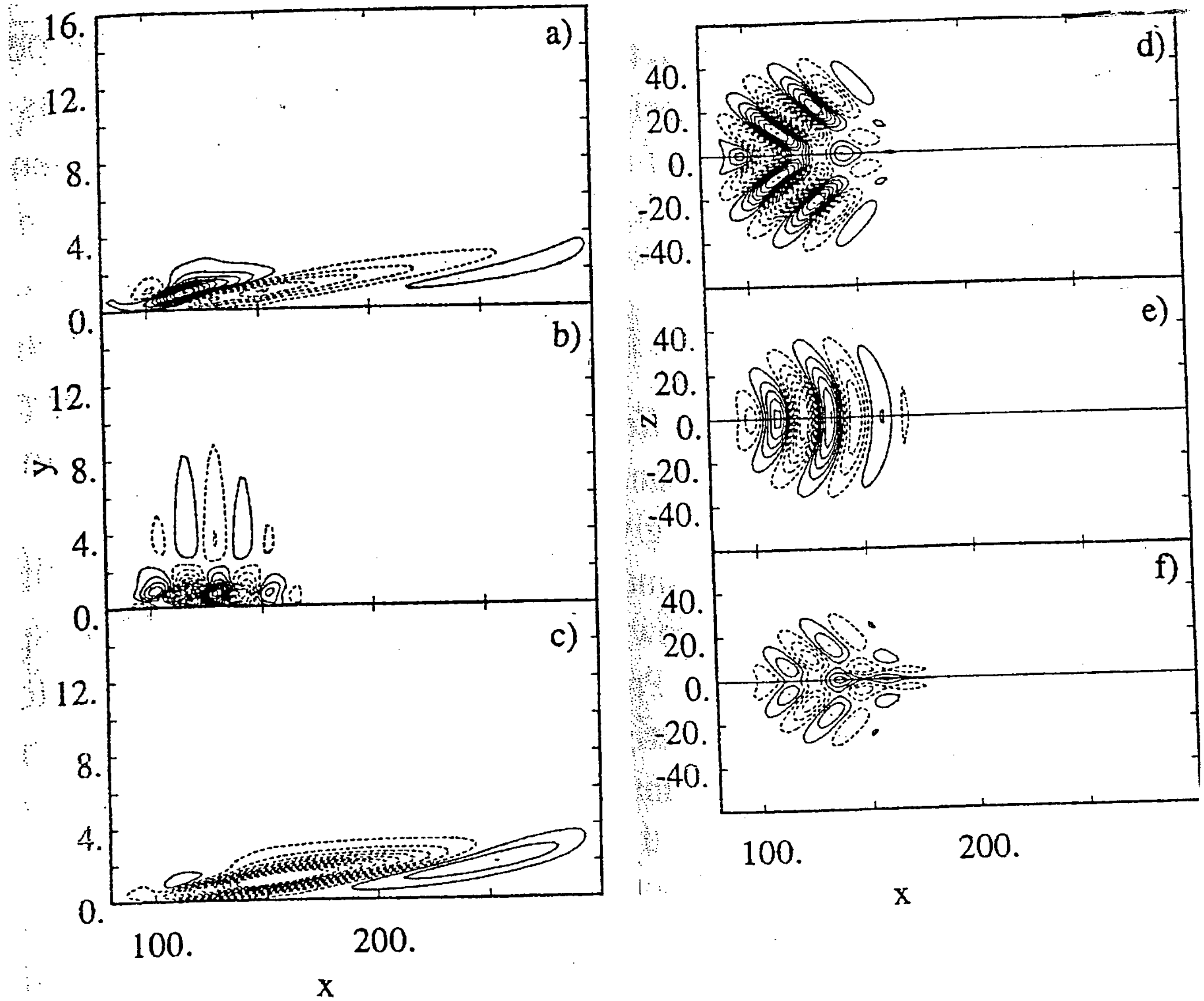

Figure 5.39. (a)-(c) Contours in the (x,z)-plane of the velocity $u(x,y,z,t)$ at $y = 0$ and $t = 300$ for three selected models of the initial velocity field. Marks (a), (b) and (c) have the same meaning as in Fig. 5.38.

(d)-(f) Contours in the (x,y)-plane of the vertical velocity $w(x,y,z,t)$ at $z=1$ and $t = 300$ for three selected models of the initial velocity field. The marks (d), (e) and (f) correspond the first, second and third models.

Given results represent computations at $Re^* = 950$ and such amplitude A that $\max_x|w(x,0)| = 10^{-5}$. All dimensional quantities are made dimensionless by scales δ^* and U_0.

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Figure 5.4. Treshold amplitude A of the plane T-S wave with streamwise wavenumber k_1 in a Blasius boundary layer for the onset of three-dimensionality with spanwise wavenumber k_2 [after Massev (1968b)]. Curve 1: Re* = 1203, k_1 = 0.43; curve 2: Re* = 519, k_1 = 0.27. All the quantities are non-dimensionalized by scales δ^* and U_0.

Figure 5.5. Examples of the amplitude spectra $S_u(f)$ (presented in lenear scale) of streamwise-velocity fluctuations $u(t)$ in a laboratory flat-plate boundary layer disturbed by a ribbon vibrating with frequency $f_0 = 2\pi\omega_0$.

(a) Typical spectrum $S_u(f)$ measured by Kachanov, Kozlov and Levchenko (1977) [after Kachanov (1994a)]. Peaks denoted as $f_1, f_{1/2}$, $3f_{1/2}$, $2f_1$, $5f_{1/2}$ and $3f_1$ correspond to frequencies f_0, $f_0/2$, $3f_0/2$, $2f_0$, $5f_0/2$ and $3f_0$.

(b) Spectra $S_u(f)$ measured inside a boundary layer at two values of the ribbon frequency f_0 and coordinate x (measured from the leading edge of a plate) but fixed values of y and z : 1. f_0 = 96.4 Hz ($F_0 = 2\pi f_0 \nu/U_0^2$ = 109×10^{-6}), x = 600 mm (Re = $(U_0x/\nu)^{1/2}$ = 608); 2. f_0 = 111.4 Hz (F_0 = 124×10^{-6}), x = 640 mm (Re = 633) [after Kachanov and Levchenko (1984)].

Figure 5.6. Dependence of the dimensionless amplitudes $A = u'/U_0$ of the primary plane wave (1) and subharmonic oblique waves (2) on the coordinate x [and Re = $(U_0x/\nu)^{1/2}$] at y = -2.5 mm, z/δ = 0.26 (where δ is the boundary-layer thickness), according to measurements by Kachanov and Levchenko (1984).

Figure 5.7. Measured dependence of phases ϕ_1 and $\phi_{1/2}$, and amplitudes A_1 and $A_{1/2}$ of the primary plane wave (1) and subharmonic waves (2) on the spanwise coordinate y [after Kachanov and Levchenko (1984)].

Figure 5.8. (a) Regular system of Λ-vortices typical for the K-regime of disturbance development in a boundary layer.
(b) Staggered system of Λ-vortices typical for the N-regime of disturbance development. The figures show flow streaklines appearing when the disturbed flow is visualized by smoke [after Herbert *et al.* (1987)].

Figure 5.9. Spanwise distributions of phases ϕ (a) and maximum (with respect to z) amplitudes A (b) for the primary plane wave of frequency f (o) and subharmonic oblique waves of frequency $f/2$ (•) for the case 3 of Corke and Mangano's measurements [after Corke and Mangano (1989)].

Figure 5.10. Streamwise developments of maximum amplitudes of streamwise velocity fluctuations for (a) subharmonic waves of frequency $f/2$, and (b) primary waves of frequency f in the cases 1, 2, and 3 [after Corke and Mangano (1989)].

Figure 5.11. Streamwise development of maximum amplitudes of artificially exited plane wave of frequency 36 Hz and oblique wave of frequency 16 Hz, together with development of the produced by their nonlinear interaction 3D wave of frequency 20 Hz [after Corke (1995)].

Figure 5.12. Streamwise development of a number of waves produced by nonlinear interactions of waves from 'detuned resonance triad' artificially excited by Corke [after Corke (1995)].

Figure 5.13. Dependence of the amplification rate $G_0 = dA(x)/Adx$ of the oblique-wave amplitude $A \equiv A_2 = A_3$ of a resonant wave triad on $K_2 = k_2 \nu / U_0$ (and k_2/k_1) for different values of the plane wave amplitude A_1 [and $F_1 \equiv \omega_1 \nu / U_0^2 = 115\times10^{-6}$, $\mathrm{Re}^+ = (U_0 x/\nu)^{1/2} = 640$]. All dimensional quantities are non-dimensionalized by scales $\delta^+ = (\nu x/U_0)^{1/2}$ and U_0. Curves 1, 2, ..., 6 correspond to A_1 = 0.14, 0.21, 0.28, 0.40, 0.53, 0.72%, dotted line shows the dependence of optimal values $(k_2/k_1)_{pr}$ and $(K_2)_{pr}$ on A_1 [after Zel'man and Maslennikova (1953a) and Kachanov (1994a)].

Figure 5.14. Dependence of the the value of $(k_2/k_1)_{pr}$ corresponding to the most amplified oblique subharmonics of the plane wave on the plane-wave amplitude A_1. Experimental points correspond to laboratory observations: 1 - of Kachanov and Levchenko (1982,1984); 2, 3 - of Saric *et al.* (1984); 4 - of Saric and Thomas (1984) [after Zel'man and Maslennikova (1993a) and Kachanov (1994a)].

Figure 5.15. (a) Comparison of the measured by Corke and Mangano (1989) maximal values A_{max} of the oblique-wave amplitudes $A_2 = A_3$ in three studied cases with the theoretical dependence of A_{max} on k_2/k_1 which follows from the application to the their experiments results of the secondary-instability theory developed by Herbert (1983b,1988a) and Herbert and Bertolotti (1985) [after Corke and Mangano (1989)].

(b) Comparison of the maximal oblique-wave growth rate G_{max} observed by Corke and Mangano in three cases studied in their experiments with theoretical estimate of the dependence of G_{max} on k_2/k_1 following from Mankbadi's theory of critical-layer nonlinearity [after Mankbadi (1993a)].

(c) Comparison of Mankbadi's theoretical estimate of the dependence of G_{max} on k_2/k_1 with the corresponding theoretical estimate by Herbert (1988a) and results of numerical simulation by Spalart and Yang (1987) of disturbance development in a boundary layer with a vibtrating ribbon in it; for $F \equiv \omega\nu/U_0^2 = 58.8\times10^{-5}$ and initial conditions $A_1(0) = 1.4\%$ and $\mathrm{Re}^+(0) = 950$ [after Mankbadi (1993a)].

Figure 5.16. Measured (*points*) and calculated (*curves*) vertical profiles of the amplitude $A_{1/2}(z)$ (*left*) and the phase $\phi_{1/2}(z)$ (*right*) of the subharmonic 3D wave of frequency $\omega/2$ resonantly amplified in the N-regime of instability development in a Blasius boundary layer. Experimental data by Kachanov and Levchenko (1982). Calculations: 1 - secondary-instability theory of Herbert (1984a); 2 - numerical simulation of Fasel *et al.* (1989); 3 - resonant-triad theory of Zel'man and Maslennikova (1989,1990, 1993a) [after Kachanov (1994a)].

Figure 5.17. (a) Resonant streamwise amplifications of the plane wave amplitude $A_1(x)$ (results 1, 2, 3) and of the amplitude $A_{1/2}(x)$ of the two subharmonic waves of twice smaller frequency (results 4, 5, 6) during the initial stage of the N-regime of instability development. Experimental data (points 1 and 4) by Kachanov and Levchenko (1982); calculations (curves): 2 and 5 - Herbert's (1884a) theory; 3 and 6 - numerical simulations by Fasel *et al.* (1989).

(b) and (c) The same resonant amplifications at more late stages of the N-regime when values of $A_{1/2}(x)$ (2, 2' and 5) overtake those of $A_1(x)$ (1, 1', 3 and 4); $\mathrm{Re}=(xU_0/\nu)^{1/2} \propto x^{1/2}$. (b): expereimental data (points 1' and 2') by Saric *et al.* (1984), calculated curves 1 and 2 - theory by Maslennikova and Zel'man (1985) and Zel'man and

Maslennikova (1993a); dotted curve 3 - theory taking non-parallelism into account. (c): experimental data (points 4 and 5) by Corke and Mangano (1989); theoretical calculations (curves) by Crouch and Herbert (1993) [all figures after Kachanov (1994a)].

Figure 5.18. Downstream-growth curves for amplitudes of five-wave disturbance system in a Blasius boundary layer. The system includes the plane wave 1 of frequency ω and wave vector $\{k, 0\}$ and oblique-wave pairs 2-3 and 4-5 with frequency-wave vector values $\{\omega/2, k_1, \pm k_2\}$ and $\{\omega/2, k_1^*, \pm k_2^*\}$ where $F = \omega\nu/U_0^2 = 230\times10^{-6}$, $K_2 = k_2\nu/U_0 = 0.171\times10^{-3}$, $K_2^* = k_2^*\nu/U_0 = 0.15\times10^{-3}$; $\mathrm{Re} = (U_0x/\nu)^{1/2}$ [after Zel'man and Maslennikova (1993a)].

Figure 5.19. Amplification curves for seven-wave system including the primary plane wave 0 with frequency-wave vector (f-w) combination $\{\omega_0, k, 0\}$, a pair of secondary oblique waves 1-2 with f-w combination $\{\omega_0/2, k_1, \pm k_2\}$ and two pairs of tertiary oblique waves 3-4 and 5-6 with f-w combinations $\{\omega_0/4, k_1', \pm k_2'\}$ and $\{\omega_0/4, k_1'', \pm k_2''\}$. Here $F_0 = \omega_0\nu/U_0^2 = 122\times10^{-6}$, $k_2/k_1 = 2$, $k_2'/k_1' = 2.8$, $k_2''/k_1'' = 3.44$, $\mathrm{Re}^* = (U_0\delta^*/\nu)^{1/2}$ [after Zel'man and Maslennikova (1993b)].

Figure 5.20. Downstream growth of spanwise modulation of the amplitude u' of streamwise disturbance velocity in a boundary layer disturbed by vibrating ribbon.

o - data for $x = 7.6$ cm; Δ - for $x = 15.2$ cm; $\times$ - for $x = 19$cm, where x is measured from the trailing edge of the ribbon; 1 - modulation 'peaks', 2 - 'valleys' [after Klebanoff *et al.* (1962)].

Figure 5.21. Typical single and double spikes in a boundary layer flows [after Klebanoff *et al.* (1962)]. 1 - 1st spike; 2 - 2nd spike; T - fundamental period of spike repetitions.

Figure 5.22. Amplification of total intensity of streamwise disturbance velocity $u(\mathbf{x},t)$ and amplitudes of its harmonics with frequencies ω_1, ω_2, ... , ω_6 (symbols and curves1, 2, ... ,6; ω_1 - fundamental frequency of primary T-S wave, $\omega_n = n\omega_1$) observed at y corresponding to peak position of spanwise modulation, fixed value of z and variable x-coordinate. Streamwise intervals 1s, 2s, 3s - places of formation of the 1st, 2nd, and 3rd spike [after Kachanov *et al.* (1984) and Kachanov (1994a)].

Figure 5.23. Vertical (*left*) and spanwise (*right*) profiles of the amplitudes A_i (*bottom*) and phases ϕ_i (*top*) of streamwise-velocity harmonics with frequencies ω_1, ω_2, ... , ω_{10} measured at the stage of developed spikes (curves 1, 2, ... , 10) [after Borodulin and Kachanov (1992) and Kachanov (1994a)]. Mean-velocity profile is added to vertical amplitude profiles to show the boundary-layer thickness.

Figure 5.24. Streamwise development of values of amplitudes $A_{n,,m}$ of (n,m)-Fourier components of velocity $u(\mathbf{x},t)$ at the heights z where these amplitudes takes maximal values [after Rist and Fasel (1995)].

(n,m)-Fourier component corresponds to frequency $\omega_n = n\omega_1$ and spanwise wavenumber $k_{2,m} = mk_0$ (where ω_1 - fundamental frequency of primary wave, k_0 - wavenumber of the fundamental spanwise periodicity (0,1) shown in Fig. 5.21). Symbols (0,0) and (1,0) correspond to amplitudes of the 'mean flow correction' and 'primary 2D wave'.

Figure 5.25. Two examples of Λ-vortices appearing at two time instants in the numerical simulation of unstability developments in a flat-plate boundary layer by Rist and Fasel. Shown here three-dimensional Λ-shaped structures are bounded by surfaces $|\eta_x|$ = constant where η_x is the x -component of flow vorticity; λ_y is the primary spanwise wavelength clearly seen in Fig. 5.20 [after Rist and Fasel (1995)].

Figure 5.26. Comparison of 'spike signal' in the (u', t, y)-space (u' is the stremwise disturbance velocity measured in parts of U_0) appearing in numerical simulation of boundary-layer instability development by Rist and Kloker at x = 500 mm, z = 8 mm (a) with the 'spike signal' observed at the same values of x and z in the corresponding laboratory experiment of Kachanov and Borodulin (b) [after Rist and Kachanov (1995)].

Figure 5.27. (a) Dependence of the dimensionless height z_s/δ (where δ is the boundary-layer thickness) of the center of an appearing spike on streamwise coordinate x during spike downstream evolution. (b) Dependence on x of the streamwise velocity c_s of a spike [after Borodulin and Kachanov (1994)].

Figure 5.28. Schematic form of a family of Ryzhov's three-parameter soliton solution of the Benjamin-Ono equation (5.16) corresponding to various values of the amplitude parameter Δ and fixed values of other two parameters (determining scales of the dependence of A on t and x). Here $A_0 = T^{-1}\int_0^T A(t)dt$ is the mean amplitude, A^-, A^+ and τ_0 are some numerical form characteristics and T is fundamental spike period [after Kachanov, Ryzhov and Smith (1993)].

Figure 5.29. Experimental (*points*) and theoretical (*curves*) dependencies of soliton form characteristics A^-, A^+ and τ_0 on the soliton magnitude $A_m = (A^+ + A^-)/2$. Theoretical curves correspond to shown in Fig. 5.28 solution of B-O equation with appropriately chosen parameter values; points - data of Borodulin and Kachanov (1988) [after Kachanov, Ryzhov and Smith (1993)].

Figure 5.30. The dependence on t of the growth curve $G(t)$ for the energy of plane-wave disturbances with $k_1 = 1$ [all physical quantities are non-dimensionalized by scales $H_1 = H/2$ and $U_0 = U(H/2)$] in a plane Poiseuille flow between walls at $z = 0$ and $z = H$. Here the curve labelled 'Unstable' corresponds to the 'optimal' 2D wave having the greatest transient growth in the linearly unstable Poiseuille flow with $\mathrm{Re} = U_0H_1/\nu = 8000$, the 'Stable' curve corresponds to the optimal 2D wave in the linearly stable Poiseuille flow with Re = 5000, while 'Modal' curve shows the growth of the unstable solution of the Orr-Sommerfeld eigenvalue problem with $k_1 = 1$ and Re = 8000. In all cases it is assumed that the initial energy at $t = 0$ is equal to 1 [after Reddy and Henningson (1993)].

Figure 5.31. (a) The dependence of the energies E of a number of (n,m)-Fourier components [i.e., waves with frequencies and spanwise wavenumbers $(n\omega_0, mk_{2,0})$] on x in a Blasius boundary layer with a pair of oblique waves with frequencies and spanwise wavenumbers $(\omega_0, \pm k_{2,0})$. (a) Dependence of E on $(x - x_0)/\delta^*$ in the case when the initial energy of (1,1)-mode is equal to 1 [after Berlin *et al.* (1994)]. (b) Dependence of E (measured in some conventional units) on x (in mm); coordinate x_0 of the 'wave generator' is here close to 186 mm [after Berlin *et al.* (1999)].

Figure 5.32. Computed energy-growth curves for a number of (n,m)-waves with streamwise and spanwise wavenumbers $(nk_{1,0}, mk_{2,0})$ in a boundary layer disturbed by 'primary oblique waves' with wavenumbers $(k_{1,0}, \pm k_{2,0})$ and a 'weak noise' consisting of supplementary small-amplitude (n,m)-waves with n = 0, 1, 2, and m = 0, 1, 2 [after Schmid and Henningson (2000)].

Figure 5.33. Dependence of the transition time T_0 on the disturbance initial energy E_0 (which is proportional to A_0^2) for three transiton scenarios: i) 'oblique transition' (labelled 'oblique waves'), ii) 'TS-wave-secondary-instability scenario' (K-regime, labelled 'TS waves'), and iii) 'streak-breakdown scenario' (labelled 'vortices' since streaks are produced by streamwise vortices). The results are for temporally growing Blasius boundary layer with initial Re* = 500 [after Schmid *et al.* (1996)].

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