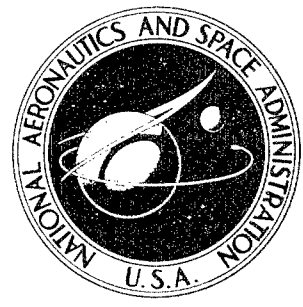


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# TABLES OF NATURAL FREQUENCIES AND NODES FOR TRANSVERSE VIBRATION OF TAPERED BEAMS

*by Han-chung Wang and Will J. Worley*

Prepared under Grant No. NsG-434 by

UNIVERSITY OF ILLINOIS

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Urbana, Illinois

SUMMARY

The natural frequencies, nodal points and mode functions for transverse vibration of tapered beams are presented in this report.

The beams considered have the cross-sectional area bounded by the curve

$$\left| \frac{y}{h} \right|^\beta + \left| \frac{z}{b} \right|^\gamma = 1$$

with the thickness  $h$  and width  $b$  varying along the beam according to the relations

$$h = h_0 \left( \frac{X}{L} \right)^\phi \quad \text{and} \quad b = b_0 \left( \frac{X}{L} \right)^\psi$$

where  $\beta$ ,  $\gamma$ ,  $\phi$  and  $\psi$  are positive constants not necessarily integers. 1  $\rightarrow$  pg 2

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## INTRODUCTION

### 1. Statement of the Problem

The geometrical properties of solid bodies generated by revolving the line defined as

$$\left| \frac{X}{l} \right|^\alpha + \left| \frac{y}{h} \right|^\beta = 1 \quad (4.1)^*$$

and of bodies bounded by the surface defined as

$$\left| \frac{X}{l} \right|^\alpha + \left| \frac{y}{h} \right|^\beta + \left| \frac{z}{b} \right|^\gamma = 1 \quad (4.2)$$

have been published in the first and second reports [1, 2]\*\* under this grant. The current report treats the dynamic response of a general class of tapered beams. The shape of beams includes the bodies generated by Eqs. (4.1) and (4.2) with the parameter  $\alpha = 1$ . Special cases of bodies of the above shapes have been tested widely in NACA and NASA reports as well as in the technical journals, [3, 4, 5, 6, 7]. Beams of special shapes also have been applied as designs for high speed machine guns as reported in a recent paper by Elder [8].

To achieve more nearly complete information on the applications of this class of bodies, the results for the mode functions and for the natural frequencies of vibration of tapered beams are presented in this report. This report includes much of the existing data as special cases. The results appear in tables and in graphical form. The complete and detailed derivations are reported by Wang [9].

### 2. Profiles of the Beams

The beams whose flexural rigidity and mass per unit length vary according to two arbitrary powers of the longitudinal coordinate are considered in this report. The relationship of the variations may be written as

$$\begin{aligned} EI &= E_0 I_0 \left( \frac{X}{l} \right)^m \\ \rho A &= \rho_0 A_0 \left( \frac{X}{l} \right)^n \end{aligned} \quad (4.3)$$

\* The notation (4.1) is adopted to aid in cross-referencing equations from the first three reports under this grant.

\*\* Number in brackets refer to the References.

where  $E_0 I_0$  is the bending rigidity and  $\rho_0 A_0$  is the mass per unit length at the larger end of the beam where  $X = \ell$ .

The relations of Eq. (4.3) can be considered as a homogeneous beam with the moment of inertia and cross-sectional area varying with powers  $m$  and  $n$  respectively. The relations can be applied for a general class of cross-sections with varying thickness and varying width. An important group of beam shapes can be considered as shown in Fig. 1. The cross-section of the beam is symmetrical and its first quadrant is bounded by the curve of the equation

$$\left(\frac{Y}{h}\right)^\beta + \left(\frac{Z}{b}\right)^\gamma = 1 \quad (4.4)$$

where  $b$  represents half of the width and  $h$  represents half of the thickness of the beam. These parameters vary according to the relations

$$b = b_0 \left(\frac{X}{\ell}\right)^\psi, \quad h = h_0 \left(\frac{X}{\ell}\right)^\phi \quad (4.5)$$

The constants  $\psi$  and  $\phi$  are positive but not necessarily integers.

The selection of different values for the parameters  $\gamma$  and  $\beta$  in Eq. (4.4) permits the cross-section of the beam to be varied from the diamond shape,  $\gamma = \beta = 1$ , through the elliptical shape,  $\gamma = \beta = 2$ , to the rectangular shape,  $\gamma$  and  $\beta \gg 1$ . The moment of inertia and the area for this group of cross-section may be expressed in terms of  $\gamma$  and  $\beta$  [10], which gives

$$I = \frac{4}{3} b_0 h_0^3 \left[ \frac{\Gamma\left(\frac{1}{\gamma} + 1\right) \Gamma\left(\frac{3}{\beta} + 1\right)}{\Gamma\left(\frac{1}{\gamma} + \frac{3}{\beta} + 1\right)} \right] \left(\frac{X}{\ell}\right)^{\psi + 3\phi} \quad (4.6)$$

$$A = 4b_0 h_0 \left[ \frac{\Gamma\left(\frac{1}{\gamma} + 1\right) \Gamma\left(\frac{1}{\beta} + 1\right)}{\Gamma\left(\frac{1}{\gamma} + \frac{1}{\beta} + 1\right)} \right] \left(\frac{X}{\ell}\right)^{\psi + \phi}$$

Comparison of Eqs. (4.6) with Eqs. (4.3) yields the relationships

$$m = \psi + 3\phi, \quad n = \psi + \phi$$

The beam described by Eqs. (4.3) can also be considered as a nonhomogeneous beam with uniform cross-section, provided the modulus of elasticity and the density vary as powers  $m$  and  $n$ . The longitudinal vibration of beams of this type has been treated by Lindholm and Doshi [11] .

### 3. Symbols

$X$	horizontal coordinate along the length of the beam
$y$	vertical coordinate perpendicular to $X$
$z$	horizontal coordinate perpendicular to $X$
$\ell$	reference length of the beam
$L$	actual beam length
$b$	half the width of the beam
$h$	half the thickness of the beam
$b_0$	the width of the beam at the larger end
$h_0$	the thickness of the beam at the larger end
$x$	dimensionless abscissa, $X/\ell$
$\alpha$	exponent of $(X/\ell)$
$\beta$	exponent of $(y/h)$
$\gamma$	exponent of $(z/b)$
$\psi$	exponent of $x$ for width variation
$\phi$	exponent of $x$ for thickness variation
$m$	exponent of $x$ for area moment of inertia variation
$n$	exponent of $x$ for cross-sectional area variation
$EI$	bending rigidity of the beam (elastic modulus times moment of inertia of the cross-section)
$\rho A$	mass per unit length of the beam
$\theta$	$= 4 - m + n$
$u$	$x^\theta$
$\delta_x$	$x \frac{d}{dx}$
$\delta_u$	$u \frac{d}{du}$
$\omega$	circular frequency
$p$	eigenvalue, $\rho A_0 \ell^4 \omega^2 / (EI_0)$
$s$	indicial root
${}_0F_3$	generalized hypergeometric function
$y(x)$	mode function

$U(x, \xi)$	influence function for beam deflections
$K(x, \xi)$	kernel of homogeneous integral equations
$c$	truncation of the beam, dimensionless; see Fig. 1
$a_r$	coefficients of the series
$P_u Q_v$	$\delta^u P \delta^v Q - \delta^v P \delta^u Q$
$r_g$	radius of gyration

#### 4. Acknowledgement

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## SOLUTIONS OF MODE FUNCTIONS

Neglecting rotatory inertia, the differential equation for the mode functions of vibrating beams is given as

$$\frac{d^2}{dX^2} EI(X) \frac{d^2 y}{dX^2} - \rho \omega^2 A(X) y = 0 \quad (4.7)$$

For a homogeneous beam, having the profile described by the relations

$$I = I_0 \left(\frac{X}{\ell}\right)^m \quad \text{and} \quad A = A_0 \left(\frac{X}{\ell}\right)^n \quad (4.8)$$

Eq. (4.7) can be expanded as

$$x^4 \frac{d^4 y}{dx^4} + 2mx^3 \frac{d^3 y}{dx^3} + m(m-1)x^2 \frac{d^2 y}{dx^2} - px^\theta y = 0 \quad (4.9)$$

where  $x = (X/\ell)$ ,  $p = \rho A_0 \ell^4 \omega^2 / (EI_0)$  and  $\theta = 4 - m + n$ .

Upon introducing the differential operator  $\delta_x$  to represent  $x \frac{d}{dx}$ , Eq. (4.9) may be written as

$$\delta_x (\delta_x - 1) (\delta_x + m - 2) (\delta_x + m - 3) y - px^\theta y = 0 \quad (4.10)$$

Let  $u = x^\theta$  and let  $\delta_u = u \frac{d}{du}$ , thus  $\delta_x^r y = \theta^r \delta_u^r y$

and Eq. (4.10) yields

$$\delta_u (\delta_u - \frac{1}{\theta}) (\delta_u + \frac{m-2}{\theta}) (\delta_u + \frac{m-3}{\theta}) y - \frac{p}{\theta^4} uy = 0 \quad (4.11)$$

Equation (4.11) is a type of generalized hypergeometric equation [12], which possesses a general solution with linear combinations of four generalized hypergeometric functions. It can be written as

$$y = A_1 y_0 + A_2 y_1 + A_3 y_{2-m} + A_4 y_{3-m} \quad (4.12)$$

where each series, in hypergeometric series notation, is defined as

$$\begin{aligned}
 y_0 &= {}_0F_3 \left( -; 1 - \frac{1}{\theta}, 1 + \frac{m-2}{\theta}, 1 + \frac{m-3}{\theta}; \frac{p}{\theta^4} u \right) \\
 y_1 &= u^{\frac{1}{\theta}} {}_0F_3 \left( -; 1 + \frac{1}{\theta}, 1 + \frac{m-1}{\theta}, 1 + \frac{m-2}{\theta}; \frac{p}{\theta^4} u \right) \\
 y_{2-m} &= u^{\frac{2-m}{\theta}} {}_0F_3 \left( -; 1 - \frac{1}{\theta}, 1 - \frac{m-1}{\theta}, 1 - \frac{m-2}{\theta}; \frac{p}{\theta^4} u \right) \\
 y_{3-m} &= u^{\frac{3-m}{\theta}} {}_0F_3 \left( -; 1 + \frac{1}{\theta}, 1 - \frac{m-2}{\theta}, 1 - \frac{m-3}{\theta}; \frac{p}{\theta^4} u \right)
 \end{aligned} \tag{4.13}$$

provided that  $m$  is not an integer. When  $m$  is an integer, then logarithmic terms appear in the general solution.

The solutions of the differential equation (4.10) also can be obtained by the standard series method, the method of Frobenius. The series in Eq. (4.12) will have the form

$$y_s = x^s \sum_{r=0}^{\infty} a_r x^{r\theta} \tag{4.14}$$

with the recurrence formular of the coefficients being

$$a_r = \frac{p}{(s+r\theta)(s+r\theta-1)(s+r\theta+m-2)(s+r\theta+m-3)} a_{r-1} \tag{4.15}$$

where  $s = 0, 1, 2-m$  and  $3-m$  are the roots of the indicial equation.

When the parameter  $m$  is an integer, then two equal roots of  $s$  may exist for the indicial equation or the denominator of the recurrence formula of Eq. (4.15) becomes zero. When equal roots occur, the series solutions of Eq. (4.13) are dependent on each other. When the denominator of the recurrence formula is zero, the series solutions are meaningless. In these cases, by applying the theorems due to Frobenius [13], the independent series solution for the repeated root,  $s = s_0$ ,

$$\begin{aligned}
 \text{is } y_{s_0} &= \log x \cdot (y)_{s=s_0} \\
 &- x^{s_0} \sum_{r=1}^{\infty} \left[ \sum_{\eta=1}^r \left( \frac{1}{s_0 + \eta\theta} + \frac{1}{s_0 + \eta\theta - 1} + \frac{1}{s_0 + \eta\theta + m - 2} + \frac{1}{s_0 + \eta\theta + m - 3} \right) \right] a_r x^{r\theta} \tag{4.16}
 \end{aligned}$$

The series solution for the root,  $s = s_1$ , which makes the recurrence formula of the coefficients undefined, is given by

$$y_{s_1} = \log x \cdot (y)_{s=s_1} + x^{s_1} \left\{ 1 + \sum_{r=1}^{\infty} \left[ \frac{1}{s-s_1} - \sum_{\eta=1}^r \left( \frac{1}{s+\eta\theta} + \frac{1}{s+\eta\theta-1} + \frac{1}{s+\eta\theta+m-2} + \frac{1}{s+\eta\theta+m-3} \right) \right]_{s=s_1} a_r x^{r\theta} \right\} \quad (4.17)$$

Again, when  $\theta = 0$ , the series of Eq. (4.14), does not apply. For this case, the recurrence formula of the coefficients is again undefined; however, Eq. (4.10) is then of the Euler-Cauchy type for which a general solution always exists of the form

$$y = A_1 x^{s_1} + A_2 x^{s_2} + A_3 x^{s_3} + A_4 x^{s_4} \quad (4.18)$$

where  $s_1, s_2, s_3,$  and  $s_4$  are the roots of the auxiliary equation

$$s(s-1)(s+m-2)(s+m-3) - p = 0 \quad (4.19)$$

Vibration can occur only when Eq. (4.19) has two or four non-real roots. This condition introduces the trigonometric function of  $\log x$  into the solution for  $y$ .

Suppose  $s_1, s_2$  are two real roots and  $s_3, s_4$  are two complex roots represented by  $\alpha_1 \pm i\alpha_2$ , then the solution for the normal function has the form

$$y = A_1 x^{s_1} + A_2 x^{s_2} + x^{\alpha_1} \left[ A_3 \cos(\alpha_2 \log x) + A_4 \sin(\alpha_2 \log x) \right] \quad (4.20)$$

## FREQUENCY EQUATIONS AND NODAL POINTS

To establish the frequency equations for tapered beams with different end conditions, two separate geometrical categories of beams are treated. First, the complete tapered beam, which is gradually narrowed toward a point, is considered. The end coordinates of the beam are 0 and  $l$ . Next, the truncated beam, with end coordinates  $c \cdot l$  and  $l$ , as indicated in Fig. 1, is considered. The origin, for the truncated beam lies beyond the end of the beam and serves as a reference point for the variation of the area moment of inertia and for the cross-sectional area of the beam.

### 1. Complete Tapered Beams

Since one end of the beam is at the origin, the arbitrary constants  $A_3$  and  $A_4$  in the general solution, Eq. (4.12), must vanish if finite values exist for the deflection, moment and shear at the end  $x = 0$ . Hence, the general solution of the mode function for beams with any combination of parameters  $m$  and  $n$  can be written as

$$y = A_1 P + A_2 Q \quad (4.21)$$

where  $P$  and  $Q$  are the series of  $y_0$  and  $y_1$  as defined in Eq. (4.13). The frequency equation is then obtained using the boundary conditions at  $X = l$ , that is, at  $x = 1$ . Three different cases are considered below.

#### A. Cantilever Beams

The base of the beam is fixed and the tip is free. Both the deflection and slope vanish at  $x = 1$ . Application of the boundary conditions to Eq. (4.21) and elimination of the constants  $A_1$  and  $A_2$  yields the frequency equation

$$P \frac{dQ}{dx} - Q \frac{dP}{dx} = 0, \quad \text{at } x = 1 \quad (4.22)$$

Representation of  $\delta^u P \cdot \delta^v Q - \delta^v P \cdot \delta^u Q$  in symbolic form as  $P_u Q_v$  yields the relations  $P_u Q_v = -P_v Q_u$  and  $\delta(P_u Q_{u+1}) = P_u Q_{u+2}$ . Using this notation, the frequency equation yields

$$\frac{1}{x} P Q_1 = 0, \quad \text{at } x = 1 \quad (4.23)$$

Since P and Q are in series form, as in Eq. (4.14), their derivatives and products are in series form as well.

Let the series be defined as  $U = PQ_1$ . Then U is a solution of a fifth order differential equation

$$(\delta-1)(\delta+m-2)(\delta+m-3)(\delta+m-4)(\delta+2m-5)U + 2(2\delta+\theta+2m-6)px^\theta U = 0 \quad (4.24)$$

Solving Eq. (4.24) and comparing the corresponding terms of U with Eq. (4.23) the series which satisfies both Eq. (4.24) and (4.23) is established as

$$U = \sum_{r=0}^{\infty} a_r p^r x^{1+r\theta} \quad (4.25)$$

Substitution of U into Eq. (4.23) yields the frequency equation as a polynomial of p

$$\sum_{r=0}^{\infty} a_r p^r = 0 \quad (4.26)$$

with the recurrence formula of coefficients

$$a_r = \frac{-2(2r\theta - \theta + 2m - 4)}{r\theta(r\theta+m-1)(r\theta+m-2)(r\theta+m-3)(r\theta+2m-4)} a_{r-1} \quad (4.27)$$

#### B. Antisymmetrical Mode Vibration of Free-Free Beams

Tapered beams with both ends free are considered to consist of two equal halves fitted together at their large ends as shown in Fig. 2. Each half is a section of a solid which has the profile shown in Fig. 1. When the beam vibrates in the antisymmetrical mode, the deflection and the bending moment are zero at the middle section where the two halves are joined together, that is

$$y = 0 \quad \text{and} \quad EI(x) \frac{d^2 y}{dx^2} = 0 \quad \text{at } x = 1$$

Substitution of these conditions into Eq. (4.21) and applying the differential operator  $P_u Q_v$ , the frequency equation becomes

$$\frac{1}{x^2} U = 0 \quad \text{at } x = 1 \quad (4.28)$$

where  $U = PQ_2 - PQ_1$ . The function U is a solution of a fifth order differential

equation

$$(\delta - \theta - 1)(\delta + m - 4)(\delta + m - 3)(\delta + 2m - 5)(\delta + m - 2)U + 2(2\delta + \theta + 2m - 6)px^\theta U = 0 \quad (4.29)$$

Hence, the frequency equation may again be represented by a polynomial of  $p$ , of the form of Eq. (4.26), with the coefficients

$$a_r = \frac{-2(2r\theta + \theta + 2m - 4)}{r\theta(r\theta + \theta + m - 1)(r\theta + \theta + m - 2)(r\theta + \theta + m - 3)(r\theta + \theta + 2m - 4)} a_{r-1} \quad (4.30)$$

The locations of the nodes for the normal mode vibration, can be obtained by substituting the corresponding natural frequency,  $p_i$ , into Eq. (4.21) and solving for  $x$  with  $y$  equal to zero. Since  $P$  and  $Q$  are generalized hypergeometric functions, the equation, which gives the nodal points for cantilever beams and free-free beams of antisymmetrical mode, can be represented by

$$\begin{aligned} & {}_0F_3\left(-; 1 + \frac{1}{\theta}, 1 + \frac{m-1}{\theta}, 1 + \frac{m-2}{\theta}; \frac{p_i}{\theta^4}\right) {}_0F_3\left(-; 1 - \frac{1}{\theta}, 1 + \frac{m-2}{\theta}, 1 + \frac{m-3}{\theta}; p_i \frac{x^\theta}{\theta^4}\right) \\ & - x {}_0F_3\left(-; 1 - \frac{1}{\theta}, 1 + \frac{m-2}{\theta}, 1 + \frac{m-3}{\theta}; \frac{p_i}{\theta^4}\right) {}_0F_3\left(-; 1 + \frac{1}{\theta}, 1 + \frac{m-1}{\theta}, 1 + \frac{m-2}{\theta}; p_i \frac{x^\theta}{\theta^4}\right) = 0 \end{aligned} \quad (4.31)$$

### C. Symmetrical Mode Vibration of Free-Free Beams

For the symmetrical mode vibration of the free-free beam, Fig. 2, the slope and shear are zero at the middle section of the beam, hence

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{d}{dx} EI(x) \frac{d^2 y}{dx^2} = 0 \quad \text{at } x = 1$$

Upon substituting the end conditions into Eq. (4.21) and using the notation  $P_u Q_v$ , the frequency equation may be written as

$$\frac{d}{dx} (x^{m-3} U) = 0 \quad (4.32)$$

at  $x = 1$ , where

$$U = PQ_3 + (m-3)PQ_2 - (m-2)PQ_1$$

As discussed in the previous sections, U may be represented by a series which is a solution of a sixth order differential equation

$$(\delta - \theta - 1)(\delta - \theta + m - 2)(\delta - \theta + m - 3)(\delta + m - 4)(\delta + m - 3)(\delta + 2m - 5)U + 2(2\delta + \theta + 2m - 6)(\delta + m - 3)px^\theta U = 0 \quad (4.33)$$

Substituting the solution U into Eq. (4.32) and evaluating at  $x = 1$ , gives the equation for the eigenvalues p as

$$\sum_{r=0}^{\infty} (r\theta + \theta + m - 2)a_r p^r = 0 \quad (4.34)$$

with the recurrence formulas for the coefficients

$$a_r = \frac{-2(2r\theta + \theta + 2m - 4)}{r\theta(r\theta + m - 1)(r\theta + \theta + m - 2)(r\theta + \theta + m - 3)(r\theta + \theta + 2m - 4)} a_{r-1} \quad (4.35)$$

The locations of the nodes for the normal mode vibration in this case can be solved from the equation

$$\left[ \frac{P_i}{(\theta + 1)(\theta + m - 1)(\theta + m - 2)} {}_0F_3 \left( -; 2 + \frac{1}{\theta}, 2 + \frac{m-1}{\theta}, 2 + \frac{m-2}{\theta}; \frac{P_i}{\theta^4} \right) + {}_0F_3 \left( -; 1 + \frac{1}{\theta}, 1 + \frac{m-1}{\theta}, 1 + \frac{m-2}{\theta}; \frac{P_i}{\theta^4} \right) \right] {}_0F_3 \left( -; 1 - \frac{1}{\theta}, 1 + \frac{m-2}{\theta}, 1 + \frac{m-3}{\theta}; P_i \frac{x^\theta}{\theta^4} \right) - \left[ \frac{P_i x^\theta}{(\theta - 1)(\theta + m - 2)(\theta + m - 3)} {}_0F_3 \left( -; 1 - \frac{1}{\theta}, 1 + \frac{m-2}{\theta}, 1 + \frac{m-3}{\theta}; \frac{P_i}{\theta^4} \right) + {}_0F_3 \left( -; 1 + \frac{1}{\theta}, 1 + \frac{m-1}{\theta}, 1 + \frac{m-2}{\theta}; P_i \frac{x^\theta}{\theta^4} \right) \right] = 0 \quad (4.36)$$

## 2. Truncated Tapered Beams

### A. Exact Solutions

A beam which is gradually reduced to a small cross-section instead of to a point, may be considered as a beam truncated at the location  $x = c$  as shown in Fig. 1. The total length of the beam is  $(1 - c) \cdot l$ , where  $l$  is a reference length for the tapering. Since the beam does not start from the origin, the general solution of the mode function must be written in the form of Eq. (4.12) with four series. By substituting the end conditions into the mode function, a fourth order determinant equation of the eigenvalues  $p$  may be obtained.

Consider the case for a cantilever beam; the end conditions are

$$\begin{aligned} y = y' = 0 & \quad \text{at } x = 1 \\ \text{and } EIy'' = (EIy'')' = 0 & \quad \text{at } x = c \end{aligned}$$

which gives the frequency equation

$$\begin{vmatrix} [y_0]_{x=1} & [y_1]_{x=1} & [y_{2-m}]_{x=1} & [y_{3-m}]_{x=1} \\ [y'_0]_{x=1} & [y'_1]_{x=1} & [y'_{2-m}]_{x=1} & [y'_{3-m}]_{x=1} \\ [y''_0]_{x=c} & [y''_1]_{x=c} & [y''_{2-m}]_{x=c} & [y''_{3-m}]_{x=c} \\ [(x^m y''_0)']_{x=c} & [(x^m y''_1)']_{x=c} & [(x^m y''_{2-m})']_{x=c} & [(x^m y''_{3-m})']_{x=c} \end{vmatrix} = 0 \quad (4.37)$$

where  $y_0$ ,  $y_1$ ,  $y_{2-m}$  and  $y_{3-m}$  are defined by Eqs. (4.13), (4.16), or (4.17).

## B. Approximate Solutions

The calculation of the natural frequencies from the characteristic equation, Eq. (4.37), involves tedious numerical computations since each element in the determinant is a combination of generalized hypergeometric series. For practical purposes and in order to supply results to check the exact solutions, two approximate methods are introduced for calculating the upper bound and the lower bound of the approximate fundamental frequencies.

The Ritz method is one of the approximate methods which has been widely used to determine the upper bound of the natural frequencies for elastic systems. The frequency of the fundamental mode is calculated by minimizing the expression for the energies, which, for beams having the prescribed profile is

$$p = \frac{\int_c^1 x^m \left( \frac{d^2 y}{dx^2} \right)^2 dx}{\int_c^1 x^n y^2 dx} \quad (4.38)$$

As an example, consider a cantilever beam. In order to satisfy the boundary conditions at free end, a one term approximation is assumed for the second derivative of the beam deflection, as

$$y'' = 12a (x - c)^2 \quad (4.39)$$

Integration of Eq. (4.39) gives the deflection curve

$$y = a \left[ (x - c)^4 + C_1 x + C_2 \right]$$

where  $C_1 = -4(1 - c)^3$  and  $C_2 = 4(1 - c)^3 - (1 - c)^4$  in order to satisfy the boundary conditions  $y = y' = 0$  at  $x = 1$ .

The lower bound approximate frequency of the fundamental mode is obtained from the expression

$$p = 1 \int_c^1 K(x, \xi) d\xi \quad (4.41)$$

where

$$K(x, \xi) = \frac{EI_0}{\ell^3} \xi^n U(x, \xi) \quad (4.42)$$

is the kernel of a homogeneous integral equation for the beam profile of current interest. The influence function for beam deflections,  $U(x, \xi)$ , is the static deflection of the beam at  $x$  with a unit load applied at a distance  $\xi$  measured from the origin. For a cantilever beam,  $U(x, \xi)$  can be expressed as

$$U(x, \xi) = \frac{\ell^3}{EI_0} \left[ \frac{1}{(2-m)(3-m)} x^{3-m} - \frac{\xi}{(1-m)(2-m)} x^{2-m} + \frac{(2-m)\xi - (1-m)}{(1-m)(2-m)} x - \frac{(3-m)\xi - (2-m)}{(2-m)(3-m)} \right] \quad (4.43)$$

for  $m \neq 1, 2$  and  $3$ , and

$$\begin{aligned} U(x, \xi) &= \frac{\ell^3}{EI_0} \left[ \frac{x^2}{2} - \xi x (\log x - 1) - x - \xi + \frac{1}{2} \right], & \text{for } m = 1 \\ U(x, \xi) &= \frac{\ell^3}{EI_0} \left[ x(\log x - 1) + \xi(\log x - x) + \xi + 1 \right], & \text{for } m = 2 \quad (4.44) \\ U(x, \xi) &= \frac{\ell^3}{EI_0} \left[ x(1 - \frac{\xi}{2}) - \log x - \frac{\xi}{2x} + \xi - 1 \right], & \text{for } m = 3 \end{aligned}$$

## NUMERICAL RESULTS AND DISCUSSION OF TABLES

### 1. Natural Frequencies and Nodal Points for Complete Tapered Beams

The natural frequencies for different vibrational modes of complete tapered beams can be calculated by solving for the roots of polynomials of Eqs. (4.26) and (4.34). The coefficients of the polynomials, for cantilever beams, for free-free beams executing antisymmetrical vibrational modes and for free-free beams executing symmetrical vibrational modes, can be generated from Eqs. (4.27), (4.30) and (4.35) respectively. The recurrence formulas indicate that the coefficients reduce rapidly as the number of terms increases. Therefore, in the numerical calculations, the first five frequencies are computed with the first sixteen terms of the series. Computer programs are written to generate the coefficients as well as to solve for the roots.

The exponents, for a uniform beam, are  $\psi = \phi = 0$  or  $m = 0$ ,  $\theta = 4$ , and the frequency equation for cantilever beams, Eq. (4.26), becomes

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4r)!} (4p)^r = 0 \quad (4.45)$$

where  $p = \omega^2 \left( \frac{\rho A_0}{E I_0} \right) \ell^4$  as defined earlier. Let the beam length be  $L$ , which is equal to  $\ell^0$  for the complete tapered beam, and let the frequency constant  $K$  be  $\sqrt{p}$ . Then the natural frequency can be represented as

$$\omega = K \sqrt{\frac{E I_0}{\rho A_0}} / L^2 \quad (4.46)$$

The first five frequency constants,  $K$ , for uniform cantilever beams are obtained from Eq. (4.45) using the first 16 terms of the series. These frequency constants are

$$3.51601, \quad 22.0345, \quad 61.6972, \quad 120.911, \quad 199.860$$

The substitution of each frequency into Eq. (4.31) establishes the locations the nodes for the corresponding natural frequency, which are given in the following table.

### Uniform Cantilever Beam

$\psi$	$\phi$	mode	frequency constant, K	locations of nodes, (X/L)				
.00	.00	1st	3.51601	1.00000				
		2nd	22.03449	.21656	1.00000			
		3rd	61.69721	.13232	.49645	1.00000		
		4th	120.90191	.09444	.35591	.64166	1.00000	
		5th	199.85953	.07345	.27678	.50009	.72125	1.00000

In the table, the first two columns display the combinations of the exponents  $\psi$  and  $\phi$ , the third and fourth columns indicate corresponding modes and their natural frequencies. The remaining columns show the locations of the nodes for different modes.

For cantilever beams with other combinations of  $\psi$  and  $\phi$ , the results for frequencies and nodes for the first five modes are listed in Table 1. Page one of Table 1 lists data for the combinations of  $\psi$  and  $\phi$  corresponding to beams of constant thickness with width varying as  $x^\psi$ . Page two of the table lists corresponding data for beams of constant width with thickness varying as  $x^\phi$ . The frequency data of these two cases are also plotted in Figs. 3 and 4 as the ratio of frequencies of tapered beams to those of uniform beams. Fig. 5 indicates the variation of the ratio of frequencies for the beams with taper both in width and thickness according to  $x^\phi$ . The ratio of frequencies for the first three modes, for 81 combinations of  $\psi$  and  $\phi$ , are plotted in Fig. 6 in three dimensional form. The figures reveal the variation of the natural frequencies of different modes as the taper of the beam varies. It is of interest to note that the frequencies of the fundamental mode increase when the beams taper either in width or in thickness. The frequencies of the higher modes increase as the taper of the width increases, that is, as  $\psi$  decreases. The higher mode frequencies decrease as the taper on the thickness increases, that is, as  $\phi$  increases.

The shapes of the first four normal modes for the vibration of tapered cantilever beams are plotted in Figs. 7, 8 and 9. Fig. 7 shows the change of mode shapes and the shifting of the nodal points for constant thickness beams as the exponent  $\psi$  increases. Fig. 8 shows those for beams with constant width and varying thickness. Fig. 9 displays those for beams with both width and thickness varying as a same power of  $x$ , that is  $\psi = \phi$ . The amplitudes of the deflections are normalized to the deflection at the free end.

For free-free beams of antisymmetrical and symmetrical mode vibration, the frequency constant  $K$  and the nodal points are also computed. For a uniform free-free beam, vibrating in antisymmetrical modes, the frequencies can be obtained from Eq. (4.26) with coefficients given by Eq. (4.30) and with  $m = 0, \nu = 4$ , that is

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4r+3)!} (4p)^r = 0 \quad (4.47)$$

The substitution of the first five roots of Eq. (4.47) into Eq. (4.31) gives the locations of the nodes for the corresponding mode. The results are listed as follows:

#### Uniform Free-Free Beam of Antisymmetrical Mode

$\psi$	$\phi$	mode	frequency constant, $K$	locations of nodes ( $X/L$ )					
.00	.00	1st	5.59332	.44832					
		2nd	30.22585	.18889	.71161				
		3rd	74.63888	.12020	.45291	.81825			
		4th	138.79131	.08814	.33213	.60005	.86666		
		5th	222.68295	.06959	.26221	.47372	.68421	.89474	

The results for other combinations of  $\psi$  and  $\phi$  are listed in Table 2 in the same order as in Table 1. The variation of the ratio of the frequencies of tapered beams to the frequencies of uniform beams are plotted in Fig. 10. The nodal points listed here are for one half of the beam length. The nodes of the other half of the beam are symmetrically located. The notation  $L$  represents half the total length of the free-free beam.

For free-free beams of symmetrical mode vibration, the frequencies and nodal points are evaluated from Eqs. (4.34) and (4.36) respectively. The results are listed in Table 3. For a uniform beam, the frequency equation is given as

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(4r+1)!} (4p)^r = 0 \quad (4.48)$$

which gives the first five roots and the corresponding nodes as follows:

### Uniform Free-Free Beam of Symmetrical Mode

$\psi$	$\phi$	mode	frequency			locations of nodes (X/L)				
			constant, K							
.00	.00	1st	15.41821	.2642	1.0000					
		2nd	49.96486	.1469	.5536	1.0000				
		3rd	104.24769	.1017	.3832	.6924	1.0000			
		4th	178.26972	.0778	.2931	.5295	.7647	1.0000		
		5th	272.03098	.0630	.2372	.4286	.6190	.8095	1.0000	

The above calculated results for frequencies and nodes for general tapered beams are checked with existing results of some special cases which appear in References [14, 15, 16, 17, 18, 19, 20].

#### 2. Natural Frequencies of Truncated Beams

For truncated cantilever beams the frequency constants,  $K$  in Eq. (4.46), are the roots of Eq. (4.37). The numerical results are obtained by using the method of regula falsi. The series involved in each of the elements of Eq. (4.37) are calculated using 16 terms. The frequency constants  $K$  for the first two vibrational modes are presented in Table 4 for beams truncated at  $0.2 \ell$  and at  $0.4 \ell$ . The combinations of the exponents  $\psi$  and  $\phi$  again include the beams with constant thickness, with constant width and with both thickness and width varying as the same power.

The upper bound and the lower bound approximations for the fundamental natural frequencies are calculated for six different degrees of truncation as well as for complete tapered beams. The values for the lower bound approximation are evaluated from Eq. (4.41) with the kernel defined in Eq. (4.42). The values of the upper bound approximation are evaluated from Eq. (4.38) with the deflection curve defined as Eq. (4.40). The approximate values of  $K$  are listed in Table 5 with 84 different combinations of  $\psi$  and  $\phi$ .

Comparisons of the correct frequencies of truncated beams with the approximate frequencies appear in Figs. 12, 13 and 14. The results for constant thickness beams with varying width appear in Fig. 12, while Fig. 13 displays results for constant width beams with varying thickness and Fig. 14 gives the results for beams for which both width and thickness vary. The frequencies of the upper bound approximation for constant thickness beams are closer to the correct results than those for constant width beams. This implies that the assumed deflection curve is more nearly correct for the constant thickness beams. The lower bound approximations also yield more nearly

correct results for beams with constant thickness than for beams with constant width. This is true because the frequencies of the higher modes for constant thickness beams are larger than those for constant width beams.

### 3. Radii of Gyration of Cross-Section

The evaluation of circular frequencies from the frequency constants  $K$ , as defined in Eq. (4.46), involves the calculation of the radius of gyration,  $(I_o/A_o)^{1/2}$ . For the beams with cross-sections bounded by Eq. (4.4), the area moments of inertia and the cross-sectional areas with different values of  $\gamma$  and  $\beta$  were listed in the first report [1]. The radius of gyration of the cross-section at the large end of the beam, calculated from Eq. (4.6), is

$$\frac{r_g}{h_o} = \frac{1}{3} \frac{\Gamma(\frac{3}{\beta} + 1) \Gamma(\frac{1}{\gamma} + \frac{1}{\beta} + 1)}{\Gamma(\frac{1}{\beta} + 1) \Gamma(\frac{1}{\gamma} + \frac{3}{\beta} + 1)} \quad (4.49)$$

The results of Eq. (4.49) are listed in Table 6 with the same combinations of  $\alpha$  and  $\beta$  as those used in the first report.

## REFERENCES

1. "Geometrical and Inertial Properties of a Class of Thin Shells of Revolution" by  
Will J. Worley and Han-chung Wang  
National Aeronautics and Space Administration, Grant No. NsG -  
434, NASA Contractor Report CR-89, September, 1964, 208  
pages.
2. "Geometrical and Inertial Properties of a Class of Thin Shells of a General  
Type" by  
Will J. Worley and Han-chung Wang  
National Aeronautics and Space Administration, Grant No. NsG -  
434, Supplement No. 1, NASA Contractor Report CR-271, Aug.  
1965, 67 pages.
3. "Landing Characteristics of a Lenticular-Shaped Reentry Vehicle" by  
Ulysse J. Blanchard  
NASA Tech. Note D-940, September, 1961, 35 pages.
4. "Estimation of the Forces and Moments Acting on Inclined Bodies of Revo-  
lution of High Fineness Ratio" by  
H. Julian Allen  
NASA, Research Memorandum, RM A9 126, Nov. 14, 1949,  
27 pages.
5. "Bodies of Revolution Having Minimum Drag at High Supersonic Airspeeds" by  
  
A. J. Eggers, Jr., Meyer M. Resnikoff and David H. Dennis  
NACA Report 1306, 1957, 12 pages.
6. "Resistance of Slender Bodies Moving with Supersonic Velocities, with  
Special Reference to Projectiles" by  
Theodore von Kármán and Norton B. Moore  
Trans. A.S.M.E., V. 54, Pt. I, N. 23, December 15, 1932,  
pp. 303-310

7. "Experimental Investigation at a Mach Number of 3.11 of the Lift, Drag and Pitching-Moment Characteristics of Five Blunt Lifting Bodies" by  
William Letko  
NASA Tech. Note D-226, April 1960, 13 pages.
8. "Transient Response of a Tapered Cantilever Beam" by  
Alexander S. Elder  
Developments in Theoretical and Applied Mechanics, Vol. 2,  
Proceedings of the Second Southeastern Conference on Theo-  
retical and Applied Mechanics, edited by W. A. Shaw, Pergamon  
Press, N.Y., N. Y., 1965, pp. 441-466.
9. A General Treatment of the Transverse Vibration of Tapered Beams, by  
Han-chung Wang  
Ph.D. Thesis, Department of Theoretical and Applied Mechanics,  
University of Illinois, Urbana, Illinois, Feb., 1966. . .
10. "Areas, Centroids and Inertias for a Family of Elliptic-Type Closed  
Curves" by  
Will J. Worley and Fred D. Breuer  
Product Engineering, V. 28, N. 8, August, 1957, pp. 141-144.
11. "Wave Propagation in an Elastic Nonhomogeneous Bar of Finite Length" by  
U. S. Lindholm and K. D. Doshi  
Journal of Applied Mechanics, Vol. 32, No. 1, March, 1965  
pp. 135-142.
12. Special Functions by  
Earl D. Rainville  
The MacMillan Company, N. Y., N. Y., 1960, 365 pages,  
pp. 73-80.
13. Mathematics in Physics and Engineering by  
J. Irving and N. Mullineux  
Academic Press, N. Y., N. Y., 1959, 883 pages, Chapter II.
14. "Transverse Vibration of a Rod of Varying Cross Section" by  
P. F. Ward  
Philosophical Magazine, Vol. 25, Series 6, Jan. 1913, pp. 85-106.

15. "The Lateral Vibration of Bars of Variable Section" by  
J. W. Nicholson  
Proceedings of the Royal Society of London, Series A, Vol. 93,  
No. A654, Sept. 1917, pp. 506-519.
16. "The Lateral Vibrations of Sharply-Pointed Bars" by  
J. W. Nicholson  
Proceedings of the Royal Society of London, Series A, Vol. 97,  
No. A683, April 1920, pp. 172-181.
17. "Lateral Vibrations of Bars of Conical Type" by  
D. M. Wrinch  
Proceedings of the Royal Society of London, Series A, Vol. 101,  
1922, pp. 493-508.
18. "On the Lateral Vibrations of Rods of Variable Cross-Section" by  
D. M. Wrinch  
Philosophical Magazine, Series 6, Vol. 46, Aug., 1923,  
pp. 273-291.
19. "Lateral Vibrations of Tapered Bars" by  
Akimasa Ono  
Journal of the Society of Mechanical Engineering in Japan, Tokyo,  
Japan, Vol. 28, No. 99, July 1925, pp. 429-441.
20. "Bending Vibrations of Variable Section Beams" by  
E. T. Cranch and Alfred A. Adler  
Journal of Applied Mechanics, Trans. A.S.M.E., Vol. 23,  
No. 1, March 1956, pp. 103-108.

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.10	.00	1ST	3.84759	1.00000						
		2ND	22.91518	.23054	1.00000					
		3RD	63.04715	.14189	.50266	1.00000				
		4TH	122.74233	.10160	.36153	.64453	1.00000			
		5TH	202.19116	.07916	.28164	.50319	.72293	1.00000		
.20	.00	1ST	4.18617	1.00000						
		2ND	23.79954	.24375	1.00000					
		3RD	64.40342	.15105	.50861	1.00000				
		4TH	124.58941	.10851	.36696	.64732	1.00000			
		5TH	204.52963	.08469	.28636	.50624	.72459	1.00000		
.30	.00	1ST	4.53188	1.00000						
		2ND	24.68792	.25626	1.00000					
		3RD	65.76606	.15985	.51432	1.00000				
		4TH	126.44310	.11519	.37221	.65005	1.00000			
		5TH	206.87497	.09006	.29095	.50922	.72621	1.00000		
.40	.00	1ST	4.88482	1.00000						
		2ND	25.58063	.26812	1.00000					
		3RD	67.13513	.16831	.51982	1.00000				
		4TH	128.30356	.12166	.37731	.65272	1.00000			
		5TH	209.22717	.09527	.29542	.51215	.72781	1.00000		
.50	.00	1ST	5.24506	1.00000						
		2ND	26.47796	.27941	1.00000					
		3RD	68.51067	.17647	.52511	1.00000				
		4TH	130.17072	.12793	.38225	.65533	1.00000			
		5TH	211.58626	.10035	.29977	.51502	.72938	1.00000		
.60	.00	1ST	5.61263	1.00000						
		2ND	27.38015	.29017	1.00000					
		3RD	69.89271	.18434	.53022	1.00000				
		4TH	132.04458	.13402	.38705	.65788	1.00000			
		5TH	213.95227	.10529	.30402	.51783	.73092	1.00000		
.80	.00	1ST	6.36984	1.00000						
		2ND	29.19990	.31028	1.00000					
		3RD	72.67645	.19930	.53992	1.00000				
		4TH	135.81270	.14570	.39625	.66281	1.00000			
		5TH	218.70500	.11482	.31221	.52331	.73394	1.00000		
1.00	.00	1ST	7.15646	1.00000						
		2ND	31.04131	.32874	1.00000					
		3RD	75.48660	.21333	.54900	1.00000				
		4TH	139.60798	.15678	.40498	.66755	1.00000			
		5TH	223.48545	.12392	.32004	.52860	.73685	1.00000		

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.00	.10	1ST	3.75540	1.00000						
		2ND	21.53823	.22510	1.00000					
		3RD	58.22532	.13605	.49123	1.00000				
		4TH	112.59111	.09607	.34845	.63288	1.00000			
		5TH	184.78438	.07402	.26843	.48864	.71254	1.00000		
.00	.20	1ST	3.98423	1.00000						
		2ND	20.99322	.23211	1.00000					
		3RD	54.81987	.13863	.48536	1.00000				
		4TH	104.54557	.09677	.34037	.62355	1.00000			
		5TH	170.27172	.07380	.25953	.47657	.70323	1.00000		
.00	.30	1ST	4.20172	1.00000						
		2ND	20.40273	.23769	1.00000					
		3RD	51.48127	.14012	.47883	1.00000				
		4TH	96.76554	.09660	.33162	.61360	1.00000			
		5TH	156.32171	.07285	.25005	.46380	.69324	1.00000		
.00	.40	1ST	4.40698	1.00000						
		2ND	19.76973	.24191	1.00000					
		3RD	48.20998	.14058	.47157	1.00000				
		4TH	89.25128	.09560	.32216	.60296	1.00000			
		5TH	142.93450	.07123	.23998	.45027	.68250	1.00000		
.00	.50	1ST	4.59896	1.00000						
		2ND	19.09684	.24483	1.00000					
		3RD	45.00642	.14003	.46353	1.00000				
		4TH	82.00310	.09381	.31196	.59156	1.00000			
		5TH	130.11026	.06896	.22928	.43590	.67089	1.00000		
.00	.60	1ST	4.77655	1.00000						
		2ND	18.38646	.24647	1.00000					
		3RD	41.87105	.13849	.45463	1.00000				
		4TH	75.02126	.09126	.30096	.57928	1.00000			
		5TH	117.84916	.06610	.21793	.42060	.65832	1.00000		
.00	.80	1ST	5.08332	1.00000						
		2ND	16.86125	.24588	1.00000					
		3RD	35.80663	.13245	.43390	1.00000				
		4TH	61.85787	.08395	.27624	.55160	1.00000			
		5TH	95.01719	.05871	.19314	.38675	.62966	1.00000		
.00	1.00	1ST	5.31510	1.00000						
		2ND	15.20717	.23980	1.00000					
		3RD	30.01981	.12230	.40833	1.00000				
		4TH	49.76335	.07376	.24730	.51858	1.00000			
		5TH	74.44003	.04931	.16529	.34763	.59497	1.00000		

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$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.10	.10	1ST	4.08637	1.00000					
		2ND	22.38253	.23812	1.00000				
		3RD	59.52071	.14497	.49731	1.00000			
		4TH	114.35302	.10272	.35396	.63583	1.00000		
		5TH	187.01330	.07930	.27318	.49183	.71431	1.00000	
.10	.20	1ST	4.31441	1.00000					
		2ND	21.80267	.24428	1.00000				
		3RD	56.06091	.14698	.49135	1.00000			
		4TH	106.22908	.10297	.34578	.62659	1.00000		
		5TH	172.39803	.07869	.26418	.47984	.70509	1.00000	
.10	.30	1ST	4.53085	1.00000					
		2ND	21.17865	.24911	1.00000				
		3RD	52.66817	.14797	.48473	1.00000			
		4TH	98.37077	.10239	.33694	.61674	1.00000		
		5TH	158.34546	.07739	.25461	.46715	.69520	1.00000	
.10	.40	1ST	4.73475	1.00000					
		2ND	20.51322	.25268	1.00000				
		3RD	49.34290	.14797	.47741	1.00000			
		4TH	90.77835	.10102	.32741	.60621	1.00000		
		5TH	144.85576	.07544	.24444	.45371	.68456	1.00000	
.10	.50	1ST	4.92505	1.00000					
		2ND	19.80887	.25501	1.00000				
		3RD	46.08556	.14700	.46932	1.00000			
		4TH	83.45211	.09889	.31714	.59491	1.00000		
		5TH	131.92909	.07287	.23366	.43943	.67307	1.00000	
.10	.60	1ST	5.10058	1.00000					
		2ND	19.06779	.25613	1.00000				
		3RD	42.89657	.14508	.46039	1.00000			
		4TH	76.39234	.09602	.30607	.58276	1.00000		
		5TH	119.56565	.06973	.22222	.42421	.66062	1.00000	
.10	.80	1ST	5.40200	1.00000					
		2ND	17.48294	.25464	1.00000				
		3RD	36.72535	.13835	.43962	1.00000			
		4TH	63.07337	.08812	.28123	.55534	1.00000		
		5TH	96.52915	.06181	.19723	.39055	.63226	1.00000	
.10	1.00	1ST	5.62660	1.00000					
		2ND	15.77062	.24780	1.00000				
		3RD	30.83211	.12757	.41406	1.00000			
		4TH	50.82352	.07737	.25215	.52262	1.00000		
		5TH	75.74766	.05191	.16916	.35161	.59792	1.00000	

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.20	.10	1ST	4.42440	1.00000					
		2ND	23.23123	.25045	1.00000				
		3RD	60.82253	.15355	.50316	1.00000			
		4TH	116.12162	.10917	.35930	.63871	1.00000		
		5TH	189.24912	.08443	.27781	.49495	.71605	1.00000	
.20	.20	1ST	4.65164	1.00000					
		2ND	22.61712	.25585	1.00000				
		3RD	57.30847	.15504	.49712	1.00000			
		4TH	107.91934	.10900	.35103	.62956	1.00000		
		5TH	174.53124	.08346	.26871	.48304	.70691	1.00000	
.20	.30	1ST	4.86701	1.00000					
		2ND	21.96005	.26001	1.00000				
		3RD	53.86166	.15556	.49043	1.00000			
		4TH	99.98279	.10804	.34212	.61980	1.00000		
		5TH	160.37614	.08183	.25905	.47043	.69712	1.00000	
.20	.40	1ST	5.06950	1.00000					
		2ND	21.26259	.26298	1.00000				
		3RD	50.48250	.15514	.48305	1.00000			
		4TH	92.31226	.10633	.33252	.60938	1.00000		
		5TH	146.78397	.07957	.24880	.45707	.68658	1.00000	
.20	.50	1ST	5.25804	1.00000					
		2ND	20.52706	.26479	1.00000				
		3RD	47.17144	.15379	.47492	1.00000			
		4TH	84.90802	.10388	.32219	.59819	1.00000		
		5TH	133.75492	.07673	.23794	.44288	.67521	1.00000	
.20	.60	1ST	5.43143	1.00000					
		2ND	19.75552	.26542	1.00000				
		3RD	43.92890	.15151	.46596	1.00000			
		4TH	77.77034	.10071	.31106	.58615	1.00000		
		5TH	121.28914	.07332	.22641	.42775	.66289	1.00000	
.20	.80	1ST	5.72734	1.00000					
		2ND	18.11131	.26311	1.00000				
		3RD	37.65095	.14413	.44517	1.00000			
		4TH	64.29584	.09224	.28610	.55898	1.00000		
		5TH	98.04816	.06490	.20125	.39427	.63480	1.00000	
.20	1.00	1ST	5.94455	1.00000					
		2ND	16.34084	.25557	1.00000				
		3RD	31.65133	.13275	.41962	1.00000			
		4TH	51.89071	.08095	.25691	.52655	1.00000		
		5TH	77.06235	.05451	.17297	.35551	.60081	1.00000	

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.30	.10	1ST	4.76957	1.00000					
		2ND	24.08461	.26218	1.00000				
		3RD	62.13082	.16182	.50879	1.00000			
		4TH	117.89695	.11542	.36447	.64153	1.00000		
		5TH	191.49183	.08942	.28231	.49801	.71775	1.00000	
.30	.20	1ST	4.99599	1.00000					
		2ND	23.43679	.26689	1.00000				
		3RD	58.56259	.16283	.50267	1.00000			
		4TH	109.61637	.11487	.35613	.63246	1.00000		
		5TH	176.67138	.08812	.27313	.48617	.70871	1.00000	
.30	.30	1ST	5.21022	1.00000					
		2ND	22.74709	.27044	1.00000				
		3RD	55.06178	.16292	.49593	1.00000			
		4TH	101.60165	.11356	.34715	.62280	1.00000		
		5TH	162.41377	.08618	.26339	.47365	.69901	1.00000	
.30	.40	1ST	5.41124	1.00000					
		2ND	22.01795	.27287	1.00000				
		3RD	51.62881	.16211	.48850	1.00000			
		4TH	93.85304	.11152	.33749	.61247	1.00000		
		5TH	148.71918	.08364	.25307	.46038	.68857	1.00000	
.30	.50	1ST	5.59794	1.00000					
		2ND	21.25152	.27418	1.00000				
		3RD	48.26409	.16040	.48034	1.00000			
		4TH	86.37083	.10877	.32711	.60139	1.00000		
		5TH	135.58774	.08053	.24213	.44627	.67731	1.00000	
.30	.60	1ST	5.76911	1.00000					
		2ND	20.44970	.27438	1.00000				
		3RD	44.96803	.15779	.47136	1.00000			
		4TH	79.15528	.10531	.31592	.58946	1.00000		
		5TH	123.01965	.07686	.23053	.43123	.66510	1.00000	
.30	.80	1ST	6.05931	1.00000					
		2ND	18.74637	.27131	1.00000				
		3RD	38.58345	.14980	.45056	1.00000			
		4TH	65.52530	.09632	.29088	.56254	1.00000		
		5TH	99.57422	.06796	.20521	.39792	.63730	1.00000	
.30	1.00	1ST	6.26895	1.00000					
		2ND	16.91781	.26312	1.00000				
		3RD	32.47746	.13786	.42504	1.00000			
		4TH	52.96490	.08451	.26158	.53040	1.00000		
		5TH	78.38411	.05711	.17672	.35934	.60365	1.00000	

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.40	.10	1ST	5.12194	1.00000					
		2ND	24.94292	.27335	1.00000				
		3RD	63.44562	.16980	.51421	1.00000			
		4TH	119.67903	.12150	.36949	.64427	1.00000		
		5TH	193.74142	.09429	.28670	.50101	.71943	1.00000	
.40	.20	1ST	5.34749	1.00000					
		2ND	24.26185	.27743	1.00000				
		3RD	59.82330	.17037	.50803	1.00000			
		4TH	111.32020	.12059	.36108	.63529	1.00000		
		5TH	178.81845	.09267	.27745	.48925	.71047	1.00000	
.40	.30	1ST	5.56052	1.00000					
		2ND	23.53993	.28043	1.00000				
		3RD	56.26857	.17007	.50124	1.00000			
		4TH	103.22733	.11895	.35205	.62572	1.00000		
		5TH	164.45835	.09045	.26764	.47681	.70086	1.00000	
.40	.40	1ST	5.75999	1.00000					
		2ND	22.77940	.28236	1.00000				
		3RD	52.78184	.16889	.49378	1.00000			
		4TH	95.40070	.11661	.34234	.61549	1.00000		
		5TH	150.66135	.08764	.25725	.46361	.69052	1.00000	
.40	.50	1ST	5.94477	1.00000					
		2ND	21.98229	.28323	1.00000				
		3RD	49.36352	.16685	.48559	1.00000			
		4TH	87.84056	.11357	.33190	.60451	1.00000		
		5TH	137.42756	.08427	.24624	.44959	.67937	1.00000	
.40	.60	1ST	6.11362	1.00000					
		2ND	21.15039	.28303	1.00000				
		3RD	46.01401	.16393	.47659	1.00000			
		4TH	80.54718	.10985	.32068	.59269	1.00000		
		5TH	124.75719	.08037	.23457	.43464	.66728	1.00000	
.40	.80	1ST	6.39792	1.00000					
		2ND	19.38815	.27925	1.00000				
		3RD	39.52286	.15537	.45579	1.00000			
		4TH	66.76175	.10034	.29555	.56602	1.00000		
		5TH	101.10733	.07100	.20910	.40151	.63975	1.00000	
.40	1.00	1ST	6.59979	1.00000					
		2ND	17.50155	.27046	1.00000				
		3RD	33.31050	.14289	.43032	1.00000			
		4TH	54.04608	.08805	.26615	.53416	1.00000		
		5TH	79.71292	.05970	.18043	.36310	.60642	1.00000	

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.50	.10	1ST	5.48156	1.00000						
		2ND	25.80637	.28400	1.00000					
		3RD	64.76696	.17751	.51944	1.00000				
		4TH	121.46787	.12741	.37437	.64696	1.00000			
		5TH	195.99795	.09905	.29098	.50395	.72107	1.00000		
.50	.20	1ST	5.70617	1.00000						
		2ND	25.09250	.28752	1.00000					
		3RD	61.09063	.17768	.51321	1.00000				
		4TH	113.03085	.12616	.36590	.63806	1.00000			
		5TH	180.97245	.09713	.28166	.49227	.71220	1.00000		
.50	.30	1ST	5.91792	1.00000						
		2ND	24.33867	.29001	1.00000					
		3RD	57.48205	.17701	.50638	1.00000				
		4TH	104.85989	.12422	.35682	.62858	1.00000			
		5TH	166.50990	.09464	.27179	.47990	.70268	1.00000		
.50	.40	1ST	6.11574	1.00000						
		2ND	23.54703	.29149	1.00000					
		3RD	53.94163	.17549	.49889	1.00000				
		4TH	96.95525	.12159	.34707	.61844	1.00000			
		5TH	152.61052	.09157	.26134	.46679	.69244	1.00000		
.50	.50	1ST	6.29851	1.00000						
		2ND	22.71946	.29195	1.00000					
		3RD	50.46977	.17314	.49068	1.00000				
		4TH	89.31721	.11829	.33659	.60756	1.00000			
		5TH	139.27439	.08797	.25027	.45285	.68139	1.00000		
.50	.60	1ST	6.46495	1.00000						
		2ND	21.85763	.29137	1.00000					
		3RD	47.06685	.16993	.48168	1.00000				
		4TH	81.94603	.11431	.32533	.59585	1.00000			
		5TH	126.50177	.08383	.23854	.43798	.66942	1.00000		
.50	.80	1ST	6.74316	1.00000						
		2ND	20.03664	.28694	1.00000					
		3RD	40.46916	.16082	.46089	1.00000				
		4TH	68.00520	.10432	.30012	.56942	1.00000			
		5TH	102.64750	.07402	.21293	.40503	.64215	1.00000		
.50	1.00	1ST	6.93705	1.00000						
		2ND	18.09205	.27760	1.00000					
		3RD	34.15046	.14784	.43545	1.00000				
		4TH	55.13427	.09155	.27065	.53784	1.00000			
		5TH	81.04880	.06228	.18408	.36679	.60915	1.00000		

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.60	.10	1ST	5.84844	1.00000						
		2ND	26.67514	.29418	1.00000					
		3RD	66.09489	.18496	.52450	1.00000				
		4TH	123.26350	.13316	.37911	.64959	1.00000			
		5TH	198.26140	.10369	.29516	.50684	.72269	1.00000		
.60	.20	1ST	6.07203	1.00000						
		2ND	25.92884	.29719	1.00000					
		3RD	62.36461	.18477	.51822	1.00000				
		4TH	114.74832	.13160	.37060	.64077	1.00000			
		5TH	183.13340	.10150	.28579	.49523	.71390	1.00000		
.60	.30	1ST	6.28241	1.00000						
		2ND	25.14344	.29922	1.00000					
		3RD	58.70225	.18375	.51135	1.00000				
		4TH	106.49930	.12937	.36147	.63138	1.00000			
		5TH	168.56846	.09875	.27586	.48294	.70447	1.00000		
.60	.40	1ST	6.47850	1.00000						
		2ND	24.32092	.30028	1.00000					
		3RD	55.10819	.18192	.50384	1.00000				
		4TH	98.51672	.12647	.35168	.62133	1.00000			
		5TH	154.56667	.09545	.26535	.46991	.69432	1.00000		
.60	.50	1ST	6.65916	1.00000						
		2ND	23.46308	.30036	1.00000					
		3RD	51.58284	.17928	.49562	1.00000				
		4TH	90.80081	.12292	.34117	.61055	1.00000			
		5TH	141.12825	.09161	.25422	.45605	.68338	1.00000		
.60	.60	1ST	6.82309	1.00000						
		2ND	22.57145	.29944	1.00000					
		3RD	48.12654	.17579	.48661	1.00000				
		4TH	83.35184	.11870	.32987	.59894	1.00000			
		5TH	128.25335	.08725	.24244	.44126	.67152	1.00000		
.60	.80	1ST	7.09501	1.00000						
		2ND	20.69188	.29441	1.00000					
		3RD	41.42238	.16618	.46584	1.00000				
		4TH	69.25565	.10825	.30461	.57274	1.00000			
		5TH	104.19474	.07702	.21669	.40848	.64451	1.00000		
.60	1.00	1ST	7.28073	1.00000						
		2ND	18.68930	.28455	1.00000					
		3RD	34.99732	.15271	.44046	1.00000				
		4TH	56.22945	.09503	.27506	.54143	1.00000			
		5TH	82.39172	.06486	.18769	.37042	.61182	1.00000		

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)				
.80	.10	1ST	6.60401	1.00000					
		2ND	28.42930	.31329	1.00000				
		3RD	68.77063	.19919	.53411	1.00000			
		4TH	126.87519	.14423	.38822	.65467	1.00000		
		5TH	202.80911	.11268	.30325	.51246	.72585	1.00000	
.80	.20	1ST	6.82529	1.00000					
		2ND	27.61914	.31539	1.00000				
		3RD	64.93265	.19833	.52777	1.00000			
		4TH	118.20383	.14211	.37963	.64601	1.00000		
		5TH	187.47620	.10999	.29377	.50099	.71721	1.00000	
.80	.30	1ST	7.03265	1.00000					
		2ND	26.77137	.31660	1.00000				
		3RD	61.16289	.19670	.52086	1.00000			
		4TH	109.79881	.13936	.37043	.63679	1.00000		
		5TH	172.70647	.10676	.28375	.48885	.70795	1.00000	
.80	.40	1ST	7.22499	1.00000					
		2ND	25.88775	.31693	1.00000				
		3RD	57.46172	.19430	.51331	1.00000			
		4TH	101.66040	.13597	.36058	.62692	1.00000		
		5TH	158.50002	.10302	.27314	.47598	.69799	1.00000	
.80	.50	1ST	7.40113	1.00000					
		2ND	24.96983	.31634	1.00000				
		3RD	53.82950	.19113	.50508	1.00000			
		4TH	93.78885	.13195	.35002	.61633	1.00000		
		5TH	144.85704	.09875	.26192	.46227	.68724	1.00000	
.80	.60	1ST	7.55975	1.00000					
		2ND	24.01897	.31481	1.00000				
		3RD	50.26657	.18714	.49608	1.00000			
		4TH	86.18438	.12729	.33867	.60492	1.00000		
		5TH	131.77768	.09399	.25003	.44766	.67560	1.00000	
.80	.80	1ST	7.81849	1.00000					
		2ND	22.02259	.30869	1.00000				
		3RD	43.34957	.17660	.47537	1.00000			
		4TH	71.77754	.11598	.31331	.57918	1.00000		
		5TH	107.31037	.08296	.22406	.41522	.64909	1.00000	
.80	1.00	1ST	7.98725	1.00000					
		2ND	19.90401	.29790	1.00000				
		3RD	36.71176	.16225	.45011	1.00000			
		4TH	58.44082	.10190	.28364	.54839	1.00000		
		5TH	85.09876	.06998	.19476	.37750	.61700	1.00000	

TABLE 1. NATURAL FREQUENCIES AND NODES FOR TAPERED CANTILEVER BEAMS

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
1.00	.10	1ST	7.38863	1.00000					
		2ND	30.20649	.33090	1.00000				
		3RD	71.47303	.21260	.54313	1.00000			
		4TH	130.51427	.15477	.39688	.65954	1.00000		
		5TH	207.38463	.12130	.31099	.51788	.72891	1.00000	
1.00	.20	1ST	7.60720	1.00000					
		2ND	29.33359	.33224	1.00000				
		3RD	67.52760	.21115	.53675	1.00000			
		4TH	121.68684	.15216	.38823	.65104	1.00000		
		5TH	191.84694	.11816	.30143	.50655	.72042	1.00000	
1.00	.30	1ST	7.81115	1.00000					
		2ND	28.42435	.33276	1.00000				
		3RD	63.65066	.20899	.52981	1.00000			
		4TH	113.12596	.14895	.37898	.64197	1.00000		
		5TH	176.87248	.11451	.29133	.49456	.71132	1.00000	
1.00	.40	1ST	7.99934	1.00000					
		2ND	27.48032	.33246	1.00000				
		3RD	59.84256	.20610	.52226	1.00000			
		4TH	104.83184	.14512	.36909	.63227	1.00000		
		5TH	162.46145	.11036	.28064	.48183	.70154	1.00000	
1.00	.50	1ST	8.17056	1.00000					
		2ND	26.50287	.33129	1.00000				
		3RD	56.10363	.20245	.51403	1.00000			
		4TH	96.80474	.14068	.35849	.62186	1.00000		
		5TH	148.61396	.10572	.26934	.46828	.69098	1.00000	
1.00	.60	1ST	8.32347	1.00000					
		2ND	25.49314	.32923	1.00000				
		3RD	52.43416	.19803	.50505	1.00000			
		4TH	89.04486	.13562	.34711	.61065	1.00000		
		5TH	135.33020	.10057	.25737	.45383	.67955	1.00000	
1.00	.80	1ST	8.56825	1.00000					
		2ND	23.38030	.32219	1.00000				
		3RD	45.30443	.18666	.48442	1.00000			
		4TH	74.32746	.12353	.32168	.58534	1.00000		
		5TH	110.45425	.08880	.23121	.42172	.65351	1.00000	
1.00	1.00	1ST	8.71926	1.00000					
		2ND	21.14566	.31058	1.00000				
		3RD	38.45377	.17150	.45930	1.00000			
		4TH	60.68014	.10866	.29193	.55506	1.00000		
		5TH	87.83399	.07507	.20166	.38434	.62200	1.00000	

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.10	.00	1ST	16.04270	.2810	1.0000						
		2ND	51.07784	.1575	.5604	1.0000					
		3RD	105.85465	.1094	.3892	.6954	1.0000				
		4TH	180.37029	.0838	.2982	.5327	.7665	1.0000			
		5TH	274.62509	.0679	.2417	.4317	.6211	.8107	1.0000		
.20	.00	1ST	16.67084	.2969	1.0000						
		2ND	52.19662	.1676	.5669	1.0000					
		3RD	107.46787	.1168	.3950	.6984	1.0000				
		4TH	182.47738	.0897	.3032	.5359	.7682	1.0000			
		5TH	277.22591	.0727	.2460	.4348	.6232	.8118	1.0000		
.30	.00	1ST	17.30292	.3118	1.0000						
		2ND	53.32124	.1774	.5732	1.0000					
		3RD	109.08740	.1240	.4007	.7013	1.0000				
		4TH	184.59102	.0953	.3080	.5390	.7699	1.0000			
		5TH	279.83343	.0774	.2502	.4378	.6252	.8129	1.0000		
.40	.00	1ST	17.93916	.3260	1.0000						
		2ND	54.45176	.1867	.5793	1.0000					
		3RD	110.71325	.1310	.4061	.7041	1.0000				
		4TH	186.71124	.1009	.3127	.5421	.7715	1.0000			
		5TH	282.44766	.0820	.2543	.4408	.6272	.8140	1.0000		
.50	.00	1ST	18.57977	.3395	1.0000						
		2ND	55.58823	.1958	.5851	1.0000					
		3RD	112.34544	.1377	.4114	.7068	1.0000				
		4TH	188.83803	.1062	.3173	.5451	.7731	1.0000			
		5TH	285.06863	.0865	.2583	.4437	.6292	.8150	1.0000		
.60	.00	1ST	19.22493	.3523	1.0000						
		2ND	56.73065	.2045	.5907	1.0000					
		3RD	113.98401	.1443	.4165	.7095	1.0000				
		4TH	190.97142	.1114	.3218	.5481	.7747	1.0000			
		5TH	287.69633	.0908	.2622	.4465	.6311	.8161	1.0000		
.80	.00	1ST	20.52951	.3762	1.0000						
		2ND	59.03360	.2210	.6013	1.0000					
		3RD	117.28031	.1568	.4263	.7147	1.0000				
		4TH	195.25801	.1215	.3304	.5538	.7778	1.0000			
		5TH	292.97194	.0992	.2698	.4521	.6349	.8182	1.0000		
1.00	.00	1ST	21.85396	.3980	1.0000						
		2ND	61.36083	.2365	.6112	1.0000					
		3RD	120.60229	.1687	.4357	.7197	1.0000				
		4TH	199.57111	.1311	.3387	.5593	.7809	1.0000			
		5TH	298.27457	.1073	.2770	.4575	.6386	.8202	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.00	.10	1ST	15.08138	.2770	1.0000				
		2ND	47.18989	.1518	.5507	1.0000			
		3RD	97.11908	.1038	.3766	.6855	1.0000		
		4TH	164.86205	.0786	.2850	.5188	.7577	1.0000	
		5TH	250.41947	.0631	.2287	.4164	.6081	.8030	1.0000
.00	.20	1ST	14.71346	.2884	1.0000				
		2ND	44.46738	.1556	.5474	1.0000			
		3RD	90.21738	.1050	.3693	.6783	1.0000		
		4TH	151.95371	.0786	.2765	.5076	.7503	1.0000	
		5TH	229.67805	.0625	.2198	.4035	.5964	.7960	1.0000
.00	.30	1ST	14.31692	.2984	1.0000				
		2ND	41.79778	.1583	.5436	1.0000			
		3RD	83.54284	.1053	.3615	.6706	1.0000		
		4TH	139.54486	.0779	.2673	.4958	.7423	1.0000	
		5TH	209.80682	.0613	.2103	.3900	.5839	.7885	1.0000
.00	.40	1ST	13.89400	.3074	1.0000				
		2ND	39.18153	.1599	.5393	1.0000			
		3RD	77.09574	.1048	.3530	.6624	1.0000		
		4TH	127.63570	.0764	.2576	.4832	.7338	1.0000	
		5TH	190.80585	.0595	.2003	.3759	.5707	.7804	1.0000
.00	.50	1ST	13.44673	.3152	1.0000				
		2ND	36.61912	.1606	.5345	1.0000			
		3RD	70.87632	.1034	.3437	.6538	1.0000		
		4TH	116.22631	.0744	.2472	.4699	.7247	1.0000	
		5TH	172.67526	.0571	.1899	.3609	.5564	.7716	1.0000
.00	.60	1ST	12.97695	.3221	1.0000				
		2ND	34.11102	.1602	.5292	1.0000			
		3RD	64.88489	.1012	.3338	.6445	1.0000		
		4TH	105.31692	.0716	.2362	.4557	.7148	1.0000	
		5TH	155.41518	.0542	.1789	.3451	.5412	.7621	1.0000
.00	.80	1ST	11.97619	.3332	1.0000				
		2ND	29.25974	.1566	.5167	1.0000			
		3RD	53.58713	.0946	.3113	.6239	1.0000		
		4TH	84.99889	.0644	.2119	.4243	.6926	1.0000	
		5TH	123.50709	.0472	.1552	.3108	.5070	.7403	1.0000
.00	1.00	1ST	10.90236	.3410	1.0000				
		2ND	24.63138	.1490	.5014	1.0000			
		3RD	43.20475	.0850	.2848	.6000	1.0000		
		4TH	66.68312	.0550	.1845	.3880	.6662	1.0000	
		5TH	95.08264	.0386	.1294	.2721	.4667	.7138	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.10	.10	1ST	15.68218	.2928	1.0000						
		2ND	48.25902	.1618	.5574	1.0000					
		3RD	98.65841	.1110	.3825	.6886	1.0000				
		4TH	166.87082	.0842	.2901	.5222	.7596	1.0000			
		5TH	252.89746	.0677	.2331	.4196	.6102	.8042	1.0000		
.10	.20	1ST	15.29172	.3032	1.0000						
		2ND	45.49287	.1649	.5540	1.0000					
		3RD	91.68921	.1118	.3752	.6815	1.0000				
		4TH	153.87080	.0838	.2814	.5111	.7522	1.0000			
		5TH	232.03995	.0667	.2240	.4068	.5987	.7973	1.0000		
.10	.30	1ST	14.87364	.3125	1.0000						
		2ND	42.77983	.1671	.5502	1.0000					
		3RD	84.94729	.1116	.3672	.6740	1.0000				
		4TH	141.37030	.0827	.2722	.4993	.7444	1.0000			
		5TH	212.05267	.0652	.2144	.3934	.5863	.7899	1.0000		
.10	.40	1ST	14.43004	.3207	1.0000						
		2ND	40.12034	.1683	.5459	1.0000					
		3RD	78.43290	.1107	.3587	.6659	1.0000				
		4TH	129.36955	.0810	.2623	.4869	.7360	1.0000			
		5TH	192.93569	.0631	.2044	.3793	.5732	.7819	1.0000		
.10	.50	1ST	13.96281	.3280	1.0000						
		2ND	37.51488	.1685	.5410	1.0000					
		3RD	72.14631	.1090	.3494	.6574	1.0000				
		4TH	117.86867	.0786	.2519	.4737	.7270	1.0000			
		5TH	174.68916	.0604	.1938	.3644	.5591	.7732	1.0000		
.10	.60	1ST	13.47367	.3343	1.0000						
		2ND	34.95393	.1678	.5357	1.0000					
		3RD	66.08781	.1065	.3394	.6483	1.0000				
		4TH	106.86784	.0756	.2408	.4596	.7173	1.0000			
		5TH	157.31317	.0573	.1827	.3487	.5440	.7638	1.0000		
.10	.80	1ST	12.43560	.3446	1.0000						
		2ND	30.02742	.1635	.5233	1.0000					
		3RD	54.65624	.0993	.3168	.6280	1.0000				
		4TH	86.36714	.0678	.2164	.4284	.6954	1.0000			
		5TH	125.17338	.0498	.1588	.3145	.5101	.7422	1.0000		
.10	1.00	1ST	11.32564	.3517	1.0000						
		2ND	25.31434	.1553	.5081	1.0000					
		3RD	44.14037	.0891	.2903	.6044	1.0000				
		4TH	67.86891	.0579	.1888	.3924	.6694	1.0000			
		5TH	96.51744	.0407	.1328	.2759	.4702	.7161	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.20	.10	1ST	16.28719	.3077	1.0000						
		2ND	49.33404	.1713	.5639	1.0000					
		3RD	100.20407	.1180	.3882	.6917	1.0000				
		4TH	168.88618	.0896	.2950	.5255	.7614	1.0000			
		5TH	255.38217	.0721	.2373	.4227	.6124	.8054	1.0000		
.20	.20	1ST	15.87465	.3173	1.0000						
		2ND	46.52435	.1739	.5604	1.0000					
		3RD	93.16742	.1183	.3808	.6847	1.0000				
		4TH	155.79447	.0889	.2862	.5144	.7541	1.0000			
		5TH	234.46858	.0708	.2281	.4100	.6009	.7985	1.0000		
.20	.30	1ST	15.43540	.3259	1.0000						
		2ND	43.76795	.1756	.5565	1.0000					
		3RD	86.35815	.1178	.3728	.6772	1.0000				
		4TH	143.20236	.0875	.2769	.5028	.7464	1.0000			
		5TH	214.30526	.0690	.2184	.3966	.5887	.7912	1.0000		
.20	.40	1ST	14.97140	.3335	1.0000						
		2ND	41.06531	.1764	.5522	1.0000					
		3RD	79.77653	.1165	.3642	.6693	1.0000				
		4TH	131.11007	.0854	.2670	.4904	.7381	1.0000			
		5TH	195.07231	.0666	.2083	.3826	.5757	.7833	1.0000		
.20	.50	1ST	14.48444	.3402	1.0000						
		2ND	38.41688	.1762	.5473	1.0000					
		3RD	73.42282	.1144	.3549	.6609	1.0000				
		4TH	119.51771	.0827	.2565	.4773	.7292	1.0000			
		5TH	176.70983	.0637	.1976	.3678	.5617	.7748	1.0000		
.20	.60	1ST	13.97613	.3461	1.0000						
		2ND	35.82312	.1752	.5420	1.0000					
		3RD	67.29729	.1117	.3448	.6519	1.0000				
		4TH	108.42548	.0794	.2453	.4634	.7197	1.0000			
		5TH	159.21796	.0604	.1864	.3522	.5467	.7655	1.0000		
.20	.80	1ST	12.90096	.3555	1.0000						
		2ND	30.80149	.1703	.5296	1.0000					
		3RD	55.73196	.1039	.3222	.6320	1.0000				
		4TH	87.74213	.0712	.2208	.4324	.6981	1.0000			
		5TH	126.84654	.0524	.1624	.3181	.5131	.7442	1.0000		
.20	1.00	1ST	11.75495	.3620	1.0000						
		2ND	26.00374	.1615	.5146	1.0000					
		3RD	45.08263	.0932	.2956	.6088	1.0000				
		4TH	69.06147	.0608	.1930	.3966	.6725	1.0000			
		5TH	97.95907	.0429	.1361	.2796	.4736	.7184	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.30	.10	1ST	16.89661	.3218	1.0000						
		2ND	50.41502	.1805	.5701	1.0000					
		3RD	101.75608	.1247	.3938	.6947	1.0000				
		4TH	170.90811	.0949	.2997	.5287	.7631	1.0000			
		5TH	257.87359	.0765	.2414	.4258	.6145	.8066	1.0000		
.30	.20	1ST	16.46240	.3307	1.0000						
		2ND	47.56186	.1827	.5666	1.0000					
		3RD	94.65202	.1246	.3863	.6878	1.0000				
		4TH	157.72475	.0938	.2909	.5177	.7560	1.0000			
		5TH	236.78396	.0749	.2321	.4131	.6031	.7998	1.0000		
.30	.30	1ST	16.00231	.3386	1.0000						
		2ND	44.75219	.1839	.5626	1.0000					
		3RD	87.77546	.1238	.3783	.6804	1.0000				
		4TH	145.04106	.0921	.2815	.5062	.7484	1.0000			
		5TH	216.56462	.0728	.2224	.3999	.5910	.7926	1.0000		
.30	.40	1ST	15.51818	.3457	1.0000						
		2ND	42.01646	.1843	.5583	1.0000					
		3RD	81.12665	.1222	.3696	.6726	1.0000				
		4TH	132.85724	.0897	.2716	.4939	.7402	1.0000			
		5TH	197.21571	.0701	.2121	.3859	.5781	.7847	1.0000		
.30	.50	1ST	15.01169	.3519	1.0000						
		2ND	39.32513	.1838	.5534	1.0000					
		3RD	74.70584	.1198	.3603	.6643	1.0000				
		4TH	121.17343	.0868	.2610	.4809	.7314	1.0000			
		5TH	178.73730	.0670	.2014	.3712	.5643	.7763	1.0000		
.30	.60	1ST	14.48436	.3573	1.0000						
		2ND	36.68863	.1824	.5481	1.0000					
		3RD	68.51333	.1168	.3502	.6555	1.0000				
		4TH	109.98981	.0833	.2497	.4671	.7220	1.0000			
		5TH	161.12957	.0634	.1901	.3556	.5494	.7671	1.0000		
.30	.80	1ST	13.37229	.3660	1.0000						
		2ND	31.58196	.1769	.5358	1.0000					
		3RD	56.81429	.1085	.3275	.6359	1.0000				
		4TH	89.12387	.0745	.2251	.4364	.7008	1.0000			
		5TH	128.52653	.0549	.1659	.3216	.5161	.7461	1.0000		
.30	1.00	1ST	12.19031	.3719	1.0000						
		2ND	26.69958	.1676	.5209	1.0000					
		3RD	46.03153	.0972	.3009	.6130	1.0000				
		4TH	70.26080	.0637	.1972	.4008	.6755	1.0000			
		5TH	99.40757	.0450	.1393	.2833	.4769	.7207	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)					
.40	.10	1ST	17.51065	.3352	1.0000				
		2ND	51.50197	.1894	.5761	1.0000			
		3RD	103.31447	.1313	.3992	.6976	1.0000		
		4TH	172.93664	.1001	.3044	.5318	.7649	1.0000	
		5TH	260.37175	.0807	.2454	.4288	.6166	.8077	1.0000
.40	.20	1ST	17.05512	.3434	1.0000				
		2ND	48.60544	.1911	.5725	1.0000			
		3RD	96.14306	.1308	.3916	.6908	1.0000		
		4TH	159.66165	.0987	.2955	.5210	.7578	1.0000	
		5TH	239.16609	.0788	.2361	.4162	.6053	.8011	1.0000
.40	.30	1ST	16.57447	.3508	1.0000				
		2ND	45.76257	.1919	.5685	1.0000			
		3RD	89.19922	.1296	.3836	.6835	1.0000		
		4TH	146.88642	.0967	.2860	.5095	.7503	1.0000	
		5TH	218.83077	.0765	.2262	.4030	.5933	.7939	1.0000
.40	.40	1ST	16.07045	.3573	1.0000				
		2ND	42.97383	.1919	.5641	1.0000			
		3RD	82.48326	.1277	.3749	.6759	1.0000		
		4TH	134.61109	.0940	.2760	.4974	.7422	1.0000	
		5TH	199.36589	.0736	.2159	.3891	.5805	.7861	1.0000
.40	.50	1ST	15.54461	.3631	1.0000				
		2ND	40.23965	.1911	.5593	1.0000			
		3RD	75.99541	.1251	.3655	.6677	1.0000		
		4TH	122.83585	.0908	.2654	.4845	.7336	1.0000	
		5TH	180.77160	.0702	.2051	.3745	.5668	.7778	1.0000
.40	.60	1ST	14.99840	.3681	1.0000				
		2ND	37.56047	.1894	.5540	1.0000			
		3RD	69.73595	.1218	.3554	.6590	1.0000		
		4TH	111.56086	.0870	.2541	.4707	.7243	1.0000	
		5TH	163.04798	.0664	.1938	.3590	.5521	.7687	1.0000
.40	.80	1ST	13.84960	.3762	1.0000				
		2ND	32.36883	.1834	.5418	1.0000			
		3RD	57.90325	.1130	.3327	.6397	1.0000		
		4TH	90.51236	.0779	.2293	.4402	.7033	1.0000	
		5TH	130.21336	.0575	.1694	.3252	.5191	.7480	1.0000
.40	1.00	1ST	12.63170	.3816	1.0000				
		2ND	27.40185	.1736	.5271	1.0000			
		3RD	46.98707	.1013	.3061	.6172	1.0000		
		4TH	71.46690	.0666	.2012	.4049	.6785	1.0000	
		5TH	100.86293	.0472	.1426	.2869	.4802	.7229	1.0000

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.50	.10	1ST	18.12946	.3480	1.0000						
		2ND	52.59495	.1979	.5818	1.0000					
		3RD	104.87926	.1376	.4044	.7004	1.0000				
		4TH	174.97178	.1051	.3089	.5349	.7666	1.0000			
		5TH	262.87666	.0849	.2493	.4318	.6187	.8089	1.0000		
.50	.20	1ST	17.65293	.3556	1.0000						
		2ND	49.65511	.1992	.5782	1.0000					
		3RD	97.64051	.1369	.3968	.6937	1.0000				
		4TH	161.60520	.1034	.2999	.5242	.7596	1.0000			
		5TH	241.55497	.0827	.2399	.4193	.6075	.8023	1.0000		
.50	.30	1ST	17.15199	.3625	1.0000						
		2ND	46.76913	.1997	.5742	1.0000					
		3RD	90.62948	.1353	.3887	.6866	1.0000				
		4TH	148.73843	.1011	.2905	.5128	.7522	1.0000			
		5TH	221.10368	.0801	.2300	.4061	.5956	.7952	1.0000		
.50	.40	1ST	16.62827	.3685	1.0000						
		2ND	43.93743	.1994	.5698	1.0000					
		3RD	83.84638	.1331	.3800	.6790	1.0000				
		4TH	136.37164	.0982	.2804	.5007	.7442	1.0000			
		5TH	201.52288	.0770	.2197	.3923	.5829	.7875	1.0000		
.50	.50	1ST	16.08326	.3739	1.0000						
		2ND	41.16046	.1982	.5650	1.0000					
		3RD	77.29152	.1303	.3706	.6710	1.0000				
		4TH	124.50496	.0948	.2697	.4879	.7357	1.0000			
		5TH	182.81269	.0734	.2088	.3777	.5693	.7793	1.0000		
.50	.60	1ST	15.51829	.3786	1.0000						
		2ND	38.43865	.1963	.5597	1.0000					
		3RD	70.96514	.1267	.3605	.6624	1.0000				
		4TH	113.13864	.0908	.2583	.4743	.7265	1.0000			
		5TH	164.97322	.0693	.1973	.3623	.5547	.7703	1.0000		
.50	.80	1ST	14.33292	.3860	1.0000						
		2ND	33.16211	.1898	.5476	1.0000					
		3RD	58.99883	.1174	.3378	.6434	1.0000				
		4TH	91.90761	.0812	.2335	.4440	.7059	1.0000			
		5TH	131.90701	.0601	.1728	.3286	.5219	.7499	1.0000		
.50	1.00	1ST	13.07914	.3909	1.0000						
		2ND	28.11056	.1795	.5330	1.0000					
		3RD	47.94925	.1053	.3111	.6212	1.0000				
		4TH	72.67976	.0695	.2053	.4090	.6814	1.0000			
		5TH	102.32514	.0493	.1458	.2905	.4834	.7251	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.60	.10	1ST	18.75316	.3602	1.0000						
		2ND	53.69398	.2062	.5874	1.0000					
		3RD	106.45044	.1438	.4095	.7032	1.0000				
		4TH	177.01355	.1101	.3133	.5379	.7683	1.0000			
		5TH	265.38832	.0889	.2532	.4347	.6207	.8100	1.0000		
.60	.20	1ST	18.25593	.3673	1.0000						
		2ND	50.71092	.2071	.5838	1.0000					
		3RD	99.14442	.1427	.4019	.6966	1.0000				
		4TH	163.55538	.1081	.3043	.5273	.7614	1.0000			
		5TH	243.95063	.0866	.2437	.4223	.6096	.8035	1.0000		
.60	.30	1ST	17.73493	.3736	1.0000						
		2ND	47.78188	.2073	.5797	1.0000					
		3RD	92.06621	.1409	.3937	.6896	1.0000				
		4TH	150.59710	.1055	.2948	.5160	.7540	1.0000			
		5TH	223.38338	.0837	.2338	.4092	.5978	.7964	1.0000		
.60	.40	1ST	17.19172	.3793	1.0000						
		2ND	44.90730	.2066	.5753	1.0000					
		3RD	85.21605	.1385	.3850	.6821	1.0000				
		4TH	138.13886	.1024	.2847	.5040	.7462	1.0000			
		5TH	203.68665	.0803	.2233	.3954	.5852	.7889	1.0000		
.60	.50	1ST	16.62768	.3843	1.0000						
		2ND	42.08759	.2052	.5705	1.0000					
		3RD	78.59420	.1353	.3756	.6742	1.0000				
		4TH	126.18083	.0987	.2739	.4913	.7378	1.0000			
		5TH	184.86059	.0765	.2124	.3809	.5717	.7807	1.0000		
.60	.60	1ST	16.04405	.3886	1.0000						
		2ND	39.32319	.2030	.5652	1.0000					
		3RD	72.20091	.1315	.3654	.6657	1.0000				
		4TH	114.72316	.0945	.2625	.4778	.7287	1.0000			
		5TH	166.90530	.0723	.2009	.3655	.5572	.7719	1.0000		
.60	.80	1ST	14.82223	.3955	1.0000						
		2ND	33.95182	.1960	.5532	1.0000					
		3RD	60.10104	.1218	.3427	.6470	1.0000				
		4TH	93.30962	.0844	.2376	.4478	.7084	1.0000			
		5TH	133.60754	.0626	.1761	.3320	.5248	.7517	1.0000		
.60	1.00	1ST	13.53261	.4000	1.0000						
		2ND	28.82571	.1853	.5388	1.0000					
		3RD	48.91807	.1092	.3161	.6252	1.0000				
		4TH	73.89939	.0723	.2093	.4129	.6843	1.0000			
		5TH	103.79419	.0515	.1490	.2940	.4865	.7273	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
.80	.10	1ST	20.01576	.3830	1.0000						
		2ND	55.91034	.2220	.5979	1.0000					
		3RD	109.61211	.1558	.4192	.7086	1.0000				
		4TH	181.11694	.1196	.3219	.5438	.7715	1.0000			
		5TH	270.43190	.0968	.2606	.4404	.6247	.8122	1.0000		
.80	.20	1ST	19.47788	.3892	1.0000						
		2ND	52.84103	.2222	.5943	1.0000					
		3RD	102.17166	.1541	.4116	.7021	1.0000				
		4TH	167.47572	.1171	.3128	.5333	.7648	1.0000			
		5TH	248.76226	.0940	.2511	.4281	.6138	.8058	1.0000		
.80	.30	1ST	18.91740	.3947	1.0000						
		2ND	49.82607	.2218	.5902	1.0000					
		3RD	94.95921	.1518	.4034	.6953	1.0000				
		4TH	154.33448	.1141	.3032	.5222	.7577	1.0000			
		5TH	227.96313	.0907	.2410	.4152	.6021	.7989	1.0000		
.80	.40	1ST	18.33567	.3996	1.0000						
		2ND	46.86588	.2206	.5859	1.0000					
		3RD	87.97500	.1488	.3946	.6881	1.0000				
		4TH	141.69343	.1105	.2930	.5105	.7500	1.0000			
		5TH	208.03464	.0869	.2304	.4015	.5898	.7915	1.0000		
.80	.50	1ST	17.73395	.4039	1.0000						
		2ND	43.96086	.2186	.5811	1.0000					
		3RD	81.21926	.1452	.3852	.6803	1.0000				
		4TH	129.55267	.1064	.2822	.4979	.7418	1.0000			
		5TH	188.97685	.0827	.2194	.3872	.5765	.7836	1.0000		
.80	.60	1ST	17.11332	.4076	1.0000						
		2ND	41.11139	.2159	.5759	1.0000					
		3RD	74.69226	.1410	.3750	.6721	1.0000				
		4TH	117.91239	.1018	.2707	.4846	.7330	1.0000			
		5TH	170.78995	.0781	.2078	.3719	.5623	.7749	1.0000		
.80	.80	1ST	15.81892	.4135	1.0000						
		2ND	35.58051	.2081	.5640	1.0000					
		3RD	62.32536	.1305	.3524	.6539	1.0000				
		4TH	96.13392	.0909	.2456	.4550	.7132	1.0000			
		5TH	137.02913	.0677	.1828	.3387	.5303	.7552	1.0000		
.80	1.00	1ST	14.45767	.4172	1.0000						
		2ND	30.27530	.1967	.5499	1.0000					
		3RD	50.87562	.1171	.3257	.6328	1.0000				
		4TH	76.35894	.0780	.2171	.4206	.6898	1.0000			
		5TH	106.75287	.0558	.1553	.3009	.4927	.7315	1.0000		

TABLE 2. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF ANTISYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K	LOCATIONS OF NODES (X/L)							
1.00	.10	1ST	21.29928	.4040	1.0000						
		2ND	58.15127	.2368	.6078	1.0000					
		3RD	112.79963	.1671	.4285	.7137	1.0000				
		4TH	185.24694	.1287	.3300	.5495	.7747	1.0000			
		5TH	275.50256	.1045	.2678	.4459	.6285	.8143	1.0000		
1.00	.20	1ST	20.72162	.4093	1.0000						
		2ND	54.99597	.2365	.6041	1.0000					
		3RD	105.22490	.1650	.4208	.7075	1.0000				
		4TH	171.42275	.1258	.3209	.5392	.7681	1.0000			
		5TH	253.60104	.1012	.2581	.4338	.6178	.8081	1.0000		
1.00	.30	1ST	20.12233	.4141	1.0000						
		2ND	51.89532	.2355	.6001	1.0000					
		3RD	97.87833	.1622	.4126	.7008	1.0000				
		4TH	158.09865	.1223	.3112	.5282	.7612	1.0000			
		5TH	232.57011	.0975	.2480	.4210	.6064	.8013	1.0000		
1.00	.40	1ST	19.50265	.4184	1.0000						
		2ND	48.84971	.2338	.5958	1.0000					
		3RD	90.76018	.1588	.4038	.6937	1.0000				
		4TH	145.27482	.1184	.3010	.5166	.7537	1.0000			
		5TH	212.40985	.0933	.2374	.4075	.5942	.7941	1.0000		
1.00	.50	1ST	18.86367	.4221	1.0000						
		2ND	45.85953	.2315	.5910	1.0000					
		3RD	83.87066	.1548	.3944	.6863	1.0000				
		4TH	132.95142	.1139	.2901	.5043	.7457	1.0000			
		5TH	193.12044	.0888	.2262	.3932	.5811	.7863	1.0000		
1.00	.60	1ST	18.20635	.4254	1.0000						
		2ND	42.92513	.2283	.5859	1.0000					
		3RD	77.21004	.1502	.3843	.6783	1.0000				
		4TH	121.12860	.1089	.2786	.4912	.7371	1.0000			
		5TH	174.70194	.0838	.2145	.3781	.5671	.7779	1.0000		
1.00	.80	1ST	16.83970	.4304	1.0000						
		2ND	37.22491	.2198	.5742	1.0000					
		3RD	64.57622	.1389	.3616	.6606	1.0000				
		4TH	98.98528	.0973	.2533	.4620	.7179	1.0000			
		5TH	140.47814	.0727	.1892	.3451	.5357	.7587	1.0000		
1.00	1.00	1ST	15.40683	.4334	1.0000						
		2ND	31.75062	.2076	.5604	1.0000					
		3RD	52.85972	.1247	.3350	.6400	1.0000				
		4TH	78.84556	.0836	.2246	.4281	.6951	1.0000			
		5TH	109.73896	.0601	.1614	.3076	.4987	.7355	1.0000		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.10	.00	1ST	5.96531	.46975						
		2ND	31.09310	.20187	.71811					
		3RD	75.99965	.12912	.45938	.82068				
		4TH	140.64556	.09492	.33769	.60327	.86795			
		5TH	225.03065	.07504	.26696	.47693	.68618	.89554		
.20	.00	1ST	6.34515	.48935						
		2ND	31.96779	.21423	.72426					
		3RD	77.35779	.13770	.46560	.82303				
		4TH	142.50713	.10146	.34306	.60642	.86922			
		5TH	227.38572	.08032	.27159	.48008	.68812	.89532		
.30	.00	1ST	6.73274	.50735						
		2ND	32.84992	.22601	.73009					
		3RD	78.74329	.14597	.47159	.82531				
		4TH	144.37609	.10780	.34828	.60950	.87045			
		5TH	229.74813	.08546	.27609	.48317	.69003	.89709		
.40	.00	1ST	7.12804	.52396						
		2ND	33.73947	.23726	.73562					
		3RD	80.12618	.15395	.47738	.82753				
		4TH	146.25239	.11395	.35335	.61251	.87166			
		5TH	232.11789	.09045	.28048	.48620	.69190	.89785		
.50	.00	1ST	7.53098	.53933						
		2ND	34.63642	.24802	.74089					
		3RD	81.51643	.16167	.48297	.82968				
		4TH	148.13605	.11993	.35827	.61546	.87284			
		5TH	234.49499	.09532	.28476	.48917	.69374	.89859		
.60	.00	1ST	7.94150	.55362						
		2ND	35.54077	.25834	.74591					
		3RD	82.91404	.16914	.48837	.83177				
		4TH	150.02705	.12574	.36306	.61834	.87400			
		5TH	236.87944	.10007	.28893	.49209	.69555	.89933		
.80	.00	1ST	8.78507	.57938						
		2ND	37.37162	.27776	.75529					
		3RD	85.73135	.18339	.49867	.83578				
		4TH	153.83114	.13691	.37227	.62392	.87624			
		5TH	241.67039	.10923	.29701	.49779	.69909	.90076		
1.00	.00	1ST	9.65836	.60201						
		2ND	39.23190	.29576	.76387					
		3RD	88.57805	.19683	.50835	.83957				
		4TH	157.66458	.14753	.38103	.62928	.87840			
		5TH	246.49069	.11799	.30474	.50328	.70252	.90214		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.00	.10	1ST	5.78131	.46626						
		2ND	28.95721	.19634	.71191					
		3RD	69.97776	.12340	.44749	.81453				
		4TH	128.81296	.08950	.32457	.59079	.86277			
		5TH	205.46280	.07000	.25386	.46207	.67480	.89113		
.00	.20	1ST	5.95253	.48209						
		2ND	27.69303	.20243	.71182					
		3RD	65.44420	.12554	.44146	.81058				
		4TH	119.18417	.08998	.31642	.58097	.85862			
		5TH	188.91291	.06966	.24498	.44980	.66476	.88726		
.00	.30	1ST	6.10602	.49608						
		2ND	26.43320	.20725	.71133					
		3RD	61.03814	.12668	.43479	.80639				
		4TH	109.90489	.08963	.30764	.57055	.85417			
		5TH	173.03329	.06863	.23555	.43686	.65403	.88309		
.00	.40	1ST	6.24080	.50845						
		2ND	25.17752	.21085	.71042					
		3RD	56.75954	.12686	.42743	.80191				
		4TH	100.97510	.08851	.29820	.55945	.84940			
		5TH	157.82389	.06695	.22557	.42319	.64252	.87858		
.00	.50	1ST	6.35582	.51939						
		2ND	23.92577	.21326	.70908					
		3RD	52.60829	.12612	.41932	.79712				
		4TH	92.39474	.08664	.28806	.54759	.84427			
		5TH	143.28472	.06467	.21501	.40871	.63015	.87369		
.00	.60	1ST	6.45002	.52904						
		2ND	22.67767	.21450	.70728					
		3RD	48.58426	.12447	.41040	.79198				
		4TH	84.16375	.08407	.27718	.53488	.83871			
		5TH	129.41568	.06182	.20384	.39335	.61679	.86836		
.00	.80	1ST	6.57120	.54491						
		2ND	20.19084	.21341	.70218					
		3RD	40.91702	.11847	.38975	.78038				
		4TH	68.74930	.07688	.25292	.50640	.82610			
		5TH	103.68770	.05459	.17959	.35957	.58656	.85609		
.00	1.00	1ST	6.59365	.55667						
		2ND	17.71250	.20723	.69469					
		3RD	33.75518	.10874	.36453	.76654				
		4TH	54.73005	.06707	.22482	.47277	.81089			
		5TH	80.63878	.04552	.15259	.32087	.55036	.84104		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.10	.10	1ST	6.15072	.48586						
		2ND	29.79616	.20850	.71824					
		3RD	71.28543	.13174	.45385	.81703				
		4TH	130.58940	.09579	.33002	.59410	.86413			
		5TH	207.70799	.07503	.25850	.46536	.67686	.89198		
.10	.20	1ST	6.31897	.50013						
		2ND	28.50364	.21387	.71800					
		3RD	66.69875	.13336	.44773	.81316				
		4TH	120.88277	.09584	.32177	.58439	.86005			
		5TH	191.05561	.07432	.24952	.45317	.66693	.88817		
.10	.30	1ST	6.46911	.51279						
		2ND	27.21538	.21805	.71738					
		3RD	62.23955	.13404	.44099	.80905				
		4TH	111.52566	.09511	.31291	.57407	.85568			
		5TH	175.07348	.07295	.24000	.44031	.65631	.88406		
.10	.40	1ST	6.60016	.52403						
		2ND	25.93119	.22107	.71637					
		3RD	57.90776	.13380	.43356	.80467				
		4TH	102.51802	.09363	.30340	.56308	.85100			
		5TH	159.76156	.07096	.22993	.42673	.64492	.87962		
.10	.50	1ST	6.71110	.53398						
		2ND	24.65083	.22297	.71495					
		3RD	53.70329	.13268	.42541	.79998				
		4TH	93.85979	.09144	.29319	.55133	.84596			
		5TH	145.11982	.06839	.21927	.41234	.63267	.87480		
.10	.60	1ST	6.80083	.54279						
		2ND	23.37400	.22375	.71310					
		3RD	49.62598	.13068	.41645	.79493				
		4TH	85.55088	.08857	.28224	.53874	.84051			
		5TH	131.14822	.06527	.20802	.39706	.61946	.86956		
.10	.80	1ST	6.91196	.55728						
		2ND	20.82924	.22185	.70794					
		3RD	41.85190	.12403	.39576	.78359				
		4TH	69.98044	.08081	.25786	.51054	.82814			
		5TH	105.21501	.05753	.18357	.36345	.58954	.85750		
.10	1.00	1ST	6.92273	.56801						
		2ND	18.29207	.21497	.70049					
		3RD	34.58267	.11370	.37051	.77005				
		4TH	55.80481	.07046	.22961	.47721	.81324			
		5TH	81.96057	.04798	.15634	.32492	.55371	.84272		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)			
.20	.10	1ST	6.52781	.50388				
		2ND	30.64254	.22011	.72424			
		3RD	72.60046	.13979	.45998	.81945		
		4TH	132.37318	.10190	.33531	.59734	.86545	
		5TH	209.96057	.07994	.26303	.46858	.67889	.89282
.20	.20	1ST	6.69293	.51680				
		2ND	29.32163	.22482	.72387			
		3RD	67.96065	.14094	.45379	.81566		
		4TH	122.58873	.10155	.32698	.58773	.86144	
		5TH	193.20567	.07887	.25396	.45647	.66906	.88906
.20	.30	1ST	6.83959	.52831				
		2ND	28.00492	.22841	.72314			
		3RD	63.44830	.14119	.44697	.81163		
		4TH	113.15379	.10046	.31805	.57751	.85715	
		5TH	177.12101	.07718	.24435	.44370	.65854	.88501
.20	.40	1ST	6.96679	.53854				
		2ND	26.69218	.23091	.72205			
		3RD	59.06333	.14056	.43950	.80734		
		4TH	104.06828	.09866	.30847	.56662	.85255	
		5TH	161.70657	.07490	.23420	.43019	.64727	.88064
.20	.50	1ST	7.07350	.54763				
		2ND	25.38318	.23234	.72056			
		3RD	54.80561	.13908	.43130	.80274		
		4TH	95.33216	.09616	.29820	.55499	.84760	
		5TH	146.96230	.07206	.22346	.41589	.63515	.87589
.20	.60	1ST	7.15866	.55568				
		2ND	24.07757	.23269	.71865			
		3RD	50.67498	.13675	.42232	.79780		
		4TH	86.94532	.09300	.28719	.54252	.84225	
		5TH	132.88810	.06869	.21212	.40070	.62207	.87073
.20	.80	1ST	7.25951	.56895				
		2ND	21.47475	.23005	.71345			
		3RD	42.79401	.12950	.40159	.78669		
		4TH	71.21884	.08471	.26269	.51458	.83012	
		5TH	106.74961	.06046	.18749	.36726	.59246	.85887
.20	1.00	1ST	7.25837	.57875				
		2ND	18.87860	.22252	.70605			
		3RD	35.41729	.11861	.37635	.77344		
		4TH	56.88678	.07385	.23431	.48154	.81552	
		5TH	83.28960	.05044	.16004	.32889	.55700	.84435

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.30	.10	1ST	6.91251	.52052						
		2ND	31.49632	.23122	.72993					
		3RD	73.92286	.14757	.46590	.82181				
		4TH	134.16430	.10784	.34045	.60051	.86675			
		5TH	212.22046	.08471	.26744	.47174	.68089	.89364		
.30	.20	1ST	7.07437	.53226						
		2ND	30.14701	.23533	.72945					
		3RD	69.22991	.14828	.45964	.81809				
		4TH	124.30202	.10712	.33205	.59099	.86281			
		5TH	195.36306	.08333	.25829	.45971	.67115	.88993		
.30	.30	1ST	7.21740	.54275						
		2ND	28.80179	.23839	.72863					
		3RD	64.66439	.14814	.45277	.81413				
		4TH	114.78924	.10570	.32305	.58087	.85859			
		5TH	179.17588	.08134	.24861	.44702	.66074	.88594		
.30	.40	1ST	7.34061	.55210						
		2ND	27.46048	.24041	.72746					
		3RD	60.22621	.14716	.44525	.80992				
		4TH	105.62587	.10358	.31341	.57009	.85407			
		5TH	163.65890	.07878	.23838	.43360	.64958	.88163		
.30	.50	1ST	7.44300	.56042						
		2ND	26.12277	.24140	.72591					
		3RD	55.91522	.14535	.43703	.80542				
		4TH	96.81186	.10080	.30309	.55857	.84921			
		5TH	148.81207	.07568	.22756	.41938	.63758	.87696		
.30	.60	1ST	7.52346	.56780						
		2ND	24.78834	.24136	.72397					
		3RD	51.73126	.14271	.42802	.80058				
		4TH	88.34706	.09737	.29203	.54622	.84396			
		5TH	134.63529	.07207	.21614	.40427	.62464	.87188		
.30	.80	1ST	7.61380	.57997						
		2ND	22.12734	.23801	.71874					
		3RD	43.74332	.13488	.40727	.78969				
		4TH	72.46450	.08857	.26743	.51853	.83205			
		5TH	108.29149	.06337	.19135	.37100	.59533	.86022		
.30	1.00	1ST	7.60053	.58894						
		2ND	19.47208	.22988	.71139					
		3RD	36.25901	.12345	.38204	.77672				
		4TH	57.97592	.07721	.23893	.48577	.81773			
		5TH	84.62586	.05290	.16369	.33280	.56022	.84595		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.40	.10	1ST	7.30476	.53593						
		2ND	32.35749	.24186	.73535					
		3RD	75.25262	.15511	.47161	.82409				
		4TH	135.96279	.11361	.34544	.60361	.86802			
		5TH	214.48772	.08938	.27175	.47485	.68285	.89444		
.40	.20	1ST	7.46322	.54664						
		2ND	30.97974	.24542	.73477					
		3RD	70.50652	.15542	.46530	.82044				
		4TH	126.02268	.11256	.33698	.59418	.86414			
		5TH	197.52780	.08769	.26253	.46290	.67321	.89079		
.40	.30	1ST	7.60249	.55623						
		2ND	29.60600	.24799	.73387					
		3RD	65.88782	.15491	.45839	.81656				
		4TH	116.43204	.11082	.32793	.58416	.86000			
		5TH	181.23810	.08542	.25278	.45028	.66290	.88685		
.40	.40	1ST	7.72161	.56480						
		2ND	28.23606	.24957	.73263					
		3RD	61.39641	.15359	.45083	.81243				
		4TH	107.19079	.10842	.31824	.57348	.85555			
		5TH	165.61856	.08260	.24247	.43694	.65185	.88261		
.40	.50	1ST	7.81954	.57244						
		2ND	26.86962	.25016	.73103					
		3RD	57.03213	.15147	.44258	.80801				
		4TH	98.29884	.10537	.30788	.56207	.85078			
		5TH	150.66916	.07926	.23159	.42280	.63997	.87801		
.40	.60	1ST	7.89520	.57922						
		2ND	25.50630	.24975	.72905					
		3RD	52.79481	.14854	.43356	.80327				
		4TH	89.75611	.10168	.29677	.54983	.84562			
		5TH	136.38979	.07541	.22010	.40778	.62715	.87301		
.40	.80	1ST	7.97481	.59040						
		2ND	22.78702	.24577	.72381					
		3RD	44.69932	.14018	.41281	.79260				
		4TH	73.71740	.09239	.27208	.52239	.83394			
		5TH	109.84064	.06627	.19515	.37468	.59814	.86153		
.40	1.00	1ST	7.94921	.59862						
		2ND	20.07249	.23707	.71653					
		3RD	37.10783	.12824	.38759	.77990				
		4TH	59.07225	.08056	.24347	.48992	.81990			
		5TH	85.96933	.05535	.16730	.33664	.56338	.84752		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE - FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.50	.10	1ST	7.70451	.55026						
		2ND	33.22605	.25206	.74052					
		3RD	76.58974	.16241	.47714	.82630				
		4TH	137.76862	.11924	.35031	.60664	.86926			
		5TH	216.76231	.09393	.27596	.47790	.68478	.89523		
.50	.20	1ST	7.85944	.56006						
		2ND	31.81981	.25513	.73985					
		3RD	71.79048	.16235	.47078	.82272				
		4TH	127.75066	.11787	.34179	.59730	.86545			
		5TH	199.69988	.09196	.26667	.46602	.67523	.89163		
.50	.30	1ST	7.99483	.56885						
		2ND	30.41752	.25725	.73888					
		3RD	67.11858	.16150	.46383	.81892				
		4TH	118.08217	.11584	.33269	.58737	.86137			
		5TH	183.30764	.08943	.25686	.45348	.66502	.88775		
.50	.40	1ST	8.10973	.57673						
		2ND	29.01891	.25842	.73758					
		3RD	62.57391	.15987	.45625	.81486				
		4TH	108.76303	.11316	.32296	.57679	.85700			
		5TH	167.58556	.08637	.24649	.44022	.65408	.88357		
.50	.50	1ST	8.20310	.58375						
		2ND	27.62368	.25864	.73594					
		3RD	58.15633	.15746	.44798	.81053				
		4TH	99.79315	.10986	.31255	.56549	.85231			
		5TH	152.53357	.08279	.23555	.42617	.64232	.87904		
.50	.60	1ST	8.27384	.59000						
		2ND	26.23144	.25790	.73394					
		3RD	53.86561	.15426	.43894	.80588				
		4TH	91.17242	.10592	.30141	.55336	.84724			
		5TH	138.15160	.07872	.22399	.41123	.62962	.87411		
.50	.80	1ST	8.34252	.60029						
		2ND	23.45375	.25331	.72869					
		3RD	45.66350	.14539	.41819	.79542				
		4TH	74.97754	.09617	.27664	.52616	.83577			
		5TH	111.39706	.06915	.19890	.37830	.60090	.86282		
.50	1.00	1ST	8.30437	.60783						
		2ND	20.67981	.24409	.72147					
		3RD	37.95372	.13296	.39300	.78298				
		4TH	60.17573	.08388	.24794	.49397	.82200			
		5TH	87.32002	.05781	.17086	.34041	.56647	.84904		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)				
.60	.10	1ST	8.11171	.56362					
		2ND	34.10197	.26187	.74545				
		3RD	77.93421	.16950	.48250	.82845			
		4TH	139.58179	.12473	.35504	.60961	.87048		
		5TH	219.04425	.09839	.28008	.48089	.68668	.89601	
.60	.20	1ST	8.26298	.57262					
		2ND	32.66723	.26448	.74471				
		3RD	73.08178	.16909	.47609	.82494			
		4TH	129.48601	.12306	.34648	.60035	.86672		
		5TH	201.87928	.09615	.27073	.46909	.67722	.89246	
.60	.30	1ST	8.39438	.58070					
		2ND	31.23633	.26619	.74367				
		3RD	68.35665	.16793	.46911	.82121			
		4TH	119.73962	.12076	.33734	.59052	.86271		
		5TH	185.38451	.09338	.26086	.45663	.66711	.88863	
.60	.40	1ST	8.50494	.58795					
		2ND	29.80902	.26698	.74232				
		3RD	63.75872	.16600	.46151	.81723			
		4TH	110.34258	.11783	.32757	.58004	.85842		
		5TH	169.55988	.09008	.25044	.44345	.65627	.88451	
.60	.50	1ST	8.59363	.59443					
		2ND	28.38496	.26686	.74065				
		3RD	59.28779	.16332	.45323	.81298			
		4TH	101.29476	.11428	.31713	.56883	.85381		
		5TH	154.40532	.08628	.23943	.42947	.64462	.88005	
.60	.60	1ST	8.65935	.60019					
		2ND	26.96374	.26580	.73862				
		3RD	54.94365	.15986	.44419	.80841			
		4TH	92.59603	.11011	.30595	.55681	.84883		
		5TH	139.92068	.08199	.22782	.41462	.63205	.87520	
.60	.80	1ST	8.71689	.60968					
		2ND	24.12752	.26065	.73338				
		3RD	46.63435	.15051	.42345	.79816			
		4TH	76.24491	.09992	.28111	.52986	.83756		
		5TH	112.96074	.07201	.20259	.38185	.60360	.86409	
.60	1.00	1ST	8.66598	.61661					
		2ND	21.29402	.25094	.72623				
		3RD	38.82667	.13762	.39829	.78597			
		4TH	61.28636	.08719	.25233	.49794	.82405		
		5TH	88.67790	.06026	.17439	.34413	.56951	.85054	

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
.80	.10	1ST	8.94826	.58784						
		2ND	35.87586	.28040	.75467					
		3RD	80.64518	.18308	.49272	.83258				
		4TH	143.23018	.13531	.36417	.61535	.87283			
		5TH	223.63016	.10703	.28805	.48672	.69038	.89753		
.80	.20	1ST	9.09185	.59547						
		2ND	34.38400	.28220	.75380					
		3RD	75.68635	.18205	.48625	.82919				
		4TH	132.97867	.13311	.35553	.60627	.86919			
		5TH	206.26014	.10431	.27860	.47507	.68109	.89407		
.80	.30	1ST	9.21490	.60236						
		2ND	32.89580	.28317	.75266					
		3RD	70.85474	.18032	.47923	.82559				
		4TH	123.07650	.13031	.34632	.59661	.86531			
		5TH	189.56025	.10107	.26863	.46276	.67117	.89035		
.80	.40	1ST	9.31647	.60854						
		2ND	31.41095	.28330	.75124					
		3RD	66.15020	.17786	.47160	.82176				
		4TH	113.52360	.12691	.33650	.58632	.86116			
		5TH	173.53047	.09734	.25811	.44973	.66054	.88634		
.80	.50	1ST	9.39549	.61408						
		2ND	29.92909	.28256	.74951					
		3RD	61.57252	.17469	.46331	.81766				
		4TH	104.31985	.12291	.32600	.57532	.85670			
		5TH	158.17068	.09313	.24701	.43591	.64911	.88201		
.80	.60	1ST	9.45086	.61901						
		2ND	28.44975	.28093	.74746					
		3RD	57.12144	.17076	.45427	.81326				
		4TH	95.46505	.11832	.31477	.56351	.85189			
		5TH	143.48078	.08844	.23529	.42122	.63678	.87730		
.80	.80	1ST	9.48552	.62712						
		2ND	25.49611	.27478	.74223					
		3RD	48.59751	.16052	.43357	.80340				
		4TH	78.80130	.10730	.28982	.53702	.84102			
		5TH	116.10986	.07768	.20982	.38878	.60887	.86654		
.80	1.00	1ST	9.40851	.63297						
		2ND	22.54303	.26417	.73523					
		3RD	40.57372	.14678	.40850	.79168				
		4TH	63.52904	.09374	.26089	.50562	.82800			
		5TH	91.41526	.06515	.18131	.35138	.57542	.85343		

TABLE 3. NATURAL FREQUENCIES AND NODES FOR TAPERED FREE-FREE BEAMS OF SYMMETRICAL MODE

$\Psi$	$\Phi$	MODE	FREQUENCY CONSTANT, K		LOCATIONS OF NODES (X/L)					
1.00	.10	1ST	9.81405	.60923						
		2ND	37.67907	.29763	.76314					
		3RD	83.38550	.19593	.50235	.83648				
		4TH	146.90791	.14543	.37287	.62086	.87509			
		5TH	228.24542	.11533	.29569	.49236	.69396	.89900		
1.00	.20	1ST	9.94950	.61577						
		2ND	36.12995	.29874	.76217					
		3RD	78.32020	.19437	.49584	.83321				
		4TH	136.50064	.14276	.36418	.61194	.87156			
		5TH	210.67030	.11217	.28616	.48084	.68484	.89563		
1.00	.30	1ST	10.06378	.62168						
		2ND	34.58428	.29908	.76095					
		3RD	73.38202	.19214	.48880	.82974				
		4TH	126.44264	.13951	.35492	.60246	.86781			
		5TH	193.76527	.10853	.27611	.46868	.67510	.89201		
1.00	.40	1ST	10.15593	.62700						
		2ND	33.04173	.29863	.75947					
		3RD	68.57077	.18922	.48116	.82604				
		4TH	116.73384	.13569	.34505	.59235	.86379			
		5TH	177.53030	.10441	.26551	.45580	.66467	.88811		
1.00	.50	1ST	10.22488	.63177						
		2ND	31.50187	.29736	.75771					
		3RD	63.88623	.18560	.47287	.82209				
		4TH	107.37406	.13129	.33451	.58154	.85947			
		5TH	161.96526	.09982	.25433	.44214	.65345	.88390		
1.00	.60	1ST	10.26950	.63601						
		2ND	29.96420	.29524	.75564					
		3RD	59.32808	.18125	.46384	.81785				
		4TH	98.35312	.12631	.32325	.56993	.85482			
		5TH	147.07000	.09477	.24252	.42761	.64134	.87933		
1.00	.80	1ST	10.28050	.64296						
		2ND	26.89264	.28820	.75047					
		3RD	50.58920	.17022	.44321	.80834				
		4TH	81.38650	.11453	.29822	.54389	.84432			
		5TH	119.28795	.08328	.21685	.39549	.61395	.86889		
1.00	1.00	1ST	10.17662	.64792						
		2ND	23.81939	.27682	.74362					
		3RD	42.34886	.15570	.41825	.79707				
		4TH	65.80021	.10021	.26919	.51299	.83176			
		5TH	94.18135	.07001	.18807	.35841	.58111	.85621		

TABLE 4. NATURAL FREQUENCIES FOR TRUNCATED BEAMS

$\Psi$	$\Phi$	C = 0.2		C = 0.4	
		1ST MODE	2ND MODE	1ST MODE	2ND MODE
.1	.0	3.67847	22.36876	3.61365	22.22533
.2	.0	3.84786	22.70905	3.71272	22.41816
.3	.0	4.02157	23.05519	3.81430	22.61243
.4	.0	4.20091	23.40771	3.91801	22.80770
.5	.0	4.38517	23.76588	4.02307	23.00573
.6	.0	4.57712	24.13146	4.13134	23.20523
.7	.0	4.77262	24.50256	4.24163	23.40584
.8	.0	4.97587	24.88047	4.35377	23.60921
.9	.0	5.18301	25.26468	4.46853	23.81437
1.0	.0	5.39691	25.65562	4.58573	24.02130
.0	.1	3.61104	21.40556	3.56582	21.56926
.0	.2	3.70551	20.77436	3.61455	21.10498
.0	.3	3.79561	20.14155	3.66175	20.64266
.0	.4	3.88164	19.50751	3.70660	20.18308
.0	.5	3.96387	18.87360	3.75005	19.72495
.0	.6	4.04053	18.24194	3.79130	19.27075
.0	.7	4.11409	17.61204	3.83125	18.82074
.0	.8	4.18075	16.98365	3.86828	18.36944
.0	.9	4.23912	16.36086	3.90247	17.97842
.0	1.0	4.29265	15.74243	3.93432	17.59643
.1	.1	3.77736	21.73113	3.66352	21.75592
.2	.2	4.04253	21.41178	3.81260	21.47170
.3	.3	4.31454	21.07969	3.96323	21.18252
.4	.4	4.58885	20.73267	4.11464	20.88849
.5	.5	4.86505	20.37156	4.26638	20.59206
.6	.6	5.14136	19.99895	4.41749	20.29139
.7	.7	5.41396	19.61430	4.56803	19.98790
.8	.8	5.68160	19.21557	4.71720	19.68703
.9	.9	5.94408	18.80675	4.86446	19.40364
1.0	1.0	6.19721	18.38376	5.00722	19.13450

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.00	.00	.00	3.46410	3.53009	.00	.60	.00	4.57891	5.13806
		.10	3.46410	3.53009			.10	4.13003	4.58113
		.20	3.46410	3.53009			.20	3.92379	4.27828
		.30	3.46410	3.53009			.30	3.79515	4.07590
		.40	3.46410	3.53009			.40	3.70554	3.92952
		.50	3.46410	3.53009			.50	3.63910	3.81882
		.75	3.46410	3.53009			.75	3.52972	3.63569
.00	.10	.00	3.68862	3.79204	.00	.80	.00	4.79400	5.69898
		.10	3.59159	3.68841			.10	4.27443	4.98299
		.20	3.55178	3.64280			.20	4.02933	4.56564
		.30	3.52681	3.61297			.30	3.87526	4.28366
		.40	3.50935	3.59139			.40	3.76697	4.07954
		.50	3.49645	3.57492			.50	3.68584	3.92557
		.75	3.47570	3.54703			.75	3.54938	3.67299
.00	.20	.00	3.89954	4.05588	.00	1.00	.00	4.89898	6.27495
		.10	3.71413	3.85364			.10	4.36092	5.40881
		.20	3.63587	3.76033			.20	4.10184	4.87170
		.30	3.58693	3.69910			.30	3.93573	4.50434
		.40	3.55281	3.65480			.40	3.81669	4.23801
		.50	3.52763	3.62107			.50	3.72577	4.03760
		.75	3.48703	3.56424			.75	3.56785	3.71133
.00	.30	.00	4.09560	4.32206	.10	.00	.00	3.78444	3.86568
		.10	3.83053	4.02565			.10	3.67314	3.75169
		.20	3.71576	3.88267			.20	3.62115	3.69739
		.30	3.64415	3.78846			.30	3.58526	3.65957
		.40	3.59431	3.72032			.40	3.55780	3.63046
		.50	3.55755	3.66854			.50	3.53561	3.60684
		.75	3.49811	3.58171			.75	3.49402	3.56231
.00	.40	.00	4.27532	4.59096	.10	.10	.00	4.00639	4.13031
		.10	3.93954	4.20432			.10	3.80288	3.91666
		.20	3.79081	4.00978			.20	3.71002	3.81446
		.30	3.69814	3.88105			.30	3.64854	3.74524
		.40	3.63371	3.78795			.40	3.60329	3.69349
		.50	3.58616	3.71732			.50	3.56805	3.65270
		.75	3.50892	3.59944			.75	3.50561	3.57944
.00	.50	.00	4.43706	4.86287	.10	.20	.00	4.21433	4.39723
		.10	4.03983	4.38953			.10	3.92704	4.08853
		.20	3.86039	4.14166			.20	3.79498	3.93642
		.30	3.74859	3.97686			.30	3.70906	3.83419
		.40	3.67084	3.85768			.40	3.64691	3.75866
		.50	3.61336	3.76741			.50	3.59927	3.69991
		.75	3.51946	3.61744			.75	3.51693	3.59684

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.10	.30	.00	4.40690	4.66685	.20	.00	.00	4.11047	4.20899
		.10	4.04439	4.26717			.10	3.88993	3.98278
		.20	3.87539	4.06324			.20	3.78311	3.87085
		.30	3.76650	3.92643			.30	3.70949	3.79299
		.40	3.68849	3.82597			.40	3.65338	3.73330
		.50	3.62920	3.74844			.50	3.60824	3.68508
		.75	3.52800	3.61450			.75	3.52414	3.59481
.10	.40	.00	4.58258	4.93946	.20	.10	.00	4.32982	4.47647
		.10	4.15365	4.45242			.10	4.02164	4.15437
		.20	3.95063	4.19490			.20	3.87305	3.99235
		.30	3.82055	4.02195			.30	3.77328	3.88149
		.40	3.72787	3.89542			.40	3.69909	3.79810
		.50	3.65777	3.79830			.50	3.64075	3.73200
		.75	3.53879	3.63242			.75	3.53571	3.61213
.10	.50	.00	4.73962	5.21532	.20	.20	.00	4.53467	4.74663
		.10	4.25347	4.64414			.10	4.14712	4.33284
		.20	4.02000	4.33135			.20	3.95874	4.11879
		.30	3.87085	4.12073			.30	3.83414	3.97333
		.40	3.76488	3.96700			.40	3.74284	3.86506
		.50	3.68489	3.84949			.50	3.67201	3.78026
		.75	3.54930	3.65061			.75	3.54703	3.62971
.10	.60	.00	4.87608	5.49465	.20	.30	.00	4.72361	5.01978
		.10	4.34245	4.84219			.10	4.26510	4.51802
		.20	4.08282	4.47258			.20	4.03955	4.25014
		.30	3.91706	4.22278			.30	3.89174	4.06849
		.40	3.79936	4.04072			.40	3.78446	3.93419
		.50	3.71049	3.90201			.50	3.70193	3.82988
		.75	3.55954	3.66906			.75	3.55808	3.64757
.10	.80	.00	5.07807	6.06430	.20	.40	.00	4.89506	5.29619
		.10	4.48208	5.25671			.10	4.37428	4.70976
		.20	4.18599	4.76927			.20	4.11480	4.38636
		.30	3.99587	4.43667			.30	3.94575	4.16697
		.40	3.86007	4.19457			.40	3.82378	4.00548
		.50	3.75683	4.01104			.50	3.73043	3.88084
		.75	3.57913	3.70675			.75	3.56885	3.66568
.10	1.00	.00	5.16633	6.64945	.20	.50	.00	5.04723	5.57605
		.10	4.56063	5.69483			.10	4.47328	4.90791
		.20	4.25446	5.08473			.20	4.18382	4.52742
		.30	4.05419	4.66363			.30	3.99583	4.26876
		.40	3.90864	4.35697			.40	3.86064	4.07894
		.50	3.79617	4.12539			.50	3.75745	3.93314
		.75	3.59752	3.74550			.75	3.57934	3.68407

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.20	.60	.00	5.17811	5.85955	.30	.30	.00	5.04571	5.38093
		.10	4.56070	5.11232			.10	4.49254	4.77819
		.20	4.24589	4.67330			.20	4.20822	4.44343
		.30	4.04162	4.37386			.30	4.01987	4.21469
		.40	3.89487	4.15456			.40	3.88223	4.04501
		.50	3.78290	3.98679			.50	3.77573	3.91288
		.75	3.58955	3.70272			.75	3.58835	3.68092
.20	.80	.00	5.36656	6.43796	.30	.40	.00	5.21275	5.66119
		.10	4.69487	5.53927			.10	4.60130	4.97633
		.20	4.34633	4.97934			.20	4.28334	4.58425
		.30	4.11895	4.59396			.30	4.07378	4.31618
		.40	3.95475	4.31229			.40	3.92146	4.11817
		.50	3.82880	4.09811			.50	3.80416	3.96495
		.75	3.60907	3.74081			.75	3.59910	3.69923
.20	1.00	.00	5.43765	7.03230	.30	.50	.00	5.35985	5.94509
		.10	4.76478	5.98943			.10	4.69915	5.18080
		.20	4.41039	5.30421			.20	4.35183	4.72995
		.30	4.17494	4.82725			.30	4.12355	4.42101
		.40	4.00209	4.47868			.40	3.95812	4.19353
		.50	3.86750	4.21482			.50	3.83106	4.01838
		.75	3.62738	3.77995			.75	3.60957	3.71781
.30	.00	.00	4.44230	4.56015	.30	.60	.00	5.48494	6.23279
		.10	4.11441	4.22339			.10	4.78465	5.39144
		.20	3.95000	4.05059			.20	4.41298	4.88048
		.30	3.83681	3.93044			.30	4.16883	4.52919
		.40	3.75086	3.83867			.40	3.99206	4.27108
		.50	3.68199	3.76484			.50	3.85634	4.07317
		.75	3.55446	3.62758			.75	3.61975	3.73666
.30	.10	.00	4.65893	4.83064	.30	.80	.00	5.65943	6.81998
		.10	4.24779	4.40155			.10	4.91267	5.83058
		.20	4.04088	4.17655			.20	4.51033	5.19589
		.30	3.90105	4.02181			.30	4.24449	4.75556
		.40	3.79676	3.90525			.40	4.05104	4.43274
		.50	3.71456	3.81282			.50	3.90175	4.18682
		.75	3.56602	3.64509			.75	3.63921	3.77515
.30	.20	.00	4.86056	5.10415	.30	1.00	.00	5.71288	7.42349
		.10	4.37426	4.58654			.10	4.97328	6.29253
		.20	4.12718	4.30751			.20	4.56961	5.53018
		.30	3.96218	4.11657			.30	4.29796	4.99526
		.40	3.84061	3.97403			.40	4.09704	4.60316
		.50	3.74583	3.86217			.50	3.93977	4.30591
		.75	3.57732	3.66287			.75	3.65743	3.81470

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.40	.00	.00	4.77996	4.91929	.40	.60	.00	5.79655	6.61438
		.10	4.34650	4.47354			.10	5.01420	5.67949
		.20	4.12184	4.23667			.20	4.58408	5.09417
		.30	3.96724	4.07198			.30	4.29869	4.68884
		.40	3.85026	3.94659			.40	4.09093	4.39031
		.50	3.75687	3.84613			.50	3.93080	4.16117
		.75	3.58497	3.66064			.75	3.65014	3.77089
.40	.10	.00	4.99375	5.19293	.40	.80	.00	5.95664	7.21036
		.10	4.48122	4.65818			.10	5.13539	6.13056
		.20	4.21353	4.36715			.20	4.67798	5.41897
		.30	4.03187	4.16627			.30	4.37251	4.92155
		.40	3.89632	4.01499			.40	4.14892	4.55595
		.50	3.78949	3.89519			.50	3.97569	4.27718
		.75	3.59653	3.67834			.75	3.66953	3.80978
.40	.20	.00	5.19199	5.46988	.40	1.00	.00	5.99199	7.82303
		.10	4.60836	4.84961			.10	5.18602	6.60404
		.20	4.30027	4.50267			.20	4.73208	5.76267
		.30	4.09320	4.26398			.30	4.42325	5.16770
		.40	3.94022	4.08561			.40	4.19350	4.73045
		.50	3.82076	3.94564			.50	4.01298	4.39868
		.75	3.60782	3.69631			.75	3.68766	3.84975
.40	.30	.00	5.37318	5.75035	.50	.00	.00	5.12348	5.28650
		.10	4.72660	5.04762			.10	4.58609	4.73319
		.20	4.38141	4.64317			.20	4.29863	4.42920
		.30	4.15091	4.36511			.30	4.10081	4.21766
		.40	3.98181	4.15846			.40	3.95158	4.05711
		.50	3.85061	3.99746			.50	3.83290	3.92898
		.75	3.61883	3.71456			.75	3.61569	3.69398
.40	.40	.00	5.53563	6.03450	.50	.10	.00	5.33427	5.56339
		.10	4.83460	5.25206			.10	4.72183	4.92424
		.20	4.45623	4.78862			.20	4.39098	4.56422
		.30	4.20463	4.46963			.30	4.16575	4.31491
		.40	4.02090	4.23353			.40	3.99776	4.12735
		.50	3.87895	4.05066			.50	3.86554	3.97914
		.75	3.62955	3.73307			.75	3.62724	3.71187
.40	.50	.00	5.67746	6.32247	.50	.20	.00	5.52896	5.84385
		.10	4.93095	5.46274			.10	4.84930	5.12200
		.20	4.52403	4.93897			.20	4.47803	4.70432
		.30	4.25400	4.57755			.30	4.22722	4.41562
		.40	4.05734	4.31081			.40	4.04169	4.19985
		.50	3.90571	4.10523			.50	3.89680	4.03069
		.75	3.63999	3.75184			.75	3.63851	3.73004

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.50	.30	.00	5.70600	6.12806	.60	.00	.00	5.47284	5.66186
		.10	4.96716	5.32627			.10	4.83306	5.00233
		.20	4.55910	4.84944			.20	4.48037	4.62823
		.30	4.28485	4.51979			.30	4.23752	4.36756
		.40	4.08321	4.27460			.40	4.05484	4.17027
		.50	3.92658	4.08364			.50	3.91006	4.01341
		.75	3.64950	3.74848			.75	3.64662	3.72760
.50	.40	.00	5.86365	6.41614	.60	.10	.00	5.68047	5.94208
		.10	5.07406	5.53688			.10	4.96949	5.19968
		.20	4.63346	4.99953			.20	4.57324	4.76782
		.30	4.33830	4.62739			.30	4.30271	4.46781
		.40	4.12213	4.35159			.40	4.10111	4.24238
		.50	3.95481	4.13798			.50	3.94272	4.06468
		.75	3.66020	3.76719			.75	3.65815	3.74569
.50	.50	.00	6.00000	6.70820	.60	.20	.00	5.87142	6.22612
		.10	5.16857	5.75364			.10	5.09695	5.40365
		.20	4.70039	5.15454			.20	4.66044	4.91253
		.30	4.38721	4.73842			.30	4.36423	4.57156
		.40	4.15829	4.43082			.40	4.14503	4.31677
		.50	3.98141	4.19370			.50	3.97394	4.11736
		.75	3.67061	3.78617			.75	3.66940	3.76406
.50	.60	.00	6.11289	7.00434	.60	.30	.00	6.04411	6.51410
		.10	5.24922	5.97637			.10	5.21410	5.61406
		.20	4.75916	5.31443			.20	4.74127	5.06228
		.30	4.43120	4.85286			.30	4.42172	4.67879
		.40	4.19150	4.51230			.40	4.18644	4.39344
		.50	4.00629	4.25082			.50	4.00364	4.17144
		.75	3.68073	3.80541			.75	3.68036	3.78269
.50	.80	.00	6.25815	7.60910	.60	.40	.00	6.19677	6.80614
		.10	5.36292	6.43911			.10	5.31955	5.83071
		.20	4.84923	5.64861			.20	4.81500	5.21703
		.30	4.50299	5.09196			.30	4.47482	4.78951
		.40	4.24840	4.68197			.40	4.22515	4.47239
		.50	4.05061	4.36921			.50	4.03174	4.22694
		.75	3.70004	3.84471			.75	3.69104	3.80160
.50	1.00	.00	6.27495	8.23093	.60	.50	.00	6.32744	7.10230
		.10	5.40292	6.92384			.10	5.41188	6.05342
		.20	4.89778	6.00168			.20	4.88089	5.37671
		.30	4.55081	5.34461			.30	4.52316	4.90368
		.40	4.29146	4.86060			.40	4.26098	4.55360
		.50	4.08714	4.49316			.50	4.05816	4.28383
		.75	3.71808	3.88509			.75	3.70143	3.82078

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
.60	.60	.00	6.43391	7.40267	.80	.30	.00	6.73607	7.31124
		.10	5.48960	6.28200			.10	5.72654	6.21672
		.20	4.93820	5.54127			.20	5.11899	5.50787
		.30	4.56637	5.02130			.30	4.70423	5.01000
		.40	4.29377	4.63708			.40	4.39840	4.63944
		.50	4.08282	4.34213			.50	4.16107	4.35199
		.75	3.71151	3.84023			.75	3.74270	3.85200
.60	.80	.00	6.56390	8.01622	.80	.40	.00	6.87814	7.61123
		.10	5.59514	6.75613			.10	5.82809	6.44505
		.20	5.02406	5.88482			.20	5.19093	5.67193
		.30	4.63594	5.26684			.30	4.75636	5.12703
		.40	4.34950	4.81082			.40	4.43656	4.72235
		.50	4.12653	4.46294			.50	4.18885	4.40984
		.75	3.73074	3.87994			.75	3.75332	3.87131
.60	1.00	.00	6.56171	8.64718	.80	.50	.00	6.99682	7.91563
		.10	5.62389	7.25185			.10	5.91505	6.67923
		.20	5.06668	6.24724			.20	5.25420	5.84094
		.30	4.68063	5.52604			.30	4.80330	5.24757
		.40	4.39093	4.99362			.40	4.47163	4.80759
		.50	4.16224	4.58935			.50	4.21485	4.46912
		.75	3.74869	3.92073			.75	3.76364	3.89089
.80	.00	.00	6.18902	6.43723	.80	.60	.00	7.08982	8.22447
		.10	5.34864	5.56883			.10	5.98592	6.91909
		.20	4.85869	5.04607			.20	5.30803	6.01484
		.30	4.52044	4.68026			.30	4.84467	5.37163
		.40	4.26723	4.40465			.40	4.50342	4.89514
		.50	4.06786	4.18708			.50	4.23899	4.52983
		.75	3.70907	3.79571			.75	3.77365	3.91075
.80	.10	.00	6.38977	6.72429	.80	.80	.00	7.18799	8.85557
		.10	5.48542	5.77840			.10	6.07323	7.41518
		.20	4.95214	5.19487			.20	5.38434	6.37703
		.30	4.58590	4.78660			.30	4.90919	5.63019
		.40	4.31356	4.48058			.40	4.55650	5.07717
		.50	4.10049	4.24061			.50	4.28135	4.65554
		.75	3.72057	3.81420			.75	3.79271	3.95129
.80	.20	.00	6.57267	7.01561	.80	1.00	.00	7.14647	9.50473
		.10	5.61184	5.99444			.10	6.07763	7.93210
		.20	5.03915	5.34883			.20	5.41391	6.75801
		.30	4.64728	4.89652			.30	4.94702	5.90259
		.40	4.35734	4.55884			.40	4.59437	5.26844
		.50	4.13159	4.29558			.50	4.31528	4.78698
		.75	3.73178	3.83296			.75	3.81046	3.99292

TABLE 5. APPROXIMATE FUNDAMENTAL FREQUENCIES

$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND	$\Psi$	$\Phi$	C	UPPER BOUND	LOWER BOUND
1.00	.00	.00	6.92820	7.24569	1.00	.60	.00	7.76392	9.07980
		.10	5.89209	6.17242			.10	6.50219	7.58991
		.20	5.25670	5.49065			.20	5.69329	6.51501
		.30	4.81608	5.01052			.30	5.13355	5.74018
		.40	4.48749	4.65004			.40	4.71991	5.16478
		.50	4.23031	4.36731			.50	4.39933	4.72442
		.75	3.77234	3.86499			.75	3.83656	3.98247
1.00	.10	.00	7.12130	7.53977	1.00	.80	.00	7.82856	9.72840
		.10	6.02787	6.39369			.10	6.56883	8.10688
		.20	5.35007	5.64868			.20	5.75852	6.89566
		.30	4.88149	5.12308			.30	5.19222	6.01191
		.40	4.53373	4.72988			.40	4.76994	5.35527
		.50	4.26284	4.42316			.50	4.44016	4.85515
		.75	3.78380	3.88388			.75	3.85545	4.02384
1.00	.20	.00	7.29537	7.83849	1.00	1.00	.00	7.74597	10.39568
		.10	6.15190	6.62122			.10	6.54654	8.64398
		.20	5.43621	5.81190			.20	5.77350	7.29493
		.30	4.94241	5.23930			.30	5.22233	6.29764
		.40	4.57721	4.81212			.40	4.80384	5.55516
		.50	4.29374	4.48049			.50	4.47214	4.99172
		.75	3.79496	3.90304			.75	3.87298	4.06632
1.00	.30	.00	7.44868	8.14184	.25	.25	.00	4.79257	5.06144
		.10	6.26280	6.85480			.10	4.31996	4.55190
		.20	5.51437	5.98022			.20	4.08352	4.27855
		.30	4.99846	5.35914			.30	3.92706	4.09233
		.40	4.61776	4.89673			.40	3.81259	3.95397
		.50	4.32292	4.53928			.50	3.72391	3.84593
		.75	3.80583	3.92248			.75	3.56771	3.65520
1.00	.40	.00	7.57934	8.44984	.75	.75	.00	7.01561	8.48425
		.10	6.35914	7.09423			.10	5.93616	7.12264
		.20	5.58379	6.15356			.20	5.27732	6.16033
		.30	5.04926	5.48258			.30	4.82584	5.47252
		.40	4.65518	4.98371			.40	4.49205	4.96365
		.50	4.35030	4.59953			.50	4.23235	4.57509
		.75	3.81638	3.94220			.75	3.77249	3.92315
1.00	.50	.00	7.68521	8.76249	.90	.90	.00	7.49528	9.61462
		.10	6.43944	7.33933			.10	6.32607	8.01807
		.20	5.64359	6.33185			.20	5.58736	6.82582
		.30	5.09442	5.60960			.30	5.07015	5.95653
		.40	4.68929	5.07306			.40	4.68243	5.31135
		.50	4.37580	4.66125			.50	4.37786	4.82073
		.75	3.82663	3.96220			.75	3.83297	4.00830

TABLE 6. RADII OF GYRATION FOR DIFFERENT CROSS-SECTIONS

$\gamma$	$\beta$	RADIUS/h	$\gamma$	$\beta$	RADIUS/h	$\gamma$	$\beta$	RADIUS/h
1.00	1.00	.40825	2.00	1.00	.47809	4.00	1.00	.52298
	1.25	.42008		1.25	.48467		1.25	.52641
	1.50	.43033		1.50	.49042		1.50	.52943
	1.75	.43930		1.75	.49549		1.75	.53212
	2.00	.44721		2.00	.50000		2.00	.53452
	2.50	.46057		2.50	.50767		2.50	.53864
	3.00	.47140		3.00	.51394		3.00	.54203
	3.50	.48038		3.50	.51918		3.50	.54488
	4.00	.48795		4.00	.52361		4.00	.54730
	5.00	.50000		5.00	.53071		5.00	.55120
7.00	.51640	7.00	.54045	7.00	.55658			
10.00	.53108	10.00	.54924	10.00	.56148			
1.25	1.00	.43355	2.50	1.00	.49507	5.00	1.00	.53300
	1.25	.44345		1.25	.50043		1.25	.53576
	1.50	.45204		1.50	.50513		1.50	.53821
	1.75	.45958		1.75	.50929		1.75	.54038
	2.00	.46625		2.00	.51299		2.00	.54233
	2.50	.47753		2.50	.51930		2.50	.54566
	3.00	.48671		3.00	.52448		3.00	.54842
	3.50	.49433		3.50	.52881		3.50	.55074
	4.00	.50076		4.00	.53248		4.00	.55271
	5.00	.51102		5.00	.53837		5.00	.55589
7.00	.52502	7.00	.54647	7.00	.56030			
10.00	.53759	10.00	.55380	10.00	.56431			
1.50	1.00	.45227	3.00	1.00	.50709	7.00	1.00	.54495
	1.25	.46075		1.25	.51161		1.25	.54694
	1.50	.46813		1.50	.51558		1.50	.54870
	1.75	.47463		1.75	.51909		1.75	.55027
	2.00	.48038		2.00	.52223		2.00	.55168
	2.50	.49014		2.50	.52759		2.50	.55410
	3.00	.49809		3.00	.53200		3.00	.55611
	3.50	.50471		3.50	.53569		3.50	.55779
	4.00	.51030		4.00	.53882		4.00	.55923
	5.00	.51924		5.00	.54385		5.00	.56156
7.00	.53145	7.00	.55079	7.00	.56478			
10.00	.54244	10.00	.55708	10.00	.56773			
1.75	1.00	.46667	3.50	1.00	.51605	10.00	1.00	.55427
	1.25	.47408		1.25	.51995		1.25	.55567
	1.50	.48055		1.50	.52338		1.50	.55691
	1.75	.48625		1.75	.52643		1.75	.55802
	2.00	.49130		2.00	.52915		2.00	.55902
	2.50	.49989		2.50	.53381		2.50	.56073
	3.00	.50691		3.00	.53764		3.00	.56216
	3.50	.51275		3.50	.54085		3.50	.56335
	4.00	.51770		4.00	.54358		4.00	.56438
	5.00	.52561		5.00	.54798		5.00	.56604
7.00	.53645	7.00	.55404	7.00	.56834			
10.00	.54622	10.00	.55955	10.00	.57045			

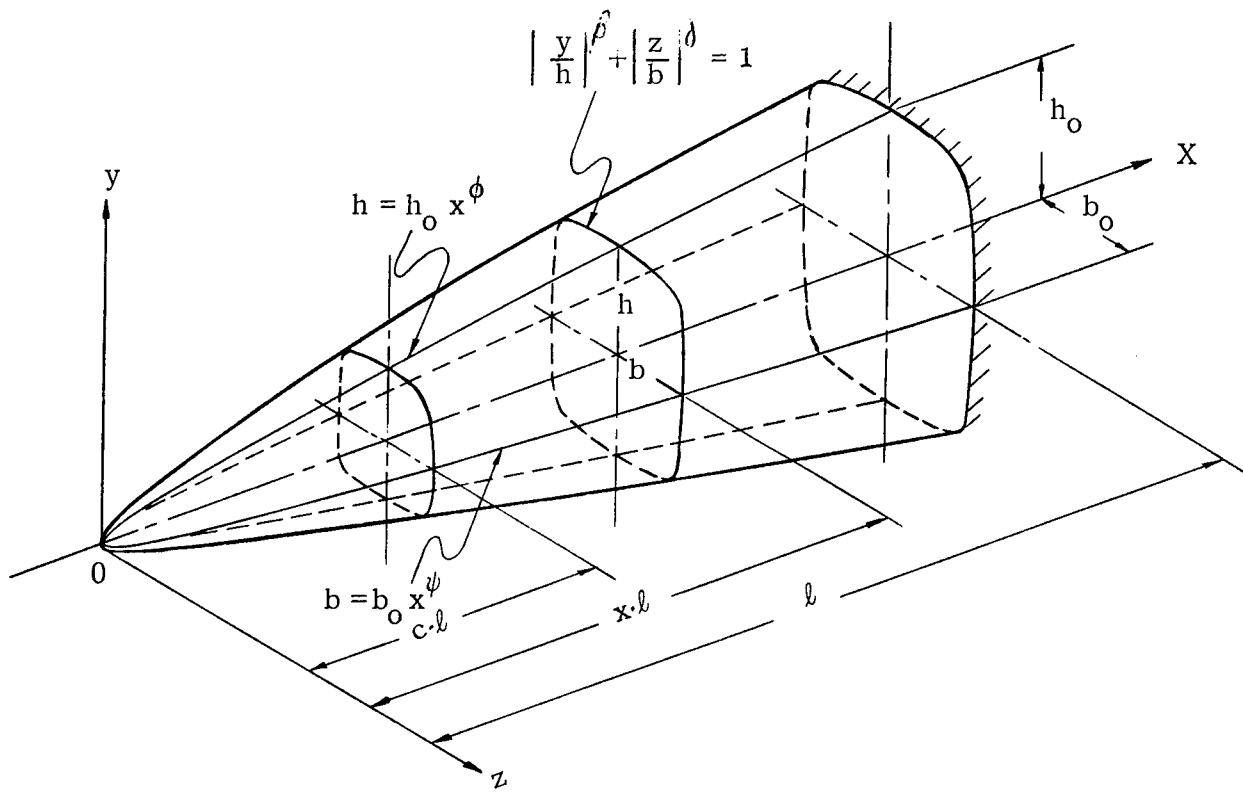


Fig. 1 Cantilever Beam

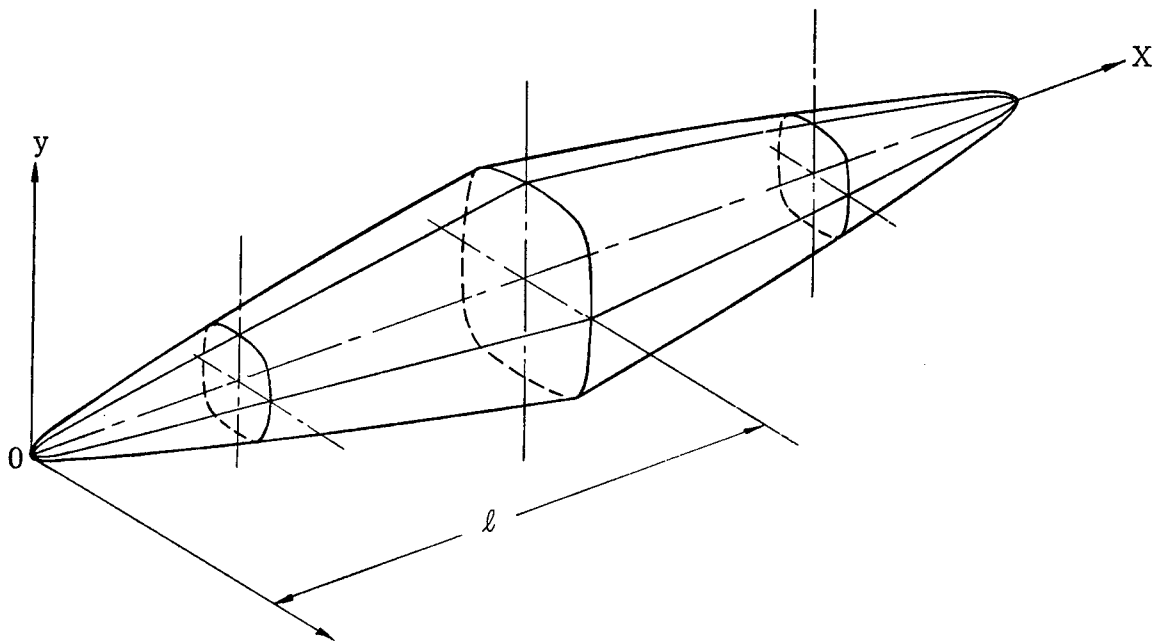


Fig. 2 Free-Free Beam

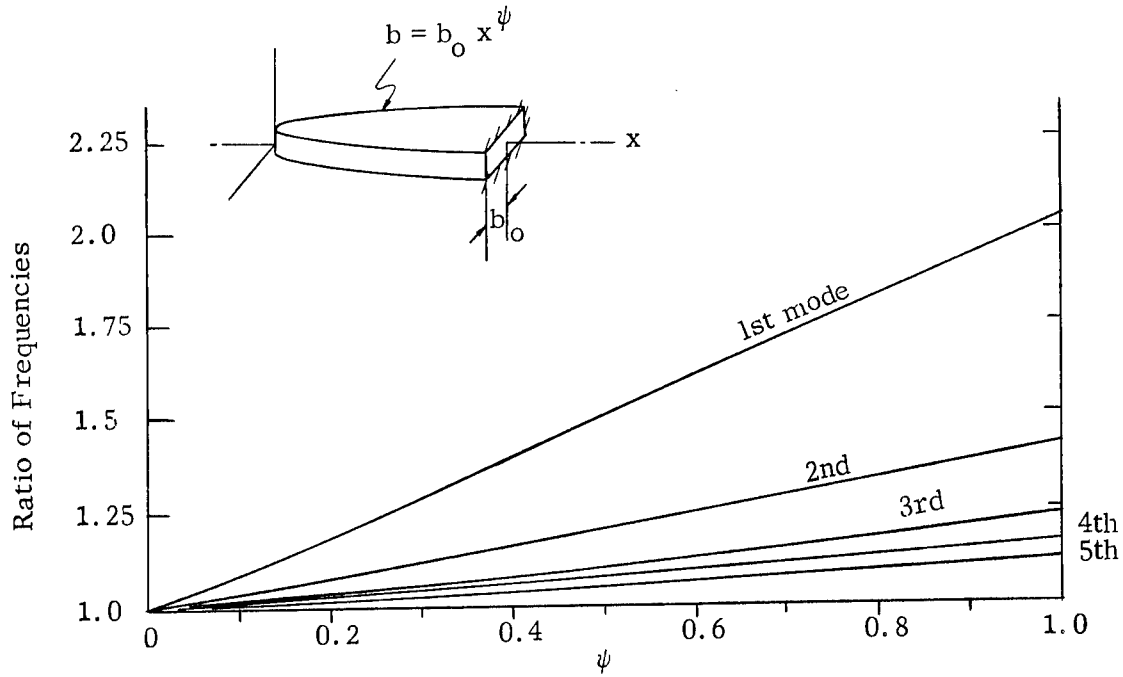


Fig. 3 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying in Width to Those of Uniform Cross-Section

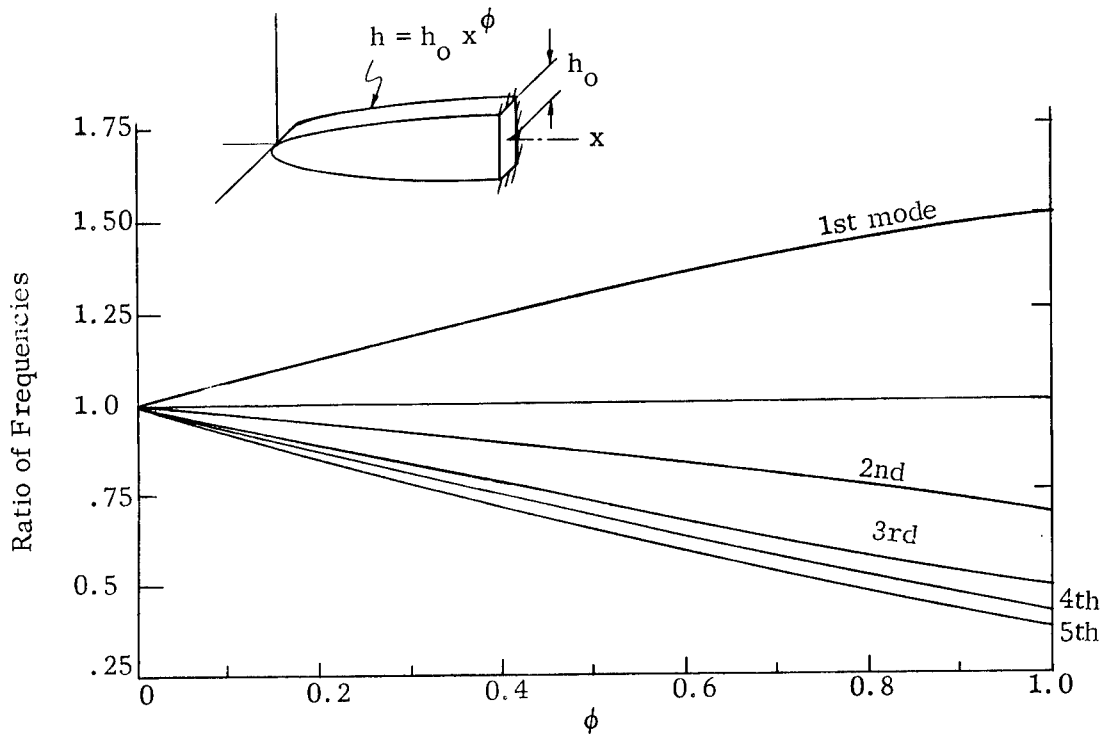


Fig. 4 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying in Thickness to Those of Uniform Cross-Section

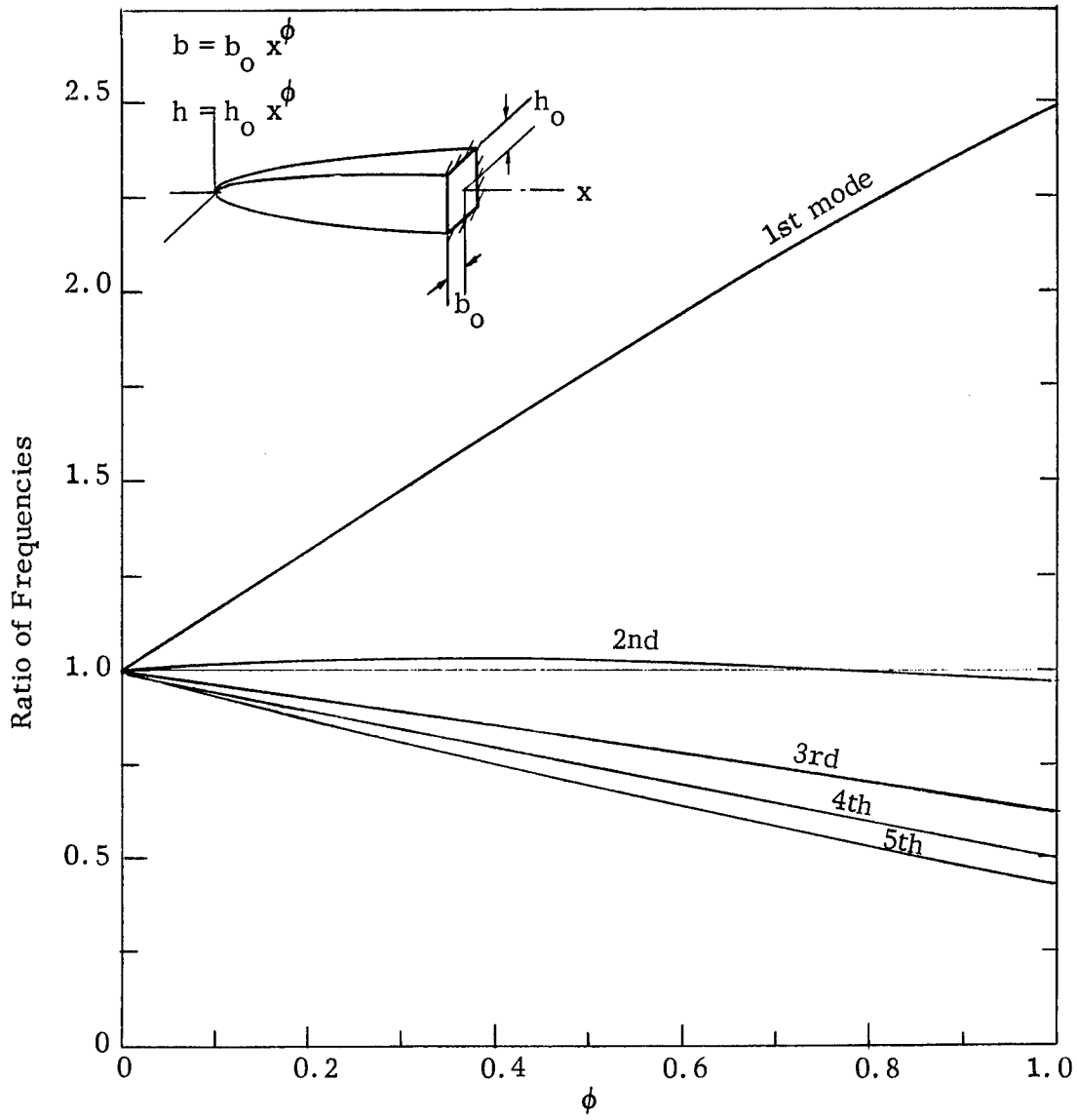


Fig. 5 Ratio of Frequencies of Cantilever Beams with Cross-Section Varying Both in Width and Thickness to Those of Uniform Cross-Section

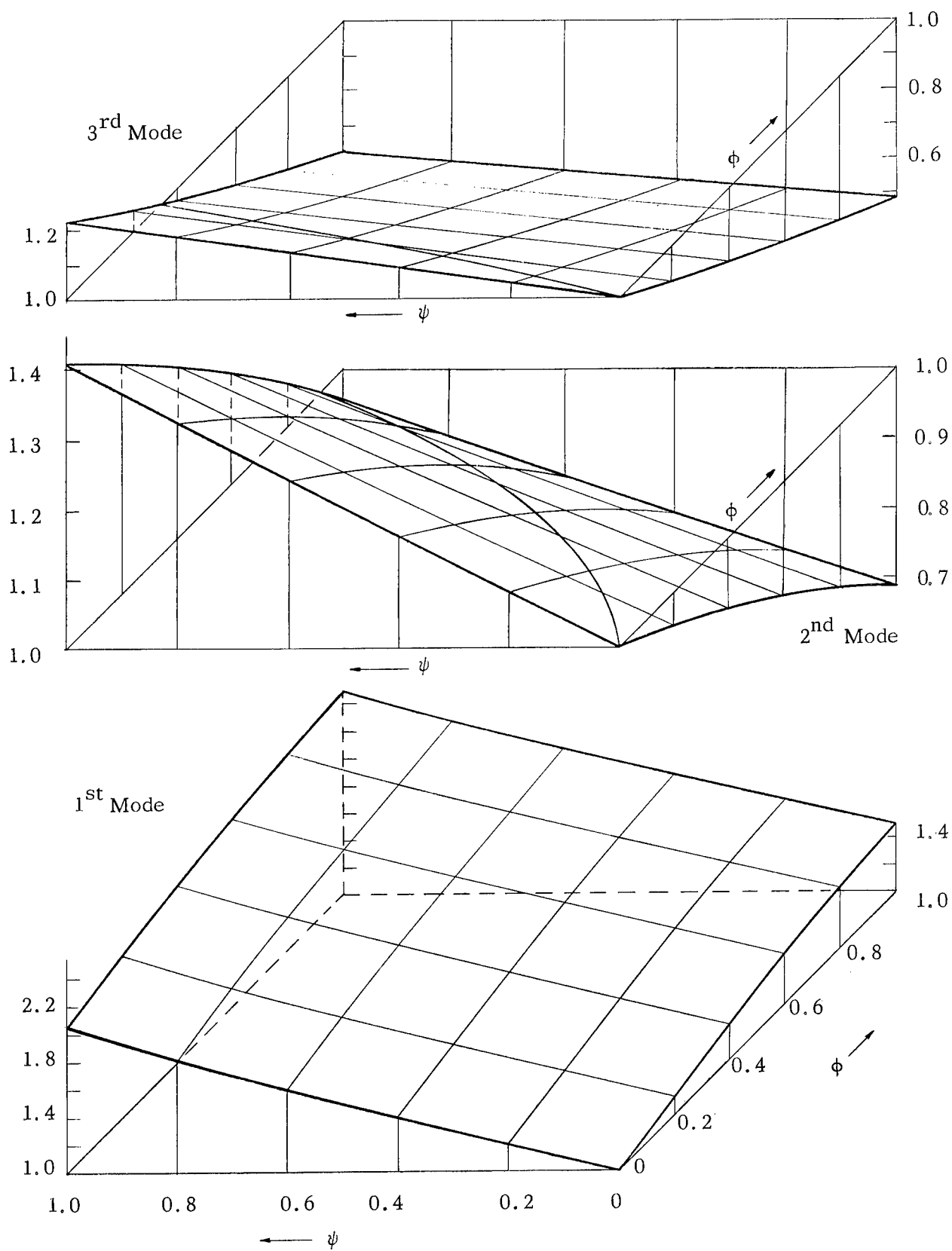


Fig. 6 Ratio of Frequencies of Tapered Cantilever Beams to Frequencies of a Uniform Beam

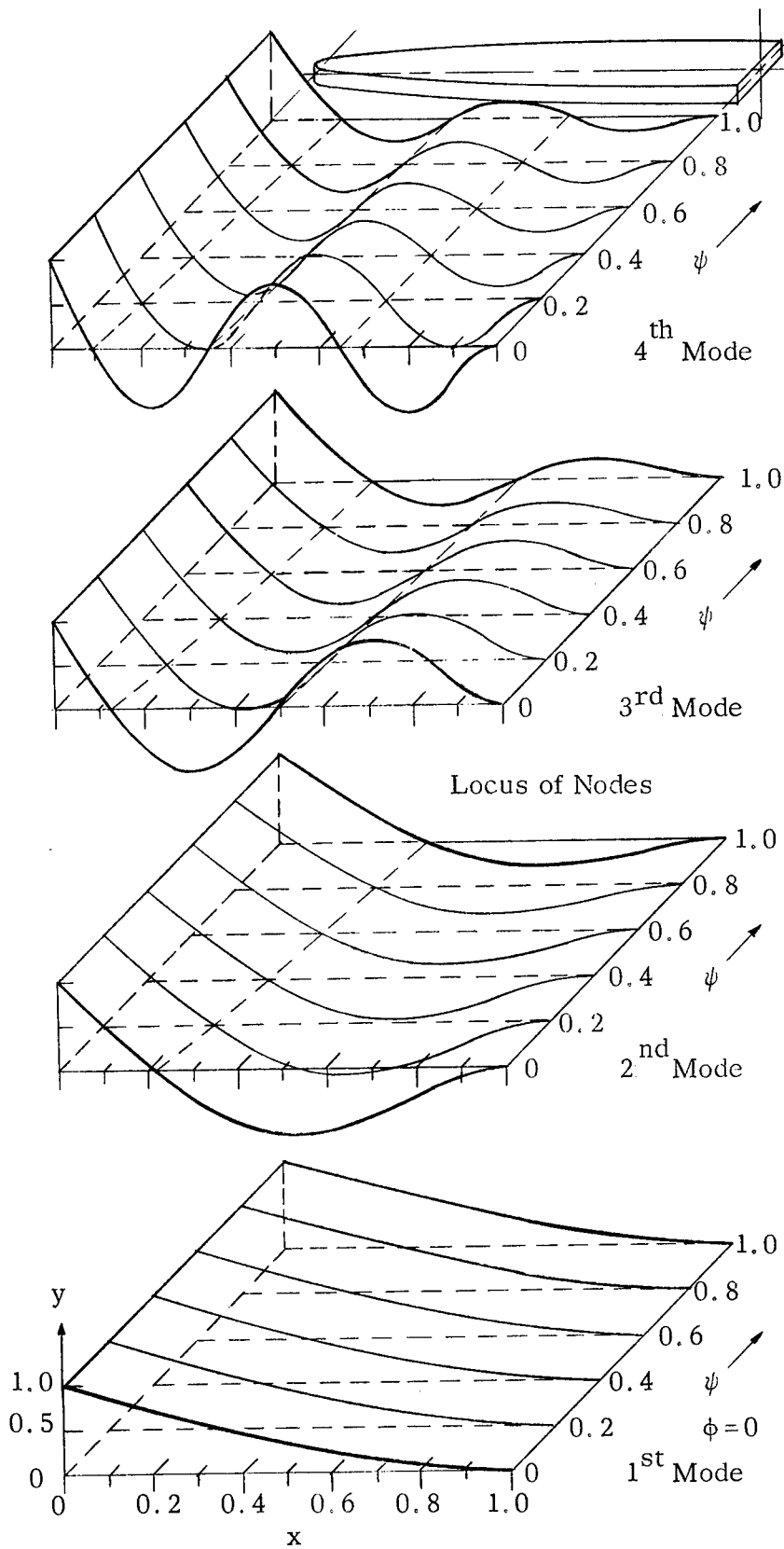


Fig. 7 Mode Shapes for Cantilever Beams with Varying Width and Constant Thickness

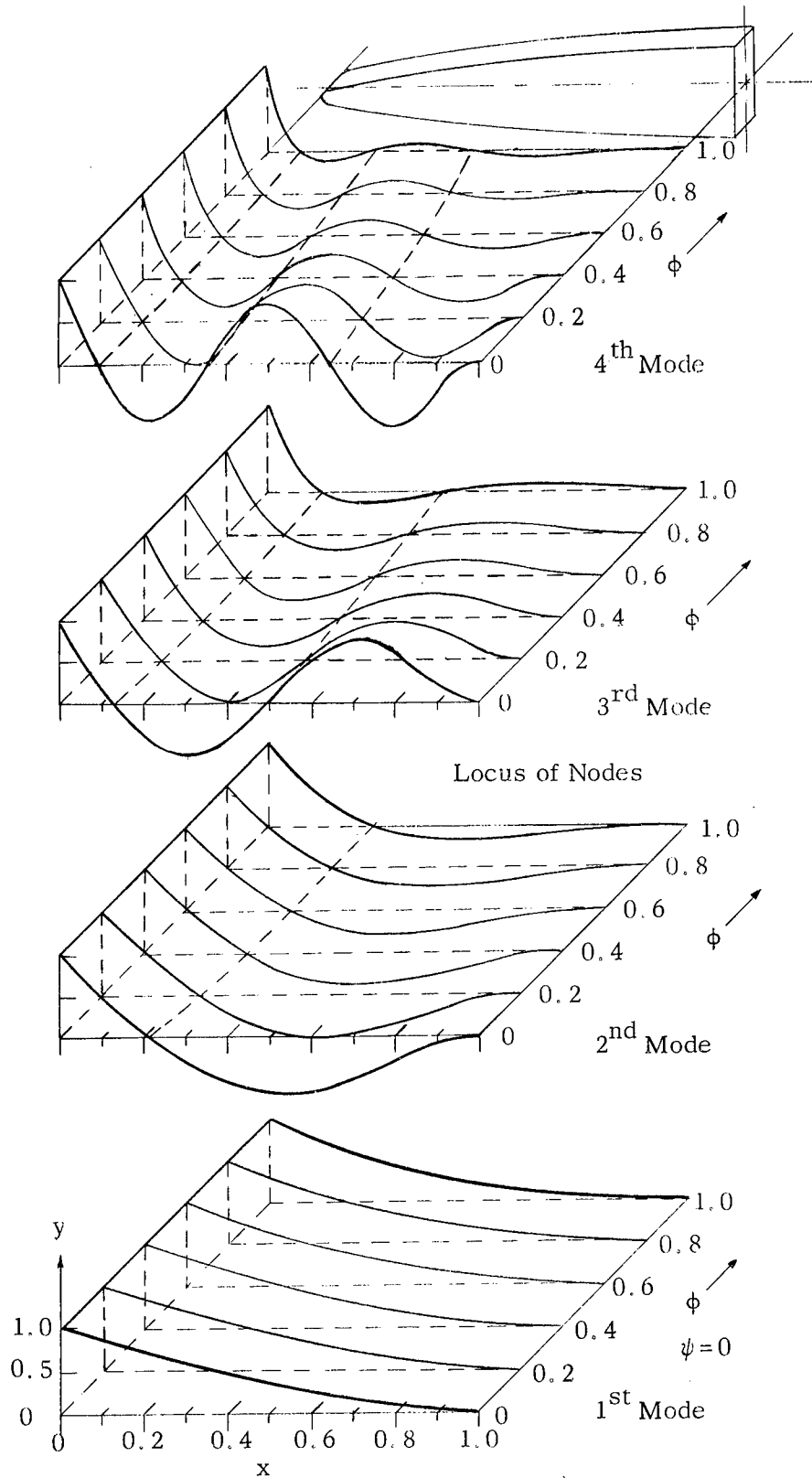


Fig. 8 Mode Shapes for Cantilever Beams with Varying Thickness and Constant Width

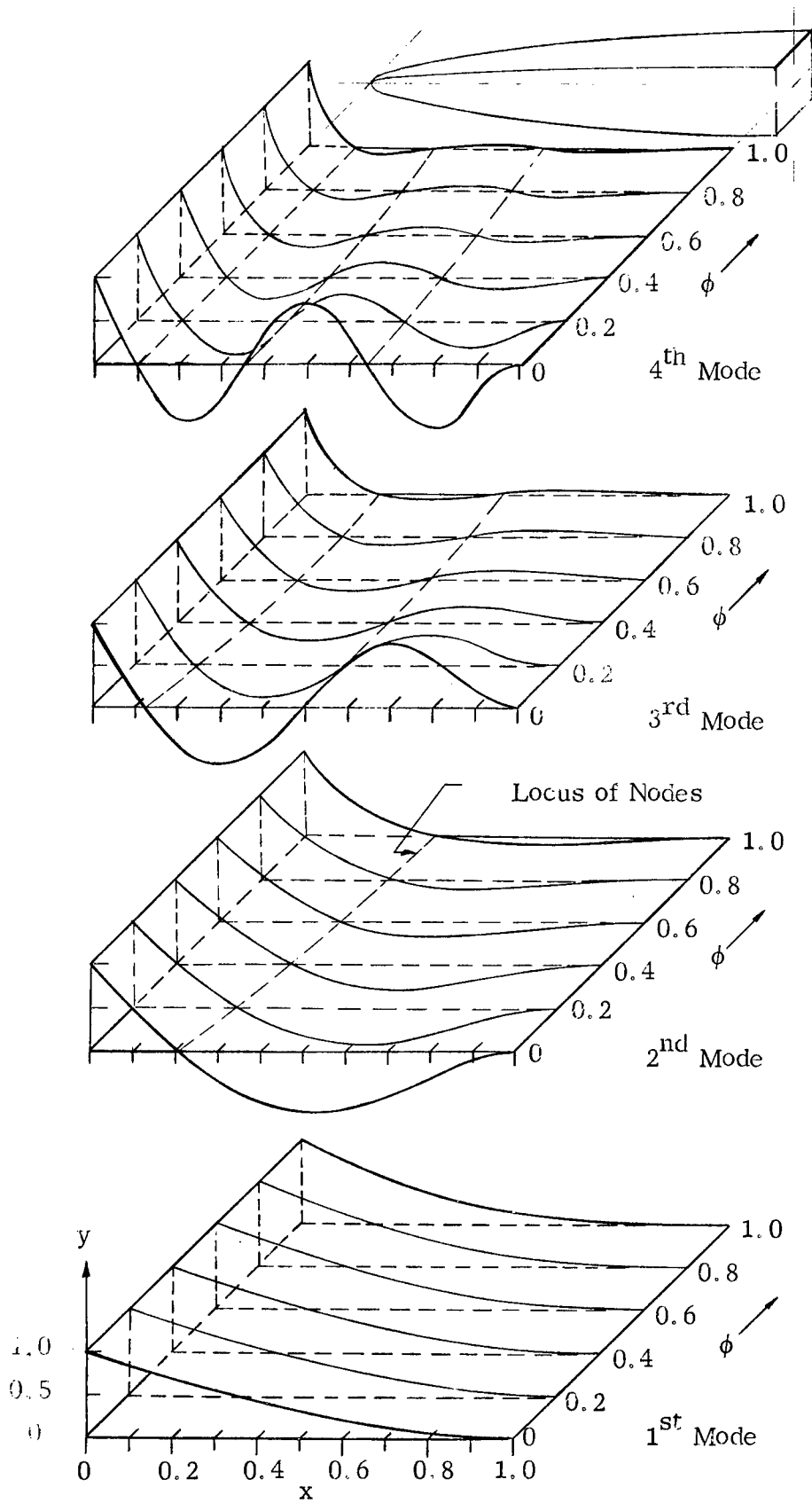


Fig. 9 Mode Shapes for Cantilever Beams with Varying Width and Thickness

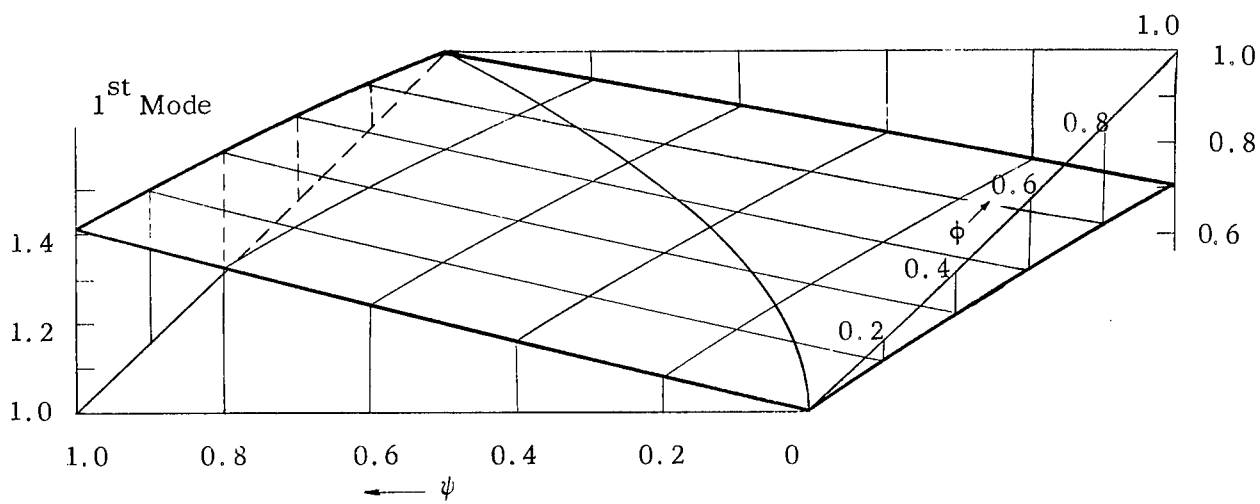
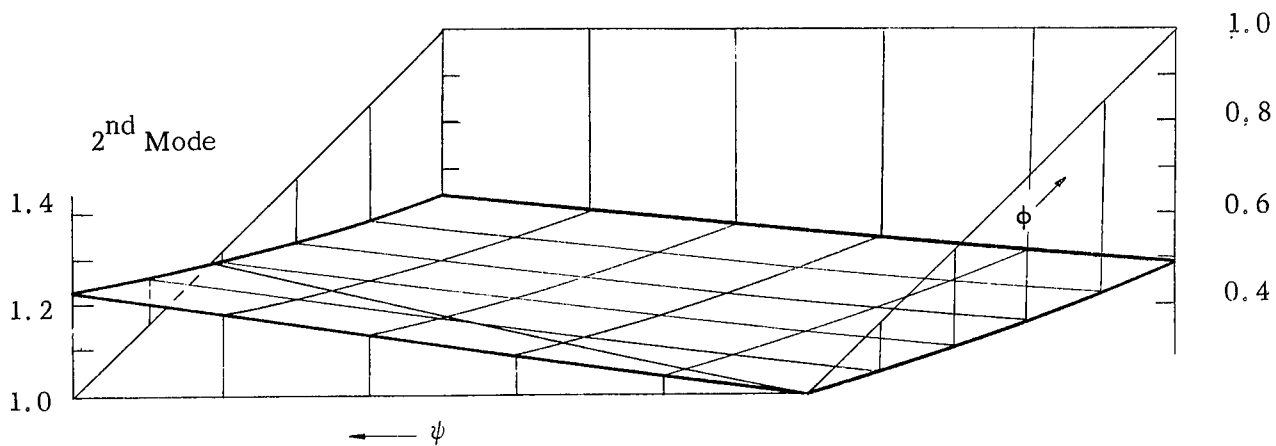
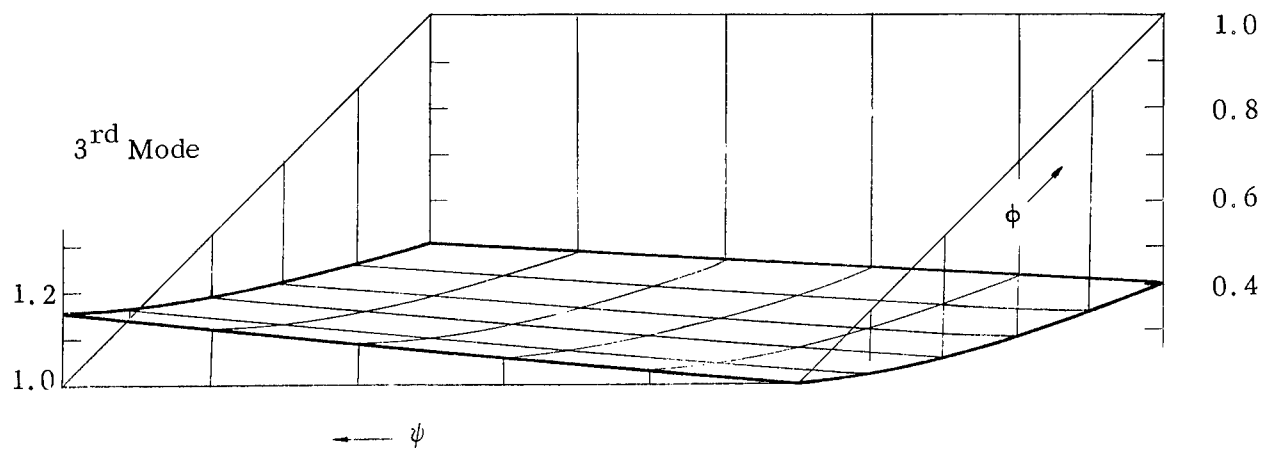


Fig.10 Ratio of Frequencies of Antisymmetrical Mode of Tapered Free-Free Beams to Frequencies of a Uniform Beam

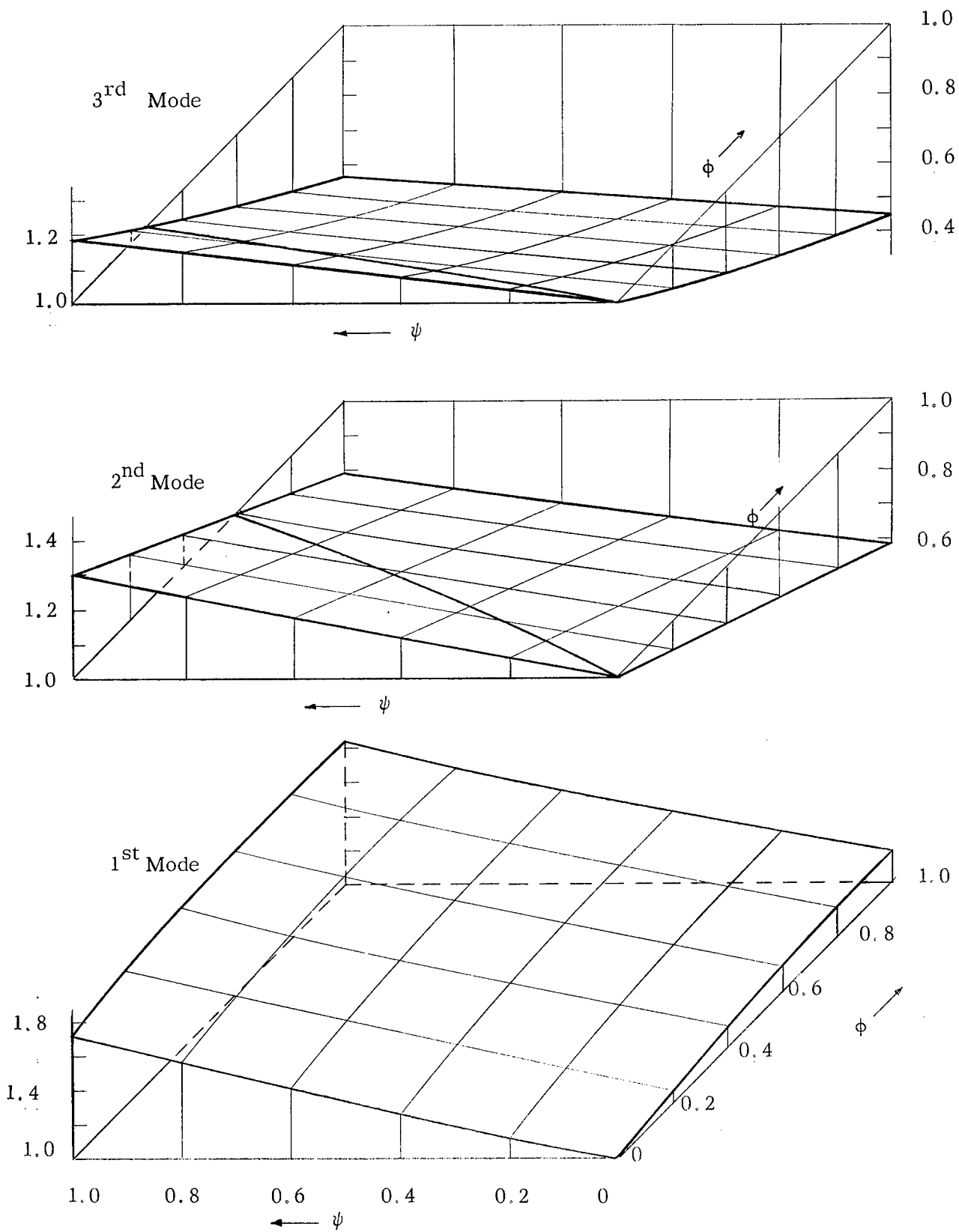


Fig. 11 Ratio of Frequencies of Symmetrical Mode of Tapered Free-Free Beams to Frequencies of a Uniform Beam

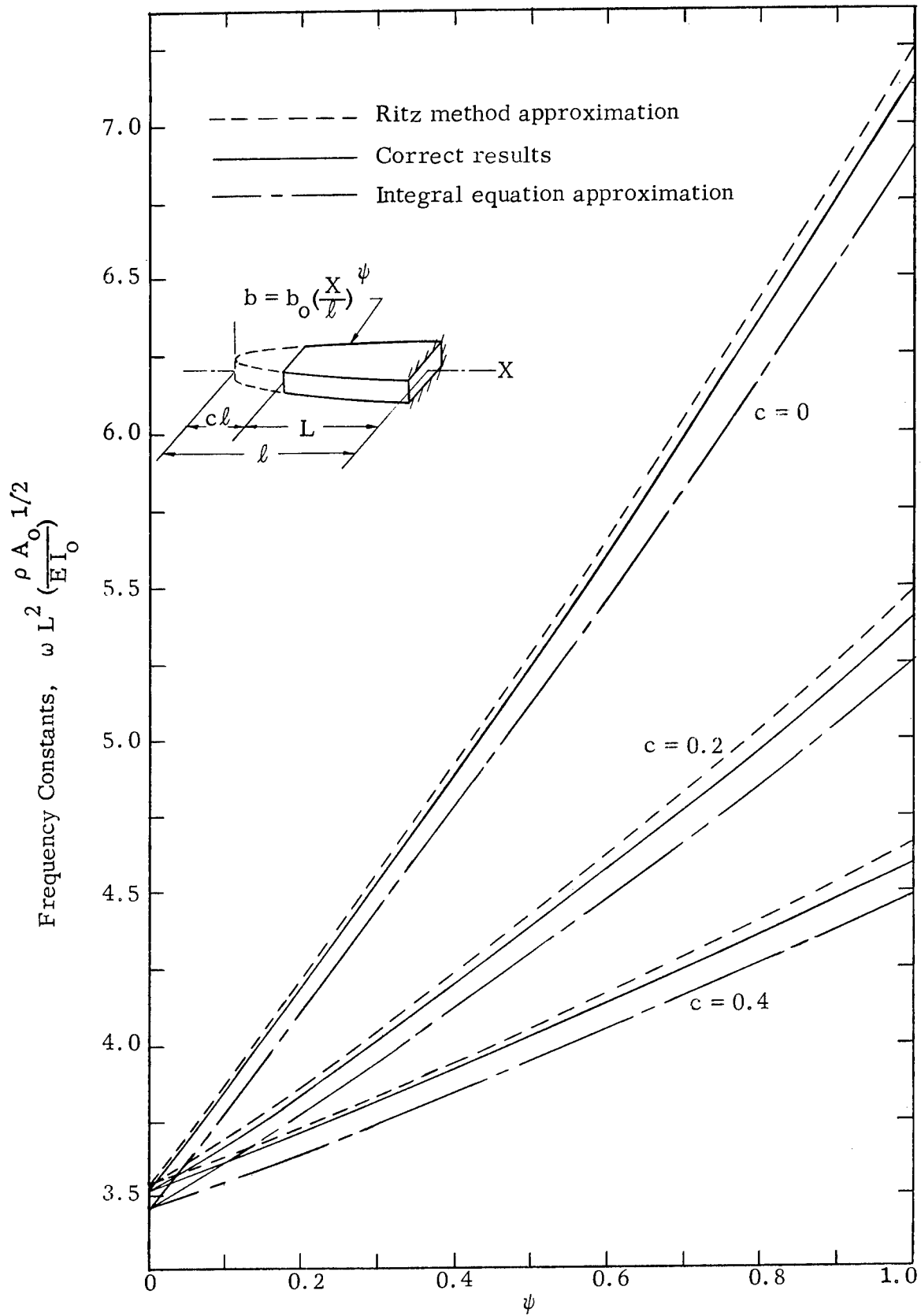


Fig.12 Fundamental Frequencies of Truncated Cantilever Beams with Varying Width

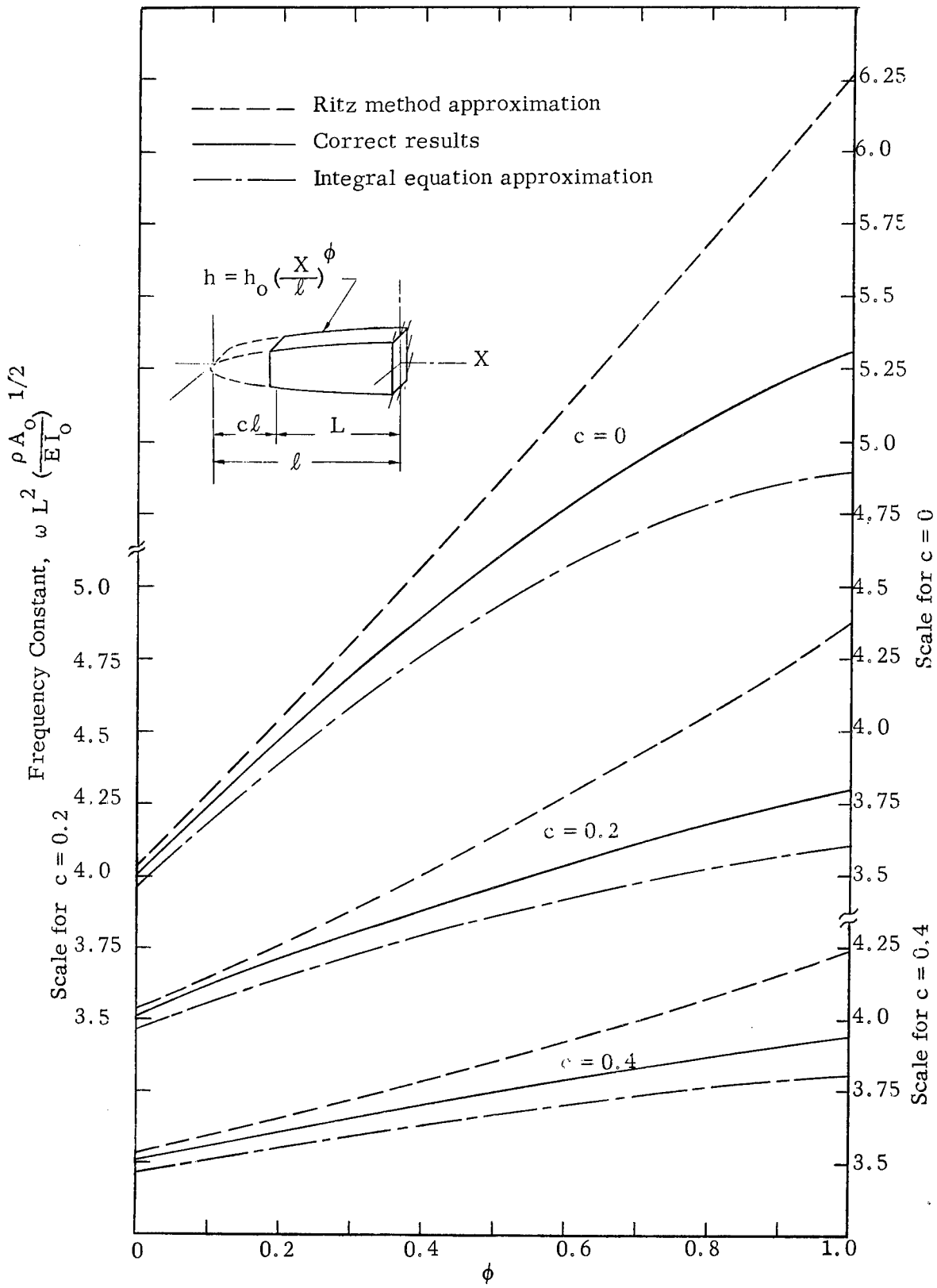


Fig. 13 Fundamental Frequencies of Truncated Cantilever Beams with Varying Thickness

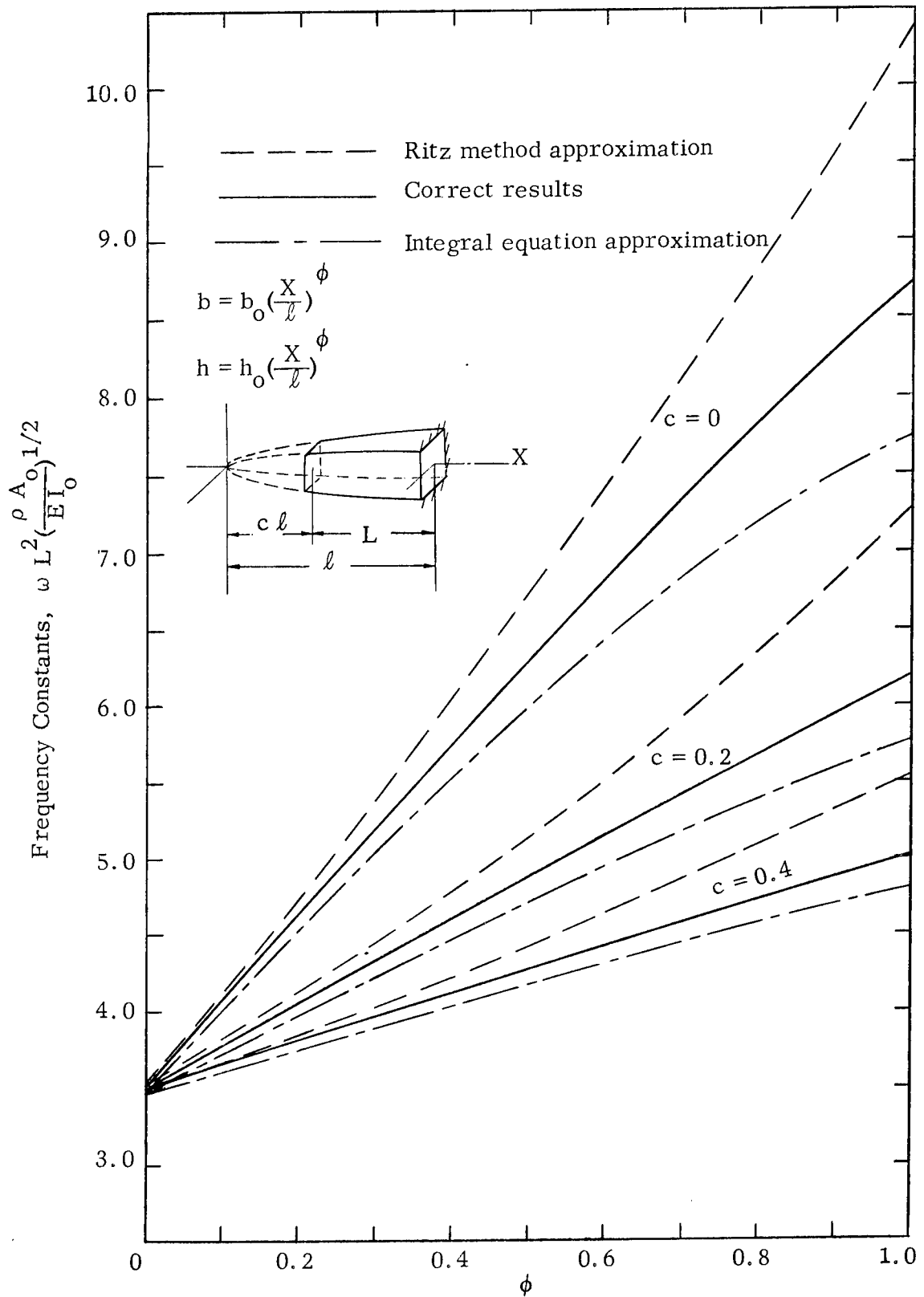


Fig. 14 Fundamental Frequencies of Truncated Cantilever Beams with Varying Width and Thickness

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