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**IMPLEMENTING FAMILIES OF IMPLICIT  
CHEBYSHEV METHODS WITH EXACT  
COEFFICIENTS FOR THE NUMERICAL  
INTEGRATION OF FIRST- AND SECOND-  
ORDER DIFFERENTIAL EQUATIONS**



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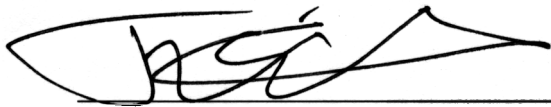
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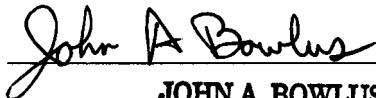
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# List of Symbols

|                 |                                                                                 |
|-----------------|---------------------------------------------------------------------------------|
| $(a, b)$        | The open interval $a, b$ with $b > a$                                           |
| $[a, b]$        | The closed interval $a, b$ with $b > a$                                         |
| $ x $           | The absolute value of $x$                                                       |
| $\mathbb{A}$    | The field of numbers that solve polynomial equations with rational coefficients |
| $\mathbb{R}$    | The field of real numbers                                                       |
| $\mathbb{Z}$    | The field of integers                                                           |
| $C^n$           | The space of continuous function with differentiability $n$                     |
| $\sum''$        | A summation with the first and last terms halved                                |
| $\mathbf{f}(x)$ | A vector function of $x$                                                        |
| $\mathbf{x}$    | A vector quantity                                                               |
| $f(x)$          | A scalar function of $x$                                                        |
| $h$             | An interval fixed stepsize                                                      |
| $x$             | A scalar quantity                                                               |

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# Chapter 1

## Background

### 1.1 Introduction

In this report, we present a method to generate the exact coefficients necessary to implement two families of implicit Chebyshev methods for numerical integration. The first family is valid for systems of first-order ordinary differential equations as presented by Richardson and Panovsky [7]. The second family is applied to systems of second-order ordinary differential equations of a special form as presented by Panovsky and Richardson [5].

Before we begin the brief review of the two families of numerical methods, we address two preliminary topics related to the implementation of these methods.

#### 1.1.1 Chebyshev polynomial interpolation

For the numerical methods discussed here, the interpolation over a particular interval is accomplished using Chebyshev polynomials. These polynomials are a clear choice of basis functions because of their minimax error properties as well as their excellent convergence characteristics. As a result, for a sample interval  $[x, x + h]$ , the interpolation points or nodes are not evenly spaced along the  $x$ -axis. However, they are evenly spaced along the arc-length of a semi-circle centered at the interval midpoint. As an example, Figure 1.1 demonstrates this by dividing the semi-circle into six equal sectors. Each angular position  $\theta_j, j \in \mathbb{Z}[0, n]$  is given as

$$\theta_j = \frac{n-j}{n} \pi. \quad (1.1)$$

Considering the unit semi-circle, the locations of the nodal points inside the unit interval, or unit *intra*-nodal points  $\xi_j$ , are given by

$$\xi_j = \frac{1}{2} (1 + \cos \theta_j). \quad (1.2)$$

Conveniently, the above expression also correctly includes the interval end points for  $j = 0, n$  so that  $\xi_j \in \mathbb{R}[0, 1]$ . Returning to the interval  $[x, x + h]$ , these intra-nodal

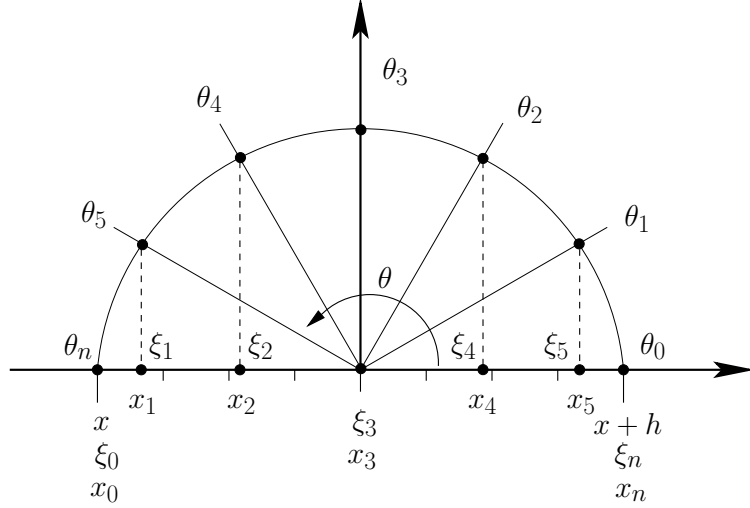


Figure 1.1: Sample interval  $[x, x + h]$  for  $n = 6$ .

points  $x_j$  are given by

$$x_j = x + h\xi_j. \quad (1.3)$$

For the example given in Figure 1.1, the upper limit of  $j$  is  $n = 6$ . In a terminology consistent with Henrici [3], this upper bound  $n$  is called the *degree* of the selected interpolation method.

### 1.1.2 Implicit methods

The implicit nature of these methods arises as the result of interpolation mechanism used to propagate a given numerical solution  $y(x)$  to  $y(x + h)$ . Obviously, we are interested in  $y(x + h)$ , i.e.  $\xi = 1$ . However, to compute  $y(x + h)$ , we must have information about the values of both the derivative and solution at the at the intra-nodal points. The fact that the derivatives are direct functions of the solution gives rise to the *implicit* nature of these methods.

## 1.2 Review of Methods

### 1.2.1 First-order differential equations

For the case of a family of implicit Chebyshev methods for the numerical integration of first-order differential equations as presented by Richardson and Panovsky [7], the form of the initial value problem of dimension  $m$  is given by

$$\frac{dy}{dx} = \mathbf{f}(\mathbf{y}(x), x), \quad \mathbf{y}(x)|_{x_0} = \mathbf{y}_0, \quad \mathbf{y}, \mathbf{f} \in \mathbb{R}^m, \quad (1.4)$$

with each  $f \in C^1[0, b]$ .

## CHAPTER 1. BACKGROUND

The fundamental final result necessary to develop the integration procedure for a specific  $m$  is given by

$$y(x + h\xi_j) = y(x) + \frac{1}{2}h \sum_{k=0}^n a_k R_{jk}, \quad (1.5)$$

with the summand terms defined as

$$a_k = \frac{2}{n} \sum_{j=0}^n f(x_j) \cos k\theta_j, \quad (1.6)$$

and

$$R_{jk} = \begin{cases} \frac{\cos(k+1)\theta_j + (-1)^k}{2(k+1)} - \frac{\cos(k-1)\theta_j + (-1)^k}{2(k-1)}, & k \geq 2, \\ \frac{1}{4}(\cos 2\theta_j - 1), & k = 1, \\ \cos \theta_j + 1, & k = 0. \end{cases} \quad (1.7)$$

The local truncation error for a given degree  $n$  induced by the discrete approximation found in Eq. (1.5) is written  $e_n(z)$ , where

$$z = x + h\xi, \quad (1.8)$$

with  $\xi \in \mathbb{R}[0, 1]$ . For  $f(\xi)$  evaluated at some point  $\beta \in \mathbb{R}[0, 1]$ , the error estimate is given by

$$e_n(z) \approx \begin{cases} \frac{nh^{n+3}D^{n+2}f(z(\beta))}{2^{2n}(n^2-1)(n^2-9)(n+1)!}, & n \text{ even}, \\ \frac{h^{n+2}D^{n+1}f(z(\beta))}{2^{2n-1}n(n^2-4)(n+1)!}, & n \text{ odd}, \end{cases} \quad (1.9)$$

where  $D^k f(z(\beta))$  represents the  $k^{\text{th}}$  derivative of  $f$  evaluated at  $z(\beta)$ .

### 1.2.2 Second-order differential equations of special form

For the case of a family of implicit Chebyshev methods for the numerical integration of second-order differential equations, as presented by Panovsky and Richardson [5], the special form of the initial value problem of dimension  $m$  is given by

$$\frac{d^2 \mathbf{y}}{dx^2} = \mathbf{f}(\mathbf{y}(x), x), \quad \mathbf{y}(x)|_{x_0} = \mathbf{y}_0, \quad \mathbf{y}'(x)|_{x_0} = \mathbf{y}'_0, \quad \mathbf{y}, \mathbf{y}', \mathbf{f} \in \mathbb{R}^m, \quad (1.10)$$

with each  $f \in C^2[0, b]$ . Examining Eq. (1.10), we quickly notice that no first derivatives appear in the arguments of  $\mathbf{f}$ , thus constituting the special form. Panovsky and Richardson [5] provide a discussion for the range of applications of this problem form.

## CHAPTER 1. BACKGROUND

The fundamental final result necessary to develop the integration procedure for a specific  $m$  is given by

$$y(x + h\xi_j) = 2y(x) - y(x - h\xi_j) + \frac{1}{4}h^2 \sum_{k=0}^n (a_k^+ + a_k^-) S_{jk}, \quad (1.11)$$

with the summand terms

$$a_k^\pm = \frac{2}{n} \sum_{j=0}^n f(x_j^\pm) \cos k\theta_j, \quad (1.12)$$

where  $x_j^\pm = x \pm h\xi_j$ , and

$$S_{jk} = \begin{cases} \frac{\cos(k+2)\theta_j}{4(k+1)(k+2)} + \frac{\cos(k-2)\theta_j}{4(k-1)(k-2)} - \frac{\cos k\theta_j}{2(k+1)(k-1)} + \\ \quad + \frac{(-1)^{k+1} \cos \theta_j}{(k+1)(k-1)} + \frac{(-1)^{k+1}}{(k+2)(k-2)}, & k \geq 3^\dagger, \\ \frac{1}{48} (-9 - 16 \cos \theta_j - 8 \cos 2\theta_j + \cos 4\theta_j), & k = 2, \\ \frac{1}{24} (8 - 9 \cos \theta_j + \cos 3\theta_j), & k = 1, \\ \frac{1}{4} (3 - 4 \cos \theta_j + \cos 2\theta_j), & k = 0. \end{cases} \quad (1.13)$$

As before, the local truncation error for a given degree  $n$  incurred by the discrete approximation found in Eq. (1.11) is written  $e_n(z)$  for  $z$  given by Eq. (1.8) with  $\xi \in \mathbb{R}[-1, 1]$ . Then, for  $f(\xi)$  evaluated at some point  $\beta \in \mathbb{R}[-1, 1]$ , the error estimate is written as

$$e_n(z) \approx \frac{h^{n+3}}{2^{2n-1} n^3 (n+1)!} |D^{n+1} f(z(\beta))|. \quad (1.14)$$

The above error estimate presumes that

$$|D^{n+1} f^+(\sigma)| \approx |D^{n+1} f^-(\eta)| \approx |D^{n+1} f(\beta)|, \quad (1.15)$$

for  $\sigma, \eta \in \mathbb{R}[-1, 1]$ .

### 1.2.3 Iteration procedure

For the family of methods discussed in § 1.2.2, Eq. (1.11) indicates that intra-nodal information from the previous step is required, viz.  $a_k^-$ . That information can be stored from the previous iteration and easily made available. Additionally, it indicates that this family of methods is not self-starting. To provide the necessary starting

---

<sup>†</sup>For  $k \geq 3$ ,  $S_{jk}$  appearing here typographically corrects  $R_{jk}$  given by Eq. (21) in Panovsky and Richardson [5, p. 39]; cf. Carpenter [2, p. 56].

---

**Algorithm 1.1** Example continuous substitution iteration

---

```

1: compute intra-nodal step sizes:  $h_j \leftarrow x_j - x_{j-1}$ 
2: while  $x < x_f$  do
3:   for all  $j \in \mathbb{Z}[1, n]$  do
4:     generate initial guesses for all  $y_j$ :  $\mathbf{y}^{(0)}$ 
5:     repeat
6:       compute  $a_k^{(\ell)}$ 
7:       compute  $y_j^{(\ell)}$ 
8:     until  $\|\mathbf{y}^{(\ell)} - \mathbf{y}^{(\ell-1)}\| < \epsilon$ 
9:   end for
10:   $\mathbf{y}^{(\ell-1)} \leftarrow \mathbf{y}^{(\ell)}$ 
11:  increment  $x$ 
12: end while

```

---

information any number of high-accuracy methods may be used. The fact that high-accuracy, fully explicit or predictor-corrector methods are computationally expensive is of little concern because it is only required to bootstrap the process. Fortunately, the family of methods for first-order systems (§ 1.2.1) is self-starting, as indicated by Eq. (1.5).

Despite their differences, both families of methods still share similar iteration schemes because derivative and solution values at the forward intra-nodal points must be known. To enhance the process, it is convenient to generate an initial guess with a simple, low-order, fully explicit method. Experimentation has shown that a fourth-order Runge-Kutta scheme is sufficient for this task. Once the initial guesses have been obtained for each  $y_j$ , a method termed *iteration by continuous substitution*<sup>1</sup> is used to until the difference between two successive  $y_j$  is sufficiently small. An example algorithm for this is given in Algorithm 1.1. This sample algorithm is by no means optimized for speed of execution, but rather illustrative of the procedure necessary to implement the families of methods.

#### 1.2.4 Method degree recommendations

For second-order systems of special form, the local truncation error analysis performed in Panovsky and Richardson [5] does not explicitly state any recommendations for choice of method degree with respect to computational efficiency. However, the analysis presented in Richardson and Panovsky [7] for first-order systems clearly favors even method degrees since for a given degree  $n$  even, the truncation errors for  $n$  and  $n + 1$  are very similar. This is clearly seen from Eq. (1.9).

---

<sup>1</sup>See Panovsky and Richardson [5] for additional detail.

### 1.3 Goals

Richardson et al. [8] provided several significant performance improvements in the implementation of the family of first-order methods. Further significant performance improvement for both families could be obtained by expanding the coefficients  $a_k$  and  $a_k^\pm$  directly into Eqs. (1.5) and (1.11), respectively. This would eliminate their explicit computation in the inner-most iteration of Algorithm 1.1 and considerably speed the process. However, as Panovsky and Richardson [5, p. 41] noted, “The explicit replacement of [these coefficients] ... would require considerable algebraic manipulations even for modest values of  $n$ .” Indeed, considerable effort would be required to perform this algebra by hand for a single interesting value of  $n$ . Fortunately, symbolic algebra systems allow us to perform algebraic operations that would ordinarily be most unproductive if performed by hand. To accomplish this task, we develop a package using the MATHEMATICA<sup>®</sup> symbolic algebra system.

In addition to automatically performing large quantities of tedious algebra, there are additional benefits to such an approach. One obvious benefit is that the required coefficients can be computed exactly as algebraic numbers<sup>2</sup>. Knowing the coefficients exactly will ensure that they can be computed externally to any accuracy desired for a given precision.

In terms of the explicit forms for first- and second-order methods, we want to write Eqs. (1.5) and (1.11) as

$$y(x + h\xi_j) = y(x) + h \sum_{k=0}^n w_{jk} f(x_k), \quad (1.16)$$

and

$$y(x + h\xi_j) = 2y(x) - y(x - h\xi_j) + h^2 \sum_{k=0}^n w_{jk} (f(x_k^+) + f(x_k^-)), \quad (1.17)$$

respectively. In each of the above equations, we seek to compute the  $w_{jk}$  exactly.

---

<sup>2</sup>See Wolfram [10, p. 1070].

## Chapter 2

# Computing the Exact Coefficients

### 2.1 Introduction

As mentioned in the previous chapter, MATHEMATICA<sup>®</sup> was selected to perform as the symbolic algebra engine. It is certainly not the only symbolic algebra system that could have been selected. Many other systems exist that could perform this task.

In the following sections, we develop and describe the MATHEMATICA<sup>®</sup> code necessary to compute the exact coefficients for both families of implicit methods previously discussed. To better document the program development, **noweb**<sup>1</sup> by Ramsey [6] was used. The coefficients themselves appear in appendices A–C. For notational convenience, all derivative evaluations are written  $f(x)$  rather than the fully explicit  $f(y(x), x)$ . Thus  $y(x)$  is always an implicit argument when writing  $f(x)$ .

We begin by considering the mathematics from Chapter 1.

### 2.2 Mathematical definitions

The common mathematics from Chapter 1 are

$$\begin{aligned} 7 \quad \langle \textit{mathematical definitions: common 7} \rangle &\equiv & (24) \\ &\langle \textit{define interval related quantities 8a} \rangle \\ &\langle \textit{define double prime summation 8b} \rangle \\ &\langle \textit{define finite-sum Chebyshev approximation coefficients 8c} \rangle \\ &\langle \textit{define abstracted summation evaluator 8d} \rangle \\ &\langle \textit{define generic method generator for given degree 9a} \rangle \end{aligned}$$

with family specific mathematics defined in code chunks that follow.

---

<sup>1</sup>The **noweb** system is a considerably simplified implementation of the *literate programming* concept presented by Knuth [4].

### 2.2.1 Interval related quantities

From § 1.1.1, the location of the points of interest on the unit interval from Eqs. (1.1) and (1.3) become

$$\begin{aligned}
 8a \quad \langle \text{define interval related quantities } 8a \rangle \equiv & \quad (7) \\
 & \text{theta}[j\_Integer, n\_Integer] := (n-j) \text{ Pi} / n; \\
 & \text{xi}[j\_Integer, n\_Integer] := (1 + \text{Cos}[\text{theta}[j, n]]) / 2;
 \end{aligned}$$

### 2.2.2 Double prime summation

$$\begin{aligned}
 8b \quad \langle \text{define double prime summation } 8b \rangle \equiv & \quad (7) \\
 & \text{sumpp}[elist\_List] := \\
 & \quad \text{Module}[\{\}, \\
 & \quad \quad \text{Return}[\text{Plus} @@ \text{MapAt}[\#/2\&, elist, \{\{1\}, \{-1\}\}]]; \\
 & \quad ];
 \end{aligned}$$

### 2.2.3 Finite-sum Chebyshev approximation coefficients

Eqs. (1.6) and (1.12) provide the finite-sum approximation Chebyshev coefficients for each family. Since the  $a_k$  from § 1.2.1 are the same as the  $a_k^+$  from § 1.2.2, the definition of  $a_k^\pm$  is used to provide  $a_k$ .

$$\begin{aligned}
 8c \quad \langle \text{define finite-sum Chebyshev approximation coefficients } 8c \rangle \equiv & \quad (7) \\
 & \text{ak}[k\_Integer /; k \geq 0, s\_Integer /; s === +1 || s === -1, \\
 & \quad n\_Integer /; n > 0] := \\
 & \quad \text{Module}[\{elist, j\}, \\
 & \quad \quad elist = \text{Table}[f[x + \text{Sign}[s] h \text{xi}[j, n]] \text{Cos}[k \text{theta}[j, n]], \\
 & \quad \quad \quad \{j, 0, n\}]; \\
 & \quad \quad \text{Return}[(2/n) \text{sumpp}[elist]]; \\
 & \quad ];
 \end{aligned}$$

The second argument to  $\text{ak}[\dots]$ , i.e.  $s$ , represents the sign required to compute each  $x_j$ . As the condition indicates,  $s$  must be given such that  $s = \pm 1$ .

### 2.2.4 Abstracted summation evaluator

$$\begin{aligned}
 8d \quad \langle \text{define abstracted summation evaluator } 8d \rangle \equiv & \quad (7) \\
 & \text{summation}[j\_Integer /; j > 0, n\_Integer /; n > 0] := \\
 & \quad \text{Module}[\{\}, \\
 & \quad \quad \text{pickupGenOptions}[\text{Equations}, \text{opts}]; \\
 & \quad \quad \text{Return}[\text{sumpp}[\text{sumtable}[j, n]]];
 \end{aligned}$$

];

For the summations appearing in Eqs. (1.5) and (1.11), each of the terms are generated and placed in a `Table` with the first and last entries of the list halved (`sumpp`), as required, and then added together. The `sumtable` and leading multipliers of the summations are handled in family specific code chunks that follow.

### 2.2.5 Generic method generator for specific degree

For both families of methods, the method for a particular degree is first generated in raw form, and will require considerable simplification.

9a  $\langle \text{define generic method generator for given degree } 9a \rangle \equiv$  (7)

```

GenEqsRaw[ n_Integer /; n > 0 && EvenQ[n], opts___Rule ] :=
Module[ {j},
  pickupGenOptions[ Equations, opts ];

  Return[ Table[ { y[ x + h xi[j,n] ],
                  leadingterms[j,n],
                  summation[j,n] },
                {j,n} ] ];
];

```

In the table returned from `GenEqsRaw`, the first list element of each table entry is the intra-nodal point of interest. The second and third elements of the list contain the corresponding leading terms and summation information, respectively, from Eqs (1.5) and (1.11). This information is family specific.

### 2.2.6 First-order system specifics

Here, we address the family specific mathematics from § 1.2.1.

9b  $\langle \text{mathematical definitions: } \mathcal{O}(1) \text{ } 9b \rangle \equiv$  (24b) 10a▷

```

R[ j_Integer /; j > 0, k_Integer /; k >= 0,
  n_Integer /; n > 0 ] :=
Module[ {tj},
  tj=theta[j,n];
  Which[
    k == 0,
      (Cos[tj] + 1),
    k == 1,
      (Cos[2 tj] - 1) / 4,
    True, (* k >= 2 *)
      ( (Cos[ (k+1)tj ] + (-1)^k) / (k+1) -
        (Cos[ (k-1)tj ] + (-1)^k) / (k-1) ) / 2
  ]
];

```

```
]
];
```

The above chunk provides  $R_{jk}$  from Eq. (1.7).

10a  $\langle \text{mathematical definitions: } \mathcal{O}(1) \text{ 9b} \rangle + \equiv$  (24b)  $\langle 9b \text{ 10b} \rangle$   
`sumcoefficient = {h, 2};`

`sumcoefficient` defines the leading multiplier of the summation of Eq. (1.5).

10b  $\langle \text{mathematical definitions: } \mathcal{O}(1) \text{ 9b} \rangle + \equiv$  (24b)  $\langle 10a \text{ 10c} \rangle$   
`leadingterms[ j_Integer, n_Integer ] := (y[x]);`

`leadingterms` represents the non-summation terms in the right hand side of Eq. (1.5).

10c  $\langle \text{mathematical definitions: } \mathcal{O}(1) \text{ 9b} \rangle + \equiv$  (24b)  $\langle 10b \rangle$   
`sumtable[ j_Integer /; j > 0, n_Integer /; n > 0 ] :=`  
`Table[ ak[k,+1,n] R[j,k,n], {k,0,n} ];`

Lastly, `sumtable` provides the summand information for Eq. (1.5).

### 2.2.7 Second-order special form system specifics

Here, we address the family specific mathematics from § 1.2.2.

10d  $\langle \text{mathematical definitions: } \mathcal{O}(2)s \text{ 10d} \rangle + \equiv$  (24c)  $\langle 11a \rangle$   
`S[ j_Integer /; j > 0, k_Integer /; k >= 0,`  
`n_Integer /; n > 0 ] :=`  
`Module[ {tj},`  
`tj=theta[j,n];`  
`Which[`  
`k == 0,`  
`1/4( 3 + 4 Cos[tj] + Cos[2 tj] ),`  
`k == 1,`  
`1/24( -8 - 9 Cos[tj] + Cos[3 tj] ),`  
`k == 2,`  
`1/48( -9 - 16 Cos[tj] - 8 Cos[2 tj] + Cos[4 tj] ),`  
`True, (* k > 2 *)`  
`Cos[ (k+2)tj ] / ( 4(k+1)(k+2) ) +`  
`Cos[ (k-2)tj ] / ( 4(k-1)(k-2) ) -`  
`(Cos[ k tj ] + 2 Cos[ k Pi ] Cos[ tj ]) / ( 2(k-1)(k+1) ) -`  
`Cos[ k Pi ] / ( (k-2)(k+2) )`  
`]`  
`];`

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The above chunk defines  $S_{jk}$  from Eq. (1.13).

11a  $\langle \text{mathematical definitions: } \mathcal{O}(2)s \text{ 10d} \rangle + \equiv$  (24c)  $\langle 10d \text{ 11b} \rangle$   
`sumcoefficient = {h^2, 4};`

`sumcoefficient` provides the leading multiplier of the summation of Eq. (1.11).

11b  $\langle \text{mathematical definitions: } \mathcal{O}(2)s \text{ 10d} \rangle + \equiv$  (24c)  $\langle 11a \text{ 11c} \rangle$   
`leadingterms[ j_Integer, n_Integer /; n > 0 ] :=`  
`(2 y[x] - y[x - 1/2 (1+Cos[(n-j) Pi/n]) h ]);`

`leadingterms` represents the non-summation terms in the right hand side of Eq. (1.11).

11c  $\langle \text{mathematical definitions: } \mathcal{O}(2)s \text{ 10d} \rangle + \equiv$  (24c)  $\langle 11b \rangle$   
`sumtable[ j_Integer /; j > 0, n_Integer /; n > 0 ] :=`  
`Table[ (ak[k,+1,n]+ak[k,-1,n]) S[j,k,n], {k,0,n} ];`

Lastly, `sumtable` computes the summand information for Eq. (1.11).

### 2.3 Simplification engine

`Equations` is the publicly exported function that users invoke to obtain the method of degree  $n$  for a given family. This function does not generate the raw equations directly, `GenEqsRaw` performs that task. However, it does perform the specific simplifications necessary to obtain the most compact form of a method.

11d  $\langle \text{simplification engine: common:head 11d} \rangle + \equiv$  (24)  
`Equations[ n_Integer /; n > 0 && EvenQ[n], opts___Rule ] :=`  
 $\langle \text{state module decls and process user opts 11e} \rangle$   
 $\langle \text{generate raw degree } n \text{ equations 12a} \rangle$   
 $\langle \text{apply } g : \cos n\pi/m \mapsto \mathbb{A} \text{ or apply } N \text{ 12b} \rangle$   
 $\langle \text{replace numeric stepsizes in function args with symbols 12c} \rangle$   
 $\langle \text{fully simplify expressions to get single weight for each } f \text{ 14c} \rangle$

The additional argument predicate `EvenQ` attached to `n` ensures that only even degree methods will be generated as recommended in § 1.2.4. For convenience, methods for second-order systems of special form are also so constrained. However, simply removing the predicate here and in `GenEqsRaw` will permit generation of methods of odd degree.

11e  $\langle \text{state module decls and process user opts 11e} \rangle + \equiv$  (11d)  
`Module[ {j, isExpandFailed=False, creplist={}, flist, hlist,`  
`hrlist, frlist, tmp, den, coef, result},`  
`pickUpGenOptions[ Equations, opts ];`

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12a  $\langle \text{generate raw degree } n \text{ equations } 12a \rangle \equiv$  (11d)

```
Print[ "n = ", n, ":" ];
Print[ " Generating equations." ];
eqs = GenEqsRaw[ n, opts ];
```

12b  $\langle \text{apply } g : \cos n\pi/m \mapsto \mathbb{A} \text{ or apply N } 12b \rangle \equiv$  (11d)

```
If[ exactp == True,
  Print[ " Expanding Cos[n Pi / m] terms." ];
  creplist = CosRepList[ eqs ];
  If[ Not[ IsDesiredForm[ creplist ] ] ,
    Print[ " ...expansion failed." ];
    $MaxExtraPrecision += 50;
    isExpandFailed = True;
  ],,
  (*else*)
  Print[ " Converting to numerical coefficients, prec = ", prec ];
  eqs = Simplify[ N[ eqs, prec ] ];
];
```

If the replacement fails, the boolean `isExpandFailed` is set to `True`. Failure is measured against `IsDesiredForm` which requires that the expressions be free of `Root`, `Complex`, and `Power[-1, _]` objects.

12c  $\langle \text{replace numeric stepsizes in function args with symbols } 12c \rangle \equiv$  (11d)

```
 $\langle \text{extract unique function calls } 12d \rangle$ 
 $\langle \text{build replacement rule for forward unit intra-nodal points } 12e \rangle$ 
 $\langle \text{handle replacement for back intra-nodal points } 13a \rangle$ 
 $\langle \text{perform stepsize replacement } 13b \rangle$ 
 $\langle \text{check if } \mathbb{A} \text{ numbers are irreducible Root objects } 13c \rangle$ 
```

12d  $\langle \text{extract unique function calls } 12d \rangle \equiv$  (12c)

```
Print[ " Simplifying intra-nodal stepsizes." ];
flist = Cases[ eqs, _[_], Infinity] // Union;
```

12e  $\langle \text{build replacement rule for forward unit intra-nodal points } 12e \rangle \equiv$  (12c)

```
hlist = Table[ xi[j,n], {j,1,n-1}] h;
hrlist = Table[ hlist[[j]] -> MakeSym[h,j], {j,Length[hlist]} ]
// Sort[ #, Greater[ LeafCount[#1], LeafCount[#2] ]& ]&&;
frlist = flist // ExpandAll;
```

The `Sort` applied to `hrlist` uses `LeafCount` to provide a sort based on expression complexity. The purpose is to move the more complex expressions to the head of the list, so that less complex expressions are not incorrectly substituted into portions of the more complex expressions. For example, for the two rules  $x3 \rightarrow a$ ,  $x4 \rightarrow a+b$ , the

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rule for **x4** must be executed first to avoid the faulty execution of rule **x3** resulting in incorrect expressions, viz. **x3+b** rather than simply **x4**.

13a *⟨handle replacement for back intra-nodal points 13a⟩*≡ (12c)  
`For[ j = 1, j <= Length[hrlist], j++,  
 frlist = frlist //. ExpandAll[{ hrlist[[j]],  
 Rule @@ (-#& /@ List @@ hrlist[[j]]) }]  
];`

The purpose of the **-#&** rule is to also replace the any back intra-nodal points with negative version of the intra-nodal stepsize, i.e.  $-h_j$ . This does not affect the family of first-order methods, since back points will not exist in the function arguments, so it is safe to apply in both cases.

13b *⟨perform stepsize replacement 13b⟩*≡ (12c)  
`hrlist[[All,1]] = hrlist[[All,1]] // Collect[#, h, Simplify] &;  
frlist = Thread[ Rule[flist, frlist] ];  
eqs = (eqs //. frlist);`

If the resulting expression does not meet the desired form, e.g. contains **Root** objects because the resulting polynomial equations are not reducible to nested square roots, a different method of labeling the **Root** objects is invoked.

13c *⟨check if  $\mathbb{A}$  numbers are irreducible Root objects 13c⟩*≡ (12c)  
`If[ isExpandFailed,  
 ⟨grab all Cos occurrences 13d⟩  
 ⟨compute maximally reduced Root objects 13e⟩  
 ⟨calculate unique numerical values with extra precision 13f⟩  
 ⟨construct replacement rule for (numerically) unique Root objects 14a⟩  
];  
⟨apply Root replacement rule 14b⟩`

13d *⟨grab all Cos occurrences 13d⟩*≡ (13c)  
`Print[ " ...labeling/replacing Root[] objects."];  
coslist = Cases[ eqs, Cos[_], Infinity] // Union;`

13e *⟨compute maximally reduced Root objects 13e⟩*≡ (13c)  
`rootlist = coslist // TrigToExp // RootReduce // FullSimplify;`

13f *⟨calculate unique numerical values with extra precision 13f⟩*≡ (13c)  
`nlist = Abs[N[rootlist, $MaxExtraPrecision]]  
// Union[#,  
 SameTest->(Abs[#1-#2]<10^(-$MaxExtraPrecision+1)&)]&;`

The **SameTest** used for the **Union** attempts to provide sufficient precision to resolve roots that may be close together. This is most likely not necessary since the

Chebyshev intra-nodal points should be sufficiently different to avoid this problem for most degrees  $n$  of interest. However, for very large  $n$ , intra-nodal clustering near the interval end points may cause difficulty depending on the required floating-point precision.

An interesting side effect of this `SameTest` is that it accounts for a slight difference in the way `N` is applied between MATHEMATICA<sup>®</sup> versions 4.0 and 4.1.

```
14a  ⟨construct replacement rule for (numerically) unique Root objects 14a⟩≡      (13c)
      clist = Table[ MakeSym[c,j], {j,Length[nlist]} ];
      nclist = Thread[Rule[nlist,clist]];
      nclist = {Table[ Rule @@ (-#& /@ List@@nclist[[k]]),
                  {k,Length[nclist]}],
                nclist} //Flatten;
      rclist = Thread[Rule[rootlist,
                           (N[rootlist,$MaxExtraPrecision]/.nclist)]];
      creplist = Thread[ Rule[ coslist, (rootlist//.rclist) ] ];
```

The meaning of the above code chunk may be difficult to perceive, initially. Its goal is to determine which `Root` objects are unique (numerically), symbolically label them (`c1,c2,...`), replace repeated `Root` objects with the appropriate label, then replace all `RootReduce`'d `Cos` expressions with the appropriate symbol.

The resulting replacement rules are only applied if `creplist` is not empty.

```
14b  ⟨apply Root replacement rule 14b⟩≡      (13c)
      If[ creplist != {},
          eqs = eqs //. creplist;
      ];
```

```
14c  ⟨fully simplify expressions to get single weight for each f 14c⟩≡      (11d)
      Print[ " Factoring common terms from integral approximation."];
      WriteString[ $Output, "    j = "];
      For[ j = 1, j <= n, j++,
          WriteString[ $Output, j, ", "];
          t1 = (eqs[[j,3]] / sumcoefficient[[2]] ) // ExpandAll
              // Collect[ #, f[_], Simplify ]&
              // Cancel // Together;
          den = t1 // Denominator;
          coef = sumcoefficient[[1]] / den;
          eqs[[j,3]] = coef;
          t1 = t1 // Numerator
              // Collect[#,f[_],
                          FullSimplify[#,ExcludedForms->{Sqrt[_]}]&]&;

          t2 = Cases[ t1, f[_], Infinity ] // Union;
          Assert[ t2[[1]] == f[x], "PRC::expr missing f[b]" ];
          t3 = Coefficient[ t1, t2 ];
```

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```
t4 = Thread[ List[t3,t2] ];
```

For the above chunk, the denominator of the summation multiplier for each equation is incorporated into the coefficients and a common denominator factored. The combined summation multiplier is then incorporated into the output list. In finding a common denominator, care is taken not to disturb any `Sqrt` objects appearing.

The family specific code for first-order methods is quite brief. At this stage the simplification is complete. The only remaining task is to rearrange the list slightly so that the coefficients appear from left to right in increasing order of intra-nodal location.

15a  $\langle \text{simplification engine: } \mathcal{O}(1) \text{ 15a} \rangle \equiv$  (24b)

```
t1 = Prepend[ RotateLeft[ Rest[t4], 1], First[t4] ];
```

The family specific code for second-order special form methods is not as brief as for the first-order methods. These expressions contain both forward and back intra-nodal location information. Given the symmetry in the problem, the summand of Eq. (1.11) can be written  $w_j(f(x+h_j) + f(x-h_j))$ . This is accomplished in the `For` block below.

Once the desired summand form is achieved, the expressions still require rearrangement for viewing convenience. This is handled in a fashion similar to the first-order methods. However, there is one notable difference. For the case of second-order special form methods, it is trivial to show that the  $n^{\text{th}}$  equation of degree  $n$  is always a corrector, thus lacking an  $f(x+h) + f(x-h)$  term. As a consistency check and for viewing convenience, this is explicitly stated in the output equations.

15b  $\langle \text{simplification engine: } \mathcal{O}(2)s \text{ 15b} \rangle \equiv$  (24c)

```
t5 = Rest[ t4 ] // Partition[#,2]&;
For[ k = 1; m = Length[t5]; t1 = {}, k <= m, k++,
  tmp = t5[[k]];
  If[ SameQ[ tmp[All,1] ],
    AppendTo[ t1, {tmp[[1,1]], Plus@@tmp[[All,2]]} ]; ,
  (*else*)
    Throw["PRC::unexpected intra-nodal asymmetry."];
];
];
If[ FreeQ[ t1, f[x+h] ],
  PrependTo[ t1, First[t4] ];
  AppendTo[ t1, {0, f[x-h]+f[x+h]} ];,
(*else*)
  t1 = Prepend[ RotateLeft[t1,1], First[t4] ];
];
```

15c  $\langle \text{simplification engine: common:tail 15c} \rangle \equiv$  (24)

```
AppendTo[ eqs[[j]], t1 ];
```

```

        If[ IsDesiredForm[ eqs[[j]] ] === False,
            WriteString[ $Output, "(" ,j,"x)" ];
        ];
    ];

    hrlist = (Reverse /@ hrlist) //. creplist // Sort[#,noLettersQ]&;
    hrlist[[All,2]] = FullSimplify[ Coefficient[hrlist[[All,2]], h] ] h;
    result = { hrlist, eqs};

    If[ isExpandFailed,
        crlist = Thread[ Rule[ clist, clist /. (Reverse /@ rclist) ] ];
        AppendTo[ result, crlist ];
    ];

    Print["\n Done."];

    Return[ result ];
];

```

## 2.4 Utility function definitions

16a     $\langle \text{utility functions 16a} \rangle \equiv$  (24)

- $\langle \text{convenient assertion device 16b} \rangle$
- $\langle \text{construct compound symbols from String+Number combinations 17a} \rangle$
- $\langle \text{predicate to validate desired expression form 17b} \rangle$
- $\langle \text{predicate to sort String by numeric values appearing 17c} \rangle$
- $\langle \text{complexity function that provides a weighted leaf count 18a} \rangle$
- $\langle \text{function that constructs and applies } g : \cos n\pi/m \mapsto \mathbb{A} \text{ 18b} \rangle$
- $\langle \text{function to handle opening output files of form } n[0-9]+-[ab].\text{tex 19a} \rangle$

### 2.4.1 Assertion device

16b     $\langle \text{convenient assertion device 16b} \rangle \equiv$  (16a)

```

Assert[ expr_, excpt_:Null ] :=
    Block[{},
        Assert::indet = "Indeterminate boolean expression '1' encountered.";
        Assert::failed = "Assertion '1' failed.";

        If[ Evaluate[expr],
            Null,
            (* True, do nothing *)

```

```

        Message[Assert::failed, expr]; (* false => fail *)
        Throw[excpt],
        Message[Assert::failed, expr]; (* other => fail differently *)
        Message[Assert::indet, expr];
        Throw[excpt];
    ];
];
SetAttributes[Assert, HoldFirst];

```

### 2.4.2 Constructing arbitrary symbol tags

17a  $\langle$ construct compound symbols from String+Number combinations 17a $\rangle \equiv$  (16a)

```

MakeSym[q_, p_Integer] :=
  Block[ {res},
    res = MakeExpression[ ToString[q] <> ToString[p], StandardForm];
    Return[ res // ReleaseHold ];
  ];

```

### 2.4.3 Expression form predicate

The predicate `IsDesiredForm` determines if an input expression is free from complex numbers, `Root` objects, and terms like  $(-1)^p$ . A result of `False` indicates that the expression is *not* of the desired form.

17b  $\langle$ predicate to validate desired expression form 17b $\rangle \equiv$  (16a)

```

IsDesiredForm[ expr_ ] := ( FreeQ[expr, Power[-1,_], Infinity] &&
                             FreeQ[expr, Complex[_,_], Infinity ] &&
                             FreeQ[expr, Root[__,_], Infinity ] );

```

### 2.4.4 Special sorting predicate

When sorting certain expressions, it is preferable to avoid lexical sorting and instead ask that the predicate use numerical ordering. The predicate `noLettersQ` does just that. While the code may look rather messy, the fundamental idea is straight forward. When comparing two expressions, the predicate first ensures both are strings, then converts both arguments to character lists and deletes any element matching `LetterQ`. From there, the remaining digits are rejoined as strings and manually converted to expressions that are subject to `OrderQ`. Of course, `noLettersQ` is expecting to sort only on the right hand sides of rules of the form `x1->1` rather than single entries.

17c  $\langle \text{predicate to sort String by numeric values appearing 17c} \rangle \equiv$  (16a)

```
noLettersQ = OrderedQ[ToExpression[((StringJoin@@#&)/@#&)[
    ((DeleteCases[#, _?LetterQ]&)/@#&)[
        (Characters/@#&)[ToString/@((#[[1]]&)/@{##})]]]]]&;
```

### 2.4.5 Weighted leaf count

The MATHEMATICA<sup>®</sup> FullSimplify function allows users to specify a complexity function to rank the complexity of various expression forms. By default, forms are ranked according to their LeafCount. To avoid the expressions specified by IsDesiredForm, their LeafCount complexity is artificially increased by a factor of  $10^{1000}$ . This approach is a brute-force application of a penalty method to direct the simplification process. It is neither elegant nor efficient, but accomplishes the task in a reasonable time. To keep efficiency issues to a minimum, it is only used in the simplification of expressions of the form  $\text{Cos}[n \text{ Pi}/m]$ .

18a  $\langle \text{complexity function that provides a weighted leaf count 18a} \rangle \equiv$  (16a)

```
WLC[iexpr_] :=
Module[{fudge = 10^1000, lc = LeafCount[iexpr]},
  If[ Not[ IsDesiredForm[ iexpr ] ],
    lc *= fudge ] ;
  Return[ lc ];
];
```

### 2.4.6 Mapping $\cos n\pi/m$ to $\mathbb{A}$

CosRepList attempts to transform expressions of the form  $\text{Cos}[n \text{ Pi}/m]$  to algebraic numbers, specifically nested square roots, within a given expression list.

18b  $\langle \text{function that constructs and applies } g : \cos n\pi/m \mapsto \mathbb{A} \text{ 18b} \rangle \equiv$  (16a)

```
CosRepList[ eqs_ ] :=
Module[ {cosines, rlist={}},
  If[ MemberQ[ eqs, Cos[_], Infinity ],
    cosines = Cases[ eqs, Cos[_], Infinity ] // Union;
    rlist = (# -> FullSimplify[TrigToExp[#],
        ComplexityFunction->WLC]) & /@ cosines;

  ];
  Return[ rlist ];
];
```

### 2.4.7 Opening specially named files

19a *(function to handle opening output files of form n[0-9]+-[ab].tex 19a)*  $\equiv$  (16a)

```

myOpenWrite[ str_String, opts___Rule ] :=
Module[ {filename},
  pickupTeXOptions[ EquationsLaTeXed, opts ];

  If[ StringQ[ofn],
    filename = ofn <> str <> ".tex";
    os = OpenWrite[filename];
    If[ os == $Failed,
      Throw[{myOpenWrite, OpenWrite, filename, $Failed}]];
    Return[ os ],
    (*else*)
    Throw[{myOpenWrite, StringQ[], $Failed}]];
];
];

```

## 2.5 L<sup>A</sup>T<sub>E</sub>X output functions

Producing L<sup>A</sup>T<sub>E</sub>X output of the methods generated here requires considerably more processing than the standard TeXForm command. The following code chunks address this need.

Essentially two options are available for the L<sup>A</sup>T<sub>E</sub>X output: tabular or standard equation form. For the tabular form, tables of coefficients with appropriate table headings, captions and labels are produced. To account for the possibility of having tables that may span multiple pages, the `longtable` package is used for every table. For the case of standard equation form, constants like `Root` objects and intra-nodal stepsizes are placed in standard `amsmath align` environments. In both output forms, the `breqn` package<sup>2</sup> is required to automatically and semi-intelligently break multi-line expressions. To use the package, insert

```

\usepackage[cmbase]{flexisym}
\usepackage{breqn}
\setkeys{breqn}{compact}

```

into the preamble of your L<sup>A</sup>T<sub>E</sub>X wrapper. Additionally, for tabular output, the command `\mx` must be defined with the form

```

\providecommand{\mx}[1]{\begin{dmath*}[style={\small}]#1\end{dmath*}}

```

for the output to be successfully processed.

---

<sup>2</sup>This package is available via FTP from [ftp.ams.org/pub/tex/](ftp://ftp.ams.org/pub/tex/).

19b  $\langle \text{\textit{L\textsubscript{A}T\textsubscript{E}X} output functions 19b} \rangle \equiv$  (24)  
 $\langle \text{\textit{function to generate longtable header 20a}} \rangle$   
 $\langle \text{\textit{function to L\textsubscript{A}T\textsubscript{E}X simple lists of constants 20b}} \rangle$   
 $\langle \text{\textit{function to L\textsubscript{A}T\textsubscript{E}X material as coefficient tables 21}} \rangle$   
 $\langle \text{\textit{function to L\textsubscript{A}T\textsubscript{E}X material as standard equations 23}} \rangle$   
 $\langle \text{\textit{publicly exported L\textsubscript{A}T\textsubscript{E}X output function 24a}} \rangle$

### 2.5.1 longtable header

The table header produced is for use with the `longtable` package.

20a  $\langle \text{\textit{function to generate longtable header 20a}} \rangle \equiv$  (19b)  
`TableHead[ caption_String, shortcaption_String, head_String ] :=`  
`StringJoin[`  
`caption,`  
`" \\hline\\hline\n",`  
`head,`  
`" \\hline\n",`  
`" \\endfirsthead\n",`  
`shortcaption, " (\emph{continued})}\\\\\\\n",`  
`" \\hline\\hline\n",`  
`head,`  
`" \\hline\n",`  
`" \\endhead\n",`  
`" \\hline\n",`  
`" \\multicolumn{2}{r}{(\emph{continues})}\\\\\\\n",`  
`" \\endfoot\n",`  
`" \\hline\\hline\n",`  
`" \\endlastfoot\n"`  
`];`

### 2.5.2 Intra-nodal stepsizes and Root objects

20b  $\langle \text{\textit{function to L\textsubscript{A}T\textsubscript{E}X simple lists of constants 20b}} \rangle \equiv$  (19b)  
`texconsts[ eqs_, tabhead_, os_, opts___Rule ] :=`  
`Module[ {tmp, lhs, rhs, j, n, environ="align",`  
`endinput="", newline="\\\\\n"},`  
`pickupTeXOptions[ EquationsLaTeXed, opts ];`  
`tmp = (List@@#)& /@ eqs;`  
`n = Length[ tmp ] + 1;`  
`If[ isTabular,`

```

        environ = "longtable";
        cols = "{p{\widtha}p{\widthb}}\n";
        endinput = "\n\\endinput\n";
        newline = "\\tabularnewline";
    ];

    WriteString[ os, "\\begin{" ,environ,"}",
        If[ isTabular, cols ],
        If[ isTabular, tabhead, "\n" ] ];
    For[ j = 1, j < n, j++,
        {lhs, rhs} = tmp[[j]];
        If[ isTabular == True,
            WriteString[ os, " \mx{ ", j, " } & \mx{ ", TeXForm[rhs],
                " }", If[ j != (n-1), newline, ""], "\n"];
        (*else*)
            WriteString[ os, " ",TeXForm[lhs]," & = ", TeXForm[rhs],
                If[ j != (n-1), newline, ""], "\n"];
    ];
];
WriteString[ os, "\\end{" ,environ,"}\n", endinput];
];

```

### 2.5.3 Tables of coefficients

```

21  <function to LATEX material as coefficient tables 21>≡ (19b)
    LaTeXEqsTable[ eqs_, opts__Rule ] :=
        Module[{aj, bj, tmp, lst, rhsb, rhsc, tabhead, j, k, m,
            n = Length[ eqs[[2]] ]},
            pickupTeXOptions[ EquationsLaTeXed, opts ];
            pickupGenOptions[ Equations, opts];

            nlabel[os_, n_] := WriteString[ os, "%% n = ", n, "\n" ];

            os = myOpenWrite[ "-a", opts ];
            nlabel[os, n];

            (* send out Root objects first if appearing *)
            If[ Length[eqs] > 2,
                tabhead = TableHead[
                    " \\caption{Unique \\texttt{Root} objects $n=" <> ToString[n] <>
                        "$.\\label{tab:c:" <>
                            ToString[n] <> "}}\\n",
                    " \\caption[] {Algebraic numbers, $n=" <> ToString[n] <> "$.",

```

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```

" \\parbox[c][20pt][c]{\\widtha}{\\centerline{$\\ell$}} & " <>
" \\parbox[c][20pt][c]{\\widthb}{\\centerline{$c_{\\ell}$}}\\\\\\n" ];
texconsts[ Sort[ eqs[[3]], noLettersQ ], tabhead, os, opts ];
];

(* send out intra-nodal stepsizes with some fancy rule sorting *)
tabhead = TableHead[
" \\caption{Intra-nodal points $h_j$, $n=$" <> ToString[n] <>
"$." <> "\\label{tab:h:" <> ToString[n] <> "}}\\\\\\n",
" \\caption[] {Intra-nodal abscissa $h_j$, $n=$" <>
ToString[n] <> "$.",
" \\parbox[c][20pt][c]{\\widtha}{\\centerline{$j$}} & " <>
" \\parbox[c][20pt][c]{\\widthb}{\\centerline{$h_j$}}\\\\\\n" ];
texconsts[ Sort[ eqs[[1]], noLettersQ ], tabhead, os, opts ];
Close[os];

os = myOpenWrite[ "-b", opts ];
nlabel[os, n];
WriteString[ os, "\\begin{longtable}{p{\\widtha}p{\\widthb}}\\n",
TableHead[
" \\caption{Integral approximation $I_j$ for $n=$" <>
ToString[n] <> "$."\\label{tab:I:" <> ToString[n] <>
"}}\\\\\\n",
" \\caption[] {Integral approximation $I_j$, $n=$" <>
ToString[n] <> "$.",
" \\parbox[c][20pt][c]{\\widtha}{" <>
" \\centerline{$d_{j,w_{jk}}$}} & " <>
" \\parbox[c][20pt][c]{\\widthb}{" <>
" \\centerline{Expression}}\\\\\\n" ]];
For[ j = 1, j <= n, j++,
{aj, lst} = eqs[[2,j,{3,4}]];
aj = Coefficient[ aj, sumcoefficient[[1]] ];
WriteString[ os, " \\mx{ d_{", j, "}} & \\mx{ ",
TeXForm[ aj ], " } \\\\", "n" ];

For[ k = 0; tmp = lst[[All,1]]; m = Length[tmp], k < m, k++,
WriteString[ os, " \\mx{ w_{", j, ",", k, "}} & \\mx{ ",
TeXForm[ tmp[[k+1]] ],
" } \\tabularnewline", "n" ];
];
];
WriteString[ os, "\\end{longtable}\\n"];
WriteString[ os, "\\n\\endinput\\n"];
Close[os];

```

];

### 2.5.4 List of equations

```

23  ⟨function to LATEX material as standard equations 23⟩≡ (19b)
    LaTeXEqsFull[ eqs_, opts__Rule ] :=
      Module[ {tmp, lhs, rhsa, rhsb, rhsc, j, n},
        pickupTeXOptions[ EquationsLaTeXed, opts ];
        pickupGenOptions[ Equations, opts ];

        os = myOpenWrite[ "", opts ];
        WriteString[ os, "%% n = ", ToString[ Length[ eqs[[2]] ] ], "\n" ];

        (* send out Root objects first if appearing *)
        If[ Length[eqs] > 2,
          WriteString[ os, "\\begin{subequations}\n"];
          texconsts[ Sort[ eqs[[3]], noLettersQ ], "", os, opts ];
          WriteString[ os, "\\end{subequations}\n\n"];
        ];

        (* send out intra-nodal stepsizes with some fancy rule sorting *)
        WriteString[ os, "\\begin{subequations}\n"];
        texconsts[ Sort[ eqs[[1]], noLettersQ ], "", os, opts ];
        WriteString[ os, "\\end{subequations}\n\n"];

        WriteString[ os, "\\begin{subequations}\n"];
        tmp = eqs[[2]];
        n = Length[tmp];
        For[ j = 1, j <= n, j++,
          {lhs, rhsa} = tmp[[j,{1,2}]];
          WriteString[ os, "  \\begin{dmath}\n"];
          WriteString[ os, "    ", TeXForm[ lhs ], " & = \n" ];
          WriteString[ os, "      ", TeXForm[ rhsa ], " + \n"];
          {rhsb, rhsc} = tmp[[j,{3,4}]];
          WriteString[ os, "        ", TeXForm[ rhsb ], " \\left[ ",
            TeXForm[ rhsc ], " \\right] \n",
            "  \\end{dmath}\n"];
        ];
        WriteString[ os, "\\end{subequations}\n\n\\endinput\n"];
        If[ StringQ[ofn], Close[os] ];
      ];

```

### 2.5.5 L<sup>A</sup>T<sub>E</sub>X production public export

24a  $\langle$ publicly exported L<sup>A</sup>T<sub>E</sub>X output function 24a $\rangle \equiv$  (19b)

```

LaTeXeqs[ eqs_, opts___Rule ] :=
Module[ {},
  pickupTeXOptions[ EquationsLaTeXed, opts ];

  If[ isTabular == True,
    LaTeXEqstable[ eqs, opts ];,
    (*else*)
    LaTeXEqFull[ eqs, opts ];
  ];
];

```

## 2.6 Root code chunks

The code chunks contained in this section represent the root chunks for both families of methods. In addition to these root chunks, a sample driver script for the family of first-order methods is provided that generates the methods, produces L<sup>A</sup>T<sub>E</sub>X output and stores the information in appropriate files.

### 2.6.1 Root chunk for family of first-order methods

24b  $\langle$ order1.mma 24b $\rangle \equiv$

```

 $\langle$ package header:  $\mathcal{O}(1)$  26b $\rangle$ 
 $\langle$ public exports to Global' context:  $\mathcal{O}(1)$  27b $\rangle$ 
 $\langle$ public exports to Global' context: common 28b $\rangle$ 
 $\langle$ begin Private context definitions 30a $\rangle$ 
 $\langle$ options processing functions 30b $\rangle$ 
 $\langle$ mathematical definitions: common 7 $\rangle$ 
 $\langle$ mathematical definitions:  $\mathcal{O}(1)$  9b $\rangle$ 
 $\langle$ simplification engine: common:head 11d $\rangle$ 
 $\langle$ simplification engine:  $\mathcal{O}(1)$  15a $\rangle$ 
 $\langle$ simplification engine: common:tail 15c $\rangle$ 
 $\langle$ utility functions 16a $\rangle$ 
 $\langle$ LATEX output functions 19b $\rangle$ 
 $\langle$ package tail: common 30c $\rangle$ 

```

### 2.6.2 Root chunk for family of special form second-order methods

24c  $\langle$ order2s.mma 24c $\rangle \equiv$

```

 $\langle$ package header:  $\mathcal{O}(2)s$  27a $\rangle$ 

```

```

⟨public exports to Global‘ context: common 28b⟩
⟨begin Private context definitions 30a⟩
⟨public exports to Global‘ context:  $\mathcal{O}(2)$ s 28a⟩
  ⟨options processing functions 30b⟩
  ⟨mathematical definitions: common 7⟩
  ⟨mathematical definitions:  $\mathcal{O}(2)$ s 10d⟩
  ⟨simplification engine: common:head 11d⟩
  ⟨simplification engine:  $\mathcal{O}(2)$ s 15b⟩
  ⟨simplification engine: common:tail 15c⟩
  ⟨utility functions 16a⟩
  ⟨ $\text{\LaTeX}$  output functions 19b⟩
⟨package tail: common 30c⟩

```

### 2.6.3 Sample driver script for family of first-order methods

As a sample, the driver below produces the methods for first-order differential equations for even degrees  $2 \leq n \leq 16$ . Once generated, the MATHEMATICA<sup>®</sup> form is stored to a file, then the  $\text{\LaTeX}$  output is generated and saved to file. Here, the chosen output form is tables of coefficients.

To produce methods for second-order differential equations, simply alter the `Get` to load the appropriate file.

```

25a  ⟨generate.mma 25a⟩≡
      Get["order1.mma"];
      For[ n = 2, n < 16, n+=2,
        tmp = Equations[ n ] // Timing;
        Put[ tmp, "n"<>ToString[n]<>".mma"];
        Print[ "time = ", tmp[[1]] ];
        LaTeXeqs[ tmp[[2]], OutputFilePrefix -> "n"<>ToString[n],
                  TeXTableForm->True ];
      ];
      Print["Done."];

```

## 2.7 Miscellaneous chunks

```

25b  ⟨file header: common:head 25b⟩≡                                     (26b 27a)
      (* $Id: package.nw,v 1.32 2002/05/29 15:01:50 mitchejw Exp $ *)

      (* :Title: *)

      (* :Author: Jason Wm. Mitchell *)

```

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26a  $\langle \text{file header: common:tail 26a} \rangle \equiv$  (26b 27a)

```
(* :Context: Global' *)

(* :Package Version: 0.1 *)

(* :Copyright:
    Copyright (C) 2001 Jason Wm. Mitchell, All rights reserved.

    This software is released under the Perl Artistic License,
    see http://www.perl.com/pub/a/language/misc/Artistic.html

    This software is OSI Certified Open Source Software.
    OSI Certified is a certification mark of the Open Source Initiative.
*)

(* :History: *)

(* :Keywords: Numerical integration, initial-value problem,
    Chebyshev methods, implicit methods,
    high-order methods
*)

(* :Mathematica Version: 3.x, 4.x *)

(* :Limitation(s): *)

(*=====*)
```

26b  $\langle \text{package header: } \mathcal{O}(1) \text{ 26b} \rangle \equiv$  (24b)

```
 $\langle \text{file header: common:head 25b} \rangle$ 
(* :Name: PanvoskyRichardsonOrder1 *)

(* :Summary: PanvoskyRichardsonOrder1 is a package that generates
    equations for the numerical integration of initial
    value problems of the form
     $y'(x) = f(y(x);x)$ ,  $y(0) = c1$ ,
    using an implicit, intra-step Chebyshev interpolation.

    The equations are generated with exact coefficients
    based on the Chebyshev weights. The user must
    implement the (implicit) iteration procedure;
    successive substitution.
```

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\*)

```
(* :Source: Richardson, D. L. and Panovsky, J., ‘‘A Family of
implicit Chebyshev methods for the numerical
integration of first-order differential equations’’,
Proceedings of the AAS/AIAA Astrodynamics Specialists
Conference, 1993.
```

\*)

*⟨file header: common:tail 26a⟩*

```
BeginPackage["PanvoskyRichardsonOrder1"];
```

```
ClearAll["PanvoskyRichardsonOrder1*"];
```

27a *⟨package header:  $\mathcal{O}(2)$ s 27a⟩* (24c)

*⟨file header: common:head 25b⟩*

```
(* :Name: PanvoskyRichardsonOrder2S *)
```

```
(* :Summary: PanvoskyRichardsonOrder2S is a package that generates
equations for the numerical integration of initial
value problems of the form
y''(x) = f(y(x);t), y(0) = c1, y'(0) = c2,
using an implicit, intra-step Chebyshev interpolation.
```

```

The equations are generated with exact coefficients
based on the Chebyshev weights. The user must
implement the (implicit) iteration procedure;
successive substitution.
```

\*)

```
(* :Source: Panovsky, J. and Richardson, D. L., ‘‘A Family of
implicit Chebyshev methods for the numerical
integration of second-order differential equations’’,
J. Comp. Appl. Math., 23, 1988, pp. 35--51.
```

\*)

*⟨file header: common:tail 26a⟩*

```
BeginPackage["PanvoskyRichardsonOrder2S"];
```

```
ClearAll["PanvoskyRichardsonOrder2S*"];
```

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- 27b  $\langle public exports to Global ' context: O(1) \rangle \equiv$  (24b)
- ```

PanvoskyRichardsonOrder1::usage =
  "PanvoskyRichardsonOrder1 is a package that generates equations "
  "for the numerical integration of initial value problems of the "
  "form y'(x) = f(y(x);x), y(0) = c1, using an implicit, "
  "intra-step Chebyshev interpolation.\n\n"
  "The equations are generated with exact coefficients based on the "
  "Chebyshev weights. The user must implement the (implicit) "
  "iteration procedure; successive substitution.";
Equations::usage =
  "Equations[ n, options... ] generates equations of order n for the "
  "numerical integration of systems of equations of the form "
  "y'(x) = f(y(x);x), y(0) = c1.";

```
- 28a  $\langle public exports to Global ' context: O(2) \rangle \equiv$  (24c)
- ```

PanvoskyRichardsonOrder2s::usage =
  "PanvoskyRichardsonOrder2s is a package that generates equations "
  "for the numerical integration of initial value problems of the "
  "form y''(x) = f(y(x);x), y(0) = c1, y'(0) = c2, using an implicit, "
  "intra-step Chebyshev interpolation.\n\n"
  "The equations are generated with exact coefficients based on the "
  "Chebyshev weights. The user must implement the (implicit) "
  "iteration procedure; successive substitution.";
Equations::usage =
  "Equations[ n, options... ] generates equations of order n for the "
  "numerical integration of systems of equations of the form "
  "y''(x) = f(y(x);x), y(0) = c1, y'(0) = c2.";

```
- 28b  $\langle public exports to Global ' context: common \rangle \equiv$  (24)
- $\langle function descriptions: common \rangle$
- $\langle function Equations option descriptions \rangle$
- $\langle function LaTeXEqs option descriptions \rangle$
- $\langle define function options and associations \rangle$
- 28c  $\langle function descriptions: common \rangle \equiv$  (28b)
- ```

LaTeXeqs::usage =
  "LaTeXeqs[ eqs, options... ] LaTeXs the output of Equations[n].";
IsDesiredForm::usage =
  "Predicate to determine if an expression is devoid of specific "
  "algebraic forms";
WLC::usage =
  "A leaf count function weighted against Root, Complex and (-1)^(n/m)...";
MakeSym::usage =

```

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```
"Routine to create symbols like x1, from tag (x) and counter."
CosRepList::usage =
  "Replace Cos[n Pi/m] with irrational (algebraic) numbers.";
GenEqsRaw::usage =
  "generate raw equation pieces";
```

29a  $\langle$ function Equations option descriptions 29a $\rangle \equiv$  (28b)

```
IndependentVariable::usage =
  "Independent variable; default:  x.";
ConstantsLabel::usage =
  "Label prefix for Root[] object constants; default:  c."
ExactCoefficients::usage =
  "Exact or floating point coefficients switch; default:  True";
NumericalPrecision::usage =
  "Number of decimal digits of precision for floating point "
  "coefficient representation; default:  16.";
```

29b  $\langle$ function LaTeXEqs option descriptions 29b $\rangle \equiv$  (28b)

```
TeXTableForm::usage =
  "Boolean LaTeX option to put results in a table friendly form; "
  "default:  False."
OutputFilePrefix::usage = "
  Output filename prefix; default:  Null."
OutputStreamList::usage =
  "Stream list to write output; overridden by OutputFilePrefix; "
  "default:  $Output";
```

The replacement rule `IndependentVariable->Global`b` may seem strange since the independent variable discussed thus far has been  $x$ . However, MATHEMATICA<sup>®</sup> relies on a lexically sorted canonical form, so that  $f(x+h)$  appears as `f[h+x]` because lexically  $h > x$ . To provide the preferred form while still subject to a lexical canonical form, `b` was arbitrarily selected as the independent variable so that  $f(b+h)$  appears as `f[b+h]`. By calling `pickupGenOptions` were appropriate, `x` can be used in code while `b` with appearing in the output.

29c  $\langle$ define function options and associations 29c $\rangle \equiv$  (28b)

```
`GenEqsOptionsList = { IndependentVariable->Global`b,
                        DependentVariable->Global`y,
                        DerivativeFunction->Global`f,
                        StepSize->Global`h,
                        ConstantsLabel->Global`c,
                        ExactCoefficients->True,
```

## CHAPTER 2. COMPUTING THE EXACT COEFFICIENTS

```

NumericalPrecision->16 };

'TeXEqOptionsList = { TeXTableForm->False,
                      OutputFilePrefix->Null,
                      OutputStreamList->$Output };

Options[Equations] = 'GenEqOptionsList;

Options[EquationsLaTeXed] = 'TeXEqOptionsList;

30a  <begin Private context definitions 30a>≡ (24)
      Begin["Private"];

      (* explicitly declare all local variables to avoid Global'* collisions *)
      { 'x, 'y, 'f, 'h, 'c, 'protected, 'exactp, 'prec, 'ofn, 'osl };

30b  <options processing functions 30b>≡ (24)
      pickupGenOptions[ from_, opts___Rule ] :=
        Module[ {},
          x = IndependentVariable /. {opts} /. Options[ from ];
          y = DependentVariable /. {opts} /. Options[ from ];
          f = DerivativeFunction /. {opts} /. Options[ from ];
          h = StepSize /. {opts} /. Options[ from ];
          c = ConstantsLabel /. {opts} /. Options[ from ];
          exactp = ExactCoefficients /. {opts} /. Options[ from ];
          prec = NumericalPrecision /. {opts} /. Options[ from ];
        ];

      pickupTeXOptions[ from_, opts___Rule ] :=
        Module[ {},
          isTabular = TeXTableForm /. {opts} /. Options[ from ];
          ofn = OutputFilePrefix /. {opts} /. Options[ from ];
          osl = OutputStreamList /. {opts} /. Options[ from ];
        ];

30c  <package tail: common 30c>≡ (24)
      (* restore protections *)
      Protect[ Evaluate[protected] ];

      End[]; (* end of private context *)
      Protect[ ]; (* protect exported symbols *)
      EndPackage[]; (* end of package *)

```

## Chapter 3

# Implementing an Integrator for First-Order Systems

### 3.1 Introduction

Implementing a numerical integrator for first-order systems for a given degree  $n$  is straight forward. In the following, an integrator of degree  $n = 8$  is constructed using the exact coefficients generated by the method described in Chapter 2 and computed numerically to double-precision.

### 3.2 Basic Requirements

To begin, it is necessary to select a programming language and compiler. For this example, C++ is the programming language of implementation. Since certain aspects of the Standard Library are specified incorrectly in the 1998 ISO standard (ISO/IEC 14882), a functional compiler must necessarily incorporate defect resolutions produced by the ISO/IEC standardization working group<sup>1</sup>. A compiler that addresses this issue and provides a functional Standard Library and Standard Template Library (STL) is the C++ compiler found in the GNU Compiler Collection, `gcc-3.1.0`.

With the programming language and compiler selected, we identify the highest level components necessary to implement an integrator of degree  $n = 8$ . From the previous chapters, we will need the associated exact coefficients computed to double-precision, a fourth-order Runge-Kutta guesser, and an iteration by continuous substitution similar to Algorithm 1.1.

As a final basic requirement, we must choose the data structure needed to manipulate the array data that will arise. Specifically, we would like a templated, fixed-size container that provides amortized constant time access, is not bounds checked, behaves like a mathematical vector, is optimized for speed, is amenable to the BLAS *slice*<sup>2</sup> abstraction, and interacts with STL algorithms. With a complete Standard

---

<sup>1</sup>see the JTC1/SC22/WG21 web page at <http://std.dkuug.dk/jtc1/sc22/wg21/>.

<sup>2</sup>see the Basic Linear Algebra Subprograms library at <http://www.netlib.org/blas/>.

Library and STL, there are many options available. Fortunately, a combination of the Standard Library container `valarray` and `slice` meets all of these requirements individually. Unfortunately, the combined requirement of interaction between slices and STL algorithms is not met by default because slices alone do not meet STL iterator requirements. This can be remedied with the addition of a `slice_iterator` class.

### 3.3 Root chunk: `slice_iterator`

The `slice_iterator` developed here is based largely on the *Slice\_iter* class presented in Stroustrup [9, § 22.4.5, p. 670]. Several alterations have been made to that original design and will be discussed as they appear.

```
32a  <slice_iterator 32a>≡
      <si: header 40c>
      #ifndef PANOVSky_RICHARDSON_SLICE_ITERATOR
      #define PANOVSky_RICHARDSON_SLICE_ITERATOR
      <si: includes 40b>

      namespace pr_cheby
      {
        <si: class decl 32b>
        <si: member defs 36a>
      }
      #endif //PANOVSky_RICHARDSON_SLICE_ITERATOR
```

In the case of `slice_iterator`, as with all the supporting code presented here, we choose to place all code in the header file for `inline` convenience.

#### 3.3.1 Class declaration

```
32b  <si: class decl 32b>≡ (32a)
      template <typename T>
      class slice_iterator
      {
      public:
        <si: forward iterator typedef decls 33c>
        <si: constructor decls 34a>
        <si: forward iterator interface decls 34b>
        <si: access operator decls 35a>
        <si: comparison & ordering decls 35b>
      private:
        <si: private data 33a>
        <si: private function decls 33b>
```

```
};
```

*⟨si: global comparison & order decls 35c⟩*

To help prevent name clashes with other existing libraries, we introduce the namespace `pr_cheby`. All remaining code to support the integrator in this example will be introduced in this namespace.

### 3.3.1.1 Private data

The implementation of `slice_iterator` will require a certain amount of aggregated data to represent its internal state. Since we are building the iterator to work with `valarray`, we will need a pointer to the desired instance. Also, we will need the desired `slice` view, as well as an index indicating the current element.

```
33a  ⟨si: private data 33a⟩≡ (32b)
      std::valarray<T>* v_;
      std::slice s_;
      size_t cur_;
```

Since each data type is `private`, the identifiers are suffixed with an underscore to draw attention to their access membership. This notation will be used in all the presented in this example.

### 3.3.1.2 Private functions

The `ref` member provides a mechanism to calculate the slice element to be accessed in `v_`.

```
33b  ⟨si: private function decls 33b⟩≡ (32b)
      T& ref( size_t k );
      const T& ref( size_t k ) const;
```

A `const`-qualified version of `ref` is provided.

### 3.3.1.3 Forward iterator declarations

Since the purpose of `slice_iterator` is to allow `slice` interaction with STL algorithms and provide element access, it must meet some minimal iterator interface. A minimal iterator interface for our needs is the `forward_iterator`.

```
33c  ⟨si: forward iterator typedef decls 33c⟩≡ (32b)
      typedef typename std::forward_iterator_tag iterator_category;
      typedef T value_type;
      typedef typename std::ptrdiff_t difference_type;
      typedef T* pointer;
```

```
typedef T& reference;
```

The `iterator_category` is declared as a forward iterator, and the remaining declarations are provided as required by the `iterator_traits` template class, Stroustrup [9, § 19.2.2].

### 3.3.1.4 Constructors

A default constructor is supplied explicitly so that arrays of empty `slices` may be constructed without difficulty. The second constructor below creates a specific instance of a `slice_iterator` given a `valarray` and its `slice`.

Since the `slice_iterator` is very simple with respect to data (§ 3.3.1.1), no copy constructor or assignment operator is explicitly defined. The default compiler-supplied behaviour will be sufficient for copy and assignment.

34a     $\langle si: \text{ constructor decls } 34a \rangle \equiv$  (32b)

```
    slice_iterator();
    slice_iterator( std::valarray<T>* v, std::slice s );
```

### 3.3.1.5 Forward iterator interface

For the case of STL iterators, the member methods `begin()` and `end()` provide the interface that specifies the range of an operation over a container. These members return the first and one-past-the-end iterators, respectively. Stroustrup [9, p. 670] seems to consider only the case when the iterator is initially constructed and so does not provide a `begin()` member. Since we anticipate that we will want to repeatedly reuse existing iterator instances, a `begin()` member is supplied to avoid the additional overhead of unnecessarily recreating iterators to obtain a starting iterator element.

34b     $\langle si: \text{ forward iterator interface decls } 34b \rangle \equiv$  (32b) 34c>

```
    slice_iterator begin();
    slice_iterator end();
```

A forward iterator does not define any new expressions beyond the input iterator category, but some of the input iterator restrictions are relaxed so that only two pre- and post-increment operators need to be provided, i.e. `(void)i++` is dropped.

34c     $\langle si: \text{ forward iterator interface decls } 34b \rangle + \equiv$  (32b) <34b

```
    slice_iterator& operator++();
    slice_iterator operator++(int);
```

### 3.3.1.6 Access operators

To permit element access, appropriate operators are declared. The `operator*()` is provided for use with STL algorithms requiring sequential access, and `operator[]()` is provided for random access and to make `slice_iterator` feel more like an array, unlike `std::slice_array`.

```

35a  <si: access operator decls 35a>≡ (32b)
      T& operator*();
      const T& operator*() const;

      T& operator[](size_t k);
      const T& operator[](size_t k) const;

```

For each of the element access operators above, a `const`-qualified version is also declared. To minimize overhead, these `const`-qualified access operators return a reference to a constant element rather than a copy of the element accessed.

With regard to random access efficiency, the `operator[]()`, the input index of type `size_t` should not be bounds checked.

### 3.3.1.7 Comparison functions

To facilitate comparison of `slice_iterator` instances, it is preferable to provide a few public member functions that accomplish this task rather than declare a large block of `friend` functions. Since these members do not alter state, they are appropriately `const`-qualified.

```

35b  <si: comparison & ordering decls 35b>≡ (32b)
      bool slice_equal( const slice_iterator<T>& in ) const;
      bool is_equal( const slice_iterator<T>& in ) const;
      bool is_less_than( const slice_iterator<T>& in ) const;

```

The member `slice_equal` should determine if the `slice` contained within `*this` is equivalent to the `slice` contained in the input `in`.

### 3.3.1.8 Global comparison and order functions

```

35c  <si: global comparison & order decls 35c>≡ (32b)
      template <typename T> inline
      bool operator==(const slice_iterator<T>& a, const slice_iterator<T>& b);

      template <typename T> inline
      bool operator!=(const slice_iterator<T>& a, const slice_iterator<T>& b);

      template <typename T> inline

```

```
bool operator<(const slice_iterator<T>& a, const slice_iterator<T>& b);

template <typename T> inline
bool operator>(const slice_iterator<T>& a, const slice_iterator<T>& b);
```

These global operators are required by the algorithms found in the STL. Since public member functions are provided for comparison and ordering, these functions will simply be expanded `inline` as forwarding functions. Consequently, their call overhead will be eliminated at compile-time.

### 3.3.2 Member definitions

36a  $\langle si: \text{member defs 36a} \rangle \equiv$  (32a)

```

     $\langle si: \text{private function defs 36b} \rangle$ 
     $\langle si: \text{constructor defs 36d} \rangle$ 
     $\langle si: \text{forward iterator interface defs 37b} \rangle$ 
     $\langle si: \text{access operator defs 38a} \rangle$ 
     $\langle si: \text{comparison \& ordering defs 39a} \rangle$ 
     $\langle si: \text{global comparison \& order defs 40a} \rangle$ 
```

#### 3.3.2.1 Private functions

Computing the *k*th element of a slice *s* is straightforward, and defined below.

36b  $\langle si: \text{private function defs 36b} \rangle \equiv$  (36a) 36c>

```

    template <typename T> inline
    T& slice_iterator<T>::ref( size_t k )
    { return( (*v_)[s_.start()+k*s_.stride()] ); }
```

The `const`-qualified `ref` is defined as

36c  $\langle si: \text{private function defs 36b} \rangle + \equiv$  (36a) <36b

```

    template <typename T> inline
    const T& slice_iterator<T>::ref( size_t k ) const
    { return( (*v_)[s_.start()+k*s_.stride()] ); }
```

#### 3.3.2.2 Constructors

In terms of constructors, we must be able to easily construct arrays of uninitialized `slice_iterator` instances, as well as instances that refer to a specific slice of a `valarray`.

The default constructor is

36d  $\langle si: \text{constructor defs 36d} \rangle \equiv$  (36a) 37a  $\triangleright$   
`template <typename T> inline  
slice_iterator<T>::slice_iterator() : v_(0), cur_(0) {}`

As defined, this constructor simply zeroes `v_` and `cur_`, indicating that the iterator is empty, allowing the easy construction of arrays of `slice_iterators`.

To create a `slice_iterator` for a `slice` of a `valarray`, we define the following constructor.

37a  $\langle si: \text{constructor defs 36d} \rangle + \equiv$  (36a)  $\triangleleft$  36d  
`template <typename T> inline  
slice_iterator<T>::slice_iterator( std::valarray<T>* v, std::slice s )  
: v_(v), s_(s), cur_(0) {}`

**Copy constructor, destructor, and assignment** For each of these cases, the default, compiler-generated definitions are used.

### 3.3.2.3 Forward iterator interface

The implementation of the `begin()` and `end()` members is quite simple. A copy of `*this` is made and `cur_` is set either to zero or `s_.size()`, respectively.

37b  $\langle si: \text{forward iterator interface defs 37b} \rangle \equiv$  (36a) 37c  $\triangleright$   
`template <typename T> inline  
slice_iterator<T> slice_iterator<T>::begin()  
{  
slice_iterator<T> tmp = *this;  
tmp.cur_ = 0;  
return( tmp );  
}  
  
template <typename T> inline  
slice_iterator<T> slice_iterator<T>::end()  
{  
slice_iterator<T> tmp = *this;  
tmp.cur_ = s_.size();  
return( tmp );  
}`

To complete the iterator interface, we define the pre- and post-increment operators.

37c *<si: forward iterator interface defs 37b>+≡* (36a) <37b

```

template <typename T> inline
slice_iterator<T>& slice_iterator<T>::operator++()
{
    cur_++;
    return( *this );
}

template <typename T> inline
slice_iterator<T> slice_iterator<T>::operator++(int)
{
    slice_iterator<T> tmp = *this;
    cur_++;
    return( tmp );
}

```

### 3.3.2.4 Access operators

Element access is provided by `operator*()` and `operator[]()` and their `const`-qualified equivalents. These member methods are implemented via the previously defined `ref` member. For the `operator*()`, `ref` is called using the current index, `cur_`.

38a *<si: access operator defs 38a>≡* (36a) 38b>

```

template <typename T> inline
T& slice_iterator<T>::operator*()
{ return( ref(cur_) ); }

template <typename T> inline
const T& slice_iterator<T>::operator*() const
{ return( ref(cur_) ); }

```

For random element access based on an input index, `operator[]()` is defined. It simply calls `ref` with the requested index.

38b *<si: access operator defs 38a>+≡* (36a) <38a

```

template <typename T> inline
T& slice_iterator<T>::operator[](size_t k)
{ return( ref(k) ); }

template <typename T> inline
const T& slice_iterator<T>::operator[](size_t k) const
{ return( ref(k) ); }

```

For each operator defined here, a `const`-qualified version is also defined.

As previously discussed in § 3.3.1.6, the access operators defined here do not check bounds on any element access request.

### 3.3.2.5 Comparison & ordering

To avoid the use of `friend` functions when providing definitions for the global comparison and ordering operators needed by STL algorithms, we choose to provide several `public` members to perform these tasks. Then, the global functions for comparison and ordering will then simply forward to these `public` members.

Equivalence between two `slice_iterator` instances will be established if their `slice`, current index, and container pointer all compare favorably. The member `slice_equal` checks that the slices are equivalent.

39a  $\langle si: comparison \ \& \ ordering \ defs \ 39a \rangle \equiv$  (36a) 39b $\triangleright$   

```
template <typename T> inline
bool slice_iterator<T>::slice_equal( const slice_iterator<T>& in ) const
{ return( s_.start()==in.s_.start() && s_.stride()==in.s_.stride() ); }
```

Two `slice_iterator`s are fully compared with the member method `is_equal`.

39b  $\langle si: comparison \ \& \ ordering \ defs \ 39a \rangle + \equiv$  (36a)  $\triangleleft 39a \ 39c \triangleright$   

```
template <typename T> inline
bool slice_iterator<T>::is_equal( const slice_iterator<T>& in ) const
{ return( v_==in.v_ && cur_==in.cur_ && slice_equal(in) ); }
```

The above definition differs from that of Stroustrup [9, § 22.4.5, p. 670] by the additional comparison of `v_`. The rationale for this is straight forward. Stroustrup considers only the equivalence of the data view provided by the `slice_iterator`. However, it would seem that both the view and the data should compare favorably before equivalence between two `slice_iterator`s is established.

Ordering is accomplished with the definition of `is_less_than`.

39c  $\langle si: comparison \ \& \ ordering \ defs \ 39a \rangle + \equiv$  (36a)  $\triangleleft 39b$   

```
template <typename T> inline
bool slice_iterator<T>::is_less_than( const slice_iterator<T>& in ) const
{ return( v_==in.v_ && slice_equal(in) && cur_<in.cur_ ); }
```

In a similar fashion as with `is_equal`, we only consider ordering to be meaningful if the data and the `slice` are equivalent.

Each of the member methods defined here are `const`-qualified since their operations do not change the state of a `slice_iterator` instance.

**3.3.2.6 Global comparison & ordering**

As mentioned in the previous section, the global functions used by STL algorithms for comparison and ordering are implemented as forwarding functions to their counterpart defined in § 3.3.2.5.

40a  $\langle si: \text{global comparison \& order defs 40a} \rangle \equiv$  (36a)

```

template <typename T> inline
bool operator==( const slice_iterator<T>& a, const slice_iterator<T>& b )
{ return( a.is_equal(b) ); }

template <typename T> inline
bool operator!=( const slice_iterator<T>& a, const slice_iterator<T>& b )
{ return( !(a==b) ); }

template <typename T> inline
bool operator<( const slice_iterator<T>& a, const slice_iterator<T>& b )
{ return( a.is_less_than(b) ); }

template <typename T> inline
bool operator>( const slice_iterator<T>& a, const slice_iterator<T>& b )
{ return( !(a<b) ); }

```

**3.3.2.7 Header & includes**

The `slice_iterator` translation unit is completed with the following include information and header.

40b  $\langle si: \text{includes 40b} \rangle \equiv$  (32a)

```

#include <cstddef>
#include <iterator>
#include <valarray>

```

40c  $\langle si: \text{header 40c} \rangle \equiv$  (32a)

```

//=====
// Project: DTIC technical report example integrator implementation
//
// Namespace: pr_cheby
//
// Purpose: slice iterator used by Chebyshev integrator of degree n = 8.
//
// $RCSfile: slice-iterator.nw,v $
// $Revision: 1.17 $
//

```

```
// $State: Exp $
// $Locker: $
//
// $Source: /home/mitchejw/work/research/tex/dtic/tr/RCS/slice-iterator.nw,v $
//
// Comments: 0. Since valarray<> is used, slice_iterators are necessary to
//             use STL functions.
//
//             1. See Stroustrup, "The C++ Programming Language, Special Ed."
//             for more details
//
// Orig. Author: Jason Wm. Mitchell
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// Orig. Date: 02-Feb-2002
// Last Changed: $Date: 2002/05/29 15:10:39 $
//
// Copyright (C) 2002 Jason Wm. Mitchell, All Rights Reserved.
// Restrictions: released under the Perl Artistic license.
// Security: unclassified
//
// This software is OSI Certified Open Source Software.
// OSI Certified is a certification mark of the Open Source Initiative.
//=====
```

### 3.4 Root chunk: default\_guesser

As previously suggested initially in § 1.2.3 and identified as a basic requirement in § 3.2, a fourth-order Runge-Kutta scheme will be used to jump-start the process described by Algorithm 1.1.

In terms of design concepts, it is useful to view the guess generator, and integrators in general, as an operator transforming an input vector to some output vector. As an implementation, treating this operator like a function is very convenient. For example, let the operation of guess generation be represented by the function  $g$ . Then, for given a first-order system of equations, stepsize  $h$ , initial independent variable  $x_0$ , and initial state  $\mathbf{y}_0$ , the next state  $\mathbf{y}$  is simply  $g : \mathbf{y}_0 \mapsto \mathbf{y}$ . Since a fourth-order Runge-Kutta method requires storage of intermediate derivatives, some contextual data is associated with the operator and we may think of it as a closure. We implement this by providing an `operator()()` member.

As with the previous code, we introduce all declarations and definitions in the

```

pr_cheby namespace.
42a  <default_guesser 42a>≡
    <dg: header 46b>
    #ifndef PANOVSky_RICHARDSON_DEFAULT_GUESSER
    #define PANOVSky_RICHARDSON_DEFAULT_GUESSER
    <dg: includes 46a>
    namespace pr_cheby
    {
        using namespace std;

        <dg: class decl 42b>

        <dg: member defs 44a>
    }
    #endif // PANOVSky_RICHARDSON_DEFAULT_GUESSER

```

### 3.4.1 Class declaration

The `default_guesser` object is parameterized by the scalar type `T`.

```

42b  <dg: class decl 42b>≡ (42a)
    template <typename T >
    class default_guesser
    {
    public:
        <dg: typedef decls 43b>
        <dg: constructors decls 43c>
        <dg: member-template operator()() decl 43d>
    private:
        <dg: private data decls 42c>
        <dg: private function decls 43a>
    };

```

#### 3.4.1.1 Private data

The internal data representation is relatively small. We need to know that the Runge-Kutta method is fourth-order `n_`, the number of first-order equations `m_`, and we must have storage for the fully explicit intermediate derivative values `k_`.

```

42c  <dg: private data decls 42c>≡ (42b)
    static const size_t n_ = 4;
    const size_t m_;
    mutable valarray<valarray<T> > k_;

```

The method order is fixed, and so `n_` is `static`, `const`, of integral type, and thus defined in the class declaration. Also, the number of first-order equations `m_` will be considered to be constant. The intermediate derivatives `k_` are stored in a `valarray` of `valarrays`. The `mutable` qualification of `k_` is necessary to permit constant reference invocations. For this case, we choose to emphasize the stateless mapping operator behaviour of the guesser, rather than an object requiring intermediate states to complete its task. This may seem at odds with the general design philosophy of using constant objects, but its use here is well understood and documented.

#### 3.4.1.2 Private functions

The only `private` function declaration given here is used to ban default construction. We prefer to disallow user attempts to create uninitialized `default_guesser` instances. This restriction is for clarity and programming convenience.

43a  $\langle dg: \text{private function decls } 43a \rangle \equiv$  (42b)  
`default_guesser();`

Since we only seek to generate compile-time errors with this mechanism, the function need not be defined.

#### 3.4.1.3 typedef declarations

The declaration of `class default_guesser` is parameterized by type `T` which is captured in a `typedef` declaration.

43b  $\langle dg: \text{typedef decls } 43b \rangle \equiv$  (42b)  
`typedef T value_type;`

#### 3.4.1.4 Constructors

The only input required to construct an instance of `default_guesser` is the number of first-order equations `m` that are necessary. A copy constructor is also provided<sup>3</sup>.

43c  $\langle dg: \text{constructors decls } 43c \rangle \equiv$  (42b)  
`default_guesser( const size_t& m );`  
`default_guesser( const default_guesser& );`

#### 3.4.1.5 Member-template operator()()

The function-like behaviour is provided by the member-template `operator()()`.

---

<sup>3</sup>See § 3.4.1.2 for the discussion regarding the lack of a public default constructor.

43d  $\langle dg: \text{member-template operator}()() \text{ decl 43d} \rangle \equiv$  (42b)

```

    template <typename V1, typename V2, typename E >
    void operator()( const T& x, const V1& y,
                    const T& h, const E& eqs,
                    V2& y_new ) const;

```

Choosing a member-template provides the necessary type flexibility while retaining type-safety. While `default_guesser` is parameterized by `T` already, this member-template adds the `V1`, `V2`, and `E` types. The types `V1` and `V2` are intended to represent vector-like quantities and that provide an `operator[]()` access method. The type `E` represents the system of first-order equations to be evaluated.

The argument signature for the member-template expects the current independent variable `x`, the current state `y`, the fixed stepsize `h`, the first-order equations `eqs`. These arguments are considered inputs and are `const`-qualified. The only output argument is `y_new` of type `V2` which is the estimate of  $\mathbf{y}(x + h)$ .

The return type of the member-template is `void` because the writable storage for the output `y_new` is passed into the method in the argument list. In addition, the method is `const`-qualified.

One additional user enforced requirement here is that the scalar types of `V1` and `V2` be of type `T`. This is not an unreasonable expectation and is a most likely scenario.

### 3.4.2 Member definitions

44a  $\langle dg: \text{member defs 44a} \rangle \equiv$  (42a)

$\langle dg: \text{constructor defs 44b} \rangle$

$\langle dg: \text{member-template operator}()() \text{ def 45b} \rangle$

#### 3.4.2.1 Constructors

Construction of the appropriate temporary storage `k_` and retention of the number first-order equations `m_` is performed by the user constructor below.

44b  $\langle dg: \text{constructor defs 44b} \rangle \equiv$  (44a) 45a▷

```

    template <typename T> inline
    default_guesser<T>::default_guesser( const size_t& m )
        : m_(m), k_( valarray<T>(m), n_ )
    {
        assert( m_ > 0 );
    }

```

The constructor above list assigns `m_` and uses `valarray`'s array constructor to allocate four `valarrays` of length `m`. The `assert` macro is used, ex post facto, to ensure that `m` is strictly positive. A `sentry` instance could be a used to enforce this requirement before construction of `k_`, however the `assert` macro is sufficient for

this example. No attempt is made to verify that the requested memory was actually allocated. In addition, any exceptions thrown from custom allocators are ignored.

The copy constructor definition attempts to make sure the appropriate objects are constructed and copied.

```

45a  <dg: constructor defs 44b>+≡ (44a) <44b
      template <typename T> inline
      default_guesser<T>::default_guesser( const default_guesser<T>& g )
      : m_(g.m_), k_( valarray<T>(g.m_), n_ ) {}

```

It is worth noting that the values found in `g.k_` are not copied because they are always computed when needed.

### 3.4.2.2 Member-template operator()()

The Runge-Kutta algorithm presented here is the original classical generalization of Simpson's rule<sup>4</sup>. The following implementation of this algorithm attempts to minimize necessary storage by using the output `y_new` as additional intermediate storage. The intermediate derivative values are stored in the previously allocated `k_`.

```

45b  <dg: member-template operator()() def 45b>≡ (44a)
      template <typename T>
      template <typename V1, typename V2, typename E > inline
      void default_guesser<T>::operator()( const T& x, const V1& y,
                                             const T& h, const E& eqs,
                                             V2& y_new ) const
      {
        const T h_on_2 = h / T(2);

        eqs( x, y, k_[0] );
        y_new = y + h_on_2 * k_[0];

        eqs( x + h_on_2, y_new, k_[1] );
        y_new = y + h_on_2 * k_[1];

        eqs( x + h_on_2, y_new, k_[2] );
        y_new = y + h * k_[2];

        eqs( x + h, y_new, k_[3] );
        y_new = y + h*( k_[0] + T(2)*(k_[1]+k_[2]) + k_[3] ) / T(6);
      }

```

To allow for the possibility that the stepsize `h` given to the algorithm may change, it is passed in at each method call. But, like all of the method arguments, it is passed

---

<sup>4</sup>For more details on the development of this algorithm, see Atkinson [1, p.423].

by reference to prevent unnecessary copying during each method call. This *hand* optimization is a concern because this method will be frequently invoked.

The algorithm as described here clearly sets the interface requirements for invoking the system of first-order equations. They must behave in a function-like fashion, and accept the arguments of independent variable  $x$ , a state value  $y$  or  $y\_new$ , and output storage for the derivative values  $k\_j$ . These arguments must be of type `const T&`, `const V1&` or `const V2&`, and `valarray<T>&`, respectively. As previously mentioned, specifying references will avoid unnecessary copying. This will be important for the derivatives because we expect them to be the most evaluated function over the integration.

### 3.5 Includes & header

The include files necessary are `cstdint`, `cassert`, and `valarray`.

```

46a  <dg: includes 46a>≡ (42a)
      #include <cstdint>
      #include <cassert>
      #include <valarray>

46b  <dg: header 46b>≡ (42a)
      //=====
      // Project: DTIC technical report example integrator implementation.
      //
      // Namespace: pr_cheby
      //
      // Purpose: Default guess generator for intra-nodal points
      //
      // $RCSfile: default-guesser.nw,v $
      // $Revision: 1.19 $
      //
      // $State: Exp $
      // $Locker: $
      //
      // $Source: /home/mitchejw/work/research/tex/dtic/tr/RCS/default-guesser.nw,v $
      //
      // Comments: 0. This represents the wrapping for the intra-nodal
      //               guess generator. The default guesser is a classical
      //               Runge-Kutta order four (4) guesser; simple coefficients.
      //
      //               1. Check operator() for the interface to y(x)\mapsto y(x+h)
      //
      // Orig. Author: Jason Wm. Mitchell
      //               US Air Force Research Laboratory

```

```

//          Air Vehicles Directorate, AFRL/VACA
//          2210 Eighth Street, Bldg. 146, Rm. 305
//          Wright-Patterson AFB, OH 45433 USA
//          E-Mail: jason+dtic_report2002@maiar.org
//          Orig. Date: 02-Feb-2002
//          Last Changed: $Date: 2002/05/29 14:56:19 $
//
// Copyright (C) 2002 Jason Wm. Mitchell, All Rights Reserved.
// Restrictions: released under Perl Artistic license
// Security:      unclassified
//
// This software is OSI Certified Open Source Software.
// OSI Certified is a certification mark of the Open Source Initiative.
//=====

```

### 3.6 Root chunk: `pr_constants`

To allow the integrator access to the MATHEMATICA<sup>®</sup> generated coefficients for degree  $n = 8$  in a type abstract manner, we introduce these coefficients in a *traits* template<sup>5</sup>. The double-precision version of these coefficients, accurate to twenty decimal places, are provided as the traits template *specialized* for type `double`.

In addition to the integrator coefficients, the traits template should include any other precision related constants that the integrator implementation may require. By isolating the floating-point constants in this way, we ensure that different precision floating-point constants can be used without altering integration algorithm. In terms of implementation, only the traits template specialized for the new type would need to be specified. For example, a `long double` specialization could be provided for the traits template rather than the `double`. Then, by choosing `long double` as the scalar type for all operations, the integrator would, at compile-time, deduce that it should use the traits template specialized for `long double`, thus making algorithm modification unnecessary.

```

47 <pr_constants 47>≡
    <prc: header 53b>
    #ifndef PANOVSky_RICHARDSON_CHEBYSHEV_8TH_DEGREE_CONSTANTS
    #define PANOVSky_RICHARDSON_CHEBYSHEV_8TH_DEGREE_CONSTANTS
    <prc: includes 53a>

    namespace pr_cheby
    {
        using namespace std;

```

---

<sup>5</sup> *Traits* template classes conveniently aggregate type dependent template parameters.

```

    <prc: traits decl 48a>
    <prc: double specialization 49a>
}
#endif // PANOVSky_RICHARDSON_CHEBYSHEV_8TH_DEGREE_CONSTANTS

```

### 3.6.1 Traits template declaration

```

48a  <prc: traits decl 48a>≡ (47)
      template <typename T>
      struct constants
      {
          static const bool is_specialized = false;

          <prc: integral type decls 48b>
          <prc: floating type decls 48c>
      };

```

The `is_specialized` member provides a compile time mechanism to verify that the appropriate specialization is provided and available.

#### 3.6.1.1 Integral type members

In addition to the integrator constants, we include constants for the maximum number of iterations permitted on an interval and the iteration residual tolerance. Also, since it may be useful to relax the residual tolerance, we include a maximum number of unconverged iterations and a multiplicative growth factor for the residual tolerance.

```

48b  <prc: integral type decls 48b>≡ (48a)
      typedef size_t size_type;
      static const size_t n = 8;
      static const size_t maximum_iterations = 0;
      static const size_t maximum_unconverged_iterations = 0;

```

Selection of the values related to the maximum number of iterations should be based on some experimentation with the desired problem.

#### 3.6.1.2 Floating type members

The interpolating weights `w` are stored as an arrays of arrays of type `T`. To facilitate accessing rows of weights for inner product operations, a `typedef` declaration is provided. Also, a `typedef` declarations is provided for the parameterizing type.

Along with the interpolation weights, two other sets of constants are useful to precompute for the iteration algorithm: interior  $\xi_j$  and  $\Delta\xi_j = \xi_j - \xi_{j-1}$  for  $0 < j \leq n + 1$ .

```

48c  <prc: floating type decls 48c>≡ (48a)
      typedef T value_type;
      typedef T const (* const array_type)[ n+1 ];
      static const T epsilon;
      static const T epsilon_grow_factor;
      static const T xi[ n-1 ];
      static const T dxi[ n ];
      static const T w[][ n+1 ];

```

In the declaration of `array_type`, we have deviated from the convenience use of the `const`-qualifier and moved to the explicit use which binds to the left.

### 3.6.2 Explicit specialization for double

The implementation precision chosen is `double` for the Intel 32-bit platform.

```

49a  <prc: double specialization 49a>≡ (47)
      template <>
      struct constants<double>
      {
          static const bool is_specialized = true;

          <prc dbl: integral type decls 49b>
          <prc dbl: floating type decls 49c>
      };

      <prc dbl: floating type defs 50a>

```

The member `is_specialized` is set to indicate explicit specialization.

#### 3.6.2.1 Specialized integral type members

```

49b  <prc dbl: integral type decls 49b>≡ (49a)
      typedef size_t size_type;
      static const size_t n = 8;
      static const size_t maximum_iterations = 16;
      static const size_t maximum_unconverged_iterations = 6;

```

The values related to the maximum number of iterations have been chosen with the problem of an harmonic oscillator in mind.

#### 3.6.2.2 Specialized floating type members

```

49c  <prc dbl: floating type decls 49c>≡ (49a)
      typedef double value_type;

```

```
typedef double const (* const array_type)[ n+1 ];
static const double epsilon;
static const double epsilon_grow_factor;
static const double xi[ n-1 ];
static const double dxi[ n ];
static const double w[][ n+1 ];
```

### 3.6.2.3 Specialized constant definitions

For the target platform, double-precision means approximately sixteen decimal places of accuracy, and so the floating-point constants given here follow the standard C++ library implementation convention of providing approximately twenty decimal places in the definitions.

**Tolerance related constants:** The iteration residual tolerance is chosen to be to be twice the *unit round* provided by `numeric_limits<double>::epsilon()`. If the decision is made to relax the tolerance requirement by using a multiplier, that factor is taken to be 2. These particular values are chosen only after relatively little experimentation for a specific and well understood problem and should not be considered a uniform choice for all problems.

```
50a  <prc dbl: floating type defs 50a>≡ (49a) 50b>
      const double constants<double>::epsilon =
      numeric_limits<double>::epsilon() * 2;

      const double constants<double>::epsilon_grow_factor = 2.0;
```

**Interior intra-nodal points:** The interior  $\xi_j$  locations are computed exactly and stored. The end-points are excluded to reduce rounding errors in the algorithm. As a result of this, evaluations at the end-points must be handled individually.

```
50b  <prc dbl: floating type defs 50a>+≡ (49a) <50a 51a>
      template <>
      const double constants<double>::xi[] =
      {
        3.8060233744356621936e-02,
        1.4644660940672623780e-01,
        3.0865828381745511414e-01,
        5.0000000000000000000e-01,
        6.9134171618254488586e-01,
        8.5355339059327376220e-01,
        9.6193976625564337806e-01,
```

};

**Intra-nodal spacing:** Algorithm 1.1 requires that guesses are generated at the intra-nodal points. Since the intra-nodal spacing is not fixed,  $\Delta\xi_j$  is computed and stored for use by the guess generating method.

51a  $\langle \text{prc dbl: floating type defs 50a} \rangle + \equiv$  (49a)  $\langle 50b \ 51b \rangle$

```

template <>
const double constants<double>::dxi[] =
{
    3.8060233744356621936e-02,
    1.0838637566236961586e-01,
    1.6221167441072887634e-01,
    1.9134171618254488586e-01,
    1.9134171618254488586e-01,
    1.6221167441072887634e-01,
    1.0838637566236961586e-01,
    3.8060233744356621936e-02
};

```

The data is obviously symmetric, however, it convenient to store all  $\Delta\xi_j$  for looping logic. The resulting redundant data is not significant in size and so their storage is not optimized.

**Integration weights:** The exact integration weights  $w_{jk}$ , written to twenty decimal places, are stored in **w** such that each row represents the weights in the inner product for one intra-nodal point as described by Eq. (1.16)

51b  $\langle \text{prc dbl: floating type defs 50a} \rangle + \equiv$  (49a)  $\langle 51a \rangle$

```

template <>
const double constants<double>::w[][n+1] =
{
    {
        1.5446942589330503001e-02, 2.4571482764589171269e-02,
        -2.7292031693117551470e-03, 1.2032717847525820515e-03,
        -7.2548587956445056873e-04, 5.1933327638929719332e-04,
        -4.1852184693699231789e-04, 3.704716357776345416e-04,
        -1.7805741066949699949e-04
    },
    {
        1.1622746778311606175e-03, 9.0507498067101649271e-02,
        6.1703929899043699124e-02, -1.0149503593034591360e-02,
        5.2541859706611991662e-03, -3.5448091021551241652e-03,

```

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```

2.7783648001647387572e-03, -2.4276059907176542285e-03,
1.1622746778311606175e-03
},
{
1.3299339017321784808e-02, 6.1002579087781576551e-02,
1.5851788614537362019e-01, 8.5788009730346129385e-02,
-1.4811162089693133896e-02, 8.1141101793756148493e-03,
-5.8380991989182836497e-03, 4.9112819285460210914e-03,
-2.3256609826782151916e-03
},
{
3.9682539682539682540e-03, 8.1654337316187620786e-02,
1.2876683494014880164e-01, 1.9930971175315382270e-01,
9.8412698412698412698e-02, -1.8450782392908931947e-02,
1.0915704742390880903e-02, -8.5450127081785432803e-03,
3.9682539682539682540e-03
},
{
1.0262168919186151699e-02, 6.8198042679463056415e-02,
1.4552063888145796619e-01, 1.7274481918086927590e-01,
2.1163655891508995929e-01, 9.5070919629898761363e-02,
-1.8835346462833937648e-02, 1.2106745520227500955e-02,
-5.3628310808138483005e-03},

{
6.7742332586767758905e-03, 7.5536930598726731734e-02,
1.3690417488237494378e-01, 1.8440373846240001491e-01,
1.9157121085473562623e-01, 1.9100843295327948211e-01,
7.7978609783495983416e-02, -1.7398173459092571765e-02,
6.7742332586767758905e-03
},

{
8.1145653471774335074e-03, 7.2738852972231314052e-02,
1.4010106152947667486e-01, 1.8033959608385559355e-01,
1.9755088270496127597e-01, 1.7965565757549230870e-01,
1.4241174285185143769e-01, 4.8537841843419906236e-02,
-7.5104346528225664926e-03
},
{
7.9365079365079365079e-03, 7.3109324608009077506e-02,
1.3968253968253968254e-01, 1.8085892936024489075e-01,
1.9682539682539682540e-01, 1.8085892936024489075e-01,
1.3968253968253968254e-01, 7.3109324608009077506e-02,

```

## CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

```
        7.9365079365079365079e-03
    }
};
```

### 3.6.3 Includes & headers

53a     $\langle \text{prc: includes 53a} \rangle \equiv$  (47)  
      `#include <limits>`  
      `#include <cstddef>`

53b     $\langle \text{prc: header 53b} \rangle \equiv$  (47)  
      `//=====`  
      `// Project: DTIC technical report example integrator implementation`  
      `//`  
      `// Namespace: pr_cheby`  
      `//`  
      `// Purpose: specialized constants for Chebyshev integrator of`  
      `// degree n = 8.`  
      `//`  
      `// $RCSfile: pr-constants.nw,v $`  
      `// $Revision: 1.15 $`  
      `//`  
      `// $State: Exp $`  
      `// $Locker: $`  
      `//`  
      `// $Source: /home/mitchejw/work/research/tex/dtic/tr/RCS/pr-constants.nw,v $`  
      `//`  
      `// Comments: 0. Change constants here for the desired type.`  
      `//`  
      `// Orig. Author: Jason Wm. Mitchell`  
      `// US Air Force Research Laboratory`  
      `// Air Vehicles Directorate, AFRL/VACA`  
      `// 2210 Eighth Street, Bldg. 146, Rm. 305`  
      `// Wright-Patterson AFB, OH 45433`  
      `// E-Mail: jason+dtic_report2002@maiar.org`  
      `// Orig. Date: 02-Feb-2002`  
      `// Last Changed: $Date: 2002/05/29 15:05:23 $`  
      `//`  
      `// Copyright (C) 2002 Jason Wm. Mitchell, All Rights Reserved.`  
      `// Restrictions: released under the Perl Artistic license`  
      `// Security: unclassified`  
      `//`  
      `// This software is OSI Certified Open Source Software.`

```
// OSI Certified is a certification mark of the Open Source Initiative.
//=====
```

### 3.7 Root chunk: pr\_cheby

With the supporting code defined, we move on to the implementation of the example integrator.

```
54a  <pr_cheby 54a>≡
      <pr: header 65>
      #ifndef PANOVSKY_RICHARDSON_CHEBYSHEV_8TH_DEGREE
      #define PANOVSKY_RICHARDSON_CHEBYSHEV_8TH_DEGREE
      <pr: includes 64c>

      namespace pr_cheby
      {
          using namespace std;

          <pr: class decl 54b>
          <pr: member defs 58a>
      }
      #endif // PANOVSKY_RICHARDSON_CHEBYSHEV_8TH_DEGREE
```

#### 3.7.1 Class declaration

```
54b  <pr: class decl 54b>≡ (54a)
      template <typename T, typename EqsT, typename GuessT = default_guesser<T> >
      class integrator
      {
      public:
          <pr: public typedef decls 56d>
          <pr: enumerated status codes 57a>
          <pr: constructor decl 57b>
          <pr: member-template operator()() decl 57c>
      private:
          <pr: private typedef decls 55a>
          <pr: private data decls 55b>
          <pr: private function decls 56b>
      };

```

The `integrator` is parameterized by the desired scalar type `T`, as well as by the type of equations and the guesser, `EqsT` and `GuessT`, respectively. The template parameter `GuessT` has as a default argument the previously defined `default_guesser`

object. Supplying a default type to `GuessT` will provide a less cumbersome interface when using the default guesser.

### 3.7.1.1 Private data

55a `<pr: private typedef decls 55a>≡` (54b)  
`typedef typename constants<T>::array_type array_type;`

The `array_type` typedef is forwarded from the traits template `constants` for local use.

55b `<pr: private data decls 55b>≡` (54b) 55c>  
`static const size_t n_ = constants<T>::n;`  
`const T* xi_;`  
`const T* dxi_;`  
`const array_type w_;`  
`size_t iter_max_;`  
`T epsilon_;`

To eliminate the scoping prefix when accessing the specialized traits template `constants`, we declare local shorthands for each identifier but with the established access membership suffix. In the case of the method degree, its value is assigned directly since it is constant and of integral type. Although `iter_max_` is of integral type, its value is not assigned directly, allowing the possibility of assignment at construction. Also, `iter_max_` is not `const`-qualified since it may be useful to alter the value based on convergence performance.

55c `<pr: private data decls 55b>+≡` (54b) <55b 55d>  
`static const size_t o_ = n_ + 1;`

As shown by Figure 1.1, the total number of nodal points in a given interval is  $n + 1$ , so that value is assigned here as `o_`.

55d `<pr: private data decls 55b>+≡` (54b) <55c 55e>  
`const size_t m_;`  
`const T h_;`

The number of first-order equations `m_` and the fixed stepsize `h_` are declared and `const`-qualified.

55e `<pr: private data decls 55b>+≡` (54b) <55d 56a>  
`valarray<T> f_arena_;`  
`slice_iterator<T> f_[ o_ ];`  
`valarray<valarray<T> > y_;`

The declarations above are responsible for storing the computed derivative and solution values, `f_arena_` and `y_` respectively, over the interval.

Regarding `f_arena_`, recall that Eq. (1.16) requires an inner product operation involving the weights  $w_{jk}$  and the derivative values  $f_k$ . These weights are stored in the traits template `constants` such that each row represents the coefficients for one first-order equation,  $y_j$ . To efficiently use the STL algorithm `inner_product`, the derivative information should also be stored in the same sense, i.e. one row of derivatives should represent the derivative values at each node point for the first-order equation of interest. However, evaluating the system of first-order equations will provide values for all the first-order equations at a given node—the transpose of our desired situation. To handle this, a block of memory large enough to hold all the derivative information is allocated as a `valarray` and identified as `f_arena_`. This memory is handed to the system of first-order equations as a `slice_iterator` whose `slice` correctly spans the arena so that every *row* of `f_arena_` represents the derivative values at the nodal points for the first-order equation of interest. Since there are  $n + 1$  points, we allocate `o_` iterators.

```
56a  <pr: private data decls 55b>+≡ (54b) <55e
      EqsT    eqs_;
      GuessT  guess_;
```

Lastly, it is convenient to contain copies of the first-order equations and the guesser. The overhead associated with retaining copies of these objects is small and only occurs at `integrator` construction.

### 3.7.1.2 Private functions

```
56b  <pr: private function decls 56b>≡ (54b) 56c>
      void show_static_coefficients_() const;
```

For debugging purposes, a private method is provided to output the coefficients imported from the traits template `constants`.

```
56c  <pr: private function decls 56b>+≡ (54b) <56b
      integrator();
      integrator( const integrator& );
```

The remaining private declarations enforce a ban on default and copy construction for `integrator`.

### 3.7.1.3 Public typedefs

```
56d  <pr: public typedef decls 56d>≡ (54b)
      typedef T      value_type;
      typedef EqsT    equation_type;
      typedef GuessT  guesser_type;
```

The input template parameters are captured in the above public `typedef` declarations.

#### 3.7.1.4 Convergence codes

From Algorithm 1.1, we must have some method to indicate that a particular iteration sequence either converged or failed to converge. This is handled with an enumeration.

57a *<pr: enumerated status codes 57a>*≡ (54b)  
`enum status_code_t { CONVERGENCE_FAILED = -1, OK = 0 };`

Convergence is given the label `OK` and assigned a value of zero. Convergence failure is indicated by a value of `-1` is given the label `CONVERGENCE_FAILED`.

#### 3.7.1.5 Constructor

To construct an instance of `integrator`, we require the fixed stepsize, the system of first-order equations, and a guesser.

57b *<pr: constructor decl 57b>*≡ (54b)  
`integrator( const T& stepsize, const EqsT& eqs,  
const GuessT& guess = default_guesser<T>(EqsT::size()) );`

With this default argument declaration, we see an additional requirement is added to the interface to the first-order equations. The object must provide the `static` member method `size()` to return the number of first-order equations involved.

#### 3.7.1.6 Member-template operator()()

The function-like behaviour here is provided by the member-template `operator()()`.

57c *<pr: member-template operator()() decl 57c>*≡ (54b)  
`template <typename V>  
int operator()( const T& x, V& yn );`

As discussed in § 3.4.1.5, the additional type `V` is intended to indicate that `yn` should behave like a mathematical vector, and provide an `operator[]()` random access method.

The argument list expects the current value of the independent variable `x` and the value of the current state `yn`. The type `V` is not `const`-qualified for `yn` since the input state will be over-written with the estimate of the state at  $x + h$ . The value of `x` is unchanged and thus `const`-qualified, thus `x` must be adjusted by the user for the next step. A return type of `int` is used to return the convergence status of the method invocation. The implicit conversion of the returned enumerated convergence status code of type `status_code_t` is understood and acceptable.

### 3.7.2 Member definitions

58a  $\langle pr: member\ defs\ 58a \rangle \equiv$  (54a)  
 $\langle pr: constructor\ def\ 58b \rangle$   
 $\langle pr: debug\ support\ def\ 59d \rangle$   
 $\langle pr: member-template\ operator()() def\ 60c \rangle$

#### 3.7.2.1 Constructor

58b  $\langle pr: constructor\ def\ 58b \rangle \equiv$  (58a)  
`template <typename T, typename EqsT, typename GuessT>`  
`inline`  
`integrator<T,EqsT,GuessT>::integrator( const T& stepsize,`  
 `const EqsT& eqs,`  
 `const GuessT& guess )`  
 $\langle pr: c'tor\ list\ 58c \rangle$   
`{`  
 $\langle pr: c'tor\ body\ 58d \rangle$   
`}`

As previously established in § 3.7.1.5, the `guess` argument is given a default value of `default_guesser<T>` if it is not explicitly provided.

58c  $\langle pr: c'tor\ list\ 58c \rangle \equiv$  (58b)  
`: xi_( constants<T>::xi ),`  
 `dxi_( constants<T>::dxi ),`  
 `w_( constants<T>::w ),`  
 `iter_max_( constants<T>::maximum_iterations ),`  
 `epsilon_( constants<T>::epsilon ),`  
 `m_( EqsT::size() ),`  
 `h_( stepsize ),`  
 `f_arena_( EqsT::size() * o_ ),`  
 `y_( valarray<T>(EqsT::size()), o_ ),`  
 `eqs_( eqs ),`  
 `guess_( guess )`

The constructor list definition is straightforward. First, each shorthand notation presented in § 3.7.1.1 is initialized with its traits template counterpart. Next, the number of first-order equations and stepsize are stored. Then, the storage for the derivative and solution values at the node points is allocated. Finally, the first-order equations and guesser are copied and stored locally.

58d  $\langle pr: c'tor\ body\ 58d \rangle \equiv$  (58b) 59a▷  
`assert( m_ > 0 );`

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

After constructor list completion, we verify that the number of first-order equations is strictly positive.

```
59a  <pr: c'tor body 58d>+≡ (58b) <58d 59b>
      for( size_t k = 0; k < o_; k++ )
          f_[k] = slice_iterator<T>(&f_arena_, slice(k,m_,o_));
```

The next construction task is to initialize the array of `slice_iterators` so that they span `f_arena_` in the desired fashion. In this case, each `slice` has a starting index of `k`, with length `m_` and stride of `m_` and `o_`, respectively.

```
59b  <pr: c'tor body 58d>+≡ (58b) <59a 59c>
      for( size_t k = 0; k < y_.size(); k++ )
          assert( y_[k].size() != 0 );
```

In the above, we verify that the length of the elements of `y_` are nonzero.

```
59c  <pr: c'tor body 58d>+≡ (58b) <59b>
      #ifdef PR_CHEBY_SHOW_STATIC_COEFFICIENTS
          show_static_coefficients_();
      #endif // PR_CHEBY_SHOW_STATIC_COEFFICIENTS
```

Lastly, we provide a preprocessor switch to trigger the output the static coefficients contained in `constants`.

#### 3.7.2.2 Debug support

```
59d  <pr: debug support def 59d>≡ (58a)
      template <class T, class EqsT, class GuessT>
      inline
      void integrator<T,EqsT,GuessT>::show_static_coefficients_() const
      {
          <pr: debug: show  $\xi_j$  59e>
          <pr: debug: show  $\Delta\xi_j$  60a>
          <pr: debug: show  $w_{jk}$  60b>
      }
```

Since the above function does not change the state of `integrator`, it is `const`-qualified.

```
59e  <pr: debug: show  $\xi_j$  59e>≡ (59d)
      for( size_t j = 0; j < n_ - 1; j++ )
          fprintf(stderr,"xi[%2i]:\t% 23.16e\n", j, xi_[j]);

      fprintf(stderr,"\n");
```

The interior intra-nodal points  $\xi_j$  are output first. These, as well as the remaining constants, are written to `stderr`.

The function `fprintf` is used here rather than `cerr` for performance reasons. Standard C++ stream output for the compiler and library version selected can be up to a factor of eight times slower than `printf`, or C style, output. This includes the case when output is redirected to a null device. In addition, stream output does not allow for the same formatting flexibility as C style output. For these reasons, C style output will be used for all output.

```
60a  <pr: debug: show  $\Delta\xi_j$  60a>≡ (59d)
      for( size_t j = 0; j < n_ - 2; j++ )
        fprintf(stderr,"dxi[%2i]:\t% 23.16e\n", j, dxi_[j]);

      fprintf(stderr,"\n");
```

Next, we output  $\Delta\xi_j$ .

```
60b  <pr: debug: show  $w_{jk}$  60b>≡ (59d)
      for( size_t j = 0; j < n_; j++ )
        for( size_t l = 0; l < o_; l++ )
          fprintf(stderr,"w[%2i,%2i]:\t% 23.16e\n", j,l,w_[j][l]);

      fprintf(stderr,"\n");
```

Lastly, we output the integration weights  $w_{jk}$ .

### 3.7.2.3 Member-template operator()()

This member-template definition implements an iteration scheme similar to that described by Algorithm 1.1. The difference, of course, being the use of precomputed and exact integration weights.

```
60c  <pr: member-template operator()() def 60c>≡ (58a)
      template <class T, class EqsT, class GuessT>
      template <class V>
      inline
      int integrator<T,EqsT,GuessT>::operator()( const T& x, V& yn )
      {
        <pr:oper: local decls 61a>
        <pr:oper: store  $y_0$  &  $f_0$  info 61b>
        <pr:oper: generate guesses for  $y_j$  if  $j \neq 0$  61c>
        <pr:oper: iteration loop 61d>
        <pr:oper: post iteration operations 63a>
        <pr:oper: return convergence status 64b>
      }
```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

The argument list above is straightforward, and the interface requirements are discussed in the declaration found in § 3.7.1.6.

61a  $\langle pr:oper: local\ decls\ 61a \rangle \equiv$  (60c)

```

    status_code_t status = OK;
    bool    is_converged = false;
    size_t iter_count    = 0;
    T      sum;
    T      residual;

```

Several locally scoped values are declared and initialized here. The first declaration `status` of type `status_code_t` is used to report the convergence status of the iteration sequence. It is initially assumes convergence and is assigned the enumerated value `OK`. The next declaration `is_converged` is a `bool` that is used as a local loop continuation flag. The `iter_count` declaration counts the number of completed iterations. The last two floating-point declarations `sum` and `residual` are used to hold the newly computed solution and the residual between the new and old solution at an intra-nodal point, respectively.

61b  $\langle pr:oper: store\ y_0\ \&\ f_0\ info\ 61b \rangle \equiv$  (60c)

```

    y_[0] = yn;
    eqs_( x, yn, f_[0] );

```

Initially, the input solution `yn` is stored in `y_[0]`, and then the derivatives are computed for `yn` and stored in `f_[0]`.

61c  $\langle pr:oper: generate\ guesses\ for\ y_j\ if\ j \neq 0\ 61c \rangle \equiv$  (60c)

```

    for( size_t j = 0; j < n_; j++ )
        guess_( x + h_*xi_[j], y_[j], h_*dxi_[j], eqs_, y_[j+1] );

```

To bootstrap the iteration process, solution guesses at the intra-nodal points are generated with `guess_`. The supplied guesser must meet the interface defined in § 3.4.1.5 and § 3.4.2.2. The intra-nodal locations and stepsizes are supplied to the guesser in terms of  $h\xi_j$ . With respect to the iteration, the solution guess computed with input `y_[j]` is stored in `y_[j+1]`.

61d  $\langle pr:oper: iteration\ loop\ 61d \rangle \equiv$  (60c)

```

    while( !is_converged && iter_count++ < iter_max_ )
    {
        is_converged = true;

         $\langle pr:oper:iter:a: compute\ derivatives\ 62a \rangle$ 
         $\langle pr:oper:iter:a: compute\ solution\ for\ each\ equation\ and\ node\ 62b \rangle$ 
    }
    yn = y_[n_];

```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

The iteration loop continues only if the solution has neither converged nor reached the maximum number of iterations specified. Inside the loop, convergence is initially assumed to be `true`. This logic may seem unconventional, but convergence fails if the residual at any nodal point for any equation is above the specified iteration tolerance. Thus, it is more convenient to register a single failure rather than success at every nodal point for all equations.

Once the iteration has completed, the new estimate for  $\mathbf{y}(x + h)$  is copied back into `yn` before returning to the calling method.

62a  $\langle \text{pr:oper:iter:a: compute derivatives 62a} \rangle \equiv$  (61d)

```

for( size_t j = 1; j < n_; j++ )
    eqs_( x + h_*xi_[j], y_[j], f_[j] );
eqs_( x + h_, y_[n_], f_[n_] );

```

Computing the derivatives is straightforward. But, we must remember to compute  $f_n$  since  $\xi_n = 1$  is not contained in `xi_`.

62b  $\langle \text{pr:oper:iter:a: compute solution for each equation and node 62b} \rangle \equiv$  (61d)

```

for( size_t l = 0; l < m_; l++ )
{
    for( size_t j = 0; j < n_; j++ )
    {
         $\langle \text{pr:oper:iter:b: compute } y_j \text{ 62c} \rangle$ 
         $\langle \text{pr:oper:iter:b: check residual and store new value 62d} \rangle$ 
    }
}

```

Within the main iteration loop, the solution for each equation `l` at each nodal point `j` is considered.

62c  $\langle \text{pr:oper:iter:b: compute } y_j \text{ 62c} \rangle \equiv$  (62b)

```

sum = y_[0][l] + h_ *
    inner_product( &w_[j][0], &w_[j][o_], &f_arena_[l*o_], T(0) );

```

The new solution at the current nodal point as prescribed by Eq. (1.16) is computed and stored in `sum`. The STL algorithm `inner_product` is used to handle the inner product between the exact coefficients  $w_{jk}$  and the derivatives  $f_k$  for each nodal point `j`.

62d  $\langle \text{pr:oper:iter:b: check residual and store new value 62d} \rangle \equiv$  (62b)

```

residual = fabs( sum - y_[j+1][l] );
if( residual > epsilon_ )
    is_converged = false;

y_[j+1][l] = sum;

```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

With the newly computed solution stored in `sum`, it is compared to the previous estimate `y_[j+1][1]` and the residual is computed. If the residual is greater than the iteration tolerance specified by `epsilon_`, then the solution is tagged as not yet converged.

After the convergence check, the newly computed solution `sum` replaces the previous value `y_[j+1][1]`.

```
63a  <pr:oper: post iteration operations 63a>≡ (60c)
      bool convergence_failed = (iter_count > iter_max_ && residual > epsilon_);

      <pr:oper: warn if convergence failed 63b>
      <pr:oper: should grow iteration tolerance? 63c>
      <pr:oper: show convergence statistics 64a>
```

After the iteration for the interval has completed, several preprocessor selectable options are provided. Two of the options provide status information regarding the iteration. The third option allows the iteration tolerance to be relaxed in a prescribed manner.

Since two of the provided options require the same boolean test, a single `bool` is defined outside of the preprocessor scoped options. The value of `convergence_failed` is considered to be `true` if the maximum iteration count is exceeded and the residual is greater than the desired tolerance.

```
63b  <pr:oper: warn if convergence failed 63b>≡ (63a)
      #ifdef PR_CHEBY_CONVERGENCE_WARN
      if( convergence_failed )
      {
          status = CONVERGENCE_FAILED;
          fprintf(stderr,
                  "...:operator(...): Warning! Convergence failure!\n");
          fprintf(stderr, " x: % 23.16e, residual: % 23.16e\n", x, residual);
      }
      #endif // PR_CHEBY_CONVERGENCE_WARN
```

If selected, the above code block will warn of convergence failure and provide the values of the independent variable and the residual. This information is written to `stderr` to avoid injecting warning information into the user data stream on `stdout`<sup>6</sup>. In addition to the report, the value of `status` is updated to reflect the current iteration state.

```
63c  <pr:oper: should grow iteration tolerance? 63c>≡ (63a)
      #if defined PR_CHEBY_GROW_EPSILON
      static size_t convergence_watch = 0;
      if( convergence_failed )
      {
```

---

<sup>6</sup>See § 3.7.2.2 for the discussion regarding the choice of C style output rather than C++ streams.

```

    if( ++convergence_watch >
        constants<T>::maximum_unconverged_iterations )
    {
        epsilon_ *= constants<T>::epsilon_grow_factor;
        fprintf(stderr,
            "...:operator(...): Warning! Growing epsilon:%23.16e\n",
            epsilon_);
    }
}
#endif // PR_CHEBY_GROW_EPSILON

```

If the above option is selected, it will permit the iteration tolerance to be relaxed for the duration of the integration. A static value labeled `convergence_watch` keeps track of the number of times convergence has failed throughout the integration. If it fails to converge a number of times specified by `maximum_unconverged_iterations` in the traits template `constants`, the iteration tolerance `epsilon_` is multiplied by `epsilon_grow_factor`, also specified in the `constants` traits template.

64a     $\langle pr:oper: show\ convergence\ statistics\ 64a \rangle \equiv$  (63a)

```

#ifdef PR_CHEBY_SHOW_STATS
    fprintf(stderr,"x: % 23.16e, Passes: %3i, Residual: % 23.16e\n",
        x, iter_count, residual);
#endif // PR_CHEBY_SHOW_STATS

```

The last preprocessor selectable option reports status information on the recently completed iteration. It provides the values of the independent variable, the number of iterations, and the final residual.

64b     $\langle pr:oper: return\ convergence\ status\ 64b \rangle \equiv$  (60c)

```

return( status );

```

Once the sequence is complete, the status is returned to the calling method.

### 3.7.3 Includes & header

64c     $\langle pr: includes\ 64c \rangle \equiv$  (54a)

```

#include <cassert>
#include <cstddef>
#include <cstdio>

#include <algorithm>
#include <numeric>
#include <limits>

#include "pr_constants"
#include "default_guesser"

```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

```
#include "slice_iterator"
```

```
65  <pr: header 65>≡ (54a)
//=====
//   Project:  DTIC technical report example integrator implementation
//
//   Namespace:  pr_cheby
//
//   Purpose:  Perform the Chebyshev integration for degree n = 8.
//
//   $RCSfile:  pr-cheby.nw,v $
//   $Revision: 1.9 $
//
//   $State:  Exp $
//   $Locker:  $
//
//   $Source:  /home/mitchejw/work/research/tex/dtic/tr/RCS/pr-cheby.nw,v $
//
//   Comments:  0. This code should (in general) not be touched.  Nothing here is
//               application specific.  If you want to change the guessing
//               mechanism, look at default_guesser.  Precision changes,
//               look at constants.
//
//               1. This code makes no attempt to automatically adjust step size
//               or method order to match truncation error tolerances.
//
//   Orig. Author:  Jason Wm. Mitchell
//                   US Air Force Research Laboratory
//                   Air Vehicles Directorate, AFRL/VACA
//                   2210 Eighth Street, Bldg. 146, Rm. 305
//                   Wright-Patterson AFB, OH 45433
//   E-Mail:  jason+dtic_report2002@maiar.org
//   Orig. Date:  02-Feb-2002
//   Last Changed:  $Date: 2002/05/29 13:45:18 $
//
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//
//   This software is OSI Certified Open Source Software.
//   OSI Certified is a certification mark of the Open Source Initiative.
//=====
```

## 3.8 Testing

To test the numerical integrators presented in § 3.4 and § 3.7, we require a system of first-order equations. The system used here represents an harmonic oscillator.

### 3.8.1 Harmonic oscillator

The interface requirements for the system of first-order equations have already been specified in § 3.4.2.2 and § 3.7.1.5.

```

66a  <harmonic_oscillator 66a>≡
      <ho: header 67a>
      #include <cstddef>

      struct harmonic_oscillator
      {
          <ho: report number of equations 66b>
          <ho: member-template operator()() 66c>
      };

66b  <ho: report number of equations 66b>≡                                     (66a)
      static size_t size() { return( 2 ); }
```

From § 3.7.1.5, the `integrator` constructor requires a `static` method named `size()` that reports the number of first-order equations as a `size_t`.

```

66c  <ho: member-template operator()() 66c>≡                                     (66a)
      template <typename T, typename V1, typename V2> inline
      void operator()( const T& x, const V1& y, V2& ydot ) const
      {
          ydot[0] = y[1];
          ydot[1] = -y[0];
      }
```

From § 3.4.2.2, invoking the system of first-order equations requires a signature given by the above. The use of a member-template allows the flexibility of accepting differing container types in a type-safe manner. Still, these containers must meet the interface requirements for use in this method. In the above, we only require that `V1` and `V2` provide random access operators, and have elements that are assignable and possess unary minus. Since the state is not changed by invoking this method, it is `const`-qualified.

In terms of operation, the value of the independent variable `x` and state `y` are passed into the method, and the programmed derivatives are computed stored in `ydot` for return to the caller.

Regarding calling efficiency, all the arguments are passed by reference for efficiency since this method will be invoked frequently.

### 3.8.1.1 Header

```

67a  <ho: header 67a>≡ (66a)
      //=====
      // Project: DTIC technical report example integrator implementation.
      //
      // Namespace: <none>
      //
      // Purpose: Harmonic oscillator equations
      //
      // $RCSfile: pr-mains.nw,v $
      // $Revision: 1.5 $
      //
      // $State: Exp $
      // $Locker: $
      //
      // $Source: /home/mitchejw/work/research/tex/dtic/tr/RCS/pr-mains.nw,v $
      //
      // Comments:
      //
      // Orig. Author: Jason Wm. Mitchell
      //                  US Air Force Research Laboratory
      //                  Air Vehicles Directorate, AFRL/VACA
      //                  2210 Eighth Street, Bldg. 146, Rm. 305
      //                  Wright-Patterson AFB, OH 45433 USA
      // E-Mail: jason+dtic_report2002@maiar.org
      // Orig. Date: 04-May-2002
      // Last Changed: $Date: 2002/05/29 15:07:12 $
      //
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      // Restrictions: released under Perl Artistic license
      // Security: unclassified
      //
      // This software is OSI Certified Open Source Software.
      // OSI Certified is a certification mark of the Open Source Initiative.
      //=====

```

### 3.8.2 Fourth-order Runge-Kutta method

Before testing the numerical integrator presented in § 3.7, we must test the fourth-order Runge-Kutta algorithm. Using the set of harmonic oscillator equations defined in § 3.8.1, we construct a small program to validate its operation.

```

67b  <gmain.cc 67b>≡
      <gm: header 70>

```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

```

    <gm: includes 69b>
    <preprocessor check for M_PI 68a>
    int main()
    {
        using namespace std;

        <gm: typedef decls 68b>
        <gm: declaration block 68c>
        <gm: integration loop 69a>

        return( 0 );
    }

```

```

68a    <preprocessor check for M_PI 68a>≡ (67b 71a)
        #ifndef M_PI
            #error "Try #define _XOPEN_SOURCE 500..."
        #endif

```

Since we are using equations for an harmonic oscillator, it is convenient to have access to the frequently provided preprocessor constant `M_PI`. The preprocessor flags that enable the `M_` category of constants vary between compilers and so we provide a minimal check as well as a suggested preprocessor flag to enable them.

```

68b    <gm: typedef decls 68b>≡ (67b)
        typedef harmonic_oscillator eqs_t;
        typedef pr_cheby::default_guesser<double> guesser_t;

```

These two declarations provide a shorthand for the descriptive but lengthy types used here.

```

68c    <gm: declaration block 68c>≡ (67b) 68d>
        const size_t m = eqs_t::size();
        const double y0[] = { 0.0, 1.0 };
        valarray<double> y(y0,m), y_new(m);

```

The first of the declarations creates a shorthand for the number of first-order equations, i.e. 2. Next, we create a standard `double` array for the initial conditions. The values above correspond to the initial conditions  $y(0) = 0$ , and  $y'(0) = 1$ . These initial conditions generate the solutions  $y(x) = \sin x$ ,  $y'(x) = \cos x$ . Knowing this information allows us to compare the numerical solution to an exact solution for error analysis. Lastly, these initial conditions are used to initialize a `valarray` representing the numerical solution vector. A second `valarray` is constructed to hold the output values from the integration.

```

68d  <gm: declaration block 68c>+≡ (67b) <68c
      const size_t p = 200;
      const double m_2pi = M_PI * 2.0;
      const double h = m_2pi / p;
      const double xf = m_2pi * 100.0;
      double x = 0.0;
      double u;
      guesser_t rk4(m);

```

Here, we choose the stepsize for the test. To do this, we divide one period of the motion into 200 steps. Since the period is  $2\pi$ , this gives us a step size of  $h = 2\pi/200$ . With `h` defined, we elect to run the integration for 100 periods starting from  $x_0 = 0$ .

To prevent the compiler supplied `sin` and `cos` algorithms from suffering error induced by large input arguments, `u` is provided to store the unwound independent variable `x`.

The last declaration in the block instantiates a `default_guesser` labeled `rk4`.

```

69a  <gm: integration loop 69a>≡ (67b)
      for( size_t k = 1; x < xf; k++ )
      {
          rk4( x, y, h, eqs_t(), y_new );
          y = y_new;
          x = k*h;
          u = h * (k%p);

          printf("% 23.16e % 23.16e % 23.16e\n", x, y[0]-sin(x), y[1]-cos(x));
      }

```

In the integration loop, the value of  $\mathbf{y}(x + h)$  is computed and stored in `y_new`. This state is then copied to `y`, and `x` is incremented to reflect its value at the new state. The value of the unwound argument `u` is computed as  $u = h \bmod(k, p)$ .

The numerical solution is then compared to the exact solution, and that difference is output. As discussed previously in § 3.7.2.2, C style output is used.

### 3.8.2.1 Results

Compiling and executing this test produces an error that is  $\mathcal{O}(h^4)$  for the duration of the integration for both  $\mathbf{y}$  and  $\mathbf{y}'$ .

### 3.8.2.2 Includes & header

```

69b  <gm: includes 69b>≡ (67b)
      #include <cstdint>
      #include <cmath>
      #include <cstdio>

```

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

```
#include <valarray>

#include "default_guesser"
#include "harmonic_oscillator"
```

```
70  <gm: header 70>≡ (67b)
//=====
//   Project:  DTIC technical report example integrator implementation.
//
//   Namespace:  <none>
//
//   Purpose:  Test default guess generator.
//
//   $RCSfile:  pr-mains.nw,v $
//   $Revision: 1.5 $
//
//   $State:  Exp $
//   $Locker:  $
//
//   $Source:  /home/mitchejw/work/research/tex/dtic/tr/RCS/pr-mains.nw,v $
//
//   Comments:  Just a quick test program to check out RK4 guesser:
//               quick, not elegant.
//
//   Orig. Author:  Jason Wm. Mitchell
//                   US Air Force Research Laboratory
//                   Air Vehicles Directorate, AFRL/VACA
//                   2210 Eighth Street, Bldg. 146, Rm. 305
//                   Wright-Patterson AFB, OH 45433 USA
//   E-Mail:  jason+dtic_report2002@maiar.org
//   Orig. Date: 02-Feb-2002
//   Last Changed:  $Date: 2002/05/29 15:07:12 $
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//
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//=====
```

### 3.8.3 Chebyshev integrator of degree $n = 8$

Finally, we consider a `main` for testing the integrator of interest: Chebyshev of degree  $n = 8$ . The system of equations used to verify this integrator will be the same harmonic oscillator equations defined in § 3.8.1 used in the previous section.

```
71a  <cmain.cc 71a>≡
      <cm: header 72c>
      <cm: includes 72b>
      <preprocessor check for M_PI 68a>
      int main()
      {
          using namespace std;

          <cm: typedef decls 71b>
          <cm: declaration block 71c>
          <cm: integration loop 72a>

          return( 0 );
      }
```

The basic layout of `cmain.cc` is quite similar to that of `gmain.cc` defined in § 3.8.2.

```
71b  <cm: typedef decls 71b>≡ (71a)
      typedef harmonic_oscillator eqs_t;
      typedef pr_cheby::integrator<double,eqs_t> integrator_t;
```

We begin by providing a few convenient `typedefs`, and as before the scalar type used is `double`.

```
71c  <cm: declaration block 71c>≡ (71a) 71d>
      const size_t p = 64;
      const double m_2pi = M_PI * 2.0;
      const double h = m_2pi / p;
      const double x0 = 0.0;
      const double xf = m_2pi * 500;
```

In the first portion of the declaration block, we set the interval and the stepsize. In this case, we are only considering 64 points in the period, rather than the 200 chosen for the Runge-Kutta algorithm, resulting in a much larger stepsize. In addition, the integration is performed over 500 periods, compared to the 100 periods for the Runge-Kutta algorithm, starting from  $x_0 = 0$ .

```
71d  <cm: declaration block 71c>+≡ (71a) <71c>
      double y0[] = {0.0, 1.0};
      valarray<double> y( y0, eqs_t::size() );
      integrator_t cheby8( h, eqs_t() );
```

With the interval and stepsize selected, the initial conditions as discussed in § 3.8.2 are set, and the Chebyshev integrator `cheby8` is instantiated. Note that since the default guessing mechanism is used, the syntax that instantiates `cheby8` is quite simple.

72a  $\langle cm: \text{integration loop 72a} \rangle \equiv$  (71a)

```
double u, x = x0;
for( size_t k = 1; x < xf; k++ )
{
    cheby8( x, y );
    x = k*h;
    u = h * (k%p);
    printf("% 23.16e % 23.16e % 23.16e\n", x, sin(u)-y[0], cos(u)-y[1]);
}
```

Before the integration loop definition, we declare the unwound angular argument `u`, and the independent variable `x`. As before, the argument `u` is restricted to  $u \in [0, 2\pi)$ .

Within the integration loop, the current independent variable and state are input into `cheby8()`. On method return, the new state  $\mathbf{y}(x + h)$  is returned in `y`. The independent variable `x` is then incremented accordingly. The current value of `x` is unwound and stored in `u`. This information is used to compare the current numerical solution to the exact solution given in § 3.8.2, with the error output on `stdout`.

### 3.8.3.1 Results

Compiling and executing this test produces a maximum error over the interval that is on the order of the unit-round for the platform chosen<sup>7</sup>.

### 3.8.3.2 Includes & header

72b  $\langle cm: \text{includes 72b} \rangle \equiv$  (71a)

```
#include <cstddef>
#include <cmath>
#include <cstdio>

#include <valarray>

#include "pr_cheby"
#include "harmonic_oscillator"
```

72c  $\langle cm: \text{header 72c} \rangle \equiv$  (71a)

```
//=====
// Project: DTIC technical report example integrator implementation
```

---

<sup>7</sup>See § 3.6.2.

### CHAPTER 3. EXAMPLE FOR FIRST-ORDER SYSTEMS

```
//
// Namespace: <none>
//
// Purpose: Driver for Chebyshev integration for degree n = 8.
//
// $RCSfile: pr-mains.nw,v $
// $Revision: 1.5 $
//
// $State: Exp $
// $Locker: $
//
// $Source: /home/mitchejw/work/research/tex/dtic/tr/RCS/pr-mains.nw,v $
//
// Comments:
//
// Orig. Author: Jason Wm. Mitchell
//               US Air Force Research Laboratory
//               Air Vehicles Directorate, AFRL/VACA
//               2210 Eighth Street, Bldg. 146, Rm. 305
//               Wright-Patterson AFB, OH 45433
//               E-Mail: jason+dtic_report2002@maiar.org
// Orig. Date: 02-Feb-2002
// Last Changed: $Date: 2002/05/29 15:07:12 $
//
// Copyright (C) 2002 Jason Wm. Mitchell, All Rights Reserved.
// Restrictions: released under the Perl artistic license
// Security:    unclassified
//
// This software is OSI Certified Open Source Software.
// OSI Certified is a certification mark of the Open Source Initiative.
//=====
```

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## Appendix A

### Intra-nodal points

The following tables contain the intra-nodal point locations, Eq. (1.2) for even degrees  $2 \leq n \leq 14$ . For each degree, the intra-nodal point locations are given as multiples of a method's fixed stepsize  $h$ , i.e.  $h_j = h\xi_j$ . The values for  $\xi_0$  and  $\xi_n$  are not shown in the following since their values are 0 and 1, respectively.

When possible, all numeric quantities in the following tables have been represented as rational functions of nested roots. Unfortunately, this explicit form is not always obtainable in general, so we must consider an implicit symbolic representation of these numbers, something MATHEMATICA<sup>®</sup> terms *algebraic numbers*<sup>1</sup>. These numbers have the property that applying algebraic operations on them results in a single algebraic number. They are written `Root[g,k]` meaning the  $k^{\text{th}}$  root of the polynomial equation  $g(x) = 0$ . For example, `Root[5 - 2 #13, 2]` represents the second root of  $2x^3 - 5 = 0$ , where `#1` is the MATHEMATICA<sup>®</sup> notation for a polynomial variable of interest. This requires a unique ordering of the roots to be meaningful. MATHEMATICA<sup>®</sup> orders roots by magnitude except that real roots come before complex ones, and complex conjugate pairs are adjacent.

The intra-nodal information presented in the following tables are common for both families of implicit Chebyshev methods.

Table A.1: Intra-nodal points  $\xi_j$ ,  $n = 2$ .

| $j$ | $\xi_j$       |
|-----|---------------|
| 1   | $\frac{1}{2}$ |

---

<sup>1</sup>See Wolfram [10, § 3.4.3].

APPENDIX A. INTRA-NODAL POINTS

Table A.2: Intra-nodal points  $\xi_j$ ,  $n = 4$ .

| $j$ | $\xi_j$                  | $j$ | $\xi_j$                  |
|-----|--------------------------|-----|--------------------------|
| 1   | $\frac{2 - \sqrt{2}}{4}$ | 3   | $\frac{2 + \sqrt{2}}{4}$ |
| 2   | $\frac{1}{2}$            |     |                          |

Table A.3: Intra-nodal points  $\xi_j$ ,  $n = 6$ .

| $j$ | $\xi_j$                  | $j$ | $\xi_j$                  |
|-----|--------------------------|-----|--------------------------|
| 1   | $\frac{2 - \sqrt{3}}{4}$ | 4   | $\frac{3}{4}$            |
| 2   | $\frac{1}{4}$            | 5   | $\frac{2 + \sqrt{3}}{4}$ |
| 3   | $\frac{1}{2}$            |     |                          |

Table A.4: Intra-nodal points  $\xi_j$ ,  $n = 8$ .

| $j$ | $\xi_j$                             | $j$ | $\xi_j$                             |
|-----|-------------------------------------|-----|-------------------------------------|
| 1   | $\frac{2 - \sqrt{2 + \sqrt{2}}}{4}$ | 5   | $\frac{2 + \sqrt{2 - \sqrt{2}}}{4}$ |
| 2   | $\frac{2 - \sqrt{2}}{4}$            | 6   | $\frac{2 + \sqrt{2}}{4}$            |
| 3   | $\frac{2 - \sqrt{2 - \sqrt{2}}}{4}$ | 7   | $\frac{2 + \sqrt{2 + \sqrt{2}}}{4}$ |
| 4   | $\frac{1}{2}$                       |     |                                     |

APPENDIX A. INTRA-NODAL POINTS

Table A.5: Intra-nodal points  $\xi_j$ ,  $n = 10$ .

| $j$ | $\xi_j$                                | $j$ | $\xi_j$                                |
|-----|----------------------------------------|-----|----------------------------------------|
| 1   | $\frac{4 - \sqrt{2(5 + \sqrt{5})}}{8}$ | 6   | $\frac{3 + \sqrt{5}}{8}$               |
| 2   | $\frac{3 - \sqrt{5}}{8}$               | 7   | $\frac{4 + \sqrt{10 - 2\sqrt{5}}}{8}$  |
| 3   | $\frac{4 - \sqrt{10 - 2\sqrt{5}}}{8}$  | 8   | $\frac{5 + \sqrt{5}}{8}$               |
| 4   | $\frac{5 - \sqrt{5}}{8}$               | 9   | $\frac{4 + \sqrt{2(5 + \sqrt{5})}}{8}$ |
| 5   | $\frac{1}{2}$                          |     |                                        |

Table A.6: Intra-nodal points  $\xi_j$ ,  $n = 12$ .

| $j$ | $\xi_j$                             | $j$ | $\xi_j$                             |
|-----|-------------------------------------|-----|-------------------------------------|
| 1   | $\frac{4 - \sqrt{2} - \sqrt{6}}{8}$ | 7   | $\frac{4 - \sqrt{2} + \sqrt{6}}{8}$ |
| 2   | $\frac{2 - \sqrt{3}}{4}$            | 8   | $\frac{3}{4}$                       |
| 3   | $\frac{2 - \sqrt{2}}{4}$            | 9   | $\frac{2 + \sqrt{2}}{4}$            |
| 4   | $\frac{1}{4}$                       | 10  | $\frac{2 + \sqrt{3}}{4}$            |
| 5   | $\frac{4 + \sqrt{2} - \sqrt{6}}{8}$ | 11  | $\frac{4 + \sqrt{2} + \sqrt{6}}{8}$ |

(continues)

APPENDIX A. INTRA-NODAL POINTS

Table A.6: Intra-nodal abscissa  $\xi_j$ ,  $n = 12$ . (*continued*)

| $j$ | $\xi_j$       | $j$ | $\xi_j$ |
|-----|---------------|-----|---------|
| 6   | $\frac{1}{2}$ |     |         |

Table A.7: Intra-nodal points  $\xi_j$ ,  $n = 14$ .

| $j$ | $\xi_j$             | $j$ | $\xi_j$             |
|-----|---------------------|-----|---------------------|
| 1   | $\frac{1 - c_6}{2}$ | 8   | $\frac{1 + c_1}{2}$ |
| 2   | $\frac{1 - c_5}{2}$ | 9   | $\frac{1 + c_2}{2}$ |
| 3   | $\frac{1 - c_4}{2}$ | 10  | $\frac{1 + c_3}{2}$ |
| 4   | $\frac{1 - c_3}{2}$ | 11  | $\frac{1 + c_4}{2}$ |
| 5   | $\frac{1 - c_2}{2}$ | 12  | $\frac{1 + c_5}{2}$ |
| 6   | $\frac{1 - c_1}{2}$ | 13  | $\frac{1 + c_6}{2}$ |
| 7   | $\frac{1}{2}$       |     |                     |

Table A.8: Unique **Root** objects  $n = 14$ .

| $\ell$ | $c_\ell$                                                  |
|--------|-----------------------------------------------------------|
| 1      | $\text{Root}[1 - 4 \# 1 - 4 \# 1^2 + 8 \# 1^3, 2]$        |
| 2      | $\text{Root}[-7 + 56 \# 1^2 - 112 \# 1^4 + 64 \# 1^6, 4]$ |

(*continues*)

APPENDIX A. INTRA-NODAL POINTS

Table A.8: Algebraic numbers,  $n = 14$ . (*continued*)

| $\ell$ | $c_\ell$                                               |
|--------|--------------------------------------------------------|
| 3      | $\text{Root}[-1 - 4 \#1 + 4 \#1^2 + 8 \#1^3, 3]$       |
| 4      | $\text{Root}[-7 + 56 \#1^2 - 112 \#1^4 + 64 \#1^6, 5]$ |
| 5      | $\text{Root}[1 - 4 \#1 - 4 \#1^2 + 8 \#1^3, 3]$        |
| 6      | $\text{Root}[-7 + 56 \#1^2 - 112 \#1^4 + 64 \#1^6, 6]$ |

## Appendix B

# Coefficients for first-order methods

In the following tables, exact coefficients for a family of implicit Chebyshev methods for the numerical integration of first-order differential equations are given. These coefficients for even degrees  $2 \leq n \leq 14$  are used to recast Eq. (1.5), such that

$$y(x + h\xi_j) = y(x) + I_j, \quad (\text{B.1})$$

where

$$I_j = h d_j \sum_{k=0}^n w_{jk} f(x_k), \quad (\text{B.2})$$

where  $d_j$  represents a common denominator for each expression.

The constants  $c_\ell$  appearing in Table B.7 are defined in Appendix A, Table A.8.

Table B.1: Integral approximation  $I_j$  for  $n = 2$ .

| $d_j, w_{jk}$ | Expression     | $w_{jk}$  | Expression |
|---------------|----------------|-----------|------------|
| $d_1$         | $\frac{1}{24}$ | $w_{1,1}$ | 8          |
| $w_{1,0}$     | 5              | $w_{1,2}$ | -1         |
| $d_2$         | $\frac{1}{6}$  | $w_{2,1}$ | 4          |
| $w_{2,0}$     | 1              | $w_{2,2}$ | 1          |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.2: Integral approximation  $I_j$  for  $n = 4$ .

| $d_j, w_{jk}$ | Expression        | $w_{jk}$  | Expression                      |
|---------------|-------------------|-----------|---------------------------------|
| $d_1$         | $\frac{1}{480}$   | $w_{1,2}$ | $24 \left(4 - 3\sqrt{2}\right)$ |
| $w_{1,0}$     | $23 + 4\sqrt{2}$  | $w_{1,3}$ | $64 - 43\sqrt{2}$               |
| $w_{1,1}$     | $64 - 13\sqrt{2}$ | $w_{1,4}$ | $-7 + 4\sqrt{2}$                |
| $d_2$         | $\frac{1}{120}$   | $w_{2,2}$ | 24                              |
| $w_{2,0}$     | 2                 | $w_{2,3}$ | $16 - 15\sqrt{2}$               |
| $w_{2,1}$     | $16 + 15\sqrt{2}$ | $w_{2,4}$ | 2                               |
| $d_3$         | $\frac{1}{480}$   | $w_{3,2}$ | $24 \left(4 + 3\sqrt{2}\right)$ |
| $w_{3,0}$     | $23 - 4\sqrt{2}$  | $w_{3,3}$ | $64 + 13\sqrt{2}$               |
| $w_{3,1}$     | $64 + 43\sqrt{2}$ | $w_{3,4}$ | $-7 - 4\sqrt{2}$                |
| $d_4$         | $\frac{1}{30}$    | $w_{4,2}$ | 12                              |
| $w_{4,0}$     | 1                 | $w_{4,3}$ | 8                               |
| $w_{4,1}$     | 8                 | $w_{4,4}$ | 1                               |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.3: Integral approximation  $I_j$  for  $n = 6$ .

| $d_j, w_{jk}$ | Expression                           | $w_{jk}$  | Expression            |
|---------------|--------------------------------------|-----------|-----------------------|
| $d_1$         | $\frac{1}{20160}$                    | $w_{1,3}$ | $2624 - 1488\sqrt{3}$ |
| $w_{1,0}$     | $424 + 72\sqrt{3}$                   | $w_{1,4}$ | $2339 - 1368\sqrt{3}$ |
| $w_{1,1}$     | $1280 - 235\sqrt{3}$                 | $w_{1,5}$ | $1280 - 725\sqrt{3}$  |
| $w_{1,2}$     | $2269 - 1368\sqrt{3}$                | $w_{1,6}$ | $-136 + 72\sqrt{3}$   |
| $d_2$         | $\frac{1}{20160}$                    | $w_{2,3}$ | $-368$                |
| $w_{2,0}$     | $72$                                 | $w_{2,4}$ | $207$                 |
| $w_{2,1}$     | $5 \left( 296 + 189\sqrt{3} \right)$ | $w_{2,5}$ | $1480 - 945\sqrt{3}$  |
| $w_{2,2}$     | $2097$                               | $w_{2,6}$ | $72$                  |
| $d_3$         | $\frac{1}{2520}$                     | $w_{3,3}$ | $328$                 |
| $w_{3,0}$     | $53$                                 | $w_{3,4}$ | $-62$                 |
| $w_{3,1}$     | $10 \left( 16 + 7\sqrt{3} \right)$   | $w_{3,5}$ | $160 - 70\sqrt{3}$    |
| $w_{3,2}$     | $638$                                | $w_{3,6}$ | $-17$                 |
| $d_4$         | $\frac{1}{2240}$                     | $w_{4,3}$ | $624$                 |
| $w_{4,0}$     | $24$                                 | $w_{4,4}$ | $279$                 |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.3: Integral approximation  $I_j$ ,  $n = 6$ . (*continued*)

| $d_j, w_{jk}$ | Expression                            | $w_{jk}$  | Expression                           |
|---------------|---------------------------------------|-----------|--------------------------------------|
| $w_{4,1}$     | $15 \left( 8 + 7 \sqrt{3} \right)$    | $w_{4,5}$ | $3 \left( 40 - 35 \sqrt{3} \right)$  |
| $w_{4,2}$     | 489                                   | $w_{4,6}$ | 24                                   |
| $d_5$         | $\frac{1}{20160}$                     | $w_{5,3}$ | $2624 + 1488 \sqrt{3}$               |
| $w_{5,0}$     | $424 - 72 \sqrt{3}$                   | $w_{5,4}$ | $2339 + 1368 \sqrt{3}$               |
| $w_{5,1}$     | $5 \left( 256 + 145 \sqrt{3} \right)$ | $w_{5,5}$ | $5 \left( 256 + 47 \sqrt{3} \right)$ |
| $w_{5,2}$     | $2269 + 1368 \sqrt{3}$                | $w_{5,6}$ | $-8 \left( 17 + 9 \sqrt{3} \right)$  |
| $d_6$         | $\frac{1}{630}$                       | $w_{6,3}$ | 164                                  |
| $w_{6,0}$     | 9                                     | $w_{6,4}$ | 144                                  |
| $w_{6,1}$     | 80                                    | $w_{6,5}$ | 80                                   |
| $w_{6,2}$     | 144                                   | $w_{6,6}$ | 9                                    |

Table B.4: Integral approximation  $I_j$  for  $n = 8$ .

| $d_j, w_{jk}$ | Expression                     |
|---------------|--------------------------------|
| $d_1$         | $\frac{1}{40320}$              |
| $w_{1,0}$     | $475 + 80 \sqrt{2 + \sqrt{2}}$ |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                     |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,1}$     | $2560 - 768\sqrt{2} + 126\sqrt{4 - 2\sqrt{2}} - 210\sqrt{2 - \sqrt{2}}$<br>$- 715\sqrt{2 + \sqrt{2}} + 330\sqrt{2(2 + \sqrt{2})}$              |
| $w_{1,2}$     | $1155\sqrt{2} - 16\left(-71 + 42\sqrt{2 - \sqrt{2}} + 80\sqrt{2 + \sqrt{2}}\right)$                                                            |
| $w_{1,3}$     | $768\sqrt{2} - 126\sqrt{4 - 2\sqrt{2}}$<br>$+ 5\left(512 + 105\sqrt{2 - \sqrt{2}} - 206\sqrt{2 + \sqrt{2}} - 150\sqrt{2(2 + \sqrt{2})}\right)$ |
| $w_{1,4}$     | $32\left(124 + 42\sqrt{2 - \sqrt{2}} - 85\sqrt{2 + \sqrt{2}}\right)$                                                                           |
| $w_{1,5}$     | $768\sqrt{2} + 294\sqrt{4 - 2\sqrt{2}}$<br>$- 5\left(-512 + 105\sqrt{2 - \sqrt{2}} + 290\sqrt{2 + \sqrt{2}} + 66\sqrt{2(2 + \sqrt{2})}\right)$ |
| $w_{1,6}$     | $-1155\sqrt{2} - 16\left(-281 + 42\sqrt{2 - \sqrt{2}} + 80\sqrt{2 + \sqrt{2}}\right)$                                                          |
| $w_{1,7}$     | $-768\sqrt{2} - 294\sqrt{4 - 2\sqrt{2}}$<br>$+ 5\left(512 + 42\sqrt{2 - \sqrt{2}} - 353\sqrt{2 + \sqrt{2}} + 150\sqrt{2(2 + \sqrt{2})}\right)$ |
| $w_{1,8}$     | $5\left(-31 + 16\sqrt{2 + \sqrt{2}}\right)$                                                                                                    |
| $d_2$         | $\frac{1}{10080}$                                                                                                                              |
| $w_{2,0}$     | $-20\left(-2 + \sqrt{2}\right)$                                                                                                                |
| $w_{2,1}$     | $1168 - 512\sqrt{2} + 105\sqrt{2 - \sqrt{2}} + 210\sqrt{2 + \sqrt{2}}$                                                                         |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                             |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,2}$          | $704 - 58 \sqrt{2}$                                                                                                                                    |
| $w_{2,3}$          | $-128 \sqrt{2} + 7 \left( 16 + 30 \sqrt{2 - \sqrt{2}} - 15 \sqrt{2 + \sqrt{2}} \right)$                                                                |
| $w_{2,4}$          | $992 - 664 \sqrt{2}$                                                                                                                                   |
| $w_{2,5}$          | $-128 \sqrt{2} + 7 \left( 16 - 30 \sqrt{2 - \sqrt{2}} + 15 \sqrt{2 + \sqrt{2}} \right)$                                                                |
| $w_{2,6}$          | $704 - 478 \sqrt{2}$                                                                                                                                   |
| $w_{2,7}$          | $1168 - 512 \sqrt{2} - 105 \sqrt{2 - \sqrt{2}} - 210 \sqrt{2 + \sqrt{2}}$                                                                              |
| $w_{2,8}$          | $-20 \left( -2 + \sqrt{2} \right)$                                                                                                                     |
| $d_3$              | $\frac{1}{40320}$                                                                                                                                      |
| $w_{3,0}$          | $475 + 80 \sqrt{2 - \sqrt{2}}$                                                                                                                         |
| $w_{3,1}$          | $2560 - 768 \sqrt{2} + 330 \sqrt{4 - 2 \sqrt{2}} - 1450 \sqrt{2 - \sqrt{2}}$<br>$+ 525 \sqrt{2 + \sqrt{2}} + 294 \sqrt{2 \left( 2 + \sqrt{2} \right)}$ |
| $w_{3,2}$          | $1155 \sqrt{2} + 16 \left( 281 - 80 \sqrt{2 - \sqrt{2}} + 42 \sqrt{2 + \sqrt{2}} \right)$                                                              |
| $w_{3,3}$          | $2560 + 768 \sqrt{2} - 330 \sqrt{4 - 2 \sqrt{2}} - 715 \sqrt{2 - \sqrt{2}}$<br>$+ 210 \sqrt{2 + \sqrt{2}} + 126 \sqrt{2 \left( 2 + \sqrt{2} \right)}$  |
| <i>(continues)</i> |                                                                                                                                                        |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                               |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,4}$     | $-32 \left( -124 + 85 \sqrt{2 - \sqrt{2}} + 42 \sqrt{2 + \sqrt{2}} \right)$                                                                              |
| $w_{3,5}$     | $768 \sqrt{2} - 294 \sqrt{2 (2 + \sqrt{2})}$<br>$- 5 \left( -512 + 150 \sqrt{4 - 2 \sqrt{2}} + 353 \sqrt{2 - \sqrt{2}} + 42 \sqrt{2 + \sqrt{2}} \right)$ |
| $w_{3,6}$     | $-1155 \sqrt{2} + 16 \left( 71 - 80 \sqrt{2 - \sqrt{2}} + 42 \sqrt{2 + \sqrt{2}} \right)$                                                                |
| $w_{3,7}$     | $2560 - 768 \sqrt{2} + 750 \sqrt{4 - 2 \sqrt{2}} - 1030 \sqrt{2 - \sqrt{2}}$<br>$- 525 \sqrt{2 + \sqrt{2}} - 126 \sqrt{2 (2 + \sqrt{2})}$                |
| $w_{3,8}$     | $5 \left( -31 + 16 \sqrt{2 - \sqrt{2}} \right)$                                                                                                          |
| $d_4$         | $\frac{1}{2520}$                                                                                                                                         |
| $w_{4,0}$     | 10                                                                                                                                                       |
| $w_{4,1}$     | $-48 \sqrt{2} + 5 \left( 32 - 21 \sqrt{2 - \sqrt{2}} + 21 \sqrt{2 + \sqrt{2}} \right)$                                                                   |
| $w_{4,2}$     | $176 + 105 \sqrt{2}$                                                                                                                                     |
| $w_{4,3}$     | $48 \sqrt{2} + 5 \left( 32 + 21 \sqrt{2 - \sqrt{2}} + 21 \sqrt{2 + \sqrt{2}} \right)$                                                                    |
| $w_{4,4}$     | 248                                                                                                                                                      |
| $w_{4,5}$     | $48 \sqrt{2} - 5 \left( -32 + 21 \sqrt{2 - \sqrt{2}} + 21 \sqrt{2 + \sqrt{2}} \right)$                                                                   |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                 |
|---------------|----------------------------------------------------------------------------------------------------------------------------|
| $w_{4,6}$     | $176 - 105\sqrt{2}$                                                                                                        |
| $w_{4,7}$     | $-48\sqrt{2} + 5 \left( 32 + 21\sqrt{2-\sqrt{2}} - 21\sqrt{2+\sqrt{2}} \right)$                                            |
| $w_{4,8}$     | $10$                                                                                                                       |
| $d_5$         | $\frac{1}{40320}$                                                                                                          |
| $w_{5,0}$     | $475 - 80\sqrt{2-\sqrt{2}}$                                                                                                |
| $w_{5,1}$     | $2560 - 768\sqrt{2} - 750\sqrt{4-2\sqrt{2}} + 1030\sqrt{2-\sqrt{2}}$<br>$+ 525\sqrt{2+\sqrt{2}} + 126\sqrt{2(2+\sqrt{2})}$ |
| $w_{5,2}$     | $1155\sqrt{2} + 16 \left( 281 + 80\sqrt{2-\sqrt{2}} - 42\sqrt{2+\sqrt{2}} \right)$                                         |
| $w_{5,3}$     | $2560 + 768\sqrt{2} + 750\sqrt{4-2\sqrt{2}} + 1765\sqrt{2-\sqrt{2}}$<br>$+ 210\sqrt{2+\sqrt{2}} + 294\sqrt{2(2+\sqrt{2})}$ |
| $w_{5,4}$     | $32 \left( 124 + 85\sqrt{2-\sqrt{2}} + 42\sqrt{2+\sqrt{2}} \right)$                                                        |
| $w_{5,5}$     | $2560 + 768\sqrt{2} + 330\sqrt{4-2\sqrt{2}} + 715\sqrt{2-\sqrt{2}}$<br>$- 210\sqrt{2+\sqrt{2}} - 126\sqrt{2(2+\sqrt{2})}$  |
| $w_{5,6}$     | $-1155\sqrt{2} + 16 \left( 71 + 80\sqrt{2-\sqrt{2}} - 42\sqrt{2+\sqrt{2}} \right)$                                         |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                         |
|---------------|------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,7}$     | $2560 - 768\sqrt{2} - 330\sqrt{4 - 2\sqrt{2}} + 1450\sqrt{2 - \sqrt{2}}$<br>$- 525\sqrt{2 + \sqrt{2}} - 294\sqrt{2(2 + \sqrt{2})}$ |
| $w_{5,8}$     | $-5(31 + 16\sqrt{2 - \sqrt{2}})$                                                                                                   |
| $d_6$         | $\frac{1}{10080}$                                                                                                                  |
| $w_{6,0}$     | $20(2 + \sqrt{2})$                                                                                                                 |
| $w_{6,1}$     | $128\sqrt{2} + 7(16 + 15\sqrt{2 - \sqrt{2}} + 30\sqrt{2 + \sqrt{2}})$                                                              |
| $w_{6,2}$     | $704 + 478\sqrt{2}$                                                                                                                |
| $w_{6,3}$     | $1168 + 512\sqrt{2} + 210\sqrt{2 - \sqrt{2}} - 105\sqrt{2 + \sqrt{2}}$                                                             |
| $w_{6,4}$     | $992 + 664\sqrt{2}$                                                                                                                |
| $w_{6,5}$     | $1168 + 512\sqrt{2} - 210\sqrt{2 - \sqrt{2}} + 105\sqrt{2 + \sqrt{2}}$                                                             |
| $w_{6,6}$     | $704 + 58\sqrt{2}$                                                                                                                 |
| $w_{6,7}$     | $128\sqrt{2} - 7(-16 + 15\sqrt{2 - \sqrt{2}} + 30\sqrt{2 + \sqrt{2}})$                                                             |
| $w_{6,8}$     | $20(2 + \sqrt{2})$                                                                                                                 |
| $d_7$         | $\frac{1}{40320}$                                                                                                                  |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                               |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{7,0}$     | $475 - 80 \sqrt{2 + \sqrt{2}}$                                                                                                                           |
| $w_{7,1}$     | $-768 \sqrt{2} + 294 \sqrt{4 - 2 \sqrt{2}}$<br>$+ 5 \left( 512 - 42 \sqrt{2 - \sqrt{2}} + 353 \sqrt{2 + \sqrt{2}} - 150 \sqrt{2 (2 + \sqrt{2})} \right)$ |
| $w_{7,2}$     | $1155 \sqrt{2} + 16 \left( 71 + 42 \sqrt{2 - \sqrt{2}} + 80 \sqrt{2 + \sqrt{2}} \right)$                                                                 |
| $w_{7,3}$     | $2560 + 768 \sqrt{2} - 294 \sqrt{4 - 2 \sqrt{2}} + 525 \sqrt{2 - \sqrt{2}}$<br>$+ 1450 \sqrt{2 + \sqrt{2}} + 330 \sqrt{2 (2 + \sqrt{2})}$                |
| $w_{7,4}$     | $32 \left( 124 - 42 \sqrt{2 - \sqrt{2}} + 85 \sqrt{2 + \sqrt{2}} \right)$                                                                                |
| $w_{7,5}$     | $768 \sqrt{2} + 126 \sqrt{4 - 2 \sqrt{2}}$<br>$+ 5 \left( 512 - 105 \sqrt{2 - \sqrt{2}} + 206 \sqrt{2 + \sqrt{2}} + 150 \sqrt{2 (2 + \sqrt{2})} \right)$ |
| $w_{7,6}$     | $-1155 \sqrt{2} + 16 \left( 281 + 42 \sqrt{2 - \sqrt{2}} + 80 \sqrt{2 + \sqrt{2}} \right)$                                                               |
| $w_{7,7}$     | $2560 - 768 \sqrt{2} - 126 \sqrt{4 - 2 \sqrt{2}} + 210 \sqrt{2 - \sqrt{2}}$<br>$+ 715 \sqrt{2 + \sqrt{2}} - 330 \sqrt{2 (2 + \sqrt{2})}$                 |
| $w_{7,8}$     | $-5 \left( 31 + 16 \sqrt{2 + \sqrt{2}} \right)$                                                                                                          |
| $d_8$         | $\frac{1}{630}$                                                                                                                                          |
| $w_{8,0}$     | 5                                                                                                                                                        |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression        |
|---------------|-------------------|
| $w_{8,1}$     | $80 - 24\sqrt{2}$ |
| $w_{8,2}$     | 88                |
| $w_{8,3}$     | $80 + 24\sqrt{2}$ |
| $w_{8,4}$     | 124               |
| $w_{8,5}$     | $80 + 24\sqrt{2}$ |
| $w_{8,6}$     | 88                |
| $w_{8,7}$     | $80 - 24\sqrt{2}$ |
| $w_{8,8}$     | 5                 |

Table B.5: Integral approximation  $I_j$  for  $n = 10$ .

| $d_j, w_{jk}$ | Expression                                                                                                                                           |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_1$         | $\frac{1}{8870400}$                                                                                                                                  |
| $w_{1,0}$     | $224 \left( 298 + 25\sqrt{2(5 + \sqrt{5})} \right)$                                                                                                  |
| $w_{1,1}$     | $386048 - 78848\sqrt{5} + 8085\sqrt{50 - 10\sqrt{5}} - 20955\sqrt{10 - 2\sqrt{5}}$<br>$- 48391\sqrt{2(5 + \sqrt{5})} + 14245\sqrt{10(5 + \sqrt{5})}$ |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                   |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,2}$     | $548226 - 64174\sqrt{5} - 26400\sqrt{50 - 10\sqrt{5}}$ $- 142800\sqrt{2(5 + \sqrt{5})} + 30800\sqrt{10(5 + \sqrt{5})}$                                       |
| $w_{1,3}$     | $386048 + 78848\sqrt{5} - 10395\sqrt{50 - 10\sqrt{5}} + 26169\sqrt{10 - 2\sqrt{5}}$ $- 74725\sqrt{2(5 + \sqrt{5})} - 32725\sqrt{10(5 + \sqrt{5})}$           |
| $w_{1,4}$     | $163966\sqrt{5} + 26400\sqrt{50 - 10\sqrt{5}}$ $- 14\left(-21141 + 10200\sqrt{2(5 + \sqrt{5})} + 2200\sqrt{10(5 + \sqrt{5})}\right)$                         |
| $w_{1,5}$     | $192\left(3624 + 220\sqrt{10 - 2\sqrt{5}} - 1085\sqrt{2(5 + \sqrt{5})}\right)$                                                                               |
| $w_{1,6}$     | $779226 - 51326\sqrt{5} + 26400\sqrt{50 - 10\sqrt{5}}$ $- 142800\sqrt{2(5 + \sqrt{5})} - 30800\sqrt{10(5 + \sqrt{5})}$                                       |
| $w_{1,7}$     | $386048 + 78848\sqrt{5} + 10395\sqrt{50 - 10\sqrt{5}} - 47289\sqrt{10 - 2\sqrt{5}}$ $- 95515\sqrt{2(5 + \sqrt{5})} - 16555\sqrt{10(5 + \sqrt{5})}$           |
| $w_{1,8}$     | $526974 - 48466\sqrt{5} - 26400\sqrt{50 - 10\sqrt{5}}$ $- 142800\sqrt{2(5 + \sqrt{5})} + 30800\sqrt{10(5 + \sqrt{5})}$                                       |
| $w_{1,9}$     | $-78848\sqrt{5} - 8085\sqrt{50 - 10\sqrt{5}} - 165\sqrt{10 - 2\sqrt{5}}$ $+ 13\left(29696 - 9373\sqrt{2(5 + \sqrt{5})} + 2695\sqrt{10(5 + \sqrt{5})}\right)$ |
| $w_{1,10}$    | $224\left(-98 + 25\sqrt{2(5 + \sqrt{5})}\right)$                                                                                                             |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                                               |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_2$              | $\frac{1}{8870400}$                                                                                                                                                                      |
| $w_{2,0}$          | $-5600 \left( -3 + \sqrt{5} \right)$                                                                                                                                                     |
| $w_{2,1}$          | $388096 - 58496 \sqrt{5} + 28875 \sqrt{50 - 10 \sqrt{5}} - 28875 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 86625 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 17325 \sqrt{10 \left( 5 + \sqrt{5} \right)}$ |
| $w_{2,2}$          | $521450 - 72690 \sqrt{5}$                                                                                                                                                                |
| $w_{2,3}$          | $80096 - 51104 \sqrt{5} - 17325 \sqrt{50 - 10 \sqrt{5}} + 86625 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 28875 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 28875 \sqrt{10 \left( 5 + \sqrt{5} \right)}$  |
| $w_{2,4}$          | $130 \left( 415 - 87 \sqrt{5} \right)$                                                                                                                                                   |
| $w_{2,5}$          | $-192 \left( -3873 + 1775 \sqrt{5} \right)$                                                                                                                                              |
| $w_{2,6}$          | $400450 - 173010 \sqrt{5}$                                                                                                                                                               |
| $w_{2,7}$          | $80096 - 51104 \sqrt{5} + 17325 \sqrt{50 - 10 \sqrt{5}} - 86625 \sqrt{10 - 2 \sqrt{5}}$<br>$- 28875 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 28875 \sqrt{10 \left( 5 + \sqrt{5} \right)}$  |
| $w_{2,8}$          | $636950 - 280590 \sqrt{5}$                                                                                                                                                               |
| $w_{2,9}$          | $388096 - 58496 \sqrt{5} - 28875 \sqrt{50 - 10 \sqrt{5}} + 28875 \sqrt{10 - 2 \sqrt{5}}$<br>$- 86625 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 17325 \sqrt{10 \left( 5 + \sqrt{5} \right)}$ |
| <i>(continues)</i> |                                                                                                                                                                                          |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                               |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,10}$    | $-5600 \left( -3 + \sqrt{5} \right)$                                                                                                                                                     |
| $d_3$         | $\frac{1}{8870400}$                                                                                                                                                                      |
| $w_{3,0}$     | $224 \left( 298 + 25 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                                     |
| $w_{3,1}$     | $386048 - 78848 \sqrt{5} + 16555 \sqrt{50 - 10 \sqrt{5}} - 95515 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 47289 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 10395 \sqrt{10 \left( 5 + \sqrt{5} \right)}$ |
| $w_{3,2}$     | $779226 + 51326 \sqrt{5} + 30800 \sqrt{50 - 10 \sqrt{5}}$<br>$- 142800 \sqrt{10 - 2 \sqrt{5}} + 26400 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                             |
| $w_{3,3}$     | $386048 + 78848 \sqrt{5} - 14245 \sqrt{50 - 10 \sqrt{5}} - 48391 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 20955 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 8085 \sqrt{10 \left( 5 + \sqrt{5} \right)}$  |
| $w_{3,4}$     | $526974 + 48466 \sqrt{5} - 30800 \sqrt{50 - 10 \sqrt{5}}$<br>$- 142800 \sqrt{10 - 2 \sqrt{5}} - 26400 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                             |
| $w_{3,5}$     | $-192 \left( -3624 + 1085 \sqrt{10 - 2 \sqrt{5}} + 220 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                                                                     |
| $w_{3,6}$     | $548226 + 64174 \sqrt{5} - 30800 \sqrt{50 - 10 \sqrt{5}}$<br>$- 142800 \sqrt{10 - 2 \sqrt{5}} - 26400 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                             |
| $w_{3,7}$     | $386048 + 78848 \sqrt{5} - 35035 \sqrt{50 - 10 \sqrt{5}} - 121849 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 165 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 8085 \sqrt{10 \left( 5 + \sqrt{5} \right)}$   |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                        |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,8}$            | $295974 - 163966 \sqrt{5} + 30800 \sqrt{50 - 10 \sqrt{5}}$ $- 142800 \sqrt{10 - 2 \sqrt{5}} + 26400 \sqrt{10 (5 + \sqrt{5})}$                                                     |
| $w_{3,9}$            | $386048 - 78848 \sqrt{5} + 32725 \sqrt{50 - 10 \sqrt{5}} - 74725 \sqrt{10 - 2 \sqrt{5}}$ $- 26169 \sqrt{2 (5 + \sqrt{5})} - 10395 \sqrt{10 (5 + \sqrt{5})}$                       |
| $w_{3,10}$           | $224 \left( -98 + 25 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                              |
| $d_4$                | $\frac{1}{354816}$                                                                                                                                                                |
| $w_{4,0}$            | $-224 \left( -5 + \sqrt{5} \right)$                                                                                                                                               |
| $w_{4,1}$            | $27680 - 8352 \sqrt{5} - 1155 \sqrt{50 - 10 \sqrt{5}} - 1155 \sqrt{10 - 2 \sqrt{5}}$ $+ 3465 \sqrt{2 (5 + \sqrt{5})} + 693 \sqrt{10 (5 + \sqrt{5})}$                              |
| $w_{4,2}$            | $40850 - 4958 \sqrt{5}$                                                                                                                                                           |
| $w_{4,3}$            | $3968 \sqrt{5} + 3 \left( 5120 + 231 \sqrt{50 - 10 \sqrt{5}} + 1155 \sqrt{10 - 2 \sqrt{5}} \right.$ $\left. + 385 \sqrt{2 (5 + \sqrt{5})} + 385 \sqrt{10 (5 + \sqrt{5})} \right)$ |
| $w_{4,4}$            | $22150 + 1598 \sqrt{5}$                                                                                                                                                           |
| $w_{4,5}$            | $25920 - 13632 \sqrt{5}$                                                                                                                                                          |
| $w_{4,6}$            | $17530 - 6718 \sqrt{5}$                                                                                                                                                           |
| ( <i>continues</i> ) |                                                                                                                                                                                   |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                                 |
|--------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,7}$          | $3968\sqrt{5} - 3 \left( -5120 + 231\sqrt{50 - 10\sqrt{5}} + 1155\sqrt{10 - 2\sqrt{5}} \right. \\ \left. + 385\sqrt{2(5 + \sqrt{5})} + 385\sqrt{10(5 + \sqrt{5})} \right)$ |
| $w_{4,8}$          | $26990 - 11426\sqrt{5}$                                                                                                                                                    |
| $w_{4,9}$          | $27680 - 8352\sqrt{5} + 1155\sqrt{50 - 10\sqrt{5}} + 1155\sqrt{10 - 2\sqrt{5}} \\ - 3465\sqrt{2(5 + \sqrt{5})} - 693\sqrt{10(5 + \sqrt{5})}$                               |
| $w_{4,10}$         | $-224(-5 + \sqrt{5})$                                                                                                                                                      |
| $d_5$              | $\frac{1}{138600}$                                                                                                                                                         |
| $w_{5,0}$          | 1043                                                                                                                                                                       |
| $w_{5,1}$          | $6032 - 1232\sqrt{5} - 1155\sqrt{10 - 2\sqrt{5}} + 1386\sqrt{2(5 + \sqrt{5})}$                                                                                             |
| $w_{5,2}$          | $5859 + 3509\sqrt{5}$                                                                                                                                                      |
| $w_{5,3}$          | $6032 + 1232\sqrt{5} + 1386\sqrt{10 - 2\sqrt{5}} + 1155\sqrt{2(5 + \sqrt{5})}$                                                                                             |
| $w_{5,4}$          | $10941 + 5269\sqrt{5}$                                                                                                                                                     |
| $w_{5,5}$          | 10872                                                                                                                                                                      |
| $w_{5,6}$          | $5859 - 3509\sqrt{5}$                                                                                                                                                      |
| <i>(continues)</i> |                                                                                                                                                                            |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                            |
|--------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,7}$          | $6032 + 1232\sqrt{5} - 1386\sqrt{10 - 2\sqrt{5}} - 1155\sqrt{2(5 + \sqrt{5})}$                                                                        |
| $w_{5,8}$          | $10941 - 5269\sqrt{5}$                                                                                                                                |
| $w_{5,9}$          | $6032 - 1232\sqrt{5} + 1155\sqrt{10 - 2\sqrt{5}} - 1386\sqrt{2(5 + \sqrt{5})}$                                                                        |
| $w_{5,10}$         | $-343$                                                                                                                                                |
| $d_6$              | $\frac{1}{8870400}$                                                                                                                                   |
| $w_{6,0}$          | $5600(3 + \sqrt{5})$                                                                                                                                  |
| $w_{6,1}$          | $80096 + 51104\sqrt{5} - 28875\sqrt{50 - 10\sqrt{5}} - 28875\sqrt{10 - 2\sqrt{5}}$<br>$+ 86625\sqrt{2(5 + \sqrt{5})} + 17325\sqrt{10(5 + \sqrt{5})}$  |
| $w_{6,2}$          | $400450 + 173010\sqrt{5}$                                                                                                                             |
| $w_{6,3}$          | $388096 + 58496\sqrt{5} + 17325\sqrt{50 - 10\sqrt{5}} + 86625\sqrt{10 - 2\sqrt{5}}$<br>$+ 28875\sqrt{2(5 + \sqrt{5})} + 28875\sqrt{10(5 + \sqrt{5})}$ |
| $w_{6,4}$          | $10(63695 + 28059\sqrt{5})$                                                                                                                           |
| $w_{6,5}$          | $192(3873 + 1775\sqrt{5})$                                                                                                                            |
| $w_{6,6}$          | $521450 + 72690\sqrt{5}$                                                                                                                              |
| <i>(continues)</i> |                                                                                                                                                       |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                         |
|----------------------|----------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,7}$            | $388096 + 58496\sqrt{5} - 17325\sqrt{50 - 10\sqrt{5}} - 86625\sqrt{10 - 2\sqrt{5}}$ $- 28875\sqrt{2(5 + \sqrt{5})} - 28875\sqrt{10(5 + \sqrt{5})}$ |
| $w_{6,8}$            | $130(415 + 87\sqrt{5})$                                                                                                                            |
| $w_{6,9}$            | $80096 + 51104\sqrt{5} + 28875\sqrt{50 - 10\sqrt{5}} + 28875\sqrt{10 - 2\sqrt{5}}$ $- 86625\sqrt{2(5 + \sqrt{5})} - 17325\sqrt{10(5 + \sqrt{5})}$  |
| $w_{6,10}$           | $5600(3 + \sqrt{5})$                                                                                                                               |
| $d_7$                | $\frac{1}{8870400}$                                                                                                                                |
| $w_{7,0}$            | $-224(-298 + 25\sqrt{10 - 2\sqrt{5}})$                                                                                                             |
| $w_{7,1}$            | $386048 - 78848\sqrt{5} - 32725\sqrt{50 - 10\sqrt{5}} + 74725\sqrt{10 - 2\sqrt{5}}$ $+ 26169\sqrt{2(5 + \sqrt{5})} + 10395\sqrt{10(5 + \sqrt{5})}$ |
| $w_{7,2}$            | $779226 + 51326\sqrt{5} - 30800\sqrt{50 - 10\sqrt{5}}$ $+ 142800\sqrt{10 - 2\sqrt{5}} - 26400\sqrt{10(5 + \sqrt{5})}$                              |
| $w_{7,3}$            | $386048 + 78848\sqrt{5} + 35035\sqrt{50 - 10\sqrt{5}} + 121849\sqrt{10 - 2\sqrt{5}}$ $- 165\sqrt{2(5 + \sqrt{5})} + 8085\sqrt{10(5 + \sqrt{5})}$   |
| $w_{7,4}$            | $526974 + 48466\sqrt{5} + 30800\sqrt{50 - 10\sqrt{5}}$ $+ 142800\sqrt{10 - 2\sqrt{5}} + 26400\sqrt{10(5 + \sqrt{5})}$                              |
| ( <i>continues</i> ) |                                                                                                                                                    |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                   |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{7,5}$     | $192 \left( 3624 + 1085 \sqrt{10 - 2\sqrt{5}} + 220 \sqrt{2(5 + \sqrt{5})} \right)$                                                                                          |
| $w_{7,6}$     | $548226 + 64174\sqrt{5} + 30800\sqrt{50 - 10\sqrt{5}}$<br>$+ 142800\sqrt{10 - 2\sqrt{5}} + 26400\sqrt{10(5 + \sqrt{5})}$                                                     |
| $w_{7,7}$     | $386048 + 78848\sqrt{5} + 14245\sqrt{50 - 10\sqrt{5}} + 48391\sqrt{10 - 2\sqrt{5}}$<br>$- 20955\sqrt{2(5 + \sqrt{5})} - 8085\sqrt{10(5 + \sqrt{5})}$                         |
| $w_{7,8}$     | $295974 - 163966\sqrt{5} - 30800\sqrt{50 - 10\sqrt{5}}$<br>$+ 142800\sqrt{10 - 2\sqrt{5}} - 26400\sqrt{10(5 + \sqrt{5})}$                                                    |
| $w_{7,9}$     | $386048 - 78848\sqrt{5} - 16555\sqrt{50 - 10\sqrt{5}} + 95515\sqrt{10 - 2\sqrt{5}}$<br>$- 47289\sqrt{2(5 + \sqrt{5})} - 10395\sqrt{10(5 + \sqrt{5})}$                        |
| $w_{7,10}$    | $-224 \left( 98 + 25\sqrt{10 - 2\sqrt{5}} \right)$                                                                                                                           |
| $d_8$         | $\frac{1}{354816}$                                                                                                                                                           |
| $w_{8,0}$     | $224(5 + \sqrt{5})$                                                                                                                                                          |
| $w_{8,1}$     | $-3968\sqrt{5} + 3 \left( 5120 + 385\sqrt{50 - 10\sqrt{5}} - 385\sqrt{10 - 2\sqrt{5}} \right.$<br>$\left. + 1155\sqrt{2(5 + \sqrt{5})} - 231\sqrt{10(5 + \sqrt{5})} \right)$ |
| $w_{8,2}$     | $17530 + 6718\sqrt{5}$                                                                                                                                                       |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                            |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,3}$     | $27680 + 8352\sqrt{5} - 693\sqrt{50 - 10\sqrt{5}} + 3465\sqrt{10 - 2\sqrt{5}}$ $+ 1155\sqrt{2(5 + \sqrt{5})} - 1155\sqrt{10(5 + \sqrt{5})}$                           |
| $w_{8,4}$     | $26990 + 11426\sqrt{5}$                                                                                                                                               |
| $w_{8,5}$     | $192(135 + 71\sqrt{5})$                                                                                                                                               |
| $w_{8,6}$     | $2(20425 + 2479\sqrt{5})$                                                                                                                                             |
| $w_{8,7}$     | $27680 + 8352\sqrt{5} + 693\sqrt{50 - 10\sqrt{5}} - 3465\sqrt{10 - 2\sqrt{5}}$ $- 1155\sqrt{2(5 + \sqrt{5})} + 1155\sqrt{10(5 + \sqrt{5})}$                           |
| $w_{8,8}$     | $22150 - 1598\sqrt{5}$                                                                                                                                                |
| $w_{8,9}$     | $-3968\sqrt{5} + 3\left(5120 - 385\sqrt{50 - 10\sqrt{5}} + 385\sqrt{10 - 2\sqrt{5}}\right.$ $\left. - 1155\sqrt{2(5 + \sqrt{5})} + 231\sqrt{10(5 + \sqrt{5})}\right)$ |
| $w_{8,10}$    | $224(5 + \sqrt{5})$                                                                                                                                                   |
| $d_9$         | $\frac{1}{8870400}$                                                                                                                                                   |
| $w_{9,0}$     | $-224\left(-298 + 25\sqrt{2(5 + \sqrt{5})}\right)$                                                                                                                    |
| $w_{9,1}$     | $-78848\sqrt{5} + 8085\sqrt{50 - 10\sqrt{5}} + 165\sqrt{10 - 2\sqrt{5}}$ $+ 13\left(29696 + 9373\sqrt{2(5 + \sqrt{5})} - 2695\sqrt{10(5 + \sqrt{5})}\right)$          |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                         |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,2}$     | $548226 - 64174\sqrt{5} + 26400\sqrt{50 - 10\sqrt{5}}$ $+ 142800\sqrt{2(5 + \sqrt{5})} - 30800\sqrt{10(5 + \sqrt{5})}$                             |
| $w_{9,3}$     | $386048 + 78848\sqrt{5} - 10395\sqrt{50 - 10\sqrt{5}} + 47289\sqrt{10 - 2\sqrt{5}}$ $+ 95515\sqrt{2(5 + \sqrt{5})} + 16555\sqrt{10(5 + \sqrt{5})}$ |
| $w_{9,4}$     | $163966\sqrt{5} - 26400\sqrt{50 - 10\sqrt{5}}$ $+ 14\left(21141 + 10200\sqrt{2(5 + \sqrt{5})} + 2200\sqrt{10(5 + \sqrt{5})}\right)$                |
| $w_{9,5}$     | $-192\left(-3624 + 220\sqrt{10 - 2\sqrt{5}} - 1085\sqrt{2(5 + \sqrt{5})}\right)$                                                                   |
| $w_{9,6}$     | $779226 - 51326\sqrt{5} - 26400\sqrt{50 - 10\sqrt{5}}$ $+ 142800\sqrt{2(5 + \sqrt{5})} + 30800\sqrt{10(5 + \sqrt{5})}$                             |
| $w_{9,7}$     | $386048 + 78848\sqrt{5} + 10395\sqrt{50 - 10\sqrt{5}} - 26169\sqrt{10 - 2\sqrt{5}}$ $+ 74725\sqrt{2(5 + \sqrt{5})} + 32725\sqrt{10(5 + \sqrt{5})}$ |
| $w_{9,8}$     | $526974 - 48466\sqrt{5} + 26400\sqrt{50 - 10\sqrt{5}}$ $+ 142800\sqrt{2(5 + \sqrt{5})} - 30800\sqrt{10(5 + \sqrt{5})}$                             |
| $w_{9,9}$     | $386048 - 78848\sqrt{5} - 8085\sqrt{50 - 10\sqrt{5}} + 20955\sqrt{10 - 2\sqrt{5}}$ $+ 48391\sqrt{2(5 + \sqrt{5})} - 14245\sqrt{10(5 + \sqrt{5})}$  |
| $w_{9,10}$    | $-224\left(98 + 25\sqrt{2(5 + \sqrt{5})}\right)$                                                                                                   |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                           |
|---------------|--------------------------------------|
| $d_{10}$      | $\frac{1}{34650}$                    |
| $w_{10,0}$    | 175                                  |
| $w_{10,1}$    | $3016 - 616\sqrt{5}$                 |
| $w_{10,2}$    | $4200 - 440\sqrt{5}$                 |
| $w_{10,3}$    | $3016 + 616\sqrt{5}$                 |
| $w_{10,4}$    | $40 \left( 105 + 11\sqrt{5} \right)$ |
| $w_{10,5}$    | 5436                                 |
| $w_{10,6}$    | $40 \left( 105 + 11\sqrt{5} \right)$ |
| $w_{10,7}$    | $3016 + 616\sqrt{5}$                 |
| $w_{10,8}$    | $4200 - 440\sqrt{5}$                 |
| $w_{10,9}$    | $3016 - 616\sqrt{5}$                 |
| $w_{10,10}$   | 175                                  |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$  for  $n = 12$ .

| $d_j, w_{jk}$ | Expression                                                                         |
|---------------|------------------------------------------------------------------------------------|
| $d_1$         | $\frac{1}{69189120}$                                                               |
| $w_{1,0}$     | $1680 \left( 215 + 18 \sqrt{2} + 18 \sqrt{6} \right)$                              |
| $w_{1,1}$     | $-101677 \sqrt{2} - 97361 \sqrt{6} - 2048 \left( -1346 + 455 \sqrt{3} \right)$     |
| $w_{1,2}$     | $-22 \left( -162392 + 38624 \sqrt{2} + 33033 \sqrt{3} + 20736 \sqrt{6} \right)$    |
| $w_{1,3}$     | $997421 \sqrt{2} - 1040303 \sqrt{6} - 2048 \left( -1346 + 455 \sqrt{3} \right)$    |
| $w_{1,4}$     | $-22 \left( -44456 + 38624 \sqrt{2} - 33033 \sqrt{3} + 20736 \sqrt{6} \right)$     |
| $w_{1,5}$     | $-16 \left( -199744 + 4261 \sqrt{2} + 78192 \sqrt{6} \right)$                      |
| $w_{1,6}$     | $-6 \left( -1027709 + 248256 \sqrt{2} + 216216 \sqrt{3} + 124704 \sqrt{6} \right)$ |
| $w_{1,7}$     | $-1833043 \sqrt{2} - 720529 \sqrt{6} + 2048 \left( 1346 + 455 \sqrt{3} \right)$    |
| $w_{1,8}$     | $64 \left( 70808 + 4117 \sqrt{2} - 31347 \sqrt{6} \right)$                         |
| $w_{1,9}$     | $-1202413 \sqrt{2} - 1086895 \sqrt{6} + 2048 \left( 1346 + 455 \sqrt{3} \right)$   |
| $w_{1,10}$    | $6 \left( 280963 - 248256 \sqrt{2} + 216216 \sqrt{3} - 124704 \sqrt{6} \right)$    |
| $w_{1,11}$    | $-16 \left( -199744 + 130387 \sqrt{2} + 6120 \sqrt{6} \right)$                     |
| $w_{1,12}$    | $1680 \left( -71 + 18 \sqrt{2} + 18 \sqrt{6} \right)$                              |

(continues)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                       |
|---------------|------------------------------------------------------------------|
| $d_2$         | $\frac{1}{69189120}$                                             |
| $w_{2,0}$     | $-60480 \left( -2 + \sqrt{3} \right)$                            |
| $w_{2,1}$     | $5115328 + 519519 \sqrt{2} - 2136256 \sqrt{3} + 291291 \sqrt{6}$ |
| $w_{2,2}$     | $2275328 - 1293534 \sqrt{3}$                                     |
| $w_{2,3}$     | $5115328 - 519519 \sqrt{2} - 2136256 \sqrt{3} - 291291 \sqrt{6}$ |
| $w_{2,4}$     | $2275328 - 176418 \sqrt{3}$                                      |
| $w_{2,5}$     | $-32 \left( -99872 + 3003 \sqrt{2} + 60804 \sqrt{3} \right)$     |
| $w_{2,6}$     | $54 \left( 73705 - 40928 \sqrt{3} \right)$                       |
| $w_{2,7}$     | $397888 - 519519 \sqrt{2} - 272576 \sqrt{3} + 291291 \sqrt{6}$   |
| $w_{2,8}$     | $128 \left( 35404 - 20133 \sqrt{3} \right)$                      |
| $w_{2,9}$     | $397888 + 519519 \sqrt{2} - 272576 \sqrt{3} - 291291 \sqrt{6}$   |
| $w_{2,10}$    | $54 \left( 71703 - 40928 \sqrt{3} \right)$                       |
| $w_{2,11}$    | $32 \left( 99872 + 3003 \sqrt{2} - 60804 \sqrt{3} \right)$       |
| $w_{2,12}$    | $-60480 \left( -2 + \sqrt{3} \right)$                            |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                   |
|---------------|--------------------------------------------------------------|
| $d_3$         | $\frac{1}{4324320}$                                          |
| $w_{3,0}$     | $105 \left( 215 + 36 \sqrt{2} \right)$                       |
| $w_{3,1}$     | $172288 - 78878 \sqrt{2} - 58240 \sqrt{3} + 62972 \sqrt{6}$  |
| $w_{3,2}$     | $176 \left( 808 + 79 \sqrt{2} - 546 \sqrt{3} \right)$        |
| $w_{3,3}$     | $172288 - 114914 \sqrt{2} - 58240 \sqrt{3} + 38948 \sqrt{6}$ |
| $w_{3,4}$     | $176 \left( 808 + 79 \sqrt{2} + 546 \sqrt{3} \right)$        |
| $w_{3,5}$     | $199744 - 11525 \sqrt{2}$                                    |
| $w_{3,6}$     | $233364 - 186192 \sqrt{2}$                                   |
| $w_{3,7}$     | $172288 - 114914 \sqrt{2} + 58240 \sqrt{3} - 38948 \sqrt{6}$ |
| $w_{3,8}$     | $283232 - 207304 \sqrt{2}$                                   |
| $w_{3,9}$     | $172288 - 78878 \sqrt{2} + 58240 \sqrt{3} - 62972 \sqrt{6}$  |
| $w_{3,10}$    | $257388 - 186192 \sqrt{2}$                                   |
| $w_{3,11}$    | $199744 - 137651 \sqrt{2}$                                   |
| $w_{3,12}$    | $105 \left( -71 + 36 \sqrt{2} \right)$                       |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                     |
|---------------|----------------------------------------------------------------|
| $d_4$         | $\frac{1}{69189120}$                                           |
| $w_{4,0}$     | 60480                                                          |
| $w_{4,1}$     | $417344 + 567567 \sqrt{2} + 495040 \sqrt{3} + 243243 \sqrt{6}$ |
| $w_{4,2}$     | $2098624 - 1135134 \sqrt{3}$                                   |
| $w_{4,3}$     | $417344 - 567567 \sqrt{2} + 495040 \sqrt{3} - 243243 \sqrt{6}$ |
| $w_{4,4}$     | $22 \left( 95392 + 51597 \sqrt{3} \right)$                     |
| $w_{4,5}$     | $32 \left( 109996 + 81081 \sqrt{2} \right)$                    |
| $w_{4,6}$     | 3746790                                                        |
| $w_{4,7}$     | $417344 - 567567 \sqrt{2} - 495040 \sqrt{3} + 243243 \sqrt{6}$ |
| $w_{4,8}$     | 343936                                                         |
| $w_{4,9}$     | $417344 + 567567 \sqrt{2} - 495040 \sqrt{3} - 243243 \sqrt{6}$ |
| $w_{4,10}$    | 179226                                                         |
| $w_{4,11}$    | $32 \left( 109996 - 81081 \sqrt{2} \right)$                    |
| $w_{4,12}$    | 60480                                                          |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                    |
|---------------|-------------------------------------------------------------------------------|
| $d_5$         | $\frac{1}{69189120}$                                                          |
| $w_{5,0}$     | $-1680 \left( -215 + 18\sqrt{2} - 18\sqrt{6} \right)$                         |
| $w_{5,1}$     | $1833043\sqrt{2} - 720529\sqrt{6} - 2048 \left( -1346 + 455\sqrt{3} \right)$  |
| $w_{5,2}$     | $22 \left( 44456 + 38624\sqrt{2} - 33033\sqrt{3} - 20736\sqrt{6} \right)$     |
| $w_{5,3}$     | $1202413\sqrt{2} - 1086895\sqrt{6} - 2048 \left( -1346 + 455\sqrt{3} \right)$ |
| $w_{5,4}$     | $22 \left( 162392 + 38624\sqrt{2} + 33033\sqrt{3} - 20736\sqrt{6} \right)$    |
| $w_{5,5}$     | $16 \left( 199744 + 130387\sqrt{2} - 6120\sqrt{6} \right)$                    |
| $w_{5,6}$     | $6 \left( 1027709 + 248256\sqrt{2} + 216216\sqrt{3} - 124704\sqrt{6} \right)$ |
| $w_{5,7}$     | $101677\sqrt{2} - 97361\sqrt{6} + 2048 \left( 1346 + 455\sqrt{3} \right)$     |
| $w_{5,8}$     | $-64 \left( -70808 + 4117\sqrt{2} + 31347\sqrt{6} \right)$                    |
| $w_{5,9}$     | $-997421\sqrt{2} - 1040303\sqrt{6} + 2048 \left( 1346 + 455\sqrt{3} \right)$  |
| $w_{5,10}$    | $6 \left( 280963 + 248256\sqrt{2} - 216216\sqrt{3} - 124704\sqrt{6} \right)$  |
| $w_{5,11}$    | $16 \left( 199744 + 4261\sqrt{2} - 78192\sqrt{6} \right)$                     |
| $w_{5,12}$    | $-1680 \left( 71 + 18\sqrt{2} - 18\sqrt{6} \right)$                           |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                 |
|---------------|------------------------------------------------------------|
| $d_6$         | $\frac{1}{1081080}$                                        |
| $w_{6,0}$     | 1890                                                       |
| $w_{6,1}$     | $43072 - 21021 \sqrt{2} - 14560 \sqrt{3} + 21021 \sqrt{6}$ |
| $w_{6,2}$     | $35552 - 18018 \sqrt{3}$                                   |
| $w_{6,3}$     | $43072 + 21021 \sqrt{2} - 14560 \sqrt{3} - 21021 \sqrt{6}$ |
| $w_{6,4}$     | $22 \left( 1616 + 819 \sqrt{3} \right)$                    |
| $w_{6,5}$     | $49936 + 39039 \sqrt{2}$                                   |
| $w_{6,6}$     | 115398                                                     |
| $w_{6,7}$     | $43072 + 21021 \sqrt{2} + 14560 \sqrt{3} + 21021 \sqrt{6}$ |
| $w_{6,8}$     | 70808                                                      |
| $w_{6,9}$     | $43072 - 21021 \sqrt{2} + 14560 \sqrt{3} - 21021 \sqrt{6}$ |
| $w_{6,10}$    | 7290                                                       |
| $w_{6,11}$    | $49936 - 39039 \sqrt{2}$                                   |
| $w_{6,12}$    | 1890                                                       |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                        |
|---------------|-----------------------------------------------------------------------------------|
| $d_7$         | $\frac{1}{69189120}$                                                              |
| $w_{7,0}$     | $1680 \left( 215 + 18 \sqrt{2} - 18 \sqrt{6} \right)$                             |
| $w_{7,1}$     | $-1202413 \sqrt{2} + 1086895 \sqrt{6} - 2048 \left( -1346 + 455 \sqrt{3} \right)$ |
| $w_{7,2}$     | $-22 \left( -44456 + 38624 \sqrt{2} + 33033 \sqrt{3} - 20736 \sqrt{6} \right)$    |
| $w_{7,3}$     | $-1833043 \sqrt{2} + 720529 \sqrt{6} - 2048 \left( -1346 + 455 \sqrt{3} \right)$  |
| $w_{7,4}$     | $22 \left( 162392 - 38624 \sqrt{2} + 33033 \sqrt{3} + 20736 \sqrt{6} \right)$     |
| $w_{7,5}$     | $16 \left( 199744 - 4261 \sqrt{2} + 78192 \sqrt{6} \right)$                       |
| $w_{7,6}$     | $6 \left( 1027709 - 248256 \sqrt{2} + 216216 \sqrt{3} + 124704 \sqrt{6} \right)$  |
| $w_{7,7}$     | $997421 \sqrt{2} + 1040303 \sqrt{6} + 2048 \left( 1346 + 455 \sqrt{3} \right)$    |
| $w_{7,8}$     | $64 \left( 70808 + 4117 \sqrt{2} + 31347 \sqrt{6} \right)$                        |
| $w_{7,9}$     | $-101677 \sqrt{2} + 97361 \sqrt{6} + 2048 \left( 1346 + 455 \sqrt{3} \right)$     |
| $w_{7,10}$    | $6 \left( 280963 - 248256 \sqrt{2} - 216216 \sqrt{3} + 124704 \sqrt{6} \right)$   |
| $w_{7,11}$    | $16 \left( 199744 - 130387 \sqrt{2} + 6120 \sqrt{6} \right)$                      |
| $w_{7,12}$    | $1680 \left( -71 + 18 \sqrt{2} - 18 \sqrt{6} \right)$                             |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                  |
|---------------|-----------------------------------------------------------------------------|
| $d_8$         | $\frac{1}{2562560}$                                                         |
| $w_{8,0}$     | 6720                                                                        |
| $w_{8,1}$     | $3 \left( 62912 + 7007 \sqrt{2} - 29120 \sqrt{3} + 3003 \sqrt{6} \right)$   |
| $w_{8,2}$     | $3 \left( 30272 - 14014 \sqrt{3} \right)$                                   |
| $w_{8,3}$     | $-3 \left( -62912 + 7007 \sqrt{2} + 29120 \sqrt{3} + 3003 \sqrt{6} \right)$ |
| $w_{8,4}$     | $66 \left( 1376 + 637 \sqrt{3} \right)$                                     |
| $w_{8,5}$     | $96 \left( 1108 + 1001 \sqrt{2} \right)$                                    |
| $w_{8,6}$     | 284178                                                                      |
| $w_{8,7}$     | $3 \left( 62912 - 7007 \sqrt{2} + 29120 \sqrt{3} + 3003 \sqrt{6} \right)$   |
| $w_{8,8}$     | 322944                                                                      |
| $w_{8,9}$     | $3 \left( 62912 + 7007 \sqrt{2} + 29120 \sqrt{3} - 3003 \sqrt{6} \right)$   |
| $w_{8,10}$    | 152046                                                                      |
| $w_{8,11}$    | $106368 - 96096 \sqrt{2}$                                                   |
| $w_{8,12}$    | 6720                                                                        |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                               |
|---------------|--------------------------------------------------------------------------|
| $d_9$         | $\frac{1}{4324320}$                                                      |
| $w_{9,0}$     | $22575 - 3780\sqrt{2}$                                                   |
| $w_{9,1}$     | $2 \left( 86144 + 57457\sqrt{2} - 29120\sqrt{3} - 19474\sqrt{6} \right)$ |
| $w_{9,2}$     | $-176 \left( -808 + 79\sqrt{2} + 546\sqrt{3} \right)$                    |
| $w_{9,3}$     | $2 \left( 86144 + 39439\sqrt{2} - 29120\sqrt{3} - 31486\sqrt{6} \right)$ |
| $w_{9,4}$     | $176 \left( 808 - 79\sqrt{2} + 546\sqrt{3} \right)$                      |
| $w_{9,5}$     | $199744 + 137651\sqrt{2}$                                                |
| $w_{9,6}$     | $233364 + 186192\sqrt{2}$                                                |
| $w_{9,7}$     | $2 \left( 86144 + 39439\sqrt{2} + 29120\sqrt{3} + 31486\sqrt{6} \right)$ |
| $w_{9,8}$     | $283232 + 207304\sqrt{2}$                                                |
| $w_{9,9}$     | $2 \left( 86144 + 57457\sqrt{2} + 29120\sqrt{3} + 19474\sqrt{6} \right)$ |
| $w_{9,10}$    | $257388 + 186192\sqrt{2}$                                                |
| $w_{9,11}$    | $199744 + 11525\sqrt{2}$                                                 |
| $w_{9,12}$    | $-105 \left( 71 + 36\sqrt{2} \right)$                                    |

*(continues)*

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                       |
|---------------|------------------------------------------------------------------|
| $d_{10}$      | $\frac{1}{69189120}$                                             |
| $w_{10,0}$    | $60480 \left(2 + \sqrt{3}\right)$                                |
| $w_{10,1}$    | $397888 + 519519 \sqrt{2} + 272576 \sqrt{3} + 291291 \sqrt{6}$   |
| $w_{10,2}$    | $22 \left(103424 + 8019 \sqrt{3}\right)$                         |
| $w_{10,3}$    | $397888 - 519519 \sqrt{2} + 272576 \sqrt{3} - 291291 \sqrt{6}$   |
| $w_{10,4}$    | $22 \left(103424 + 58797 \sqrt{3}\right)$                        |
| $w_{10,5}$    | $32 \left(99872 - 3003 \sqrt{2} + 60804 \sqrt{3}\right)$         |
| $w_{10,6}$    | $54 \left(73705 + 40928 \sqrt{3}\right)$                         |
| $w_{10,7}$    | $5115328 - 519519 \sqrt{2} + 2136256 \sqrt{3} + 291291 \sqrt{6}$ |
| $w_{10,8}$    | $128 \left(35404 + 20133 \sqrt{3}\right)$                        |
| $w_{10,9}$    | $5115328 + 519519 \sqrt{2} + 2136256 \sqrt{3} - 291291 \sqrt{6}$ |
| $w_{10,10}$   | $54 \left(71703 + 40928 \sqrt{3}\right)$                         |
| $w_{10,11}$   | $32 \left(99872 + 3003 \sqrt{2} + 60804 \sqrt{3}\right)$         |
| $w_{10,12}$   | $60480 \left(2 + \sqrt{3}\right)$                                |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                    |
|---------------|-------------------------------------------------------------------------------|
| $d_{11}$      | $\frac{1}{69189120}$                                                          |
| $w_{11,0}$    | $-1680 \left( -215 + 18\sqrt{2} + 18\sqrt{6} \right)$                         |
| $w_{11,1}$    | $-997421\sqrt{2} + 1040303\sqrt{6} - 2048 \left( -1346 + 455\sqrt{3} \right)$ |
| $w_{11,2}$    | $22 \left( 162392 + 38624\sqrt{2} - 33033\sqrt{3} + 20736\sqrt{6} \right)$    |
| $w_{11,3}$    | $101677\sqrt{2} + 97361\sqrt{6} - 2048 \left( -1346 + 455\sqrt{3} \right)$    |
| $w_{11,4}$    | $22 \left( 44456 + 38624\sqrt{2} + 33033\sqrt{3} + 20736\sqrt{6} \right)$     |
| $w_{11,5}$    | $16 \left( 199744 + 130387\sqrt{2} + 6120\sqrt{6} \right)$                    |
| $w_{11,6}$    | $6 \left( 1027709 + 248256\sqrt{2} - 216216\sqrt{3} + 124704\sqrt{6} \right)$ |
| $w_{11,7}$    | $1202413\sqrt{2} + 1086895\sqrt{6} + 2048 \left( 1346 + 455\sqrt{3} \right)$  |
| $w_{11,8}$    | $64 \left( 70808 - 4117\sqrt{2} + 31347\sqrt{6} \right)$                      |
| $w_{11,9}$    | $1833043\sqrt{2} + 720529\sqrt{6} + 2048 \left( 1346 + 455\sqrt{3} \right)$   |
| $w_{11,10}$   | $6 \left( 280963 + 248256\sqrt{2} + 216216\sqrt{3} + 124704\sqrt{6} \right)$  |
| $w_{11,11}$   | $16 \left( 199744 + 4261\sqrt{2} + 78192\sqrt{6} \right)$                     |
| $w_{11,12}$   | $-1680 \left( 71 + 18\sqrt{2} + 18\sqrt{6} \right)$                           |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                             |
|---------------|----------------------------------------|
| $d_{12}$      | $\frac{1}{270270}$                     |
| $w_{12,0}$    | 945                                    |
| $w_{12,1}$    | $21536 - 7280\sqrt{3}$                 |
| $w_{12,2}$    | 17776                                  |
| $w_{12,3}$    | $21536 - 7280\sqrt{3}$                 |
| $w_{12,4}$    | 17776                                  |
| $w_{12,5}$    | 24968                                  |
| $w_{12,6}$    | 30672                                  |
| $w_{12,7}$    | $16 \left( 1346 + 455\sqrt{3} \right)$ |
| $w_{12,8}$    | 35404                                  |
| $w_{12,9}$    | $16 \left( 1346 + 455\sqrt{3} \right)$ |
| $w_{12,10}$   | 30672                                  |
| $w_{12,11}$   | 24968                                  |
| $w_{12,12}$   | 945                                    |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$  for  $n = 14$ .

| $d_j, w_{jk}$      | Expression                                                                                                                                                                     |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_1$              | $\frac{1}{10090080}$                                                                                                                                                           |
| $w_{1,0}$          | $132 (293 + 98 c_6)$                                                                                                                                                           |
| $w_{1,1}$          | $143 c_1 (-64 + 119 c_2) + 7 (-28 (-1848 + 208 c_3 + 429 c_4 + 1200 c_5) + 13 (11 c_2 (-21 + 22 c_3) + 6 c_4 (61 c_3 + 115 c_5))) + 66 (-3027 + 2765 c_5) c_6$                 |
| $w_{1,2}$          | $-105105 c_1^2 + 286 c_3 (-817 + 392 c_2 + 294 c_5) + 294 c_1 (344 + 429 c_3 + 520 c_4 - 1320 c_6) + 49 (26 c_5 (-32 + 88 c_2 - 120 c_4 + 165 c_5) - 231 (-33 + 32 c_6))$      |
| $w_{1,3}$          | $-2002 c_2 (42 + 43 c_3 - 91 c_5) + 1568 (231 + 26 c_5) + 390 c_4 (-407 + 161 c_5) - 105 c_1 (2240 + 832 c_4 - 3025 c_6) - 379533 c_6 + 286 c_3 (32 + 189 c_6)$                |
| $w_{1,4}$          | $91 c_1 (2819 + 1232 c_2 + 231 c_3 - 1680 c_4 - 2310 c_5) + 21 c_3 (-13539 + 6006 c_3 - 7280 c_4 + 18480 c_6) - 11 (26 (593 + 392 c_2 - 735 c_5) c_5 + 21 (-1487 + 1568 c_6))$ |
| $w_{1,5}$          | $143 c_1 (-64 + 329 c_2) - 1001 c_2 (-21 + 86 c_3) + 98 (3696 - 416 c_3 + 858 c_4 + 1443 c_3 c_4) - 210 (1120 + 1131 c_4) c_5 + 66 (-7837 + 7315 c_5) c_6$                     |
| $w_{1,6}$          | $637 c_1 (-229 + 176 c_2 - 33 c_3 - 240 c_4 + 330 c_5) - 147 c_3 (1403 + 858 c_3 + 1040 c_4 - 2640 c_6) - 11 (26 c_5 (-817 + 392 c_2 + 735 c_5) + 1029 (-33 + 32 c_6))$        |
| $w_{1,7}$          | $9152 c_3 + 390 c_4 (407 - 609 c_5) + 2002 c_2 (42 + 11 c_3 - 59 c_5) + 1568 (231 + 26 c_5) - 105 c_1 (2240 + 832 c_4 - 3311 c_6) - 693 (487 + 78 c_3) c_6$                    |
| <i>(continues)</i> |                                                                                                                                                                                |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                     |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,8}$     | $169598 c_3 + 7 (21632 c_5 + 33 (1487 - 1568 c_6))$<br>$+ 7 (15015 c_1^2 + 182 (88 c_2 (c_3 + c_5) - 3 c_5 (22 c_3 + 40 c_4 + 55 c_5))$<br>$- 6 c_1 (-9272 + 3003 c_3 - 3640 c_4 + 9240 c_6))$ |
| $w_{1,9}$     | $362208 + 235200 c_3 + 91 c_1 (448 + 352 c_2 - 1125 c_4)$<br>$- 21021 c_4 + 141414 c_3 c_4 - 9152 c_5$<br>$- 286 c_2 (-555 + 413 c_5) - 462 (594 + 603 c_3 + 325 c_5) c_6$                     |
| $w_{1,10}$    | $-143 c_1 (-899 + 784 c_2 + 2205 c_5)$<br>$+ 49 (2574 c_3^2 - 52 c_3 (-16 + 44 c_2 - 60 c_4 + 33 c_5)$<br>$+ 3 (2541 + 1403 c_5 + 1040 c_4 c_5 - 176 (14 + 15 c_5) c_6))$                      |
| $w_{1,11}$    | $-16 (-35362 + 4004 c_2 - 10920 c_4 + 43197 c_6)$                                                                                                                                              |
| $w_{1,12}$    | $-143 c_1 (451 + 784 c_2 - 2205 c_5)$<br>$- 7 (18018 c_3^2 + 52 c_3 (416 + 308 c_2 - 420 c_4 - 231 c_5)$<br>$- 3 (16357 + (13539 + 7280 c_4) c_5) + 3696 (14 + 15 c_5) c_6)$                   |
| $w_{1,13}$    | $362208 + 235200 c_3 + 91 c_1 (448 + 352 c_2 - 795 c_4)$<br>$+ 21021 c_4 + 33306 c_3 c_4 - 9152 c_5$<br>$+ 286 c_2 (-555 + 637 c_5) - 462 (958 + 837 c_3 - 325 c_5) c_6$                       |
| $w_{1,14}$    | $132 (-97 + 98 c_6)$                                                                                                                                                                           |
| $d_2$         | $\frac{1}{10090080}$                                                                                                                                                                           |
| $w_{2,0}$     | $-12936 (-1 + c_5)$                                                                                                                                                                            |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                                                                      |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,1}$            | $  \begin{aligned}  & -26 c_1 (-3608 + 4851 c_2 - 1568 c_3 - 4851 c_4 + 5880 c_5) \\  & + 49 (429 c_2 (-1 + 5 c_5) + 2 (3696 - 858 c_4 + 24 c_5 (-254 + 165 c_5) \\  & + 13 c_3 (-32 + 88 c_3 + 165 c_4 - 165 c_6) + 2145 c_6))  \end{aligned}  $               |
| $w_{2,2}$            | $  \begin{aligned}  & 7 (53361 + 4758 c_1^2 + c_1 (29463 - 50133 c_5) \\  & - 54895 c_5 + 286 c_3 (-91 + 59 c_3 + 59 c_5))  \end{aligned}  $                                                                                                                    |
| $w_{2,3}$            | $  \begin{aligned}  & -152880 c_1^2 + 42042 c_2 (-2 + 5 c_3) \\  & + 176 (2058 + 13 c_3 (-41 + 49 c_3)) \\  & + 210210 (-1 + c_3) c_4 - 784 (410 + 143 c_3) c_5 \\  & + 294 c_1 (429 c_2 + 800 (-1 + c_5) - 429 c_6) + 21021 (-1 + 5 c_5) c_6  \end{aligned}  $ |
| $w_{2,4}$            | $  \begin{aligned}  & -7 (-49071 + 40617 c_3 + 26 c_1 (-832 + 777 c_1 + 1778 c_3) \\  & + 70235 c_5 - 75691 c_3 c_5)  \end{aligned}  $                                                                                                                          |
| $w_{2,5}$            | $  \begin{aligned}  & 26 c_1 (3608 + 4851 c_2 + 1568 c_3 - 4851 c_4 - 5880 c_5) \\  & + 49 (-429 c_2 (-1 + 5 c_5) \\  & + 2 (3 (1232 + 286 c_4 + 8 c_5 (-254 + 165 c_5) - 715 c_6) \\  & + 13 c_3 (-32 + 88 c_3 - 165 c_4 + 165 c_6)))  \end{aligned}  $        |
| $w_{2,6}$            | $  -7 (-53361 + 26 c_1 (224 + 183 c_1 - 466 c_3) + 29463 c_3 + 23045 c_5 - 28501 c_3 c_5)  $                                                                                                                                                                    |
| $w_{2,7}$            | $  \begin{aligned}  & -152880 c_1^2 - 93808 c_3 - 294 c_1 (429 c_2 - 800 (-1 + c_5) - 429 c_6) \\  & + 49 (7392 + 4290 c_4 - 6560 c_5 + 429 c_6) \\  & - 7007 (-16 c_3^2 + 6 c_2 (-2 + 5 c_3) + 30 c_3 c_4 + 16 c_3 c_5 + 15 c_5 c_6)  \end{aligned}  $         |
| $w_{2,8}$            | $  \begin{aligned}  & 7 (49071 + 20202 c_1^2 + c_1 (40617 - 69867 c_5) \\  & - 31729 c_5 - 286 c_3 (-59 + 91 c_3 + 91 c_5))  \end{aligned}  $                                                                                                                   |
| ( <i>continues</i> ) |                                                                                                                                                                                                                                                                 |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                        |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,9}$     | $  \begin{aligned}  & -152880 c_1^2 + 235200 c_3 + 1274 c_1 (32 + 32 c_3 - 99 c_4 - 99 c_6) \\  & - 539 (390 c_2 (-1 + c_3) + 928 c_3 c_5 - 195 c_4 c_5 + 390 c_3 c_6) \\  & - 11 (1911 c_4 + 4 (6100 c_5 - 147 (56 + 13 c_6)))  \end{aligned}  $ |
| $w_{2,10}$    | $  \begin{aligned}  & -7 (26 c_3 (-224 + 649 c_3) + 26 c_1 (-1001 + 466 c_3 - 183 c_5) \\  & + 3 (-17787 + c_5 (6536 + 15125 c_5)))  \end{aligned}  $                                                                                             |
| $w_{2,11}$    | $16 (28927 + 19110 c_1 + 14014 c_3 - 46893 c_5)$                                                                                                                                                                                                  |
| $w_{2,12}$    | $  \begin{aligned}  & 7 (338 c_3 (-64 + 77 c_3) + 26 c_1 (-649 + 1778 c_3 + 777 c_5) \\  & - 3 (-16357 + c_5 (4248 + 16555 c_5)))  \end{aligned}  $                                                                                               |
| $w_{2,13}$    | $  \begin{aligned}  & -152880 c_1^2 + 235200 c_3 + 11 (32928 + 1911 c_4 - 24400 c_5 - 7644 c_6) \\  & + 1274 c_1 (32 + 32 c_3 + 99 c_4 + 99 c_6) \\  & + 539 (390 c_2 (-1 + c_3) - 928 c_3 c_5 - 195 c_4 c_5 + 390 c_3 c_6)  \end{aligned}  $     |
| $w_{2,14}$    | $-12936 (-1 + c_5)$                                                                                                                                                                                                                               |
| $d_3$         | $\frac{1}{10090080}$                                                                                                                                                                                                                              |
| $w_{3,0}$     | $132 (293 + 98 c_4)$                                                                                                                                                                                                                              |
| $w_{3,1}$     | $  \begin{aligned}  & -273 c_2 (77 + 265 c_3 - 122 c_5) \\  & - 14 (-25872 + 2912 c_3 + 31614 c_4 + 16800 c_5 - 27621 c_4 c_5) \\  & + 286 c_1 (-32 + 525 c_4 - 637 c_6) + 286 (555 + 112 c_3) c_6  \end{aligned}  $                              |
| $w_{3,2}$     | $  \begin{aligned}  & -128557 c_3 - 147 c_1 (-1403 + 1040 c_2 - 2145 c_3 + 2640 c_4 - 572 c_5) \\  & + 49 (7623 - 7392 c_4 + 26 c_5 (-32 + 120 c_2 + 99 c_5)) \\  & - 112112 (c_3 + c_5) c_6  \end{aligned}  $                                    |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                      |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,3}$     | $22 (16464 - 3822 c_2 + 416 c_3 - 23511 c_4)$ $+ 210 c_1 (-1120 + 1131 c_2 + 2299 c_4)$ $+ 1274 (32 + 111 c_2) c_5 + 1001 (-21 + 47 c_3 - 86 c_5) c_6$                                          |
| $w_{3,4}$     | $-21 (-16357 + 18544 c_3 + 17248 c_4) - 169598 c_5$ $+ 7 (-30030 c_1^2 + 52 c_1 (416 + 420 c_2 + 231 c_5 - 308 c_6))$ $+ 7 (3120 c_2 c_3 + 33 c_3 (65 c_3 + 240 c_4 - 78 c_5) + 2288 c_5 c_6))$ |
| $w_{3,5}$     | $-273 c_2 (-77 + 375 c_3 - 518 c_5)$ $- 14 (2912 c_3 + 297 c_4 (66 - 67 c_5) + 336 (-77 + 50 c_5))$ $- 286 c_1 (32 + 525 c_4 - 413 c_6) + 286 (-555 + 112 c_3) c_6$                             |
| $w_{3,6}$     | $210210 c_1^2 - 11319 (-33 + 32 c_4) + 233662 c_5$ $+ 2548 c_1 (-16 + 60 c_2 - 33 c_5 - 44 c_6) + 49 (-2145 c_3^2$ $+ 6 c_3 (-344 + 520 c_2 + 1320 c_4 + 429 c_5) + 2288 c_5 c_6)$              |
| $w_{3,7}$     | $22 (16464 + 3822 c_2 + 416 c_3 - 9081 c_4)$ $- 210 c_1 (1120 + 299 c_2 - 869 c_4)$ $+ 182 (224 + 183 c_2) c_5 + 1001 (21 + 17 c_3 + 22 c_5) c_6$                                               |
| $w_{3,8}$     | $-21 c_1 (-13539 + 7280 c_2 + 15015 c_3 + 18480 c_4 + 4004 c_5)$ $+ 7 (49071 - 51744 c_4 + 26 (832 + 840 c_2 - 693 c_5) c_5)$ $+ 143 (451 c_3 - 784 (c_3 + c_5) c_6)$                           |
| $w_{3,9}$     | $158730 c_2 + 235200 c_3 - 7 (12480 c_2 c_3 + 33 c_4 (1375 c_3 + 234 c_5)$ $+ 12298 c_5 c_6 + 26 c_1 (-224 + 345 c_2 + 1001 c_6))$ $+ 11 (-34503 c_4 + 4 (-208 c_5 + 147 (56 + 13 c_6)))$       |
| $w_{3,10}$    | $-210210 c_1^2 - 49 (3 (-2541 + 176 c_4 (14 + 15 c_5)$ $+ c_5 (-1403 + 1040 c_2 + 858 c_5))$ $+ 13 c_3 (-229 + 240 c_2 + 33 c_5 - 176 c_6))$ $- 286 c_1 (-817 + 735 c_3 - 392 c_6)$             |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                           |
|----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,11}$           | $-16 (-35362 + 10920 c_2 + 43197 c_4 - 4004 c_6)$                                                                                                                                                    |
| $w_{3,12}$           | $210210 c_1^2 - 7 (3 (-16357 + (-13539 + 7280 c_2 - 6006 c_5) c_5$<br>$+ 1232 c_4 (14 + 15 c_5)) + 13 c_3 (2819 + 1680 c_2 - 231 c_5 - 1232 c_6))$<br>$+ 286 c_1 (-593 + 735 c_3 + 392 c_6)$         |
| $w_{3,13}$           | $362208 + 235200 c_3 - 390 c_2 (407 + 224 c_3)$<br>$- 337491 c_4 - 347655 c_3 c_4 - 9152 c_5 + 54054 c_4 c_5$<br>$+ 2002 (-42 + 11 c_5) c_6 + 182 c_1 (224 + 1305 c_2 + 649 c_6)$                    |
| $w_{3,14}$           | $132 (-97 + 98 c_4)$                                                                                                                                                                                 |
| $d_4$                | $\frac{1}{1441440}$                                                                                                                                                                                  |
| $w_{4,0}$            | $-1848 (-1 + c_3)$                                                                                                                                                                                   |
| $w_{4,1}$            | $-7 (2288 c_1^2 + 429 c_2 (1 + 5 c_3 + 6 c_5)$<br>$- 286 c_1 (-8 + 8 c_3 - 15 c_4 + 15 c_6) - 2 (16 c_3 (-257 + 150 c_5)$<br>$+ 3 (1232 + 143 c_4 (-2 + 3 c_5) + 40 c_5 (-20 + 13 c_5) + 715 c_6)))$ |
| $w_{4,2}$            | $-7 (-7623 + 11341 c_3 + 26 (32 - 111 c_5) c_5 + c_1 (-4209 + 10813 c_3 + 6604 c_5))$                                                                                                                |
| $w_{4,3}$            | $14 (3696 - 2400 c_1 - 858 c_2 - 2145 c_1 c_2 - 2552 c_3 + 5104 c_1 c_3$<br>$- 2145 (1 + c_1) c_4 + 13 (32 + 32 c_1 + 99 c_2) c_5 + 1560 c_5^2)$<br>$- 3003 (1 + 5 c_3 + 6 c_5) c_6$                 |
| $w_{4,4}$            | $49071 + 16874 c_1^2 - 16874 c_5 - 52 c_1 (-416 + 233 c_5)$<br>$+ 3 c_3 (-29896 + 15125 c_3 + 1586 c_5)$                                                                                             |
| ( <i>continues</i> ) |                                                                                                                                                                                                      |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                      |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,5}$            | $-7 (2288 c_1^2 - 429 c_2 (1 + 5 c_3 + 6 c_5)$ $- 286 c_1 (-8 + 8 c_3 + 15 c_4 - 15 c_6) + 2 (-16 c_3 (-257 + 150 c_5)$ $+ 3 (-1232 + 40 (20 - 13 c_5) c_5 + 143 c_4 (-2 + 3 c_5) + 715 c_6)))$ |
| $w_{4,6}$            | $7 (7623 - 3718 c_1^2 + 3718 c_5 + 52 c_1 (-16 + 127 c_5)$ $+ 3 c_3 (-3944 + 2365 c_3 + 962 c_5))$                                                                                              |
| $w_{4,7}$            | $14 (3696 - 2552 c_3 + 2145 c_4 + 416 c_5 + 1560 c_5^2 - 429 c_2 (-2 + 3 c_5)$ $+ c_1 (-2400 + 2145 c_2 + 5104 c_3 + 2145 c_4 + 416 c_5))$ $+ 3003 (1 + 5 c_3 + 6 c_5) c_6$                     |
| $w_{4,8}$            | $49071 - 32197 c_3 + 26 c_5 (832 + 183 c_5) + c_1 (40617 - 28501 c_3 + 12116 c_5)$                                                                                                              |
| $w_{4,9}$            | $-7 (-4290 c_2 + 3 c_3 (864 + 2640 c_3 + 715 c_4)$ $+ 11 (39 c_4 + 4 (-168 + 52 c_5 - 39 c_6)) + 26 (88 c_1^2$ $- c_1 (32 + 165 c_2 + 32 c_5 + 165 c_6) + 3 c_5 (40 c_3 + 33 (c_4 + c_6))))$    |
| $w_{4,10}$           | $7 (7623 - 6791 c_3 + 3718 c_1 (1 + c_1 + c_3) + 4209 c_5 - 9981 c_3 c_5 - 2886 c_5^2)$                                                                                                         |
| $w_{4,11}$           | $112 (853 + 286 c_1 - 957 c_3 - 390 c_5)$                                                                                                                                                       |
| $w_{4,12}$           | $49071 - 70703 c_3 - 16874 c_1 (1 + c_1 + c_3) + 40617 c_5 - 50133 c_3 c_5 - 4758 c_5^2$                                                                                                        |
| $w_{4,13}$           | $-7 (4290 c_2 + 3 c_3 (864 + 2640 c_3 - 715 c_4)$ $+ 11 (-39 c_4 + 4 (-168 + 52 c_5 + 39 c_6)) + 26 (88 c_1^2$ $+ c_1 (165 c_2 - 32 (1 + c_5) + 165 c_6) + 3 c_5 (40 c_3 - 33 (c_4 + c_6))))$   |
| $w_{4,14}$           | $-1848 (-1 + c_3)$                                                                                                                                                                              |
| ( <i>continues</i> ) |                                                                                                                                                                                                 |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                 |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_5$         | $\frac{1}{10090080}$                                                                                                                                                                       |
| $w_{5,0}$     | $132 (293 + 98 c_2)$                                                                                                                                                                       |
| $w_{5,1}$     | $-286 c_1 (32 + 189 c_2 - 301 c_4) + 231 c_2 (-1643 + 1375 c_5)$<br>$- 98 (-3696 + 416 c_3 + 858 c_4 + 1859 c_3 c_4 + 2400 c_5)$<br>$+ 390 (407 + 161 c_3 + 224 c_5) c_6$                  |
| $w_{5,2}$     | $-362208 c_2 - 286 c_3 (817 + 735 c_3 - 392 c_4) + 49 (7623 - 2977 c_5)$<br>$- 49 (2574 c_1^2 + 26 c_5 (165 c_3 - 88 c_4 - 120 c_6)$<br>$+ 3 c_1 (-1403 + 2640 c_2 - 143 c_5 + 1040 c_6))$ |
| $w_{5,3}$     | $352 (1029 + 26 c_3) - 462 c_2 (958 + 325 c_3)$<br>$- 286 (555 + 637 c_3) c_4 + 2912 (14 + 11 c_4) c_5$<br>$+ 273 (-77 + 265 c_5) c_6 + 42 c_1 (-5600 + 9207 c_2 + 793 c_6)$               |
| $w_{5,4}$     | $343497 - 126126 c_1^2 - 284319 c_3 - 64493 c_5$<br>$+ 364 c_1 (416 + 231 c_3 + 308 c_4 + 420 c_6)$<br>$+ 49 (528 c_2 (-14 + 15 c_3) - 143 (45 c_3 + 16 c_4) c_5 + 3120 c_3 c_6)$          |
| $w_{5,5}$     | $286 c_1 (-32 + 189 c_2 - 77 c_4)$<br>$+ 588 (616 + 143 c_4 - 400 c_5) + 231 c_2 (-1461 + 1505 c_5)$<br>$+ 182 c_3 (-224 + 649 c_4 - 1305 c_6) + 390 (-407 + 224 c_5) c_6$                 |
| $w_{5,6}$     | $-147 (-2541 + 2464 c_2 + 1403 c_3) + 128557 c_5$<br>$+ 49 (2574 c_1^2 + 7920 c_2 c_3 - 52 c_1 (16 + 33 c_3 - 44 c_4 - 60 c_6)$<br>$+ 13 (495 c_3 c_5 - 176 c_4 c_5 + 240 c_3 c_6))$       |
| $w_{5,7}$     | $22 (16464 + 21 c_2 (-594 + 325 c_3) + 7215 c_4 + 13 c_3 (32 + 413 c_4))$<br>$+ 2912 (14 + 11 c_4) c_5 + 273 (77 + 375 c_5) c_6$<br>$+ 42 c_1 (-5600 + 6633 c_2 + 3367 c_6)$               |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                             |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,8}$     | $343497 - 362208 c_2 + 286 c_3 (593 + 735 c_3 + 392 c_4)$ $+ 256529 c_5 + 7 (18018 c_1^2 + 182 c_5 (165 c_3 + 88 c_4 + 120 c_6))$ $- 3 c_1 (-13539 + 18480 c_2 + 1001 c_5 + 7280 c_6)$ |
| $w_{5,9}$     | $-66 c_2 (3027 + 2765 c_3) - 182 c_1 (-224 + 121 c_4 - 183 c_6)$ $+ 210 c_3 (1120 + 299 c_6)$ $+ 11 (91 c_4 (-21 + 17 c_5) + 4 (8232 - 208 c_5 + 1911 c_6))$                           |
| $w_{5,10}$    | $286 c_1 (817 + 294 c_3 - 392 c_4 - 441 c_5)$ $+ 49 (7623 + 26 c_3 (32 + 165 c_3 - 88 c_4) + 2064 c_5$ $- 2145 c_5^2 - 528 c_2 (14 + 15 c_5) - 3120 (c_3 + c_5) c_6)$                  |
| $w_{5,11}$    | $-16 (-35362 + 43197 c_2 + 4004 c_4 + 10920 c_6)$                                                                                                                                      |
| $w_{5,12}$    | $-286 c_1 (593 + 294 c_3 + 392 c_4 - 441 c_5)$ $- 7 (26 c_3 (832 + 1155 c_3 + 616 c_4) + 3696 c_2 (14 + 15 c_5)$ $- 3 (16357 + c_5 (18544 + 5005 c_5)) + 21840 (c_3 + c_5) c_6)$       |
| $w_{5,13}$    | $-66 c_2 (7837 + 7315 c_3) + 210 c_3 (1120 - 1131 c_6)$ $+ 182 c_1 (224 + 473 c_4 + 777 c_6)$ $+ 11 (91 c_4 (21 + 47 c_5) - 4 (-8232 + 208 c_5 + 1911 c_6))$                           |
| $w_{5,14}$    | $132 (-97 + 98 c_2)$                                                                                                                                                                   |
| $d_6$         | $\frac{1}{10090080}$                                                                                                                                                                   |
| $w_{6,0}$     | $-12936 (-1 + c_1)$                                                                                                                                                                    |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                              |
|----------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,1}$            | $11 c_1 (-24400 + 9555 c_2 + 45472 c_5)$ $+ 49 (429 c_2 (-1 + 6 c_3) - 2 (1560 c_3^2 + 13 c_3 (32 + 99 c_4 - 32 c_5))$ $+ 429 c_4 (2 + 5 c_5) + 48 (-77 + 50 c_5)) + 4290 (1 + c_5) c_6)$               |
| $w_{6,2}$            | $-7 (-53361 + 45375 c_1^2 + 6 c_1 (3268 + 793 c_3)$ $+ 26 c_3 (1001 + 466 c_5) + 26 c_5 (224 + 649 c_5))$                                                                                               |
| $w_{6,3}$            | $388080 c_1^2 - 93808 c_3 - 98 (-3696 + 2145 c_4 - 416 c_5)$ $- 1274 ((-32 c_3 + 165 c_4 - 88 c_5) c_5 + 33 c_2 (2 + 3 c_3 + 5 c_5))$ $+ 21021 (-1 + 6 c_3) c_6 + 147 c_1 (-4064 + 1040 c_3 + 715 c_6)$ |
| $w_{6,4}$            | $7 (3 c_3 (-13539 + 6734 c_3) + c_1 (-31729 + 69867 c_3 + 26026 c_5)$ $- 11 (-4461 + 26 c_5 (59 + 91 c_5)))$                                                                                            |
| $w_{6,5}$            | $-11 c_1 (24400 + 9555 c_2 - 45472 c_5) + 49 (-429 c_2 (-1 + 6 c_3)$ $+ 2 (-1560 c_3^2 + 13 c_3 (99 c_4 + 32 (-1 + c_5))$ $+ 3 (1232 - 800 c_5 + 143 c_4 (2 + 5 c_5) - 715 (1 + c_5) c_6)))$            |
| $w_{6,6}$            | $7 (183 c_3 (-161 + 26 c_3) + c_1 (-54895 + 50133 c_3 - 16874 c_5)$ $+ 11 (4851 + 26 c_5 (91 + 59 c_5)))$                                                                                               |
| $w_{6,7}$            | $388080 c_1^2 - 93808 c_3 + 98 (3696 + 2145 c_4 + 416 c_5)$ $+ 1274 (33 c_2 (2 + 3 c_3 + 5 c_5) + c_5 (32 c_3 + 165 c_4 + 88 c_5))$ $+ 147 c_1 (-4064 + 1040 c_3 - 715 c_6) - 21021 (-1 + 6 c_3) c_6$   |
| $w_{6,8}$            | $-7 (-49071 + 49665 c_1^2 + 6 c_1 (2124 + 3367 c_3)$ $- 338 c_5 (64 + 77 c_5) - 26 c_3 (649 + 1778 c_5))$                                                                                               |
| ( <i>continues</i> ) |                                                                                                                                                                                                         |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                                                                  |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,9}$            | $  \begin{aligned}  & -49 c_1 (6560 + 4800 c_3 - 2145 c_4 - 2288 c_5) \\  & + 294 c_3 (800 - 520 c_3 + 429 c_4 + 429 c_6) \\  & + 14014 (15 c_2 (1 + c_5) + c_5 (8 c_5 + 15 c_6)) \\  & + 11 (-1911 c_4 + 4 (8232 + 2132 c_5 + 1911 c_6))  \end{aligned}  $ |
| $w_{6,10}$           | $  \begin{aligned}  & -7 (26 c_3 (-224 + 183 c_3 - 466 c_5) \\  & - 21 (2541 + 1403 c_5) + 11 c_1 (2095 + 2591 c_5))  \end{aligned}  $                                                                                                                      |
| $w_{6,11}$           | $-16 (-28927 + 46893 c_1 + 19110 c_3 + 14014 c_5)$                                                                                                                                                                                                          |
| $w_{6,12}$           | $  \begin{aligned}  & -7 (26 c_3 (832 + 777 c_3 + 1778 c_5) \\  & + 11 c_1 (6385 + 6881 c_5) - 3 (16357 + 13539 c_5))  \end{aligned}  $                                                                                                                     |
| $w_{6,13}$           | $  \begin{aligned}  & -49 c_1 (6560 + 4800 c_3 + 2145 c_4 - 2288 c_5) - 210210 c_2 (1 + c_5) \\  & + 11 (1911 c_4 + 4 (52 c_5 (41 + 49 c_5) + 147 (56 - 13 c_6))) \\  & - 294 (c_3 (-800 + 520 c_3 + 429 c_4) + 143 (3 c_3 + 5 c_5) c_6)  \end{aligned}  $  |
| $w_{6,14}$           | $-12936 (-1 + c_1)$                                                                                                                                                                                                                                         |
| $d_7$                | $\frac{1}{2522520}$                                                                                                                                                                                                                                         |
| $w_{7,0}$            | $9669$                                                                                                                                                                                                                                                      |
| $w_{7,1}$            | $-4 (572 c_1 - 3003 c_2 + 2548 c_3 + 42 (-539 + 429 c_4 + 350 c_5) - 19305 c_6)$                                                                                                                                                                            |
| $w_{7,2}$            | $109368 c_1 - 110968 c_3 - 3626 (-33 + 26 c_5)$                                                                                                                                                                                                             |
| $w_{7,3}$            | $-4 (14700 c_1 + 18018 c_2 - 572 c_3 + 19305 c_4 - 7 (3234 + 364 c_5 + 429 c_6))$                                                                                                                                                                           |
| ( <i>continues</i> ) |                                                                                                                                                                                                                                                             |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                        |
|---------------|-----------------------------------------------------------------------------------|
| $w_{7,4}$     | $59598 + 121940 c_1 - 13272 c_3 - 94952 c_5$                                      |
| $w_{7,5}$     | $-4 (572 c_1 + 7 (-3234 + 429 c_2 + 364 c_3 - 2574 c_4 + 2100 c_5) + 19305 c_6)$  |
| $w_{7,6}$     | $-98 (-1221 + 962 c_1 + 1116 c_3) + 110968 c_5$                                   |
| $w_{7,7}$     | $4 (22638 - 14700 c_1 + 18018 c_2 + 572 c_3 + 19305 c_4 + 2548 c_5 - 3003 c_6)$   |
| $w_{7,8}$     | $59598 + 13272 c_1 + 94952 c_3 + 121940 c_5$                                      |
| $w_{7,9}$     | $4 (2548 c_1 + 19305 c_2 + 14700 c_3 + 11 (2058 + 273 c_4 - 52 c_5 + 1638 c_6))$  |
| $w_{7,10}$    | $110968 c_1 + 98 (1221 + 962 c_3 + 1116 c_5)$                                     |
| $w_{7,11}$    | $141448$                                                                          |
| $w_{7,12}$    | $-94952 c_1 - 121940 c_3 + 42 (1419 + 316 c_5)$                                   |
| $w_{7,13}$    | $4 (2548 c_1 - 19305 c_2 + 14700 c_3 - 11 (-2058 + 273 c_4 + 52 c_5 + 1638 c_6))$ |
| $w_{7,14}$    | $-3201$                                                                           |
| $d_8$         | $\frac{1}{1441440}$                                                               |
| $w_{8,0}$     | $1848 (1 + c_1)$                                                                  |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                   |
|----------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,1}$            | $7 (429 c_2 (-1 + 6 c_3) + 11 c_1 (195 c_2 + 464 (1 - 2 c_5))$ $+ 2 (3696 + 1560 c_3^2 - 2400 c_5 - 429 c_4 (2 + 5 c_5)$ $- 13 c_3 (99 c_4 + 32 (1 + c_5)) + 2145 (1 + c_5) c_6))$           |
| $w_{8,2}$            | $7 (7623 + 7095 c_1^2 + 6 c_1 (1972 + 481 c_3)$ $- 26 c_5 (32 + 143 c_5) - 26 c_3 (143 + 254 c_5))$                                                                                          |
| $w_{8,3}$            | $-14 (3960 c_1^2 + 24 c_1 (-54 + 65 c_3) - 11 (336 + 104 c_3 - 195 c_4)$ $+ 13 (-32 + 32 c_3 + 165 c_4) c_5 + 1144 c_5^2 + 429 c_2 (2 + 3 c_3 + 5 c_5))$ $+ 3003 (-1 + 5 c_1 + 6 c_3) c_6$   |
| $w_{8,4}$            | $-3 c_3 (13539 + 1586 c_3) + c_1 (70703 - 50133 c_3 + 16874 c_5)$ $- 11 (-4461 + 1534 c_5 (1 + c_5))$                                                                                        |
| $w_{8,5}$            | $-7 (429 c_2 (-1 + 6 c_3) + 11 c_1 (195 c_2 + 464 (-1 + 2 c_5))$ $+ 2 (-1560 c_3^2 + c_3 (-1287 c_4 + 416 (1 + c_5))$ $+ 3 (-1232 + 800 c_5 - 143 c_4 (2 + 5 c_5) + 715 (1 + c_5) c_6)))$    |
| $w_{8,6}$            | $-7 (3 c_3 (1403 + 962 c_3) + c_1 (-6791 + 9981 c_3 + 3718 c_5)$ $- 11 (693 + 338 c_5 (1 + c_5)))$                                                                                           |
| $w_{8,7}$            | $-7 (7920 c_1^2 - 22 (336 + 104 c_3 + 39 c_2 (2 + 3 c_3) + 195 c_4)$ $- 26 (32 + 165 c_2 - 32 c_3 + 165 c_4) c_5 + 2288 c_5^2$ $+ 429 (-1 + 6 c_3) c_6 + c_1 (-2592 + 3120 c_3 + 2145 c_6))$ |
| $w_{8,8}$            | $49071 + 16874 c_3 + 3 c_1 (29896 + 15125 c_1 + 1586 c_3)$ $+ 21632 c_5 + 12116 c_3 c_5 + 16874 c_5^2$                                                                                       |
| ( <i>continues</i> ) |                                                                                                                                                                                              |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,9}$            | $7 (6 c_3 (800 + 520 c_3 + 429 c_4) + c_1 (8224 + 4800 c_3 + 2145 c_4 - 2288 c_5) + 4290 c_2 (1 + c_5) + 858 (3 c_3 + 5 c_5) c_6 - 11 (39 c_4 + 4 (52 c_5 (1 + c_5) - 3 (56 + 13 c_6))))$ |
| $w_{8,10}$           | $7 (26 c_3 (32 + 111 c_3 + 254 c_5) + 11 c_1 (1031 + 983 c_5) + 3 (2541 + 1403 c_5))$                                                                                                     |
| $w_{8,11}$           | $112 (853 + 957 c_1 + 390 c_3 + 286 c_5)$                                                                                                                                                 |
| $w_{8,12}$           | $26 c_3 (-832 + 183 c_3 - 466 c_5) + 11 c_1 (2927 + 2591 c_5) + 3 (16357 + 13539 c_5)$                                                                                                    |
| $w_{8,13}$           | $7 (6 c_3 (800 + 520 c_3 - 429 c_4) + c_1 (8224 + 4800 c_3 - 2145 c_4 - 2288 c_5) - 4290 c_2 (1 + c_5) - 858 (3 c_3 + 5 c_5) c_6 + 11 (39 c_4 - 4 (-168 + 52 c_5 (1 + c_5) + 39 c_6)))$   |
| $w_{8,14}$           | $1848 (1 + c_1)$                                                                                                                                                                          |
| $d_9$                | $\frac{1}{10090080}$                                                                                                                                                                      |
| $w_{9,0}$            | $132 (293 - 98 c_2)$                                                                                                                                                                      |
| $w_{9,1}$            | $-286 c_1 (32 + 189 c_2 - 77 c_4) - 231 c_2 (-1461 + 1505 c_5) - 14 (13 c_3 (224 + 649 c_4) + 42 (-616 + 143 c_4 + 400 c_5)) + 390 (407 + 609 c_3 - 224 c_5) c_6$                         |
| $w_{9,2}$            | $362208 c_2 - 286 c_3 (817 + 735 c_3 + 392 c_4) + 49 (7623 - 2977 c_5) - 49 (2574 c_1^2 + 26 c_5 (165 c_3 + 88 c_4 + 120 c_6) - 3 c_1 (1403 + 2640 c_2 + 143 c_5 + 1040 c_6))$            |
| ( <i>continues</i> ) |                                                                                                                                                                                           |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                            |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,3}$     | $352 (1029 + 26 c_3) - 462 c_2 (-594 + 325 c_3)$ $- 286 (555 + 413 c_3) c_4 - 2912 (-14 + 11 c_4) c_5$ $- 273 (77 + 375 c_5) c_6 - 42 c_1 (5600 + 6633 c_2 + 3367 c_6)$               |
| $w_{9,4}$     | $343497 - 126126 c_1^2 - 25872 c_2 (-14 + 15 c_3)$ $- 64493 c_5 + 364 c_1 (416 + 231 c_3 - 308 c_4 - 420 c_6)$ $- 7 (-16016 c_4 c_5 + 3 c_3 (13539 + 15015 c_5 + 7280 c_6))$          |
| $w_{9,5}$     | $286 c_1 (-32 + 189 c_2 - 301 c_4)$ $+ 98 (3696 - 416 c_3 + 858 c_4 + 1859 c_3 c_4 - 2400 c_5)$ $- 231 c_2 (-1643 + 1375 c_5) - 390 (407 + 161 c_3 + 224 c_5) c_6$                    |
| $w_{9,6}$     | $147 (2541 + 2464 c_2 - 1403 c_3) + 128557 c_5$ $+ 49 (2574 c_1^2 - 7920 c_2 c_3 - 52 c_1 (16 + 33 c_3 + 44 c_4 + 60 c_6)$ $+ 13 (495 c_3 c_5 + 176 c_4 c_5 - 240 c_3 c_6))$          |
| $w_{9,7}$     | $22 (16464 + 416 c_3 + 21 c_2 (958 + 325 c_3) + 7215 c_4 + 8281 c_3 c_4)$ $- 2912 (-14 + 11 c_4) c_5 - 273 (-77 + 265 c_5) c_6$ $- 42 c_1 (5600 + 9207 c_2 + 793 c_6)$                |
| $w_{9,8}$     | $343497 + 362208 c_2 + 286 c_3 (593 + 735 c_3 - 392 c_4)$ $+ 256529 c_5 + 7 (18018 c_1^2 + 182 c_5 (165 c_3 - 88 c_4 - 120 c_6)$ $+ 3 c_1 (13539 + 18480 c_2 - 1001 c_5 + 7280 c_6))$ |
| $w_{9,9}$     | $66 c_2 (7837 + 7315 c_3) - 182 c_1 (-224 + 473 c_4 + 777 c_6)$ $+ 210 c_3 (1120 + 1131 c_6)$ $- 11 (91 c_4 (21 + 47 c_5) + 4 (208 c_5 - 147 (56 + 13 c_6)))$                         |
| $w_{9,10}$    | $286 c_1 (817 + 294 c_3 + 392 c_4 - 441 c_5)$ $+ 49 (7623 + 26 c_3 (32 + 165 c_3 + 88 c_4) + 2064 c_5$ $- 2145 c_5^2 + 528 c_2 (14 + 15 c_5) + 3120 (c_3 + c_5) c_6)$                 |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                 |
|----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,11}$           | $16 (35362 + 43197 c_2 + 4004 c_4 + 10920 c_6)$                                                                                                                                                            |
| $w_{9,12}$           | $-286 c_1 (593 + 294 c_3 - 392 c_4 - 441 c_5)$<br>$+ 7 (-26 c_3 (832 + 1155 c_3 - 616 c_4) + 3696 c_2 (14 + 15 c_5)$<br>$+ 3 (16357 + c_5 (18544 + 5005 c_5)) + 21840 (c_3 + c_5) c_6)$                    |
| $w_{9,13}$           | $66 c_2 (3027 + 2765 c_3) + 210 c_3 (1120 - 299 c_6)$<br>$+ 182 c_1 (224 + 121 c_4 - 183 c_6)$<br>$- 11 (91 c_4 (-21 + 17 c_5) + 4 (-8232 + 208 c_5 + 1911 c_6))$                                          |
| $w_{9,14}$           | $-132 (97 + 98 c_2)$                                                                                                                                                                                       |
| $d_{10}$             | $\frac{1}{10090080}$                                                                                                                                                                                       |
| $w_{10,0}$           | $12936 (1 + c_3)$                                                                                                                                                                                          |
| $w_{10,1}$           | $112112 c_1^2 - 286 c_1 (-328 + 392 c_3 + 735 c_4 - 735 c_6)$<br>$- 49 (429 c_2 (1 + 5 c_3 + 6 c_5) + 2 (-429 c_4 (-2 + 3 c_5)$<br>$+ 120 c_5 (20 + 13 c_5) + 80 c_3 (-41 + 30 c_5) - 33 (112 + 65 c_6)))$ |
| $w_{10,2}$           | $7 (53361 + 23045 c_3 + c_1 (29463 + 28501 c_3 - 12116 c_5) - 26 c_5 (224 + 183 c_5))$                                                                                                                     |
| $w_{10,3}$           | $-98 c_1 (2400 + 2145 c_2 + 5104 c_3 + 2145 c_4 + 416 c_5) + 2 (134200 c_3$<br>$+ 21021 c_2 (-2 + 3 c_5) - 49 (-3696 + 2145 c_4 - 416 c_5 + 1560 c_5^2))$<br>$- 21021 (1 + 5 c_3 + 6 c_5) c_6$             |
| $w_{10,4}$           | $7 (49071 + 21632 c_1 + 26026 c_1^2 + 12744 c_3$<br>$- 49665 c_3^2 - 26 (649 + 1778 c_1 + 777 c_3) c_5)$                                                                                                   |
| ( <i>continues</i> ) |                                                                                                                                                                                                            |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                    |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{10,5}$    | $112112 c_1^2 - 286 c_1 (-328 + 392 c_3 - 735 c_4 + 735 c_6)$<br>$+ 49 (429 c_2 (1 + 5 c_3 + 6 c_5) - 2 (80 c_3 (-41 + 30 c_5)$<br>$+ 3 (-1232 + 143 c_4 (-2 + 3 c_5) + 40 c_5 (20 + 13 c_5) + 715 c_6)))$    |
| $w_{10,6}$    | $7 (-16874 c_1^2 + 52 c_1 (-112 + 233 c_5)$<br>$+ 77 (693 + 338 c_5) - 3 c_3 (-6536 + 15125 c_3 + 1586 c_5))$                                                                                                 |
| $w_{10,7}$    | $98 c_1 (-2400 + 2145 c_2 - 5104 c_3 + 2145 c_4 - 416 c_5) + 2 (134200 c_3$<br>$- 21021 c_2 (-2 + 3 c_5) + 49 (3696 + 2145 c_4 + 104 (4 - 15 c_5) c_5))$<br>$+ 21021 (1 + 5 c_3 + 6 c_5) c_6$                 |
| $w_{10,8}$    | $7 (49071 + 70235 c_3 + 26 (832 - 777 c_5) c_5 + c_1 (40617 + 75691 c_3 + 46228 c_5))$                                                                                                                        |
| $w_{10,9}$    | $210210 c_2 + 147 c_3 (4064 + 2640 c_3 - 715 c_4) + 1274 (88 c_1^2$<br>$+ c_1 (32 + 165 c_2 - 32 c_5 + 165 c_6) + 3 c_5 (40 c_3 - 33 (c_4 + c_6)))$<br>$+ 11 (-1911 c_4 + 4 (2132 c_5 + 147 (56 + 13 c_6)))$  |
| $w_{10,10}$   | $7 (53361 + 54895 c_3 + 286 c_1 (91 + 59 c_1 + 59 c_3)$<br>$+ 29463 c_5 + 50133 c_3 c_5 + 4758 c_5^2)$                                                                                                        |
| $w_{10,11}$   | $16 (28927 - 14014 c_1 + 46893 c_3 + 19110 c_5)$                                                                                                                                                              |
| $w_{10,12}$   | $-7 (-31729 c_3 + 286 c_1 (59 + 91 c_1 + 91 c_3)$<br>$- 69867 c_3 c_5 - 20202 c_5^2 - 3 (16357 + 13539 c_5))$                                                                                                 |
| $w_{10,13}$   | $-210210 c_2 + 147 c_3 (4064 + 2640 c_3 + 715 c_4)$<br>$+ 11 (1911 c_4 + 4 (2132 c_5 + 147 (56 - 13 c_6))) + 1274 (88 c_1^2$<br>$- c_1 (-32 + 165 c_2 + 32 c_5 + 165 c_6) + 3 c_5 (40 c_3 + 33 (c_4 + c_6)))$ |
| $w_{10,14}$   | $12936 (1 + c_3)$                                                                                                                                                                                             |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                      |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_{11}$             | $\frac{1}{10090080}$                                                                                                                                                                            |
| $w_{11,0}$           | $132 (293 - 98 c_4)$                                                                                                                                                                            |
| $w_{11,1}$           | $273 c_2 (-77 + 375 c_3 - 518 c_5)$<br>$+ 286 c_1 (-32 + 525 c_4 - 413 c_6) - 2912 c_3 (14 + 11 c_6)$<br>$+ 6 (60368 + 693 c_4 (66 - 67 c_5) - 39200 c_5 + 26455 c_6)$                          |
| $w_{11,2}$           | $-128557 c_3 + 147 c_1 (1403 + 1040 c_2 + 2145 c_3 + 2640 c_4 + 572 c_5)$<br>$+ 49 (7623 + 7392 c_4 + 26 c_5 (-32 - 120 c_2 + 99 c_5))$<br>$+ 112112 (c_3 + c_5) c_6$                           |
| $w_{11,3}$           | $210 c_1 (-1120 + 299 c_2 - 869 c_4) + 22 (16464 + 416 c_3 + 9081 c_4)$<br>$+ 40768 c_5 - 546 c_2 (154 + 61 c_5) - 1001 (21 + 17 c_3 + 22 c_5) c_6$                                             |
| $w_{11,4}$           | $-210210 c_1^2 + 105105 c_3^2$<br>$- 42 c_3 (9272 + 3640 c_2 + 9240 c_4 + 3003 c_5)$<br>$- 364 c_1 (-416 + 420 c_2 - 231 c_5 - 308 c_6)$<br>$+ 11 (31227 + 32928 c_4 - 26 c_5 (593 + 392 c_6))$ |
| $w_{11,5}$           | $273 c_2 (77 + 265 c_3 - 122 c_5)$<br>$- 286 c_1 (32 + 525 c_4 - 637 c_6) - 2912 c_3 (14 + 11 c_6)$<br>$- 6 (-154 (392 + 479 c_4) + 7 (5600 + 9207 c_4) c_5 + 26455 c_6)$                       |
| $w_{11,6}$           | $210210 c_1^2 + 11319 (33 + 32 c_4) + 233662 c_5$<br>$- 2548 c_1 (16 + 60 c_2 + 33 c_5 - 44 c_6)$<br>$+ 49 (-3 c_3 (688 + 1040 c_2 + 715 c_3 + 2640 c_4 - 858 c_5) - 2288 c_5 c_6)$             |
| $w_{11,7}$           | $-210 c_1 (1120 + 1131 c_2 + 2299 c_4) + 22 (16464 + 416 c_3 + 23511 c_4)$<br>$+ 40768 c_5 - 3822 c_2 (-22 + 37 c_5) - 1001 (-21 + 47 c_3 - 86 c_5) c_6$                                        |
| ( <i>continues</i> ) |                                                                                                                                                                                                 |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                       |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{11,8}$    | $21 c_1 (13539 + 7280 c_2 - 15015 c_3 + 18480 c_4 - 4004 c_5)$<br>$+ 7 (49071 + 51744 c_4 - 26 c_5 (-832 + 840 c_2 + 693 c_5))$<br>$+ 143 (451 c_3 + 784 (c_3 + c_5) c_6)$                       |
| $w_{11,9}$    | $235200 c_3 + 390 c_2 (407 + 224 c_3) - 182 c_1 (-224 + 1305 c_2 + 649 c_6)$<br>$+ 77 (4515 c_3 c_4 - 26 c_5 (27 c_4 + 11 c_6))$<br>$+ 11 (30681 c_4 + 4 (8232 - 208 c_5 + 1911 c_6))$           |
| $w_{11,10}$   | $-210210 c_1^2$<br>$+ 49 (3 (2541 + (1403 + 1040 c_2 - 858 c_5) c_5 + 176 c_4 (14 + 15 c_5))$<br>$+ 13 c_3 (229 + 240 c_2 - 33 c_5 - 176 c_6))$<br>$- 286 c_1 (-817 + 735 c_3 + 392 c_6)$        |
| $w_{11,11}$   | $16 (35362 + 10920 c_2 + 43197 c_4 - 4004 c_6)$                                                                                                                                                  |
| $w_{11,12}$   | $210210 c_1^2$<br>$+ 7 (3 (16357 + 1232 c_4 (14 + 15 c_5) + c_5 (13539 + 7280 c_2 + 6006 c_5))$<br>$+ 13 c_3 (-2819 + 1680 c_2 + 231 c_5 - 1232 c_6))$<br>$+ 286 c_1 (-593 + 735 c_3 - 392 c_6)$ |
| $w_{11,13}$   | $362208 + 235200 c_3 + 390 c_2 (-407 + 224 c_3)$<br>$+ 379533 c_4 + 317625 c_3 c_4 - 9152 c_5 + 54054 c_4 c_5$<br>$+ 2002 (-42 + 43 c_5) c_6 + 182 c_1 (224 + 345 c_2 + 1001 c_6)$               |
| $w_{11,14}$   | $-132 (97 + 98 c_4)$                                                                                                                                                                             |
| $d_{12}$      | $\frac{1}{1441440}$                                                                                                                                                                              |
| $w_{12,0}$    | $1848 (1 + c_5)$                                                                                                                                                                                 |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                  |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{12,1}$    | $-7 (26 c_1 (88 + 99 c_2 + 32 c_3 - 99 c_4 - 120 c_5)$ $- 429 c_2 (-1 + 5 c_5) + 2 (13 c_3 (32 + 88 c_3 - 165 c_4 + 165 c_6)$ $+ 3 (286 c_4 + 24 c_5 (-18 + 55 c_5) - 11 (112 + 65 c_6))))$ |
| $w_{12,2}$    | $7 (7623 - 2886 c_1^2 + 6791 c_5 + 3718 c_3 (-1 + c_3 + c_5) + 3 c_1 (1403 + 3327 c_5))$                                                                                                    |
| $w_{12,3}$    | $14 (3696 + 1560 c_1^2 + 429 c_2 (-2 + 5 c_3) - 2145 c_4 + 4112 c_5$ $+ 143 c_3 (8 - 8 c_3 + 15 c_4 + 8 c_5) + 3 c_1 (429 c_2 - 800 (1 + c_5)))$ $- 3003 (1 + 6 c_1 - 5 c_5) c_6$           |
| $w_{12,4}$    | $49071 + 26 c_1 (832 + 183 c_1 - 466 c_3) - 40617 c_3 + 32197 c_5 - 28501 c_3 c_5$                                                                                                          |
| $w_{12,5}$    | $7 (-429 c_2 (-1 + 5 c_5) + 26 c_1 (-88 + 99 c_2 - 32 c_3 - 99 c_4 + 120 c_5)$ $+ 2 (3 (1232 + 286 c_4 + 24 (18 - 55 c_5) c_5 - 715 c_6)$ $- 13 c_3 (32 + 88 c_3 + 165 c_4 - 165 c_6)))$    |
| $w_{12,6}$    | $7 (7623 - 4209 c_3 + 26 c_1 (-32 + 111 c_1 + 254 c_3) + 11341 c_5 - 10813 c_3 c_5)$                                                                                                        |
| $w_{12,7}$    | $14 (3696 + 1560 c_1^2 - 429 c_2 (-2 + 5 c_3) + 2145 c_4 + 4112 c_5$ $+ 143 c_3 (8 - 8 c_3 - 15 c_4 + 8 c_5) - 3 c_1 (429 c_2 + 800 (1 + c_5)))$ $+ 3003 (1 + 6 c_1 - 5 c_5) c_6$           |
| $w_{12,8}$    | $49071 - 4758 c_1^2 + 70703 c_5 - 16874 c_3 (-1 + c_3 + c_5) + 3 c_1 (13539 + 16711 c_5)$                                                                                                   |
| $w_{12,9}$    | $7 (7392 + 3120 c_1^2 - 4290 c_2 (-1 + c_3) + 4800 c_3$ $- 429 c_4 + 5104 c_5 + 10208 c_3 c_5 + 2145 c_4 c_5$ $- 858 (-2 + 5 c_3) c_6 - 26 c_1 (-32 + 32 c_3 + 99 c_4 + 99 c_6))$           |
| $w_{12,10}$   | $-7 (26 c_3 (-32 + 143 c_3) + 26 c_1 (-143 + 254 c_3 + 111 c_5)$ $- 3 (2541 + c_5 (3944 + 2365 c_5)))$                                                                                      |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                 |
|----------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{12,11}$          | $-112 (-853 + 390 c_1 + 286 c_3 - 957 c_5)$                                                                                                                                                |
| $w_{12,12}$          | $26 c_3 (-832 + 649 c_3) + 26 c_1 (-649 + 466 c_3 - 183 c_5)$<br>$+ 3 (16357 + c_5 (29896 + 15125 c_5))$                                                                                   |
| $w_{12,13}$          | $7 (7392 + 3120 c_1^2 + 4290 c_2 (-1 + c_3) + 4800 c_3$<br>$+ 429 c_4 + 5104 c_5 + 10208 c_3 c_5 - 2145 c_4 c_5$<br>$- 26 c_1 (-32 + 32 c_3 - 99 c_4 - 99 c_6) + 858 (-2 + 5 c_3) c_6)$    |
| $w_{12,14}$          | $1848 (1 + c_5)$                                                                                                                                                                           |
| $d_{13}$             | $\frac{1}{10090080}$                                                                                                                                                                       |
| $w_{13,0}$           | $132 (293 - 98 c_6)$                                                                                                                                                                       |
| $w_{13,1}$           | $-143 c_1 (64 + 329 c_2) + 1001 c_2 (-21 + 86 c_3)$<br>$- 196 (-1848 + 208 c_3 + 429 c_4 + 1200 c_5) + 517242 c_6$<br>$- 42 (3367 c_3 c_4 + 5 c_5 (-1131 c_4 + 2299 c_6))$                 |
| $w_{13,2}$           | $-105105 c_1^2 - 286 c_3 (817 + 392 c_2 - 294 c_5)$<br>$+ 294 c_1 (344 + 429 c_3 - 520 c_4 + 1320 c_6)$<br>$+ 49 (26 c_5 (-32 - 88 c_2 + 120 c_4 + 165 c_5) + 231 (33 + 32 c_6))$          |
| $w_{13,3}$           | $-2002 c_2 (42 + 11 c_3 - 59 c_5) + 1568 (231 + 26 c_5)$<br>$+ 390 c_4 (-407 + 609 c_5) + 337491 c_6$<br>$+ 286 c_3 (32 + 189 c_6) + 105 c_1 (832 c_4 - 7 (320 + 473 c_6))$                |
| $w_{13,4}$           | $-91 c_1 (-2819 + 1232 c_2 - 231 c_3 - 1680 c_4 + 2310 c_5)$<br>$+ 21 c_3 (-13539 + 6006 c_3 + 7280 c_4 - 18480 c_6)$<br>$+ 11 (26 c_5 (-593 + 392 c_2 + 735 c_5) + 21 (1487 + 1568 c_6))$ |
| ( <i>continues</i> ) |                                                                                                                                                                                            |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                          |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{13,5}$    | $-143 c_1 (64 + 119 c_2)$<br>$- 196 (-1848 + 208 c_3 - 429 c_4 + 1200 c_5) + 199782 c_6$<br>$- 7 (143 c_2 (-21 + 22 c_3) + 78 c_4 (61 c_3 + 115 c_5) + 26070 c_5 c_6)$                              |
| $w_{13,6}$    | $-637 c_1 (229 + 176 c_2 + 33 c_3 - 240 c_4 - 330 c_5)$<br>$- 147 c_3 (1403 + 858 c_3 - 1040 c_4 + 2640 c_6)$<br>$+ 11 (26 (817 + 392 c_2 - 735 c_5) c_5 + 1029 (33 + 32 c_6))$                     |
| $w_{13,7}$    | $2002 c_2 (42 + 43 c_3 - 91 c_5) + 1568 (231 + 26 c_5)$<br>$- 390 c_4 (-407 + 161 c_5) + 105 c_1 (-2240 + 832 c_4 - 3025 c_6)$<br>$+ 379533 c_6 - 286 c_3 (-32 + 189 c_6)$                          |
| $w_{13,8}$    | $169598 c_3$<br>$+ 7 (15015 c_1^2 - 182 (88 c_2 (c_3 + c_5) + 3 c_5 (22 c_3 - 40 c_4 + 55 c_5))$<br>$- 6 c_1 (-9272 + 3003 c_3 + 3640 c_4 - 9240 c_6))$<br>$+ 7 (21632 c_5 + 33 (1487 + 1568 c_6))$ |
| $w_{13,9}$    | $362208 + 235200 c_3 - 91 c_1 (-448 + 352 c_2 - 795 c_4)$<br>$- 21021 c_4 - 33306 c_3 c_4 - 9152 c_5$<br>$- 286 c_2 (-555 + 637 c_5) + 462 (958 + 837 c_3 - 325 c_5) c_6$                           |
| $w_{13,10}$   | $143 c_1 (899 + 784 c_2 - 2205 c_5)$<br>$+ 49 (2574 c_3^2 + 52 c_3 (16 + 44 c_2 - 60 c_4 - 33 c_5)$<br>$+ 3 (2541 + 1403 c_5 - 1040 c_4 c_5 + 176 (14 + 15 c_5) c_6))$                              |
| $w_{13,11}$   | $16 (35362 + 4004 c_2 - 10920 c_4 + 43197 c_6)$                                                                                                                                                     |
| $w_{13,12}$   | $143 c_1 (-451 + 784 c_2 + 2205 c_5)$<br>$+ 7 (-18018 c_3^2 + 52 c_3 (-416 + 308 c_2 - 420 c_4 + 231 c_5)$<br>$+ 3 (16357 + 13539 c_5 - 7280 c_4 c_5 + 1232 (14 + 15 c_5) c_6))$                    |

(*continues*)

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                  |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{13,13}$          | $362208 + 235200 c_3 - 91 c_1 (-448 + 352 c_2 - 1125 c_4)$<br>$+ 21021 c_4 - 141414 c_3 c_4 - 9152 c_5$<br>$+ 286 c_2 (-555 + 413 c_5) + 462 (594 + 603 c_3 + 325 c_5) c_6$ |
| $w_{13,14}$          | $-132 (97 + 98 c_6)$                                                                                                                                                        |
| $d_{14}$             | $\frac{1}{630630}$                                                                                                                                                          |
| $w_{14,0}$           | 1617                                                                                                                                                                        |
| $w_{14,1}$           | $-4 (286 c_1 + 49 (-231 + 26 c_3 + 150 c_5))$                                                                                                                               |
| $w_{14,2}$           | $14 (3201 + 2190 c_1 - 286 c_3 + 494 c_5)$                                                                                                                                  |
| $w_{14,3}$           | $4 (-7350 c_1 + 286 c_3 + 49 (231 + 26 c_5))$                                                                                                                               |
| $w_{14,4}$           | $14 (3201 + 494 c_1 - 2190 c_3 + 286 c_5)$                                                                                                                                  |
| $w_{14,5}$           | $-4 (286 c_1 + 49 (-231 + 26 c_3 + 150 c_5))$                                                                                                                               |
| $w_{14,6}$           | $14 (3201 + 494 c_1 - 2190 c_3 + 286 c_5)$                                                                                                                                  |
| $w_{14,7}$           | $4 (-7350 c_1 + 286 c_3 + 49 (231 + 26 c_5))$                                                                                                                               |
| $w_{14,8}$           | $14 (3201 + 2190 c_1 - 286 c_3 + 494 c_5)$                                                                                                                                  |
| $w_{14,9}$           | $196 (231 + 26 c_1 + 150 c_3) - 1144 c_5$                                                                                                                                   |
| ( <i>continues</i> ) |                                                                                                                                                                             |

APPENDIX B. COEFFICIENTS FOR FIRST-ORDER METHODS

Table B.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                 |
|---------------|--------------------------------------------|
| $w_{14,10}$   | $14 (3201 + 286 c_1 - 494 c_3 + 2190 c_5)$ |
| $w_{14,11}$   | 70724                                      |
| $w_{14,12}$   | $14 (3201 + 286 c_1 - 494 c_3 + 2190 c_5)$ |
| $w_{14,13}$   | $196 (231 + 26 c_1 + 150 c_3) - 1144 c_5$  |
| $w_{14,14}$   | 1617                                       |

## Appendix C

# Coefficients for second-order methods

In the following tables, exact coefficients for a family of implicit Chebyshev methods for the numerical integration of second-order differential equations are given. These coefficients for even degrees  $2 \leq n \leq 14$  are used to recast Eq. (1.11), such that

$$y(x + h\xi_j) = 2y(x) + y(x - h\xi_j) + I_j, \quad (\text{C.1})$$

where

$$I_j = h^2 d_j \sum_{k=0}^n w_{jk} (f(x_k^+) + f(x_k^-)), \quad (\text{C.2})$$

where  $d_j$  represents a common denominator for each expression.

The constants  $c_\ell$  appearing in Table C.7 are defined in Appendix A, Table A.8.

Table C.1: Integral approximation  $I_j$  for  $n = 2$ .

| $d_j, w_{jk}$ | Expression     | $w_{jk}$  | Expression |
|---------------|----------------|-----------|------------|
| $d_1$         | $\frac{1}{96}$ | $w_{1,1}$ | 6          |
| $w_{1,0}$     | 14             | $w_{1,2}$ | -1         |
| $d_2$         | $\frac{1}{3}$  | $w_{2,1}$ | 1          |
| $w_{2,0}$     | 1              | $w_{2,2}$ | 0          |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.2: Integral approximation  $I_j$  for  $n = 4$ .

| $d_j, w_{jk}$ | Expression          | $w_{jk}$  | Expression          |
|---------------|---------------------|-----------|---------------------|
| $d_1$         | $\frac{1}{1920}$    | $w_{1,2}$ | $134 - 96\sqrt{2}$  |
| $w_{1,0}$     | $66 - 30\sqrt{2}$   | $w_{1,3}$ | $182 - 128\sqrt{2}$ |
| $w_{1,1}$     | 10                  | $w_{1,4}$ | $1 - \sqrt{2}$      |
| $d_2$         | $\frac{1}{480}$     | $w_{2,2}$ | 10                  |
| $w_{2,0}$     | 18                  | $w_{2,3}$ | $20 - 16\sqrt{2}$   |
| $w_{2,1}$     | $4(5 + 4\sqrt{2})$  | $w_{2,4}$ | 1                   |
| $d_3$         | $\frac{1}{1920}$    | $w_{3,2}$ | $134 + 96\sqrt{2}$  |
| $w_{3,0}$     | $6(11 + 5\sqrt{2})$ | $w_{3,3}$ | 10                  |
| $w_{3,1}$     | $182 + 128\sqrt{2}$ | $w_{3,4}$ | $1 + \sqrt{2}$      |
| $d_4$         | $\frac{1}{15}$      | $w_{4,2}$ | 3                   |
| $w_{4,0}$     | 1                   | $w_{4,3}$ | $2 - \sqrt{2}$      |
| $w_{4,1}$     | $2 + \sqrt{2}$      | $w_{4,4}$ | 0                   |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.3: Integral approximation  $I_j$  for  $n = 6$ .

| $d_j, w_{jk}$      | Expression              | $w_{jk}$  | Expression              |
|--------------------|-------------------------|-----------|-------------------------|
| $d_1$              | $\frac{1}{322560}$      | $w_{1,3}$ | $18206 - 10496\sqrt{3}$ |
| $w_{1,0}$          | $4706 - 2240\sqrt{3}$   | $w_{1,4}$ | $6710 - 3884\sqrt{3}$   |
| $w_{1,1}$          | 350                     | $w_{1,5}$ | $17750 - 10240\sqrt{3}$ |
| $w_{1,2}$          | $25142 - 14548\sqrt{3}$ | $w_{1,6}$ | $49 - 32\sqrt{3}$       |
| $d_2$              | $\frac{1}{322560}$      | $w_{2,3}$ | -418                    |
| $w_{2,0}$          | 2466                    | $w_{2,4}$ | 234                     |
| $w_{2,1}$          | $3530 + 2140\sqrt{3}$   | $w_{2,5}$ | $3530 - 2140\sqrt{3}$   |
| $w_{2,2}$          | 1890                    | $w_{2,6}$ | 81                      |
| $d_3$              | $\frac{1}{10080}$       | $w_{3,3}$ | 70                      |
| $w_{3,0}$          | 142                     | $w_{3,4}$ | -8                      |
| $w_{3,1}$          | $40(7 + 4\sqrt{3})$     | $w_{3,5}$ | $40(7 - 4\sqrt{3})$     |
| $w_{3,2}$          | 568                     | $w_{3,6}$ | -1                      |
| $d_4$              | $\frac{1}{35840}$       | $w_{4,3}$ | 2286                    |
| $w_{4,0}$          | 786                     | $w_{4,4}$ | 210                     |
| <i>(continues)</i> |                         |           |                         |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.3: Integral approximation  $I_j$ ,  $n = 6$ . (*continued*)

| $d_j, w_{jk}$ | Expression                           | $w_{jk}$  | Expression                            |
|---------------|--------------------------------------|-----------|---------------------------------------|
| $w_{4,1}$     | $90 \left( 17 + 10 \sqrt{3} \right)$ | $w_{4,5}$ | $3 \left( 510 - 300 \sqrt{3} \right)$ |
| $w_{4,2}$     | 4122                                 | $w_{4,6}$ | 9                                     |
| $d_5$         | $\frac{1}{322560}$                   | $w_{5,3}$ | $18206 + 10496 \sqrt{3}$              |
| $w_{5,0}$     | $4706 + 2240 \sqrt{3}$               | $w_{5,4}$ | $6710 + 3884 \sqrt{3}$                |
| $w_{5,1}$     | $17750 + 10240 \sqrt{3}$             | $w_{5,5}$ | 350                                   |
| $w_{5,2}$     | $25142 + 14548 \sqrt{3}$             | $w_{5,6}$ | $49 + 32 \sqrt{3}$                    |
| $d_6$         | $\frac{1}{315}$                      | $w_{6,3}$ | 41                                    |
| $w_{6,0}$     | 9                                    | $w_{6,4}$ | 18                                    |
| $w_{6,1}$     | $10 \left( 2 + \sqrt{3} \right)$     | $w_{6,5}$ | $-10 \left( -2 + \sqrt{3} \right)$    |
| $w_{6,2}$     | 54                                   | $w_{6,6}$ | 0                                     |

Table C.4: Integral approximation  $I_j$  for  $n = 8$ .

| $d_j, w_{jk}$ | Expression                                                    |
|---------------|---------------------------------------------------------------|
| $d_1$         | $\frac{1}{161280}$                                            |
| $w_{1,0}$     | $2 \left( 643 + 4 \sqrt{2} - 315 \sqrt{2 + \sqrt{2}} \right)$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                               |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,1}$     | $-6 \left( -49 + 28 \sqrt{2} + 128 \sqrt{2 - \sqrt{2}} + 128 \sqrt{2 + \sqrt{2}} - 128 \sqrt{2 (2 + \sqrt{2})} \right)$                  |
| $w_{1,2}$     | $3256 \sqrt{2} + 168 \sqrt{4 - 2 \sqrt{2}} - 11 \left( -416 + 256 \sqrt{2 + \sqrt{2}} + 145 \sqrt{2 (2 + \sqrt{2})} \right)$             |
| $w_{1,3}$     | $30 \sqrt{2} + 4 \left( 977 + 448 \sqrt{2 - \sqrt{2}} - 448 \sqrt{2 + \sqrt{2}} - 192 \sqrt{2 (2 + \sqrt{2})} \right)$                   |
| $w_{1,4}$     | $5146 + 1544 \sqrt{2} - 3968 \sqrt{2 + \sqrt{2}}$                                                                                        |
| $w_{1,5}$     | $4 \left( 1391 + 701 \sqrt{2} - 448 \sqrt{2 - \sqrt{2}} - 832 \sqrt{2 + \sqrt{2}} - 192 \sqrt{2 (2 + \sqrt{2})} \right)$                 |
| $w_{1,6}$     | $1595 \sqrt{2 (2 + \sqrt{2})} - 8 \left( -572 + 297 \sqrt{2} + 21 \sqrt{4 - 2 \sqrt{2}} + 352 \sqrt{2 + \sqrt{2}} \right)$               |
| $w_{1,7}$     | $2 \left( 2765 - 29 \sqrt{2} + 384 \sqrt{2 - \sqrt{2}} - 2176 \sqrt{2 + \sqrt{2}} + 384 \sqrt{2 (2 + \sqrt{2})} \right)$                 |
| $w_{1,8}$     | $3 + 4 \sqrt{2} - 5 \sqrt{2 + \sqrt{2}}$                                                                                                 |
| $d_2$         | $\frac{1}{40320}$                                                                                                                        |
| $w_{2,0}$     | $324 - 160 \sqrt{2}$                                                                                                                     |
| $w_{2,1}$     | $1360 - 853 \sqrt{2} + 159 \sqrt{4 - 2 \sqrt{2}} - 192 \sqrt{2 - \sqrt{2}}$<br>$+ 448 \sqrt{2 + \sqrt{2}} - 266 \sqrt{2 (2 + \sqrt{2})}$ |
| $w_{2,2}$     | 84                                                                                                                                       |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                     |
|----------------------|--------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,3}$            | $592 - 427\sqrt{2} - 266\sqrt{4 - 2\sqrt{2}} + 448\sqrt{2 - \sqrt{2}}$ $+ 192\sqrt{2 + \sqrt{2}} - 159\sqrt{2(2 + \sqrt{2})}$  |
| $w_{2,4}$            | $1412 - 992\sqrt{2}$                                                                                                           |
| $w_{2,5}$            | $592 - 427\sqrt{2} + 266\sqrt{4 - 2\sqrt{2}} - 448\sqrt{2 - \sqrt{2}}$ $- 192\sqrt{2 + \sqrt{2}} + 159\sqrt{2(2 + \sqrt{2})}$  |
| $w_{2,6}$            | $1996 - 1408\sqrt{2}$                                                                                                          |
| $w_{2,7}$            | $1360 - 853\sqrt{2} - 159\sqrt{4 - 2\sqrt{2}} + 192\sqrt{2 - \sqrt{2}}$ $- 448\sqrt{2 + \sqrt{2}} + 266\sqrt{2(2 + \sqrt{2})}$ |
| $w_{2,8}$            | 2                                                                                                                              |
| $d_3$                | $\frac{1}{161280}$                                                                                                             |
| $w_{3,0}$            | $-2 \left( -643 + 4\sqrt{2} + 315\sqrt{2 - \sqrt{2}} \right)$                                                                  |
| $w_{3,1}$            | $4 \left( 1391 - 701\sqrt{2} + 192\sqrt{4 - 2\sqrt{2}} - 832\sqrt{2 - \sqrt{2}} + 448\sqrt{2 + \sqrt{2}} \right)$              |
| $w_{3,2}$            | $-168\sqrt{2(2 + \sqrt{2})} + 11 \left( 416 + 216\sqrt{2} - 145\sqrt{4 - 2\sqrt{2}} - 256\sqrt{2 - \sqrt{2}} \right)$          |
| $w_{3,3}$            | $6 \left( 49 + 28\sqrt{2} - 128\sqrt{4 - 2\sqrt{2}} - 128\sqrt{2 - \sqrt{2}} + 128\sqrt{2 + \sqrt{2}} \right)$                 |
| ( <i>continues</i> ) |                                                                                                                                |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                        |
|----------------------|-------------------------------------------------------------------------------------------------------------------|
| $w_{3,4}$            | $5146 - 1544\sqrt{2} - 3968\sqrt{2 - \sqrt{2}}$                                                                   |
| $w_{3,5}$            | $2 \left( 2765 + 29\sqrt{2} - 384\sqrt{4 - 2\sqrt{2}} - 2176\sqrt{2 - \sqrt{2}} - 384\sqrt{2 + \sqrt{2}} \right)$ |
| $w_{3,6}$            | $4576 - 3256\sqrt{2} + 1595\sqrt{4 - 2\sqrt{2}} - 2816\sqrt{2 - \sqrt{2}} + 168\sqrt{2(2 + \sqrt{2})}$            |
| $w_{3,7}$            | $-30\sqrt{2} + 4 \left( 977 + 192\sqrt{4 - 2\sqrt{2}} - 448\sqrt{2 - \sqrt{2}} - 448\sqrt{2 + \sqrt{2}} \right)$  |
| $w_{3,8}$            | $3 - 4\sqrt{2} - 5\sqrt{2 - \sqrt{2}}$                                                                            |
| $d_4$                | $\frac{1}{10080}$                                                                                                 |
| $w_{4,0}$            | 82                                                                                                                |
| $w_{4,1}$            | $-48\sqrt{2 - \sqrt{2}} + 56 \left( 3 + 2\sqrt{2 + \sqrt{2}} \right)$                                             |
| $w_{4,2}$            | $4 \left( 63 + 44\sqrt{2} \right)$                                                                                |
| $w_{4,3}$            | $8 \left( 21 + 14\sqrt{2 - \sqrt{2}} + 6\sqrt{2 + \sqrt{2}} \right)$                                              |
| $w_{4,4}$            | 42                                                                                                                |
| $w_{4,5}$            | $-8 \left( -21 + 14\sqrt{2 - \sqrt{2}} + 6\sqrt{2 + \sqrt{2}} \right)$                                            |
| $w_{4,6}$            | $252 - 176\sqrt{2}$                                                                                               |
| ( <i>continues</i> ) |                                                                                                                   |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                              |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,7}$     | $8 \left( 21 + 6 \sqrt{2 - \sqrt{2}} - 14 \sqrt{2 + \sqrt{2}} \right)$                                                                  |
| $w_{4,8}$     | 1                                                                                                                                       |
| $d_5$         | $\frac{1}{161280}$                                                                                                                      |
| $w_{5,0}$     | $1286 - 8 \sqrt{2} + 630 \sqrt{2 - \sqrt{2}}$                                                                                           |
| $w_{5,1}$     | $-30 \sqrt{2} + 4 \left( 977 - 192 \sqrt{4 - 2 \sqrt{2}} + 448 \sqrt{2 - \sqrt{2}} + 448 \sqrt{2 + \sqrt{2}} \right)$                   |
| $w_{5,2}$     | $168 \sqrt{2 \left( 2 + \sqrt{2} \right)} + 11 \left( 416 + 216 \sqrt{2} + 145 \sqrt{4 - 2 \sqrt{2}} + 256 \sqrt{2 - \sqrt{2}} \right)$ |
| $w_{5,3}$     | $2 \left( 2765 + 29 \sqrt{2} + 384 \sqrt{4 - 2 \sqrt{2}} + 2176 \sqrt{2 - \sqrt{2}} + 384 \sqrt{2 + \sqrt{2}} \right)$                  |
| $w_{5,4}$     | $5146 - 1544 \sqrt{2} + 3968 \sqrt{2 - \sqrt{2}}$                                                                                       |
| $w_{5,5}$     | $6 \left( 49 + 28 \sqrt{2} + 128 \sqrt{4 - 2 \sqrt{2}} + 128 \sqrt{2 - \sqrt{2}} - 128 \sqrt{2 + \sqrt{2}} \right)$                     |
| $w_{5,6}$     | $4576 - 3256 \sqrt{2} - 1595 \sqrt{4 - 2 \sqrt{2}} + 2816 \sqrt{2 - \sqrt{2}} - 168 \sqrt{2 \left( 2 + \sqrt{2} \right)}$               |
| $w_{5,7}$     | $-4 \left( -1391 + 701 \sqrt{2} + 192 \sqrt{4 - 2 \sqrt{2}} - 832 \sqrt{2 - \sqrt{2}} + 448 \sqrt{2 + \sqrt{2}} \right)$                |
| $w_{5,8}$     | $3 - 4 \sqrt{2} + 5 \sqrt{2 - \sqrt{2}}$                                                                                                |
| $d_6$         | $\frac{1}{40320}$                                                                                                                       |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                               |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,0}$     | $324 + 160 \sqrt{2}$                                                                                                                     |
| $w_{6,1}$     | $592 + 427 \sqrt{2} - 159 \sqrt{4 - 2 \sqrt{2}} - 192 \sqrt{2 - \sqrt{2}}$<br>$+ 448 \sqrt{2 + \sqrt{2}} + 266 \sqrt{2 (2 + \sqrt{2})}$  |
| $w_{6,2}$     | $4 (499 + 352 \sqrt{2})$                                                                                                                 |
| $w_{6,3}$     | $1360 + 853 \sqrt{2} + 266 \sqrt{4 - 2 \sqrt{2}} + 448 \sqrt{2 - \sqrt{2}}$<br>$+ 192 \sqrt{2 + \sqrt{2}} + 159 \sqrt{2 (2 + \sqrt{2})}$ |
| $w_{6,4}$     | $4 (353 + 248 \sqrt{2})$                                                                                                                 |
| $w_{6,5}$     | $1360 + 853 \sqrt{2} - 266 \sqrt{4 - 2 \sqrt{2}} - 448 \sqrt{2 - \sqrt{2}}$<br>$- 192 \sqrt{2 + \sqrt{2}} - 159 \sqrt{2 (2 + \sqrt{2})}$ |
| $w_{6,6}$     | 84                                                                                                                                       |
| $w_{6,7}$     | $592 + 427 \sqrt{2} + 159 \sqrt{4 - 2 \sqrt{2}} + 192 \sqrt{2 - \sqrt{2}}$<br>$- 448 \sqrt{2 + \sqrt{2}} - 266 \sqrt{2 (2 + \sqrt{2})}$  |
| $w_{6,8}$     | 2                                                                                                                                        |
| $d_7$         | $\frac{1}{161280}$                                                                                                                       |
| $w_{7,0}$     | $2 (643 + 4 \sqrt{2} + 315 \sqrt{2 + \sqrt{2}})$                                                                                         |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                   |
|---------------|------------------------------------------------------------------------------------------------------------------------------|
| $w_{7,1}$     | $-2 \left( -2765 + 29 \sqrt{2} + 384 \sqrt{2 - \sqrt{2}} - 2176 \sqrt{2 + \sqrt{2}} + 384 \sqrt{2 (2 + \sqrt{2})} \right)$   |
| $w_{7,2}$     | $3256 \sqrt{2} - 168 \sqrt{4 - 2 \sqrt{2}} + 11 \left( 416 + 256 \sqrt{2 + \sqrt{2}} + 145 \sqrt{2 (2 + \sqrt{2})} \right)$  |
| $w_{7,3}$     | $4 \left( 1391 + 701 \sqrt{2} + 448 \sqrt{2 - \sqrt{2}} + 832 \sqrt{2 + \sqrt{2}} + 192 \sqrt{2 (2 + \sqrt{2})} \right)$     |
| $w_{7,4}$     | $5146 + 1544 \sqrt{2} + 3968 \sqrt{2 + \sqrt{2}}$                                                                            |
| $w_{7,5}$     | $30 \sqrt{2} + 4 \left( 977 - 448 \sqrt{2 - \sqrt{2}} + 448 \sqrt{2 + \sqrt{2}} + 192 \sqrt{2 (2 + \sqrt{2})} \right)$       |
| $w_{7,6}$     | $-2376 \sqrt{2} + 168 \sqrt{4 - 2 \sqrt{2}} + 11 \left( 416 + 256 \sqrt{2 + \sqrt{2}} - 145 \sqrt{2 (2 + \sqrt{2})} \right)$ |
| $w_{7,7}$     | $-6 \left( -49 + 28 \sqrt{2} - 128 \sqrt{2 - \sqrt{2}} - 128 \sqrt{2 + \sqrt{2}} + 128 \sqrt{2 (2 + \sqrt{2})} \right)$      |
| $w_{7,8}$     | $3 + 4 \sqrt{2} + 5 \sqrt{2 + \sqrt{2}}$                                                                                     |
| $d_8$         | $\frac{1}{315}$                                                                                                              |
| $w_{8,0}$     | 5                                                                                                                            |
| $w_{8,1}$     | $20 - 6 \sqrt{2} - 3 \sqrt{2 - \sqrt{2}} + 7 \sqrt{2 + \sqrt{2}}$                                                            |
| $w_{8,2}$     | $11 (2 + \sqrt{2})$                                                                                                          |
| $w_{8,3}$     | $20 + 6 \sqrt{2} + 7 \sqrt{2 - \sqrt{2}} + 3 \sqrt{2 + \sqrt{2}}$                                                            |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.4: Integral approximation  $I_j$ ,  $n = 8$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                     |
|---------------|----------------------------------------------------------------|
| $w_{8,4}$     | 31                                                             |
| $w_{8,5}$     | $20 + 6\sqrt{2} - 7\sqrt{2 - \sqrt{2}} - 3\sqrt{2 + \sqrt{2}}$ |
| $w_{8,6}$     | $-11(-2 + \sqrt{2})$                                           |
| $w_{8,7}$     | $20 - 6\sqrt{2} + 3\sqrt{2 - \sqrt{2}} - 7\sqrt{2 + \sqrt{2}}$ |
| $w_{8,8}$     | 0                                                              |

Table C.5: Integral approximation  $I_j$  for  $n = 10$ .

| $d_j, w_{jk}$ | Expression                                                                                                                                                                          |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_1$         | $\frac{1}{35481600}$                                                                                                                                                                |
| $w_{1,0}$     | $14 \left( 12861 + 25\sqrt{5} - 3168\sqrt{2(5 + \sqrt{5})} \right)$                                                                                                                 |
| $w_{1,1}$     | $-154 \left( -201 + 75\sqrt{5} + 512\sqrt{10 - 2\sqrt{5}} \right. \\ \left. + 256\sqrt{2(5 + \sqrt{5})} - 256\sqrt{10(5 + \sqrt{5})} \right)$                                       |
| $w_{1,2}$     | $2 \left( 453362 + 121188\sqrt{5} + 4950\sqrt{50 - 10\sqrt{5}} - 4125\sqrt{10 - 2\sqrt{5}} \right. \\ \left. - 148694\sqrt{2(5 + \sqrt{5})} - 20619\sqrt{10(5 + \sqrt{5})} \right)$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                    |
|----------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,3}$            | $-4 \left( -131931 + 13283 \sqrt{5} - 38400 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. + 28544 \sqrt{2 (5 + \sqrt{5})} + 9856 \sqrt{10 (5 + \sqrt{5})} \right)$                                 |
| $w_{1,4}$            | $834974 + 374354 \sqrt{5} + 9900 \sqrt{50 - 10 \sqrt{5}} + 8250 \sqrt{10 - 2 \sqrt{5}} \\ - 240212 \sqrt{2 (5 + \sqrt{5})} - 97558 \sqrt{10 (5 + \sqrt{5})}$                                  |
| $w_{1,5}$            | $6 \left( 178759 + 18715 \sqrt{5} - 57984 \sqrt{2 (5 + \sqrt{5})} \right)$                                                                                                                    |
| $w_{1,6}$            | $-2 \left( -545487 + 53463 \sqrt{5} + 4950 \sqrt{50 - 10 \sqrt{5}} + 4125 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. + 148694 \sqrt{2 (5 + \sqrt{5})} - 20619 \sqrt{10 (5 + \sqrt{5})} \right)$ |
| $w_{1,7}$            | $4 \left( 235331 + 88271 \sqrt{5} - 38400 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. - 67968 \sqrt{2 (5 + \sqrt{5})} - 9856 \sqrt{10 (5 + \sqrt{5})} \right)$                                   |
| $w_{1,8}$            | $650724 - 238904 \sqrt{5} - 9900 \sqrt{50 - 10 \sqrt{5}} + 8250 \sqrt{10 - 2 \sqrt{5}} \\ - 240212 \sqrt{2 (5 + \sqrt{5})} + 97558 \sqrt{10 (5 + \sqrt{5})}$                                  |
| $w_{1,9}$            | $1060794 - 117542 \sqrt{5} + 78848 \sqrt{10 - 2 \sqrt{5}} \\ - 346624 \sqrt{2 (5 + \sqrt{5})} + 39424 \sqrt{10 (5 + \sqrt{5})}$                                                               |
| $w_{1,10}$           | $7 \left( 61 + 25 \sqrt{5} - 32 \sqrt{2 (5 + \sqrt{5})} \right)$                                                                                                                              |
| ( <i>continues</i> ) |                                                                                                                                                                                               |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                                                                               |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_2$              | $\frac{1}{35481600}$                                                                                                                                                                                                     |
| $w_{2,0}$          | $-350 \left( -389 + 129 \sqrt{5} \right)$                                                                                                                                                                                |
| $w_{2,1}$          | $529692 - 211000 \sqrt{5} + 30800 \sqrt{50 - 10 \sqrt{5}} - 61446 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 115200 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 46650 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                               |
| $w_{2,2}$          | $-11550 \left( -5 + \sqrt{5} \right)$                                                                                                                                                                                    |
| $w_{2,3}$          | $260962 - 118446 \sqrt{5} - 46650 \sqrt{50 - 10 \sqrt{5}} + 115200 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 61446 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 30800 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                               |
| $w_{2,4}$          | $3800 - 240 \sqrt{5}$                                                                                                                                                                                                    |
| $w_{2,5}$          | $422442 - 189858 \sqrt{5}$                                                                                                                                                                                               |
| $w_{2,6}$          | $-400 \left( -2307 + 1030 \sqrt{5} \right)$                                                                                                                                                                              |
| $w_{2,7}$          | $260962 - 118446 \sqrt{5} + 46650 \sqrt{50 - 10 \sqrt{5}} - 115200 \sqrt{10 - 2 \sqrt{5}}$<br>$- 61446 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 30800 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                               |
| $w_{2,8}$          | $823750 - 367910 \sqrt{5}$                                                                                                                                                                                               |
| $w_{2,9}$          | $-211000 \sqrt{5} - 30800 \sqrt{50 - 10 \sqrt{5}} + 6 \left( 88282 + 10241 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. - 19200 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 7775 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$ |
| <i>(continues)</i> |                                                                                                                                                                                                                          |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                                                                              |
|--------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,10}$         | $-175 \left( -5 + \sqrt{5} \right)$                                                                                                                                                                                     |
| $d_3$              | $\frac{1}{35481600}$                                                                                                                                                                                                    |
| $w_{3,0}$          | $-14 \left( -12861 + 25 \sqrt{5} + 3168 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                                                 |
| $w_{3,1}$          | $4 \left( 235331 - 88271 \sqrt{5} + 9856 \sqrt{50 - 10 \sqrt{5}} \right.$<br>$\left. - 67968 \sqrt{10 - 2 \sqrt{5}} + 38400 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                               |
| $w_{3,2}$          | $2 \left( 545487 + 53463 \sqrt{5} - 20619 \sqrt{50 - 10 \sqrt{5}} - 148694 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. + 4125 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 4950 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$ |
| $w_{3,3}$          | $154 \left( 201 + 75 \sqrt{5} - 256 \sqrt{50 - 10 \sqrt{5}} \right.$<br>$\left. - 256 \sqrt{10 - 2 \sqrt{5}} + 512 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                                        |
| $w_{3,4}$          | $650724 + 238904 \sqrt{5} - 97558 \sqrt{50 - 10 \sqrt{5}} - 240212 \sqrt{10 - 2 \sqrt{5}}$<br>$- 8250 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 9900 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                |
| $w_{3,5}$          | $-6 \left( -178759 + 18715 \sqrt{5} + 57984 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                                             |
| $w_{3,6}$          | $906724 - 242376 \sqrt{5} + 41238 \sqrt{50 - 10 \sqrt{5}} - 297388 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 8250 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 9900 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                |
| <i>(continues)</i> |                                                                                                                                                                                                                         |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                   |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,7}$          | $2 \left( 530397 + 58771 \sqrt{5} - 19712 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. - 173312 \sqrt{10 - 2 \sqrt{5}} - 39424 \sqrt{2 (5 + \sqrt{5})} \right)$ |
| $w_{3,8}$          | $834974 - 374354 \sqrt{5} + 97558 \sqrt{50 - 10 \sqrt{5}} - 240212 \sqrt{10 - 2 \sqrt{5}} \\ - 8250 \sqrt{2 (5 + \sqrt{5})} + 9900 \sqrt{10 (5 + \sqrt{5})}$ |
| $w_{3,9}$          | $4 \left( 131931 + 13283 \sqrt{5} + 9856 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. - 28544 \sqrt{10 - 2 \sqrt{5}} - 38400 \sqrt{2 (5 + \sqrt{5})} \right)$   |
| $w_{3,10}$         | $-7 \left( -61 + 25 \sqrt{5} + 32 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                            |
| $d_4$              | $\frac{1}{1419264}$                                                                                                                                          |
| $w_{4,0}$          | $9030 - 1778 \sqrt{5}$                                                                                                                                       |
| $w_{4,1}$          | $41650 - 13858 \sqrt{5} + 1232 \sqrt{50 - 10 \sqrt{5}} - 3850 \sqrt{10 - 2 \sqrt{5}} \\ + 7680 \sqrt{2 (5 + \sqrt{5})} - 1866 \sqrt{10 (5 + \sqrt{5})}$      |
| $w_{4,2}$          | $43160 - 4496 \sqrt{5}$                                                                                                                                      |
| $w_{4,3}$          | $20860 - 3848 \sqrt{5} - 1866 \sqrt{50 - 10 \sqrt{5}} + 7680 \sqrt{10 - 2 \sqrt{5}} \\ + 3850 \sqrt{2 (5 + \sqrt{5})} - 1232 \sqrt{10 (5 + \sqrt{5})}$       |
| $w_{4,4}$          | $462 (5 + \sqrt{5})$                                                                                                                                         |
| <i>(continues)</i> |                                                                                                                                                              |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,5}$     | $44730 - 20238 \sqrt{5}$                                                                                                                                  |
| $w_{4,6}$     | $53430 - 23786 \sqrt{5}$                                                                                                                                  |
| $w_{4,7}$     | $20860 - 3848 \sqrt{5} + 1866 \sqrt{50 - 10 \sqrt{5}} - 7680 \sqrt{10 - 2 \sqrt{5}}$<br>$- 3850 \sqrt{2 (5 + \sqrt{5})} + 1232 \sqrt{10 (5 + \sqrt{5})}$  |
| $w_{4,8}$     | $59440 - 26528 \sqrt{5}$                                                                                                                                  |
| $w_{4,9}$     | $41650 - 13858 \sqrt{5} - 1232 \sqrt{50 - 10 \sqrt{5}} + 3850 \sqrt{10 - 2 \sqrt{5}}$<br>$- 7680 \sqrt{2 (5 + \sqrt{5})} + 1866 \sqrt{10 (5 + \sqrt{5})}$ |
| $w_{4,10}$    | $7 (5 + \sqrt{5})$                                                                                                                                        |
| $d_5$         | $\frac{1}{554400}$                                                                                                                                        |
| $w_{5,0}$     | $2786$                                                                                                                                                    |
| $w_{5,1}$     | $2400 \sqrt{2 (5 + \sqrt{5})} - 308 (-27 + 3 \sqrt{5} + 4 \sqrt{10 - 2 \sqrt{5}})$                                                                        |
| $w_{5,2}$     | $4 (2579 + 1171 \sqrt{5})$                                                                                                                                |
| $w_{5,3}$     | $4 \left( 2079 + 231 \sqrt{5} + 600 \sqrt{10 - 2 \sqrt{5}} + 308 \sqrt{2 (5 + \sqrt{5})} \right)$                                                         |
| $w_{5,4}$     | $6316 + 2836 \sqrt{5}$                                                                                                                                    |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                           |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,5}$     | 1386                                                                                                                                                                                 |
| $w_{5,6}$     | $10316 - 4684\sqrt{5}$                                                                                                                                                               |
| $w_{5,7}$     | $4 \left( 2079 + 231\sqrt{5} - 600\sqrt{10 - 2\sqrt{5}} - 308\sqrt{2(5 + \sqrt{5})} \right)$                                                                                         |
| $w_{5,8}$     | $6316 - 2836\sqrt{5}$                                                                                                                                                                |
| $w_{5,9}$     | $-2400\sqrt{2(5 + \sqrt{5})} - 308 \left( -27 + 3\sqrt{5} - 4\sqrt{10 - 2\sqrt{5}} \right)$                                                                                          |
| $w_{5,10}$    | -7                                                                                                                                                                                   |
| $d_6$         | $\frac{1}{35481600}$                                                                                                                                                                 |
| $w_{6,0}$     | $350(389 + 129\sqrt{5})$                                                                                                                                                             |
| $w_{6,1}$     | $2 \left( 130481 + 59223\sqrt{5} - 15400\sqrt{50 - 10\sqrt{5}} - 30723\sqrt{10 - 2\sqrt{5}} \right. \\ \left. + 57600\sqrt{2(5 + \sqrt{5})} + 23325\sqrt{10(5 + \sqrt{5})} \right)$  |
| $w_{6,2}$     | $400(2307 + 1030\sqrt{5})$                                                                                                                                                           |
| $w_{6,3}$     | $2 \left( 264846 + 105500\sqrt{5} + 23325\sqrt{50 - 10\sqrt{5}} + 57600\sqrt{10 - 2\sqrt{5}} \right. \\ \left. + 30723\sqrt{2(5 + \sqrt{5})} + 15400\sqrt{10(5 + \sqrt{5})} \right)$ |
| $w_{6,4}$     | $823750 + 367910\sqrt{5}$                                                                                                                                                            |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,5}$     | $6 \left( 70407 + 31643 \sqrt{5} \right)$                                                                                                                                                                                 |
| $w_{6,6}$     | $11550 \left( 5 + \sqrt{5} \right)$                                                                                                                                                                                       |
| $w_{6,7}$     | $2 \left( 264846 + 105500 \sqrt{5} - 23325 \sqrt{50 - 10 \sqrt{5}} - 57600 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. - 30723 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 15400 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$ |
| $w_{6,8}$     | $40 \left( 95 + 6 \sqrt{5} \right)$                                                                                                                                                                                       |
| $w_{6,9}$     | $2 \left( 130481 + 59223 \sqrt{5} + 15400 \sqrt{50 - 10 \sqrt{5}} + 30723 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. - 57600 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 23325 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$  |
| $w_{6,10}$    | $175 \left( 5 + \sqrt{5} \right)$                                                                                                                                                                                         |
| $d_7$         | $\frac{1}{35481600}$                                                                                                                                                                                                      |
| $w_{7,0}$     | $14 \left( 12861 - 25 \sqrt{5} + 3168 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                                                     |
| $w_{7,1}$     | $4 \left( 131931 + 13283 \sqrt{5} - 9856 \sqrt{50 - 10 \sqrt{5}} \right.$<br>$\left. + 28544 \sqrt{10 - 2 \sqrt{5}} + 38400 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                                 |
| $w_{7,2}$     | $2 \left( 545487 + 53463 \sqrt{5} + 20619 \sqrt{50 - 10 \sqrt{5}} + 148694 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. - 4125 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 4950 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$   |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                     |
|----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{7,3}$            | $2 \left( 530397 + 58771 \sqrt{5} + 19712 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. + 173312 \sqrt{10 - 2 \sqrt{5}} + 39424 \sqrt{2 (5 + \sqrt{5})} \right)$                                   |
| $w_{7,4}$            | $650724 + 238904 \sqrt{5} + 97558 \sqrt{50 - 10 \sqrt{5}} + 240212 \sqrt{10 - 2 \sqrt{5}} \\ + 8250 \sqrt{2 (5 + \sqrt{5})} + 9900 \sqrt{10 (5 + \sqrt{5})}$                                   |
| $w_{7,5}$            | $6 \left( 178759 - 18715 \sqrt{5} + 57984 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                      |
| $w_{7,6}$            | $-2 \left( -453362 + 121188 \sqrt{5} + 20619 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. - 148694 \sqrt{10 - 2 \sqrt{5}} + 4125 \sqrt{2 (5 + \sqrt{5})} + 4950 \sqrt{10 (5 + \sqrt{5})} \right)$ |
| $w_{7,7}$            | $154 \left( 201 + 75 \sqrt{5} + 256 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. + 256 \sqrt{10 - 2 \sqrt{5}} - 512 \sqrt{2 (5 + \sqrt{5})} \right)$                                              |
| $w_{7,8}$            | $834974 - 374354 \sqrt{5} - 97558 \sqrt{50 - 10 \sqrt{5}} + 240212 \sqrt{10 - 2 \sqrt{5}} \\ + 8250 \sqrt{2 (5 + \sqrt{5})} - 9900 \sqrt{10 (5 + \sqrt{5})}$                                   |
| $w_{7,9}$            | $-4 \left( -235331 + 88271 \sqrt{5} + 9856 \sqrt{50 - 10 \sqrt{5}} \right. \\ \left. - 67968 \sqrt{10 - 2 \sqrt{5}} + 38400 \sqrt{2 (5 + \sqrt{5})} \right)$                                   |
| $w_{7,10}$           | $7 \left( 61 - 25 \sqrt{5} + 32 \sqrt{10 - 2 \sqrt{5}} \right)$                                                                                                                                |
| ( <i>continues</i> ) |                                                                                                                                                                                                |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                             |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_8$         | $\frac{1}{1419264}$                                                                                                                                                                    |
| $w_{8,0}$     | $9030 + 1778 \sqrt{5}$                                                                                                                                                                 |
| $w_{8,1}$     | $20860 + 3848 \sqrt{5} - 1232 \sqrt{50 - 10 \sqrt{5}} - 3850 \sqrt{10 - 2 \sqrt{5}}$<br>$+ 7680 \sqrt{2 (5 + \sqrt{5})} + 1866 \sqrt{10 (5 + \sqrt{5})}$                               |
| $w_{8,2}$     | $53430 + 23786 \sqrt{5}$                                                                                                                                                               |
| $w_{8,3}$     | $2 \left( 20825 + 6929 \sqrt{5} + 933 \sqrt{50 - 10 \sqrt{5}} + 3840 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. + 1925 \sqrt{2 (5 + \sqrt{5})} + 616 \sqrt{10 (5 + \sqrt{5})} \right)$ |
| $w_{8,4}$     | $16 (3715 + 1658 \sqrt{5})$                                                                                                                                                            |
| $w_{8,5}$     | $6 (7455 + 3373 \sqrt{5})$                                                                                                                                                             |
| $w_{8,6}$     | $43160 + 4496 \sqrt{5}$                                                                                                                                                                |
| $w_{8,7}$     | $2 \left( 20825 + 6929 \sqrt{5} - 933 \sqrt{50 - 10 \sqrt{5}} - 3840 \sqrt{10 - 2 \sqrt{5}} \right.$<br>$\left. - 1925 \sqrt{2 (5 + \sqrt{5})} - 616 \sqrt{10 (5 + \sqrt{5})} \right)$ |
| $w_{8,8}$     | $-462 (-5 + \sqrt{5})$                                                                                                                                                                 |
| $w_{8,9}$     | $20860 + 3848 \sqrt{5} + 1232 \sqrt{50 - 10 \sqrt{5}} + 3850 \sqrt{10 - 2 \sqrt{5}}$<br>$- 7680 \sqrt{2 (5 + \sqrt{5})} - 1866 \sqrt{10 (5 + \sqrt{5})}$                               |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                             |
|----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,10}$           | $-7 \left( -5 + \sqrt{5} \right)$                                                                                                                                                                                      |
| $d_9$                | $\frac{1}{35481600}$                                                                                                                                                                                                   |
| $w_{9,0}$            | $14 \left( 12861 + 25 \sqrt{5} + 3168 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                                                                                                                    |
| $w_{9,1}$            | $-2 \left( -530397 + 58771 \sqrt{5} + 39424 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. - 173312 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 19712 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$                              |
| $w_{9,2}$            | $2 \left( 453362 + 121188 \sqrt{5} - 4950 \sqrt{50 - 10 \sqrt{5}} + 4125 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. + 148694 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 20619 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$ |
| $w_{9,3}$            | $4 \left( 235331 + 88271 \sqrt{5} + 38400 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. + 67968 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 9856 \sqrt{10 \left( 5 + \sqrt{5} \right)} \right)$                                  |
| $w_{9,4}$            | $834974 + 374354 \sqrt{5} - 9900 \sqrt{50 - 10 \sqrt{5}} - 8250 \sqrt{10 - 2 \sqrt{5}} \\ + 240212 \sqrt{2 \left( 5 + \sqrt{5} \right)} + 97558 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                 |
| $w_{9,5}$            | $6 \left( 178759 + 18715 \sqrt{5} + 57984 \sqrt{2 \left( 5 + \sqrt{5} \right)} \right)$                                                                                                                                |
| $w_{9,6}$            | $1090974 - 106926 \sqrt{5} + 9900 \sqrt{50 - 10 \sqrt{5}} + 8250 \sqrt{10 - 2 \sqrt{5}} \\ + 297388 \sqrt{2 \left( 5 + \sqrt{5} \right)} - 41238 \sqrt{10 \left( 5 + \sqrt{5} \right)}$                                |
| ( <i>continues</i> ) |                                                                                                                                                                                                                        |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                   |
|----------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,7}$            | $4 \left( 131931 - 13283 \sqrt{5} - 38400 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. + 28544 \sqrt{2 (5 + \sqrt{5})} + 9856 \sqrt{10 (5 + \sqrt{5})} \right)$  |
| $w_{9,8}$            | $650724 - 238904 \sqrt{5} + 9900 \sqrt{50 - 10 \sqrt{5}} - 8250 \sqrt{10 - 2 \sqrt{5}} \\ + 240212 \sqrt{2 (5 + \sqrt{5})} - 97558 \sqrt{10 (5 + \sqrt{5})}$ |
| $w_{9,9}$            | $-154 \left( -201 + 75 \sqrt{5} - 512 \sqrt{10 - 2 \sqrt{5}} \right. \\ \left. - 256 \sqrt{2 (5 + \sqrt{5})} + 256 \sqrt{10 (5 + \sqrt{5})} \right)$         |
| $w_{9,10}$           | $7 \left( 61 + 25 \sqrt{5} + 32 \sqrt{2 (5 + \sqrt{5})} \right)$                                                                                             |
| $d_{10}$             | $\frac{1}{17325}$                                                                                                                                            |
| $w_{10,0}$           | 175                                                                                                                                                          |
| $w_{10,1}$           | $754 - 154 \sqrt{5} - 77 \sqrt{10 - 2 \sqrt{5}} + 150 \sqrt{2 (5 + \sqrt{5})}$                                                                               |
| $w_{10,2}$           | $25 (47 + 5 \sqrt{5})$                                                                                                                                       |
| $w_{10,3}$           | $754 + 154 \sqrt{5} + 150 \sqrt{10 - 2 \sqrt{5}} + 77 \sqrt{2 (5 + \sqrt{5})}$                                                                               |
| $w_{10,4}$           | $925 + 345 \sqrt{5}$                                                                                                                                         |
| $w_{10,5}$           | 1359                                                                                                                                                         |
| ( <i>continues</i> ) |                                                                                                                                                              |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.5: Integral approximation  $I_j$ ,  $n = 10$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                  |
|---------------|---------------------------------------------------------------------------------------------|
| $w_{10,6}$    | $25 \left( 47 - 5 \sqrt{5} \right)$                                                         |
| $w_{10,7}$    | $754 + 154 \sqrt{5} - 150 \sqrt{10 - 2 \sqrt{5}} - 77 \sqrt{2 \left( 5 + \sqrt{5} \right)}$ |
| $w_{10,8}$    | $925 - 345 \sqrt{5}$                                                                        |
| $w_{10,9}$    | $754 - 154 \sqrt{5} + 77 \sqrt{10 - 2 \sqrt{5}} - 150 \sqrt{2 \left( 5 + \sqrt{5} \right)}$ |
| $w_{10,10}$   | 0                                                                                           |

Table C.6: Integral approximation  $I_j$  for  $n = 12$ .

| $d_j, w_{jk}$ | Expression                                                                         |
|---------------|------------------------------------------------------------------------------------|
| $d_1$         | $\frac{1}{276756480}$                                                              |
| $w_{1,0}$     | $-48 \left( -20198 + 5005 \sqrt{2} - 54 \sqrt{3} + 5005 \sqrt{6} \right)$          |
| $w_{1,1}$     | $-13728 \left( -17 + 9 \sqrt{3} \right)$                                           |
| $w_{1,2}$     | $11 \left( 378646 + 61123 \sqrt{2} - 194832 \sqrt{3} - 52111 \sqrt{6} \right)$     |
| $w_{1,3}$     | $4 \left( 1120088 + 9728 \sqrt{2} - 9447 \sqrt{3} - 456192 \sqrt{6} \right)$       |
| $w_{1,4}$     | $-11 \left( -378646 + 267971 \sqrt{2} - 218864 \sqrt{3} + 154737 \sqrt{6} \right)$ |
| $w_{1,5}$     | $16 \left( 244901 + 99872 \sqrt{2} - 81653 \sqrt{3} - 99872 \sqrt{6} \right)$      |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                          |
|---------------|-------------------------------------------------------------------------------------|
| $w_{1,6}$     | $9667086 - 2626719 \sqrt{2} + 1064016 \sqrt{3} - 3182703 \sqrt{6}$                  |
| $w_{1,7}$     | $3899462 - 2756608 \sqrt{2} + 1317506 \sqrt{3} - 931840 \sqrt{6}$                   |
| $w_{1,8}$     | $-32 \left( -172555 + 70808 \sqrt{2} - 58321 \sqrt{3} + 70808 \sqrt{6} \right)$     |
| $w_{1,9}$     | $2 \left( 3377373 - 1397760 \sqrt{2} + 1140637 \sqrt{3} - 1378304 \sqrt{6} \right)$ |
| $w_{1,10}$    | $1815054 - 1299297 \sqrt{2} + 1064016 \sqrt{3} - 743313 \sqrt{6}$                   |
| $w_{1,11}$    | $-16 \left( -424107 + 299616 \sqrt{2} - 141025 \sqrt{3} + 99872 \sqrt{6} \right)$   |
| $w_{1,12}$    | $-24 \left( -38 + 35 \sqrt{2} - 54 \sqrt{3} + 35 \sqrt{6} \right)$                  |
| $d_2$         | $\frac{1}{276756480}$                                                               |
| $w_{2,0}$     | $864 \left( 1123 - 560 \sqrt{3} \right)$                                            |
| $w_{2,1}$     | $7678608 - 1520593 \sqrt{2} - 4304518 \sqrt{3} + 970017 \sqrt{6}$                   |
| $w_{2,2}$     | $44 \left( 179199 - 103424 \sqrt{3} \right)$                                        |
| $w_{2,3}$     | $7678608 + 1520593 \sqrt{2} - 4304518 \sqrt{3} - 970017 \sqrt{6}$                   |
| $w_{2,4}$     | $123552$                                                                            |
| $w_{2,5}$     | $32 \left( 172545 + 99872 \sqrt{2} - 99872 \sqrt{3} - 57801 \sqrt{6} \right)$       |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                      |
|---------------|---------------------------------------------------------------------------------|
| $w_{2,6}$     | $10733958 - 6190182 \sqrt{3}$                                                   |
| $w_{2,7}$     | $2087568 - 1481681 \sqrt{2} - 1208698 \sqrt{3} + 854751 \sqrt{6}$               |
| $w_{2,8}$     | $32 \left( 245457 - 141616 \sqrt{3} \right)$                                    |
| $w_{2,9}$     | $2087568 + 1481681 \sqrt{2} - 1208698 \sqrt{3} - 854751 \sqrt{6}$               |
| $w_{2,10}$    | $54 \left( 53369 - 30775 \sqrt{3} \right)$                                      |
| $w_{2,11}$    | $-32 \left( -172545 + 99872 \sqrt{2} + 99872 \sqrt{3} - 57801 \sqrt{6} \right)$ |
| $w_{2,12}$    | 1296                                                                            |
| $d_3$         | $\frac{1}{17297280}$                                                            |
| $w_{3,0}$     | $60594 - 30030 \sqrt{2}$                                                        |
| $w_{3,1}$     | $282380 - 173504 \sqrt{2} - 141850 \sqrt{3} + 115264 \sqrt{6}$                  |
| $w_{3,2}$     | $-22 \left( -11177 + 6464 \sqrt{2} + 6464 \sqrt{3} - 3736 \sqrt{6} \right)$     |
| $w_{3,3}$     | $242484 - 171072 \sqrt{2} - 1930 \sqrt{3} + 1216 \sqrt{6}$                      |
| $w_{3,4}$     | $-22 \left( -11177 + 6464 \sqrt{2} - 6464 \sqrt{3} + 3736 \sqrt{6} \right)$     |
| $w_{3,5}$     | 14586                                                                           |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                     |
|---------------|----------------------------------------------------------------|
| $w_{3,6}$     | $591162 - 419556 \sqrt{2}$                                     |
| $w_{3,7}$     | $242484 - 171072 \sqrt{2} + 1930 \sqrt{3} - 1216 \sqrt{6}$     |
| $w_{3,8}$     | $400022 - 283232 \sqrt{2}$                                     |
| $w_{3,9}$     | $282380 - 173504 \sqrt{2} + 141850 \sqrt{3} - 115264 \sqrt{6}$ |
| $w_{3,10}$    | $100410 - 71196 \sqrt{2}$                                      |
| $w_{3,11}$    | $565190 - 399488 \sqrt{2}$                                     |
| $w_{3,12}$    | $57 - 105 \sqrt{2}$                                            |
| $d_4$         | $\frac{1}{276756480}$                                          |
| $w_{4,0}$     | $491616$                                                       |
| $w_{4,1}$     | $509296 + 383665 \sqrt{2} + 315770 \sqrt{3} + 212949 \sqrt{6}$ |
| $w_{4,2}$     | $-418 \left( -4013 + 2305 \sqrt{3} \right)$                    |
| $w_{4,3}$     | $509296 - 383665 \sqrt{2} + 315770 \sqrt{3} - 212949 \sqrt{6}$ |
| $w_{4,4}$     | $418 \left( 4013 + 2305 \sqrt{3} \right)$                      |
| $w_{4,5}$     | $32 \left( 40583 + 28915 \sqrt{2} \right)$                     |

*(continues)*

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                          |
|--------------------|-------------------------------------------------------------------------------------|
| $w_{4,6}$          | 370656                                                                              |
| $w_{4,7}$          | $509296 - 383665 \sqrt{2} - 315770 \sqrt{3} + 212949 \sqrt{6}$                      |
| $w_{4,8}$          | 26720                                                                               |
| $w_{4,9}$          | $509296 + 383665 \sqrt{2} - 315770 \sqrt{3} - 212949 \sqrt{6}$                      |
| $w_{4,10}$         | 12204                                                                               |
| $w_{4,11}$         | $1298656 - 925280 \sqrt{2}$                                                         |
| $w_{4,12}$         | 3888                                                                                |
| $d_5$              | $\frac{1}{276756480}$                                                               |
| $w_{5,0}$          | $48 \left( 20198 + 5005 \sqrt{2} - 54 \sqrt{3} - 5005 \sqrt{6} \right)$             |
| $w_{5,1}$          | $2 \left( 1949731 + 1378304 \sqrt{2} - 658753 \sqrt{3} - 465920 \sqrt{6} \right)$   |
| $w_{5,2}$          | $11 \left( 378646 + 267971 \sqrt{2} - 218864 \sqrt{3} - 154737 \sqrt{6} \right)$    |
| $w_{5,3}$          | $2 \left( 3377373 + 1397760 \sqrt{2} - 1140637 \sqrt{3} - 1378304 \sqrt{6} \right)$ |
| $w_{5,4}$          | $-11 \left( -378646 + 61123 \sqrt{2} - 194832 \sqrt{3} + 52111 \sqrt{6} \right)$    |
| $w_{5,5}$          | $16 \left( 424107 + 299616 \sqrt{2} - 141025 \sqrt{3} - 99872 \sqrt{6} \right)$     |
| <i>(continues)</i> |                                                                                     |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                        |
|--------------------|-----------------------------------------------------------------------------------|
| $w_{5,6}$          | $3 \left( 3222362 + 875573 \sqrt{2} - 354672 \sqrt{3} - 1060901 \sqrt{6} \right)$ |
| $w_{5,7}$          | $13728 \left( 17 + 9 \sqrt{3} \right)$                                            |
| $w_{5,8}$          | $32 \left( 172555 + 70808 \sqrt{2} - 58321 \sqrt{3} - 70808 \sqrt{6} \right)$     |
| $w_{5,9}$          | $4 \left( 1120088 - 9728 \sqrt{2} + 9447 \sqrt{3} - 456192 \sqrt{6} \right)$      |
| $w_{5,10}$         | $1815054 + 1299297 \sqrt{2} - 1064016 \sqrt{3} - 743313 \sqrt{6}$                 |
| $w_{5,11}$         | $-16 \left( -244901 + 99872 \sqrt{2} - 81653 \sqrt{3} + 99872 \sqrt{6} \right)$   |
| $w_{5,12}$         | $24 \left( 38 + 35 \sqrt{2} - 54 \sqrt{3} - 35 \sqrt{6} \right)$                  |
| $d_6$              | $\frac{1}{4324320}$                                                               |
| $w_{6,0}$          | 15282                                                                             |
| $w_{6,1}$          | $16 \left( 2145 - 19 \sqrt{2} + 891 \sqrt{6} \right)$                             |
| $w_{6,2}$          | $-176 \left( -351 + 202 \sqrt{3} \right)$                                         |
| $w_{6,3}$          | $16 \left( 2145 + 19 \sqrt{2} - 891 \sqrt{6} \right)$                             |
| $w_{6,4}$          | $176 \left( 351 + 202 \sqrt{3} \right)$                                           |
| $w_{6,5}$          | $4 \left( 17589 + 12484 \sqrt{2} \right)$                                         |
| <i>(continues)</i> |                                                                                   |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                          |
|--------------------|-------------------------------------------------------------------------------------|
| $w_{6,6}$          | 123120                                                                              |
| $w_{6,7}$          | $16 \left( 2145 + 19 \sqrt{2} + 891 \sqrt{6} \right)$                               |
| $w_{6,8}$          | 7722                                                                                |
| $w_{6,9}$          | $-16 \left( -2145 + 19 \sqrt{2} + 891 \sqrt{6} \right)$                             |
| $w_{6,10}$         | 432                                                                                 |
| $w_{6,11}$         | $70356 - 49936 \sqrt{2}$                                                            |
| $w_{6,12}$         | 81                                                                                  |
| $d_7$              | $\frac{1}{276756480}$                                                               |
| $w_{7,0}$          | $-48 \left( -20198 + 5005 \sqrt{2} + 54 \sqrt{3} - 5005 \sqrt{6} \right)$           |
| $w_{7,1}$          | $2 \left( 3377373 - 1397760 \sqrt{2} - 1140637 \sqrt{3} + 1378304 \sqrt{6} \right)$ |
| $w_{7,2}$          | $-11 \left( -378646 + 267971 \sqrt{2} + 218864 \sqrt{3} - 154737 \sqrt{6} \right)$  |
| $w_{7,3}$          | $3899462 - 2756608 \sqrt{2} - 1317506 \sqrt{3} + 931840 \sqrt{6}$                   |
| $w_{7,4}$          | $11 \left( 378646 + 61123 \sqrt{2} + 194832 \sqrt{3} + 52111 \sqrt{6} \right)$      |
| $w_{7,5}$          | $16 \left( 244901 + 99872 \sqrt{2} + 81653 \sqrt{3} + 99872 \sqrt{6} \right)$       |
| <i>(continues)</i> |                                                                                     |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                        |
|---------------|-----------------------------------------------------------------------------------|
| $w_{7,6}$     | $9667086 - 2626719 \sqrt{2} - 1064016 \sqrt{3} + 3182703 \sqrt{6}$                |
| $w_{7,7}$     | $4 \left( 1120088 + 9728 \sqrt{2} + 9447 \sqrt{3} + 456192 \sqrt{6} \right)$      |
| $w_{7,8}$     | $-32 \left( -172555 + 70808 \sqrt{2} + 58321 \sqrt{3} - 70808 \sqrt{6} \right)$   |
| $w_{7,9}$     | $13728 \left( 17 + 9 \sqrt{3} \right)$                                            |
| $w_{7,10}$    | $1815054 - 1299297 \sqrt{2} - 1064016 \sqrt{3} + 743313 \sqrt{6}$                 |
| $w_{7,11}$    | $-16 \left( -424107 + 299616 \sqrt{2} + 141025 \sqrt{3} - 99872 \sqrt{6} \right)$ |
| $w_{7,12}$    | $-24 \left( -38 + 35 \sqrt{2} + 54 \sqrt{3} - 35 \sqrt{6} \right)$                |
| $d_8$         | $\frac{1}{10250240}$                                                              |
| $w_{8,0}$     | 54048                                                                             |
| $w_{8,1}$     | $-9 \left( -24784 + 1739 \sqrt{2} + 6370 \sqrt{3} - 6633 \sqrt{6} \right)$        |
| $w_{8,2}$     | $230670 - 132858 \sqrt{3}$                                                        |
| $w_{8,3}$     | $9 \left( 24784 + 1739 \sqrt{2} - 6370 \sqrt{3} - 6633 \sqrt{6} \right)$          |
| $w_{8,4}$     | $198 \left( 1165 + 671 \sqrt{3} \right)$                                          |
| $w_{8,5}$     | $288 \left( 989 + 703 \sqrt{2} \right)$                                           |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                 |
|---------------|----------------------------------------------------------------------------|
| $w_{8,6}$     | 582084                                                                     |
| $w_{8,7}$     | $9 \left( 24784 + 1739 \sqrt{2} + 6370 \sqrt{3} + 6633 \sqrt{6} \right)$   |
| $w_{8,8}$     | 336672                                                                     |
| $w_{8,9}$     | $-9 \left( -24784 + 1739 \sqrt{2} - 6370 \sqrt{3} + 6633 \sqrt{6} \right)$ |
| $w_{8,10}$    | 13728                                                                      |
| $w_{8,11}$    | $-288 \left( -989 + 703 \sqrt{2} \right)$                                  |
| $w_{8,12}$    | 144                                                                        |
| $d_9$         | $\frac{1}{17297280}$                                                       |
| $w_{9,0}$     | $60594 + 30030 \sqrt{2}$                                                   |
| $w_{9,1}$     | $242484 + 171072 \sqrt{2} - 1930 \sqrt{3} - 1216 \sqrt{6}$                 |
| $w_{9,2}$     | $22 \left( 11177 + 6464 \sqrt{2} - 6464 \sqrt{3} - 3736 \sqrt{6} \right)$  |
| $w_{9,3}$     | $282380 + 173504 \sqrt{2} - 141850 \sqrt{3} - 115264 \sqrt{6}$             |
| $w_{9,4}$     | $22 \left( 11177 + 6464 \sqrt{2} + 6464 \sqrt{3} + 3736 \sqrt{6} \right)$  |
| $w_{9,5}$     | $565190 + 399488 \sqrt{2}$                                                 |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                    |
|---------------|-------------------------------------------------------------------------------|
| $w_{9,6}$     | $6 \left( 98527 + 69926 \sqrt{2} \right)$                                     |
| $w_{9,7}$     | $282380 + 173504 \sqrt{2} + 141850 \sqrt{3} + 115264 \sqrt{6}$                |
| $w_{9,8}$     | $2 \left( 200011 + 141616 \sqrt{2} \right)$                                   |
| $w_{9,9}$     | $242484 + 171072 \sqrt{2} + 1930 \sqrt{3} + 1216 \sqrt{6}$                    |
| $w_{9,10}$    | $6 \left( 16735 + 11866 \sqrt{2} \right)$                                     |
| $w_{9,11}$    | 14586                                                                         |
| $w_{9,12}$    | $57 + 105 \sqrt{2}$                                                           |
| $d_{10}$      | $\frac{1}{276756480}$                                                         |
| $w_{10,0}$    | $864 \left( 1123 + 560 \sqrt{3} \right)$                                      |
| $w_{10,1}$    | $2087568 + 1481681 \sqrt{2} + 1208698 \sqrt{3} + 854751 \sqrt{6}$             |
| $w_{10,2}$    | 123552                                                                        |
| $w_{10,3}$    | $2087568 - 1481681 \sqrt{2} + 1208698 \sqrt{3} - 854751 \sqrt{6}$             |
| $w_{10,4}$    | $44 \left( 179199 + 103424 \sqrt{3} \right)$                                  |
| $w_{10,5}$    | $32 \left( 172545 + 99872 \sqrt{2} + 99872 \sqrt{3} + 57801 \sqrt{6} \right)$ |

*(continues)*

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                       |
|--------------------|----------------------------------------------------------------------------------|
| $w_{10,6}$         | $162 \left( 66259 + 38211 \sqrt{3} \right)$                                      |
| $w_{10,7}$         | $7678608 + 1520593 \sqrt{2} + 4304518 \sqrt{3} + 970017 \sqrt{6}$                |
| $w_{10,8}$         | $32 \left( 245457 + 141616 \sqrt{3} \right)$                                     |
| $w_{10,9}$         | $7678608 - 1520593 \sqrt{2} + 4304518 \sqrt{3} - 970017 \sqrt{6}$                |
| $w_{10,10}$        | $54 \left( 53369 + 30775 \sqrt{3} \right)$                                       |
| $w_{10,11}$        | $-32 \left( -172545 + 99872 \sqrt{2} - 99872 \sqrt{3} + 57801 \sqrt{6} \right)$  |
| $w_{10,12}$        | 1296                                                                             |
| $d_{11}$           | $\frac{1}{276756480}$                                                            |
| $w_{11,0}$         | $48 \left( 20198 + 5005 \sqrt{2} + 54 \sqrt{3} + 5005 \sqrt{6} \right)$          |
| $w_{11,1}$         | $4 \left( 1120088 - 9728 \sqrt{2} - 9447 \sqrt{3} + 456192 \sqrt{6} \right)$     |
| $w_{11,2}$         | $-11 \left( -378646 + 61123 \sqrt{2} + 194832 \sqrt{3} - 52111 \sqrt{6} \right)$ |
| $w_{11,3}$         | $-13728 \left( -17 + 9 \sqrt{3} \right)$                                         |
| $w_{11,4}$         | $11 \left( 378646 + 267971 \sqrt{2} + 218864 \sqrt{3} + 154737 \sqrt{6} \right)$ |
| $w_{11,5}$         | $16 \left( 424107 + 299616 \sqrt{2} + 141025 \sqrt{3} + 99872 \sqrt{6} \right)$  |
| <i>(continues)</i> |                                                                                  |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                          |
|--------------------|-------------------------------------------------------------------------------------|
| $w_{11,6}$         | $3 \left( 3222362 + 875573 \sqrt{2} + 354672 \sqrt{3} + 1060901 \sqrt{6} \right)$   |
| $w_{11,7}$         | $2 \left( 3377373 + 1397760 \sqrt{2} + 1140637 \sqrt{3} + 1378304 \sqrt{6} \right)$ |
| $w_{11,8}$         | $32 \left( 172555 + 70808 \sqrt{2} + 58321 \sqrt{3} + 70808 \sqrt{6} \right)$       |
| $w_{11,9}$         | $2 \left( 1949731 + 1378304 \sqrt{2} + 658753 \sqrt{3} + 465920 \sqrt{6} \right)$   |
| $w_{11,10}$        | $1815054 + 1299297 \sqrt{2} + 1064016 \sqrt{3} + 743313 \sqrt{6}$                   |
| $w_{11,11}$        | $-16 \left( -244901 + 99872 \sqrt{2} + 81653 \sqrt{3} - 99872 \sqrt{6} \right)$     |
| $w_{11,12}$        | $24 \left( 38 + 35 \sqrt{2} + 54 \sqrt{3} + 35 \sqrt{6} \right)$                    |
| $d_{12}$           | $\frac{1}{135135}$                                                                  |
| $w_{12,0}$         | 945                                                                                 |
| $w_{12,1}$         | $5384 - 19 \sqrt{2} - 1820 \sqrt{3} + 891 \sqrt{6}$                                 |
| $w_{12,2}$         | $-2222 \left( -2 + \sqrt{3} \right)$                                                |
| $w_{12,3}$         | $5384 + 19 \sqrt{2} - 1820 \sqrt{3} - 891 \sqrt{6}$                                 |
| $w_{12,4}$         | $2222 \left( 2 + \sqrt{3} \right)$                                                  |
| $w_{12,5}$         | $3121 \left( 2 + \sqrt{2} \right)$                                                  |
| <i>(continues)</i> |                                                                                     |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.6: Integral approximation  $I_j$ ,  $n = 12$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                       |
|---------------|--------------------------------------------------|
| $w_{12,6}$    | 11502                                            |
| $w_{12,7}$    | $5384 + 19\sqrt{2} + 1820\sqrt{3} + 891\sqrt{6}$ |
| $w_{12,8}$    | 8851                                             |
| $w_{12,9}$    | $5384 - 19\sqrt{2} + 1820\sqrt{3} - 891\sqrt{6}$ |
| $w_{12,10}$   | 3834                                             |
| $w_{12,11}$   | $-3121(-2 + \sqrt{2})$                           |
| $w_{12,12}$   | 0                                                |

Table C.7: Integral approximation  $I_j$  for  $n = 14$ .

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                             |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $d_1$         | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                  |
| $w_{1,0}$     | $66(12561 + 49c_5 - 12480c_6)$                                                                                                                                                                                                                                                         |
| $w_{1,1}$     | $-2(188188c_1^2 + 156156c_2^2 + 11(-197253 + 896c_4(112 + 39c_4) + 27603c_5)$<br>$+ 940800c_6 - 572c_1(-147 + 273c_3 + 128c_6) - 104c_2(-1920 + 2128c_4 + 1617c_6)$<br>$+ 4(55328c_3^2 + 245280c_5^2 - 3003c_3(-35 + 32c_5)$<br>$- 56(1456c_3 + 7383c_4 + 8400c_5)c_6 + 551694c_6^2))$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|----------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,2}$            | $ \begin{aligned} & -2 \left( -49 \left( 52 c_3 \left( 22 c_3 - 3 \left( 283 + 30 c_4 \right) \right) + 55 \left( 847 + 19 c_5 \right) \right. \right. \\ & \quad \left. \left. + 52 \left( 2 \left( 32 c_3 - 66 c_4 - 45 c_5 \right) c_5 + 11 c_2 \left( -1 + 2 c_5 \right) \right) \right) \right. \\ & \quad \left. - 8 \left( 296725 c_3 + 49 \left( -7623 + 2647 c_5 \right) \right) c_6 \right. \\ & \quad \left. + 4 c_1 \left( -195186 + 21021 c_2 + 33761 c_3 - 26754 c_4 - 228193 c_5 + 412482 c_6 \right) \right) \end{aligned} $                                     |
| $w_{1,3}$            | $ \begin{aligned} & 2 \left( 88088 c_2^2 - 7 \left( 12584 c_3^2 - 52 c_3 \left( 231 + 608 c_5 \right) \right. \right. \\ & \quad \left. \left. + 3 \left( -103323 + 104 c_4 \left( 896 + 115 c_4 \right) + c_5 \left( -54703 + 11960 c_5 \right) \right) \right) \right. \\ & \quad \left. - 308 c_2 \left( 3584 + 1248 c_4 - 4047 c_6 \right) - 32 \left( 96792 + 2288 c_3 - 73125 c_4 + 10192 c_5 \right) c_6 \right. \\ & \quad \left. + 168168 c_6^2 + 4 c_1 \left( -257994 + 80353 c_3 - 262297 c_5 + 470400 c_6 \right) \right) \end{aligned} $                            |
| $w_{1,4}$            | $ \begin{aligned} & -2 \left( -2026409 + 28028 c_1^2 + 1479624 c_3 + 28028 c_2 \left( -1 + 3 c_3 \right) \right. \\ & \quad \left. + 897603 c_5 + 392 \left( -273 c_3 \left( c_3 + c_4 \right) + \left( 2543 c_3 + 585 c_4 \right) c_5 \right) \right. \\ & \quad \left. + 2747976 c_6 - 8 \left( 284319 c_3 + 232661 c_5 \right) c_6 \right. \\ & \quad \left. + 28 c_1 \left( -62249 + 2002 c_2 - 5824 c_3 - 12012 c_4 + 7189 c_5 + 68674 c_6 \right) \right) \end{aligned} $                                                                                                  |
| $w_{1,5}$            | $ \begin{aligned} & -2 \left( 188188 c_1^2 - 156156 c_2^2 - 11 \left( 197253 + 896 c_4 \left( 112 + 39 c_4 \right) - 27603 c_5 \right) \right. \\ & \quad \left. + 4854528 c_6 - 572 c_1 \left( -147 + 273 c_3 + 128 c_6 \right) \right. \\ & \quad \left. + 104 c_2 \left( -1920 + 2128 c_4 + 1617 c_6 \right) + 4 \left( 55328 c_3^2 + 245280 c_5^2 \right. \right. \\ & \quad \left. \left. - 3003 c_3 \left( -35 + 32 c_5 \right) - 56 \left( 1456 c_3 - 7383 c_4 + 8400 c_5 \right) c_6 - 551694 c_6^2 \right) \right) \end{aligned} $                                      |
| $w_{1,6}$            | $ \begin{aligned} & -2 \left( 28028 c_2 + 780744 c_3 + 196 \left( 143 c_1^2 - 39 c_3 \left( 11 c_2 + 14 \left( c_3 - c_4 \right) \right) \right. \right. \\ & \quad \left. \left. + 2 \left( 2543 c_3 - 585 c_4 \right) c_5 + c_1 \left( 3377 - 286 c_2 - 832 c_3 + 1716 c_4 + 1027 c_5 \right) \right) \right. \\ & \quad \left. - 8 \left( 129703 c_1 + 206241 c_3 - 296725 c_5 \right) c_6 - 49 \left( 58701 c_5 - 847 \left( -55 + 72 c_6 \right) \right) \right) \end{aligned} $                                                                                            |
| $w_{1,7}$            | $ \begin{aligned} & 2 \left( -88088 c_2^2 + 21 \left( 103323 + 104 c_4 \left( 896 + 115 c_4 \right) + 54703 c_5 \right) \right. \\ & \quad \left. + 308 c_2 \left( 3584 + 1248 c_4 - 4047 c_6 \right) - 2697984 c_6 \right. \\ & \quad \left. + 4 c_1 \left( -257994 + 80353 c_3 - 262297 c_5 + 470400 c_6 \right) \right. \\ & \quad \left. - 52 \left( 7 \left( 11 c_3 \left( -21 + 22 c_3 \right) - 608 c_3 c_5 + 690 c_5^2 \right) \right. \right. \\ & \quad \left. \left. + 8 \left( 176 c_3 + 5625 c_4 + 784 c_5 \right) c_6 + 3234 c_6^2 \right) \right) \end{aligned} $ |
| ( <i>continues</i> ) |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                  |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{1,8}$     | $2 (7 (52 c_3 (154 c_3 + 15 (295 + 42 c_4)) + 11 (26317 + 31897 c_5) \\ - 364 (11 c_2 (-1 + 2 c_5) + 2 c_5 (-32 c_3 - 66 c_4 + 45 c_5))) \\ + 4 c_1 (369906 + 21021 c_2 - 33761 c_3 - 26754 c_4 + 228193 c_5 - 568638 c_6) \\ - 8 (343497 + 232661 c_3 + 240359 c_5) c_6)$                  |
| $w_{1,9}$     | $-2 (68068 c_1^2 - 1031976 c_3 + 312 (640 - 427 c_4) c_4 \\ - 1092 c_2 (1792 + 87 c_4) - 77 (28179 + 8367 c_5 - 23296 c_6) \\ - 364 c_1 (1155 + 608 c_3 - 1207 c_5 - 896 c_6) + 8 (427 c_3 (39 c_3 - 313 c_5) \\ + (286858 c_2 + 235200 c_3 - 143 (483 c_4 + 64 c_5)) c_6 + 175329 c_6^2))$ |
| $w_{1,10}$    | $2 (28028 c_1^2 + 1493841 c_5 - 28028 c_2 (1 + 2 c_3 + 3 c_5) - 196 (832 c_3^2 \\ - c_3 (3377 + 1716 c_4 + 624 c_5 - 5294 c_6) - 6 c_5 (91 c_4 + 800 c_5 - 1403 c_6)) \\ - 41503 (-55 + 72 c_6) - 52 c_1 (539 c_3 - 147 (283 + 30 c_4) + 45650 c_6))$                                       |
| $w_{1,11}$    | $2 (2865423 + 256256 c_1 - 442624 c_3 + 1960623 c_5 - 4526336 c_6)$                                                                                                                                                                                                                         |
| $w_{1,12}$    | $2 (2026409 + 28028 c_1^2 + 2192721 c_5 + 28028 c_2 (1 + 2 c_3 + 3 c_5) \\ - 52 c_1 (30975 + 539 c_3 + 4410 c_4 - 35794 c_6) - 2747976 c_6 - 28 (5824 c_3^2 \\ + c_3 (62249 + 12012 c_4 - 4368 c_5 - 68674 c_6) + 6 c_5 (637 c_4 - 5600 c_5 + 13539 c_6)))$                                 |
| $w_{1,13}$    | $-2 (68068 c_1^2 + 312 c_4 (-640 + 427 c_4) - 77 (28179 + 8367 c_5 - 51968 c_6) \\ + 52 c_2 (37632 + 1827 c_4 - 44132 c_6) - 364 c_1 (1155 + 608 c_3 - 1207 c_5 - 896 c_6) \\ + 8 (16653 c_3^2 + 11 (6279 c_4 - 832 c_5 - 15939 c_6) c_6 \\ + c_3 (-128997 - 133651 c_5 + 235200 c_6)))$    |
| $w_{1,14}$    | $33 (17 + 49 c_5 - 64 c_6)$                                                                                                                                                                                                                                                                 |
| $d_2$         | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                       |
| $w_{2,0}$     | $-3234 (-257 + c_3 + 256 c_5)$                                                                                                                                                                                                                                                              |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                                                           |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,1}$     | $  \begin{aligned}  & -2 \left( 56056 c_1^2 - 156 c_2 (-1335 + 539 c_3 + 1078 c_5) \right. \\  & \quad - 52 c_1 (-1617 + 2058 c_2 + 3136 c_3 - 6468 c_4 + 219 c_5 + 4410 c_6) \\  & \quad - 49 (4680 c_3^2 + c_3 (5973 - 1144 c_4 - 13116 c_5) + 88 c_4 (-256 + 321 c_5) \\  & \quad \left. + 3 (14861 + 8 c_5 (-3355 + 1600 c_5 - 1859 c_6) + 13312 c_6)) \right)  \end{aligned}  $ |
| $w_{2,2}$     | $  \begin{aligned}  & 14 \left( 315403 + 38064 c_1^2 + 4 c_1 (25173 + 28501 c_3 - 80558 c_5) \right. \\  & \quad \left. + 19 c_3 (-10277 + 15964 c_5) + 4 c_5 (-130361 + 40149 c_5) \right)  \end{aligned}  $                                                                                                                                                                        |
| $w_{2,3}$     | $  \begin{aligned}  & -2 \left( 107016 c_1^2 + 4312 c_2 (256 + 13 c_3 - 321 c_5) \right. \\  & \quad + 392 c_1 (2673 + 858 c_2 + 2257 c_3 + 585 c_4 - 4384 c_5 - 273 c_6) \\  & \quad + 3 (49 (-14861 + 13312 c_4 + 16852 c_5) + 2548 c_5 (-286 c_4 + 39 c_5 - 22 c_6) \\  & \quad \left. + 69420 c_6 + c_3 (115908 c_5 - 539 (493 + 52 c_6))) \right)  \end{aligned}  $             |
| $w_{2,4}$     | $  \begin{aligned}  & 14 \left( 295955 + 248996 c_1 + 23296 c_1^2 - 123367 c_3 - 23296 c_1 c_3 \right. \\  & \quad \left. - 117852 c_3^2 - 4 (155667 + 72727 c_1 - 81234 c_3) c_5 + 259116 c_5^2 \right)  \end{aligned}  $                                                                                                                                                           |
| $w_{2,5}$     | $  \begin{aligned}  & -2 \left( 56056 c_1^2 + 156 c_2 (-1335 + 539 c_3 + 1078 c_5) \right. \\  & \quad + 52 c_1 (1617 + 2058 c_2 - 3136 c_3 - 6468 c_4 - 219 c_5 + 4410 c_6) \\  & \quad - 49 (4680 c_3^2 + c_3 (5973 + 1144 c_4 - 13116 c_5) - 88 c_4 (-256 + 321 c_5) \\  & \quad \left. + 3 (14861 - 13312 c_6 + 8 c_5 (-3355 + 1600 c_5 + 1859 c_6))) \right)  \end{aligned}  $  |
| $w_{2,6}$     | $  \begin{aligned}  & -14 \left( -315403 + 86528 c_1^2 - 13081 c_3 + 162468 c_3^2 \right. \\  & \quad \left. + 4 c_1 (23639 + 21632 c_3 - 52265 c_5) - 117852 (-1 + 2 c_3) c_5 + 351780 c_5^2 \right)  \end{aligned}  $                                                                                                                                                              |
| $w_{2,7}$     | $  \begin{aligned}  & -2 \left( 107016 c_1^2 - 147 (14861 + 5423 c_3 + 13312 c_4 - 16852 c_5) \right. \\  & \quad - 392 c_1 (-2673 + 858 c_2 - 2257 c_3 + 585 c_4 + 4384 c_5 - 273 c_6) \\  & \quad - 208260 c_6 + 4 (-1078 c_2 (256 + 13 c_3 - 321 c_5) \\  & \quad \left. + 39 c_5 (2229 c_3 + 14014 c_4 + 1911 c_5) + 21021 (c_3 + 2 c_5) c_6) \right)  \end{aligned}  $          |
| $w_{2,8}$     | $  \begin{aligned}  & 14 \left( 295955 + 217692 c_1 + 324425 c_3 + 141148 c_1 c_3 \right. \\  & \quad \left. - 4 (35893 + 75410 c_1 + 70603 c_3) c_5 - 267612 c_5^2 \right)  \end{aligned}  $                                                                                                                                                                                        |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                             |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{2,9}$     | $  \begin{aligned}  & -2 (163072 c_1^2 + 3 (-49 (14861 + 13312 c_2 + c_3 (11979 + 6400 c_3)) \\  & - 52 (-1335 + 539 c_3) c_4 + 196 (5071 + 3718 c_2 + 3200 c_3 - 286 c_4) c_5 - 39468 c_5^2) \\  & - 4312 (256 + 13 c_3 - 321 c_5) c_6 \\  & - 2548 c_1 (165 + 90 c_2 - 48 c_3 + 42 c_4 - 139 c_5 + 132 c_6))  \end{aligned}  $                       |
| $w_{2,10}$    | $  \begin{aligned}  & 14 (315403 + 48464 c_1^2 + 48464 c_3^2 + 52 c_1 (5943 + 1664 c_3 - 6033 c_5) \\  & + c_3 (208329 + 1872 c_5) - 12 c_5 (27183 + 19642 c_5))  \end{aligned}  $                                                                                                                                                                     |
| $w_{2,11}$    | $98 (59103 + 6656 c_1 + 38433 c_3) - 9199104 c_5$                                                                                                                                                                                                                                                                                                      |
| $w_{2,12}$    | $  \begin{aligned}  & -14 (-295955 + 23296 c_1^2 - 23296 c_3^2 + c_3 (154671 - 385464 c_5) \\  & + 52 c_1 (4425 + 448 c_3 - 4983 c_5) + 12 c_5 (14573 + 27078 c_5))  \end{aligned}  $                                                                                                                                                                  |
| $w_{2,13}$    | $  \begin{aligned}  & -2 (163072 c_1^2 - 15288 c_2 (-128 + 143 c_5) \\  & + 3 (-49 (14861 + c_3 (11979 + 6400 c_3)) + 52 (-1335 + 539 c_3) c_4 \\  & + 196 (5071 + 3200 c_3 + 286 c_4) c_5 - 39468 c_5^2) + 4312 (256 + 13 c_3 - 321 c_5) c_6 \\  & + 2548 c_1 (-165 + 90 c_2 + 48 c_3 + 42 c_4 + 139 c_5 + 132 c_6))  \end{aligned}  $                |
| $w_{2,14}$    | $-1617 (-1 + c_3)$                                                                                                                                                                                                                                                                                                                                     |
| $d_3$         | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                                                                  |
| $w_{3,0}$     | $66 (12561 + 49 c_1 - 12480 c_4)$                                                                                                                                                                                                                                                                                                                      |
| $w_{3,1}$     | $  \begin{aligned}  & 2 (-133224 c_2^2 + c_1 (644259 + 439348 c_3 + 73216 c_4 - 1069208 c_5) \\  & - 11 (7 (-28179 + 52 c_3 (105 + 17 c_3) + 51968 c_4) + 93816 c_5) \\  & + 56 (25047 c_4^2 + 33600 c_4 c_5 - 2379 c_5^2 + 208 c_3 (28 c_4 + 19 c_5)) \\  & + 156 c_2 (-1280 + 3542 c_4 - 609 c_6) + 1956864 c_6 - 2294864 c_4 c_6)  \end{aligned}  $ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                       |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,2}$     | $2 (940800 c_1^2 + 28028 c_3^2 + 52 c_3 (4410 c_2 + 45650 c_4 - 49 (849 + 11 c_5))$ $- 3 c_1 (-497947 + 35672 c_2 + 549976 c_4 + 40768 c_5 - 28028 c_6)$ $+ 49 (8 c_4 (-7623 + 2647 c_5) + 11 (4235 + 52 c_6))$ $- 4 c_5 (3377 - 1716 c_2 + 832 c_5 + 286 c_6)))$                                                |
| $w_{3,3}$     | $-2 (981120 c_1^2 - 384384 c_2^2 + 21 (-103323 + 231168 c_4 - 20020 c_5)$ $+ c_1 (-1881600 c_4 + 33 (9201 + 11648 c_5)) + 199680 c_6$ $+ 52 (3619 c_3^2 - 42438 c_4^2 + 11 c_3 (-147 + 128 c_4 - 273 c_5) + 6272 c_4 c_5$ $+ 4256 c_5^2 - 3234 c_4 c_6 - 3003 c_6^2) - 224 c_2 (7383 c_4 - 4 (1232 + 247 c_6)))$ |
| $w_{3,4}$     | $-2 (229320 c_1^2 + 4 (-26 c_5 (2205 c_2 + 17897 c_4 + 539 c_5)$ $+ c_3 (369906 + 26754 c_2 - 568638 c_4 + 33761 c_5 - 21021 c_6))$ $+ 7 c_1 (-350867 + 48048 c_2 + 130396 c_3 + 274696 c_4 + 23296 c_5 - 8008 c_6)$ $+ 7 (-289487 + 392568 c_4 + 230100 c_5 + 4004 c_6)))$                                      |
| $w_{3,5}$     | $2 (312 c_2 (640 + 427 c_2) + c_1 (644259 + 439348 c_3 + 73216 c_4 - 1069208 c_5)$ $- 11 (-197253 + 38220 c_3 + 163072 c_4 + 93816 c_5) - 28 (2431 c_3^2$ $- 416 c_3 (28 c_4 + 19 c_5) + 6 (253 c_4 (13 c_2 + 33 c_4) - 11200 c_4 c_5 + 793 c_5^2))$ $- 1956864 c_6 + 52 (1827 c_2 + 44132 c_4) c_6)$            |
| $w_{3,6}$     | $-2 (229320 c_1^2 - 49 c_1 (1045 + 6864 c_2 - 18628 c_3 + 21176 c_4 - 3328 c_5 - 1144 c_6)$ $+ 49 (-46585 + 60984 c_4 - 44148 c_5 - 572 c_6) + 4 (26 (2205 c_2 + 22825 c_4 - 539 c_5) c_5$ $+ c_3 (195186 - 26754 c_2 - 412482 c_4 + 33761 c_5 + 21021 c_6)))$                                                   |
| $w_{3,7}$     | $-2 (981120 c_1^2 + 384384 c_2^2 + 21 (-103323 + 44800 c_4 - 20020 c_5)$ $+ c_1 (-1881600 c_4 + 33 (9201 + 11648 c_5)) - 199680 c_6$ $+ 52 (3619 c_3^2 + 42438 c_4^2 + 11 c_3 (-147 + 128 c_4 - 273 c_5) + 6272 c_4 c_5$ $+ 4256 c_5^2 + 3234 c_4 c_6 + 3003 c_6^2) + 224 c_2 (7383 c_4 - 4 (1232 + 247 c_6)))$  |
| $w_{3,8}$     | $2 (940800 c_1^2 + 28028 c_3^2 - 52 c_3 (-30975 + 4410 c_2 + 35794 c_4 + 539 c_5)$ $+ 3 c_1 (730907 + 35672 c_2 - 758184 c_4 - 40768 c_5 - 28028 c_6) + 7 (289487$ $- 8 c_4 (49071 + 34337 c_5) - 4004 c_6 + 4 c_5 (62249 - 12012 c_2 - 5824 c_5 + 2002 c_6)))$                                                  |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                                           |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{3,9}$     | $ \begin{aligned} & -2 (251160 c_1^2 + 251160 c_2^2 + 3097344 c_4 \\ & - 7 c_1 (164109 + 149884 c_3 - 46592 c_4 - 31616 c_5) \\ & + 77 (-28179 + 1092 c_5 - 14336 c_6) + 1248 c_2 (-1568 + 1875 c_4 + 308 c_6) \\ & + 4 (c_3 (-257994 + 470400 c_4 - 80353 c_5) \\ & - 11 (3822 c_4^2 + c_4 (1664 c_5 - 28329 c_6) - 2002 (c_5 - c_6) (c_5 + c_6)))) \end{aligned} $ |
| $w_{3,10}$    | $ \begin{aligned} & 2 (2282665 - 2988216 c_4 + 780744 c_5 \\ & + c_1 (2876349 - 229320 c_2 + 201292 c_3 - 2373800 c_4 + 996856 c_5) \\ & - 196 (143 c_3^2 + c_3 (-3377 + 1716 c_2 + 5294 c_4 - 832 c_5 - 286 c_6) \\ & + 3 c_5 (182 c_2 + 2806 c_4 - 182 c_5 - 143 c_6)) + 28028 c_6) \end{aligned} $                                                                |
| $w_{3,11}$    | $2 (2865423 + 1960623 c_1 - 256256 c_3 - 4526336 c_4 + 442624 c_5)$                                                                                                                                                                                                                                                                                                  |
| $w_{3,12}$    | $ \begin{aligned} & 2 (2026409 - 2747976 c_4 + 1479624 c_5 \\ & + c_1 (-897603 + 229320 c_2 + 201292 c_3 + 1861288 c_4 + 996856 c_5) \\ & + 28 (-1001 c_3^2 + c_3 (-62249 + 12012 c_2 + 68674 c_4 + 5824 c_5 - 2002 c_6) \\ & + 3 c_5 (1274 c_2 - 27078 c_4 + 1274 c_5 - 1001 c_6)) - 28028 c_6) \end{aligned} $                                                     |
| $w_{3,13}$    | $ \begin{aligned} & -2 (251160 c_1^2 - 251160 c_2^2 + 2697984 c_4 \\ & - 7 c_1 (164109 + 149884 c_3 - 46592 c_4 - 31616 c_5) \\ & - 1248 c_2 (-1568 + 1875 c_4 + 308 c_6) + 77 (-28179 + 1092 c_5 + 14336 c_6) \\ & + 4 (c_3 (-257994 + 470400 c_4 - 80353 c_5) \\ & + 11 (3822 c_4^2 - c_4 (1664 c_5 + 28329 c_6) + 2002 (c_5^2 + c_6^2)))) \end{aligned} $         |
| $w_{3,14}$    | $33 (17 + 49 c_1 - 64 c_4)$                                                                                                                                                                                                                                                                                                                                          |
| $d_4$         | $\frac{1}{46126080}$                                                                                                                                                                                                                                                                                                                                                 |
| $w_{4,0}$     | $462 (257 + c_1 - 256 c_3)$                                                                                                                                                                                                                                                                                                                                          |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                    |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,1}$     | $14 \left( 44583 - 67716 c_3 + 7228 c_3^2 - 22528 c_4 + 28248 c_3 c_4 \right. \\ + 156 c_2 (-25 + 22 c_3 - 14 c_5) - 21384 c_5 + 41728 c_3 c_5 + 6864 c_4 c_5 \\ - 2184 c_5^2 + c_1 (-16269 + 1716 c_2 - 4108 c_3 - 1144 c_4 + 18056 c_5) \\ \left. - 312 (-128 + 143 c_3 + 15 c_5) c_6 \right)$              |
| $w_{4,2}$     | $-14 \left( -47509 + 16836 c_1^2 - 47476 c_3^2 + c_3 (105132 - 23492 c_5) \right. \\ \left. + c_1 (-3361 + 33672 c_3 - 3328 c_5) + 4 (3377 - 832 c_5) c_5 \right)$                                                                                                                                            |
| $w_{4,3}$     | $14 \left( 19200 c_1^2 - 3 (-14861 + 19140 c_3 + 13312 c_4 - 2860 c_5 + 1300 c_6) \right. \\ + c_1 (-35937 - 1144 c_2 + 38400 c_3 + 2496 c_5 + 1716 c_6) \\ + 4 (22 c_2 (-256 + 321 c_3 + 78 c_5) \\ \left. - 13 (11 c_3^2 + 2 c_5 (-45 c_4 + 32 c_5 + 21 c_6) - 3 c_3 (286 c_4 - 39 c_5 + 22 c_6))) \right)$ |
| $w_{4,4}$     | $2 \left( 278795 + 48464 c_1^2 + 324936 c_3^2 + c_1 (135223 - 124592 c_3 - 86528 c_5) \right. \\ \left. + 52 c_5 (-4425 + 932 c_5) + 100 c_3 (-5931 + 2405 c_5) \right)$                                                                                                                                      |
| $w_{4,5}$     | $-14 \left( -44583 + 67716 c_3 - 7228 c_3^2 - 22528 c_4 + 28248 c_3 c_4 \right. \\ + c_1 (16269 + 1716 c_2 + 4108 c_3 - 1144 c_4 - 18056 c_5) + 156 c_2 (-25 + 22 c_3 - 14 c_5) \\ \left. + 21384 c_5 - 41728 c_3 c_5 + 6864 c_4 c_5 + 2184 c_5^2 - 312 (-128 + 143 c_3 + 15 c_5) c_6 \right)$                |
| $w_{4,6}$     | $14 \left( 47509 + 3328 c_1^2 + 33672 c_3^2 + 52 (849 - 64 c_5) c_5 \right. \\ \left. + c_1 (-26983 + 37000 c_3 + 3328 c_5) - 4 c_3 (19455 + 11869 c_5) \right)$                                                                                                                                              |
| $w_{4,7}$     | $14 \left( 19200 c_1^2 - 88 c_2 (-256 + 321 c_3 + 78 c_5) \right. \\ + c_1 (-35937 + 1144 c_2 + 38400 c_3 + 2496 c_5 - 1716 c_6) \\ + 3 (14861 + 13312 c_4 + 2860 c_5 + 1300 c_6) \\ \left. - 4 (143 c_3^2 + 26 c_5 (45 c_4 + 32 c_5 - 21 c_6) + 3 c_3 (4785 + 3718 c_4 + 507 c_5 + 286 c_6)) \right)$        |
| $w_{4,8}$     | $-2 \left( -278795 + 162468 c_1^2 + 278564 c_3^2 + c_1 (-86759 + 324936 c_3 - 86528 c_5) \right. \\ \left. + 4 c_5 (-62249 + 21632 c_5) + 4 c_3 (40617 + 83881 c_5) \right)$                                                                                                                                  |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$        | Expression                                                                                                                                                                                                                                                                                                    |
|----------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{4,9}$            | $14 (44583 + 4680 c_1^2 - 37752 c_3 - 38400 c_3^2 - 3900 c_4$ $+ 3432 c_3 c_4 - 1716 c_5 + 2756 c_3 c_5 - 2184 c_4 c_5 - 1144 c_5^2$ $- 312 c_2 (-128 + 143 c_3 + 15 c_5) - 88 (-256 + 321 c_3 + 78 c_5) c_6$ $+ c_1 (-5973 - 26428 c_3 + 1716 c_4 - 3328 c_5 + 1144 c_6))$                                   |
| $w_{4,10}$           | $-14 (-47509 - 16836 c_5 + 4 c_3 (11869 + 5041 c_3 + 9250 c_5)$ $+ c_1 (-30673 + 50804 c_3 + 20164 c_5))$                                                                                                                                                                                                     |
| $w_{4,11}$           | $14 (59103 - 38433 c_1 - 90880 c_3 + 6656 c_5)$                                                                                                                                                                                                                                                               |
| $w_{4,12}$           | $2 (278795 + 287060 c_3^2 + c_1 (-343873 + 230100 c_3 - 114004 c_5)$ $+ 12 c_5 (16711 + 3172 c_5) - 4 c_3 (160391 + 59602 c_5))$                                                                                                                                                                              |
| $w_{4,13}$           | $14 (44583 + 4680 c_1^2 + 3900 c_4 - 24 c_3 (1600 c_3 + 143 (11 + c_4)) - 1716 c_5$ $+ 52 (53 c_3 + 42 c_4) c_5 - 1144 c_5^2 + 312 c_2 (-128 + 143 c_3 + 15 c_5) - 22528 c_6$ $+ 264 (107 c_3 + 26 c_5) c_6 - c_1 (5973 + 26428 c_3 + 1716 c_4 + 3328 c_5 + 1144 c_6))$                                       |
| $w_{4,14}$           | $231 (1 + c_1)$                                                                                                                                                                                                                                                                                               |
| $d_5$                | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                         |
| $w_{5,0}$            | $-66 (-12561 + 12480 c_2 + 49 c_3)$                                                                                                                                                                                                                                                                           |
| $w_{5,1}$            | $-2 (88088 c_1^2 - 168168 c_2^2 - 77 (28179 + 8 c_4 (-1792 + 143 c_4))$ $- 52 c_1 (1408 c_2 + 7 (-231 + 608 c_3 - 883 c_5))$ $+ 7 c_3 (164109 + 35880 c_3 - 149884 c_5) + 1031976 c_5$ $- 4 c_2 (81536 c_3 + 3 (-258112 + 103873 c_4 + 156800 c_5 - 195000 c_6))$ $- 34944 (56 + 11 c_4) c_6 + 251160 c_6^2)$ |
| ( <i>continues</i> ) |                                                                                                                                                                                                                                                                                                               |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                               |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,2}$     | $2 (107016 c_1^2 + 8 c_2 (296725 c_3 + 49 (-7623 + 2647 c_5))$ $- 4 c_1 (-195186 + 412482 c_2 + 249214 c_3 + 21021 c_4 + 40768 c_5 + 26754 c_6)$ $+ 49 (11 (4235 + 52 c_4 (-1 + 2 c_5) - 4 c_5 (307 + 13 c_5 - 156 c_6))$ $+ c_3 (-58701 + 4108 c_5 + 4680 c_6)))$                                       |
| $w_{5,3}$     | $-2 (133224 c_1^2 - 8 c_1 (-128997 + 235200 c_2 + 133651 c_3 - 27664 c_5)$ $+ 21 (-103323 + 30679 c_3 + 93184 c_4 - 20020 c_5) + 199680 c_6$ $- 4 (350658 c_2^2 - 4 c_2 (250096 + 4576 c_3 - 143429 c_4 + 20384 c_5)$ $+ 138138 c_2 c_6 + 91 (1207 c_3 c_5 - 187 c_5^2 + 261 c_4 c_6 - 366 c_6^2)))$     |
| $w_{5,4}$     | $-2 (163072 c_1^2 - 3 c_3 (-730907 + 313600 c_3 - 28028 c_4)$ $- 8 c_2 (-343497 + 284319 c_3 + 232661 c_5)$ $- 7 (289487 + 4004 c_4 + 52 c_5 (-4425 + 77 c_5)) + 15288 (7 c_3 - 15 c_5) c_6$ $+ 28 c_1 (-62249 + 68674 c_2 - 4368 c_3 + 2002 c_4 - 1001 c_5 + 12012 c_6))$                               |
| $w_{5,5}$     | $-2 (88088 c_1^2 + 168168 c_2^2 - 52 c_1 (1408 c_2 + 7 (-231 + 608 c_3 - 883 c_5))$ $+ 7 c_3 (164109 + 35880 c_3 - 149884 c_5)$ $+ 11 (-197253 + 56 c_4 (-1792 + 143 c_4) + 93816 c_5) + 34944 (56 + 11 c_4) c_6$ $- 251160 c_6^2 - 4 c_2 (-674496 + 81536 c_3 - 311619 c_4 + 470400 c_5 + 585000 c_6))$ |
| $w_{5,6}$     | $2 (-163072 c_1^2 + 3 c_3 (-497947 + 313600 c_3 + 28028 c_4)$ $+ 8 c_2 (-373527 + 206241 c_3 - 296725 c_5)$ $+ 49 (46585 - 572 c_4 + 52 c_5 (849 + 11 c_5)) + 15288 (7 c_3 - 15 c_5) c_6$ $+ 196 c_1 (-3377 + 5294 c_2 + 624 c_3 + 286 c_4 + 143 c_5 + 1716 c_6))$                                       |
| $w_{5,7}$     | $-2 (133224 c_1^2 - 8 c_1 (-128997 + 235200 c_2 + 133651 c_3 - 27664 c_5)$ $+ 21 (-103323 + 30679 c_3 - 93184 c_4 - 20020 c_5) - 199680 c_6$ $+ 4 (350658 c_2^2 + 26 c_2 (17248 + 704 c_3 + 22066 c_4 + 3136 c_5 + 5313 c_6)$ $- 91 (1207 c_3 c_5 - 187 c_5^2 - 261 c_4 c_6 + 366 c_6^2)))$              |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                          |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{5,8}$     | $2 \left( 107016 c_1^2 - 8 c_2 (343497 + 232661 c_3 + 240359 c_5) \right.$ $- 4 c_1 (-369906 + 568638 c_2 + 249214 c_3 - 21021 c_4 + 40768 c_5 - 26754 c_6)$ $+ 7 (11 (26317 + 364 c_4 + 22636 c_5)$ $+ c_3 (128229 + 28756 c_5 - 32760 c_6) - 4004 c_5 (2 c_4 + c_5 + 12 c_6)))$                                   |
| $w_{5,9}$     | $-2 \left( 221312 c_1^2 + 2206776 c_2^2 + 3 c_3 (-101211 + 327040 c_3) \right.$ $+ 364 c_1 (896 c_2 - 33 (35 + 32 c_3 - 13 c_5))$ $+ 77 (-28179 + 52 c_5 (21 + 47 c_5) - 14336 c_6)$ $+ 8 c_2 (117600 + 235200 c_3 - 21021 c_4 - 9152 c_5 + 206724 c_6)$ $+ 52 (3003 c_4^2 + 7392 c_6^2 + 32 c_4 (120 + 133 c_6)))$ |
| $w_{5,10}$    | $2 \left( 56056 c_1^2 - 539 (-4235 + 95 c_3 + 52 c_4) + 780744 c_5 \right.$ $- 52 c_1 (45650 c_2 - 49 (849 + 64 c_3 + 53 c_5 - 90 c_6)) - 196 (1170 c_3^2 + 286 c_3 c_4$ $+ 4657 c_3 c_5 + 429 c_4 c_5 + 2 c_2 (7623 + 2647 c_3 + 4209 c_5) + 78 (22 c_3 + 7 c_5) c_6))$                                            |
| $w_{5,11}$    | $2 (2865423 + 442624 c_1 - 4526336 c_2 - 1960623 c_3 + 256256 c_5)$                                                                                                                                                                                                                                                 |
| $w_{5,12}$    | $2 \left( 2026409 + 56056 c_1^2 - 2456069 c_3 + 28028 c_4 + 1479624 c_5 \right.$ $+ 52 c_1 (35794 c_2 + 7 (-4425 + 448 c_3 + 371 c_5 + 630 c_6))$ $+ 28 (c_2 (-98142 + 68674 c_3 - 81234 c_5)$ $+ 7 (-1170 c_3^2 + 286 c_3 c_4 - 4657 c_3 c_5 + 429 c_4 c_5 + 78 (22 c_3 + 7 c_5) c_6)))$                           |
| $w_{5,13}$    | $-2 \left( 221312 c_1^2 - 2206776 c_2^2 + 3 c_3 (-101211 + 327040 c_3) \right.$ $+ 364 c_1 (896 c_2 - 33 (35 + 32 c_3 - 13 c_5))$ $+ 8 c_2 (606816 + 235200 c_3 + 21021 c_4 - 9152 c_5 - 206724 c_6)$ $+ 77 (-28179 + 52 c_5 (21 + 47 c_5) + 14336 c_6)$ $- 52 (3003 c_4^2 + 7392 c_6^2 + 32 c_4 (120 + 133 c_6)))$ |
| $w_{5,14}$    | $-33 (-17 + 64 c_2 + 49 c_3)$                                                                                                                                                                                                                                                                                       |
| $d_6$         | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                               |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                     |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,0}$     | $-3234 (-257 + 256 c_1 - c_5)$                                                                                                                                                                                                                                                                 |
| $w_{6,1}$     | $2 (118404 c_1^2 - 156 c_2 (1335 + 686 c_3 + 539 c_5)$<br>$+ 196 c_1 (858 c_2 + 1807 c_3 + 3 (-5071 + 2354 c_4 + 3200 c_5 - 3718 c_6))$<br>$- 49 (3328 c_3^2 - 88 c_4 (-256 + 13 c_5) - 156 c_3 (-55 + 44 c_4 - 16 c_5 - 30 c_6)$<br>$- 3 (14861 + c_5 (-11979 + 6400 c_5) + 13312 c_6)))$     |
| $w_{6,2}$     | $-14 (235704 c_1^2 - 156 c_1 (-2091 + 2011 c_3 - 12 c_5)$<br>$+ 11 (-28673 + 18939 c_5) - 52 (c_3 (-5943 + 932 c_3) + 1664 c_3 c_5 + 932 c_5^2))$                                                                                                                                              |
| $w_{6,3}$     | $2 (1881600 c_1^2 + 49 (88 c_2 (-256 + 78 c_3 + 13 c_5) - 3 (-14861 + 13312 c_4 + 1991 c_5)$<br>$+ 52 (-22 c_3^2 + 90 c_5^2 + c_3 (33 + 90 c_4 + 64 c_5))) - 156 (1335 + 686 c_3 + 539 c_5) c_6$<br>$+ 12 c_1 (115346 c_2 - 949 c_3 + 49 (-6710 + 3718 c_4 + 1093 c_5 + 286 c_6)))$            |
| $w_{6,4}$     | $-14 (-295955 + 4 c_1 (35893 + 66903 c_1 - 75410 c_3)$<br>$+ 217692 c_3 + 324425 c_5 - 4 (70603 c_1 + 35287 c_3) c_5)$                                                                                                                                                                         |
| $w_{6,5}$     | $2 (118404 c_1^2 + 156 c_2 (1335 + 686 c_3 + 539 c_5)$<br>$- 196 c_1 (858 c_2 - 1807 c_3 + 3 (5071 + 2354 c_4 - 3200 c_5 - 3718 c_6))$<br>$- 49 (3328 c_3^2 + 88 c_4 (-256 + 13 c_5) + 156 c_3 (55 + 44 c_4 + 16 c_5 - 30 c_6)$<br>$+ 3 (-14861 + (11979 - 6400 c_5) c_5 + 13312 c_6)))$       |
| $w_{6,6}$     | $14 (315403 + 160596 c_1^2 + 4 c_1 (-130361 + 80558 c_3 - 75829 c_5)$<br>$+ 195263 c_5 + 4 c_3 (-25173 + 9516 c_3 + 28501 c_5))$                                                                                                                                                               |
| $w_{6,7}$     | $2 (1103872 c_2 + 147 (14861 + 572 c_3 + 13312 c_4 - 1991 c_5)$<br>$+ 208260 c_6 + 4 (470400 c_1^2$<br>$- 637 (22 c_3^2 + 22 c_2 (6 c_3 + c_5) + 2 c_3 (45 c_4 - 32 c_5 - 21 c_6) - 3 c_5 (30 c_5 + 11 c_6))$<br>$- 3 c_1 (115346 c_2 + 949 c_3 + 49 (6710 + 3718 c_4 - 1093 c_5 + 286 c_6)))$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                      |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{6,8}$     | $-14 \left( -230100 c_3 - 11 (26905 + 14061 c_5) \right. \\ \left. + 4 (81234 c_1^2 + 3 c_1 (14573 + 21593 c_3 + 32122 c_5) + 5824 (c_3^2 + c_3 c_5 - c_5^2)) \right)$                                                                                                                                                                          |
| $w_{6,9}$     | $-2 \left( 298116 c_1^2 - 1617 (1351 + 648 c_3) + 208260 c_4 \right. \\ \left. + 15288 (c_2 (-128 + 15 c_3) + 7 c_3 (c_3 + c_4)) \right. \\ \left. + 4 c_1 (49 (12639 + 11154 c_2 + 8768 c_3 - 858 c_4) - 86931 c_5) \right. \\ \left. + 49 (18056 c_3 + 33 (493 + 52 c_4)) c_5 \right) - 8624 (-256 + 321 c_1 + 78 c_3 + 13 c_5) c_6$          |
| $w_{6,10}$    | $-14 \left( -315403 + 351780 c_1^2 + 86528 c_3^2 + 4 c_3 (-23639 + 21632 c_5) \right. \\ \left. + c_5 (13081 + 162468 c_5) + 4 c_1 (52265 c_3 + 29463 (1 + 2 c_5)) \right)$                                                                                                                                                                     |
| $w_{6,11}$    | $-2 (4599552 c_1 + 49 (-59103 + 6656 c_3 + 38433 c_5))$                                                                                                                                                                                                                                                                                         |
| $w_{6,12}$    | $14 \left( 295955 + 259116 c_1^2 + 23296 c_3^2 + 4 c_1 (-155667 + 72727 c_3 - 81234 c_5) \right. \\ \left. + (123367 - 117852 c_5) c_5 - 4 c_3 (62249 + 5824 c_5) \right)$                                                                                                                                                                      |
| $w_{6,13}$    | $2 \left( -298116 c_1^2 + 15288 c_2 (-128 + 15 c_3) \right. \\ \left. + 208260 c_4 - 539 (-4053 + 1479 c_5 + 2048 c_6) \right. \\ \left. + 4 c_1 (-619311 + 546546 c_2 - 429632 c_3 - 42042 c_4 + 86931 c_5 + 346038 c_6) \right. \\ \left. + 196 (-546 c_3^2 + 143 c_5 (3 c_4 + 2 c_6) + 2 c_3 (2673 + 273 c_4 - 2257 c_5 + 858 c_6)) \right)$ |
| $w_{6,14}$    | $1617 (1 + c_5)$                                                                                                                                                                                                                                                                                                                                |
| $d_7$         | $\frac{1}{10090080}$                                                                                                                                                                                                                                                                                                                            |
| $w_{7,0}$     | $25806$                                                                                                                                                                                                                                                                                                                                         |
| $w_{7,1}$     | $2 (6006 c_1 - 6240 c_2 - 11 (-4095 + 2730 c_3 + 3136 c_4 - 234 c_5) + 61152 c_6)$                                                                                                                                                                                                                                                              |
| $w_{7,2}$     | $-16692 c_1 - 98 (-1001 + 1326 c_3 + 154 c_5)$                                                                                                                                                                                                                                                                                                  |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                         |
|---------------|------------------------------------------------------------------------------------|
| $w_{7,3}$     | $2 (2574 c_1 - 34496 c_2 - 39 (7 (22 c_3 + 224 c_4 - 55 (3 + 2 c_5)) + 160 c_6))$  |
| $w_{7,4}$     | $135212 c_1 - 26 (-3157 + 1038 c_3 + 4074 c_5)$                                    |
| $w_{7,5}$     | $2 (6006 c_1 + 6240 c_2 - 30030 c_3 + 11 (4095 + 3136 c_4 + 234 c_5) - 61152 c_6)$ |
| $w_{7,6}$     | $-15092 c_1 + 26 (3773 + 642 c_3 + 4998 c_5)$                                      |
| $w_{7,7}$     | $2 (2574 c_1 + 34496 c_2 + 39 (1155 - 154 c_3 + 1568 c_4 + 770 c_5 + 160 c_6))$    |
| $w_{7,8}$     | $26 (3157 + 1038 c_1 + 4074 c_3) + 135212 c_5$                                     |
| $w_{7,9}$     | $78 (1155 + 770 c_1 + 1568 c_2 - 66 c_3 - 160 c_4 + 154 c_5) + 68992 c_6$          |
| $w_{7,10}$    | $129948 c_1 + 7546 (13 + 2 c_3) - 16692 c_5$                                       |
| $w_{7,11}$    | 12870                                                                              |
| $w_{7,12}$    | $82082 - 105924 c_1 - 135212 c_3 + 26988 c_5$                                      |
| $w_{7,13}$    | $78 (1155 + 770 c_1 - 1568 c_2 - 66 c_3 + 160 c_4 + 154 c_5) - 68992 c_6$          |
| $w_{7,14}$    | -33                                                                                |
| $d_8$         | $\frac{1}{46126080}$                                                               |
| $w_{8,0}$     | $462 (257 + 256 c_1 + c_5)$                                                        |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                            |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,1}$     | $ \begin{aligned} & -14 (572 c_1^2 + 11 (-4053 + 780 c_3 + 2048 c_4 + 3267 c_5) \\ & + 4 (-39 c_2 (-25 + 14 c_3 + 11 c_5) \\ & + 2 (416 c_3^2 + (143 c_4 - 2400 c_5) c_5 + 39 c_3 (22 c_4 + 8 c_5 - 15 c_6))) \\ & + 12 c_1 (-4785 + 286 c_2 + 507 c_3 + 2354 c_4 + 3200 c_5 - 3718 c_6) - 39936 c_6) \end{aligned} $ |
| $w_{8,2}$     | $ \begin{aligned} & 14 (47509 - 44148 c_3 - 26983 c_5 \\ & + 4 (8418 c_1^2 + c_1 (19455 - 11869 c_3 - 9250 c_5) - 832 (c_3^2 + c_3 c_5 - c_5^2))) \end{aligned} $                                                                                                                                                     |
| $w_{8,3}$     | $ \begin{aligned} & -14 (38400 c_1^2 + 88 c_2 (256 + 78 c_3 + 13 c_5) \\ & + 4 c_1 (7062 c_2 - 689 c_3 + 11154 c_4 - 6607 c_5 + 858 (-11 + c_6)) \\ & + 3 (-14861 + 13312 c_4 + 1991 c_5 + 1300 c_6) \\ & + 52 (22 c_3^2 + c_3 (-33 + 90 c_4 - 64 c_5 - 42 c_6) - 3 c_5 (30 c_5 + 11 c_6))) \end{aligned} $           |
| $w_{8,4}$     | $ \begin{aligned} & 2 (278795 + 287060 c_1^2 - 343873 c_5 + 4 c_3 (-50133 + 9516 c_3 + 28501 c_5) \\ & - 4 c_1 (-160391 + 59602 c_3 + 57525 c_5)) \end{aligned} $                                                                                                                                                     |
| $w_{8,5}$     | $ \begin{aligned} & -14 (572 c_1^2 + 11 (-4053 + 780 c_3 - 2048 c_4 + 3267 c_5) \\ & + 4 (39 c_2 (-25 + 14 c_3 + 11 c_5) \\ & + 2 (416 c_3^2 - c_5 (143 c_4 + 2400 c_5) - 39 c_3 (22 c_4 - 8 c_5 - 15 c_6))) \\ & - 12 c_1 (4785 + 286 c_2 - 507 c_3 + 2354 c_4 - 3200 c_5 - 3718 c_6) + 39936 c_6) \end{aligned} $   |
| $w_{8,6}$     | $ \begin{aligned} & -14 (-47509 + 20164 c_1^2 + c_3 (16836 - 20164 c_5) \\ & + 4 c_1 (-11869 + 9250 c_3 - 12701 c_5) - 30673 c_5) \end{aligned} $                                                                                                                                                                     |
| $w_{8,7}$     | $ \begin{aligned} & -14 (38400 c_1^2 - 88 c_2 (256 + 78 c_3 + 13 c_5) \\ & - 3 (14861 + 13312 c_4 - 1991 c_5 + 1300 c_6) \\ & - 4 c_1 (7062 c_2 + 689 c_3 + 11154 c_4 + 6607 c_5 + 858 (11 + c_6)) \\ & + 52 (22 c_3^2 - c_3 (33 + 90 c_4 + 64 c_5 - 42 c_6) + 3 c_5 (-30 c_5 + 11 c_6))) \end{aligned} $             |
| $w_{8,8}$     | $ \begin{aligned} & 2 (278795 + 230100 c_3 + 135223 c_5 + 4 (81234 c_1^2 \\ & + c_1 (148275 + 60125 c_3 + 31148 c_5) + 52 (233 c_3^2 + 416 c_3 c_5 + 233 c_5^2))) \end{aligned} $                                                                                                                                     |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{8,9}$     | $14 (44583 + 7228 c_1^2 + 21384 c_3 - 2184 c_3^2 + 312 c_2 (128 + 15 c_3) - 3900 c_4$<br>$+ 2184 c_3 c_4 - 16269 c_5 - 18056 c_3 c_5 + 1716 c_4 c_5 + 88 (256 + 78 c_3 + 13 c_5) c_6$<br>$+ 4 c_1 (16929 + 11154 c_2 + 10432 c_3 - 858 c_4 + 1027 c_5 + 7062 c_6))$                                                       |
| $w_{8,10}$    | $14 (47509 + 47476 c_1^2 + 3328 c_3^2 + c_3 (13508 - 3328 c_5)$<br>$+ (3361 - 16836 c_5) c_5 + 4 c_1 (26283 + 5873 c_3 + 8418 c_5))$                                                                                                                                                                                      |
| $w_{8,11}$    | $14 (59103 + 90880 c_1 - 6656 c_3 - 38433 c_5)$                                                                                                                                                                                                                                                                           |
| $w_{8,12}$    | $-2 (-278795 + 278564 c_1^2 + 86528 c_3^2 + 4 c_3 (62249 + 21632 c_5)$<br>$+ c_5 (-86759 + 162468 c_5) + 4 c_1 (83881 c_3 - 40617 (1 + 2 c_5)))$                                                                                                                                                                          |
| $w_{8,13}$    | $14 (44583 + 7228 c_1^2 + 21384 c_3 + 3900 c_4 - 16269 c_5 - 22528 c_6$<br>$- 4 c_1 (-16929 + 11154 c_2 - 10432 c_3 - 858 c_4 - 1027 c_5 + 7062 c_6) - 4 (546 c_3^2$<br>$+ 78 c_2 (128 + 15 c_3) + 546 c_3 c_4 + 4514 c_3 c_5 + 429 c_4 c_5 + 286 (6 c_3 + c_5) c_6))$                                                    |
| $w_{8,14}$    | $231 (1 + c_5)$                                                                                                                                                                                                                                                                                                           |
| $d_9$         | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                                     |
| $w_{9,0}$     | $66 (12561 + 12480 c_2 - 49 c_3)$                                                                                                                                                                                                                                                                                         |
| $w_{9,1}$     | $-2 (88088 c_1^2 + 168168 c_2^2 + 7 c_3 (164109 + 35880 c_3 - 149884 c_5)$<br>$+ 11 (-197253 + 56 c_4 (1792 + 143 c_4) + 93816 c_5)$<br>$+ 52 c_1 (1408 c_2 + 7 (231 - 608 c_3 + 883 c_5))$<br>$+ 4 c_2 (81536 c_3 + 3 (-224832 + 103873 c_4 + 156800 c_5 - 195000 c_6))$<br>$+ 34944 (-56 + 11 c_4) c_6 - 251160 c_6^2)$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                            |
|---------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,2}$     | $2 \left( 107016 c_1^2 - 8 c_2 (296725 c_3 + 49 (-7623 + 2647 c_5)) \right. \\ + 4 c_1 (195186 + 412482 c_2 - 249214 c_3 + 21021 c_4 - 40768 c_5 + 26754 c_6) \\ + 49 (11 (4235 + 52 c_4 - 1228 c_5) - 572 c_5 (2 c_4 + c_5 + 12 c_6) \\ \left. + c_3 (4108 c_5 - 3 (19567 + 1560 c_6))) \right)$                                     |
| $w_{9,3}$     | $-2 \left( 133224 c_1^2 + 21 (-103323 + 30679 c_3 + 93184 c_4 - 20020 c_5) \right. \\ + 8 c_1 (128997 + 235200 c_2 - 133651 c_3 + 27664 c_5) + 199680 c_6 \\ + 4 (350658 c_2^2 - 26 c_2 (17248 + 704 c_3 - 22066 c_4 + 3136 c_5 - 5313 c_6) \\ \left. - 91 (1207 c_3 c_5 - 187 c_5^2 - 261 c_4 c_6 + 366 c_6^2)) \right)$             |
| $w_{9,4}$     | $-2 \left( -2026409 + 163072 c_1^2 + 2192721 c_3 + 28028 c_4 + 1610700 c_5 \right. \\ - 28 c_1 (62249 + 68674 c_2 + 4368 c_3 + 2002 c_4 + 1001 c_5 + 12012 c_6) \\ + 4 (2 c_2 (-343497 + 284319 c_3 + 232661 c_5) \\ \left. - 49 (4800 c_3^2 + 429 c_3 c_4 + 143 c_5^2 + 78 (7 c_3 - 15 c_5) c_6)) \right)$                           |
| $w_{9,5}$     | $-2 \left( 88088 c_1^2 - 168168 c_2^2 - 77 (28179 + 8 c_4 (1792 + 143 c_4)) \right. \\ + 7 c_3 (164109 + 35880 c_3 - 149884 c_5) + 1031976 c_5 \\ + 52 c_1 (1408 c_2 + 7 (231 - 608 c_3 + 883 c_5)) - 34944 (-56 + 11 c_4) c_6 \\ \left. + 251160 c_6^2 + 4 c_2 (-774336 + 81536 c_3 - 311619 c_4 + 470400 c_5 + 585000 c_6) \right)$ |
| $w_{9,6}$     | $-2 \left( 163072 c_1^2 - 3 c_3 (-497947 + 313600 c_3 - 28028 c_4) \right. \\ + 8 c_2 (-373527 + 206241 c_3 - 296725 c_5) \\ - 49 (46585 + 572 c_4 + 52 c_5 (849 + 11 c_5)) + 15288 (7 c_3 - 15 c_5) c_6 \\ \left. + 196 c_1 (3377 + 5294 c_2 - 624 c_3 + 286 c_4 - 143 c_5 + 1716 c_6) \right)$                                      |
| $w_{9,7}$     | $2 \left( -133224 c_1^2 - 8 c_1 (128997 + 235200 c_2 - 133651 c_3 + 27664 c_5) \right. \\ + 3 (723261 - 214753 c_3 + 652288 c_4 + 140140 c_5 + 66560 c_6) \\ + 4 (350658 c_2^2 + c_2 (18304 c_3 + 573716 c_4 + 3136 (319 + 26 c_5) + 138138 c_6) \\ \left. + 91 (1207 c_3 c_5 - 187 c_5^2 + 261 c_4 c_6 - 366 c_6^2)) \right)$        |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                             |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{9,8}$     | $2 \left( 107016 c_1^2 + 8 c_2 (343497 + 232661 c_3 + 240359 c_5) \right. \\ + 4 c_1 (369906 + 568638 c_2 - 249214 c_3 - 21021 c_4 - 40768 c_5 - 26754 c_6) \\ + 7 (c_3 (128229 + 28756 c_5 + 32760 c_6) \\ \left. + 11 (26317 + 22636 c_5 + 364 (-c_4 + 2 c_4 c_5 - c_5^2 + 12 c_5 c_6))) \right)$                                    |
| $w_{9,9}$     | $-2 \left( 221312 c_1^2 - 2206776 c_2^2 + 3 c_3 (-101211 + 327040 c_3) \right. \\ - 364 c_1 (896 c_2 + 33 (35 + 32 c_3 - 13 c_5)) \\ + 77 (-28179 + 52 c_5 (21 + 47 c_5) - 14336 c_6) \\ - 8 c_2 (606816 + 235200 c_3 - 21021 c_4 - 9152 c_5 + 206724 c_6) \\ \left. - 52 (3003 c_4^2 + 7392 c_6^2 + 32 c_4 (-120 + 133 c_6)) \right)$ |
| $w_{9,10}$    | $2 \left( 56056 c_1^2 - 539 (-4235 + 95 c_3 - 52 c_4) + 780744 c_5 \right. \\ + 196 (-1170 c_3^2 + 286 c_3 c_4 - 4657 c_3 c_5 + 429 c_4 c_5 + 2 c_2 (7623 + 2647 c_3 + 4209 c_5) \\ \left. + 78 (22 c_3 + 7 c_5) c_6) + 52 c_1 (45650 c_2 + 49 (849 + 64 c_3 + 53 c_5 + 90 c_6)) \right)$                                              |
| $w_{9,11}$    | $2 (2865423 + 442624 c_1 + 4526336 c_2 - 1960623 c_3 + 256256 c_5)$                                                                                                                                                                                                                                                                    |
| $w_{9,12}$    | $2 \left( 56056 c_1^2 - 77 (-26317 + 31897 c_3 + 364 c_4) + 1479624 c_5 \right. \\ - 52 c_1 (35794 c_2 + 7 (4425 - 448 c_3 - 371 c_5 + 630 c_6)) \\ - 28 (c_2 (-98142 + 68674 c_3 - 81234 c_5) \\ \left. + 7 (1170 c_3^2 + 286 c_3 c_4 + 4657 c_3 c_5 + 429 c_4 c_5 + 78 (22 c_3 + 7 c_5) c_6)) \right)$                               |
| $w_{9,13}$    | $-2 \left( 221312 c_1^2 + 2206776 c_2^2 + 3 c_3 (-101211 + 327040 c_3) \right. \\ - 364 c_1 (896 c_2 + 33 (35 + 32 c_3 - 13 c_5)) \\ - 8 c_2 (117600 + 235200 c_3 + 21021 c_4 - 9152 c_5 - 206724 c_6) \\ + 77 (-28179 + 52 c_5 (21 + 47 c_5) + 14336 c_6) \\ \left. + 52 (3003 c_4^2 + 7392 c_6^2 + 32 c_4 (-120 + 133 c_6)) \right)$ |
| $w_{9,14}$    | $33 (17 + 64 c_2 - 49 c_3)$                                                                                                                                                                                                                                                                                                            |
| $d_{10}$      | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                                                  |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                            |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{10,0}$    | $3234 (257 + c_1 + 256 c_3)$                                                                                                                                                                                                                                                                          |
| $w_{10,1}$    | $-2 (156 c_2 (1335 + 1078 c_3 - 686 c_5)$<br>$+ c_1 (84084 c_2 + 347724 c_3 - 49 (-16269 + 1144 c_4 + 18056 c_5))$<br>$+ 49 (-44583 + 22528 c_4 + 21384 c_5 - 39936 c_6$<br>$+ 4 (1521 c_3^2 + 78 c_5 (22 c_4 + 7 c_5 - 15 c_6) + c_3 (7062 c_4 + 8768 c_5 - 33 (383 + 338 c_6))))))$                 |
| $w_{10,2}$    | $-14 (-315403 + 162468 c_1^2 + 351780 c_3^2 + c_1 (13081 - 235704 c_3 - 86528 c_5)$<br>$+ 4 c_5 (23639 + 21632 c_5) + 4 c_3 (-29463 + 52265 c_5))$                                                                                                                                                    |
| $w_{10,3}$    | $2 (940800 c_1^2 + 147 (14861 + 20284 c_3 - 13312 c_4 + 2860 c_5)$<br>$- 208260 c_6 + 49 c_1 (1144 c_2 - 3 (11979 + 12800 c_3 - 832 c_5 + 572 c_6))$<br>$- 4 (1078 c_2 (256 + 321 c_3 + 78 c_5)$<br>$- 13 (2277 c_3^2 - 98 c_5 (45 c_4 + 32 c_5 - 21 c_6) - 49 c_3 (858 c_4 - 139 c_5 + 66 c_6))))))$ |
| $w_{10,4}$    | $14 (295955 + 23296 c_1^2 - 324936 c_3^2 - 52 c_5 (4425 + 448 c_5)$<br>$- 156 c_3 (-1121 + 1661 c_5) + c_1 (154671 + 385464 c_3 + 23296 c_5))$                                                                                                                                                        |
| $w_{10,5}$    | $2 (c_1 (84084 c_2 - 347724 c_3 - 49 (16269 + 1144 c_4 - 18056 c_5))$<br>$+ 156 c_2 (1335 + 1078 c_3 - 686 c_5) + 49 (44583 + 22528 c_4 - 21384 c_5 - 39936 c_6$<br>$- 4 (1521 c_3^2 + 78 c_5 (-22 c_4 + 7 c_5 + 15 c_6) + c_3 (-12639 - 7062 c_4 + 8768 c_5 + 11154 c_6))))))$                       |
| $w_{10,6}$    | $14 (315403 + 48464 c_1^2 - 235704 c_3^2 + c_1 (-208329 + 1872 c_3 - 86528 c_5)$<br>$+ 52 c_5 (5943 + 932 c_5) + 156 c_3 (2091 + 2011 c_5))$                                                                                                                                                          |
| $w_{10,7}$    | $2 (940800 c_1^2 + 4312 c_2 (256 + 321 c_3 + 78 c_5) + 147 (14861 + 13312 c_4 + 2860 c_5)$<br>$- 49 c_1 (35937 + 1144 c_2 + 38400 c_3 - 2496 c_5 - 1716 c_6) + 208260 c_6 + 4 (29601 c_3^2$<br>$- 1274 c_5 (-45 c_4 + 32 c_5 + 21 c_6) + 49 c_3 (15213 + 11154 c_4 + 1807 c_5 + 858 c_6))))$          |
| $w_{10,8}$    | $-14 (-295955 - 123367 c_1 + 117852 c_1^2 - 622668 c_3 - 324936 c_1 c_3$<br>$- 259116 c_3^2 - 4 (62249 + 5824 c_1 + 72727 c_3) c_5 - 23296 c_5^2)$                                                                                                                                                    |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                                   |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{10,9}$    | $2 \left( 2184567 + 229320 c_1^2 + 3945480 c_3 + 1881600 c_3^2 - 208260 c_4 \right.$ $\left. - 168168 c_3 c_4 - 84084 c_5 - 11388 c_3 c_5 + 107016 c_4 c_5 - 56056 c_5^2 \right.$ $\left. + 15288 c_2 (128 + 143 c_3 + 15 c_5) + 4312 (256 + 321 c_3 + 78 c_5) c_6 \right.$ $\left. - 49 c_1 (5973 + 13116 c_3 + 1716 c_4 + 3328 c_5 + 1144 c_6) \right)$    |
| $w_{10,10}$   | $14 \left( 315403 + 160596 c_3^2 + c_1 (195263 + 303316 c_3 - 114004 c_5) \right.$ $\left. + 12 c_5 (8391 + 3172 c_5) + 4 c_3 (130361 + 80558 c_5) \right)$                                                                                                                                                                                                  |
| $w_{10,11}$   | $5792094 - 3766434 c_1 + 9199104 c_3 + 652288 c_5$                                                                                                                                                                                                                                                                                                           |
| $w_{10,12}$   | $-14 \left( -295955 + 4 c_3 (-35893 + 66903 c_3 - 75410 c_5) \right.$ $\left. - 217692 c_5 + c_1 (324425 + 282412 c_3 + 141148 c_5) \right)$                                                                                                                                                                                                                 |
| $w_{10,13}$   | $2 \left( 229320 c_1^2 + 3945480 c_3 + 208260 c_4 \right.$ $\left. - 49 c_1 (5973 + 13116 c_3 - 1716 c_4 + 3328 c_5 - 1144 c_6) \right.$ $\left. - 539 (-4053 + 156 c_5 + 2048 c_6) - 4 (-470400 c_3^2 + 3822 c_2 (128 + 143 c_3 + 15 c_5) \right.$ $\left. + c_3 (-42042 c_4 + 2847 c_5 + 346038 c_6) + 1274 c_5 (21 c_4 + 11 (c_5 + 6 c_6))) \right)$      |
| $w_{10,14}$   | $1617 (1 + c_1)$                                                                                                                                                                                                                                                                                                                                             |
| $d_{11}$      | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                                                                        |
| $w_{11,0}$    | $66 (12561 + 49 c_1 + 12480 c_4)$                                                                                                                                                                                                                                                                                                                            |
| $w_{11,1}$    | $2 \left( 133224 c_2^2 + c_1 (644259 + 439348 c_3 - 73216 c_4 - 1069208 c_5) \right.$ $\left. - 11 (7 (-28179 + 52 c_3 (105 + 17 c_3) - 23296 c_4) + 93816 c_5) \right.$ $\left. - 56 (25047 c_4^2 + 208 c_3 (28 c_4 - 19 c_5) + 33600 c_4 c_5 + 2379 c_5^2) \right.$ $\left. - 156 c_2 (1280 + 3542 c_4 - 609 c_6) + 1956864 c_6 + 2294864 c_4 c_6 \right)$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                         |
|---------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{11,2}$    | $2 (940800 c_1^2 + 28028 c_3^2 - 52 c_3 (41601 + 4410 c_2 + 45650 c_4 + 539 c_5)$ $+ 3 c_1 (497947 + 35672 c_2 + 549976 c_4 - 40768 c_5 - 28028 c_6)$ $- 49 (8 c_4 (-7623 + 2647 c_5)$ $+ 4 c_5 (3377 + 1716 c_2 + 832 c_5 - 286 c_6) + 11 (-4235 + 52 c_6)))$                                                     |
| $w_{11,3}$    | $-2 (981120 c_1^2 + 384384 c_2^2 - 21 (103323 + 44800 c_4 + 20020 c_5)$ $+ 3 c_1 (101211 + 627200 c_4 + 128128 c_5) + 224 c_2 (4928 + 7383 c_4 - 988 c_6)$ $+ 199680 c_6 + 52 (3619 c_3^2 + 42438 c_4^2 - 6272 c_4 c_5 + 4256 c_5^2$ $- 11 c_3 (147 + 128 c_4 + 273 c_5) + 3234 c_4 c_6 + 3003 c_6^2))$            |
| $w_{11,4}$    | $-2 (229320 c_1^2 + 88 (-31227 c_4 + 21151 c_4 c_5 - 637 c_5^2)$ $- 7 c_1 (350867 + 48048 c_2 - 130396 c_3 + 274696 c_4 - 23296 c_5 - 8008 c_6)$ $+ 4 c_3 (369906 - 26754 c_2 + 568638 c_4 + 33761 c_5 + 21021 c_6)$ $+ 7 (780 (295 + 42 c_2) c_5 - 11 (26317 + 364 c_6)))$                                        |
| $w_{11,5}$    | $2 (-133224 c_2^2 + c_1 (644259 + 439348 c_3 - 73216 c_4 - 1069208 c_5)$ $- 11 (7 (-28179 + 52 c_3 (105 + 17 c_3) - 51968 c_4) + 93816 c_5)$ $- 56 (-25047 c_4^2 + 208 c_3 (28 c_4 - 19 c_5) + 33600 c_4 c_5 + 2379 c_5^2)$ $+ 156 c_2 (1280 + 3542 c_4 - 609 c_6) - 1956864 c_6 - 2294864 c_4 c_6)$               |
| $w_{11,6}$    | $-2 (229320 c_1^2 + 780744 c_3 + 4 (3822 c_2 (7 c_3 - 15 c_5)$ $- 286 c_5 (2075 c_4 + 49 c_5) + 49 c_3 (8418 c_4 + 689 c_5 - 429 c_6))$ $+ 49 c_1 (-1045 + 6864 c_2 + 18628 c_3 + 21176 c_4 + 3328 c_5 - 1144 c_6)$ $- 49 (46585 + 60984 c_4 + 44148 c_5 - 572 c_6)))$                                             |
| $w_{11,7}$    | $2 (-981120 c_1^2 - 4004 c_3 (-21 + 47 c_3) + 21 (103323 + 231168 c_4 + 20020 c_5)$ $- 3 c_1 (101211 + 627200 c_4 + 128128 c_5) + 199680 c_6$ $+ 4 (96096 c_2^2 + 56 c_2 (4928 + 7383 c_4 - 988 c_6)$ $+ 13 (1408 c_3 c_4 + 42438 c_4^2 + 3003 c_3 c_5 + 6272 c_4 c_5 - 4256 c_5^2 + 3234 c_4 c_6 + 3003 c_6^2)))$ |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                                                                     |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{11,8}$    | $2 \left( 940800 c_1^2 + 4 \left( 7007 c_3^2 + 13 c_3 (30975 + 4410 c_2 + 35794 c_4 - 539 c_5) \right. \right.$ $\left. \left. + 14 c_5 (6006 c_2 + 34337 c_4 - 2912 c_5 - 1001 c_6) \right) \right.$ $\left. - 3 c_1 (-730907 + 35672 c_2 - 758184 c_4 + 40768 c_5 - 28028 c_6) \right.$ $\left. + 77 (26317 + 35688 c_4 + 22636 c_5 + 364 c_6) \right)$                                      |
| $w_{11,9}$    | $2 \left( -251160 c_1^2 + 2184 c_2 (896 + 115 c_2) + 1031976 c_3 \right.$ $\left. + 2697984 c_4 + 7 c_1 (164109 + 149884 c_3 + 46592 c_4 - 31616 c_5) \right.$ $\left. - 77 (-28179 + 1092 c_5 - 14336 c_6) \right.$ $\left. + 4 (7 c_3 (67200 c_4 + 11479 c_5) + 312 c_2 (1875 c_4 + 308 c_6) \right.$ $\left. - 11 (3822 c_4^2 + c_4 (1664 c_5 - 28329 c_6) + 2002 (c_5^2 + c_6^2)) \right)$ |
| $w_{11,10}$   | $2 \left( 2282665 + 2988216 c_4 + 780744 c_5 \right.$ $\left. + c_1 (2876349 + 229320 c_2 + 201292 c_3 + 2373800 c_4 + 996856 c_5) \right.$ $\left. + 196 (-143 c_3^2 + c_3 (3377 + 1716 c_2 + 5294 c_4 + 832 c_5 - 286 c_6) \right.$ $\left. + 3 c_5 (182 c_2 + 2806 c_4 + 182 c_5 - 143 c_6) \right) - 28028 c_6$                                                                            |
| $w_{11,11}$   | $2 (2865423 + 1960623 c_1 - 256256 c_3 + 4526336 c_4 + 442624 c_5)$                                                                                                                                                                                                                                                                                                                            |
| $w_{11,12}$   | $-2 (-2026409 - 2747976 c_4$ $+ c_1 (897603 + 229320 c_2 - 201292 c_3 + 1861288 c_4 - 996856 c_5) - 1479624 c_5$ $- 28028 c_6 + 28 (1001 c_3^2 + c_3 (62249 + 12012 c_2 + 68674 c_4 - 5824 c_5 - 2002 c_6)$ $- 3 c_5 (-1274 c_2 + 27078 c_4 + 1274 c_5 + 1001 c_6)))$                                                                                                                          |
| $w_{11,13}$   | $-2 (251160 c_1^2 + 251160 c_2^2 - 3097344 c_4$ $- 7 c_1 (164109 + 149884 c_3 + 46592 c_4 - 31616 c_5)$ $+ 1248 c_2 (1568 + 1875 c_4 + 308 c_6) + 77 (-28179 + 1092 c_5 + 14336 c_6)$ $- 4 (c_3 (257994 + 470400 c_4 + 80353 c_5)$ $+ 11 (3822 c_4^2 - 2002 (c_5 - c_6) (c_5 + c_6) - c_4 (1664 c_5 + 28329 c_6)))$                                                                            |
| $w_{11,14}$   | $33 (17 + 49 c_1 + 64 c_4)$                                                                                                                                                                                                                                                                                                                                                                    |
| $d_{12}$      | $\frac{1}{46126080}$                                                                                                                                                                                                                                                                                                                                                                           |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                   |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{12,0}$    | $-462 (-257 + c_3 - 256 c_5)$                                                                                                                                                                                                                                                |
| $w_{12,1}$    | $-14 (1144 c_1^2 - 11 (4053 + 543 c_3 - 2048 c_4 + 3432 c_5)$<br>$+ 4 (-1170 c_3^2 + 39 c_2 (25 + 11 c_3 + 22 c_5) + c_3 (-286 c_4 + 6607 c_5)$<br>$+ 6 c_5 (1177 c_4 + 1600 c_5 - 1859 c_6)) - 39936 c_6$<br>$+ 52 c_1 (33 + 42 c_2 - 64 c_3 - 132 c_4 + 53 c_5 + 90 c_6))$ |
| $w_{12,2}$    | $14 (47509 + 16836 c_1 - 30673 c_3 + 20164 c_1 c_3 + 4 (11869 + 9250 c_1 - 12701 c_3) c_5 - 20164 c_5^2)$                                                                                                                                                                    |
| $w_{12,3}$    | $-14 (22528 c_2 - 3 (14861 + 5423 c_3 - 13312 c_4 + 22572 c_5 - 1300 c_6)$<br>$+ 4 (546 c_1^2 - 22 c_2 (13 c_3 - 321 c_5) + 13 (79 c_3 + 858 c_4 - 139 c_5) c_5$<br>$+ 429 (c_3 + 2 c_5) c_6 + 2 c_1 (2673 - 858 c_2 + 2257 c_3 - 585 c_4 + 5216 c_5 + 273 c_6)))$           |
| $w_{12,4}$    | $-2 (-278795 + 86528 c_1^2 + 86759 c_3 + 162468 c_3^2$<br>$+ 4 c_1 (-62249 + 21632 c_3 - 83881 c_5) + 162468 (-1 + 2 c_3) c_5 + 278564 c_5^2)$                                                                                                                               |
| $w_{12,5}$    | $-14 (1144 c_1^2 - 11 (4053 + 543 c_3 + 2048 c_4 + 3432 c_5)$<br>$+ 39936 c_6 - 52 c_1 (-33 + 42 c_2 + 64 c_3 - 132 c_4 - 53 c_5 + 90 c_6)$<br>$- 4 (1170 c_3^2 + 39 c_2 (25 + 11 c_3 + 22 c_5)$<br>$- c_3 (286 c_4 + 6607 c_5) - 6 c_5 (-1177 c_4 + 1600 c_5 + 1859 c_6)))$ |
| $w_{12,6}$    | $14 (47509 + 3328 c_1^2 - 16836 c_3^2 - 4 c_1 (3377 + 832 c_3 + 5873 c_5)$<br>$+ 4 c_5 (26283 + 11869 c_5) - c_3 (3361 + 33672 c_5))$                                                                                                                                        |
| $w_{12,7}$    | $-14 (2184 c_1^2 + 8 c_1 (2673 + 858 c_2 + 2257 c_3 + 585 c_4 + 5216 c_5 - 273 c_6)$<br>$- 3 (14861 + 5423 c_3 + 13312 c_4 + 22572 c_5 + 1300 c_6)$<br>$+ 4 (22 c_2 (-256 + 13 c_3 - 321 c_5)$<br>$+ 13 (79 c_3 - 858 c_4 - 139 c_5) c_5 - 429 (c_3 + 2 c_5) c_6))$          |
| $w_{12,8}$    | $2 (278795 + 38064 c_1^2 + 4 c_5 (160391 + 71765 c_5)$<br>$+ c_3 (343873 + 230100 c_5) + 4 c_1 (28501 c_3 + 17 (2949 + 3506 c_5)))$                                                                                                                                          |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$      | Expression                                                                                                                                                                                                                                                                                                                                                                              |
|--------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{12,9}$         | $ \begin{aligned} & -14 \left( -3 (14861 + 13312 c_2 + 11979 c_3 - 1300 c_4 + 19140 c_5) - 22528 c_6 \right. \\ & \quad + 4 (832 c_1^2 - 4800 c_3^2 + 143 c_5 (-78 c_2 + 6 c_4 + c_5) - 7062 c_5 c_6 \\ & \quad \left. + 39 c_1 (-55 + 30 c_2 + 16 c_3 + 14 c_4 - 39 c_5 + 44 c_6) + c_3 (429 c_4 - 9600 c_5 + 286 c_6) \right) \end{aligned} $                                         |
| $w_{12,10}$        | $ \begin{aligned} & -14 \left( -47509 - 26983 c_3 - 77820 c_5 \right. \\ & \quad \left. + 4 (832 c_1^2 + 13 c_1 (-849 + 64 c_3 - 913 c_5) - 2 (c_3 + c_5) (416 c_3 + 4209 c_5)) \right) \end{aligned} $                                                                                                                                                                                 |
| $w_{12,11}$        | $14 (59103 + 6656 c_1 + 38433 c_3 + 90880 c_5)$                                                                                                                                                                                                                                                                                                                                         |
| $w_{12,12}$        | $ \begin{aligned} & 2 (278795 + 48464 c_1^2 + 48464 c_3^2 + 36 c_5 (16475 + 9026 c_5) \\ & \quad - c_3 (135223 + 124592 c_5) + 52 c_1 (1664 c_3 - 25 (177 + 185 c_5))) \end{aligned} $                                                                                                                                                                                                  |
| $w_{12,13}$        | $ \begin{aligned} & -14 (39936 c_2 - 3 (14861 + 11979 c_3 + 1300 c_4 + 19140 c_5) + 22528 c_6 \\ & \quad + 4 (832 c_1^2 - 4800 c_3^2 + 143 c_5 (78 c_2 - 6 c_4 + c_5) + 7062 c_5 c_6 \\ & \quad - 39 c_1 (55 + 30 c_2 - 16 c_3 + 14 c_4 + 39 c_5 + 44 c_6) - c_3 (429 c_4 + 9600 c_5 + 286 c_6))) \end{aligned} $                                                                       |
| $w_{12,14}$        | $-231 (-1 + c_3)$                                                                                                                                                                                                                                                                                                                                                                       |
| $d_{13}$           | $\frac{1}{322882560}$                                                                                                                                                                                                                                                                                                                                                                   |
| $w_{13,0}$         | $66 (12561 + 49 c_5 + 12480 c_6)$                                                                                                                                                                                                                                                                                                                                                       |
| $w_{13,1}$         | $ \begin{aligned} & -2 (-156156 c_2^2 + 1664 c_2 (120 + 133 c_4) \\ & \quad + 11 (7 (-28179 + 5460 c_3 + 128 (112 - 39 c_4) c_4) + 27603 c_5) \\ & \quad - 4854528 c_6 + 4 (47047 c_1^2 + 224 (247 c_3^2 - 429 c_3 c_5 + 1095 c_5^2) \\ & \quad - 143 c_1 (-147 + 273 c_3 - 128 c_6) \\ & \quad + 14 (3003 c_2 + 5824 c_3 + 29532 c_4 + 33600 c_5) c_6 - 551694 c_6^2)) \end{aligned} $ |
| <i>(continues)</i> |                                                                                                                                                                                                                                                                                                                                                                                         |

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                 |
|---------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{13,2}$    | $2 \left( -28028 c_2 (-1 + 2 c_5) + 49 (55 (847 + 19 c_5) \right.$ $+ 52 (c_3 (-849 + 22 c_3 + 90 c_4) + 4 (16 c_3 + 33 c_4) c_5 - 90 c_5^2))$ $- 8 (-373527 + 296725 c_3 + 129703 c_5) c_6$ $+ 4 c_1 (195186 + 21021 c_2 - 33761 c_3 - 26754 c_4 + 228193 c_5 + 412482 c_6))$                                             |
| $w_{13,3}$    | $2 \left( -88088 c_2^2 + 21 (103323 + 104 c_4 (-896 + 115 c_4) + 54703 c_5) \right.$ $+ 4 c_1 (-257994 + 80353 c_3 - 262297 c_5 - 470400 c_6)$ $+ 308 c_2 (-3584 + 1248 c_4 - 4047 c_6) + 2697984 c_6$ $- 52 (7 (11 c_3 (-21 + 22 c_3) - 608 c_3 c_5 + 690 c_5^2)$ $- 8 (176 c_3 - 5625 c_4 + 784 c_5) c_6 + 3234 c_6^2))$ |
| $w_{13,4}$    | $-2 \left( -2026409 + 28028 c_1^2 + 1479624 c_3 - 2548 (11 c_2 (-1 + 3 c_3) + 42 c_3 (c_3 - c_4)) \right.$ $+ 897603 c_5 + 392 (2543 c_3 - 585 c_4) c_5 - 2747976 c_6 + 8 (284319 c_3 + 232661 c_5) c_6$ $- 28 c_1 (62249 + 2002 c_2 + 5824 c_3 - 12012 c_4 - 7189 c_5 + 68674 c_6))$                                      |
| $w_{13,5}$    | $-2 \left( 188188 c_1^2 + 156156 c_2^2 + 11 (-197253 + 896 c_4 (-112 + 39 c_4) + 27603 c_5) \right.$ $- 572 c_1 (-147 + 273 c_3 - 128 c_6) - 940800 c_6 - 104 c_2 (1920 + 2128 c_4 + 1617 c_6)$ $+ 4 (55328 c_3^2 + 245280 c_5^2 - 3003 c_3 (-35 + 32 c_5)$ $+ 56 (1456 c_3 - 7383 c_4 + 8400 c_5) c_6 + 551694 c_6^2))$   |
| $w_{13,6}$    | $-2 \left( -28028 c_2 + 780744 c_3 + 196 (143 c_1^2 + 429 c_2 c_3 - 546 c_3 (c_3 + c_4)) \right.$ $+ 2 (2543 c_3 + 585 c_4) c_5 + c_1 (3377 + 286 c_2 - 832 c_3 - 1716 c_4 + 1027 c_5))$ $+ 8 (129703 c_1 + 206241 c_3 - 296725 c_5) c_6 - 49 (58701 c_5 + 847 (55 + 72 c_6)))$                                            |
| $w_{13,7}$    | $2 \left( 88088 c_2^2 + 4 c_1 (-257994 + 80353 c_3 - 262297 c_5 - 470400 c_6) \right.$ $- 308 c_2 (-3584 + 1248 c_4 - 4047 c_6)$ $+ 3 (723261 + 728 (896 - 115 c_4) c_4 + 382921 c_5 + 1032448 c_6)$ $+ 52 (-7 (11 c_3 (-21 + 22 c_3) - 608 c_3 c_5 + 690 c_5^2)$ $+ 8 (176 c_3 + 5625 c_4 + 784 c_5) c_6 + 3234 c_6^2))$  |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                                                                                                                                                                                                                                                                                                                                                                                                     |
|---------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $w_{13,8}$    | $2 \left( 7 \left( -4004 c_2 + 230100 c_3 + 11 \left( 26317 + 31897 c_5 + 35688 c_6 \right) \right. \right. \\ + 4 \left( 1274 \left( 11 c_3^2 + \left( 11 \left( c_2 - 6 c_4 \right) - 45 c_5 \right) c_5 + c_3 \left( -45 c_4 + 32 c_5 \right) \right) \right. \\ + 2 \left( 232661 c_3 + 240359 c_5 \right) c_6 \\ \left. \left. + c_1 \left( 369906 - 21021 c_2 - 33761 c_3 + 26754 c_4 + 228193 c_5 + 568638 c_6 \right) \right) \right)$ |
| $w_{13,9}$    | $-2 \left( 68068 c_1^2 + 312 c_4 \left( 640 + 427 c_4 \right) - 364 c_1 \left( 1155 + 608 c_3 - 1207 c_5 + 896 c_6 \right) \right. \\ - 77 \left( 28179 + 8367 c_5 + 51968 c_6 \right) + 52 c_2 \left( 1827 c_4 - 4 \left( 9408 + 11033 c_6 \right) \right) \\ + 8 \left( 16653 c_3^2 + 11 \left( 6279 c_4 + 832 c_5 - 15939 c_6 \right) c_6 \\ \left. \left. - c_3 \left( 128997 + 133651 c_5 + 235200 c_6 \right) \right) \right)$           |
| $w_{13,10}$   | $2 \left( 28028 c_1^2 + 1493841 c_5 + 28028 c_2 \left( 1 + 2 c_3 + 3 c_5 \right) + 41503 \left( 55 + 72 c_6 \right) \right. \\ + 52 c_1 \left( -49 \left( -849 + 11 c_3 + 90 c_4 \right) + 45650 c_6 \right) - 196 \left( 832 c_3^2 \right. \\ \left. \left. + c_3 \left( -3377 + 1716 c_4 - 624 c_5 - 5294 c_6 \right) - 6 c_5 \left( -91 c_4 + 800 c_5 + 1403 c_6 \right) \right) \right)$                                                   |
| $w_{13,11}$   | $2 \left( 2865423 + 256256 c_1 - 442624 c_3 + 1960623 c_5 + 4526336 c_6 \right)$                                                                                                                                                                                                                                                                                                                                                               |
| $w_{13,12}$   | $2 \left( 2026409 + 28028 c_1^2 + 2192721 c_5 - 28028 c_2 \left( 1 + 2 c_3 + 3 c_5 \right) \right. \\ + 2747976 c_6 - 52 c_1 \left( 30975 + 539 c_3 - 4410 c_4 + 35794 c_6 \right) - 28 \left( 5824 c_3^2 \right. \\ \left. \left. - 6 c_5 \left( 637 c_4 + 5600 c_5 + 13539 c_6 \right) + c_3 \left( 62249 - 12012 c_4 - 4368 c_5 + 68674 c_6 \right) \right) \right)$                                                                        |
| $w_{13,13}$   | $2 \left( -68068 c_1^2 + 1031976 c_3 + 1092 c_2 \left( -1792 + 87 c_4 \right) \right. \\ + 312 c_4 \left( 640 + 427 c_4 \right) + 364 c_1 \left( 1155 + 608 c_3 - 1207 c_5 + 896 c_6 \right) \\ + 77 \left( 28179 + 8367 c_5 + 23296 c_6 \right) + 8 \left( -16653 c_3^2 \right. \\ \left. \left. - 11 c_6 \left( 26078 c_2 - 6279 c_4 + 832 c_5 + 15939 c_6 \right) + 7 c_3 \left( 19093 c_5 + 33600 c_6 \right) \right) \right)$             |
| $w_{13,14}$   | $33 \left( 17 + 49 c_5 + 64 c_6 \right)$                                                                                                                                                                                                                                                                                                                                                                                                       |
| $d_{14}$      | $\frac{1}{315315}$                                                                                                                                                                                                                                                                                                                                                                                                                             |
| $w_{14,0}$    | $1617$                                                                                                                                                                                                                                                                                                                                                                                                                                         |

(*continues*)

APPENDIX C. COEFFICIENTS FOR SECOND-ORDER METHODS

Table C.7: Integral approximation  $I_j$ ,  $n = 14$ . (*continued*)

| $d_j, w_{jk}$ | Expression                                                               |
|---------------|--------------------------------------------------------------------------|
| $w_{14,1}$    | $-286 c_1 - 780 c_2 - 49 (-231 + 26 c_3 + 88 c_4 + 150 c_5 - 156 c_6)$   |
| $w_{14,2}$    | $28 (418 + 225 c_1 - 299 c_3 - 106 c_5)$                                 |
| $w_{14,3}$    | $11319 - 7350 c_1 - 4312 c_2 + 286 c_3 - 7644 c_4 + 1274 c_5 - 780 c_6$  |
| $w_{14,4}$    | $7 (1529 + 918 c_1 - 1290 c_3 - 910 c_5)$                                |
| $w_{14,5}$    | $-286 c_1 + 780 c_2 - 49 (-231 + 26 c_3 - 88 c_4 + 150 c_5 + 156 c_6)$   |
| $w_{14,6}$    | $-28 (-418 + 106 c_1 + 225 c_3 - 299 c_5)$                               |
| $w_{14,7}$    | $-7350 c_1 + 4312 c_2 + 286 c_3 + 49 (231 + 156 c_4 + 26 c_5) + 780 c_6$ |
| $w_{14,8}$    | $7 (1529 + 1290 c_1 + 910 c_3 + 918 c_5)$                                |
| $w_{14,9}$    | $11319 + 1274 c_1 + 7644 c_2 + 7350 c_3 - 780 c_4 - 286 c_5 + 4312 c_6$  |
| $w_{14,10}$   | $28 (418 + 299 c_1 + 106 c_3 + 225 c_5)$                                 |
| $w_{14,11}$   | 17681                                                                    |
| $w_{14,12}$   | $-7 (-1529 + 910 c_1 + 918 c_3 - 1290 c_5)$                              |
| $w_{14,13}$   | $11319 + 1274 c_1 - 7644 c_2 + 7350 c_3 + 780 c_4 - 286 c_5 - 4312 c_6$  |
| $w_{14,14}$   | 0                                                                        |