REPORT DOCUMENTATION PAGE

AFRL-SR-AR-TR-02-

0327

1. REPORT DATE *(DD–MM–YYYY)*
30–04–2001

2. REPORT TYPE
Final

3. DATES COVERED *(From — To)*
1 April 1998 31 January 2002

4. TITLE AND SUBTITLE
Input to state stability and related notions (AASERT grant)

5a. CONTRACT NUMBER
F49620-98-1-0421

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
61102F

5d. PROJECT NUMBER
2304

5e. TASK NUMBER

5f. WORK UNIT NUMBER

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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AFOSR/NM, Building 410
Bolling AFB, DC 20332-6448

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
Approval for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the authors and should not be construed as an official U.S. o vernment position, policy or decision, unless so designated by other documentation.

14. ABSTRACT
This AASERT grant provided graduate student support to complement the research carried out under grant F49620-98-1-0242. The main emphasis of this subproject was the complete characterization of IOSS, which a notion of well-posed detectability, and related concepts, resulting in necessary and sufficient characterizations in terms of dissipation inequalities, as well as the construction of norm-observers for finite-dimensional nonlinear systems.

15. SUBJECT TERMS
keywords: detectability of nonlinear systems, input to state stability

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON / 19b. TELEPHONE NUMBER *(include area code)*
U	U	U	UU	5	Dr. Marc Jacobs (202) 767-5027

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Input to state stability and related notions

F49620-98-1-0421, Final Technical Report, 1 April 1998 - 31 January 2002

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ABSTRACT

This AASERT grant provided graduate student support to complement the research carried out under grant F49620-98-1-0242. The main emphasis of this subproject was the complete characterization of IOSS, which is a notion of well-posed detectability, and related concepts, resulting in necessary and sufficient characterizations in terms of dissipation inequalities, as well as the construction of norm-observers for finite-dimensional nonlinear systems.

1 Summary of Main Accomplishments

The main accomplishment of this grant was the completion of a program of research to show that detectability of nonlinear systems is equivalent to a dissipation condition with smooth storage functions (a problem on in which we had been working for several years, was finally settled in our student Krichman's thesis, supported under this AASERT grant). The IOSS (input/output to state stability) property is central to the analysis of observers and detectability. One of the main tools in the application of this notion to adaptive control an other fields is the use of a dissipation sufficient condition provided in previous work by the PI and Wang. It is a natural mathematical question, and in principle one of great interest in order to understand the applicability of the theory, if such a condition always holds when the system is IOSS. We first obtained a nonsmooth dissipation characterization (expressed in terms of viscosity subgradients), and later a smooth one, of the IOSS property.

In addition, we have obtained a general proof of existence (and a construction in terms of Lyapunov functions) of norm observers for arbitrary IOSS systems. Two papers were submitted, one of which has appeared and the other one is in press; see [3] and [4]. In another line of work partially supported by the grant, we worked on ISS-type small-gain results. In [5], we presented a small-gain theorem for ISS operators, recovering the classical statement for ISS systems in state-space form. This paper also explained applications to incrementally stable systems and to detectable systems, as well as interconnections of stable systems.

We next sketch some of the concepts and main results. Further details on this project can be found in the reports for the "parent" grant F49620-98-1-0242, and in the published papers.

2 Technical Details

The thesis work of our student Krichman, supported under this grant, concerned the following "zero detectability" question: *is it possible to estimate, on the basis of external information provided by past input and output signals, the magnitude of the internal state $x(t)$ at time t?*

State estimation is central to control theory (Kalman filters, observers). By and large, the theory of state estimation is well-understood for linear systems, but it is still poorly developed for more general classes of systems. An outstanding open question is the derivation of useful necessary and sufficient conditions for the existence of observers, i.e., "algorithms" (dynamical systems) which converge to an estimate $\hat{x}(t)$ of the state $x(t)$ of the system of interest, using the information provided by $\{u(s), s \leq t\}$, the set of past input values, and by $\{y(s), s \leq t\}$, the set of past output measurements. In the context of stabilization to an equilibrium, let us say to the zero state $x = 0$ if we are working in an Euclidean space, a weaker type of estimate is sometimes enough: it may suffice to have a norm-estimate, that is to say, an upper bound $\hat{x}(t)$ on the *magnitude* (norm) $|x(t)|$ of the state $x(t)$. Indeed, it is often the case that norm-estimates suffice for control applications. To be more precise, one wishes that $\hat{x}(t)$ becomes eventually an upper bound on $|x(t)|$ as $t \to \infty$. We are thus interested in *norm-estimators* which, when driven by the i/o data generated by the system, produce such an upper bound $\hat{x}(t)$, cf. Figure 1. In order to understand the issues that arise, let us start by considering the very special case when the external data (inputs u and outputs y) vanish identically. The obvious estimate (assuming, as we will, that everything is normalized so that the zero state is an equilibrium for the unforced system, and the output is zero when $x = 0$) is $\hat{x}(t) \equiv 0$. However, the only way that this estimate fulfills the goal of upper bounding the norm of the true state as $t \to \infty$ is if

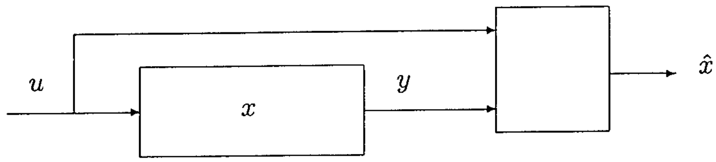

Figure 1: Norm-Estimator

$x(t) \to 0$. In other words, one obvious necessary property for the possibility of norm-estimation is that the origin must be a globally asymptotically stable state with respect to the "subsystem" consisting of those states for which the input $u \equiv 0$ produces the output $y \equiv 0$. One says in this case that the original system is *zero-detectable*. For *linear systems*, zero detectability is equivalent to detectability, that is to say, the property that if any two trajectories produce the same output, then they approach each other. Zero-detectability is a central property in the general theory of nonlinear stabilization on the basis of output measurements, and is classically found in references such as Isidori's book). Our work can be seen as a contribution towards the better characterization and understanding of this fundamental concept.

However, zero-detectability by itself is far from being sufficient for our purposes, since it fails to be "well-posed" enough. One easily sees that, at the least, one should ask that, when inputs and outputs are small, states should also be small, and if inputs and outputs converge to zero as $t \to \infty$, states do too, cf. Figure 2. Moreover, when defining formally the notion

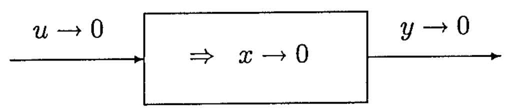

Figure 2: State Converges to Zero if External Data does

of norm-estimator and the natural necessary and sufficient conditions for its existence, other requirements appear: the existence of asymptotic bounds on states, as a function of bounds on input/output data, and the need to describe the "overshoot" (transient behavior) of the state.

One way to approach the formal definition, so as to incorporate all the above characteristics in a simple manner, is to dualizing the definition of ISS, arriving at the notion of detectability called *output to state stability* (OSS), or. more generally if there are both inputs and outputs, *input/output to state stability* (IOSS): A system $\dot{x} = f(x,u)$ with measurement (output) map $y = h(x)$ is IOSS if there exist functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that the estimate:

$$|x(t)| \;\leq\; \beta(|x(0)|\,,t) + \gamma_1\left(\left\|u|_{[0,t]}\right\|\right) + \gamma_2\left(\left\|y|_{[0,t]}\right\|\right)$$

holds for any initial state $x(0)$ and any input $u(\cdot)$, where $x(\cdot)$ is the ensuing trajectory and $y(t) = h(x(t))$ the respective output function. The terminology IOSS is self-explanatory: formally there is "stability from the i/o data to the state". It represents a natural combination of the notions of "strong" observability (cf. the PI's 1989 coprime factorizations paper) and ISS, and was called simply "detectability" in one of our 1989 papers (where it is phrased in input/output, as opposed to state space, terms, and applied to questions of parameterization of controllers) and was called "strong unboundedness observability" by Jiang, Teel, and Praly. (more precisely, this last notion allows also an additive nonnegative constant in the right-hand side of the estimate). In E.D. Sontag, Y. Wang, "Output-to-state stability and detectability of nonlinear systems," *Systems and Control Letters* **29**(1997): 279-290, we described relationships between the existence of full state observers and the IOSS property, or more precisely, a property which we called *incremental IOSS.* (The study of incremental IOSS is still in its begining stages, and represents a major direction for further work.) The use of ISS-like formalism for studying observers, and hence implicitly the IOSS property, has also appeared several times in other authors' work.

One of the main results of [3] is that a system is IOSS if and only if it admits a norm-estimator. This result is in turn a consequence of a necessary and sufficient characterization of the IOSS property in terms of smooth dissipation functions, namely, there is a proper (radially unbounded) and positive definite smooth function V of states (a "storage function" in the language of dissipative systems) such that a dissipation inequality

$$\frac{d}{dt}V(x(t)) \;\leq\; -\sigma_1(|x(t)|) + \sigma_2(|y(t)|) + \sigma_3(|u(t)|) \tag{1}$$

holds along all trajectories, with the functions σ_i of class $\mathcal{K}_\infty$. This provides an "infinitesimal" description of IOSS, and a norm-observer is easily built from V. Such a characterization in dissipation terms was conjectured for several years, but only recently we were able to obtain the complete solution.

It is worth pointing out that several authors (Lu, Morse, etc.) had independently suggested that one should *define* "detectability" in dissipation terms. In some work one finds detectability defined by the requirement that there should exist a differentiable storage function V satisfying our dissipation inequality but with the special choice $\sigma_2(r) := r^2$ (there were no inputs in the class of systems considered there). A variation of this is to weaken the dissipation inequality, to require merely $x \neq 0 \Rightarrow \frac{d}{dt}V(x(t)) < \sigma_2(|y(t)|)$ (again, with no inputs), as done for instance in the definition of detectability given by Morse. Observe that this represents a slight weakening of our property, in so far as there is no "margin" of stability $-\sigma_1(|x(t)|)$. One of our contributions is to show that such alternative definitions (when posed in the right generality) are in fact equivalent to IOSS. (We have recently started work with Morse and Liberzon which ties together the notion of IOSS with the applications in adaptive control which motivated Morse's work, and in a recent paper (IEEE Trans. Autom. Control, 2002) we showed how to formulate the requisite "minimum phase" type of property as an IOSS property for an extended system whose outputs are derivatives of the original output.)

A key preliminary step in the construction of V, just as it was for the analogous result for the ISS property, is the characterization of the IOSS property in robustness terms, by means of a "small gain" argument. The IOSS property is shown to be equivalent to the existence of a "robustness margin" $\rho \in \mathcal{K}_\infty$. This means that every system obtained by closing the loop with a feedback law Δ (even dynamic and/or time-varying) for which $|\Delta(t)| \leq \rho(|x(t)|)$ for all t, cf. Figure 3, is OSS (i.e., is IOSS as a system with no inputs). The core of the paper [3] is, thus, the construction of V for "robustly detectable" (more precisely, "robust IOSS") systems $\dot{x} = g(x,d)$ which are obtained by substituting $u = d\rho(|x|)$ in the original system, and letting $d = d(\cdot)$ be an arbitrary measurable function taking values in a unit ball. The function V must satisfy a differential inequality of the form $\dot{V}(x(t)) \leq -\sigma_1(|x(t)|) + \sigma_2(|y(t)|)$ along all trajectories, that is to say, the following partial differential inequality:

$$\nabla V(x) \cdot g(x,d) \;\leq\; -\sigma_1(|x|) + \sigma_2(|y|)\,,$$

for some functions σ_1 and σ_2 of class $\mathcal{K}_\infty$. But one last reduction consists of turning this problem into one of building Lyapunov functions for "relatively asymptotically stable" systems. Indeed,

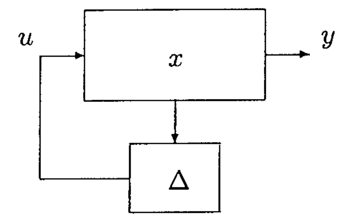

Figure 3: Robust Detectability

one observes that the main property needed for V is that it should *decrease along trajectories as long as $y(t)$ is sufficiently smaller than $x(t)$*. This leads us to the notion of "global asymptotic stability modulo outputs" and its Lyapunov-theoretic characterization.

The construction of V relies upon the solution of an appropriate optimal control problem, for which V is the value function. This problem is obtained by "fuzzifying" the dynamics near the set where $y \ll x$, so as to obtain a problem whose value function is continuous. Several elementary facts about relaxed controls are used in deriving the conclusions. The last major ingredient is the use of techniques from nonsmooth analysis, and in particular inf-convolutions, in order to obtain a Lipschitz, and from there by a standard regularization argument, a smooth, function V, starting from the continuous V that was obtained from the solution of the optimal control problem.

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Papers Supported by this Grant

1. M. Krichman and E. Sontag, "A version of a converse Lyapunov theorem for input-output to state stability," in *Proc. IEEE Conf. Decision and Control, Tampa, Dec. 1998*, IEEE Publications, 1998, pp. 4121-4126.

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