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13. ABSTRACT (Maximum 200 words)

This work of this proposal was concerned with the multidisciplinary field "controlled active vision" which involves the synthesis of techniques from control and computer vision to treat a number of fundamental problems including visual tracking. A key theme of our research was the development of techniques for using visual information in feedback control systems. Controlled active vision is leading to enhanced man-machine interfaces for interactions with computers and more complicated systems such as remote controlled weapons and vehicles. Our work has drawn upon our extensive experience in robust control, and the methods we have been developing for various problems in image processing and computer vision utilizing the theory of geometric variational evolution equations. These techniques have already been applied to visual tracking, automatic target recognition, and problems in biomedical engineering including image-guided surgery. It is important to note that many of these methods were derived from ideas in optimal control. In particular, the geometric variational techniques have been very influenced by concepts from optimal control, and the resulting concept of "viscosity solution" is a direct consequence of Hamiton-Jacobi theory. For some time now, the role of control theory in vision has been recognized. In particular, the branches of control that deal with system uncertainty, namely adaptive and robust, have been proposed as essential tools in coming to grips with the problems of both biological and machine vision. These problems all become manifest when one attempts to use a visual sensor in an uncertain environment, and to feed back in some manner the information.

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Final Report for "Visual Information in a Feedback Loop: A Control/Computer Vision Synthesis": DAAG55-98-1-0169

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1 Introduction

This work of this proposal is concerned with the multidisciplinary field *controlled active vision* which involves the synthesis of techniques from control and computer vision to treat a number of fundamental problems including visual tracking. Thus, a key theme of our research is the development of techniques for using visual information in feedback control systems. Controlled active vision is leading to enhanced man-machine interfaces for interactions with computers and more complicated systems such as remote controlled weapons and vehicles.

Our work has drawn upon our extensive experience in robust control, and the methods we have been developing for various problems in image processing and computer vision utlilizing the theory of geometric variational evolution equations. These techniques have already been applied to visual tracking, automatic target recognition, and problems in biomedical engineering including image-guided surgery. It is important to note that many of these methods were derived from ideas in optimal control [59]. In particular, the geometric variational techniques have been very influenced by concepts from optimal control, and the resulting concept of *viscosity solution* is a direct consequence of Hamiton-Jacobi theory [33].

Vision is a key sensor modality in both the natural and man-made domains. The prevalence of biological vision in even very simple organisms, indicates its utility in man-made machines. More practically, cameras are in general rather simple, reliable passive sensing devices which are quite inexpensive per bit of data. Furthermore, vision can offer information at a high rate with high resolution with a wide field of view and accuracy capturing multispectral information. Finally cameras can be used in a more active manner. Namely, one can include motorized lenses mounted on mobile platforms which can actively explore the surroundings and suitably adapt their sensing capilities. For some time now, the role of control theory in vision has been recognized. In particular, the branches of control that deal with system uncertainty, namely adaptive and robust, have been proposed as essential tools in coming to grips with the problems of both biological and machine vision. These problems all become manifest when one attempts to use a visual sensor in an uncertain environment, and to feed back in some manner the information.

2 Research Summary

In this section, we will summarize some our key findings in our just completed ARO contract.

2.1 Robust Nonlinear Control and Causality

Under ARO support, we have worked extensively in nonlinear robust control. Besides the theoretical and practical questions involved in finding an implementable nonlinear design methodology, it is interesting to note that certain associated problems of causality have arisen in this area, which we would like to briefly indicate as well. In fact, as a result of this effort, we have been able to put an explicit causality constraint in commutant lifting theory for the first time [35, 37, 42].

There have been several attempts to extend dilation theoretic techniques to nonlinear input/output operators, especially those which admit a Volterra series expansion (see [36] and the references therein). Typically, one is reduced to applying the classical (linear) commutant lifting theorem to an H^2-space defined on some D^n (where D denotes the unit disc). Now when one applies the classical result to D^n $(n \geq 2)$, even though time-invariance is preserved, causality may be lost. Indeed, for analytic functions on the disc D, time-invariance implies causality. For analytic functions on the $n-$disc $(n > 1)$, this is not necessarily the case. Consequently, for a dilation result in $H^2(D^n)$ we need to include the causality constraint explicitly in the set-up of the dilation problem. We will discuss a way of doing this now based on [42, 36, 37].

2.1.1 Control Causal Commutant Lifting Theorem

We now formulate a Causal Commutant Lifting Theorem that is suitable for control applications, in particular the full standard problem.

For the standard problem in robust control theory we may extract the following mathematical set-up. We are given complex separable Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ equipped with the unilateral shifts $S_{\mathcal{E}_1}, S_{\mathcal{E}_2}, S_{\mathcal{F}_1}, S_{\mathcal{F}_2}$, respectively. Let $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$ be a co-isometry intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{F}_1}$ (i.e., $\Theta_1 S_{\mathcal{E}_1} = S_{\mathcal{F}_1}\Theta_1$), and let $\Theta_2 : \mathcal{F}_2 \to \mathcal{E}_2$ be an isometry intertwining $S_{\mathcal{E}_2}$ with $S_{\mathcal{F}_2}$. We let $U_{\mathcal{E}_1}$, be the minimal unitary dilation of $S_{\mathcal{E}_1}$ on $\mathcal{K}_{\mathcal{E}_1}$, and similarly for $U_{\mathcal{E}_2}$ on $\mathcal{K}_{\mathcal{E}_2}$, $U_{\mathcal{F}_1}$ on $\mathcal{K}_{\mathcal{F}_1}$, and $U_{\mathcal{F}_2}$ on $\mathcal{K}_{\mathcal{F}_2}$.

Now let

$$P_{\mathcal{E}_2}^{(n)} := (I - S_{\mathcal{E}_2}^n S_{\mathcal{E}_2}^{*n}), \quad P_{\mathcal{F}_2}^{(n)} := (I - S_{\mathcal{F}_2}^n S_{\mathcal{F}_2}^{*n}), \quad n \geq 0.$$

We let the sequence $P_{\mathcal{E}_2}^{(n)}$ define the causal structure on $\mathcal{E}_2$, and similarly the causal structure of $\mathcal{F}_2$ is defined by the sequence $P_{\mathcal{F}_2}^{(n)}$. Moreover, the causal structure on $\mathcal{E}_1$ is defined by a general sequence of operators $P_1^{(n)}$, $n \geq 0$, satisfying the standard causal structure conditions [42], and similarly the causal structure on $\mathcal{F}_1$ is defined by a sequence of operators $P_2^{(n)}$, $n \geq 0$, satisfying these conditions as well. We assume that the input/putput operators Θ_1, Θ_2, are causal with respect to the above structures. We let $W : \mathcal{E}_1 \to \mathcal{E}_2$ denote a causal operator intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{E}_2}$, and let $Q : \mathcal{F}_1 \to \mathcal{F}_2$ be a causal operator intertwining $S_{\mathcal{F}_1}$ with $S_{\mathcal{F}_2}$.

Define

$$\mathcal{E}_1^{(n)} := (I - P_1^{(n)})\mathcal{E}_1, \quad \forall n \geq 0,$$

and

$$W_n := S_{\mathcal{E}_2}^{*n} W | \mathcal{E}_1^{(n)}.$$

Moreover, let

$$\mathcal{E}_1^{(c)} = \overline{\mathcal{E}_1^{(co)}}$$

where

$$\mathcal{E}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_1}^{*j} \mathcal{E}_1^{(j)} \subset \mathcal{K}_{\mathcal{E}_1}, \quad S_{\mathcal{E}_1}^{(c)} := U_{\mathcal{E}_1} | \mathcal{E}_1^{(c)}.$$

Finally, we define $W_c : \mathcal{E}_1^{(co)} \to \mathcal{E}_2$, by

$$W_c g := W_n g_n,$$

for $g = U_{\mathcal{E}_1}^{*n} g_n;\ g_n \in \mathcal{E}_1^{(n)},\ \ n \geq 0$.

Note that we can make a similar construction on the spaces $\mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$. In particular, for a causal $Q : \mathcal{F}_1 \to \mathcal{F}_2$, such that $QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q$, we can define $Q_c : \mathcal{F}_1^{(co)} \to \mathcal{F}_2$, where

$$\mathcal{F}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{F}_1}^{*j} \mathcal{F}_1^{(j)}.$$

Next, it is easy to see both W_c and Q_c extend by continuity to the closure $\mathcal{E}_1^{(c)}$, respectively $\mathcal{F}_1^{(c)} = \overline{\mathcal{F}_1^{(co)}}$. Clearly, we also have

$$\|W_c\| = \|W\|,\ \ W_c|\mathcal{E}_1 = W,\ \ W_c S_{\mathcal{E}_1}^{(c)} = S_{\mathcal{E}_2} W_c,$$

and $\|W - \Theta_2 Q \Theta_1\| = \|(W - \Theta_2 Q \Theta_1)_c\|$. Now set

$$\mu(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This corresponds to the *classical standard control problem.* We also set

$$\mu_c(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : Q \text{ causal},\ \ QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This is the *causal standard control problem.*

Let $\hat{\Theta}_1 : \mathcal{K}_{\mathcal{E}_1} \to \mathcal{K}_{\mathcal{F}_1}$ denote the extension of the co-isometry $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$, that is uniquely defined by

$$\hat{\Theta}_1 U_{\mathcal{E}_1}^{*n} e_1 = U_{\mathcal{F}_1}^{*n} \Theta_1 e_1,\ \ \forall e_1 \in \mathcal{E}_1.$$

Note that $\hat{\Theta}_1$ is also isometric and $\hat{\Theta}_1 U_{\mathcal{E}_1} = U_{\mathcal{F}_1} \hat{\Theta}_1$.

We can now state the following key result [38]:

Theorem 1 (Control Causal Commutant Lifting Theorem) *Notation as above.*

1. $\mu_c(W, \Theta_1, \Theta_2) = \mu(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2)$.
2. Q_{opt} *is a causal optimal solution, i.e.,*

$$\mu_c(W, \Theta_1, \Theta_2) = \|W - \Theta_1 Q_{opt} \Theta_2\|$$

if and only if $Q_{opt,c}$ *is such that*

$$\mu(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2) = \|W_c - \Theta_2 Q_{opt,c} \hat{\Theta}_1|\mathcal{E}_1^{(c)}\|.$$

Finally, let us recall how the classical standard problem can be solved using the commutant lifting theorem. Set

$$\begin{aligned} \mathcal{H}_1 &:= \mathcal{E}_1^{(c)} \ominus (\hat{\Theta}_1|\mathcal{E}_1^{(c)})^* \mathcal{E}_1^{(c)}, \\ \mathcal{H}_2 &:= \mathcal{E}_2 \ominus \Theta_2 \mathcal{F}_2. \end{aligned}$$

Let $P : \mathcal{E}_2 \to \mathcal{H}_2$ denote orthogonal projection. Then we define the operator

$$\Lambda = \Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2) : \mathcal{H}_1 \to \mathcal{H}_2, \tag{1}$$

by

$$\Lambda h := PW_c h,\ \ h \in \mathcal{H}_1. \tag{2}$$

Then using the commutant lifting theorem, one may show that

$$\|\Lambda\| = \mu(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2).$$

Thus from the above theorem, we have the following result:

Corollary 1 *Notation as above. Then*

$$\mu_c(W, \Theta_1, \Theta_2) = \|\Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2)\|.$$

Thus we see that Theorem 1 and Corollary 1, allow one to reduce a causal optimization problem to one involving classical interpolation.

This leads to an explicit computable solution of the nonlinear standard problem based on an iterative interpolation procedure. The computations are based on our previous skew Toeplitz methodology that we developed for distributed H^∞ control. See [36, 37, 38].

2.2 Saddle Points, Game Theory, and Nonlinear Optimization

We have described above a local analytic method for nonlinear system synthesis. We have also been exploring a very different approach applicable to certain systems with saturations (and "hard" noninvertible nonlinearities) based on a game-theoretic interpretation of the classical commutant lifting theorem [19]. This motivates us to formulate a nonlinear commutant lifting result in such a saddle-point, game-theoretic framework.

A related approach to nonlinear design has already been employed by a number of researchers; see [13, 14, 50, 51, 52, 93, 15] and the references therein. As is well known, game theoretic ideas have already been extensively applied in linear H^∞ theory. (See also [17] for an extensive discussion of the game theoretic approach to H^∞ theory, as well as a long list of references on the subject.)

In our research, instead of considering general nonlinear systems we have limited ourselves to the concrete (but certainly interesting case) of linear systems with input saturations. Such systems are, of course, essential for many practical applications. We should add that a similar approach is valid for many of the hard, memoryless, noninvertible nonlinearities which appear in control.

In order to motivate our game-theoretic approach to nonlinear H^∞, we will first give a "saddle-point" interpretation of the classical Sarason theorem in a special case. We let $w, m \in H^\infty$ with m inner. Set $H(m) := H^2 \ominus mH^2$, we let $P_{H(m)} : H^2 \to H(m)$ denote orthogonal projection, and $S(m)$ denote the compressed shift. We let $\| \ \|$ denote the 2-norm $\| \ \|_2$ on H^2 as well as the associated induced operator norm. In [19], we prove that

$$\inf_{q\in H^\infty} \sup_{\|f\|\leq 1} \|(w - mq)f\| = \sup_{\|f\|\leq 1} \inf_{q\in H^\infty} \|(w - mq)f\| = \sup_{\|f\|\leq 1} \inf_{g\in H^2} \|w - mg\|.$$

Now it is easy to show there always an optimal q_o; see e.g., [19]. We now assume that

$$\|w(S(m))\|_{ess} < \|w(S(m))\|,$$

where $\| \ \|_{ess}$ denotes the essential norm. Then there exists $f_o \in H^2$, $\|f_o\| = 1$ (a maximal vector), such that

$$\|(w - mq_o)(S(m))f_o\| = \|w(S(m))f_o\| = \|w(S(m))\| = \|(w - mq_o)(S(m))\|.$$

Now

$$P_{H(m)}(w - mq_o)f_o = (w - mq_o)(S(m))f_o = w(S(m))f_o,$$
$$(w - mq_o)f_o = w(S(m))f_o.$$

So

$$\|(w - mq_o)f\| \leq \|(w - mq_o)(S(m))f_o\| = \|(w - mq_o)f_o\|$$

for all $f \in H^2$, $\|f\| \leq 1$. Moreover,

$$\|w(S(m))f_o\| = \|(w - mq)(S(m))f_o\| \leq \|(w - mq)f_o\|.$$

Hence, we get that

$$\|(w - mq_o)f\| \le \|(w - mq_o)f_o\| \le \|(w - mq)f_o\| \tag{3}$$

for all $f \in H^2$, $\|f\| \le 1$, and for all $q \in H^\infty$. It is a nonlinear analogue of the saddle-point condition (3) that we want to analyze for saturated systems. Indeed, assuming the saddle-point condition (3), in [19] we derive all of the standard consequences of the Sarason theorem. Thus it is precisely the existence of a saddle-point which we would like to explore in the nonlinear setting.

By virtue of interpretation of the commuting lifting theorem as asserting the existence of a saddle-point, we have derived a global approach to sensitivity minimization for input saturated systems. Thus for σ_θ, a saturation of magnitude $\theta < 1$ (see [19] for all the precise definitions), and $m \in H^\infty$ inner, we want to know when there exist $f_o \in H^2$, $\|f_o\| \le 1$, q_o continuous, causal, time-invariant, such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \le \|(w - m\sigma_\theta \circ q_o)f_o\| \le \|(w - m\sigma_\theta \circ q)f_o\|$$

for all $f \in H^2$, $\|f\| \le 1$, q continuous, causal, time-invariant. Such a q_o (when it exists) will correspond to the optimal compensator, and

$$\mu := \|(w - m\sigma_\theta \circ q_o)f_o\|$$

will be the optimal performance in the weighted sensitivity minimization problem. But this is equivalent to finding $g_o = q_o(f_o) \in H^2$ such that

$$\|(w - m\sigma_\theta \circ q_o)f\| \le \|w(f_o) - m\sigma_\theta(g_o)\| \le \|(w - m\sigma_\theta \circ q)f_o\|. \tag{4}$$

Our approach then has been to follow an analogous line of reasoning which we just outlined in our analysis of the saddle-point condition in the linear case. This leads to nonlinear commutant lifting theorem valid on a *convex space* which can be used to develop a global robust design procedure for nonlinear plants with hard nonlinearities [19].

2.3 Distributed Parameter Systems

We have been improved and extended our algorithms for the computation of optimal H^∞ controllers for infinite dimensional systems. Our method is based on an explicit calculation of $\|f(T)\|$ with f rational, where we allow T to be an arbitrary contraction of class $C_{\cdot 0}$ [21]. We will now outline this method.

This method has led to a very simple way of designing multivariable controllers for distributed multivariable systems, and has important implications for extensions to the *time-varying* case as well. The approach oulined below has been reported in [21].

2.3.1 Approximate Eigenvalues of Skew Toeplitz Operators

Let T be a bounded operator on a Hilbert space $\mathcal{H}$, and let $f(\lambda) = p(\lambda)/q(\lambda)$ be a rational function with poles off the spectrum $\sigma(T)$ of T, i.e., $q(\lambda) \ne 0$ for $\lambda \in \sigma(T)$. Further, denote $A = f(T) = p(T)q(T)^{-1}$. We will be interested in the effective calculation of the norm $\|A\|$ in the case when T is a contraction represented as a functional model, and q has no zeros in the closed unit disk. However, some simple observations can be made in the general case. Thus, for instance, $\|A\|$ is greater than the spectral radius $\|A\|_{\mathrm{sp}}$, hence

$$\|A\|_{\mathrm{sp}} = \sup\left\{\left|\frac{p(\lambda)}{q(\lambda)}\right| : \lambda \in \sigma(T)\right\} \le \|A\|.$$

Next, if ρ denotes $\|A\|$, then the operator $\rho^2 I - AA^*$ is positive definite but not invertible, and hence it has zero as an approximate eigenvalue. Since

$$\rho^2 I - AA^* = q(T)^{-1}(\rho^2 q(T)q(T)^* - p(T)p(T)^*)q(T)^{*-1},$$

we deduce that the operator

$$Q = \rho^2 q(T)q(T)^* - p(T)p(T)^*$$

is positive definite and not invertible. If $p(\lambda) = \sum_{j=0}^{n} p_j\lambda^j$ and $q(\lambda) = \sum_{j=0}^{n} q_j\lambda^j$, then Q can be written as

$$Q = \sum_{i,j=0}^{n} c_{ij}T^iT^{*j},$$

where the coefficients $c_{ij} = \rho^2 q_i\bar{q}_j - p_i\bar{p}_j$ satisfy the condition $c_{ij} = \bar{c}_{ji}$, $0 \leq i, j \leq n$. Now, given an arbitrary polynomial in two variables

$$\omega(\lambda, \mu) = \sum_{i,j=0}^{n} c_{ij}\lambda^i\mu^j,$$

one can introduce an operator

$$Q_\omega = \omega(T, T^*) = \sum_{i,j=0}^{n} c_{ij}T^iT^{*j}.$$

The problem of deciding whether $\rho^2 I - AA^*$ has zero as an approximate eigenvalue is equivalent to the corresponding question for an operator of the form Q_ω such that $c_{ij} = \bar{c}_{ji}$, $0 \leq i, j \leq n$. Since the calculation of $\rho = \|A\|$ is only a problem when $\|A\|_{\text{sp}} < \rho$, we may restrict ourselves to the case in which $\omega(\lambda, \bar{\lambda}) \neq 0$ for every $\lambda \in \sigma(T)$.

In the cases of interest in control, more information is available about T and f. More precisely, T is a contraction with inner characteristic function, and f belongs to the algebra H^∞ of bounded analytic functions in the unit disk $D = \{\lambda : |\lambda| < 1\}$. This means that q has no zeros in the closure $\overline{D}$ of D, and then von Neumann's inequality implies that

$$\|A\|_{\text{sp}} = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\} \leq \|A\| \leq \sup\{|f(\zeta)| : |\zeta| = 1\}.$$

These inequalities become equalities if $\sigma(T)$ contains the entire unit circle $\partial D = \{\zeta : |\zeta| = 1\}$. Hence we will assume throughout that $\sigma(T)$ does not contain the unit circle. We have arrived at the following:

Distributed Control Problem (DCP): We are given a contraction T and a polynomial $\omega(\lambda, \mu) = \sum_{i,j=0}^{n} c_{ij}\lambda^i\mu^j$ such that

(i) the characteristic function of T is inner;

(ii) $\sigma(T)$ does not contain the unit circle;

(iii) $c_{ij} = \bar{c}_{ji}$, $0 \leq i, j \leq n$; and

(iv) $\omega(\lambda, \bar{\lambda}) \neq 0$ for every $\lambda \in \sigma(T)$.

Determine whether zero is an approximate eigenvalue of $Q_\omega = \omega(T, T^*) = \sum_{i,j=0}^{n} c_{ij}T^iT^{*j}$.

We recall that the operators Q_ω considered in Problem DCP are exactly the skew Toeplitz operators defined in [39]. In [21], a new approach to solving Problem DCP is given. This approach has the appealing property that it is extendable to *time-varying* systems. See also [34] for some recent results as well as a large set of references on time-varying versions of interpolation.

2.4 Geometric Evolutions in Vision and Image Processing

In our ARO contract, we devoted a large part of our research work to visual tracking. This is a central area in which the multivariable control methods developed over the past twenty five years could have a major impact. In order to successfully work on this problem, it is essential to incorporate and greatly extend key techniques from image processing and computer vision. We have found that the theory of geometric invariant flows is very relevant for a number of problems in controlled active vision. Interestingly these flows themselves are very much motivated by the calculas of variations and ideas in *optimal control*; see [59].

2.4.1 Background on Curve and Surface Evolution

In this section we will review some of the basic results on curvature driven flows. Full details may be found in the very recent text [79]. For simplicity, we will focus here on the case of planar curves.

A geometric set or shape can be defined by its boundary. In the case of bounded planar shapes for example, this boundary consists of closed planar curves. We will only deal with closed planar curves, keeping in mind that these curves are boundaries of planar shapes. A curve may be regarded as a trajectory of a point moving in the plane. Formally, we define a curve $\mathcal{C}(\cdot)$ as the map $\mathcal{C}(p) : S^1 \to \mathbf{R}^2$ (where S^1 denotes the unit circle). We assume that our curves are have no self-intersections, i.e., are embedded.

We now consider plane curves deforming in time. Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ denote a family of closed embedded curves, where t parametrizes the family, and p parametrizes each curve. Assume that this family evolves according to the following equation:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}} \\ \mathcal{C}(p,0) = \mathcal{C}_0(p) \end{cases} \tag{5}$$

where $\vec{\mathcal{N}}$ is the inward Euclidean unit normal, $\vec{\mathcal{T}}$ is the unit tangent, and α and β are the tangent and normal components of the evolution velocity $\vec{v}$, respectively. In fact, it is easy to show that $\text{Img}[\mathcal{C}(p,t)] = \text{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}} \text{ and } \hat{\mathcal{C}}_t = \bar{\beta} \mathcal{N},$$

respectively. (Here $\text{Img}[\cdot]$ denotes the image of the given parametrized curve in $\mathbf{R}^2$.) Thus the tangential component affects only the parametrization, and not $\text{Img}[\cdot]$. Therefore, assuming that the normal component β of $\vec{v}$ (the curve evolution velocity) in (5) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \vec{\mathcal{N}}, \tag{6}$$

where $\beta = \vec{v} \cdot \vec{\mathcal{N}}$.

The evolution (6) was studied by different researchers for different functions β. This type of flow was introduced into the theory of shape in [56, 57, 58]. One of the key cases is obtained for $\beta = \kappa$, where κ is the Euclidean curvature:

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \vec{\mathcal{N}}. \tag{7}$$

Equation (7) is called the *geometric heat equation* or the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible (using only local information) when the curve evolves according to (7). Gage and Hamilton [43] proved that a planar embedded convex curve converges to a round point when evolving according to (7). Grayson [45] proved that a planar embedded non-convex curve converges to a convex one, and from there to a round point from Gage and Hamilton result. For other results related to the Euclidean shortening flow, see [8, 9, 43, 45].

Another important example is obtained when one sets $\beta = 1$ in equation (6):

$$\frac{\partial \mathcal{C}}{\partial t} = \vec{\mathcal{N}}. \tag{8}$$

This equation simulates, under certain conditions, the grassfire flow [26, 85]. (More precisely, the unique weak solution of (8) which satisfies the *entropy* condition [85] gives the grassfire flow.) This grassfire flow is also the basis of the morphological scale-space defined by the disk as structuring element. Moreover, one can prove that with different selections of β, other morphological scale-spaces are obtained [59].

In [56, 58], we have studied the following equation in order to develop a hierarchy of shape,

$$\frac{\partial \mathcal{C}}{\partial t} = (1 + \epsilon\kappa)\vec{\mathcal{N}}. \tag{9}$$

This equation was introduced by Osher and Sethian [76] in the level set framework. If $\epsilon \to 0$ in (9), the grassfire flow is obtained, and this introduces singularities (*shocks*) in the evolving curve. (The shocks define the well-known skeleton.) On the other hand, if $\epsilon \to \infty$, equation (9) reduces to the classical Euclidean curve shortening flow, which smoothes the curve [86]. The combination of these two opposite features gives very interesting properties. When a curve evolves according to (9), the evolution of the curve slope satisfies a reaction-diffusion equation [89]. The reaction term, which tends to create singularities, competes with the diffusion term which tends to smooth the curve. For each different value of ϵ, a scale-space is obtained by looking at the solution of (9), and considering the time t as the scale parameter. We have called the set of all the scale-spaces obtained for all values of ϵ, the *reaction-diffusion scale-space* [56]. In particular, we see that the Euclidean shortening flow (equation (7)) defines an Euclidean invariant scale-space (the equation admits Euclidean invariant solutions). In contrast with other scale-spaces, like the one obtained from the classical linear heat equation, this one is a full geometric scale-space. The progressive smoothing given by κ is geometrically intrinsic to the curve.

We now discuss the affine analogue of the Euclidean shortening flow. (The affine group SA_2 is the group generated by unimodular transformations and translations of $\mathbf{R}^2$. Under certain natural conditions, it provides a good approximation to the full group of perspective projective transformations.) Then in [73, 80], we show that the simplest non-trivial affine invariant flow in the plane is given by

$$\mathcal{C}_t = \kappa^{1/3}\vec{\mathcal{N}}. \tag{10}$$

This equation was introduced independently in [6] in the level set framework where it was called the "fundamental equation of image processing." The question now is what happens when a non-convex curve evolves according to (10). The following result answers this question [10]:

Theorem 2 *Let $\mathcal{C}(\cdot, 0) : S^1 \to \mathbf{R}^2$ be a smooth embedded curve in the plane. Then there exists a family $\mathcal{C} : S^1 \times [0, T) \to \mathbf{R}^2$ satisfying*

$$\mathcal{C}_t = \kappa^{1/3}\vec{\mathcal{N}},$$

such that $\mathcal{C}(\cdot, t)$ is smooth for all $t < T$, and moreover there is a $t_0 < T$ such that for all $t > t_0$, $\mathcal{C}(\cdot, t)$ is smooth and convex.

Theorem 2 means that just as in the Euclidean case, a non-convex curve first becomes convex when evolving according to (10). After this, the curve converges to an ellipse from our results in [80]. Because of this, and other related properties (see [81]), we can conclude that equation (10) is the affine analogue of (7) for smooth embedded curves, and thus is called the *affine shortening flow.* (It is also the affine invariant formulation of the geometric heat equation.) One can use it to construct an *affine invariant scale-space* for planar shapes [81].

2.4.2 Geometric Active Contours

In this section, we will describe a paradigm for *snakes* or *active contours* based on principles from curvature driven flows and the calculus of variations.

Active contours may be regarded as autonomous processes which employ image coherence in order to track various features of interest over time. Such deformable contours have the ability to conform to various object shapes and motions. Snakes have been utilized for segmentation, edge detection, shape modeling, and visual tracking. The books [24, 79] contain excellent discussions on the state-of-the-art of the subject.

In the classical theory of snakes, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain contraints set by the user. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters. Moreover, natural criteria for the splitting and merging of contours (or for the treatment of multiple contours) are not readily available in this framework.

In [55], we propose a deformable contour model to successfully solve such problems, and which will become one of our key techniques for tracking. (A similar approach was independently formulated in [31, 87].) Our method is based on the Euclidean curve shortening evolution (see Section 2.4.1) which defines the gradient direction in which a given curve is shrinking as fast as possible relative to Euclidean arc-length, and on the theory of conformal metrics. We multiply the Euclidean arc-length by a conformal factor defined by the features of interest which we want to extract, and then we compute the corresponding gradient evolution equations. The features which we want to capture therefore lie at the bottom of a potential well to which the initial contour will flow. Moreover, our model may be easily extended to extract 3D and 4D surfaces based on motion by mean curvature [55, 64].

The starting point of this work is [30, 67] in which a snake model based on the level set formulation of the Euclidean curve shortening equation is proposed. More precisely, the model is

$$\frac{\partial \Psi}{\partial t} = \phi(x,y)\|\nabla\Psi\|(\mathrm{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nu). \tag{11}$$

Here the function $\phi(x,y)$ depends on the given image and is used as a "stopping term." For example, the term $\phi(x,y)$ may chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. One may take [30, 67]

$$\phi := \frac{1}{1 + \|\nabla G_\sigma * I\|^2}, \tag{12}$$

where I is the (grey-scale) image and G_σ is a Gaussian (smoothing filter) filter. The function $\Psi(x,y,t)$ evolves in (11) according to the associated level set flow for planar curve evolution in the normal direction with speed a function of curvature which was introduced in [76, 85, 86].

It is important to note that the Euclidean curve shortening part of this evolution, namely

$$\frac{\partial \Psi}{\partial t} = \|\nabla\Psi\|\mathrm{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) \tag{13}$$

is derived as a gradient flow for shrinking the perimeter as quickly as possible. As is explained in [30], the constant *inflation term* ν is added in (11) in order to keep the evolution moving in the proper direction. Note that we are taking Ψ to be negative in the interior and positive in the exterior of the zero level set.

We would like to modify the model (11) in a manner suggested by the curve shortening flow. We change the ordinary arc-length function along a curve $C = (x(p), y(p))^T$ with parameter p given by

$$ds = (x_p^2 + y_p^2)^{1/2}dp,$$

to

$$ds_\phi = (x_p^2 + y_p^2)^{1/2}\phi dp,$$

where $\phi(x, y)$ is a positive differentiable function. Then we want to compute the corresponding gradient flow for shortening length relative to the new metric ds_ϕ. Setting

$$L_\phi(t) := \int_0^1 \|\frac{\partial C}{\partial p}\|\phi dp,$$

and taking the first variation of the modified length function L_ϕ, and using integration by parts (see [55]), we get that

$$L'_\phi(t) \quad = \quad -\int_0^{L_\phi(t)} \langle \frac{\partial C}{\partial t}, \phi\kappa\vec{\mathcal{N}} - (\nabla\phi \cdot \vec{\mathcal{N}})\vec{\mathcal{N}}\rangle ds$$

which means that the direction in which the L_ϕ perimeter is shrinking as fast as possible is given by

$$\frac{\partial C}{\partial t} = (\phi\kappa - (\nabla\phi \cdot \vec{\mathcal{N}}))\vec{\mathcal{N}}. \tag{14}$$

This is precisely the gradient flow corresponding to the miminization of the length functional L_ϕ. The level set version of this is

$$\frac{\partial \Psi}{\partial t} = \phi\|\nabla\Psi\|\text{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nabla\phi \cdot \nabla\Psi. \tag{15}$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the potential well described by the gradient flow (15). As in [30, 67], we may also add a constant inflation term, and so derive a modified model of (11) given by

$$\frac{\partial \Psi}{\partial t} = \phi\|\nabla\Psi\|(\text{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nu) + \nabla\phi \cdot \nabla\Psi. \tag{16}$$

Notice that for ϕ as in (12), $\nabla\phi$ will look like a doublet near an edge. Of course, one may choose other candidates for ϕ in order to pick out other features.

We now have very fast implementations of these snake algorithms based on level set methods [76, 85]. Clearly, the ability of the snakes to change topology, and quickly capture the desired features will make them an indispensable tool for our visual tracking algorithms. See also [92] for more details about this.

We are also studying an affine invariant snake model for tracking based on our work in [75]. (The evolution itself works using a level set model of $\kappa^{1/3}\vec{\mathcal{N}}$ as discussed in Section 2.4.1.) We have developed affine invariant volumetric smoothers in [74], and have employed affine smoothers in movies as a preprocessing tool for motion estimation. We are now working on the incorporation of more global information for the active contours as well as utilizing Bayesian statistical models.

2.4.3 Invariant Flows

In this section, we will summarize some of our recent work on the classification of invariant geometric flows. It is interesting to note how the calculus of variations and thus optimal control type techniques plays such a fundamental role in solving this problem. This is based on our work reported in [74].

Consider the evolution of hypersurfaces which are assumed to be represented by the graph of a function. We let the $p+1$-dimensional Euclidean space $E \simeq \mathbf{R}^p \times \mathbf{R}$, with coordinates $x = (x^1, \ldots, x^p)$ representing the independent variables, and $u \in \mathbf{R}$ the dependent variable.

The hypersurface $\mathcal{S} \subset E$ will be identified with the graph of a function $u(x)$, defined on a domain $x \in D \subset \mathbf{R}^p$. The symmetry group G will be a finite-dimensional, connected transformation group acting on E. Each group transformation $g \in G$ will map hypersurfaces to hypersurfaces by pointwise transformation.

In Lie's theory of symmetry groups, one replaces the actual group transformations by their infinitesimal generators, which are vector fields on the domain E, taking the general form

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \varphi(x,u)\frac{\partial}{\partial u} = \xi^1(x,u)\frac{\partial}{\partial x^1} + \cdots + \xi^p(x,u)\frac{\partial}{\partial x^p} + \varphi(x,u)\frac{\partial}{\partial u}. \tag{17}$$

Each vector field generates a local one-parameter group of transformations (or flow) on E, obtained by integrating the associated system of ordinary differential equations

$$\frac{dx}{d\varepsilon} = \xi(x,u), \qquad \frac{du}{d\varepsilon} = \varphi(x,u), \tag{18}$$

where ε represents the group parameter. In other words, the group transformations have the Taylor expansion

$$x(\varepsilon) = x + \varepsilon\xi(x,u) + \cdots, \qquad u(\varepsilon) = u + \varepsilon\varphi(x,u) + \cdots. \tag{19}$$

The order ε terms in (19) are known as the *infinitesimal group transformations*, and can be identified with the generating vector field (17). The different one-parameter groups combine to generate the entire connected group action of G.

Fixing the vector field (17), let $u(x,\varepsilon)$ denote the one-parameter family of hypersurfaces (functions) obtained from a given hypersurface $u(x,0) = u(x)$ by applying the group transformation with parameter ε. The infinitesimal change in the hypersurface is found by expanding in powers of ε using Taylor's Theorem and the chain rule. Thus, the value of the transformed function u at the new point $x(\varepsilon)$ is given by

$$u(x(\varepsilon),\varepsilon) = u(x) + \varepsilon\varphi(x,u(x)) + \cdots. \tag{20}$$

On the other hand, if we are interested in the value of the transformed function at the original point $x = x(0)$, we substitute (19) into (20) to deduce the alternative expansion

$$u(x,\varepsilon) = u(x) + \varepsilon Q[u(x)] + \cdots. \tag{21}$$

The function

$$Q[u] = \varphi(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u}{\partial x^i}, \tag{22}$$

is known as the *characteristic* of the vector field (17). The characteristic Q depends on first order derivatives $u_i = \partial u/\partial x^i$ because the group transformations are acting on the independent variables x as well as the dependent variable u. In particular, a G–invariant hypersurface is independent of the group parameter ε, and hence satisfies the first order partial differential equation $Q(x,u^{(1)}) = 0$, indicating its "infinitesimal invariance" under the vector field $\mathbf{v}$. Conversely, any infinitesimally invariant function, i.e., any solution to the characteristic equation $Q = 0$, is, in fact, invariant under the entire connected transformation group.

Consider the function $F[u] = F(x,u^{(n)})$ depending on x, u, and the derivatives of u, denoted by $u_J = D_J u$. Here $D_J = D_{j_1}D_{j_2}\cdots D_{j_k}$ are the total derivative operators, which differentiate treating u as a function of x. The infinitesimal variation in the function $F[u]$ under the group generated by the vector field $\mathbf{v}$ is then given by

$$\left.\frac{d}{d\varepsilon}F[u(x,\varepsilon)]\right|_{\varepsilon=0} = \sum_J \frac{\partial F}{\partial u_J} D_J Q. \tag{23}$$

In (23) we evaluate F and u at the original point x. If we are interested in the value at the transformed point $x(\varepsilon)$, we must include an additional term arising from the change of independent variable, as in the passage from (21) to (20). We deduce the expansion

$$F(x(\varepsilon), u^{(n)}(x,\varepsilon)) = F(x, u^{(n)}) + \varepsilon \mathrm{pr}\, \mathbf{v}(F) + \cdots, \tag{24}$$

where

$$\mathrm{pr}\, \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F \tag{25}$$

defines the "prolongation" of the vector field $\mathbf{v}$, denoted pr $\mathbf{v}$, which forms the infinitesimal generator of the prolonged group action on the space of derivatives.

A function $F(x, u^{(n)})$ is called a *differential invariant* if its value is not affected by the group transformations. Thus we require that the left hand side of (24) be independent of ε. The infinitesimal invariance condition is obtained by differentiating with respect to ε. This produces

$$0 = \mathrm{pr}\, \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F. \tag{26}$$

Condition (26), for $\mathbf{v}$ an arbitrary infinitesimal generator of G, is necessary and sufficient for F to be a differential invariant.

A transformation group G is called a *symmetry group* of a differential equation

$$F(x, u^{(n)}) = 0 \tag{27}$$

if it maps solutions to solutions. The differential equation (27) admits G as a symmetry group if and only if the infinitesimal invariance condition

$$\mathrm{pr}\, \mathbf{v}[F] = 0 \quad \text{whenever} \quad F = 0 \tag{28}$$

holds for all infinitesimal generators of G.

Invariant Hypersurface Flows:

The goal is to determine the general form that a G–invariant evolution equation

$$u_t = K(x, u^{(n)}) \tag{29}$$

must take. Here we have introduced an additional variable t — the time or scale parameter — which is not affected by our group transformations.

Thus, for $p = 1$, we will determine all possible invariant curve evolutions in the plane under a given transformation group, while for $p = 2$ we find the invariant surface evolutions. According to (25), the infinitesimal change in the t-derivative of u at the transformed point is

$$\frac{d}{d\varepsilon} u_t(x,t,\varepsilon)\bigg|_{\varepsilon=0} = D_t Q + \sum_{i=1}^{p} \xi^i D_i u_t = Q_u u_t, \tag{30}$$

where

$$Q_u = \frac{\partial Q}{\partial u} = \frac{\partial \varphi}{\partial u} - \sum_{i=1}^{p} \frac{\partial \xi^i}{\partial u} \frac{\partial u}{\partial x^i}. \tag{31}$$

Therefore, using the infinitesimal condition (28), and substituting for u_t according to the equation (29), we deduce the basic invariance condition that an evolution equation must satisfy in order to admit a prescribed symmetry group.

Lemma 1 *A connected transformation group G is a symmetry group of the evolution equation $u_t = K[u]$ if and only if the infinitesimal condition*

$$\text{pr } \mathbf{v}(K) = Q_u K \tag{32}$$

holds for every infinitesimal generator $\mathbf{v}$ *of the group G with associated characteristic Q.*

To discover a G–invariant evolution equation for an arbitrary group, we consider the G–invariant functionals. An n–th order *variational problem* consists of finding the extremals (maxima or minima) of a *functional*

$$\mathcal{L}_D[u] = \int_D L(x, u^{(n)})\, dx = \int_D L(x, u^{(n)})\, dx^1 \wedge \ldots \wedge dx^p, \tag{33}$$

subject to certain boundary conditions.

The integrand $L[u] = L(x, u^{(n)})$, known as the *Lagrangian*, is a smooth function depending on x, u and the derivatives of u. A transformation group G is a symmetry group of a variational problem provided it leaves the functional (33) invariant.

More precisely, given a function $u(x)$ defined on a domain D, and a one-parameter subgroup of G, we let $u(x, \varepsilon)$ denote the transformed function, which is defined on a transformed domain $D(\varepsilon)$. Invariance of the functional requires that $\mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] = \mathcal{L}_D[u(x)]$. Using the standard Jacobian change of variables formula for multiple integrals, the infinitesimal invariance condition is then found by differentiating:

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] \bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \int_D L[u(x(\varepsilon), \varepsilon)] \det \left[\frac{\partial x(\varepsilon)}{\partial x} \right] dx \bigg|_{\varepsilon=0} \\ &= \int_D [\text{pr } \mathbf{v}(L) + L \,\text{div } \xi]\, dx\,. \end{aligned} \tag{34}$$

Here div $\xi = \sum D_i \xi^i$ is the total divergence arising from the infinitesimal change in the independent variables.

Lemma 2 *A connected transformation group G a symmetry group of the variational problem $\int L\, dx$ if and only if every infinitesimal generator* $\mathbf{v}$ *satisfies the infinitesimal condition*

$$\text{pr } \mathbf{v}(L) + L \,\text{div } \xi = 0. \tag{35}$$

The smooth extremals (maxima and minima) of a variational problem are known to satisfy the classical Euler-Lagrange equation,

$$E(L) := \sum_{\#J=0}^{n} (-D)_J \frac{\partial L}{\partial u_J} = 0, \qquad \alpha = 1, \ldots, q. \tag{36}$$

where $(-D)_J = (-D_{j_1})(-D_{j_2}) \cdots (-D_{j_k})$ is the signed total derivative. This condition is the infinite-dimensional analog of the vanishing gradient condition for maxima and minima of ordinary functions. The Euler-Lagrange equation is obtained by taking the variational derivative of the functional. For example, if $\mathcal{L}$ represents the G-invariant arc-length or surface area functional, the Euler-Lagrange equation will describe the G–invariant minimal curves or surfaces. In general, the invariance of a variational problem under a given transformation group implies the invariance of its Euler-Lagrange equation. (The converse, however, is not true.) We will be interested in precisely how the Euler-Lagrange equation varies, and this is the result of the following key lemma.

Lemma 3 *Let* pr **v** *be the prolonged vector field (25). Let* $L(x, u^{(n)})$ *be a Lagrangian. Then*

$$E(\mathrm{pr}\ \mathbf{v}(L) + L\,\mathrm{div}\ \xi) = \mathrm{pr}\ \mathbf{v}(E(L)) + (Q_u + \mathrm{div}\ \xi)E(L). \tag{37}$$

From this, we can construct invariant evolution equations. Suppose that L is a G–invariant Lagrangian, e.g., defining the group invariant arc length or area. Then L satisfies the infinitesimal invariance condition (35), and hence (37) implies the identity

$$\mathrm{pr}\ \mathbf{v}[E(L)] + (\mathrm{div}\ \xi + Q_u)E(L) = 0. \tag{38}$$

Equation (38) means that $E(L)$ is a relative differential invariant of weight $-\mathrm{div}\ \xi - Q_u$. In particular, the Euler-Lagrange equation $E(L) = 0$ is invariant under G, as claimed. On the other hand L itself is a relative invariant of weight $-\mathrm{div}\ \xi$. Since the prolonged vector field pr **v** acts as a derivation, the ratio $E(L)/L$ is a relative differential invariant weight $-Q_u$, i.e., it satisfies

$$\mathrm{pr}\ \mathbf{v}\left[\frac{E(L)}{L}\right] + Q_u\left[\frac{E(L)}{L}\right] = 0. \tag{39}$$

Consequently, its reciprocal $L/E(L)$ (assuming $E(L) \neq 0$) satisfies (32) and defines a G-invariant evolution equation. We have therefore deduced our fundamental theorem from [74]:

Theorem 3 *Let G be a transformation group, and let $L\,dx$ be a G-invariant Lagrangian with non-identically zero Euler-Lagrange derivative $E(L)$. Then every G–invariant evolution equation has the form*

$$u_t = \frac{L}{E(L)}\,I, \tag{40}$$

where I is a arbitrary differential invariant of G.

Although (40) defines the most general class of invariant evolution equations, the case when the differential invariant I is constant is not necessarily the simplest one. In the planar Euclidean case, $L = \sqrt{1 + u_x^2}$ is the Euclidean arc length Lagrangian, so that

$$E(L) = -D_x \frac{\partial L}{\partial u_x} = -\frac{u_{xx}}{(1 + u_x^2)^{3/2}} = -\kappa.$$

Thus the general Euclidean-invariant evolution equation has the form

$$u_t = -\sqrt{1 + u_x^2}\,\frac{I}{\kappa},$$

where I is an arbitrary function of κ and its arc length derivatives. Choosing $I = \kappa$ produces the simplest one (eikonal equation), while $I = \kappa^2$ produces the Euclidean curve shortening flow.

One can also deduce the following:

Proposition 1 *Suppose G is a connected transformation group, and $L\,dx$ a G–invariant p-form such that $E(L) \neq 0$. Then $E(L)$ is a differential invariant if and only if G is volume–preserving.*

Corollary 2 *Let G be a connected volume preserving transformation group. Then, up to constant multiple, the G–invariant flow of lowest order has the form*

$$u_t = L, \tag{41}$$

where $\omega = L\,dx^1 \wedge \ldots \wedge dx^p$ is the invariant p–form of minimal order such that $E(L) \neq 0$.

Affine Invariant Surface Flows:

We apply the preceding results to describe the simplest possible affine invariant surface evolution. This gives, for convex surfaces, the surface version of the affine shortening flow for curves. The group G is the *(special) affine group* $\mathrm{SL}(3, \mathbb{R})$, consisting of all 3×3 matrices with determinant 1, combined with the translations. Let $\mathcal{S}$ be a smooth strictly convex surface in $\mathbf{R}^3$, which we write locally as a graph $u = u(x, y)$.

The simplest affine-invariant area-form is constructed from the affine-invariant metric, which is given by [74]

$$L\, dx \wedge dy = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2}\, dx \wedge dy,$$

where

$$\kappa = \frac{u_{xx} u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2},$$

denotes the usual Gaussian curvature of $\mathcal{S}$. Corollary 2 allows us to conclude:

Corollary 3 *Up to constant multiple, the simplest affine-invariant evolution equation has the form*

$$u_t = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2}. \qquad (42)$$

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4 Students Supported

1. Steven Haker (Ph.D. 1999)
2. Matthew Montminy (Ph.D. 2001)
3. Andrew Stein (M.S. 2002)

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