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Dynamics of a Gravity Gradient Anchored Tethered Space Antenna

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An analysis of the attitude dynamics of a tether-stabilized orbiting space antenna is presented. For many missions it is desirable to fly an antenna in space with its collecting surface nadir directed. Oriented with its large spatial dimensions perpendicular to the gravitational radius vector, an antenna body alone could well be unstable. Adding tethered masses to the antenna could stabilize the attitude by altering the system's inertial properties. The motion equations are derived for a system comprised of a rigid body and two tethered point masses in orbit about an inertially spherical gravitational primary. An arbitrary number of tethers connect the point masses to the rigid body attaching at distinct locations. A single tether is allowed to connect the respective masses. The tethers are assumed to be massless tensile members of fixed unstrained length possessing viscoelastic constitutive character. The nonlinear motion equations for the system are derived from Newton-Euler momentum principles. A steady state solution to the motion equations is proposed for the system centroid in a circular orbit. Considering small motions about the steady state configuration, the linearized motion equations are derived. The linearized equations are brought to a standard canonical form suitable for application of the stability theorems for linear stationary second order systems.

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Dynamics of a Gravity Gradient Anchored Tethered Space Antenna

Introduction

The intrinsic attitude stability of a vehicle in orbit is fundamental to the viability and efficacy of any configurational concept. While virtually any spacecraft can be stabilized with sufficient effort, designers' attention is most naturally drawn to those configurations which require the least such efforts. This report begins an analysis of the unassisted attitude stability of a somewhat unconventional spacecraft configuration, but one for which a number of practical applications have been suggested. The concept studied is that of an orbiting antenna structure of laminal geometry, which flies with the large dimension collector surface oriented to the NADIR direction and is anchored in that attitude via tethered endmasses.

Figure-1 illustrates the notional system. In addition to the antenna body, two tethered endmasses are considered which can be attached to the antenna by any number of tethers. The tethers themselves are considered massless tensile members with viscoelastic constitutive character. The nonlinear equations of motion are derived for the system subject only to the gravitational attraction of an inertially spherical primary. An equilibrium solution corresponding to a circular orbit is determined and the motion equations are linearized about that NADIR oriented steady state. The linearized equations are shown to be stationary and are brought to a canonical second order form to which the theorems of linear stability theory can be applied.

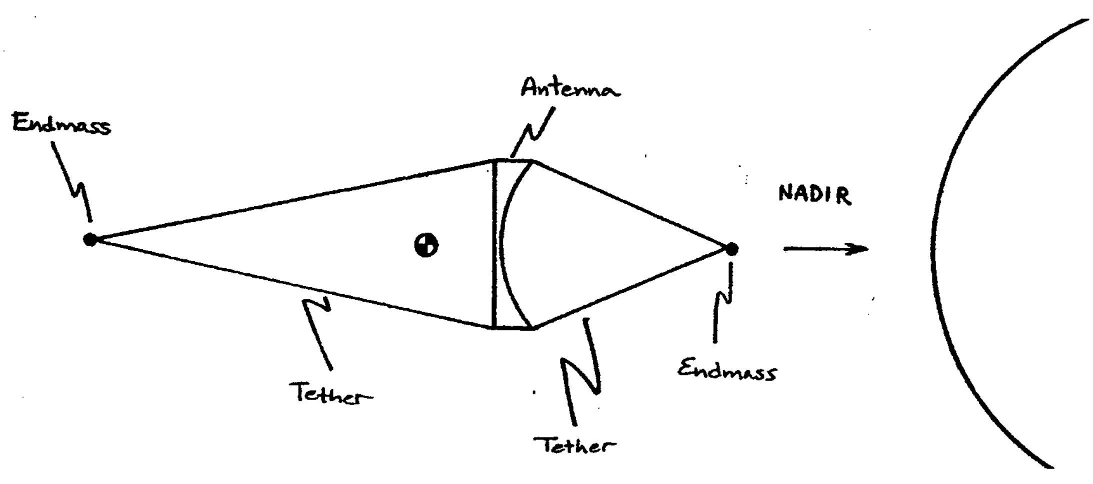

Figure-1. Gravity Gradient Anchored Tethered Space Antenna.

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System Definition

The idealized system $\mathcal{S}$ is comprised of a rigid body $\mathcal{R}$ and two point masses $\mathcal{P}_i$, (i=1,2). Particle $\mathcal{P}_i$ is assumed to be connected to $\mathcal{R}$ by N_i tethers attached at arbitrary positions. $\mathcal{P}_1$ can be connected to $\mathcal{P}_2$ by only a single tether. The tethers are idealized as massless structural members capable of exerting only tensile force on the respective endbodies directed along the straight line connecting their attachment points. The tethers are considered to be of fixed unstrained length (i.e. deployment/retraction is not allowed) but they can stretch longitudinally. The constitutive character of the tethers is taken to be linear viscoelastic; essentially they behave as massless parallel spring-damper pairs. The system is taken to be free in space subject only to the gravitational attraction of a large inertially spherical primary. The assembly possesses twelve degrees of freedom and constitutes a flexible multibody system, albeit a relatively simple one. Figure-2 shows the idealized system.

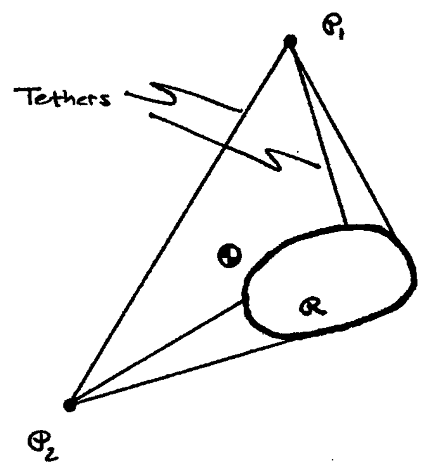

Figure-2. Flexible Multibody System.

An inertial reference frame denoted $\mathcal{F}_I$ is established with its origin, point-I, at the centroid of the gravitational primary. The position vector $\vec{R}_c$ locates the centroid of $\mathcal{S}$, point-c, relative to point-I. An orbital reference frame, $\mathcal{F}_o$, is defined having its origin at point-c and orthogonal unit basis vectors $\hat{o}_1, \hat{o}_2, \hat{o}_3$. An additional reference frame $\mathcal{F}_R$ is defined as fixed to $\mathcal{R}$, with its origin at the mass center of $\mathcal{R}$, point-0, and unit basis vectors $\hat{i}, \hat{j}, \hat{k}$. Figure-3 illustrates $\mathcal{S}$ and the basic reference points, reference frames and vector geometry. The position $\vec{r}$ locates point-0 relative to point-c. The position of $\mathcal{P}_i$ relative to point-0 is given by the vector $\vec{p}_i$.

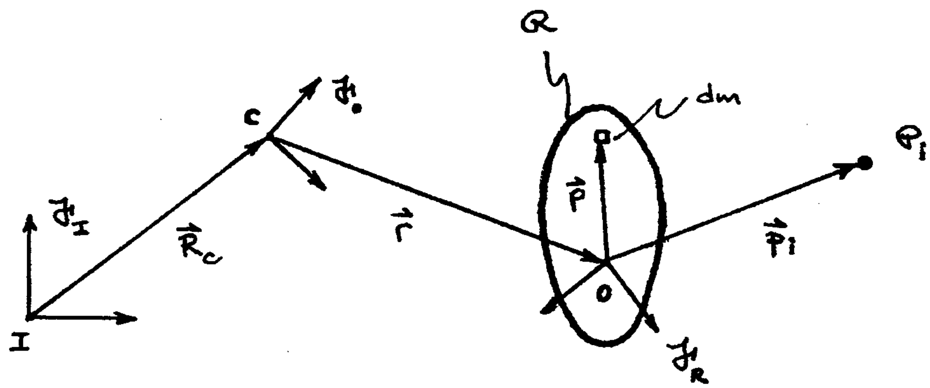

Figure-3. Reference Points, Frames and Position Vectors.

Denoting the masses of $\mathcal{R}$ and $\mathcal{P}_i$ as m_R and m_i respectively, the total mass of $\mathcal{S}$ is

$$m_T = m_R + \sum_{i=1}^{2} m_i$$

Definition of the center of mass of $\mathcal{S}$ provides the relation

$$m_T \vec{r} + \sum_{i=1}^{2} m_i \vec{p}_i = \vec{0} \tag{1}$$

System Momenta

Denoting the position vector from point-0 to an arbitrary material point of $\mathcal{R}$ as $\vec{p}$, the linear momentum of $\mathcal{R}$ is

$$\vec{L}_R = \int_{\mathcal{R}} \frac{d}{dt}(\vec{R}_c + \vec{r} + \vec{p})\, dm = m_R \frac{d}{dt}(\vec{R}_c + \vec{r}) \tag{2}$$

The linear momentum for $\mathcal{P}_i$ is simply

$$\vec{L}_i = m_i \frac{d}{dt}(\vec{R}_c + \vec{r} + \vec{p}_i) \tag{3}$$

The total linear momentum of the system is

$$\vec{L} = \vec{L}_R + \sum_{i=1}^{2} \vec{L}_i = m_T \dot{\vec{R}}_c \tag{4}$$

where the notation $\dot{\vec{v}}$ denotes the time derivative of the vector $\vec{v}$ as observed in the inertial frame.

The moment of momentum of $\mathcal{S}$ with respect to point-c is defined as

$$\vec{H} = \int_{\mathcal{R}} (\vec{r}+\vec{p})\times(\dot{\vec{R}}_c+\dot{\vec{r}}+\dot{\vec{p}})\, dm + \sum_{i=1}^{2} m_i(\vec{r}+\vec{p}_i)\times(\dot{\vec{R}}_c+\dot{\vec{r}}+\dot{\vec{p}}_i) \tag{5}$$

Introducing the angular velocity of $\mathcal{F}_R$ relative to $\mathcal{F}_I$, $\vec{\omega}_R$, we have from the law of Coriolis

$$\dot{\vec{p}} = \vec{\omega}_R \times \vec{p} \tag{6}$$

Expanding Eq.(5) and using Eqs.(1) and (6) we obtain

$$\vec{H} = \vec{\vec{I}}_R \cdot \vec{\omega}_R + \sum_{i=1}^{2}\sum_{j=1}^{2} \mu_{ij}\, \vec{p}_i \times \dot{\vec{p}}_j \tag{7}$$

where we have introduced the centroidal inertia dyadic of $\mathcal{R}$,

$$\vec{\vec{I}}_R = \int_{\mathcal{R}} (p^2 \vec{\vec{1}} - \vec{p}\,\vec{p})\, dm$$

with $\vec{\vec{1}}$ being the unit dyadic, and

$$\mu_{ij} = m_i\left(\delta_{ij} - \frac{m_j}{m_T} \right) = \mu_{ji}$$

δ_{ij} is the Kronecker delta. The law of Coriolis allows us to write

$$\dot{\vec{p}}_i = \frac{{}^{\mathcal{F}_R} d}{dt}\vec{p}_i + \vec{\omega}_R \times \vec{p}_i = \mathring{\vec{p}}_i + \vec{\omega}_R \times \vec{p}_i \tag{8}$$

where the notation of a circle over a vector implies the time derivative of that vector observed in the frame $\mathcal{F}_R$. Substituting Eq.(8) into (7), expanding and collecting terms, the systems centroidal moment of momentum can be written as

$$\vec{H} = \vec{\vec{I}}_\oplus \cdot \vec{\omega}_R + \sum_{i=1}^{2}\sum_{j=1}^{2} \mu_{ij}\, \vec{p}_i \times \mathring{\vec{p}}_j \tag{9}$$

where

$$\vec{\vec{I}}_\oplus = \vec{\vec{I}}_R + \sum_{i=1}^{2}\sum_{j=1}^{2} \mu_{ij} (\vec{p}_i \cdot \vec{p}_j \vec{\vec{1}} - \vec{p}_i\, \vec{p}_j)$$

is identified as the centroidal inertia dyadic of the system.

System Dynamics

The forces acting on the bodies of the system arise from the tether interconnections and the gravitational attraction of the primary. It will be necessary to develop explicit expressions for these forces, but presently we shall only define them symbolically. Let $\vec{f}_i$ be the gravitational force acting on $\mathcal{P}_i$, and let $\vec{f}_R$ be the resultant of the gravitational force distribution acting on $\mathcal{R}$. We define $\vec{g}$ to be the resultant moment about point-c of all gravitation forces acting on $\mathcal{S}$. Acting internal to the system are the tether forces. Define $\vec{T}_i$ to be the resultant of all tether forces acting on $\mathcal{P}_i$. Since the particles connect to each other as well as to $\mathcal{R}$ we distinguish the components of $\vec{T}_i$ as :

$$\vec{T}_i = \vec{T}_{Pi} + \vec{T}_{Ri} \tag{10}$$

where $\vec{T}_{Pi}$ is the force on $\mathcal{P}_i$ due to its connection to $\mathcal{P}_j$, $(i \neq j)$ and $\vec{T}_{Ri}$ is the resultant force on $\mathcal{P}_i$ from all the tethers connecting it to $\mathcal{R}$. With these definitions we can write the dynamic equilibrium equations.

Newton's second law applied to $\mathcal{P}_i$ requires

$$m_i(\ddot{\vec{R}}_c + \ddot{\vec{r}} + \ddot{\vec{p}}_i) = \vec{f}_i + \vec{T}_i$$

Using Eq.(1) to eliminate $\vec{r}$, the above equation becomes

$$m_i\ddot{\vec{R}}_c + \sum_{j=1}^{2} \mu_{ij}\, \ddot{\vec{p}}_j = \vec{f}_i + \vec{T}_i$$

Expressing the time derivatives of $\vec{p}_j$ in terms of those observed in $\mathcal{F}_R$ we write

$$m_i\ddot{\vec{R}}_c - \vec{c}_i \times \dot{\vec{\omega}}_R + \sum_{j=1}^{2} \mu_{ij}\, \mathring{\mathring{\vec{p}}}_j = \vec{f}_i + \vec{T}_i - \vec{N}_i \tag{11}$$

where

$$\vec{N}_i = 2\vec{\omega}_R \times \mathring{\vec{c}}_i - \vec{\omega}_R \times (\vec{\omega}_R \times \vec{c}_i) \tag{12}$$

and

$$\vec{c}_i = \sum_{j=1}^{2} \mu_{ij} \vec{p}_j$$

Newton's second law applied to the system as a whole states that

$$m_T \ddot{\vec{R}}_c = \vec{f}_R + \sum_{i=1}^{2} \vec{f}_i \tag{13}$$

The balance law for the moment of momentum of $\mathcal{S}$ expressed in terms of the system centroid states

$$\dot{\vec{H}} = \overset{\circ}{\vec{H}} + \vec{\omega}_R \times \vec{H} = \vec{g}$$

Substituting Eq.(9) into the above equation and carrying out the indicated differentiation, we obtain

$$\overline{\overline{I}}_{\oplus} \cdot \dot{\vec{\omega}}_R + \sum_{i=1}^{2} \vec{c}_i \times \overset{\circ\circ}{\vec{p}}_i = \vec{g} - \vec{N}_s \tag{14}$$

where

$$\vec{N}_s = \overset{\circ}{\overline{\overline{I}}}_{\oplus} \cdot \vec{\omega}_R + \vec{\omega}_R \times \vec{H} \tag{15}$$

Equations (11), (13) and (14) constitute four independent vector-dyadic motion equations for the system.

Gravitational Forces and Moments

In this section we develop approximate expressions for the gravitational forces acting on the system components as well as the resultant moment of those forces about the system centroid.

The gravitational force acting on $\mathcal{P}_i$ is given by

$$\vec{f}_i = - \frac{m_i \mu(\vec{R}_c + \vec{r} + \vec{p}_i)}{| \vec{R}_c + \vec{r} + \vec{p}_i |^3} \tag{16}$$

where μ is the gravitational constant of the primary. The resultant of the gravitational force distribution acting on $\mathcal{R}$ is

$$\vec{f}_R = - \int_{\mathcal{R}} \frac{\mu(\vec{R}_c + \vec{r} + \vec{p})}{| \vec{R}_c + \vec{r} + \vec{p} |^3} dm \tag{17}$$

Consider the expression

$$|\vec{R}_c+\vec{\Delta}|^{-3} = [(\vec{R}_c+\vec{\Delta})\cdot(\vec{R}_c+\vec{\Delta})]^{-3/2} = R_c^{-3}(1+2\frac{\vec{R}_c\cdot\vec{\Delta}}{R_c^2}+\frac{\Delta^2}{R_c^2})^{-3/2}$$

For the problems under consideration here $R_c \gg \Delta$. Applying the binomial expansion and retaining terms to only first order in (Δ/R_c), we can write

$$|\vec{R}_c+\vec{\Delta}|^{-3} \cong R_c^{-3}(1-3\frac{\vec{R}_c\cdot\vec{\Delta}}{R_c^2}) \tag{18}$$

Using the approximation of Eq.(18) in Eqs.(16) and (17) we obtain

$$\vec{f}_i \cong \frac{m_i\mu}{R_c^2}\{\hat{o}_3 - \frac{1}{R_c}[(\vec{r}+\vec{p}_i) - 3\hat{o}_3\hat{o}_3\cdot(\vec{r}+\vec{p}_i)]\} \tag{19}$$

$$\vec{f}_R \cong \frac{m_R\mu}{R_c^2}[\hat{o}_3 - \frac{1}{R_c}(\vec{r} - 3\hat{o}_3\hat{o}_3\cdot\vec{r})] \tag{20}$$

where the unit vector $\hat{o}_3$ is established in the NADIR direction; $\hat{o}_3 = -\vec{R}_c / R_c$. The resultant of the moments of the gravitational forces acting on $\mathcal{S}$ about point-c is

$$\vec{g} = -\int_{\mathcal{R}}(\vec{r}+\vec{p})\times\frac{\mu(\vec{R}_c+\vec{r}+\vec{p})}{|\vec{R}_c+\vec{r}+\vec{p}|^3}dm + \sum_{i=1}^{2}(\vec{r}+\vec{p}_i)\times\vec{f}_i \tag{21}$$

Using the approximation of Eq.(18) in Eq.(21) and ignoring terms of second or higher order in (Δ/R_c) in relation to terms of zero or first order, we obtain

$$\vec{g} \cong \frac{m_R\mu}{R_c^2}\vec{r}\times\hat{o}_3 + \frac{3\mu}{R_c^3}\hat{o}_3\times\vec{\vec{J}}_R.\hat{o}_3 + \sum_{i=1}^{2}(\vec{r}+\vec{p}_i)\times\vec{f}_i \tag{22}$$

where

$$\vec{\vec{J}}_R = \vec{\vec{I}}_R + m_R(r^2\vec{\vec{1}} - \vec{r}\vec{r})$$

is the inertia dyadic of $\mathcal{R}$ with respect to point-c. Substituting Eq.(19) into (22) and simplifying we achieve

$$\vec{g} \cong \frac{3\mu}{R_c^3}\hat{o}_3\times\vec{\vec{I}}_{\oplus}\cdot\hat{o}_3 \tag{23}$$

Tether Forces

Figure-4 illustrates the tether forces acting within $\mathcal{S}$. Consider first the single tether connecting $\mathcal{P}_1$ to $\mathcal{P}_2$. Define $\vec{s}_p$ to be the position vector from $\mathcal{P}_1$ to $\mathcal{P}_2$

$$\vec{s}_p = \vec{p}_2 - \vec{p}_1$$

The distance between the particles is

$$s_p = (\vec{s}_p \cdot \vec{s}_p)^{1/2}$$

and the unit vector in the direction of $\vec{s}_p$ is denoted $\hat{s}_p$. The unstrained length of the tether connecting the particles is denoted as the constant l_p. k_p is defined as the longitudinal stiffness of the tether and is constant assuming uniform material and geometric

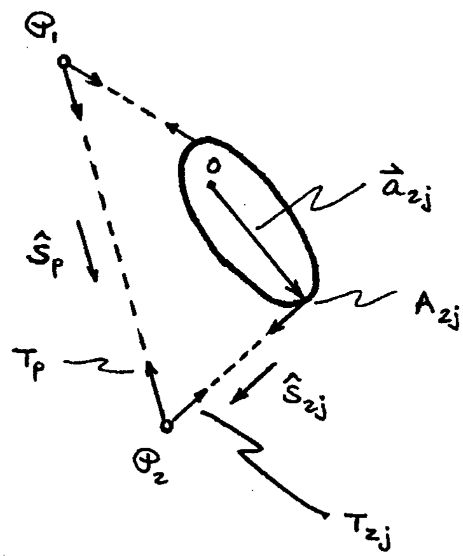

Figure-4. Tether Forces.

properties ($k_p = EA / l_p$, where E is the modulus of elasticity and A is the effective cross-sectional area). The inter-particle tether forces are

$$\vec{T}_{P1} = -\vec{T}_{P2} = T_p \, \hat{s}_p \tag{24}$$

where the tether tension is defined as

$$T_p = \begin{cases} k_p(s_p - l_p) + d_p \dot{s}_p & T_p \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

The tension must be non-negative. The stretch rate coefficient d_p governs the viscous damping of the tether.

Let τ_{ij} designate the jth tether connecting $\mathcal{P}_i$ to $\mathcal{R}$, $(i=1,2)$, $(j=1,2,...,N_i)$. τ_{ij} attaches to $\mathcal{R}$ at point-A_{ij}. The position vector $\vec{a}_{ij}$ locates A_{ij} relative to 0 (see Figure-4). The position vector from A_{ij} to $\mathcal{P}_i$ is denoted as $\vec{s}_{ij}$, and its associated magnitude and unit vector are s_{ij} and $\hat{s}_{ij}$ respectively. Let T_{ij} be the tension force developed in τ_{ij}, and define k_{ij} and d_{ij} as the tether's stiffness and damping coefficients respectively. Then we define the constitutive law for τ_{ij} as

$$T_{ij} = \begin{cases} k_{ij}(s_{ij} - l_{ij}) + d_{ij} \dot{s}_{ij} & T_{ij} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

where l_{ij} is the unstrained length of τ_{ij}. The force applied to $\mathcal{P}_i$ by τ_{ij} is

$$\vec{T}_{ij} = -T_{ij}\, \hat{s}_{ij}$$

and the resultant of all such forces is

$$\vec{T}_{Ri} = -\sum_{j=1}^{N_i} T_{ij}\, \hat{s}_{ij} = \sum_{j=1}^{N_i} T_{ij} \frac{(\vec{a}_{ij} - \vec{p}_i)}{s_{ij}} \tag{27}$$

where

$$s_{ij} = [(\vec{p}_i - \vec{a}_{ij})\cdot(\vec{p}_i - \vec{a}_{ij})]^{1/2}$$

Equations of Motion

Having developed explicit expressions for the internal and external forces acting on $\mathcal{S}$ we can now complete the specification of the nonlinear motion equations embodied by Eqs.(11),(13), and (14).

Consider first the translational equilibrium equation for the system, Eq.(13). Substituting Eqs.(19) and (20) into (13) we find that

$$\ddot{\vec{R}}_C = -\frac{\mu}{R_C^3}\vec{R}_C \tag{28}$$

Equation (28) is recognized as the classical two-body motion equations. To the level of approximation made here, the motion of the systems' centroid is independent of the other degrees of freedom. Since the solution of Eq.(28) is fully known we shall not consider it further.

The appearance of $\ddot{\vec{R}}_C$ in Eq.(11) can be eliminated by substituting Eq.(28) into (11). Further simplification can be achieved by using Eq.(1) to eliminate the explicit appearance of $\vec{r}$ in Eq.(19), and in turn from Eq.(11), leading to

$$\sum_{j=1}^{2}\mu_{ij}\,\overset{\circ\circ}{\vec{p}}_j - \vec{c}_i\times\dot{\vec{\omega}}_R + = \vec{T}_{Pi} + \vec{T}_{Ri} - \frac{\mu}{R_C^3}(\,\vec{1}-3\hat{o}_3\hat{o}_3\,)\cdot\vec{c}_i - \vec{N}_i \tag{29}$$

To obtain the scalar form of the motion equations all the vector and dyadic quantities appearing in Eqs.(14) and (29) are resolved into their components expressed in $\mathcal{F}_R$. The 3x1 column matrix of the scalar components of a vector, say $\vec{p}_i$, resolved in $\mathcal{F}_R$ are denoted by the under-bar notation, $\underline{p}_i$. The inertia dyadics $\vec{I}_\oplus$ and $\vec{I}_R$ resolved into components in $\mathcal{F}_R$, are represented by the 3x3 matrices $[I_\oplus]$ and $[I_R]$. We also introduce the matrix cross product notation;

$$\vec{v}\times\vec{u} \quad\Rightarrow\quad \underline{v}^{\times}\underline{u}$$

where

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad\Rightarrow\quad \underline{v}^{\times} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

With this notation established the nonlinear motion equations are expressed as

$$\begin{bmatrix} [I_\oplus] & \underline{c}_1^{\times} & \underline{c}_2^{\times} \\ -\underline{c}_1^{\times} & \mu_{11}[1] & \mu_{12}[1] \\ -\underline{c}_2^{\times} & \mu_{21}[1] & \mu_{22}[1] \end{bmatrix}\begin{pmatrix} \dot{\underline{\omega}}_R \\ \ddot{\underline{p}}_1 \\ \ddot{\underline{p}}_2 \end{pmatrix} = \begin{pmatrix} \underline{g} \\ \underline{g}_1 \\ \underline{g}_2 \end{pmatrix} + \begin{pmatrix} \underline{0} \\ \underline{T}_{P1}+\underline{T}_{R1} \\ \underline{T}_{P2}+\underline{T}_{R2} \end{pmatrix} - \begin{pmatrix} \underline{N}_s \\ \underline{N}_1 \\ \underline{N}_2 \end{pmatrix} \tag{30a}$$

where we have introduced the 3x3 identity matrix [1], and

$$\underline{N}_s = \underline{\omega}_R^{\times}[\,I_\oplus\,]\underline{\omega}_R + 2\sum_{i=1}^{2}(\,\underline{\omega}_R\underline{c}_i^T - \underline{c}_i^T\underline{\omega}_R[\,1\,]\,)\,\dot{\underline{p}}_i \tag{30b}$$

$$\underline{N}_i = 2\,\underline{\omega}_R^{\times}\sum_{j=1}^{2}\mu_{ij}\,\dot{\underline{p}}_j + \underline{\omega}_R^{\times}\,\underline{\omega}_R^{\times}\,\underline{c}_i \tag{30c}$$

$$\underline{g} = \frac{3\mu}{R_c^3}\,\underline{o}_3^{\times}[\,I_\oplus\,]\underline{o}_3 \tag{30d}$$

$$\underline{g}_i = -\frac{\mu}{R_c^3}(\,[\,1\,] - 3\underline{o}_3\underline{o}_3^T\,)\,\underline{c}_i \tag{30e}$$

$$[\,I_\oplus\,] = [\,I_R\,] + \sum_{i=1}^{2}\sum_{j=1}^{2}\mu_{ij}(\,\underline{p}_i^T\underline{p}_j[\,1\,] - \underline{p}_i\underline{p}_j^T\,) \tag{30f}$$

Attitude Kinematics

A primary interest of this study is the attitude behavior of the system relative to a suitable orbital reference frame. The orbital frame selected is the usual one used for spacecraft attitude dynamics, namely a local-vertical local-horizontal frame with axes defined by;

$$\hat{o}_3 = -\frac{\vec{R}_c}{|\vec{R}_c|} \qquad \hat{o}_2 = \frac{\dot{\vec{R}}_c\times\vec{R}_c}{|\dot{\vec{R}}_c\times\vec{R}_c|} \qquad \hat{o}_1 = \hat{o}_2\times\hat{o}_3$$

The motion of the orbital reference frame $\mathcal{F}_o$ is completely determined from the solution of the two-body problem governing the motion of the system centroid Eq.(28).

The angular velocity of $\mathcal{F}_o$ relative to $\mathcal{F}_I$ denoted $\vec{\omega}_o$, is known in terms of the true anomaly $\nu(t)$

$$\vec{\omega}_o = -\dot{\nu}\,\hat{o}_2 \tag{31}$$

The angular velocity $\mathcal{F}_R$ relative to $\mathcal{F}_o$ is denoted as $\vec{\Omega}$, so we have

$$\vec{\omega}_R = \vec{\omega}_o + \vec{\Omega} \tag{32}$$

The angular acceleration of $\mathcal{F}_R$ is noted as

$$\dot{\vec{\omega}}_R = \dot{\vec{\omega}}_o + \overset{\circ}{\vec{\Omega}} + \vec{\omega}_o \times \vec{\Omega} \tag{33}$$

The attitude of $\mathcal{F}_R$ relative to $\mathcal{F}_o$ is described in terms of a 1-2-3 Euler angle sequence. The signs of the single axis rotations defined below are positive according to the right hand rule. Let $\mathcal{F}_o'$ be an intermediate frame achieved from $\mathcal{F}_o$ by a rotation of angle θ_1 about the $\hat{o}_1$ axis. Let frame $\mathcal{F}_R'$ be achieved from $\mathcal{F}_o'$ by a rotation of angle θ_2 about the axis $\hat{o}_2'$. $\mathcal{F}_R$ is achieved from $\mathcal{F}_R'$ by a rotation of angle θ_3 about the $\hat{k}$ axis. The direction cosine matrix transforming vector components from $\mathcal{F}_o$ to $\mathcal{F}_R$ is obtained by concatenating the sequence of single axis rotations yielding

$$[C_{Ro}] = \begin{bmatrix} c\theta_2\, c\theta_3 & c\theta_1\, s\theta_3 + s\theta_1\, s\theta_2\, c\theta_3 & s\theta_1\, s\theta_3 - c\theta_1\, s\theta_2\, c\theta_3 \\ -c\theta_2\, s\theta_3 & c\theta_1\, c\theta_3 - s\theta_1\, s\theta_2\, s\theta_3 & s\theta_1\, c\theta_3 + c\theta_1\, s\theta_2\, s\theta_3 \\ s\theta_2 & -s\theta_1\, c\theta_2 & c\theta_1\, c\theta_2 \end{bmatrix} \tag{34}$$

where we have used the shorthand notation: $c\theta_i = \cos\theta_i$, $s\theta_i = \sin\theta_i$. The direction of the transformation of Eq.(34) corresponds to

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = [\, C_{Ro}\,] \begin{pmatrix} \hat{o}_1 \\ \hat{o}_2 \\ \hat{o}_3 \end{pmatrix}$$

The angular velocity of $\mathcal{F}_R$ with respect to $\mathcal{F}_o$, expressed in terms of the Euler angle rates is

$$\vec{\Omega} = \dot{\theta}_1 \hat{o}_1 + \dot{\theta}_2 \hat{o}_2' + \dot{\theta}_3 \hat{k}$$

Resolved into components referred to $\mathcal{F}_R$ we have

$$\underline{\Omega} = \begin{bmatrix} c\theta_2 c\theta_3 & s\theta_3 & 0 \\ -c\theta_2 s\theta_3 & c\theta_3 & 0 \\ s\theta_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = [\,\Pi\,]\, \underline{\dot{\theta}} \tag{35}$$

Inverting the above equation yields

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} c\theta_3/c\theta_2 & -\, s\theta_3/c\theta_2 & 0 \\ s\theta_3 & c\theta_3 & 0 \\ -\, c\theta_3\, t\theta_2 & s\theta_3\, t\theta_2 & 1 \end{bmatrix} \underline{\Omega}$$

or more compactly,

$$\underline{\dot{\theta}} = [\Pi]^{-1}\,\underline{\Omega}$$

where $t\theta_2 = \tan\theta_2$, and the singularity at $\theta_2 = \pm\pi/2$ is noted.

Nonlinear Equations - First Order Form

Equations (32) and (33) can now be expressed in scalar form as

$$\underline{\omega}_R = \underline{\Omega} - \dot{v}\,[\,C_{Ro}\,]\,\underline{e}_2 \tag{36}$$

$$\underline{\dot{\omega}}_R = \underline{\dot{\Omega}} - \ddot{v}\,[\,C_{Ro}\,]\,\underline{e}_2 - \dot{v}\,[\,C_{Ro}\,]\,\underline{e}_2^{\times}\,[\,C_{Ro}\,]^T\,\underline{\Omega} \tag{37}$$

where $\underline{e}_2^T = (\,0\,1\,0\,)$.

Equations (30a) are now written as

$$[\,M\,]\{\,\dot{v}\,\} = \{\,g\,\} + \{\,T\,\} - \{\,N\,\} \tag{38}$$

where $\{\,v\,\}^T = (\,\underline{\omega}_R^T\ \ \underline{\dot{p}}_1^T\ \ \underline{\dot{p}}_2^T\,)$. Here we introduce the vector of generalized coordinates

$$\{\,q\,\} = \begin{pmatrix} \underline{\theta} \\ \underline{p}_1 \\ \underline{p}_2 \end{pmatrix}$$

The mechanical state vector is defined as

$$\{\,Y\,\} = \begin{Bmatrix} \{\,v\,\} \\ \cdots\cdots \\ \{\,q\,\} \end{Bmatrix}$$

The equations of state for the system can now be expressed as

$$\{\,\dot{Y}\,\} = \begin{Bmatrix} \{\,\dot{v}\,\} \\ \cdots\cdots \\ \{\,\dot{q}\,\} \end{Bmatrix} = \begin{Bmatrix} [M]^{-1}(\,\{g\} + \{T\} - \{N\}\,) \\ \cdots\cdots\cdots\cdots \\ \begin{Bmatrix} [\Pi]^{-1}(\,\underline{\omega}_R + \dot{v}[C_{Ro}]\underline{e}_2 \\ \underline{\dot{p}}_1 \\ \underline{\dot{p}}_2 \end{Bmatrix} \end{Bmatrix} = \{\,f(Y,t)\,\} \tag{39}$$

Equations (28) and (39) complete the specification of the nonlinear system.

Steady State Solution and Linearization

We propose the existence of a steady state solution corresponding to the system configuration shown in Figure-1 with the system centroid in a circular orbit. The equilibrium solution considered has

$$\underline{\theta}(t) = \underline{0} \quad , \quad \text{(constant)} \tag{40a}$$

$$\underline{p}_i(t) = \underline{p}_i^0 \quad , \quad (i = 1, 2) \quad , \quad (\underline{p}_i^0 \text{ constant}) \tag{40b}$$

$$\vec{R}_c = -R_c\, \hat{o}_3 \quad , \quad (R_c \text{ constant}) \tag{40c}$$

$$\vec{\omega}_o = -\omega_c\, \hat{o}_2 \quad , \quad (\omega_c = \dot{v} \text{ constant}) \tag{40d}$$

$$\dot{\vec{R}}_c = \vec{\omega}_o \times \vec{R}_c \tag{40e}$$

For this case the configuration is fixed relative to $\mathcal{F}_o$, and $\mathcal{F}_R$ is parallel to $\mathcal{F}_o$. The explicit relationship between $\mathcal{F}_I$ and $\mathcal{F}_o$ need not be considered for this problem.

To study the attitude stability of the system the motion equations are linearized about the steady state solution. Let

$$\underline{\theta}(t) = \underline{\Delta\theta}(t) \tag{41a}$$

$$\underline{p}_i(t) = \underline{p}_i^0 + \underline{\Delta p}_i(t) \tag{41b}$$

The Δ quantities measure displacements from the equilibrium state and are considered to be of sufficiently small magnitude that terms of second or higher order in them, or their derivatives, can be ignored in relation to those of first (or lower) order.

Substituting Eq.(41a) into (34) yields

$$[\,C_{Ro}\,] \cong [\,1\,] - \underline{\Delta\theta}^{\times}$$

To first order Eq.(35) gives

$$\underline{\Omega} \cong \underline{\Delta\dot{\theta}}$$

Equations (36) and (37) simplify to

$$\underline{\omega}_R \cong -\omega_c\, \underline{e}_2 + \underline{\Delta\dot{\theta}} - \omega_c\, \underline{e}_2^{\times}\, \underline{\Delta\theta}$$

$$\underline{\dot{\omega}}_R \cong \underline{\Delta\ddot{\theta}} - \omega_c \, \underline{e}_2^{\times} \, \underline{\Delta\dot{\theta}}$$

Continuing our expansions of terms into zeroth and first order terms we record

$$\underline{c}_i \cong \underline{c}_i^0 + \sum_{j=1}^{2} \mu_{ij} \underline{\Delta p}_j \tag{42}$$

with

$$\underline{c}_i^0 = \sum_{j=1}^{2} \mu_{ij} \underline{p}_j^0$$

and

$$[\, I_{\oplus} \,] \cong [\, I_{\oplus}^0 \,] + [\, I_{\oplus}^1 \,] \tag{43}$$

where

$$[\, I_{\oplus}^0 \,] = [\, I_R \,] + \sum_{i=1}^{2} \sum_{j=1}^{2} \mu_{ij} \, [\, (\underline{p}_i^0)^T \underline{p}_j^0 [\, 1 \,] - \underline{p}_j^0 \, (\underline{p}_i^0)^T \,]$$

$$[\, I_{\oplus}^1 \,] = \sum_{i=1}^{2} \sum_{j=1}^{2} \mu_{ij} \, [\, (\, (\underline{p}_i^0)^T \underline{\Delta p}_j + (\underline{p}_j^0)^T \underline{\Delta p}_i \,) \, [\, 1 \,] - (\, \underline{p}_j^0 (\underline{\Delta p}_i)^T + \underline{\Delta p}_j (\underline{p}_i^0)^T \,) \,]$$

Expressing $\hat{o}_3$ in $\mathcal{F}_R$ we have

$$\underline{o}_3 \cong \underline{e}_3 + \underline{e}_3^{\times} \, \underline{\Delta\theta}$$

where $\underline{e}_3^T = (\, 0 \; 0 \; 1 \,)$. Using the above results Eq.(30d) is approximated as

$$\underline{g} \cong \underline{g}^0 + \underline{g}^1 \tag{44}$$

where

$$\underline{g}^0 = 3\omega_c^2 \, \underline{e}_3^{\times} \, [\, I_{\oplus}^0 \,] \, \underline{e}_3$$

$$\underline{g}^1 = 3\omega_c^2 \, [\, \underline{e}_3^{\times} [\, I_{\oplus}^0 \,] - (\, [\, I_{\oplus}^0 \,] \, \underline{e}_3 \,)^{\times} \,] \, \underline{e}_3^{\times} \, \underline{\Delta\theta} \; - 3\omega_c^2 \sum_{i=1}^{2} [\, \underline{e}_3^{\times} \, \underline{c}_i^0 \, \underline{e}_3^T + \underline{e}_3^T \, \underline{c}_i^0 \, \underline{e}_3^{\times} \,] \, \underline{\Delta p}_i$$

Note that we have used the relation $\mu / R_c^3 = \omega_c^2$. For Eq.(30e) we derive,

$$\underline{g}_i \cong \underline{g}_i^0 + \underline{g}_i^1 \tag{45}$$

where

$$\underline{g}_i^0 = -\omega_c^2 ([1] - 3 \underline{e}_3 \underline{e}_3^T) \underline{c}_i^0$$

$$\underline{g}_i^1 = 3\omega_c^2 [\underline{e}_3 (\underline{c}_i^0)^T + \underline{e}_3^T \underline{c}_i^0] \underline{e}_3^\times \underline{\Delta\theta} - \omega_c^2 \sum_{j=1}^{2} \mu_{ij} ([1] - 3 \underline{e}_3 \underline{e}_3^T) \underline{\Delta p}_j$$

To form the linearized approximations for the inter-particle tether force we note the following

$$\underline{s}_p \cong \underline{s}_p^0 + \underline{s}_p^1$$

where

$$\underline{s}_p^0 = (\underline{p}_2^0 - \underline{p}_1^0) \quad ; \quad \underline{s}_p^1 = (\underline{\Delta p}_2 - \underline{\Delta p}_1)$$

Also

$$s_p \cong s_p^0 + s_p^1$$

where

$$s_p^0 = \sqrt{ (\underline{s}_p^0)^T \underline{s}_p^0 } \quad ; \quad s_p^1 = \frac{ (\underline{s}_p^0)^T \underline{s}_p^1 }{ s_p^0 } \quad ; \quad (s_p)^{-1} \cong \frac{1}{s_p^0} - \frac{s_p^1}{(s_p^0)^2}$$

Expressing $\hat{s}_p$ resolved in $\mathcal{F}_R$ as $\underline{u}_p$ we have

$$\underline{u}_p \cong \underline{u}_p^0 + \underline{u}_p^1$$

$$\underline{u}_p^0 = (s_p^0)^{-1} \underline{s}_p^0 \quad ; \quad \underline{u}_p^1 = (s_p^0)^{-3} [(s_p^0)^2 [1] - \underline{s}_p^0 (\underline{s}_p^0)^T] (\underline{\Delta p}_2 - \underline{\Delta p}_1)$$

Using the expansions above Eq.(24) is now written as

$$\underline{T}_{Pi} \cong \underline{T}_{Pi}^0 + \underline{T}_{Pi}^1 \tag{46}$$

where

$$\underline{T}^0_{Pi} = (2\delta_{i1} - 1)\, k_p \,(1 - \frac{l_p}{s^0_p})\, \underline{s}^0_p$$

$$\underline{T}^1_{Pi} = -(2\delta_{i1} - 1)[K_p](\underline{\Delta p}_1 - \underline{\Delta p}_2) - (2\delta_{i1} - 1)[D_p](\underline{\Delta \dot{p}}_1 - \underline{\Delta \dot{p}}_2)$$

and

$$[K_p] = k_p[(1 - \frac{l_p}{s^0_p})(\underline{u}^0_p)^T \underline{u}^0_p [1] + \frac{l_p}{s^0_p}\underline{u}^0_p (\underline{u}^0_p)^T] = [K_p]^T$$

$$[D_p] = d_p\, \underline{u}^0_p (\underline{u}^0_p)^T = [D_p]^T$$

Development of the linearized expressions for the tether forces connecting $\mathcal{P}_i$ to $\mathcal{R}$ requires, as above, the following relations

$$\underline{s}_{ij} \cong \underline{s}^0_{ij} + \underline{\Delta p}_i \quad ; \quad \underline{s}^0_{ij} = \underline{p}^0_i - \underline{a}_{ij}$$

$$s_{ij} \cong s^0_{ij} + s^1_{ij} \quad ; \quad s^0_{ij} = [\underline{a}^T_{ij}(\underline{a}_{ij} - 2\underline{p}^0_i) + (\underline{p}^0_i)^T \underline{p}^0_i]^{1/2}$$

$$s^1_{ij} = \frac{2}{s^0_{ij}}(\underline{s}^0_{ij})^T \underline{\Delta p}_i \quad ; \quad (s_{ij})^{-1} \cong \frac{1}{s^0_{ij}} - \frac{s^1_{ij}}{(s^0_{ij})^2} \quad ; \quad \frac{\underline{s}_{ij}}{s_{ij}} \cong \underline{u}^0_{ij} + \underline{u}^1_{ij}$$

$$\underline{u}^0_{ij} = \frac{\underline{s}^0_{ij}}{s^0_{ij}} \quad ; \quad \underline{u}^1_i = (s^0_{ij})^{-3}[(s^0_{ij})^2[1] - 2\,\underline{s}^0_{ij}(\underline{s}^0_{ij})^T]\,\underline{\Delta p}_i$$

With the above expressions it follows from Eq.(27) that

$$\underline{T}_{Ri} \cong \underline{T}^0_{Ri} + \underline{T}^1_{Ri} \tag{47}$$

$$\underline{T}^0_{Ri} = -\sum_{j=1}^{N_i} k_{ij}(1 - \frac{l_{ij}}{s^0_{ij}})\,\underline{s}^0_{ij} \quad ; \quad \underline{T}^1_{Ri} = -[K_{Ri}]\,\underline{\Delta p}_i - [D_{Ri}]\,\underline{\Delta \dot{p}}_i$$

$$[K_{Ri}] = \sum_{j=1}^{N_i} (s^0_{ij})^{-3}[(s^0_{ij})^2 k_{ij}(s^0_{ij} - l_{ij})[1] + 2\,k_{ij}\, l_{ij}\, \underline{s}^0_{ij}(\underline{s}^0_{ij})^T] = [K_{Ri}]^T$$

$$[D_{Ri}] = 2\sum_{j=1}^{N_i} (s^0_{ij})^{-2} d_{ij}\, \underline{s}^0_{ij}(\underline{s}^0_{ij})^T = [D_{Ri}]^T$$

Expanding Eq.(30b) into zeroth and first order terms we obtain

$$\underline{N}_s \cong \underline{N}_s^0 + \underline{N}_s^1 \tag{48}$$

where

$$\underline{N}_s^0 = \omega_c^2 \, \underline{e}_2^\times [\, I_\oplus^0 \,] \, \underline{e}_2$$

$$\underline{N}_s^1 = \omega_c [\, ([\, I_\oplus^0 \,] \underline{e}_2)^\times - \underline{e}_2^\times [\, I_\oplus^0 \,] \,] \, \underline{\Delta\dot{\theta}} + 2\omega_c \sum_{i=1}^{2} [\, \underline{e}_2^T \underline{c}_i^0 [\, 1 \,] - \underline{e}_2 (\underline{c}_i^0)^T \,] \, \underline{\Delta\dot{p}}_i$$
$$- \omega_c^2 [\, ([\, I_\oplus^0 \,] \underline{e}_2)^\times - \underline{e}_2^\times [\, I_\oplus^0 \,] \,] \, \underline{e}_2^\times \, \underline{\Delta\theta} - \omega_c^2 \sum_{i=1}^{2} [\, \underline{e}_2^\times \underline{c}_i^0 \underline{e}_2^T + \underline{e}_2^T \underline{c}_i^0 \underline{e}_2^\times \,] \, \underline{\Delta p}_i$$

Expanding Eq.(30c) in a manner similar to that above,

$$\underline{N}_i \cong \underline{N}_i^0 + \underline{N}_i^1 \tag{49}$$

where

$$\underline{N}_i^0 = \omega_c^2 \, \underline{e}_2^\times \underline{e}_2^\times \underline{c}_i^0$$

$$\underline{N}_i^1 = \omega_c [\, (\underline{e}_2^\times \underline{c}_i^0)^\times + \underline{e}_2^\times (\underline{c}_i^0)^\times \,] \, \underline{\Delta\dot{\theta}} - 2\omega_c \, \underline{e}_2^\times \sum_{j=1}^{2} \mu_{ij} \, \underline{\Delta\dot{p}}_j$$
$$- \omega_c^2 [\, (\underline{e}_2^\times \underline{c}_i^0)^\times + \underline{e}_2^\times (\underline{c}_i^0)^\times \,] \, \underline{e}_2^\times \, \underline{\Delta\theta} + \omega_c^2 \, \underline{e}_2^\times \underline{e}_2^\times \sum_{j=1}^{2} \mu_{ij} \, \underline{\Delta p}_j$$

The vector of generalized coordinates is decomposed as

$$\{\, q \,\} = \{\, q^0 \,\} + \{\, \Delta q \,\} \tag{50}$$

where

$$\{\, q^0 \,\} = \begin{pmatrix} \underline{0} \\ \underline{p}_1^0 \\ \underline{p}_2^0 \end{pmatrix} \quad ; \quad \{\, \Delta q \,\} = \begin{pmatrix} \underline{\Delta\theta} \\ \underline{\Delta p}_1 \\ \underline{\Delta p}_2 \end{pmatrix}$$

The vector of accelerations appearing in Eq.(38) is related to the derivatives of the generalized coordinates by

$$\{\, \dot{v} \,\} = \{\, \Delta\ddot{q} \,\} + [\, B \,] \{\, \Delta\dot{q} \,\} \tag{51}$$

where

$$[B] = \begin{bmatrix} -\omega_c \underline{e}_2^{\times} & [0] & [0] \\ [0] & [0] & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Substituting Eqs.(42-51) into Eq.(38) and retaining terms to first order only, the motion equations can be written as

$$[M_0]\{\Delta\ddot{q}\} + [M_0][B]\{\Delta\dot{q}\} = \{g^0\} + \{g^1\} + \{T^0\} + \{T^1\} - \{N^0\} - \{N^1\} \quad (52)$$

where

$$[M_0] = \begin{bmatrix} [I_{\oplus}^0] & (\underline{c}_1^0)^{\times} & (\underline{c}_2^0)^{\times} \\ -(\underline{c}_1^0)^{\times} & \mu_{11}[1] & \mu_{12}[1] \\ -(\underline{c}_2^0)^{\times} & \mu_{21}[1] & \mu_{22}[1] \end{bmatrix}$$

$$\{g^0\}^T = \left(\underline{g}^{0^T} \;\; \underline{g}_1^{0^T} \;\; \underline{g}_2^{0^T} \right) \quad ; \quad \{g^1\}^T = \left(\underline{g}^{1^T} \;\; \underline{g}_1^{1^T} \;\; \underline{g}_2^{1^T} \right)$$

$$\{T^0\}^T = \left(\underline{0}^T \;\; \underline{T}_1^{0^T} \;\; \underline{T}_2^{0^T} \right) \quad ; \quad \{T^1\}^T = \left(\underline{0}^T \;\; \underline{T}_1^{1^T} \;\; \underline{T}_2^{1^T} \right)$$

$$\{N^0\}^T = \left(\underline{N}_s^{0^T} \;\; \underline{N}_1^{0^T} \;\; \underline{N}_2^{0^T} \right) \quad ; \quad \{N^1\}^T = \left(\underline{N}_s^{1^T} \;\; \underline{N}_1^{1^T} \;\; \underline{N}_2^{1^T} \right)$$

All the first order terms on the RHS of Eq.(52) are proportional the Δq's and their time derivatives. Since the steady state solution prevails when all the Δq quantities are zero, the equilibrium conditions to be satisfied by the steady state are;

$$\begin{pmatrix} \underline{N}_s^0 \\ \underline{N}_1^0 \\ \underline{N}_2^0 \end{pmatrix} = \begin{pmatrix} \underline{g}^0 \\ \underline{g}_1^0 \\ \underline{g}_2^0 \end{pmatrix} + \begin{pmatrix} \underline{0} \\ \underline{T}_1^0 \\ \underline{T}_2^0 \end{pmatrix} \quad (53)$$

The first partition of Eq.(53) requires that $\mathcal{F}_R$ (parallel to $\mathcal{F}_o$) correspond to the centroidal principal axes of the system. The second and third partitions of Eq.(53) equilibrate the steady state accelerations of the endmasses with the net gravitational and internal tether forces.

Expanding the first order terms on the RHS of Eq.(52) and recognizing Eq.(53), the linearized motion equations can be written as

$$[\,M_0\,]\{\,\Delta\ddot{q}\,\}+(\,[\,G\,]+[\,D\,]\,)\{\,\Delta\dot{q}\,\}+(\,[\,K\,]+[\,C\,]+[\,g\,]\,)\{\,\Delta q\,\} = \{\,0\,\} \qquad (54)$$

where

$$\left[\,G\,\right] = \omega_c \begin{bmatrix} [\,([I_{\oplus}^0]\underline{e}_2)^{\times}-[I_{\oplus}^0]\underline{e}_2^{\times}-\underline{e}_2^{\times}[I_{\oplus}^0]\,] & 2\,[\underline{e}_2^T\underline{c}_1^0\,[1]-\underline{e}_2(\underline{c}_1^0)^T] & 2\,[\underline{e}_2^T\underline{c}_2^0\,[1]-\underline{e}_2(\underline{c}_2^0)^T] \\ [\,\underline{e}_2^{\times}(\underline{c}_1^0)^{\times}+(\underline{c}_1^0)^{\times}\underline{e}_2^{\times}+(\underline{e}_2^{\times}\underline{c}_1^0)^{\times}] & -2\mu_{11}\,\underline{e}_2^{\times} & -2\mu_{12}\,\underline{e}_2^{\times} \\ [\,\underline{e}_2^{\times}(\underline{c}_2^0)^{\times}+(\underline{c}_2^0)^{\times}\underline{e}_2^{\times}+(\underline{e}_2^{\times}\underline{c}_2^0)^{\times}] & -2\mu_{21}\,\underline{e}_2^{\times} & -2\mu_{22}\,\underline{e}_2^{\times} \end{bmatrix}$$

$$\left[\,D\,\right] = \begin{bmatrix} [\,0\,] & [\,0\,] & [\,0\,] \\ [\,0\,] & [\,D_p\,]+[\,D_{R1}\,] & -[\,D_p\,] \\ [\,0\,] & -[\,D_p\,] & [\,D_p\,]+[\,D_{R2}\,] \end{bmatrix}$$

$$\left[\,g\,\right] = 3\omega_c^2 \begin{bmatrix} [\,([I_{\oplus}^0]\underline{e}_3)^{\times}-\underline{e}_3^{\times}[I_{\oplus}^0]\,]\,\underline{e}_3^{\times} & [\,\underline{e}_3^{\times}\underline{c}_1^0\underline{e}_3^T+\underline{e}_3^T\underline{c}_1^0\underline{e}_3^{\times}\,] & [\,\underline{e}_3^{\times}\underline{c}_2^0\underline{e}_3^T+\underline{e}_3^T\underline{c}_2^0\underline{e}_3^{\times}\,] \\ -(\,\underline{e}_3(\underline{c}_1^0)^T+\underline{e}_3^T\underline{c}_1^0\,)\,\underline{e}_3^{\times} & \frac{1}{3}\mu_{11}(\,[\,1\,]-3\underline{e}_3\underline{e}_3^T\,) & \frac{1}{3}\mu_{12}(\,[\,1\,]-3\underline{e}_3\underline{e}_3^T\,) \\ -(\,\underline{e}_3(\underline{c}_2^0)^T+\underline{e}_3^T\underline{c}_2^0\,)\,\underline{e}_3^{\times} & \frac{1}{3}\mu_{21}(\,[\,1\,]-3\underline{e}_3\underline{e}_3^T\,) & \frac{1}{3}\mu_{22}(\,[\,1\,]-3\underline{e}_3\underline{e}_3^T\,) \end{bmatrix}$$

$$\left[\,C\,\right] = -\omega_c^2 \begin{bmatrix} [\,([I_{\oplus}^0]\underline{e}_2)^{\times}-\underline{e}_2^{\times}[I_{\oplus}^0]\,]\,\underline{e}_2^{\times} & [\,\underline{e}_2^{\times}\underline{c}_1^0\underline{e}_2^T+\underline{e}_2^T\underline{c}_1^0\underline{e}_2^{\times}\,] & [\,\underline{e}_2^{\times}\underline{c}_2^0\underline{e}_2^T+\underline{e}_2^T\underline{c}_2^0\underline{e}_2^{\times}\,] \\ [\,\underline{e}_2^{\times}(\underline{c}_1^0)^{\times}+(\underline{e}_2^{\times}\underline{c}_1^0)^{\times}\,]\,\underline{e}_2^{\times} & -\mu_{11}\underline{e}_2^{\times}\underline{e}_2^{\times} & -\mu_{12}\underline{e}_2^{\times}\underline{e}_2^{\times} \\ [\,\underline{e}_2^{\times}(\underline{c}_2^0)^{\times}+(\underline{e}_2^{\times}\underline{c}_2^0)^{\times}\,]\,\underline{e}_2^{\times} & -\mu_{21}\underline{e}_2^{\times}\underline{e}_2^{\times} & -\mu_{22}\underline{e}_2^{\times}\underline{e}_2^{\times} \end{bmatrix}$$

$$\left[\,K\,\right] = \begin{bmatrix} [\,0\,] & [\,0\,] & [\,0\,] \\ [\,0\,] & [\,K_p\,]+[\,K_{R1}\,] & -[\,K_p\,] \\ [\,0\,] & -[\,K_p\,] & [\,K_p\,]+[\,K_{R2}\,] \end{bmatrix}$$

The coefficient matrices in Eq.(54) are each constant and trace their respective origins to specific dynamical effects;

$[\,M_0\,]$ - generalized mass matrix

$[\,K\,]$ - elastic stiffness matrix from the tethers

$[\,D\,]$ - viscous damping matrix from the tethers

[G] - gyric matrix capturing Coriolis effects from system rotation

[C] - centrifugal matrix from system rotation

[g] - gravitational effects matrix for orbiting system

It is immediately evident that:

$$[M_0] = [M_0]^T \quad ; \quad [D] = [D]^T \quad ; \quad [K] = [K]^T$$

While less apparent it can be shown that:

$$[G] = -[G]^T \quad ; \quad [C] = [C]^T \quad ; \quad [g] = [g]^T$$

With these properties noted, Eq.(54) are seen to be of a second order stationary damped gyric form[1]. Important aspects of the attitude stability of the orbiting assembly can be determined from the further study of the properties of the coefficient matrices.

[1] Hughes, P.C., 'Spacecraft Attitude Dynamics', Appendix-A, John Wiley & Sons, 1986.