Indian Head Division
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Indian Head, MD 20640-5035

IHTR 2503
5 May 2003

A BAYESIAN MODEL FOR THE ANALYSIS OF QUANTAL RESPONSE DATA

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REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
5 May 2003	Final Report	

4. TITLE AND SUBTITLE
A BAYESIAN MODEL FOR THE ANALYSIS OF QUANTAL RESPONSE DATA

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
0603782N

6. AUTHOR(S)
William W. McDonald

5d. PROJECT NUMBER
N0001499WX301410

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Indian Head Division
Naval Surface Warfare Center
Indian Head, MD 20640-5035

8. PERFORMING ORGANIZATION REPORT NUMBER
IHTR 2503

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of Naval Research
Arlington, VA 22217-5000

10. SPONSOR/MONITOR'S ACRONYM(S)
ONR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

Inference on the failure probabilities of ordered binomial trials conducted at M differing stress levels is considered. It is shown that a general joint prior may be constructed as a mixture of ordered M-variate Dirichlet distributions, which possesses marginals of nearly arbitrary shapes. Posterior marginals at both observational and non-observational stresses are shown to consist of sums of beta distributions. Recursive relationships are developed that permit the rapid and exact computation of the posterior marginal distributions. The model is attractive for use in successive Bayesian analyses.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON / 19b. TELEPHONE NUMBER (Include area code)
U	U	U	SAR	28	Susan Simpson / (301) 744-4284

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39-18

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FOREWORD

The work reported herein was performed at the Indian Head Division, Naval Surface Warfare Center (NSWC). The first part of this paper, which concerns the derivation of posterior marginal distributions for a prior consisting of a mixture of ordered Dirichlet distributions, was presented on 6 November 1982 at the Virginia Polytechnic and State University symposium, "Reflections on Bayesian Approaches in Operations Research, Probability, and Statistics," in Blacksburg, VA. At the symposium the author learned that the posterior marginals for an ordered Dirichlet prior had been published a year earlier by Damon Disch (1981). In November 1984 the author revised the original paper and included recursive relationships that enable the posterior marginal distributions derived earlier to be calculated. These results were not published, but they have been applied in an interactive computer code MBR written by the author and Mr. Patrick O'Neal at NSWC (White Oak Laboratory) circa 1988. This code was recently revised and rewritten in Mathcad and published as IHTR 2323.

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INTRODUCTION

This paper concerns the problem commonly associated with bioassay, but there are broader areas of application such as accelerated life testing, sensitivity testing, and military damage analysis. We will be concerned with items subjected to differing levels of stress and with the problem of estimating the probabilities of failure associated with the stress levels. Emphasis is placed on the attainment of interval estimates. The approach we take is Bayesian and the model we develop is an extension of that published by Ramsey (l972), whose work was brought to the attention of the author by Professor N. D. Singpurwalla of George Washington University.

The number of failures y at the ith stress level S is taken to be binomially distributed

$$y \sim b(n_i, p_i) \tag{1}$$

where n_i is the number of tests and p_i is the unknown (random) probability of failure. We let the total number of stresses involved be M. It is assumed that the unknown p values underlying the tests satisfy the same (complete) ordering restrictions as the stresses. These we are free to write as

$$S_1 < S_2 < \cdots < S_M \tag{2}$$

and

$$p_1 < p_2 < \cdots < p_M \ . \tag{3}$$

In the discussion that follows we develop a joint prior for the p values that is consistent with the ordering (3). This prior is related to that proposed by Ramsey (l972), but is considerably less restrictive. Specification of the prior is achieved by specifying its marginals, which can be accomplished in a variety of ways. The method currently used by the author is to obtain the user's judgments as to the modes (most likely values) and 5th and 95th percentiles (uncertainty limits) at each of the stress values. Usually these can be obtained in the forms of modal and limits curves spanning the stress values of interest. The complete marginals can then be supplied by the statistician in a manner that is consistent with the user's choices. As will be shown, the marginals must satisfy certain conditions that amount to very natural restrictions on the forms of the marginal distribution functions.

In the final sections we develop expressions for the posterior marginals. From these expressions, the user can obtain new values of the marginal modes and percentiles as modified by the data.

CONSTRUCTION OF THE PRIOR

Ramsey's prior was a form of the ordered M-variate Dirichlet distribution, which, following Wilks (l962, p. l82), we can write generally as

$$g_j(p_1 < p_2 < \cdots < p_M) = \frac{\Gamma(\alpha_{1j} + \cdots + \alpha_{M+1,j})}{\Gamma(\alpha_{1j}) \cdots \Gamma(\alpha_{M+1,j})} \prod_{i=1}^{M+1} (p_i - p_{i-1})^{\alpha_{ij}-1}, \tag{4}$$

where $0 \equiv p_0 < p_1 < p_2 < \cdots < p_M < p_{M+1} \equiv 1$ and $\alpha_{ij} > 0$ for all i,j. Our prior will consist of a mixture, or convex combination, of (4), viz.,

$$g(\underline{p}) \equiv \sum_{j=1}^{J} g_j(\underline{p}), \tag{5}$$

where $\phi_j > 0$, j=1, 2, …, J, and $\sum_j \phi_j = 1$.[1]

Now, we want to assign values to the parameters $\{\alpha_{ij}\}$ in (4) and (5) by choosing the shapes of the M marginals of $g(\underline{p})$. These we require to be of the form

$$g(p_i) \equiv \sum_{j=1}^{J} \phi_j \, g_j(p_i), \quad i = 1, 2, \ldots, M, \tag{6}$$

where

$$g_j(p_i) \equiv \frac{\Gamma(a_{ij} + b_{ij})}{\Gamma(a_{ij}) \Gamma(b_{ij})} \; p_i^{a_{ij}-1} (1 - p_i)^{b_{ij}-1} \quad . \tag{7}$$

Hence, our joint prior consists of a mixture of ordered M-variate Dirichlet distributions with marginals that are mixtures of betas. So that (6) and (7) are bounded, we will require

$$a_{ij} > 1 \text{ and } b_{ij} > 1. \tag{8}$$

[1]This choice was motivated by Mazzuchi's success (Mazzuchi and Singpurwalla, l98l) in representing the moments of the posterior marginals for (4), i.e., Ramsey's prior, and the need for a richer class of priors.

For the above forms to hold, Wilks (l962, theorem 7.7.6) gives the following conditions

$$a_{ij} = \sum_{\ell=1}^{i} \alpha_{\ell j} \tag{9}$$

and

$$b_{ij} = \sum_{\ell=i+1}^{M+1} \alpha_{\ell j} \tag{10}$$

for $i = 1,2,..., M$ and $j = 1,2,..., J$. From (9) we find

$$\begin{aligned} \alpha_{1j} &= a_{1j} \\ \alpha_{2j} &= a_{2j} - a_{1j} \\ &\vdots \\ \alpha_{Mj} &= a_{Mj} - a_{M-1,j} \end{aligned} \tag{11}$$

Hence $\alpha_{1j}, \alpha_{2j}, ..., \alpha_{Mj}$ are fully determined by $a_{1j}, a_{2j}, ..., a_{Mj}$. And from (10) we find

$$\alpha_{M+1,j} = b_{Mj} \tag{12}$$

and

$$b_{ij} = b_{Mj} + a_{Mj} - a_{ij}\ ,\ \ i = 1,2,...,M. \tag{13}$$

The latter result is found by substituting (11) and (12) into (10). Also, within the additive constant, b_{Mj}, the b parameters are determined by the a parameters. From these results we find that the beta densities of (7) are necessarily of the form

$$g_j(p_i) \propto p_i^{a_{ij}-1} (1-p_i)^{b_{Mj}+a_{Mj}-a_{ij}-1} . \tag{14}$$

It is convenient to reparameterize (14) in terms of its mode p_{ij}^* and precision index β , which are expressed as

$$p_{ij}^* = \frac{\alpha_{ij} - 1}{\beta_j} \tag{15}$$

$$\beta_j = b_{Mj} + a_{Mj} - 2 , \tag{16}$$

from (8). Substituting these into (14), we obtain

$$g_j(p_i) \propto \left(p^{p_{ij}^*} (1-p_i)^{1-p_{ij}^*} \right)^{\beta_j}. \tag{17}$$

Expression of the Dirichlet parameters in terms of β_j and p_{ij}^* yields

$$\alpha_{ij} = \beta_j (p_{ij}^* - p_{i-1,j}^*) + \delta_{i,M+1} + \delta_{i1} \tag{18}$$

for i = 1,2,..., M+1 and j = 1,2..., J. Here, as above, we have defined $p_{0j}^* \equiv 0$ and $p_{M+1,j}^* \equiv 1$. Note that the presence of the Kronecker δ 's in (18) makes $\alpha_{M+1,j}$ and α_{1j} depart from the values assigned by Ramsey, which appear to be in error (cf. p. 844).

Thus, by the above construction we are able to specify a suitable form for the joint prior if we are able to represent our marginal priors by functions of the form

$$g(p_i) = \sum_{j=1}^{J} \phi_j \, B^{-1}\left(\beta_j p_{ij}^* + 1, \beta_j (1-p_{ij}^*) + 1 \right) \left(p^{p_{ij}^*} (1-p_i)^{1-p_{ij}^*} \right)^{\beta_j} \tag{19}$$

where $B^{-1}(u,v) \equiv \dfrac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)}$, and i = 1,2,..., M.

We now show a method for choosing the parameters of (19) that will permit the representation of a wide class of marginal distributions.

Assignment Of Parameters

The assignment problem can be stated as follows. We wish to assign values to the parameters ϕ_j, β_j, j = 1,2,..., J and $p_{i1}^*, p_{i2}^*, \ldots, p_{iJ}^*$, i = 1,2,…, M in such a manner that the constraints $\phi_j > 0$, $\beta_j > 0$, $\alpha_{ij} = \beta_j (p_{ij}^* - p_{i-1,j}^*) + \delta_{i,M+1} + \delta_{i1} > 0$, j = 1,2,…, J, i = 1,2,…, M+1 and $\sum \phi_j = 1$ are satisfied and $g(p_i)$, i = 1,2,…,M, are close representations of the actual prior marginals. In the following let us denote the densities and distribution functions of the actual prior marginals by $g_i^*(\cdot)$ and $G_i^*(\cdot)$, respectively. It is assumed that such functions are available upon consultation with the user as described earlier.

It is noted that choices of the ϕ_j and β_j parameters affect all of the marginals represented by (19) simultaneously, but that sets of p_{ij}^* parameters may be independently assigned. This and the recognition that distributions may be regarded as describing the concentrations or densities of units of probability suggests the following assignment plan. For j = 1,2, …, J, let

$$\phi_j = 1/J \tag{20}$$

and

$$\beta_j = \beta \tag{21}$$

where β is a large number (e.g., 500 or 1000). Then for the ith marginal we choose the p_{ij}^*, j = 1,2, ..., J as percentiles of $g_i^*(\cdot)$ as follows:

$$G_i^*(p_{ij}^*) = j/(J+1)\,. \tag{22}$$

This process is repeated for all marginals $g_i^*(\cdot)$, i = 1,2,…, M. The interval between adjacent p_{ij}^* values corresponds to equal and constant units of probability.

This method of approximating the desired distribution shapes is very similar to that used in pattern recognition theory and found in the theory of Parzen estimators (see, e.g., Fukunaga, l972, p. 166). By increasing the value of J, the accuracy of the approximation can be increased arbitrarily. The size of the β parameter, which controls the width of the beta “kernels” (i.e., components of the sum), should be chosen so that all kernels overlap to some extent (see Meisel, l972, p. l07). In the author’s experience reasonable choices for J may be less than 50.

A review of the constraint requirements stated at the beginning of this section shows that the only constraints requiring our attention under this plan are those on the α_{ij}’s, namely $\alpha_{ij} = \beta_j(p_{ij}^* - p_{i-1,j}^*) + \delta_{i,M+1} + \delta_{i1} > 0$, $i = 1,2,...,M+1$, $j = 1,2,...,J$. It is of interest to see how these affect the shapes of the marginals that can be represented by (19).

An obvious set of sufficient constraint conditions is given by

$$p_{i-1j}^* < p_{ij}^*\,, \quad i = 1,2,...,M+1, \quad j = 1,2,...,J\,. \tag{23}$$

Now, proceeding pairwise, the plan requires that $G_{i-1}^*(p_{i-1,j}^*) = G_i^*(p_{ij}^*) = j/(J=1)$, which is illustrated in Figure 1. Hence, the constraint $p_{i-1j}^* < p_{ij}^*$ implies

$$G_{i-1}^{*}(p) > G_i^{*}(p)\,, \quad \text{and} \quad p \in \left\{p_{i-1,j}^{*}, p_{ij}^{*}\right\}. \tag{24}$$

This may be equivalently stated as

$$Pr(p_{i-1} > p) < Pr(p_i - p)\ , \quad p \in \left\{p_{i-1,j}^{*}, p_{ij}^{*}\right\}. \tag{25}$$

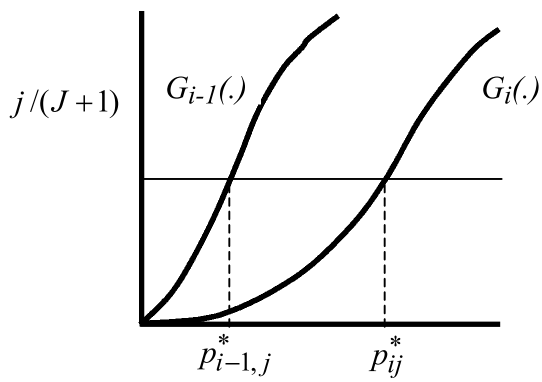

Figure 1. Illustration of Modes Assignments

By considering other values of j while allowing $J, \beta \to \infty$, we can extend the region of validity of (24) and (25) to the entire unit interval (0,1). Thus, the full set of constraints given in (23) suggests the following restrictions on the marginals that can be approximated by (19):

$$G_1^{*}(p) > G_2^{*}(p) > \cdots > G_M^{*}(p) \ \text{ for all } \ p \in (0,1)\,, \tag{26}$$

which is equivalent to

$$\Pr(p_1 > p) < \Pr(p_2 > p) < \cdots < \Pr(p_M > p), \ \text{for all}\, p \in (0,1). \tag{27}$$

Both relations (26) and (27) express conditions that must be satisfied by the prior marginals. The latter, which are referred to as the conditions for stochastic ordering, have an intuitively meaningful interpretation.

Other Features Of The Prior

Interesting properties of Ramsey's prior that were listed in his paper (l972) appear to carry over to the mixed prior. The conditional distribution of p_i given p_{i-1} and p_{i+1} is a mixture of translated betas on the interval (p_{i-1}, p_{i+1}). And in the limit as $J \to \infty$ and $\beta \to \infty$, $g(\underline{p})$ can be written as

$$g(\underline{p}) = g(p_1)\, g(p_2 \mid p_1)\, g(p_3 \mid p_2) \cdots g(p_M \mid p_{M-1}), \tag{28}$$

where the conditionals are sums of translated betas respectively over $(p_1, 1), (p_2, 1), \ldots, (p_{M-1}, 1)$ (also see Kraft and Van Eeden, l964).

THE POSTERIOR DISTRIBUTION

Since the data are assumed to be binomially distributed and acquired independently, the likelihood function is

$$L(\underline{p} \mid \underline{y}) = \prod_{i=1}^{M} \binom{n_i}{y_i} p_i^{y_i} (1 - p_i)^{n_i - y_i} \tag{29}$$

Hence, the joint posterior density is written as

$$f(\underline{p} \mid \underline{y}) \propto L(\underline{p} \mid \underline{y})\, g(\underline{p}) \ , \tag{30}$$

where $g(\underline{p})$ is obtained from (6). Substituting this, we get

$$f(\underline{p} \mid \underline{y}) \propto \sum_{j=1}^{J} \phi_j K_j \left\{ (1 - p_M)^{\alpha_{M+1,j-1}} \prod_{i=1}^{M} p_i^{y_i} (1 - p_i)^{\overline{n}_i} (p_i - p_{i-1})^{\alpha_{ij} - 1} \right\}, \tag{31}$$

where

$$K_j = \frac{\Gamma(\alpha_{1j} + \cdots + \alpha_{M+1,j})}{\Gamma(\alpha_{1j}) \cdots \Gamma(\alpha_{M+1,j})} \quad . \tag{32}$$

The marginals of the posterior distribution are thus obtained by performing the integrations indicated in

$$f\left(\underline{p} \middle| \underline{y}\right) \propto \sum_{j=1}^{J} \phi_j K_j \int_0^{p_1} dp_{i-1} \cdots \int_0^{p_2} dp_1 \int_{p_i}^{1} dp_{i+1} \cdots \int_{p_{M-1}}^{1} dp_M \{\cdot\} \tag{33}$$

where $\{\cdot\}$ is the bracketed term in (31) above.

Integration of (33) can be achieved by expanding the various binomial terms in a systematic fashion. The process has been described by Disch (1981) in considerable detail (for J = 1). Using a notation similar to that of Disch, the result, for i = 1, 2, …, M, can be expressed as

$$f(\underline{p}\,|\,\underline{y}) \propto \sum_{j=1}^{J} \phi_j K_j \sum_{\ell_1=0}^{\bar{n}_1} \cdots \sum_{\ell_{i-1}=0}^{\bar{n}_{i-1}} (-1)^{\lambda_{i-1}} \prod_{r=1}^{i-1} \binom{\bar{n}_r}{\ell_r} B^{-1}(\xi_{rj}+\lambda_r, \alpha_{r+1,j})$$

$$\times \sum_{\ell_{i+1}=0}^{y_{i+1}} \cdots \sum_{\ell_M=0}^{y_M} (-1)^{\Lambda_{i-1}} \prod_{s=i+1}^{M} \binom{y_s}{\ell_s} B^{-1}(\eta_{sj}+\Lambda_s, \alpha_{sj}) p_i^{\xi_{ij}+\lambda_{i-1}-1} q_i^{\eta_{ij}+\Lambda_{i+1}-1} \tag{34}$$

where $q_i \equiv 1-p_i$, $\bar{n}_k \equiv n_k - y_k$

$$\xi_{rj} \equiv \sum_{k=1}^{r} (y_k + \alpha_{kj}),\ \lambda_r \equiv \sum_{k=1}^{r} \ell_k,\ \lambda_0 \equiv 0$$

$$\eta_{rj} \equiv \sum_{k=s}^{M} (\bar{n}_k + \alpha_{k+1,j}),\ \Lambda_s \equiv \sum_{k=s}^{M} \ell_k,\ \Lambda_{M+1} \equiv 0$$

and the B^{-1} function was defined earlier in connection with (19). Clearly, from (34) one can easily obtain forms for the M posterior marginal distribution functions in terms of incomplete beta functions.

Interpolation and Extrapolation

Expression (34) is readily extended to non-observational stresses (i.e., stresses for which no data exist) by setting the associated values of $\bar{n}_i$ and y_i to zero. In this way we can interpolate or extrapolate the forms of the posterior marginals at stress values other than those at which data were collected provided the prior marginals at these points have been specified.

Computation of the Posterior Density

In this section we develop a recursive procedure by which (34) can be rapidly calculated. Disch (l98l) explored approximate methods of computation after pointing out the profound inefficiency of straightforward, brute-force approaches. Antoniak (1974) reported similar difficulties.

Equation (34) may be viewed as the sum, over j, of the product of two sums, the first being a sum over all $(i-1)$-tuples $(\ell_1, \ell_2, ..., \ell_{i-1})$ and the second a sum over all $(M-i)$-tuples $(\ell_{i+1}, \ell_{i+2}, ..., \ell_M)$. A useful alternative representation involves the summations in which the indices $\ell_1, \ell_2, ..., \ell_{i-1}$ and $\ell_{i+1}, \ell_{i+2}, ..., \ell_M$ are constrained. We then write (34) as

$$f(p_i \mid \underline{y}) \propto \sum_{j=1}^{J} \phi_j K_{ij} \left\{ \sum_{k=0}^{\bar{n}_{i-1}^{\leq}} (-1)^k C_p(i,j,k) p_i^{\xi_{ij}+k-1} \right\} \left\{ \sum_{k'=0}^{y_{i+1}^{\geq}} (-1)^{k'} C_q(i,j,k') q_i^{\eta_{ij}+k'-1} \right\} \tag{35}$$

where
$$C_p(i,j,k) \equiv \sum_{\ell_1=0}^{\bar{n}_1} \cdots \sum_{\ell_{i-1}=0}^{\bar{n}_{i-1}} \prod_{r=1}^{i-1} \binom{\bar{n}_r}{\ell_r} \frac{(\xi_{rj})_{\lambda_r}}{(\zeta_{rj})_{\lambda_r}} \tag{36}$$

subject to the constraint $\lambda_{i-1} = k$, and

$$C_q(i,j,k') \equiv \sum_{\ell_{i+1}=0}^{y_{i+1}} \cdots \sum_{\ell_M=0}^{y_M} \prod_{s=1}^{M} \binom{y_s}{\ell_s} \frac{(\eta_{sj})_{\Lambda_s}}{(\nu_{sj})_{\Lambda_s}} \tag{37}$$

subject to the constraint $\Lambda_{i+1} = k'$. Here, we have made use of Pochhammer's symbol

$$(z)_k \equiv \Gamma(z+k)/\Gamma(z) = z(z+1)\cdots(z+k-1), \quad (z)_0 = 1 \ ,$$

where k is an integer, and we have defined

$$\zeta_{rj} \equiv \xi_{rj} + \alpha_{r+1,j}\ , \ \nu_{rj} \equiv \eta_{sj} + \alpha_{sj}\ ,$$

$$\bar{n}_{i-1}^{\leq} \equiv \sum_{r=1}^{i-1} \bar{n}_r\ ,\ y_{i+1}^{\geq} \equiv \sum_{s=i+1}^{M} y_s\ ,$$

and
$$K_{ij} = K_j \prod_{r=1}^{i-1} \frac{\Gamma(\xi_{rj})\Gamma(\alpha_{r+1j})}{\Gamma(\xi_{rj}+\alpha_{r+1j})} \prod_{s=i+1}^{M} \frac{\Gamma(\eta_{sj})\Gamma(\alpha_{sj})}{\Gamma(\eta_{sj}+\alpha_{sj})}\ .$$

As a consequence of (36) and (37), we find $C_p(1,j,0) = C_q(M,j,0) = 1$.

The simplification of equations (36) and (37) can be effected by making index transformations which permit the factoring of the summands through the summations. We first examine equation (36). Consider the 1-to-1 and onto transformation of summation indices from the set $\ell_1, \ell_2, ..., \ell_{i-1}$ to the set

$\lambda_1, \lambda_2, ..., \lambda_{i-1}$ with transformation relations $\ell_r = \lambda_r - \lambda_{r-1}$, $r = 1, 2, ..., i-1$, $\lambda_0 = 0$. Under these transformations (36) can be expressed as

$$C_p(i,j,k) = \sum_{\lambda_{i-2}=\max(k-\overline{n}_{i-1},0)}^{\min(k,\overline{n}_{i-2}^{\leq})} \; \sum_{\lambda_{i-3}=\max(\lambda_{i-2}-\overline{n}_{i-2},0)}^{\min(\lambda_{i-2},\overline{n}_{i-3}^{\leq})} \cdots$$

$$\sum_{\lambda_1=\max(\lambda_2-\overline{n}_2,0)}^{\min(\lambda_2,\overline{n}_1^{\leq})} \; \prod_{r=1}^{i-1} \binom{\overline{n}_r}{\lambda_r - \lambda_{r-1}} \frac{(\xi_{rj})_{\lambda_r}}{(\zeta_{rj})_{\lambda_r}} \; , \tag{38}$$

where $\lambda_{i-1} = k$ and the summation limits arise from the constraints

$$\lambda_r - \overline{n}_r \;\leq\; \lambda_{r-1} \;\leq\; \lambda_r$$

$$\lambda_r \geq 0$$

$$\lambda_r \leq \; \overline{n}_r^{\leq} \; .$$

By factoring out the product terms, we obtain

$$C_p(i,j,k) \;=\; \frac{(\xi_{i-1,j})_k}{(\zeta_{i-1,j})_k} \sum_{\lambda_{i-2}=\max(k-\overline{n}_{i-1},0)}^{\min(k,\overline{n}_{i-2}^{\leq})} \binom{\overline{n}_{i-1}}{k-\lambda_{i-2}}$$

$$\times \; \frac{(\xi_{i-2,j})_{\lambda_{i-2}}}{(\zeta_{i-2,j})_{\lambda_{i-2}}} \sum_{\lambda_{i-3}=\max(\lambda_{i-2}-\overline{n}_{i-2},0)}^{\min(\lambda_{i-2},\overline{n}_{i-3}^{\leq})} \binom{\overline{n}_{i-2}}{\lambda_{i-2}-\lambda_{i-3}}$$

$$\times \cdots \times \; \frac{(\xi_{2,j})_{\lambda_2}}{(\zeta_{2,j})_{\lambda_2}} \sum_{\lambda_1=\max(\lambda_2-\overline{n}_2,0)}^{\min(\lambda_2,\overline{n}_1^{\leq})} \binom{\overline{n}_2}{\lambda_2-\lambda_1} \; \frac{(\xi_{1,j})_{\lambda_1}}{(\zeta_{1,j})_{\lambda_1}} \binom{\overline{n}_1}{\lambda_1} \tag{39}$$

By inspection of (39), we find the following recursive relationship between the C_p coefficients:

$$C_p(i,j,k) = \frac{\left(\xi_{i-1,j}\right)_k}{\left(\zeta_{i-1,j}\right)_k} \sum_{r=\max(k-\overline{n}_{i-1},0)}^{\min(k,\overline{n}_{i-2}^{\le})} \binom{\overline{n}_{i-1}}{k-r} C_p(i-1,j,r) \;,\; k=0,1,...,\overline{n}_{i-1}^{\le} \;,\; i=3,...,M$$

$$C_p(2,j,k) = \frac{\left(\xi_{1,j}\right)_k}{\left(\zeta_{1,j}\right)_k} \binom{\overline{n}_1}{k} \;,\; k=0,1,...,\overline{n}_1^{\le}, \tag{40}$$

$$C_p(1,j,k) = 1 \quad , \; k=0$$

Recursive relationships for the C_q coefficients follow in a similar manner upon transformation from the indices $\ell_{i+1}, \ell_{i+2}, ..., \ell_M$ to the set $\Lambda_{i+1}, \Lambda_{i+2}, ..., \Lambda_M$. We obtain

$$C_q(i,j,k) = \frac{\left(\eta_{i+1,j}\right)_k}{\left(\nu_{i+1,j}\right)_k} \sum_{s=\max(k-y_{i+1},0)}^{\min(k,y_{i+2}^{\ge})} \binom{y_{i+1}}{k-s} C_q(i+1,j,s) \;,k=0,1,...,y_{i+1}^{\ge} \;,\; i=1,2,...,M-2$$

$$C_q(M-1,j,k) = \frac{\left(\eta_{M,j}\right)_k}{\left(\nu_{M,j}\right)_k} \binom{y_M}{k} \;,\; k=0,1,...,y_M^{\ge} \tag{41}$$

$$C_q(M,j,k) = 1 \quad , \; k=0$$

We note from (9) that ξ_{ij} can be expressed as

$$\xi_{ij} \equiv \sum_{k=1}^{i} (y_k + \alpha_{kj}) = a_{ij} + y_i^{\le}$$

where $y_i^{\le}$ has been defined (in a manner analogous to that of $\overline{n}_i^{\le}$) as the sum of all y_k values having k less than or equal to i. Similarly, from (10) we note that

$$\eta_{ij} \equiv \sum_{k=i}^{M} (\overline{n}_k + \alpha_{k+1,j}) = b_{ij} + \overline{n}_i^{\ge} ,$$

where $\overline{n}_i^{\geq}$ is the sum of all $\overline{n}_k$ having k greater than or equal to i, analogously to $y_i^{\geq}$. Substituting these expressions into (35) and rearranging yields

$$f(p_i \mid \underline{y}) \propto \sum_{j=1}^{J} \phi_j K_{ij} \sum_{k=0}^{\overline{n}_{i-1}^{\leq}} (-1)^k C_p(i,j,k) \sum_{k'=0}^{y_{i+1}^{\geq}} (-1)^{k'} C_q(i,j,k') p_i^{a_{ij}+y_i^{\leq}+k-1} q_i^{b_{ij}+\overline{n}_i^{\geq}+k'-1} . \qquad (42)$$

The constant of proportionality, if desired, can be obtained from the condition

$$\int_{-\infty}^{\infty} f(p_i \mid \underline{y}) dp_i = 1 \ ,$$

which holds for any value of i.

Equation (42) is a particularly useful computational form for the posterior marginals. It is cast in terms of the marginal beta coefficients $\{a_{ij}, b_{ij}\}$ rather than the joint distribution coefficients $\{\alpha_{ij}\}$. This avoids possible round off problems associated with the calculation of α_{ij} in equation (18) involving p_{ij}^* differences.

A computer program that uses a closely related formulation and contains an example was recently published as McDonald (2003).

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