REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From – To)*
15-09-2003	Final Report	16 September 2002 - 15-Sep-03

4. TITLE AND SUBTITLE	**5a. CONTRACT NUMBER**
Filtering and Control of Systems with Nonrational Signals Spectra	FA8655-02-M4082
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	**5d. PROJECT NUMBER**
Professor Arunabha Bagchi	**5d. TASK NUMBER**
	5e. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	**8. PERFORMING ORGANIZATION REPORT NUMBER**
University of Twente Postbus 217 Enschede 7500 AE The Netherlands	N/A
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	**10. SPONSOR/MONITOR'S ACRONYM(S)**
EOARD PSC 802 BOX 14 FPO 09499-0014	**11. SPONSOR/MONITOR'S REPORT NUMBER(S)** SPC 02-4082

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

This report results from a contract tasking University of Twente as follows: Contractor will provide research in control theory of with nonrational spectra. He will perform a total of 220 hours research and provide associated publications and reports.

20040210 106

15. SUBJECT TERMS

EOARD, Control, Stochastic systems

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18, NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**	UL	42	NEAL D. GLASSMAN
UNCLAS	UNCLAS	UNCLAS			**19b. TELEPHONE NUMBER** *(Include area code)* +1 703-696-9548

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39-18

University of Twente

Department of Applied Mathematics

Interim Report : **September 15, 2003**
EAORD Project : **Filtering and Control of Systems with Nonrational System Spectra**
Contract No. : **FA8655-02-M-4082**

Rational Approximations of the Power Spectral Density of Atmospheric Turbulence Arising in Adaptive Optics

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September 15, 2003

[1]This work is supported by the EOARD under contract no. FA8655-02-M4082

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Contents

List of Abbreviations

Common abbreviations which are used throughout the report are the following:

w.r.t. (with respect to)

s.t. (such that)

a.e. (almost everywhere)

iff (if and only if)

LTI (linear time invariant)

PSD (power spectral density or power spectra)

WSS (wide sense stationary)

List of Notation

Below we list some of the notation that appear frequently throughout this report. It is by no means extensive since more notation will be introduced in later chapters as deemed necessary.

$\mathbb{Z}_+$ denotes the set of nonnegative integers.

$\mathbb{R}$ denotes the real numbers.

$\mathbb{R}_e$ denotes the extended real numbers, i.e. $[-\infty, \infty]$.

$\mathbb{C}$ denotes the complex numbers.

$\mathbb{C}_e$ denotes the complex numbers augmented with complex infinity (the point on the north pole of the Riemann sphere), i.e. $\mathbb{C}_e = \mathbb{C} \cup \{\infty\}$

$L_1(A)$ denotes the space of complex valued functions defined on a set A and is integrable w.r.t. the Lebesque measure defined on A.

$L_2(A)$ denotes the space of complex valued functions defined on a set A and is square integrable w.r.t. the Lebesque measure defined on A.

$L_1^{m\times n}(A)$ denotes the space of complex $m \times n-$matrix valued function defined on A with entries belonging to $L_1(A)$.

For $\omega < 0$, $L_{1,\omega}^{m\times n}(\mathbb{R}) = \{f(\cdot) \in L_1^{m\times n}(\mathbb{R}) : e^{-\omega|\cdot|}f(\cdot) \in L_1^{m\times n}(\mathbb{R})\} \subset L_1^{m\times n}(\mathbb{R})$ (following the notation in [4], p. 47).

$\Pi_\alpha = \{s \in \mathbb{C} : Im(s) > \alpha\}$.

$-\Pi_\alpha = \{s \in \mathbb{C} : Im(s) < \alpha\}$.

For $\alpha > 0$, S_α denotes the set $\{s \in \mathbb{C} : |Im(s)| < \alpha\}$.

If $c \in \mathbb{C}$ and $A \subset \mathbb{C}$, $cA = \{y \in \mathbb{C} : \exists x \in A \text{ s.t. } y = cx\}$.

$\mathfrak{R}_{m\times n}^0$ denotes the space of functions $f(\cdot)$ defined on $\mathbb{R}_e$ which can be written as $f(\lambda) = \int_{-\infty}^{\infty} k(t)e^{i\lambda t}dt$ for some $k(\cdot) \in L_1^{m\times n}(\mathbb{R})$. Note that $f(\pm\infty) = \lim_{\lambda\to\pm\infty} f(\lambda) = 0$.

$\mathfrak{R}_{m\times n}$ denotes the space of functions $f(\cdot)$ defined on $\mathbb{R}_e$ which can be written as $f(\lambda) = c + g(\lambda)$ for some $g(\cdot) \in \mathfrak{R}^0_{m\times n}$ and some $c \in \mathbb{C}^{m\times n}$.

$\mathfrak{R}^{0+}_{m\times n}$ denotes the space of functions $f(\cdot)$ defined on $\mathbb{R}_e$ which can be written as $f(\lambda) = \int_0^\infty k(t)e^{i\lambda t}dt$ for some $k(\cdot) \in L_1^{m\times n}([0,\infty))$. Note that $f(\infty) = \lim_{\lambda\to\infty} f(\lambda) = 0$.

$\mathfrak{R}^{+}_{m\times n}$ denotes the space of functions $f(\cdot)$ defined on $\mathbb{R}_e$ which can be written as $f(\lambda) = c + g(\lambda)$ for some $g(\cdot) \in \mathfrak{R}^{0+}_{m\times n}$ and $c \in \mathbb{C}^{m\times n}$.

$\hat{k}(\cdot)$ denotes the Fourier transform of a function $k(\cdot) \in L_1^{m\times n}(\mathbb{R})$.

$\lim_{t\to\infty} f(t) = O(g(t))$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = M$ for some $M \in \mathbb{C}$.

$\lim_{t\to\infty} f(t) = o(g(t))$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$.

Chapter 1

Introduction

1.1 Overview

This is the second part of the report of the EOARD funded research project "Filtering and control of systems with nonrational signals spectra". It concentrates on the general problem of approximating a given irrational scalar power spectral density (PSD) with rational functions. Although the issues arising are generic for any turbulence, our interest in this study arose from the problem of modeling atmospheric turbulence in adaptive optics (refer to the mid-year project report [8]). In adaptive optics, the PSD is typically a matrix valued function. Our study of the scalar case presented in this report is a first step towards a theory of approximations of general multidimensional PSDs. The main approximation tool that is considered in this report is the Pade approximation and the particular PSDs to which we apply the theory is the well-known Kolmogorov and von Karman spectra. Results in this report, whether they are well known or new, were independently derived by the authors unless stated otherwise.

1.2 Organization of the report

In chapter 2, we begin the discussion by recalling some facts about analytic extension of functions. The next chapter is devoted to a short exposition of the important Paley-Wiener theorem on the realizability of PSD. Chapter 4 covers an introduction to the theory of Pade approximation and a direct application of it for constructing rational approximants of both the Kolmogorov and von Karman spectra. Based on the approximations of Chapter 4, in Chapter 5 approximate state-space generator models for the Kolmogorov and von Karman spectra are derived. Finally, the main conclusions that can be drawn from this study is presented in Chapter 6.

Chapter 2

Analytic Extension of Functions

As a starting point for our discussion, we consider the conditions under which a function defined on the real axis will have an extension which is analytic everywhere in an open neighbourhood of that axis. In particular, we are interested in the case where the defined function is the Fourier transform of some impulse response. Analytic extensions are important because in systems theory, the transfer function of a system, which is a key tool in the analysis of LTI systems, is defined on a part of the complex plane rather than just on the real or imaginary axis. The relevance of the discussion to follow will become more apparent in later sections of the report.

Remark 1 *We are considering the extension of functions defined on the real axis to the complex plane because this is the context in which extension problems originally appeared. The results that follow can be easily adjusted for the extension of functions defined on the imaginary axis.*

We first present a series of basic results:

Lemma 2 *Let $H(\cdot)$ be a complex valued function defined everywhere on the real axis and suppose that it has an extension to an open subset of the complex plane including the real axis. Then for the extension to be analytic it is necessary that $H(\omega)$ be infinitely differentiable w.r.t. ω.*

Proof. The necessity is immediate due to the hypothesis that extension is analytic on the real axis and the well known fact that a function of a complex variable is infinitely differentiable at every point where it is analytic. ■

Theorem 3 *Let $H(\cdot)$ be a infinitely differentiable complex valued function defined on the real axis. Suppose that for each $\omega_0 \in \mathbb{R}$ the Taylor series expansion* *of* $H(\omega)$ *about* ω_0, *i.e.*

$$H(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n H(\lambda)}{d\lambda^n} \right]_{\lambda=\omega_0} (\omega - \omega_0)^n$$

exists and has radius of convergence $R(\omega_0) > 0$. *Then* $H(\cdot)$ *has an analytic extension to an open subset of the complex plane containing the real axis.*

Proof. Recall that a real valued power series about a point ω_0 converges absolutely within its radius of convergence. This implies that when the real and imaginary part of a complex valued power series about a point ω_0 converges simultaneously within some radius of convergence, the series will converge absolutely within that radius of convergence. It follows that for a fixed $\omega_0 \in \mathbb{R}$ the Taylor series expansion about ω_0 also converges for all $s \in \mathbb{C}$ satisfying $|s - \omega_0| < R(\omega_0)$. The series in turn defines an analytic function on the open set $S = \{s \in \mathbb{C} : \exists \omega_0 \in \mathbb{R} \text{ s.t. } |s - \omega_0| < R(\omega_0)\}$ containing the real axis. ■

Before proceeding further we wish to introduce a couple of definitions.

Definition 4 *A matrix valued function* $H(\cdot) = \{H_{ij}(\cdot)\} \in L_1^{m \times n}(\mathbb{R})$ *is said to be causally generated of type* $-\alpha$ $(\alpha > 0)$ *if it posseses the following properties:*

1. $H(\cdot)$ *has an inverse Fourier transform* $h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$ *which satisfies* $\int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = H(\omega)\ \forall \omega$ *and* $h(t) = 0\ \forall t < 0$.

2. *For* $\forall i, j$, $h_{ij}(t) \overset{t \to \infty}{\to} O(f_{ij}(t) e^{-\alpha t})$ *and, for some* $M_{ij} \geq 0$, $f_{ij}(\cdot)$ *satisfies* $\int_{M_{ij}}^{\infty} |f_{ij}(t)| e^{-\beta t} dt < \infty \quad \forall \beta > 0$.

Definition 5 *A matrix valued function* $H(\cdot) = \{H_{ij}(\cdot)\} \in L_1^{m \times n}(\mathbb{R})$ *is said to be of type* $-\alpha$ $(\alpha > 0)$ *if:*

1. *The inverse Fourier transform* $h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$ *exists and satisfies* $\int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt = H(\omega)\ \forall \omega$.

2. *For* $\forall i, j$, $h_{ij}(t) \overset{t \to \pm\infty}{\to} O(f_{ij}(t) e^{-\alpha |t|})$ *and, for some* $M_{ij} \geq 0$, $f_{ij}(\cdot)$ *satisfies* $\int_{-\infty}^{-M_{ij}} |f_{ij}(t)| e^{\beta t} dt < \infty$ *and* $\int_{M_{ij}}^{\infty} |f_{ij}(t)| e^{-\beta t} dt < \infty \quad \forall \beta > 0$.

Remark 6 *The condition on the* $f_{ij}(\cdot)$*'s is to ensure that they do not grow exponentially as* $t \to \pm\infty$.

Remark 7 *It is clear from the definitions that every function that is causally generated of type* $-\alpha$ *is also of type* $-\alpha$.

The following proposition describes a method to construct an analytic extension of $H(\cdot)$ when it is causally generated of type $-\alpha$. Before stating the proposition, we need to introduce a few more definitions.

Definition 8 *For* $\alpha \in \mathbb{R}$, $\Pi_\alpha = \{s \in \mathbb{C} : Im(s) > \alpha\}$.

Definition 9 *For* $\alpha \in \mathbb{R}$, $-\Pi_\alpha = \{s \in \mathbb{C} : Im(s) < \alpha\}$.

Definition 10 *For* $\alpha > 0$, S_α *denotes the set* $\{s \in \mathbb{C} : |Im(s)| < \alpha\}$.

Proposition 11 *Suppose that* $H(\cdot) \in L_1^{m\times n}(\mathbb{R})$ *is causally generated of type* $-\alpha$ *then* $H(\omega)$ *is infinitely differentiable w.r.t.* ω *and* $H(\cdot)$ *has an analytic extension to* $\Pi_{-\alpha}$. *In particular, on* S_α *the extension is given by*

$$H(\omega - iv) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega)}{d\omega^n} (v/i)^n$$

Proof. Let $h(\cdot) = \{h_{ij}(\cdot)\}$ denote the inverse Fourier transform of $H(\cdot)$. By the integrability of $|H_{ij}(\cdot)|$ and the properties of causally generated functions of type $-\alpha$ we have that for any $\varepsilon > 0$ and some $0 < K, N < \infty$

$$\begin{aligned}
\int_0^\infty |h_{ij}(t)|\, e^{-vt} dt &= \int_0^\infty \left| \int_{-\infty}^{\infty} H_{ij}(\omega) e^{i\omega t} d\omega \right| e^{vt} dt \\
&= \int_0^N \left| \int_{-\infty}^{\infty} H_{ij}(\omega) e^{i\omega t} d\omega \right| e^{vt} dt + \int_N^\infty \left| \int_{-\infty}^{\infty} H_{ij}(\omega) e^{i\omega t} d\omega \right| e^{vt} dt \\
&\leq \int_0^N \int_{-\infty}^{\infty} |H_{ij}(\omega)|\, d\omega e^{vt} dt + \int_N^\infty (K+\varepsilon)\, |f_{ij}(t)|\, e^{-(\alpha+v)t} dt \\
&= \left(\int_{-\infty}^{\infty} |H_{ij}(\omega)|\, d\omega \right) \left(\int_0^N e^{vt} dt \right) + (K+\varepsilon) \int_N^\infty |f_{ij}(t)|\, e^{-(\alpha+v)t} dt \\
&< \infty \text{ when } v > -\alpha
\end{aligned}$$

implying that $\int_0^\infty h(t) e^{-vt} e^{-i\omega t} dt$ exists for $v > -\alpha$. Since $\int_0^\infty |h_{ij}(t)|\, e^{\beta t} dt < \infty$ for $0 < \beta < \alpha$ it follows that

$$\begin{aligned}
\int_0^\infty |h_{ij}(t)|\, t^n dt &\leq \frac{n!}{\beta^n} \int_0^\infty |h_{ij}(t)|\, e^{\beta t} dt \\
&< \infty
\end{aligned}$$

for all finite n. Hence $\frac{d^n H(\omega)}{d\omega^n} = (-i)^n \int_0^\infty h(t) t^n e^{-i\omega t} dt$ exists for $\forall\omega$ and $\forall n$.

Let us now place the constraint $|v| < \alpha$. Define

$$
\begin{aligned}
H(\omega - iv) &= \int_0^\infty h(t) e^{-i(\omega - iv)t} dt \\
&= \int_0^\infty h(t) e^{-vt} e^{-i\omega t} dt \\
&= \int_0^\infty \lim_{n\to\infty} h(t) \left(\sum_{k=0}^{n} \frac{(-vt)^k}{k!} \right) e^{-i\omega t} dt
\end{aligned}
$$

Since

$$
\begin{aligned}
|h_{ij}(t)| \left| \sum_{k=0}^{n} \frac{(-vt)^k}{k!} \right| &\leq |h_{ij}(t)| \sum_{k=0}^{n} \frac{|v|^k t^k}{k!} \\
&\leq |h_{ij}(t)| \, e^{|v|t} \text{ for } \forall n
\end{aligned}
$$

and

$$
\int_0^\infty |h_{ij}(t)| \, e^{|v|t} dt < \infty \text{ if } |v| < \alpha
$$

Lebesque's dominated convergence theorem gives:

$$
\begin{aligned}
H(\omega - iv) &= \lim_{n\to\infty} \int_0^\infty h(t) \left(\sum_{k=0}^{n} \frac{(-vt)^k}{k!} \right) e^{-i\omega t} dt \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-v)^k}{k!} \int_0^\infty h(t) t^k e^{-i\omega t} dt \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k H(\omega)}{d\omega^k} (v/i)^k
\end{aligned}
$$

Let $s = \omega - iv$ and notice that when $H(s)$ exists at a point s and on an open neighbourhood of s it is analytic on that neighbourhood. Since $H(s)$ is well defined on S_α it is also analytic there. Moreover, we already know that $H(s)$ exists everywhere on $\Pi_{-\alpha}$ and must therefore be analytic there. Thus we may extend $H(s)$ from S_α to $\Pi_{-\alpha} = S_\alpha \cup \overline{\Pi_\alpha}$ by analytic continuation (see [9], pp. 96-97). We conclude that $H(s)$ is an analytic extension of $H(\omega)$ to $\Pi_{-\alpha}$. ■

In a similar fashion we can show the following more general result:

Proposition 12 *Suppose that $H(\cdot) \in L_1^{m\times n}(\mathbb{R})$ is of type $-\alpha$ then $H(\omega)$ is infinitely differentiable w.r.t. ω and $H(\cdot)$ has an analytic extension to S_α given by*

$$H(\omega - iv) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega)}{d\omega^n} (v/i)^n$$

Proof. The proof goes along the same lines as in the proof of preceeding proposition, only now the analysis is applied to a double sided integral since we are no longer assuming causality of the inverse Fourier transform. ■

The following result is rather obvious and is in fact a by-product of the "proof" of Proposition 12.

Lemma 13 *If $\hat{k}(\cdot) \in L_1^{m\times n}(\mathbb{R})$ is of type $-\alpha$ then $k(\cdot) \in L_{1,\omega}^{m\times n}(\mathbb{R})$ for $\forall\omega$ s.t. $0 > \omega > -\alpha$.*

From the lemma we can make the following statements:

Corollary 14 *The class of functions in $L_1^{m\times n}(\mathbb{R})$ which are causally generated of type $-\alpha$ is a subset of $\mathfrak{R}_{m\times n}^{0+}$.*

Corollary 15 *The class of functions in $L_1^{m\times n}(\mathbb{R})$ which are of type $-\alpha$ is a subset of $\mathfrak{R}_{m\times n}^{0}$.*

An interesting fact about the last two propositions is that the extension of $H(\cdot)$ to S_α has a form like a Taylor series. Indeed it can be interpreted as a Taylor series and we will now formally demonstrate it. Suppose that $F(\cdot)$ is a function of a complex variable which is analytic on an open subset of the complex plane containing the real axis such that $F(s)]_{s=\omega} = H(\omega)$. Then the Taylor series of $F(s)$ about the point $s_0 = \omega$ exists and is given by:

$$\begin{aligned} F(s) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{d^n F(s)}{ds^n}\right]_{s=\omega} (s-\omega)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega)}{d\omega^n} (s-\omega)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega)}{d\omega^n} (s-\omega)^n \end{aligned}$$

when $|s-\omega| < R(\omega)$ ($R(\omega) > 0$ is the radius of convergence of the Taylor series of $F(s)$ about ω). After inserting $s = \omega - iv$ with $|v| < R(\omega)$ we obtain the series:

$$F(\omega - iv) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n H(\omega)}{d\omega^n} (v/i)^n$$

The last proposition then implies that when $H(\cdot)$ is of type $-\alpha$ the Taylor expansion of $F(s)$ along the real axis can be guaranteed to have a uniform radius of convergence which is at least equal to α, i.e. we may set $R(\omega) = \alpha$ for $\forall\omega$.

Appealing to the proof of the above results we can make the following inference on the relation between the Fourier and Laplace transform and the analytic extension of functions defined *on the imaginary axis.*

Corollary 16 *Let $\hat{h}(\cdot) \in L_1^{m\times n}(\mathbb{R})$ be causally generated of type $-\alpha$ (of type $-\alpha$) and let $G(i\omega) = \hat{h}(\omega)$ for $\forall\omega \in \mathbb{R}$. Then $G(\cdot)$ is a function defined on the imaginary axis and has an analytic extension to $-i\Pi_{-\alpha}$ $(-iS_\alpha)$. In particular, on $-iS_\alpha$ the extension is given by:*

$$G(v + i\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n G(i\omega)}{d\omega^n} (v/i)^n$$

and if $s = v + i\omega$ then $G(s)$ for $s \in -i\Pi_{-\alpha}$ $(-iS_\alpha)$ coincides with the the one (double) sided Laplace transform of $h(\cdot)$.

We conclude this chapter with a final remark.

Remark 17 *By Lemma 13, if $\hat{k}(\cdot) \in L_1^{m\times n}(\mathbb{R})$ is of type $-\alpha$ then we can find an ω satisfying $0 > \omega > -\alpha$ such that $k(\cdot) \in L_{1,\omega}^{m\times n}(\mathbb{R}) \subset L_1^{m\times n}(\mathbb{R})$. Thus a complex matrix valued function of type $-\alpha$ can be viewed as the Fourier transform of a stable (not necessarily causal) impulse response, stable being in the sense that the impulse response decays at an "exponential" rate as $|t| \to \infty$. In connection with PSD (power spectral density) functions, the "type $-\alpha$" property is desirable since we would expect a "physical" PSD function to be associated with some second order process having an autocorrelation function decaying on both the left and the right side of the origin. Thus it is natural to consider the class of PSD functions which is a subset of complex square matrix valued functions of type $-\alpha$.*

Chapter 3

Spectral Factorization

3.1 The Paley-Wiener Theorem

An important result on the factorization of positive semidefinite matrix functions is the Paley-Wiener theorem [10][12][7] (R. Paley and N. Wiener actually formulated the scalar version of the theorem, the general multidimensional result we state here is mainly due to Wiener and other co-workers but we shall also refer to the general version as the Paley-Wiener theorem). We start with the following:

Definition 18 $\mathbb{D} = \{s \in \mathbb{C} : |s| < 1\}$

Definition 19 $\mathbb{C}_0 = \{s \in \mathbb{C} : \Re(s) > 0\}$

Definition 20 *The Hardy space* $\mathcal{H}_2^{n \times n}(\mathbb{D})$ *consists of all* $n \times n$ *matrix-valued analytic functions* $f(\cdot)$ *defined on* $\mathbb{D}$ *satisfying*

$$\int_{-\pi}^{\pi} \operatorname{tr}\left(f(re^{i\theta})^* f(re^{i\theta})\right) d\theta < \infty \; \forall r \in [0, 1)$$

and

$$\lim_{r \uparrow 1} \int_{-\pi}^{\pi} \operatorname{tr}\left(f(re^{i\theta})^* f(re^{i\theta})\right) d\theta < \infty$$

$\mathcal{H}_2^{n \times n}(\mathbb{D})$ *is a Hilbert space with the inner product:*

$$< f(\cdot), g(\cdot) >= \int_{-\pi}^{\pi} \operatorname{tr}\left(f(e^{i\theta})^* g(e^{i\theta})\right) d\theta \;\; f, g \in \mathcal{H}_2^{n \times n}(\mathbb{D})$$

Definition 21 *The Hardy space* $\mathcal{H}_2^{n\times n}(\mathbb{C}_0)$ *consists of all* $n \times n$ *matrix-valued analytic functions* $f(\cdot)$ *defined on* $\mathbb{C}_0$ *satisfying*

$$\int_{-\infty}^{\infty} \operatorname{tr}\left(f(\sigma+i\omega)^* f(\sigma+i\omega)\right) d\omega < \infty \ \forall \sigma > 0$$

and

$$\sup_{\sigma>0} \int_{-\infty}^{\infty} \operatorname{tr}\left(f(\sigma+i\omega)^* f(\sigma+i\omega)\right) d\omega < \infty$$

$\mathcal{H}_2^{n\times n}(\mathbb{C}_0)$ *is a Hilbert space with the inner product:*

$$< f(\cdot), g(\cdot) >= \int_{-\infty}^{\infty} \operatorname{tr}\left(f(i\omega)^* g(i\omega)\right) d\omega \ f, g \in \mathcal{H}_2^{n\times n}(\mathbb{C}_0)$$

Theorem 22 (Paley-Wiener) *Let* n *be a positive integer. A positive semidefinite Hermitian matrix valued function* $P(\cdot) \in L_1^{n\times n}([-\pi,\pi))$ *has a factorization of the form*

$$P(\theta) = H(\theta)\,(H(\theta))^* \ -\pi \leq \theta < \pi$$

with $H(\cdot) \in \mathcal{H}_2^{n\times n}(\mathbb{D})$ *if and only if the Paley-Wiener condition* $\ln\det P(\cdot) \in L_1([-\pi,\pi))$*, i.e.*

$$\int_{-\pi}^{\pi} |\ln\det P(\theta)| d\theta < \infty,$$

holds.

We may also restate the theorem in the following form:

Theorem 23 (Paley-Wiener) *Let* n *be a positive integer. A positive semidefinite Hermitian matrix valued function* $P(\cdot) \in L_1^{n\times n}(\mathbb{R})$ *has a factorization of the form*

$$P(\omega) = H(\omega)\,(H(\omega))^* \ -\infty < \omega < \infty$$

with $H(\cdot) \in \mathcal{H}_2^{n\times n}(\mathbb{C}_0)$ *if and only if the Paley-Wiener condition* $\frac{\ln\det P(\cdot)}{1+(\cdot)^2} \in L_1(\mathbb{R})$*, i.e.*

$$\int_{-\infty}^{\infty} \frac{|\ln\det P(\omega)|}{1+\omega^2} d\omega < \infty,$$

holds.

Remark 24 $H(\cdot)$ *is called a spectral factor of* $P(\cdot)$*. Spectral factors are only unique up to multiplication on the right by a constant unitary matrix. Thus, if* $H(\cdot)$ *is a spectral factor, so is* $H(\cdot)U$ *for some constant unitary matrix* U*.*

Remark 25 *Note that* $L_2(0,\infty)$ $(l_2(\mathbb{Z}_+))$ *is isomorphic to* $H_2(\mathbb{C}_0)$ $(H_2(\mathbb{D}))$ *(see [5], Lemma A.6.21), i.e. there is a bijective mapping from* $L_2(0,\infty)$ $(l_2(\mathbb{Z}_+))$ *to* $H_2(\mathbb{C}_0)$ $(H_2(\mathbb{D}))$*. This means for any* $x \in L_2(0,\infty)$ $(l_2(\mathbb{Z}_+))$ *there corresponds a unique* $y \in H_2(\mathbb{C}_0)$ $(H_2(\mathbb{D}))$ *and vice-versa.*

Thus given an arbitrary integrable multidimensional PSD function $P(\cdot)$, the Paley-Wiener theorem tells us precisely whether or not there exists a causal LTI system with an impulse response in $L_2(0,\infty)$ $(l_2(\mathbb{Z}_+))$ that generates $P(\cdot)$. In Wiener's own words [11] there cannot exist a causal LTI system whose frequency response $H(\cdot)$ is zero for any subset of $\mathbb{R}$ $([-\pi,\pi))$ of positive measure, i.e. a causal LTI system can never provide infinite attenuation (attenuation is defined as $|\ln|\det H(\cdot)||$) over any disjoint union of finite intervals.

3.2 Necessity of Spectral Factorization

In deriving a dynamical model for a scalar or multidimensional stochastic process Y whose only known characteristic is its PSD function $P_Y(\cdot)$, spectral factorization is necessary because a spectral factor $H_Y(\cdot)$ is the frequency response of an LTI system which generates the process Y when driven by a white noise process. However, given an arbitrary non-scalar and irrational PSD function $P_Y(\cdot)$ satisfying the Paley-Wiener condition, there is not yet available a general method for analytically deriving a spectral factor and is in fact, quite a difficult affair (e.g. see [7]). However, if $P_Y(\cdot)$ is rational then it is possible to compute a spectral factor after a finite number of iterations and this spectral factor will also be rational [13]. Thus for practical purposes it is easier to first construct a good rational approximant $\hat{P}_Y(\cdot)$ of $P_Y(\cdot)$ and use the LTI system obtained from the rational spectral factor of $\hat{P}_Y(\cdot)$ as an approximate dynamical model for *generating* the stochastic process Y.

In the next chapter we will show one method of realizing rational approximants for the scalar Kolmogorov and von Karman spectra using so-called Pade approximants. Based on these rational approximations we shall construct low order LTI systems as approximate generators for both spectra.

Remark 26 *We will often refer to an LTI system that produces a second order WSS stochastic process* Y *with PSD* $P_Y(\cdot)$ *when driven by white noise as the generator of* Y *or the generator of* $P_Y(\cdot)$*.*

Chapter 4

Rational Approximations

4.1 Overview

In this chapter, rational approximations to two scalar spectral densities that are frequently encountered when dealing with turbulence, i.e. the Kolmogorov and vor Karman spectrum, are presented.

The Kolmogorov spectrum is given by:

$$P_K(\omega) = \frac{\sigma^2}{(1+\omega^2)^{\frac{5}{6}}}$$

where σ is a positive constant. A graph of the Kolmogorov spectrum is given in Fig. 4.1 and Fig. 4.2.

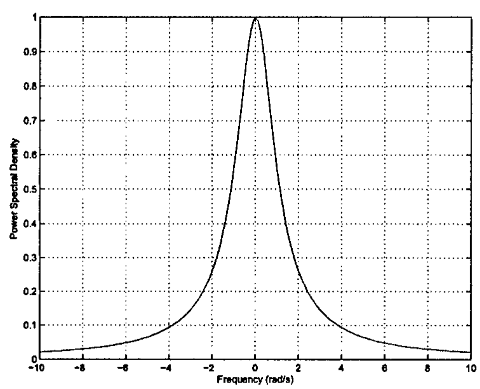

Figure 4.1: The Kolmogorov spectrum

The slightly more complicated von Karman spectrum is given by:

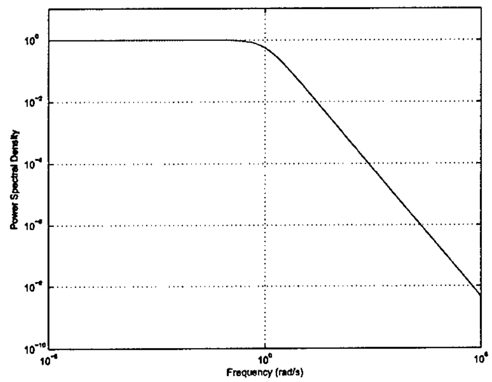

Figure 4.2: The Kolmogorov spectrum (log-log scale)

$$P_{vK}(\omega) = \frac{2k(1 + \frac{8}{3}k^2(1.339)^2\omega^2)}{(1 + k^2(1.339)^2\omega^2)^{\frac{11}{6}}}$$

where k is a positive scale parameter. A graph of the von Karman spectrum is shown in Fig. 4.3 and Fig. 4.4.

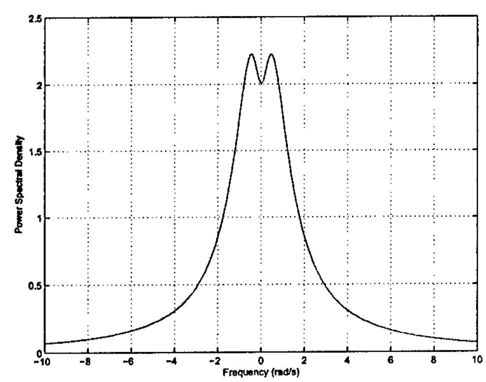

Figure 4.3: The von Karman spectrum

The most important facts to note about both spectra is that they are:

($S1$) Irrational.

($S2$) Symmetric about the vertical axis.

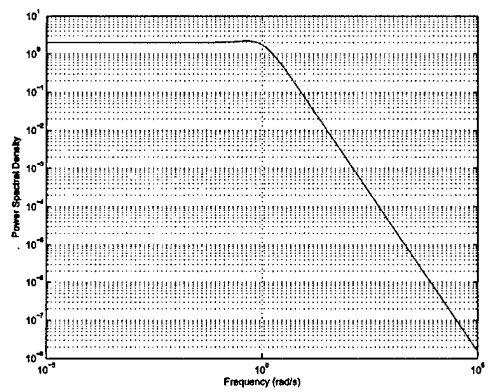

Figure 4.4: The von Karman spectrum (log-log scale)

($S3$) Decaying to zero as $\omega \to \pm\infty$.

($S4$) Infinitely differentiable with respect to ω.

It is also easy to check that the two spectra satisfy the Paley-Wiener condition and thus each has a spectral factor in $H_2(\mathbb{R})$. Furthermore, both spectra have a meromorphic extension to the complex plane. In the remainder of this report we will interpret $P_K(\cdot)$ and $P_{vK}(\cdot)$ dually either as functions defined on the real axis or as functions defined on the complex plane. In particular, $P_K(\omega)$ and $P_{vK}(\omega)$ will be viewed as the restriction of $P_K(z)$ and $P_{vK}(z)$, respectively, to the real line.

4.2 Pade Approximation

Given a scalar complex function f(z) analytic around $z = z_0$ and its Taylor series expansion about z_0

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

we may construct a rational function $\frac{P(z)}{Q(z)}$ with a polynomial numerator $P(z)$ of degree L and a polynomial denominator $Q(z)$ of degree M such that:

$$f(z)Q(z) - P(z) = O(z^{L+M+1})$$

The quotient $\frac{P(z)}{Q(z)}$ having the property above is called a *Pade approximant of $f(z)$ about z_0*. The coefficients of $P(z)$ and $Q(z)$ can be computed by solving a set of linear equations. We will not give a treatment of the theory of Pade approximation of analytic functions, suffice that in this chapter we will see some features of the Pade technique. The reader should refer to an excellent monograph by Baker and Graves-Morris [2] for a thorough discussion on the theoretical aspects of Pade approximation.

Notation 27 *Pade[L/M] is used to denote a Pade approximation with the numerator polynomial having degree L and the denominator polynomial having degree M.*

Remark 28 *It should be observed that all scalar real valued positive semidefinite proper (the degree of the numerator is equal to the degree of the denominator) or strictly proper (the degree of the numerator is less than the degree of the denominator) rational functions satisfy the Paley-Wiener condition. Therefore, any rational Pade approximants having these properties possess spectral factors in* $H_2(\mathbb{R})$.

4.3 Rational Approximation of the Kolmogorov Spectrum

In current practice, the Kolmogorov spectrum is approximated using the so-called Dryden approximation. The Dryden appoximation to the Kolmogorov spectrum is given by:

$$\hat{P}_{D-K}(\omega) = C\frac{1}{1+\omega^2}$$

where the positive constant C is chosen such that

$$C\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}d\omega = \int_{-\infty}^{\infty}P_K(\omega)d\omega$$

A visualization of the fit of the Dryden approximation to the Kolmogorov spectrum when $\sigma = 1$ is shown in Fig. 4.5 and Fig. 4.6. In this case the value of C that was computed (using a numerical integration routine on Matlab®) is $C = 1.1571$.

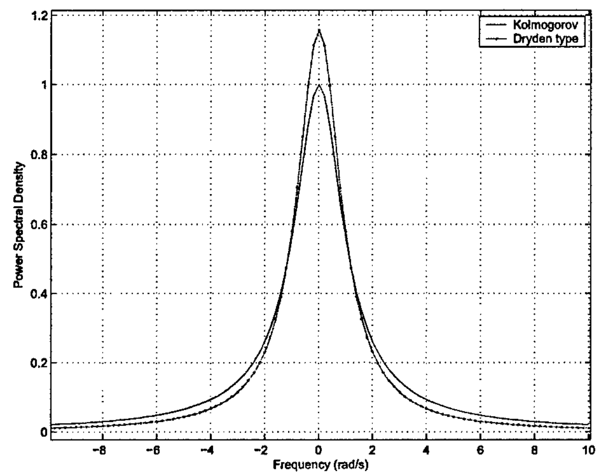

Figure 4.5: The Kolmogorov spectrum and its Dryden approximation

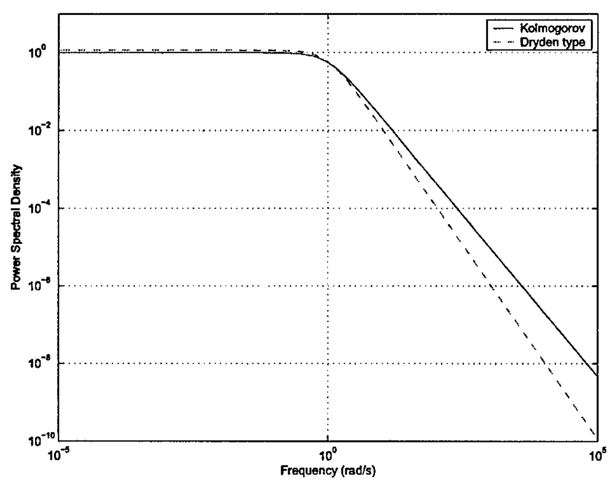

Figure 4.6: The Kolmogorov spectrum and its Dryden approximation (log-log scale)

We may, however, also construct various Pade approximations to the Kolmogorov spectra based on the Taylor expansion of $P_K(\cdot)$ about $z = 0$. Table 4.1 shows the first 25 coefficients of the Taylor series expansion.

n	c_n (n^{th} Taylor series coefficient) $c_n = 0$ for n positive odd
0	1
2	$-\frac{5}{6}$
4	$\frac{55}{72}$
6	$-\frac{935}{1296}$
8	$\frac{21505}{31104}$
10	$-\frac{124729}{186624}$
12	$\frac{4365515}{6718464}$
14	$-\frac{25569445}{40310784}$
16	$\frac{1201763915}{1934917632}$
18	$-\frac{63693487495}{104485552128}$
20	$\frac{751583152441}{1253826625536}$
22	$-\frac{4441173173515}{7522959753216}$
24	$\frac{315323295319565}{541653102231552}$

Table 4.1: Coefficients of the Taylor series expansion of $P_K(\cdot)$ about $z = 0$

The Taylor series coeffients are used to construct various Pade approximations to the Kolmogorov spectrum. These approximations were computed using Maple© version 8.0 and are shown in Table 4.2.

M	Pade[M/M+1] approximant
1	$\frac{1}{1+\frac{5}{6}z^2}$
2	$\frac{1+\frac{1}{12}z^2}{1+\frac{11}{12}z^2}$
3	$\frac{1+\frac{7}{18}z^2}{1+\frac{11}{9}z^2+\frac{55}{216}z^4}$
4	$\frac{1+\frac{7}{12}z^2+\frac{7}{432}z^4}{1+\frac{17}{12}z^2+\frac{187}{432}z^4}$
5	$\frac{1+\frac{13}{15}z^2+\frac{91}{720}z^4}{1+\frac{17}{10}z^2+\frac{187}{240}z^4+\frac{187}{2592}z^6}$
10	$\frac{1+\frac{25}{12}z^2+\frac{475}{324}z^4+\frac{6175}{15552}z^6+\frac{6175}{186624}z^8+\frac{1235}{6718464}z^{10}}{1+\frac{35}{12}z^2+\frac{1015}{324}z^4+\frac{23345}{15552}z^6+\frac{56695}{186624}z^8+\frac{124729}{6718464}z^{10}}$
11	$\frac{1+\frac{155}{66}z^2+\frac{775}{396}z^4+\frac{14725}{21384}z^6+\frac{191425}{2052864}z^8+\frac{38285}{12317184}z^{10}}{1+\frac{35}{11}z^2+\frac{1015}{264}z^4+\frac{23345}{10692}z^6+\frac{396865}{684288}z^8+\frac{11339}{186624}z^{10}+\frac{56695}{40310784}z^{12}}$

Table 4.2: Various Pade approximants of the Kolmogorov spectrum ($\sigma = 1$)

The result of using a Pade[1/2] approximation is shown in Fig. 4.7 and Fig. 4.8. It can be seen visually that at least in the region of the dominant frequency components the Pade[1/2] approximation alone, which is the simplest approximation in the table, already outperforms the Dryden approximation. Fig. 4.9 and Fig. 4.10 show the result of using Pade[11/12], this high order approximation fits the original spectrum quite well over a wide frequency range.

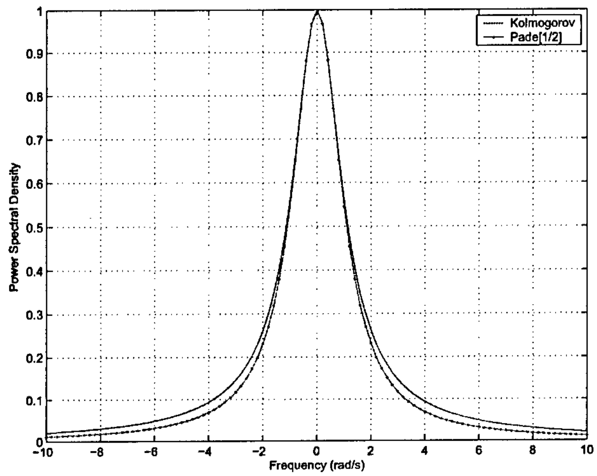

Figure 4.7: The Kolmogorov spectrum and its Pade[1/2] approximation

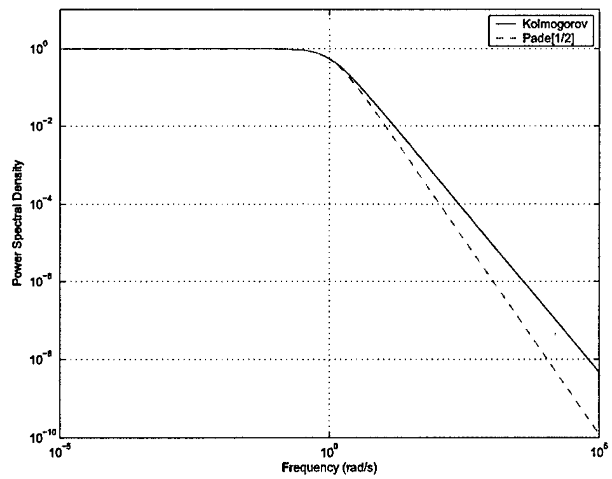

Figure 4.8: The Kolmogorov spectrum and its Pade[1/2] approximation (log-log scale)

From the table, it is seen that for odd M, the $[M/M+1]$ approximants approach zero as $\omega \rightarrow \pm\infty$. Recall that this is one of the properties of the Kolmogorov spectrum. However, for even M, this property does not hold since the order of the numerator and denominator are the same (i.e. the coefficient of the odd powered leading term of the denominator is zero) hence at $\pm\infty$ the approximants do not approach zero (although the values achieved at ∞ become smaller as M goes to ∞ through a sequence of even integers). Further notice how all the approximants surprisingly have the properties S2 and S3 of the original spectrum. As a consequence we find that:

- The numerator and denominator of the Pade approximants contain only even powers of z.
- For even M, the order of the denominator is the same as the order of the numerator because the highest order term in the numerator, which is odd, has a zero coefficient.

We want to use an approximant that behaves as closely as possible to the original spectrum (i.e. at least reproducing the properties S2, S3 and S4) so it is best to choose $[M/M+1]$ approximants with odd M.

In Table. 4.6 the locations of the poles and zeros of various Pade approximants are given. These values can give us some insight into the location and types of poles and zeros of the function that is being approximated over the complex plane (see Section 2.2 of [2]). One can notice a persistent crowding

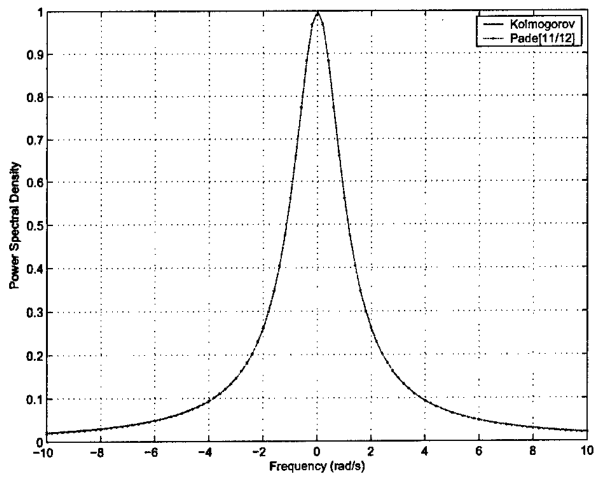

Figure 4.9: The Kolmogorov spectrum and its Pade[11/12] approximation

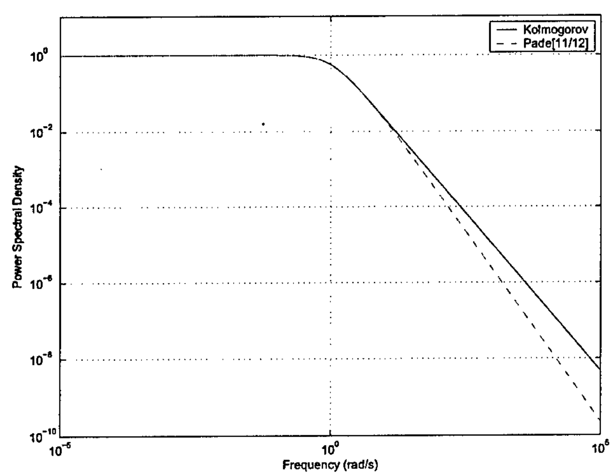

Figure 4.10: The Kolmogorov spectrum and its Pade[11/12] approximation (log-log scale)

of poles and zeros around the points z=-1 and z=1 which suggests that there may be present an essential singularity in the vicinity of each point. Furthermore, some pole-zero pairs are very close indicating that it may be permissible to cancel out each of these pole-zero pairs. However, at this stage we will not be scrutinizing this aspect of the approximation.

[M/M+1]	Zeros	Poles
[1/2]		1.0954i
		1.0954i
[3/4]	1.6036i	1.9375i
	-1.6036i	- 1.9375i
		1.0228i
		- 1.0228i
[5/6]	2.3214i	2.8466i
	-2.3214i	- 2.8466i
	1.2117i	1.2948i
	-1.2117i	- 1.2948i
		1.0101i
		- 1.0101i
[11/12]	4.5468i	5.6241i
	4.5468i	-5.6241i
	2.1563i	2.3544i
	-2.1563i	-2.3544i
	1.4766i	1.5517i
	-1.4766i	-1.5517i
	1.1835i	1.2184i
	-1.1835i	-1.2184i
	1.0469i	1.0625i
	- 1.0469i	-1.0625i
		1.0025i
		-1.0025i

[M/M+1]	Zeros	Poles
[29/30]	11.3012i	14.0127i
	-11.3012i	-14.0127i
	5.2228i	5.7351i
	-5.2228i	-5.7351i
	3.4150i	3.6217i
	-3.4150i	-3.6217i
	2.5569i	2.6668i
	-2.5569i	-2.6668i
	2.0613i	2.1286i
	-2.0613i	-2.1286i
	1.7426i	1.7874i
	-1.7426i	-1.7874i
	1.5236i	1.5550i
	-1.5236i	-1.5550i
	1.3668i	1.3897i
	-1.3668i	-1.3897i
	1.2517i	1.2686i
	-1.2517i	-1.2686i
	1.1663i	1.1788i
	-1.1663i	-1.1788i
	1.1031i	1.1123i
	-1.1031i	-1.1123i
	1.0573i	1.0639i
	- 1.0573i	-1.0639i
	1.0260i	1.0303i
	-1.0260i	-1.0303i
	1.0072i	1.0095i
	-1.0072i	- 1.0096i
		1.0004i
		-1.0004i

Table 4.3: Location of the poles and zeros of various Pade approximants of the Kolmogorov spectrum ($\sigma = 1$)

4.4 Rational Approximation of the von Karman Spectrum

This section is basically a repeat for the von Karman spectrum of what was done in the previous section for the Kolmogorov spectrum. In particular we compute some Pade approximants of the von Karman spectrum. Table 4.4 gives the values of the few coefficients of the Taylor series expansion of $P_{vK}(z)$ about $z = 0$ while in Table 4.5 the [1/2], [3/4], [5/6], [11/12] approximants of $P_{vK}(z)$ are given.

n	c_n (n^{th} Taylor series coefficient) $c_n = 0$ for n positive odd
0	2
2	2.988201668
4	-14.73342618
6	41.58053484
8	-100.0220740
10	222.8836556
12	-474.8485274
14	982.2035130
16	-1989.606379
18	3967.967356
20	-7818.702442
22	15259.17150
24	-29547.89192

Table 4.4: Coefficients of the Taylor series expansion of $P_{vK}(\cdot)$ about $z = 0$

M	Pade[M/M+1] approximant
1	$\frac{2}{1-1.494100834z^2}$
2	$\frac{2+12.84926716z^2}{1+4.930532747z^2}$
3	$\frac{2+9.613206908z^2}{1+3.312502620z^2+2.417500163z^4}$
4	$\frac{2+9.748125582z^2+0.8668030464z^4}{1+3.379961957z^2+2.750110628z^4}$
5	$\frac{2+10.54610211z^2+4.702359769z^4}{1+3.778950221z^2+4.071760295z^4+0.9645541894z^6}$
10	$\frac{2+14.51800241z^2+27.22248206z^4+17.55236429z^6+3.272024554z^8+0.03909183803z^{10}}{1+5.764900372z^2+12.36461167z^4+11.98030516z^6+4.979991582z^8+0.6376420927z^{10}}$
11	$\frac{2+15.37973535z^2+33.13447800z^4+27.08429984z^6+7.990599385z^8+0.5532223591z^{10}}{1+6.195766840z^2+14.67685168z^4+16.46562285z^6+8.713541785z^8+1.834446150z^{10}+0.08586731442z^{12}}$

Table 4.5: Various Pade approximants of the von Karman spectrum for $k = 1$

The same remarks in the previous section for the Pade approximants of the Kolmogorov spectrum goes also for the Pade approximants of the von Karman spectrum, except that in the present case the Pade[1/2] approximant performs very poorly since it has a pole on the real line and possesess none of the properties S2, S3, and S4. Thus the simplest Pade approximant that is useful for the von Karman spectrum is the Pade[3/4] approximant. The corresponding Dryden type approximation is given by:

$$P_{D-vK}(\omega) = \frac{2k(1 + \frac{8}{3}k^2(1.339)^2\omega^2)}{(1 + k^2(1.339)^2\omega^2)^2}$$

where the positive constant C is chosen such that

$$C\int_{-\infty}^{\infty} \frac{2k(1 + \frac{8}{3}k^2(1.339)^2\omega^2)}{(1 + k^2(1.339)^2\omega^2)^2}d\omega = \int_{-\infty}^{\infty} P_{vK}(\omega)d\omega$$

For $k = 1$, $C = 1.2086$. Fig. 4.11 and Fig. 4.12 show the fit of the Dryden type approximation while Fig. 4.13 and Fig. 4.14 show the fit of the Pade[3/4]. Again, visually it is clear that the Pade[3/4] is superior to the Dryden approximation in the main frequency range, although it is achieved at a higher order. The result of using Pade[11/12] is shown in Fig. 4.15 and Fig. 4.16. Once again the latter higher order approximation fits the original spectrum very nicely over a wide frequency range.

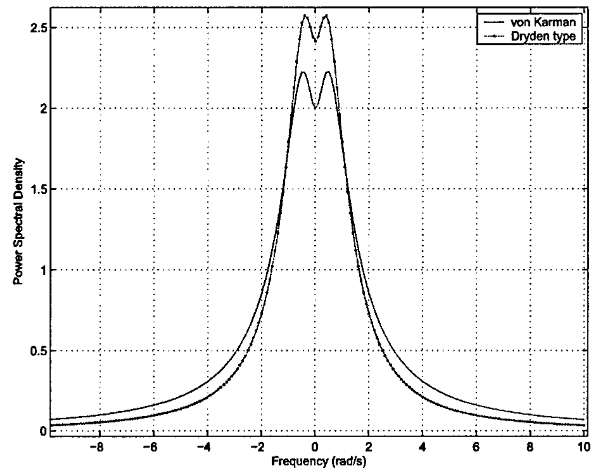

Figure 4.11: The von Karman spectrum and its Dryden approximation

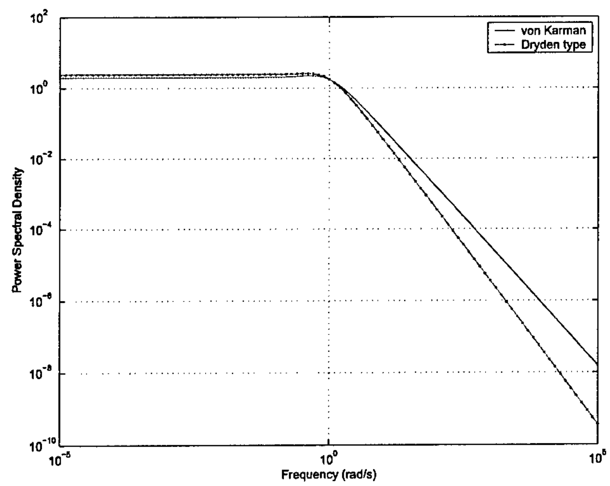

Figure 4.12: The von Karman spectrum and its Dryden approximation (log-log scale)

In Table. 4.6 the locations of the poles and zeros of various Pade approximants are given. There seems to be a crowding of poles and zeros around the pair of points { z=-0.6, z=-0.7 } and { z=0.6, z=0.7 }, suggesting that there may be present an essential singularity in the vicinity of each pair. Again, we can see that some pole-zero pairs are very close indicating that it may be permissible to cancel out these pairs.

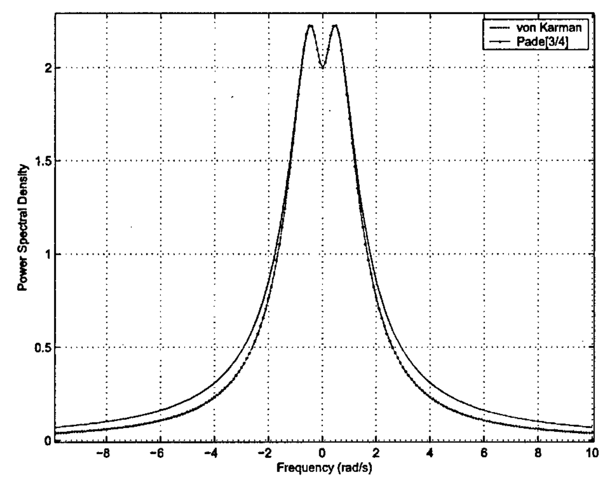

Figure 4.13: The von Karman spectrum and its Pade[3/4] approximation

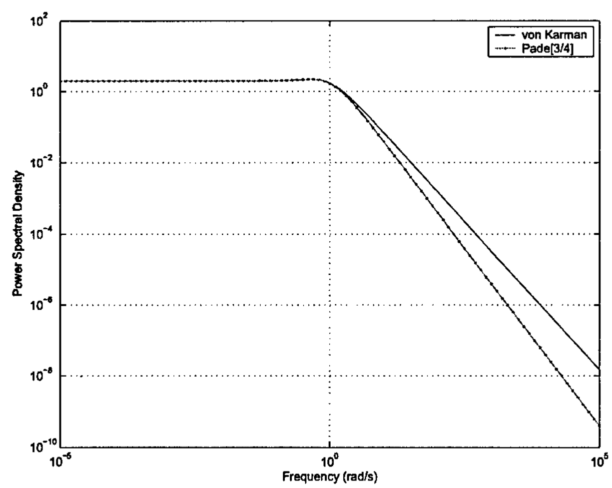

Figure 4.14: The von Karman spectrum and its Pade[3/4] approximation (log-log scale)

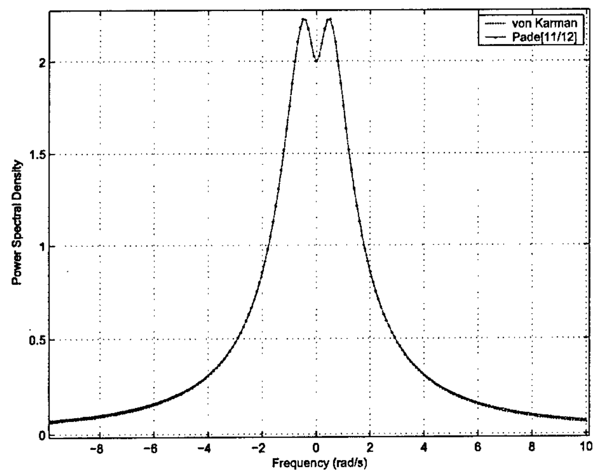

Figure 4.15: The von Karman spectrum and its Pade[11/12] approximation

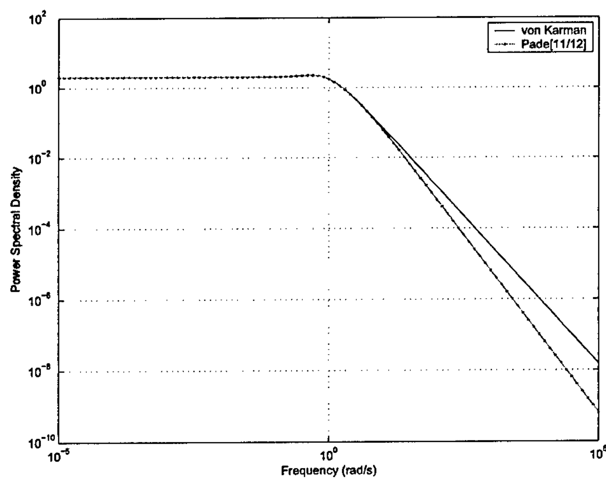

Figure 4.16: The von Karman spectrum and its Pade[11/12] approximation (log-log scale)

[M/M+1]	Zeros	Poles
[1/2]		0.4504i
		-0.4504i
[3/4]	0.4561i	0.9598i
	-0.4561i	-0.9598i
		0.6701i
		-0.6701i
[5/6]	1.4260i	1.7457i
	-1.4260i	-1.7457i
	0.4573i	0.8081i
	-0.4573i	-0.8081i
		0.7218i
		-0.7218i
[11/12]	3.1922i	3.9502i
	-3.1922i	-3.9502i
	1.5097i	1.6490i
	-1.5097i	-1.6490i
	1.0339i	1.0861i
	-1.0339i	-1.0861i
	0.8344i	0.8575i
	-0.8344i	-0.8575i
	0.4573i	0.7582i
	-0.4573i	-0.7582i
		0.7419i
		-0.7419i

[M/M+1]	Zeros	Poles
[29/30]	3.5616i	4.4072i
	-3.5616i	-4.4072i
	1.6932i	1.8465i
	-1.6932i	-1.8465i
	1.1739i	1.2307i
	-1.1739i	-1.2307i
	0.6425 + 0.5681i	-0.6425 + 0.5681i
	0.6425 - 0.5681i	-0.6425 - 0.5681i
	0.7850	-0.7850
	0.6852 + 0.1813i	-0.6852 + 0.1813i
	0.6852 - 0.1813i	-0.6852 - 0.1813i
	0.5351 + 0.3938i	-0.5351 + 0.3938i
	0.5351 - 0.3938i	-0.5351 - 0.3938i
	-0.6425 + 0.5681i	0.6425 + 0.5681i
	-0.6425 - 0.5681i	0.6425 - 0.5681i
	-0.7850	0.7850
	-0.6852 + 0.1813i	0.6852 + 0.1813i
	-0.6852 - 0.1813i	0.6852 - 0.1813i
	-0.5351 + 0.3938i	0.5351 + 0.3938i
	-0.5351 - 0.3938i	0.5351 - 0.3938i
	0.9509i	0.9780i
	-0.9509i	-0.9780i
	0.8371i	0.8515i
	-0.8371i	-0.8515i
	0.7765i	0.7840i
	-0.7765i	-0.7840i
	0.4573i	0.7508i
	-0.4573i	-0.7508i
		0.7451i
		-0.7451i

Table 4.6: Location of the poles and zeros of various Pade approximants of the von Karman spectrum ($k = 1$)

Chapter 5

Approximate Linear State-Space Models

5.1 Overview

In this chapter we will use the approximation constructed in the previous chapter to obtain approximate linear models for the actual (infinite dimensional) linear (time-invariant) system which generated the stochastic process having the Kolmogorov or the von Karman spectrum as its PSD.

5.2 Approximate Generator for the Kolmogorov Power Spectrum

As we had seen in the last chapter, the Pade[1/2] approximant is a better fit to the Kolmogorov power spectrum along the frequency domain than the simple Dryden approximation. The Pade[1/2] approximant is given by:

$$\text{Pade}[1/2] = \frac{1}{1 + \frac{5}{6}\omega^2}$$

However, in systems theory a frequency response is considered as a function defined on the imaginary axis. Thus we will work with the function

$$\begin{aligned} P_{K,[1/2]}(i\omega) &= \frac{1}{1 + \frac{5}{6}(\omega)^2} \\ &= \frac{1}{1 - \frac{5}{6}(i\omega)^2} \end{aligned}$$

instead. However the above frequency domain characteristic are still that of a continuous time system. For practical purposes, one usually uses a digital computer/controller, thus we need to discretize the rational approximant given a sampling time T. Note that choosing an optimal value for T must be done carefully by taking into consideration the profile of $P_K(\cdot)$ and the sampling theory for stochastic signals (e.g. see [3], Chapter 6). Assume that we have chosen an optimal T, then we discretize as follows. First, determine the discrete autocorrelation function $R_{K,[\frac{1}{2}]}(\cdot)$ associated with $P_{K,[1/2]}(\cdot)$ by the relation

$$\begin{aligned} R_{K,[1/2]}(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{K,[1/2]}(v)e^{ivnT}dv \\ &= \sqrt{\frac{3}{10}}e^{-\sqrt{\frac{6}{5}}|n|T} \end{aligned}$$

From $R_{K,[1/2]}(\cdot)$ we can construct the discrete PSD $D_{K,[1/2]}(\cdot)$ on the unit circle defined by:

$$\begin{aligned} D_{K,[1/2]}(e^{i\theta}) &= \sum_{n=-\infty}^{\infty} R_{K,[1/2]}(n)e^{in\theta} \quad \text{for} \quad -\pi \le \theta < \pi \\ &= \sqrt{\frac{3}{10}}\left(\sum_{n=-\infty}^{-1} e^{\sqrt{\frac{6}{5}}nT}e^{-in\theta} + 1 + \sum_{1}^{\infty} e^{-\sqrt{\frac{6}{5}}nT}e^{-in\theta}\right) \\ &= \sqrt{\frac{3}{10}}\left(\frac{e^{i\theta}}{e^{\sqrt{\frac{6}{5}}T} - e^{i\theta}} + 1 + \frac{e^{-\sqrt{\frac{6}{5}}T}}{e^{i\theta} - e^{-\sqrt{\frac{6}{5}}T}}\right) \\ &= \sqrt{\frac{3}{10}}\frac{(e^{\sqrt{\frac{6}{5}}T} - e^{-\sqrt{\frac{6}{5}}T})e^{i\theta}}{(e^{\sqrt{\frac{6}{5}}T} - e^{i\theta})(e^{i\theta} - e^{-\sqrt{\frac{6}{5}}T})} \end{aligned}$$

We may then easily factorize $D_{K,[1/2]}(e^{i\cdot})$ as:

$$D_{K,[1/2]}(e^{i\theta}) = \left(\sqrt{\sqrt{\frac{3}{10}}(1 - e^{-2\sqrt{\frac{6}{5}}T})}\frac{1}{1 - e^{i\theta}e^{-\sqrt{\frac{6}{5}}T}}\right)\left(\sqrt{\sqrt{\frac{3}{10}}(1 - e^{-2\sqrt{\frac{6}{5}}T})}\frac{1}{1 - e^{-i\theta}e^{-\sqrt{\frac{6}{5}}T}}\right)$$

Finally let us now define

$$H_{K,[1/2]}(e^{i\theta}) = \sqrt{\sqrt{\frac{3}{10}}(1 - e^{-2\sqrt{\frac{6}{5}}T})}\frac{1}{1 - e^{-i\theta}e^{-\sqrt{\frac{6}{5}}T}}$$

then $|H_{K,[1/2]}(e^{i\theta})|^2 = D_{K,[1/2]}(e^{i\theta})$ and $H_{K,[1/2]}(e^{i\theta})$ is the frequency response of a stable causal discrete linear system. It is clear that if a (discrete-time) white noise process is passed through a filter with frequency response $H_{K,[1/2]}(e^{i\theta})$ then the output of the filter will be a second order (stationary process) having the power spectrum $D_{K,[1/2]}(e^{i\theta})$. This is the approximate filter that we are looking for.

A minimal degree state space representation for a discrete LTI system having the frequency response $H_{K,[1/2]}(e^{i\cdot})$ is easily found to be:

$$\begin{aligned} x(k+1) &= e^{-\sqrt{\frac{6}{5}}T}x(k)+u(k) \\ y(k) &= \sqrt{\sqrt{\frac{3}{10}}(1-e^{-2\sqrt{\frac{6}{5}}T})}x(k) \end{aligned}$$

where $k \geq 0$. We assume that the input signal $\{u(0), u(1), ..., u(n)\}$ is a standard white noise process with unit variance, x(0) is uncorrelated with $\{u(k)\}_{k\in\mathbb{Z}_+}$ and we let the system run long enough so that it is virtually in statistical steady state.

5.3 Approximate Generator for the von Karman Power Spectrum

In this section we will repeat the procedure of the last section to derive an approximate generator for a stochastic process having the von Karman spectrum as it PSD. The main difference here is that the simplest approximation we can take is the Pade[3/4] approximation (see the previous chapter). Thus in the end we will have a higher order model with two states rather than the single state model that we derived for the Kolmogorov spectrum. Since the procedure is identical we will reuse the notation of the last section and will omit most details of the derivation.

First of all we have that:

$$\begin{aligned} R_{vK,[3/4]}(n) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P_{vK,[1/2]}(v)e^{ivnT}dv \\ &= 7.564513400e^{-0.921168420|n|T} - 2.997334813e^{-0.670111735|n|T} \end{aligned}$$

from which we find

$$\begin{aligned} D_{vK,[3/4]}(e^{i\theta}) &= 7.564513400\frac{0.841553271}{(1-0.398053675e^{i\theta})(1-0.398053675e^{-i\theta})} \\ &\quad -2.997334813\frac{0.738212839}{(1-0.511651405e^{i\theta})(1-0.511651405e^{-i\theta})} \end{aligned}$$

To avoid clutter and since our purpose here is for demonstration let us take $T=1$. For this particular value of T we have:

$$\begin{aligned} D_{vK,[3/4]}(e^{i\theta}) &= \frac{e^{-i\theta}(-2.37638081521232e^{i2\theta}+4.57350384251358e^{i\theta}-2.37638081521232)}{(1-\alpha e^{i\theta})(1-\alpha e^{-i\theta})(1-\beta e^{i\theta})(1-\beta e^{-i\theta})} \\ &= \frac{e^{-i\theta}(e^{i\theta}-\lambda^{-1})(e^{i\theta}-\lambda)}{(1-\alpha e^{i\theta})(1-\alpha e^{-i\theta})(1-\beta e^{i\theta})(1-\beta e^{-i\theta})} \end{aligned}$$

where

$$\begin{aligned}\alpha &= 0.398053675\\ \beta &= 0.511651405\\ \lambda &= 1.720133060\end{aligned}$$

Most of the complicated algebraic manipulations above were done with Maple© version 8. Finally we may factor $D_{vK,[3/4]}(e^{i\cdot})$ as:

$$D_{vK,[3/4]}(e^{i\theta}) = \left(\frac{1-\lambda^{-1}e^{-i\theta}}{(1-\alpha e^{-i\theta})(1-\beta e^{-i\theta})}\right)\left(\frac{e^{i\theta}-\lambda}{(1-\alpha e^{i\theta})(1-\beta e^{i\theta})}\right)$$

We now define

$$H_{vK,[3/4]}(e^{i\theta}) = \frac{1-\lambda e^{-i\theta}}{(1-\alpha e^{-i\theta})(1-\beta e^{-i\theta})}$$

which is basically the frequency response of a stable causal system generating the discrete power spectrum $D_{vK,[3/4]}(e^{i\cdot})$. A minimal degree state space representation of $H_{vK,[3/4]}(e^{i\cdot})$ is given by:

$$\begin{aligned}x(k+1) &= \begin{bmatrix}\alpha+\beta & \frac{\alpha\beta}{\alpha+\beta}\\ \alpha+\beta & 0\end{bmatrix}x(k) + \begin{bmatrix}1-\frac{\lambda}{\alpha+\beta}\\ 1\end{bmatrix}u(k)\\ y(k) &= \begin{bmatrix}\alpha+\beta & 0\end{bmatrix}x(k) + u(k)\end{aligned}$$

5.4 Stochastic Control Using Approximate generators

When one is given the task of designing a controller for attenuating/cancelling a stochastic disturbance, one usually needs a dynamical model for the "physical process" which is generating the disturbance. For a disturbance which is assumed to be second order wide sense stationary (WSS), the standard model is that of a causal LTI system driven by a white-noise process [13][1]. Once a dynamical model has been determined then one can proceed to design a controller based on the theory of stochastic optimal control (for finite dimensional systems, see [1]). However, if the dynamical model is infinite dimensional then the optimal contoller will also be infinite-dimensional in general. This situation is problematic from an applications point of view since a infinite dimensional controllers does not lend itself to instrumentation. In this situation, the alternative is to approximate infinite dimensional dynamical model with one which is finite dimensional (i.e. substitute an approximate finite dimensional model for the actual infinite dimensional model) and then design a finite-dimensional optimal controller based on the standard theory as described in [1]. However, in the problem that is being considered in this report, the scenario is slightly different. The characteristic of the disturbance that is known is its PSD function. This does not immediately provide us with a dynamical model, but must be obtained through spectral factorization (see Chapter 3). From this point one may proceed in two ways:

1. Explicitly determine a spectral factor and then construct a rational approximation to it.

2. Directly construct a rational approximation for the PSD and then utilize a spectral factor of the approximant as a finite dimensional approximation of a spectral factor of the true PSD.

In this report we have opted to take the second approach, and the results have been shown in the previous two sections. However, it is difficult to verify just how a control system based on the approximate generators will perform without real experimental data. Actually generating a stochastic process Y with PSD function which is that of the Kolmogorov or von Karman spectra on a digital computer is difficult, if not impossible, due to the infinite-dimensionality of its true generator.

Remark 29 *We noticed that there has recently been a similar effort towards constructing rational approximants of the von Karman spectrum in [6]. In that paper a rather complicated "diffusive representation" approach was used. It seems that our approach is better than that approach in terms of simplicity, the insights it gives into the behaviour of the transfer function throughout the complex plane and the possibility of constructing good low order approximants.*

Chapter 6

Conclusions

In this report we have constructed Pade approximants for the Kolmogorov and von Karman spectra. These approximants are able to give a good fit of the spectra, even the lower order ones. In particular, the Pade[1/2] approximant for the Kolmogorov spectrum performs better than the typical Dryden approximation that is used in practice today. In this report we have used [M/M+1] Pade approximants with odd M for the purpose of preserving some important properties of the original spectra in their Pade approximants. Based on the Pade approximants, approximate finite-dimensional LTI generators of the spectra can be obtained in lieu of the actual generators of the Kolmogorov and von Karman spectra which are necessarily infinite-dimensional. The finite-dimensional LTI generators can in turn be used as a substitute for the actual infinite dimensional generators for the control of stochastic disturbances having either the Kolmogorov or the von Karman spectrum as its PSD function (e.g. wind gust). In order to assess how these approximate generators will affect a real control system, data samples coming from an actual stochastic process characterized by the Kolmogorov or von Karman spectrum will be required.

Bibliography

[1] A. Bagchi. *Optimal Control of Stochastic Systems.* Prentice-Hall, 1993.

[2] G.A. Baker and P. Graves-Morris. *Pade Approximants.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, London, 2 edition, 1996.

[3] A.V. Balakrishnan. *Introduction to Random Processes in Engineering.* Wiley-Interscience. John Wiley & Sons, 1995.

[4] H. Bart, I. Gohberg, and M.A. Kaashoek. Wiener-Hopf equations with symbols analytic on a strip. In H. Bart and M.A. Kaashoek, editors, *Constructive Methods of Wiener-Hopf Factorization*, pages 39–74. Birkhäuser, Basel, 1986.

[5] R. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory.* Text in Applied Mathematics. Springer-Verlag, New York, 1995.

[6] N. Imbert, Ph. Mouyon, and G. Montseny. State representation and simulation of 2D turbulent wind. In *Proceedings of the AIAA Conference on Modeling and Simulation Technologies (AIAA-MST) 2000.* American Institute of Aeronautics and Astronautics, 2000.

[7] G. Janashia and E. Lagvilava. A method of approximate factorization of positive definite matrix functions. *Studia Mathematica*, 137(1):93–100, 1999.

[8] H.I. Nurdin and A. Bagchi. Overview of control problems in adaptive optics. Interim Report, EOARD Project "Filtering and control of systems with nonrational signals spectra", March 2003.

[9] E.T. Whittaker and G.N. Watson. *A Course of Modern Analysis.* Cambridge University Press, London, 1962.

[10] N. Wiener. On the factorization of matrices. *Commentarii Mathematici Helvetici*, 29:97–111, 1955.

[11] N. Wiener. *Extrapolation, Interpolation and Smoothing of Stationary Time Series: With Engineering Applications.* Technology Press Books in Engineering and Science. Technology Press MIT, Cambridge, MA, 1960.

[12] N. Wiener and P. Masani. The prediction theory of stochastic processes I. *Acta Mathematica*, 98:111–150, 1957.

[13] E. Wong and B. Hajek. *Stochastic Processes in Engineering Systems.* Springer-Verlag, New York, 1985.