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LINEAR FEEDBACK CONTROL AND NUMERICAL APPROXIMATION FOR A SYSTEM GOVERNED BY THE TWO-DIMENSIONAL BURGERS' EQUATION

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This paper is concerned with developing distributed parameter control laws for the governing equations of fluid flow. This paper illustrates the efficiency of these control laws on a simpler problem.

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Linear Feedback Control and Numerical Approximation for a System Governed by the Two-Dimensional Burgers' Equation

R. Chris Camphouse, James H. Myatt

Abstract

In this paper, we consider the problem of controlling a system governed by a two-dimensional nonlinear partial differential equation. Motivation for the problem is the development of control methodologies for fluid flow, where the dynamics of the system are governed by the nonlinear Navier-Stokes equations. A control problem for the two-dimensional Burgers' equation is considered. An initial boundary value problem is formulated to model a right-traveling shock over an obstacle. We focus on implementing a feedback control via Dirichlet boundary control on the obstacle. Numerical approximations are developed to numerically simulate solutions of the problem with and without control. Numerical examples are presented to illustrate the efficacy, as well as the shortcomings, of the control method. Additionally, the influence of boundary condition on the functional gains, and the resulting controls, will be demonstrated through numerical examples. Avenues of future work are presented.

Introduction

One of the commonly studied problems of flow control is the manipulation of flow over an obstacle. Typical control objectives include velocity tracking, drag reduction, lift increase, and separation postponement [13, 16]. Due to the complexity of the Navier-Stokes equations, control laws are often determined on a system-by-system basis. Experimental data are taken for a particular system, and a model of the system is developed from those data. Control methods are then developed for the data-induced model. A major drawback of this approach is that control methods developed for one particular configuration and flow condition may not directly apply to a different configuration and flow condition. In addition, the effectiveness of the control developed in this way depends on the accuracy and reliability of the experimental data. By their very nature, experimental data can only provide a sampling of the flow behavior for the sampled conditions. As a result, dynamics that the control needs to suppress or enhance may be missed in the data collection process. The result is likely to be a control that is not as effective as was desired, as well as a control method that doesn't generalize to different conditions.

Developing control methods directly from a partial differential equation model describing flow behavior has many advantages. It is widely accepted that the Navier-Stokes equations provide a good description of fluid behavior, even at high Reynolds number. As a result, control techniques developed from the Navier-Stokes equations have the possibility of being relevant to a variety of flow conditions and configurations. In addition, general optimality results can often be obtained when formulating controls directly from governing equations, rather than equations obtained via sets of data.

There are many difficulties inherent in developing feedback control laws for the Navier-Stokes equations. One difficulty is due to the nonlinearities involved. There is an extensive number of control techniques available for linear distributed parameter systems. However, the theory needed to control highly nonlinear problems is far from complete. Another difficulty is that numerical simulation of fluid flows can be very expensive due to the large number of states needed to resolve a given flow field.

The scalar Burgers' equation, which has a nonlinearity similar to that in Navier-Stokes, is often used for the development of control methods relevant to flow control [7, 8, 11]. These developments are often done for the one-dimensional case. Optimal actuator and sensor placement, as well as the control objective, are more complicated issues to consider in the case of higher dimensions. In this paper, we implement a feedback control for a system whose dynamics are governed by the two-dimensional Burgers' equation over a geometry similar to those of interest in flow control problems involving an obstacle. Applying these control techniques to an aerodynamic flow control problem is the subject of a future paper.

Problem Description

In this section, we develop a partial differential equation model for the system under consideration. To this end, let $\Omega_1 \subseteq \mathbb{R}^2$ be the open rectangle given by $(a, b) \times (c, d)$ where $a, b, c, d \in \mathbb{R}$. Let $\Omega_2 \subseteq \mathbb{R}^2$ be the rectangle given by $[a_1, a_2] \times [b_1, b_2]$ where $a < a_1 < a_2 < b$ and $c < b_1 < b_2 < d$, i.e., $\Omega_2 \subset \Omega_1$. The problem domain, Ω, is given by $\Omega = \Omega_1 \setminus \Omega_2$. In this configuration, Ω_2 is the obstacle on which we implement Dirichlet boundary control.

The dynamics of the system are given by the two-dimensional Burgers' equation

$$\frac{\partial}{\partial t}w(t,x,y) + K_1\frac{\partial}{\partial x}\left(\frac{1}{2}w(t,x,y)^2\right) + K_2\frac{\partial}{\partial y}\left(\frac{1}{2}w(t,x,y)^2\right) = \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}w(t,x,y) + \frac{\partial^2}{\partial y^2}w(t,x,y)\right) \quad (1)$$

for $t > 0$ and $(x, y) \in \Omega$. Behaviors described by Burgers' equation include shock formation, shock propagation, and rarefaction wave formation. In (1), K_1 and K_2 are constants used to scale the nonlinear terms. The quantity Re is a nonnegative constant, and is analogous to the Reynolds number in the Navier-Stokes equations.

In order to fully specify the model, we need to specify conditions on $\partial\Omega_1$ and $\partial\Omega_2$, as well as an initial condition. As we are interested in implementing control on $\partial\Omega_2$, denote the sides of $\partial\Omega_2$ as

$$\Gamma_1 = \{(x, b_1) \mid a_1 \le x \le a_2\}, \qquad \Gamma_2 = \{(a_1, y) \mid b_1 \le y \le b_2\}, \quad (2)$$

$$\Gamma_3 = \{(x, b_2) \mid a_1 \le x \le a_2\}, \qquad \Gamma_4 = \{(a_2, y) \mid b_1 \le y \le b_2\}.$$

For simplicity, we assume that the controls on $\partial\Omega_2$ are separable, i.e., they are the product of a function of time and a function of the spatial variables. With this assumption, we specify conditions on $\partial\Omega_2$ of the form

$$w(t,\Gamma_1) = \sum_{i=1}^{C_1} u_{1,i}(t)\phi_{1,i}(x), \qquad w(t,\Gamma_2) = \sum_{i=1}^{C_2} u_{2,i}(t)\phi_{2,i}(y), \quad (3)$$

$$w(t,\Gamma_3) = \sum_{i=1}^{C_3} u_{3,i}(t)\phi_{3,i}(x), \qquad w(t,\Gamma_4) = \sum_{i=1}^{C_4} u_{4,i}(t)\phi_{4,i}(y).$$

In (3), C_j is the number of controls on Γ_j. The control $u_{j,i}$ is the i-th control on Γ_j. The function $\phi_{j,i}(\cdot)$ is a function describing the influence of the i-th control on Γ_j.

On $\partial\Omega_1$, we specify Dirichlet and Neumann conditions of the form

$$w(t,a,y) = f(y), \qquad \frac{\partial}{\partial x}w(t,b,y) = 0, \quad (4)$$

$$w(t,x,c) = 0, \qquad w(t,x,d) = 0.$$

In (4), $f(y)$ is an inlet condition on the left, and is analogous to the inflow condition specified in many channel flow problems. The Neumann condition is an exit condition similar to those specified in fluid problems where the flow leaves the domain. The remaining Dirichlet conditions are analogous to the no-slip, no penetration conditions specified in channel flow configurations.

To complete the model, we specify an initial condition of the form

$$w(0,x,y) = w_0(x,y) \in L^2(\Omega). \quad (5)$$

The Control Problem

To facilitate discussion of the control problem, the initial boundary value problem given by (1), (3), (4), and (5) is linearized about $w \equiv 0$. The result is written in abstract form. For the sake of simplicity, we assume that the inlet condition $f(y)$ satisfies $f(y) \equiv 0$ in the following discussion of the abstract formulation.

Denote the total number of controls as N_C, i.e. $N_C = C_1 + C_2 + C_3 + C_4$. Specify Γ_N according to $\Gamma_N = \{(b, y) : c \le y \le d\}$, the portion of $\partial\Omega_1$ on which the Neumann outflow condition is specified. Finally, let the space V be defined according to

$$V = H^1_D(\Omega) \quad (6)$$

$$= \left\{v \in H^1(\Omega) : v\mid_{\partial\Omega_2} = 0 = v\mid_{\partial\Omega_1\setminus\Gamma_N} \text{ and } v_x(\Gamma_N) = 0\right\}. \quad (7)$$

Define $a(\cdot,\cdot)$ to be the symmetric bilinear form $a(z,v) = -\frac{1}{Re}\int_\Omega \nabla z \cdot \nabla v d\boldsymbol{x}$. We define the operator $A_0 : V \to V'$ according to $[A_0 z](v) = a(z,v)$.

Let $W = H^2(\Omega) \cap H^1_D(\Omega)$ and define $A_{-1} : W \to H$, where H is the state space, to be the self-adjoint restriction operator $[A_{-1}] w(\cdot) = w_{xx}(\cdot) + w_{yy}(\cdot)$. By defining $\|z(\cdot)\|_W$ as the graph norm

$$\|z(\cdot)\|_W = \| [A_{-1}] z(\cdot)\|_H + \|z(\cdot)\|_H, \tag{8}$$

it follows that the injections $W \subset V \subset H = H' \subset V' \subset W'$ are all continuous and dense.

We now lift the operator $A_0 : V \to V'$ to an operator $A_1 : H \to W'$. Formally, we perform an integration by parts and define $A_1 : H \to W'$ according to

$$[A_1 z](w) = \int_\Omega z(x,y) \Delta w(x,y) d\boldsymbol{x} \tag{9}$$

for all $w \in W'$.

Utilizing the Dirichlet map $D : \mathbb{R}^{N_C} \to L^2(\Omega)$ as discussed in [19], we define the operator $B : \mathbb{R}^{N_C} \to W'$ according to

$$B = -A_1 D. \tag{10}$$

The linearized system is then formulated in W' as

$$\dot{w}(t) = [A_1] w(t) + Bu(t) \tag{11}$$

$$w(0) = w_0. \tag{12}$$

We are now in a position to present the control strategy utilized for the system given by (1), (3), (4), and (5). We construct a linear feedback control law from the system given in (11)-(12).

For $\alpha \geq 0$, define the cost functional

$$J_\alpha(w_0, u) = \int_0^\infty \{\langle Qw, w\rangle_H + \langle Ru, u\rangle_U\} e^{2\alpha t} dt, \tag{13}$$

where Q is a self-adjoint state weight operator satisfying $Q \geq 0$. R is a self-adjoint control weight operator satisfying $R > 0$. Discussion and theoretical results for control problems with the cost functional given in (13) can be found in [7, 8].

In this work, the operator Q in (13) is specified to place a larger weight over a particular region of Ω. Define $\Omega_Q \subseteq \Omega$ to be the region in which control of the state is most important. Specify $q(\cdot,\cdot) : \Omega \to \mathbb{R}$ according to

$$q(x,y) = \begin{cases} q_1, & (x,y) \in \Omega \setminus \Omega_Q, \\ q_2, & (x,y) \in \Omega_Q, \end{cases} \tag{14}$$

where q_1, q_2 are positive constants. By specifying $q_1 \ll q_2$ and defining Q in accordance with (14), states in Ω_Q are weighted more heavily in the cost functional.

In a similar fashion, the control weight operator R allows for larger weights on particular controls. Define $r(u)$ according to

$$r(u) = \begin{cases} r_1, & u \textit{ a control on } \Gamma_1, \\ r_2, & u \textit{ a control on } \Gamma_2, \\ r_3, & u \textit{ a control on } \Gamma_3, \\ r_4, & u \textit{ a control on } \Gamma_4, \end{cases} \tag{15}$$

where r_1, r_2, r_3, and r_4 are positive constants. By defining R in accordance with (15), controls on different regions of $\partial\Omega_2$ can be weighted differently in the cost functional.

Now that Q and R are specified, the optimal control problem we consider is to minimize (13) subject to the constraint

$$\frac{\partial}{\partial t} w = Aw + Bu, \tag{16}$$

$$w(0) = w_0, \tag{17}$$

where A and B are the operators from (11). As can be seen, we find the feedback control law from the linearization of the nonlinear problem.

As shown in [7, 8], for an α-controllable system, the α-LQR problem has a unique solution of the form

$$u_{opt} = -Kw \tag{18}$$

$$= -R^{-1}B^*Pw, \tag{19}$$

where P is the unique, symmetric, non-negative solution of the algebraic Ricatti equation

$$(A+\alpha)^*P + P(A+\alpha) - PBR^{-1}B^*P + Q = 0. \tag{20}$$

It can be shown that the action of the gain operator K in (18) has an integral representation of the form

$$Kw(t,x,y) = \int_\Omega k(x,y)w(t,x,y)dxdy, \tag{21}$$

where the functional gain $k(\cdot,\cdot) \in L^2(\Omega)$ [9]. The functional gains indicate how much information each state contributes to the control and provide information about optimal sensor placement.

Once the gain operator K is obtained, the linear control law is placed into the nonlinear system according to

$$\frac{\partial}{\partial t}w = (A - BK)w + G(w) + F, \tag{22}$$

$$w(0) = w_0, \tag{23}$$

where $G(w)$ is a nonlinear operator resulting from the convective terms in (1) and F is a forcing term resulting from the inlet condition $f(y)$.

Numerical Implementation

We utilize a finite difference approximation to numerically solve the initial boundary value problem with and without control. In the work presented here, we specify a uniform grid on Ω with step-size h.

For the second order derivatives in the x and y directions, we implement the standard second order approximations

$$\frac{\partial^2}{\partial x^2}w(x_i,y_j) \simeq \frac{1}{h^2}\left(w_{i+1,j} - 2w_{i,j} + w_{i-1,j}\right), \tag{24}$$

$$\frac{\partial^2}{\partial y^2}w(x_i,y_j) \simeq \frac{1}{h^2}\left(w_{i,j+1} - 2w_{i,j} + w_{i,j-1}\right). \tag{25}$$

Extra care is needed when discretizing the nonlinear terms. At high Re, the convective terms dominate the dynamics of the system and spurious oscillations around the shock occur if an inappropriate numerical scheme is chosen. As discussed in [14], we utilize a mixture of central differences and a donor cell discretization for the convective terms. These approximations are of the form

$$\begin{aligned}\frac{\partial}{\partial x}w^2(x_i,y_j) \simeq{}& \frac{1}{4h}[(w_{i,j}+w_{i+1,j})^2 - (w_{i-1,j}+w_{i,j})^2] \\ &+ \frac{\gamma}{4h}\left[|w_{i,j}+w_{i+1,j}|\,(w_{i,j}-w_{i+1,j}) - |w_{i-1,j}+w_{i,j}|\,(w_{i-1,j}-w_{i,j})\right], \text{ and}\end{aligned} \tag{26}$$

$$\begin{aligned}\frac{\partial}{\partial y}w^2(x_i,y_j) \simeq{}& \frac{1}{4h}[(w_{i,j}+w_{i,j+1})^2 - (w_{i,j-1}+w_{i,j})^2] \\ &+ \frac{\gamma}{4h}\left[|w_{i,j}+w_{i,j+1}|\,(w_{i,j}-w_{i,j+1}) - |w_{i,j-1}+w_{i,j}|\,(w_{i,j-1}-w_{i,j})\right],\end{aligned} \tag{27}$$

where $\gamma \in [0, 1]$.

After applying the finite difference approximations given by (24)-(27) and incorporating the conditions on $\partial\Omega_1$ and $\partial\Omega_2$, we obtain a semi-discrete approximation of the form

$$\frac{\partial}{\partial t} w^N = A^N w^N + B^N u + G^N\left(w^N\right) + F^N, \tag{28}$$

$$w^N(0) = w_0^N, \tag{29}$$

where we have utilized superscript N to indicate that we have a finite-dimensional approximation to the infinite-dimensional distributed parameter system.

The finite-dimensional α-LQR problem becomes

$$\min_u \int_0^\infty \left[\left(w^N\right)^T Q\left(w^N\right) + u^T R u\right] e^{2\alpha t} dt \tag{30}$$

$$\text{subject to } \frac{\partial}{\partial t} w^N = A^N w^N + B^N u, \tag{31}$$

$$w^N(0) = w_0^N. \tag{32}$$

In the finite-dimensional case, Q is a diagonal, symmetric, positive semi-definite matrix consisting of state weights. R is a diagonal, symmetric, positive definite matrix of control weights.

Solving the finite-dimensional α-LQR problem yields a gain matrix K^N, resulting in an optimal control of the form $u_{opt} = -K^N w^N$. The control is placed into the finite-dimensional system according to

$$\frac{\partial}{\partial t} w^N = \left(A^N - B^N K^N\right) w^N + G^N\left(w^N\right) + F^N, \tag{33}$$

$$w^N(0) = w_0^N. \tag{34}$$

The system (33)-(34) is solved via a 4th order Runge-Kutta method.

Numerical Examples

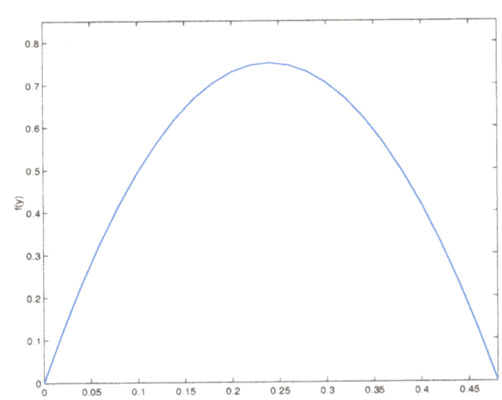

Figure 1: The inlet condition $f(y)$.

We now illustrate the effectiveness of the control strategy. First, we illustrate the system behavior without control by specifying that the obstacle boundary, $\partial\Omega_2$, be held at zero as time evolves. We then implement the control strategy, and present results, for different values of α in the cost functional. We conclude this section with a discussion of the influence of the outflow boundary condition on the functional gains.

In all examples presented, the regions Ω_1 and Ω_2 are specified as $\Omega_1 = (0, 1.5) \times (0, .48)$ and $\Omega_2 = [.16, .24] \times [.12, .32]$. The step-size used in the construction of the uniform grid is $h = .02$. The inlet condition, $f(y)$, is defined as in Figure 1.

The values we specify for K_1 and K_2 in the governing equation are $K_1 = 1$ and $K_2 = 0$. By specifying K_1 and K_2 in this way, the inlet condition will result in solutions which propagate from left to right. The value of γ that we specify in the discretization of the nonlinear terms is $\gamma = 1$. In all results presented, the initial condition is defined by $w_0(x, y) \equiv 0$.

Example 1

For the first problem, we illustrate the behavior of the system in the absence of control. This will provide a baseline against which to judge the effectiveness of the control in the later examples. In this example, we set $Re = 500$ and simulate the solution to time $T = 20$.

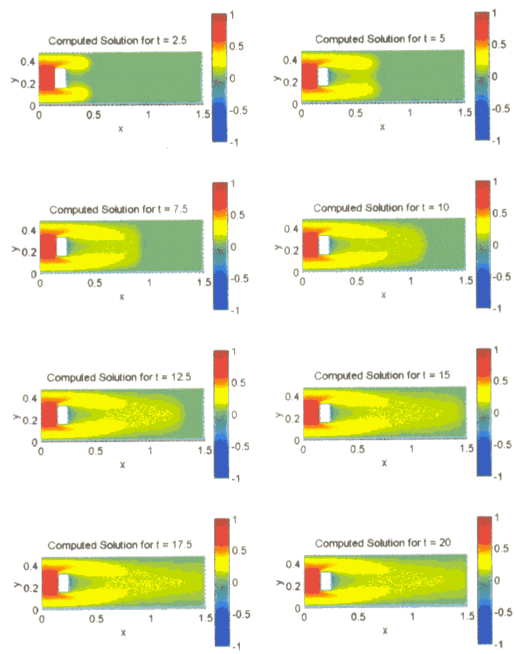

Figure 2: The solution in the absence of control.

As can be seen in Figure 2, the specification of the inlet condition $f(y)$ results in a solution propagating from left to right. In particular, the solution is a traveling shock. The shock encounters the obstacle, and a portion of it propagates above and below. As time evolves, solution values increase downstream of the obstacle until a steady-state solution is reached.

Example 2

We now implement boundary control on Ω_2. In particular, we specify that there are two controls each on the front and back of the obstacle. Similarly, we specify that there is one control each on the bottom and top of the obstacle. For simplicity, we specify that the control influence functions $\{\phi_{j,i}(\cdot)\}$ are piecewise constant. The control weights r_1, r_2, r_3, and r_4 in (15) are specified to be 500, 100, 500, and 1000, respectively.

The control objective is to drive nonzero solution values in a region downstream of the obstacle to zero. To this end, we specify

$$\Omega_Q = [.5, .75] \times [.16, .32]. \tag{35}$$

To severely penalize nonzero states in Ω_Q, we specify that $q_1 = 1$ and $q_2 = 1000$. These values result in states in Ω_Q being weighted much more heavily in the cost functional $J_\alpha(w_0, u(t))$. In order to make a comparison of the impact of α on the effectiveness of the control, we specify that $\alpha = 0$ in this example. Of course, $\alpha = 0$ corresponds to the traditional LQR control formulation.

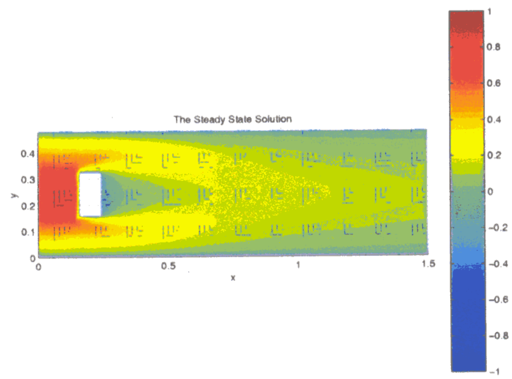

Figure 3: The steady-state solution for $\alpha = 0$.

As can be seen in Figure 3, the traditional LQR formulation of the control problem is not very effective in this case. Nonzero values in Ω_Q are only slightly reduced, even though a severe weight is placed on these values in the cost functional. The heavy penalty on states in Ω_Q has a direct influence on the functional gains, as seen in Figure 4. The functional gains for controls on the bottom, top, and rear of the obstacle have large magnitudes over Ω_Q. As a result, states in Ω_Q contribute significantly to those controls. Conversely, states in Ω_Q do not have a significant contribution to controls on the front of the obstacle. For controls on the front, the functional gains have large magnitudes near the control location.

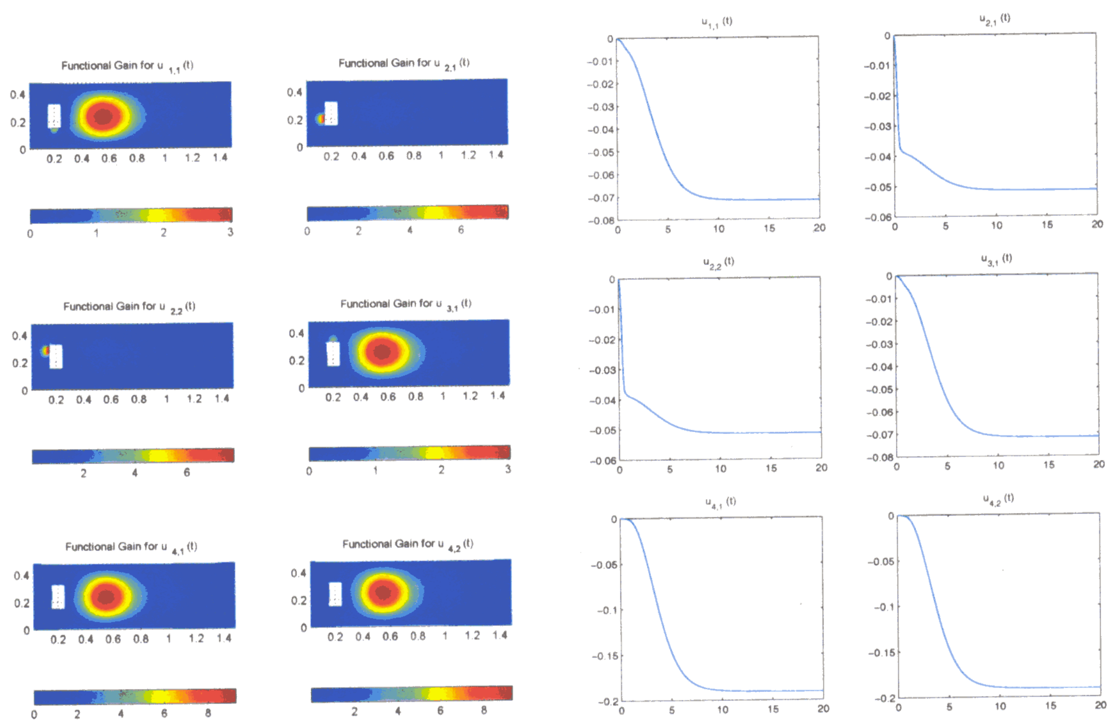

Figure 4: Results for $\alpha = 0$. Functional Gains (left), Controls (right).

Controls resulting from the traditional LQR formulation have very small magnitude, as evident in Figure 4. They are not effective at driving nonzero states in Ω_Q to zero.

Example 3

We now investigate the effectiveness of the control in the case of nonzero α. In this example, the parameters defining the problem are specified as in Example 2 with the exception that we set $\alpha = 0.2$. As

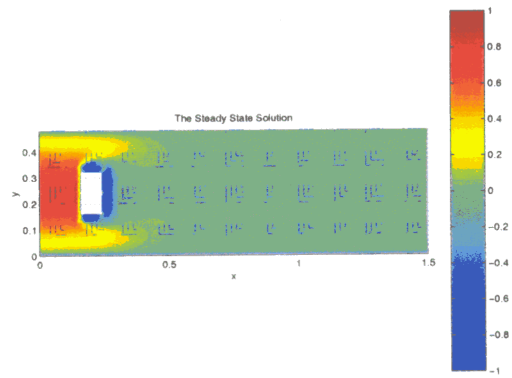

Figure 5: The steady-state solution for $\alpha = 0.2$.

can be seen in Figure 5, the control is much more effective at suppressing nonzero states in Ω_Q for the case $\alpha = 0.2$. A majority of the nonzero states in Ω_Q are greatly reduced when compared to their corresponding values in Figure 2. The control is much more effective in this case.

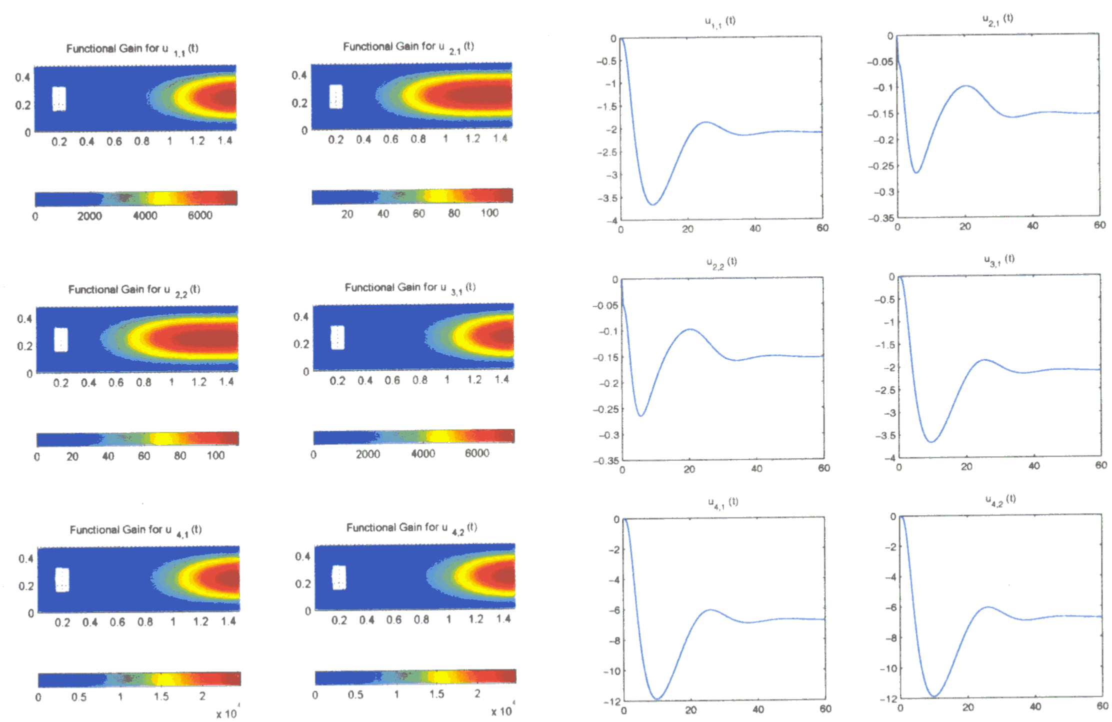

Figure 6: Results for $\alpha = 0.2$. Functional Gains (left), Controls (right).

Specifying nonzero α has a direct impact on the functional gains. From Figure 6, the functional gains have much larger magnitudes than was the case for $\alpha = 0$. In addition, the gains are large for states in Ω_Q, as well as for many states outside of Ω_Q. As a result, a majority of states downstream of the obstacle contribute to the control. The resulting controls are of larger magnitude than was the case for $\alpha = 0$, as evident in Figure 6. However, the magnitudes of the controls are not unreasonably large. There is a tradeoff in the construction of Q, R, and the value specified for α. Obviously, the state and control weights, as well as the parameter α, need to be chosen carefully in order to yield a control that is effective, but not too large.

Curiously, all of the functional gains of this example have their maximum values at the outflow on the right of the domain. At the right boundary, a Neumann condition is specified in an analogous way to the outflow condition specified in many flow problems where the flow leaves the domain. This condition is specified in order to allow the numerical solution to freely pass through the right boundary. From this example, it appears that a large portion of the control effort is directed at forcing the numerical outflow condition. We further investigate this issue in the next example.

Example 4

For the final example, we investigate the influence of the outflow boundary condition on the functional gains. As seen in the previous example, it appears that much of the control effort is aimed at forcing the Neumann condition specified at the right boundary. To illustrate the influence of the outflow condition on the functional gains, in this example we specify all parameters as they were in Example 3 with the exception that a Dirichlet condition is specified on the right boundary of the form

$$w(t, \Gamma_N) = 0. \tag{36}$$

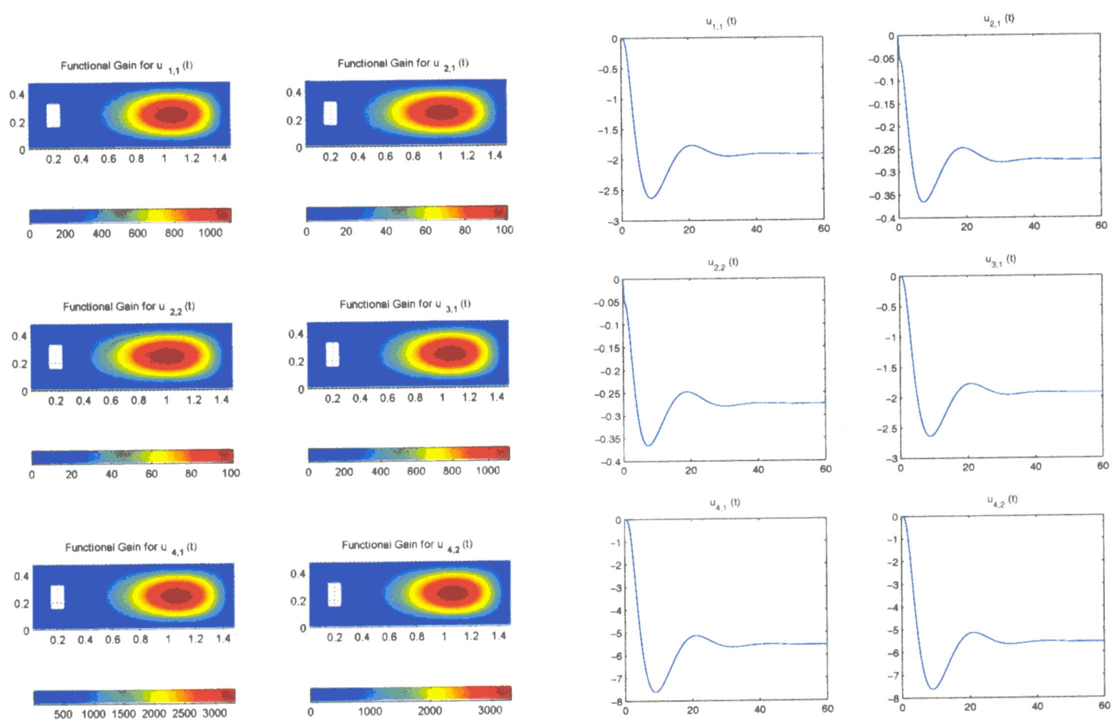

Figure 7: Results for $\alpha = 0.2$, Dirichlet Condition. Functional Gains (left), Controls (right).

In the case of $\alpha = 0$, modifying the boundary condition along Γ_N has no effect on the functional gains. The gains obtained with the condition specified in (36) are identical to those obtained in the Neumann case, i.e., those of Figure 4. However, modifying the right boundary condition has a significant effect on the functional gains for the case $\alpha = 0.2$, as evident by comparing the gains in Figure 7 with those of Figure 6. The gains computed with the Dirichlet condition given in (36) have much smaller magnitude and obtain their maxima on the interior of the domain. Moreover, the controls obtained in the Dirichlet case have much smaller amplitudes than they did in the Neumann case. Clearly, the outflow condition needs to be chosen carefully. The outflow condition is primarily specified as a numerical convenience in order to allow the computed solution to freely exit the computational domain. However, as evident in this example, the condition specified for outflow can have a dramatic effect on the functional gains and the resulting controls, depending on the cost functional used in the formulation of the control problem.

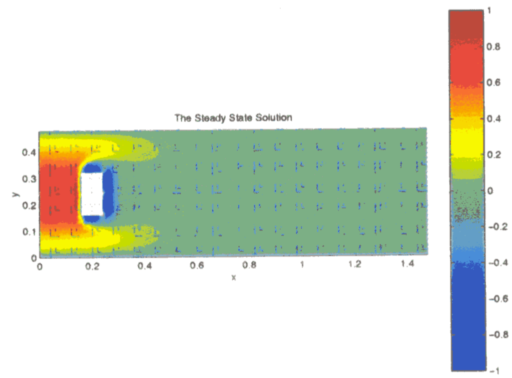

Figure 8: Dirichlet gains applied with Neumann outflow for $\alpha = 0.2$.

To verify that a large portion of the control effort of Example 3 is aimed at forcing the Neumann outflow condition, we apply the functional gains of this example to the system of Example 3 where a Neumann outflow condition is specified. The resulting steady-state solution is shown is Figure 8. The steady-state solutions shown in Figure 5 and Figure 8 are practically identical. The gains obtained in this example, and their corresponding controls, are very effective at suppressing nonzero states in Ω_Q for the system of Example 3. Moreover, the functional gains and controls of this example have much smaller magnitudes than their counterparts in Example 3. The controls of this example attain the control objective with much less effort. Clearly, careful consideration of the outflow condition must be done in order to avoid controls that are unnecessarily large for the sake of a convenient numerical outflow condition.

Conclusions and Future Work

In this paper, we have investigated the effectiveness of a particular linear feedback control strategy when applied to a system where the dynamics are governed by the two-dimensional Burgers' equation. We specifically constructed the system in such a way as to incorporate analogous conditions as those specified in many fluid flow configurations. The strategy chosen to control the system was very effective. However, as seen in the results, care must be taken in specifying quantities used in the control formulation. In particular, the construction of the state and control weights Q and R must be done carefully in order to place appropriate emphasis on regions of interest, and to correctly formulate the control objective. Moreover, the value specified for the additional performance parameter α needs to be chosen carefully to ensure that the resulting control is not prohibitively large for real-world applications.

As seen in the computational results, the numerical outflow condition can have a dramatic effect on the functional gains and their corresponding controls. The outflow condition needs to be chosen carefully so that unnecessary control effort is not expended solely for the sake of numerical convenience.

As our motivation for considering the problem presented in this paper was an investigation into the utility of linear control techniques to fluid flow problems, future work includes investigating these techniques for the Navier-Stokes equations. There are several issues that must be addressed in that situation. The case of large Reynolds number in the Navier-Stokes equations leads to systems where the dynamics are dominated by nonlinearities. More energy is required to obtain the control objective if the control enters into the problem linearly. One aspect of future work involves developing nonlinear feedback control methods to overcome this difficulty.

Obviously, the large number of grid points needed to resolve a flow field is a significant obstacle in implementing the control techniques in this paper to a fluid flow problem. In this paper, we investigated the effectiveness of the control under optimal circumstances, .i.e., full state information was available for feedback. However, full state information will not be made available in a fluid flow problem as the resulting control problem will be prohibitively large. As a result, an investigation into optimal sensor placement is also an area of future work.

Finally, more sophisticated methods of finding the gain operator should be utilized in the case of optimal control of fluid flow. In that case, finding the gains via an algebraic Ricatti equation is prohibitively expensive. Methods exploiting the structure of the problem, e.g. methods based on the Chandrasekhar equations [6, 17], have the potential of providing significant savings in the case of flow control.

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