TECHNICAL REPORT 2004-002

SENSITIVITY OF TRACK ACCURACY TO NAVIGATION, SENSOR, AND TIME ERRORS WITHIN A DISTRIBUTED SYSTEM

MAY 2004

Joint Single Integrated Air Picture System Engineering Organization (JSSEO) Analysis Branch

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FOREWORD

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EXECUTIVE SUMMARY

The success of achieving a Single Integrated Air Picture (SIAP) is a function of the accuracy with which the positional estimates of aerospace objects can be made. This report presents a detailed analysis of how track accuracy is affected by navigation, sensor, and time errors, which include errors due to time synchronization, time stamping, time latency, etc. Although the analysis assumes a shipborne sensor system for illustrative purposes, the analysis is general and can be applied to all sensing systems that comprise the distributed system that develops and maintains the SIAP.

Navigation errors are the result of errors in the accelerometers and gyroscopes forming the inertial navigation system. The focus here is not on the errors in the individual inertial instruments but on their performance as they affect the uncertainty in the position of the inertial navigation system and the orientation of the navigation reference frame. Navigation positional uncertainty or error is described by error in the north position, east position, and height of the navigation system. Error in the orientation of the navigation reference frame is described by three misalignments that represent the cumulative effect of the gyroscopes' inability to maintain a perfect inertial reference. Errors in navigation position translate directly into errors in track measurement. A 300 m east position error results in a 300 m track error. Navigation misalignments also have a significant effect on track accuracy. A 1 mrad misalignment about all three navigation reference axes results in a track error of 260 m for objects 100 nm away. The misalignment contribution is proportional to range to the object. Thus, for an object 10 nm from the ship, the 1 mrad misalignment contribution is 26 m.

Sensor error is also divided into two groups of error. First are errors in the reported location of the object relative to the sensor location. This error is quantified by error in the reported range to the object and its elevation and bearing relative to the sensor's reference axes. The second group of sensor errors consists of those relating to the uncertainty in the knowledge of the sensor's reference axes. Conceptually these errors are similar to navigation misalignments in that they represent the difference between the computer's representation of the orientation of the sensor axes and their true orientation. Sensor range error translates directly into track error. Sensor elevation error contribution to track error is proportional to the range to the object while the bearing contribution is proportional to range to the object multiplied by the cosine of elevation. Track accuracy sensitivity to sensor misalignment is almost identical to the sensitivity to navigation misalignments because the distance from the sensor to the navigation center is small compared to the range to the object.

Within the integrated architecture, potential contributors to time error include clock synchronization error (both within an interfacing unit and across interfacing units), errors in the time stamping of measurements, tracklets, and track messages, processing delays, communication network (i.e., Link 16) delays, and uncertainty in the knowledge of the nominal time latency. For simplicity, composite time errors ranging from 0.001 to 1 sec are considered. The nominal time or processing latency of the system, which is defined to be the nominal time the track information is used after it is observed by the radar, is shown to be an important factor for objects with high acceleration. For this analysis a range of potential threats is considered from relatively low-speed objects (200 kt) to high-speed combat aircraft (800 to 1,600 kt) having an acceleration capability on the order of 9 g.

The sensitivity of track error to time errors depends on the object's velocity and acceleration. The sensitivity is found to equal the vector sum of the velocity and the product of the acceleration and the time latency of the system. For a high-speed combat aircraft and a network latency of 5 sec, the time error is magnified by a factor of 956 m/sec. Thus, a time error of 1 sec is translated into a track error of 956 m. The time latency of the system has the effect of increasing or decreasing the contribution of the object's acceleration to the sensitivity. For example, reducing the time latency by a factor of two can reduce the track error to 735 m.

The parametric analysis shows that worst case conditions of 1 sec total composite time error and 30 sec network latency lead to track errors of approximately 3,000 m for high-speed aircraft. Composite time errors on the order of only a millisecond lead to track errors of less than 4 m for these worst case conditions.

A nominal track error budget is constructed illustrating the error contributions from navigation, sensor, and time. This budget shows a balance between the navigation and sensor contributions to track error. The contribution from time error is a distant third for the nominal error parameters assumed. The time error contribution becomes more important at closer ranges as the other range-dependent contributions decrease. Sensitivity curves are presented which show how quickly track error changes as a function of each of the individual error terms.

The results of the parametric model and the system specific error parameters can be applied to using and testing the Integrated Architecture Behavior Model (IABM) within the Joint Distributed Engineering Plant (JDEP) Technical Framework. In particular, the system specific error parameters define the statistical error distribution for each error term. The parameter values coupled with the corresponding distribution define the level of bias that should be introduced into every sensor that is associated with an instantiation of the IABM. The magnitude of the contribution of each error to track accuracy provides a hierarchy for the sequence of adding errors and systems into the IABM for testing. Since larger error contributors dominate smaller contributors, the errors with smaller contributions should be introduced into the IABM first when only one system is being considered. A similar hierarchy should also be applied to testing the IABM within a distributed system. The more accurate systems should be introduced into the distributed system before the less accurate systems.

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1. INTRODUCTION

1.1 Problem Statement

A critical element for achieving a Single Integrated Air Picture (SIAP) is attaining the condition of correct geodetic alignment of positional data in the 1984 World Geodetic System (WGS 84) coordinate frame. The degree of attaining this geodetic alignment can be measured by the accuracy by which the track positional information is known.

Track accuracy is a key ingredient in how well units in a distributed system can perform their various functions. For example, improvements in track accuracy improve the success ratio of correlation schemes, which in turn leads to a reduction in duals and a reduction in misidentifications. Reductions in track error result in higher values of track quality thus improving reporting responsibility throughout the distributed system. Improvements in track accuracy and its subsequent affect on improving warfighter effectiveness in a distributed system are required to enable advanced concepts of netcentric warfare such as cueing, engage on remote, and post before processing.

In this technical report, we present a methodology and results for investigating and understanding track accuracy. In particular, the methodology provides a straightforward means of identifying the major elements that contribute to errors in track positional estimates and in quantifying their contribution to track error. Because precise information on the performance of sensors is unknown, we present the results of a parametric study that illustrates how track accuracy changes as the errors in the major contributing subsystems change. The particular problem we address is that of a ship carrying a radar sensor that senses an aerospace object, Figure 1-1.

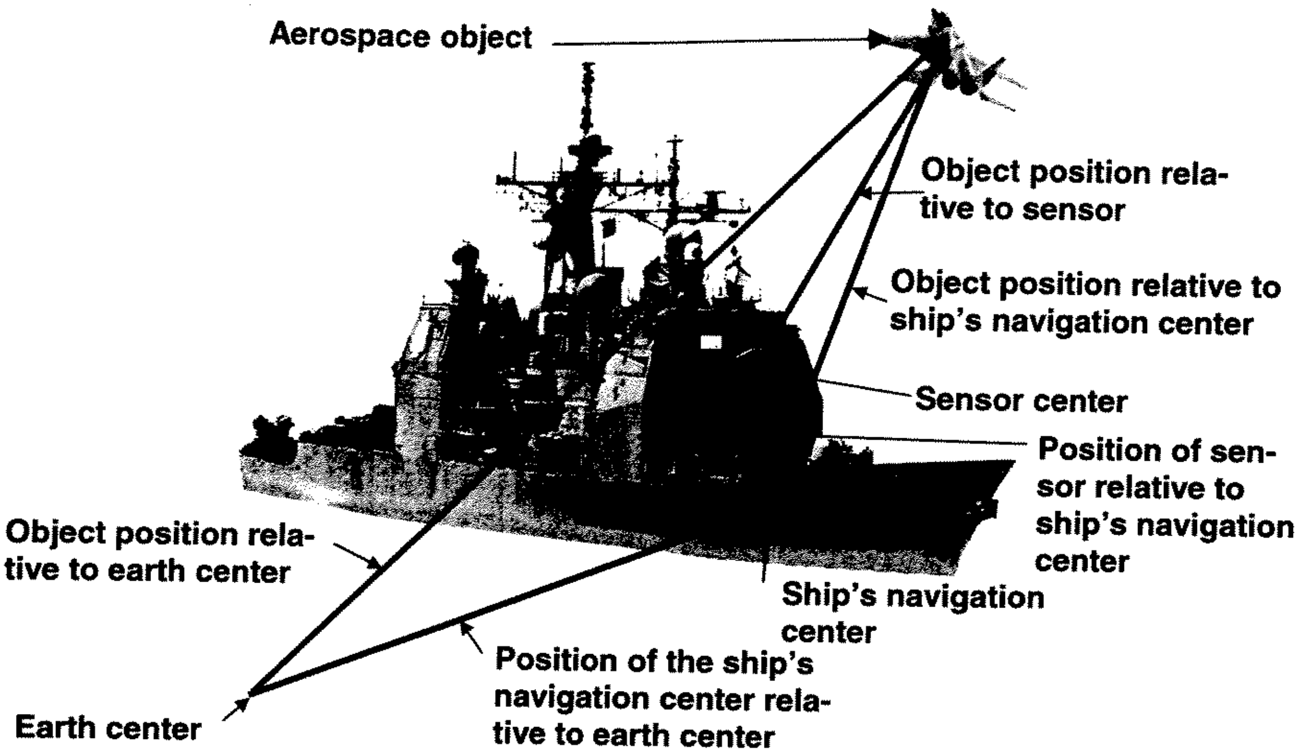


Figure 1-1 SIAP Track Accuracy Problem

1.2 Technical Approach

The analysis approach is to develop the equation that maps the location of the aerospace object expressed in the sensor frame to the location of the object relative to the Earth center expressed in the WGS 84 coordinate frame, and to consider how measurement errors and misalignments affect the estimate of track position.

In general, sensor measurements are made relative to a specified coordinate frame. For example, radar sensor measurements are made relative to the sensor coordinate frame, the ship's inertial navigation system (INS) provides measurements of position and orientation relative to an inertial coordinate frame, and ship structures are referenced to a body or vehicle frame that can rotate relative to the inertial navigation frame. Meaningful information can be exchanged between a single unit's various sensors, or the ship and other units in a distributed system only if the location and orientation of the various coordinate frames are known relative to each other.

An examination of the mapping of an object's position, expressed in the sensor frame, to its position expressed in the WGS 84 frame identifies three groups of error contributors:

- Navigation position (north, east, and height) and misalignments
- Sensor measurement (range, bearing, and elevation) and misalignments
- Time

After the error mechanisms are identified, a first-order linear perturbation analysis provides the sensitivities of track accuracy to the identified error contributors. Once the sensitivities are calculated, the error budgets and parametric sensitivities are determined.

The approach is general and can be applied equally well to land-based and airborne sensor systems. The analysis presented in the remainder of the report is intended to be complete, including statements of assumptions and technical details. An examination of the assumptions provides insight into how the analysis can be extended to other factors that affect track accuracy.

1.3 Motivation

The error budget approach and analysis presented in this report is a powerful technique for characterizing and understanding the performance of a family of systems such as the distributed architecture representation of the SIAP. It can be applied to existing systems to understand what drives current performance and suggest improvements, and it can be applied to systems in the design phase to predict performance and perform trade-off studies.

The error analysis process is a rigorous, quantitative examination into how a family of systems performs. Its power comes from the fact that it can be used to examine the total family of systems, as well as individual systems, subsystems, and single components. The process involves identifying all mechanisms that introduce error, tracing the propagation of the error throughout the family of systems, and quantifying the effect on system performance.

The error is a combination of two parts: the deterministic error, which can be compensated, and the random error, which in most cases can be reduced through improvements in the build process or through external measurements and corrections. Error budget analysis provides an assessment of the performance of the family of systems before improvements are made. It can identify candidate improvement options and provide projections of performance if compensation techniques are applied or improved build processes are initiated.

2. POSITION ERROR ANALYSIS FOR AEROSPACE OBJECTS

2.1 Introduction to Position Error Analysis

Imagine some weapon system that carries a sensor system that senses an aerospace object—think of a ship that carries a radar that senses an aircraft. The position and attitude of the navigation reference frame relative to the Earth are measured by an Inertial Navigation System (INS). There is a lever arm between the center of the INS and the sensor system, for they may be located a considerable distance apart on the ship. The sensor system measures the range and angles to the object, angles with respect to some sensor coordinate frame, which, in turn, are related to some ship referenced coordinate frame. Given the sensor system errors, the sensor-to-ship orientation errors, the lever arm errors, and the INS errors in both position and attitude, what are the resulting errors in absolute object location? That is, what are the errors in the location of the object with respect to the Earth, not just with respect to the ship?

The analysis here is basic and generic. It does not apply, in detail, to any specific set of operational systems, but it is applicable, in general, to many such combinations. The objective here is a set of definitions and a consistent notation that together lead to an understanding of error propagation and to equations useful for error estimation and for simulation.

In the operational world, the process begins with the sensor measurements of an object to relate the object's location to the sensor, then the object's location is related to the ship's INS, and finally, the object's location is related to the Earth. The discussion of the process begins in reverse, starting with the Earth, then turning to the ship's INS, then to the sensor, and finally to the object. In the development of the results, the definitions and the error relationships are introduced as they are needed. The equations are in a logical order for development and discussion, but not necessarily for use. Summary sections are included to compile the definitions and the measurement processing steps into a more compact reference form that may assist in understanding, and to restate the equations in a more usable way. Finally, the equations are reordered again, with additions, to outline a sequence of computations for use in simulation and analysis.

2.2 The Earth

Consider an ellipsoid model of the Earth geoid[1], in particular, the one used as geodetic datum by the World Geodetic System 1984 (WGS 84). Parameters related to this ellipsoid are listed in Table 2-1.

Table 2-1 WGS 84 Ellipsoid Parameters

Parameter	Notation	Formula	Value
Angular Velocity	ω_e		$7.2921151467 \times 10^{-5}$ rad/sec 15.04106718 deg/hr
Semimajor Axis	a_e		6378137 m
Flattening	f		1/298.2572236
Semiminor Axis	b_e	$a_e(1-f)$	6356752.3 m
Eccentricity	e	$\sqrt{f(2-f)}$	0.08181919084
Eccentricity Squared	e^2	e^2	0.00669437990
Axis Ratio	b_e / a_e	$\sqrt{1-e^2}$	0.9966471893
Axis Ratio Squared	$(b_e / a_e)^2$	$1-e^2$	0.9933056199
Linear Eccentricity	E	ea_e	521854.01 m
Minor Eccentricity	e'	$e/(1-f)$	0.08209443796

Let the geodetic coordinates of a point B with respect to this datum be denoted as (ϕ, λ, h), which are the geodetic latitude, the longitude, and the altitude of B above the ellipsoid. The geodetic latitude and altitude are illustrated in Figure 2-1.

[1] The WGS 84 geoid is that particular equipotential surface of the earth that coincides with mean sea level over the oceans and extends hypothetically beneath all land surfaces. The geoid serves in land areas (if conventional leveling is not available) as the vertical datum or reference surface to which height-above-mean sea level values are referenced. In ocean areas the geoid serves as the geodetic height of points on the ocean's surface.

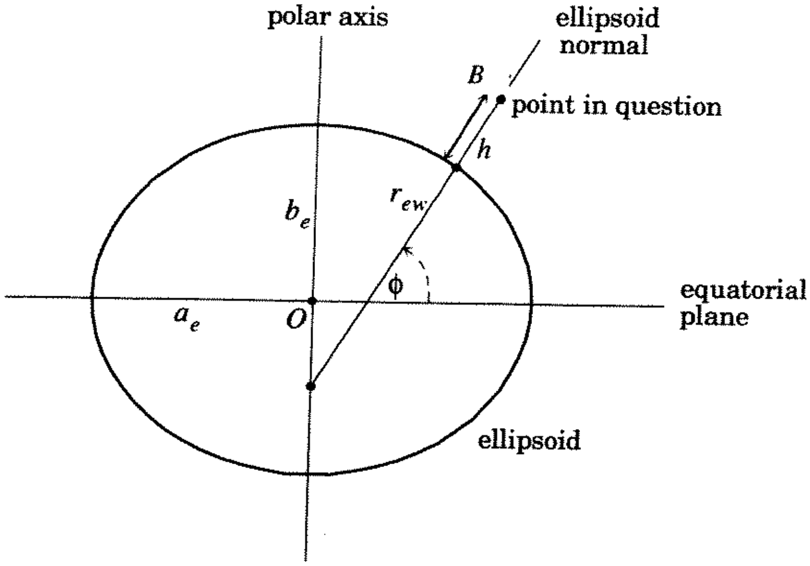


Figure 2-1 Geodetic Latitude and Altitude

Define the usual Cartesian coordinate system, called the e-frame, or the Earth frame, to be the Conventional Terrestrial System (CTS) associated with the WGS 84 datum. (All Cartesian coordinate systems used here are orthogonal, right-handed systems.) This frame is fixed to the ellipsoid, and so to the Earth. Its center is at the ellipsoid center (called here the center of the Earth, or the point O). Its z-axis is along the ellipsoid axis of symmetry (called here the Earth rotation axis, or north pole, which ignores polar motion). Its x-axis and y-axis are in the plane of ellipsoid symmetry known as the equatorial plane. The x-axis is in the Greenwich meridian (the meridian of zero longitude). Earth coordinates are depicted in Figure 2-2.

Coordinate systems here are assigned definite centers for ease of conceptualization, but they are not used as systems of positional coordinates. Instead, they are used as systems of vector coordinates, so their centers do not enter into transformations from one system to another. It is only the directions of the coordinate axes that matter. That is, all the vectors of "position" here are, in fact, displacement vectors that go from one specified point in space to another. Vectors conceptualized without regard to coordinate systems are called geometric vectors.

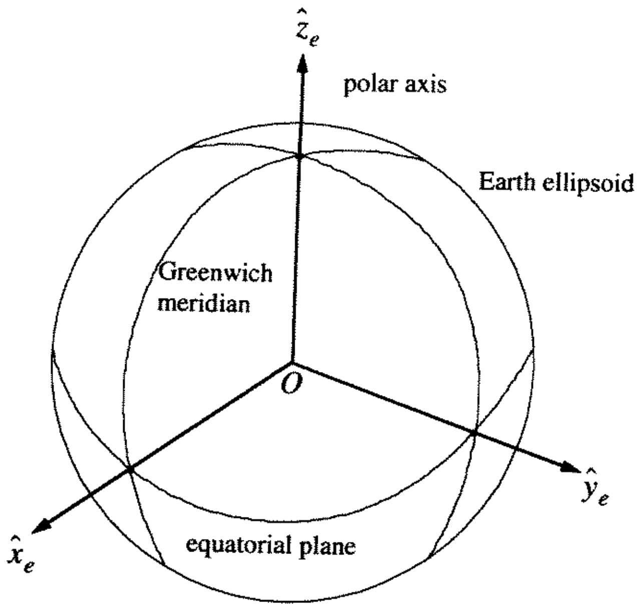


Figure 2-2 Earth Coordinates: the e-Frame

A geometric vector is never equated to its coordinate system representation as a mathematical 3-vector. That relationship is expressed by the notation $\vec{r} \leftrightarrow (\vec{r})_e$, which means that the geometric vector $\vec{r}$ is expressed by or is represented by the coordinate vector $(\vec{r})_e$ in the e-frame. This approach adds some notational complexity, but it ultimately serves to reduce confusion when several coordinate systems are in use simultaneously, as they are here.

Let $\vec{r}_b$ be the vector from the center of Earth to B. The components of $\vec{r}_b$ in the e-frame are given by

$$(\vec{r}_b)_e = \begin{bmatrix} (r_{ew}+h)C_\phi C_\lambda \\ (r_{ew}+h)C_\phi S_\lambda \\ [(1-e^2)r_{ew}+h]S_\phi \end{bmatrix}_e \tag{2.2-1}$$

where C_ϕ and S_ϕ are shorthand for $\cos\phi$ and $\sin\phi$, etc.; and r_{ew} is the east-west radius of ellipsoid curvature, a function of the latitude, given by

$$r_{ew} = a_e(1-e^2 S_\phi^2)^{-1/2} \tag{2.2-2}$$

2.3 The Ship

Consider a ship, fixed or moving, in the vicinity of the Earth. Define a point B, fixed to that ship, called the center of the ship's INS, with geodetic coordinates, as above.

The geodetic coordinates of the INS are provided by the inertial instruments forming the INS, and are subject to error. Let $(\tilde{\phi}, \tilde{\lambda}, \tilde{h})$ be the measured geodetic coordinates of B, and define the geodetic coordinate errors $(\delta\phi, \delta\lambda, \delta h)$, so that the measured quantities are the true quantities plus the errors, or

$$\tilde{\phi} = \phi + \delta\phi, \qquad \tilde{\lambda} = \lambda + \delta\lambda, \qquad \tilde{h} = h + \delta h \tag{2.3-1}$$

Measured values here are denoted with a tilde (~); true values are "plain" symbols, and error values have a preceding δ.

The measured coordinates of $(\vec{r}_b)_e$, in terms of the true values and the errors, are

$$(\tilde{\vec{r}}_b)_e = (\vec{r}_b)_e + (\delta\vec{r}_b)_e \tag{2.3-2}$$

where, approximately (to first order in the δ quantities)

$$(\delta\vec{r}_b)_e = (r_{ns} + h)\delta\phi \begin{bmatrix} -S_\phi C_\lambda \\ -S_\phi S_\lambda \\ C_\phi \end{bmatrix}_e + (r_{ew} + h)C_\phi \delta\lambda \begin{bmatrix} -S_\lambda \\ C_\lambda \\ 0 \end{bmatrix}_e + \delta h \begin{bmatrix} C_\phi C_\lambda \\ C_\phi S_\lambda \\ S_\phi \end{bmatrix}_e \tag{2.3-3}$$

where r_{ns} is the north-south radius of ellipsoid curvature, a function of the latitude, given by

$$r_{ns} = a_e(1-e^2)(1-e^2 S_\phi^2)^{-3/2} = r_{ew}(1-e^2)(1-e^2 S_\phi^2)^{-1} \tag{2.3-4}$$

Define a second Cartesian coordinate system, called the l-frame, or the local level frame, to have its center at B, and its x-, y-, and z-axes along the north, east, and down directions at B. That is, the axes of the local level frame have directions along the unit vectors $(\hat{N}, \hat{E}, \hat{D})$. Down, here, means orthogonal to the ellipsoid at the surface of the ellipsoid. In the Earth frame, the components of the unit vectors along the local level frame axes are

$$(\hat{N})_e = \begin{bmatrix} -S_\phi C_\lambda \\ -S_\phi S_\lambda \\ C_\phi \end{bmatrix}_e, \qquad (\hat{E})_e = \begin{bmatrix} -S_\lambda \\ C_\lambda \\ 0 \end{bmatrix}_e, \qquad (\hat{D})_e = \begin{bmatrix} -C_\phi C_\lambda \\ -C_\phi S_\lambda \\ -S_\phi \end{bmatrix}_e \tag{2.3-5}$$

The local level system is not fixed to the Earth (unless the ship is fixed to the Earth), and it is not fixed to the ship; its axes rotate as the ship's position changes. Figure 2-3 illustrates the local level frame.

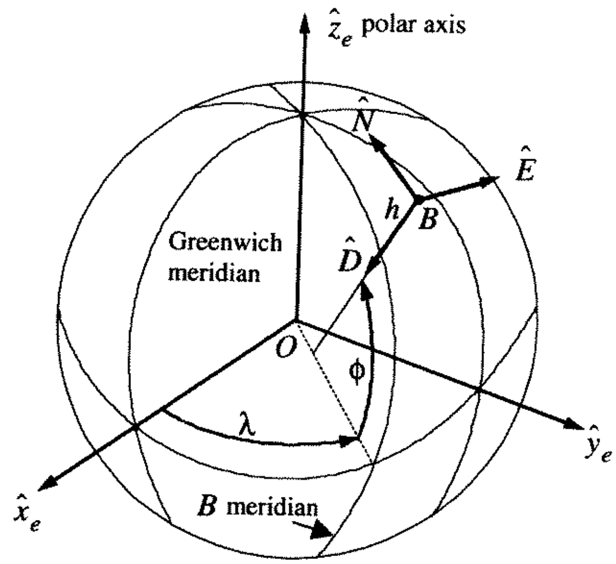


Figure 2-3 Local Level Coordinates: the l-Frame

The errors in the measured components of the INS position vector are

$$(\delta\vec{r}_b)_e = (r_{ns} + h)\delta\phi(\hat{N})_e + (r_{ew} + h)C_\phi\delta\lambda(\hat{E})_e - \delta h(\hat{D})_e \qquad (2.3\text{-}6)$$

That is, an error in latitude produces an error in the northward direction, an error in longitude produces an error in the eastward direction, and an error in altitude produces an error in the (negative) downward direction.

The rotation matrix that transforms vectors from expression in the e-frame to expression in the l-frame is given by

$$R_{le} = R_2(-\frac{\pi}{2} - \phi)R_3(\lambda) = \begin{bmatrix} -S_\phi C_\lambda & -S_\phi S_\lambda & C_\phi \\ -S_\lambda & C_\lambda & 0 \\ -C_\phi C_\lambda & -C_\phi S_\lambda & -S_\phi \end{bmatrix} \qquad (2.3\text{-}7)$$

where the standard trio of single-axis rotation matrices is defined as

$$R_1(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix},\ R_2(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix},\ R_3(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad (2.3\text{-}8)$$

Notice that the rows of R_{le} are the elements of the three local level frame axis unit vectors expressed in the Earth frame.

For example, the INS position vector, expressed in local level coordinates, is given by

$$(\vec{r}_b)_l = R_{le}(\vec{r}_b)_e = \begin{bmatrix} -e^2 r_{ew} S_\phi C_\phi \\ 0 \\ -r_{ew}(1-e^2 S_\phi^2)-h \end{bmatrix}_l \quad (2.3\text{-}9)$$

This vector has a small northward component (because the Earth is elliptical, not spherical), no eastward component, and a large, negative, downward component, as might be expected.

The errors in latitude and longitude produce errors in the transformation from the Earth frame to local level coordinates. That is

$$\tilde{R}_{le} = R_{le} + \delta R_{le} \quad (2.3\text{-}10)$$

To first order in the small angle quantities, we relate the error in the transformation to the measured transformation by the antisymmetric matrix, $\delta\Omega_{le}$, such that

$$\delta R_{le} = \delta\Omega_{le}\tilde{R}_{le} \quad (2.3\text{-}11)$$

and

$$\delta\Omega_{le} = -\delta\phi\Omega_2 + \delta\lambda R_2(-\frac{\pi}{2}-\tilde{\phi})\Omega_3 R_2^T(-\frac{\pi}{2}-\tilde{\phi}) \quad (2.3\text{-}12)$$

where the standard trio of single-axis antisymmetric matrices are defined as

$$\Omega_1=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix},\quad \Omega_2=\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix},\quad \Omega_3=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (2.3\text{-}13)$$

It helps in the derivation of the result above to note that to first order in $\delta\theta$

$$R_i(\theta+\delta\theta) = R_i(\theta)R_i(\delta\theta) = R_i(\theta)+R_i(\theta)\Omega_i\delta\theta, \qquad R_i(\theta)\Omega_i = \Omega_i R_i(\theta) \quad (2.3\text{-}14)$$

In total then, the small antisymmetric matrix $\delta\Omega_{le}$ is associated with the small 3-vector $(\delta\vec{\phi})_l$, where

$$(\delta\vec{\phi})_l = \begin{bmatrix} \delta\phi_1 \\ \delta\phi_2 \\ \delta\phi_3 \end{bmatrix}_l = -\delta\phi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_l + \delta\lambda \begin{bmatrix} C_\phi \\ 0 \\ -S_\phi \end{bmatrix}_l \qquad (2.3\text{-}15)$$

(It makes no meaningful difference when computing small error quantities, such as the one above, whether the true or the measured values of the parameters are used.) That is

$$\delta\Omega_{le} \Leftrightarrow (\delta\vec{\phi})_l, \qquad \delta\Omega_{le} = \begin{bmatrix} 0 & \delta\phi_3 & -\delta\phi_2 \\ -\delta\phi_3 & 0 & \delta\phi_1 \\ \delta\phi_2 & -\delta\phi_1 & 0 \end{bmatrix} \qquad (2.3\text{-}16)$$

(The symbol $\Leftrightarrow$ denotes that an antisymmetric matrix and a 3-vector are associated, as they are in the relationship above.)

A warning to the reader is in order. When two (or more) coordinate systems are involved, say the a-frame and the b-frame, and there is some orthonormal transformation matrix, say R_{ba}, that transforms vectors from expression in the a-frame to expression in the b-frame, then, for any "whole" vector, say $\vec{v}$, it is true that its two coordinate expression 3-vectors satisfy the relationship

$$(\vec{v})_b = R_{ba}(\vec{v})_a \qquad (2.3\text{-}17)$$

When there are errors in the coordinates of this vector, those errors are typically denoted as $(\delta\vec{v})_a$, and $(\delta\vec{v})_b$, but it is not generally true that

$$(\delta\vec{v})_b = R_{ba}(\delta\vec{v})_a \qquad \text{NOT GENERALLY TRUE} \quad (2.3\text{-}18)$$

This discrepancy happens because there are two sources of error in the components. One source is the error in the initial coordinate system. This error does transform according to the equation above. The other source is due to errors in the transformation itself, which adds an extra term to $(\delta\vec{v})_b$, so that the correct relationship is usually

$$(\delta\vec{v})_b = R_{ba}(\delta\vec{v})_a + \delta R_{ba}(\vec{v})_a \qquad (2.3\text{-}19)$$

That is, $(\delta\vec{v})_a$ and $(\delta\vec{v})_b$ are not the coordinates of the same vector, although the notation makes them look so. The notation could be modified to accommodate this issue, but the result is so ugly and confusing itself that it is not done here.

Fixed to the ship, centered at the ship's INS center, B, is yet another Cartesian coordinate system, called the b-frame, or the body frame, having the subscript b to denote its coordinate vectors. It is convenient that this body coordinate system be coincident with the local level frame under some "nominal" conditions. For a moving system such as a ship, the nominal conditions are that the ship be "level" and be pointed in the north direction. So the x-axis would point forward, the y-axis would be starboard (out the right side), and the z-axis would point toward the bottom of the ship. The b-frame is illustrated in Figure 2-4.

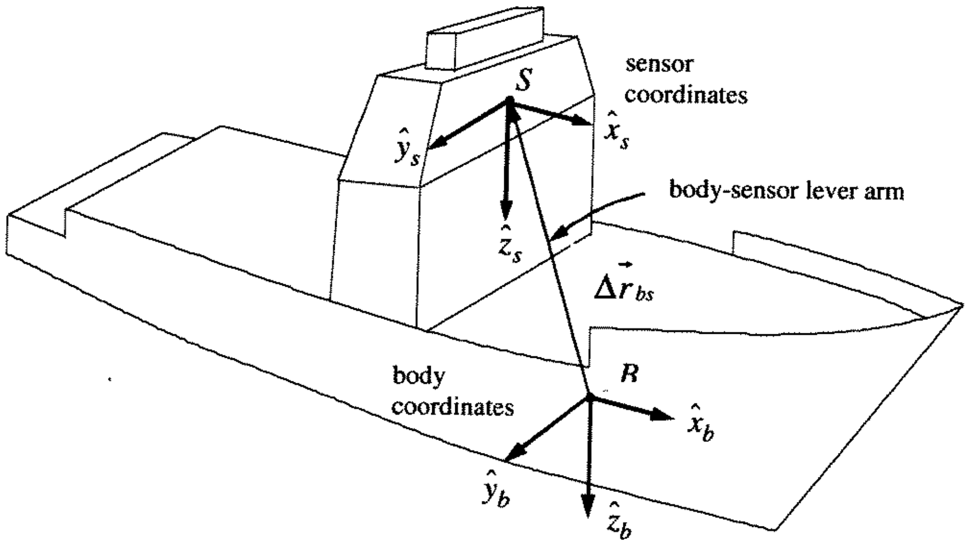


Figure 2-4 Body and Sensor Coordinate Frames

The exact relationship of the body frame to the local level frame is provided by three Euler angles, $(\psi_b, \theta_b, \phi_b)$, which are called the (true) body heading, body pitch, and body roll.

The rotation matrix that transforms vector coordinates from the l-frame to the b-frame is

$$R_{bl} = R_1(\phi_b)R_2(\theta_b)R_3(\psi_b) \tag{2.3-20}$$

$$R_{bl} = \begin{bmatrix} C_{\theta_b}C_{\psi_b} & C_{\theta_b}S_{\psi_b} & -S_{\theta_b} \\ S_{\phi_b}S_{\theta_b}C_{\psi_b} - C_{\phi_b}S_{\psi_b} & S_{\phi_b}S_{\theta_b}S_{\psi_b} + C_{\phi_b}C_{\psi_b} & S_{\phi_b}C_{\theta_b} \\ C_{\phi_b}S_{\theta_b}C_{\psi_b} + S_{\phi_b}S_{\psi_b} & C_{\phi_b}S_{\theta_b}S_{\psi_b} - S_{\phi_b}C_{\psi_b} & C_{\phi_b}C_{\theta_b} \end{bmatrix} \tag{2.3-21}$$

Like the geodetic coordinates of the body frame, the body Euler angles are measured quantities, presumably provided by the INS, and are subject to error, so the measured quantities, the true quantities, and the error quantities are related by

$$\tilde{\psi}_b = \psi_b + \delta\psi_b, \qquad \tilde{\theta}_b = \theta_b + \delta\theta_b, \qquad \tilde{\phi}_b = \phi_b + \delta\phi_b, \qquad \tilde{R}_{bl} = R_{bl} + \delta\tilde{R}_{bl} \tag{2.3-22}$$

It follows that, to first order in the small angle quantities, there is some anti-symmetric matrix, $\delta\Omega_{bl}$, such that

$$\delta R_{bl} = \delta\Omega_{bl}\tilde{R}_{bl} \tag{2.3-23}$$

and

$$\delta\Omega_{bl} = \delta\phi_b\Omega_1 + \delta\theta_b R_1(\tilde{\phi}_b)\Omega_2 R_1^T(\tilde{\phi}_b) + \delta\psi_b R_1(\tilde{\phi}_b)R_2(\tilde{\theta}_b)\Omega_3 R_1^T(\tilde{\phi}_b)R_2^T(\tilde{\theta}_b) \tag{2.3-24}$$

In total then, the small antisymmetric matrix $\delta\Omega_{bl}$ is associated with the small 3-vector $(\delta\vec{\theta})_b$ of navigation misalignments or tilt errors, where

$$(\delta\vec{\theta})_b = \begin{bmatrix} \delta\theta_N \\ \delta\theta_E \\ \delta\theta_D \end{bmatrix}_b = \delta\phi_b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_b + \delta\theta_b \begin{bmatrix} 0 \\ C_{\phi_b} \\ -S_{\phi_b} \end{bmatrix}_b + \delta\psi_b \begin{bmatrix} -S_{\theta_b} \\ C_{\theta_b}S_{\phi_b} \\ C_{\theta_b}C_{\phi_b} \end{bmatrix}_b \tag{2.3-25}$$

That is

$$\delta\Omega_{bl} \Leftrightarrow (\delta\vec{\theta})_b, \qquad \delta\Omega_{bl} = \begin{bmatrix} 0 & \delta\theta_D & -\delta\theta_E \\ -\delta\theta_D & 0 & \delta\theta_N \\ \delta\theta_E & -\delta\theta_N & 0 \end{bmatrix} \tag{2.3-26}$$

2.4 The Sensor System

Also (nominally) fixed to the ship is some sensor system, for example, a radar. Define the point S to be the "center" of the sensor system. The displacement vector from B to S will be denoted by $\Delta\vec{r}_{bs}$, and is called the body-sensor lever arm vector. This vector is assumed measured in body coordinates. Its errors will be considered later.

The sensor system is presumed to have its own Cartesian coordinate system, called the s-frame, or sensor coordinates. The rotation matrix that transforms vectors from expression in sensor coordinates to expression in body coordinates is denoted as R_{bs}. Nominally, this rotation matrix is constant, because the sensor system is assumed fixed to the ship. In practice, however, some systems are subject to deformation. Ships, as an example, deform for many reasons, including the rearrangement of fuel, stores, or cargo; temperature differences caused by the sun or engines; and stresses caused by propulsion, sea state, wind, current, or even armament usage. So the relationship between the sensor frame and the system's body frame might not be constant. Constant or not, its measurement is considered to be prone to error.

Figure 2-4, above, illustrates the s-frame and the body-sensor lever arm. In the diagram, the sensor frame and the body frame are nominally identical (in axes directions). These two coordinate frames might have any directional relationship to one another, and that relationship will not, itself, be parameterized here by Euler angles, just by the direction cosine (transformation) matrix mentioned above.

Denote the measured transformation matrix from sensor to body coordinates, $\tilde{R}_{bs}$, and the error in this transformation, δR_{bs}, so that

$$\tilde{R}_{bs} = R_{bs} + \delta R_{bs}, \qquad \delta R_{bs} = \tilde{R}_{bs} - R_{bs} \tag{2.4-1}$$

It must be that

$$\delta R_{bs} = (I - R_{bs}\tilde{R}_{bs}^T)\tilde{R}_{bs} = \delta\Omega_{bs}\tilde{R}_{bs} \tag{2.4-2}$$

where we have defined the matrix

$$\delta\Omega_{bs} = (I - R_{bs}\tilde{R}_{bs}^T) = \delta Rb_{bs}\tilde{R}_{bs}^T \tag{2.4-3}$$

It follows that

$$\delta\Omega_{bs} + \delta\Omega_{bs}^T = \delta R_{bs}\delta R_{bs}^T \tag{2.4-4}$$

so $\delta\Omega_{bs}$ is antisymmetric to first order in the error quantities, and we may associate it with a 3-vector, as

$$\begin{bmatrix} 0 & \delta\psi_3 & -\delta\psi_2 \\ -\delta\psi_3 & 0 & \delta\psi_1 \\ \delta\psi_2 & -\delta\psi_1 & 0 \end{bmatrix} = \delta\Omega_{bs} \Leftrightarrow (\delta\bar{\psi})_b = \begin{bmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_3 \end{bmatrix}_b \tag{2.4-5}$$

It is the 3-vector above that will characterize the errors in the relationship of sensor coordinates to body coordinates. These errors will be called the sensor misalignments.

2.5 The Aerospace Object

Somewhere, remote from the ship and the sensor system it carries, is an aerospace object that is sensed. Let T be a point fixed to the object, called the object center, presumably the very point sensed by the sensor system. Let the vector $\Delta\vec{r}_{st}$ be the displacement from the center of the sensor to the center of the object, called the sensor-object displacement vector. Assume that the sensor system measures the range, the bearing, and the elevation of the object with respect to sensor coordinates, Figure 2-5. Name (the true values of) these parameters (ρ, ψ_s, θ_s).

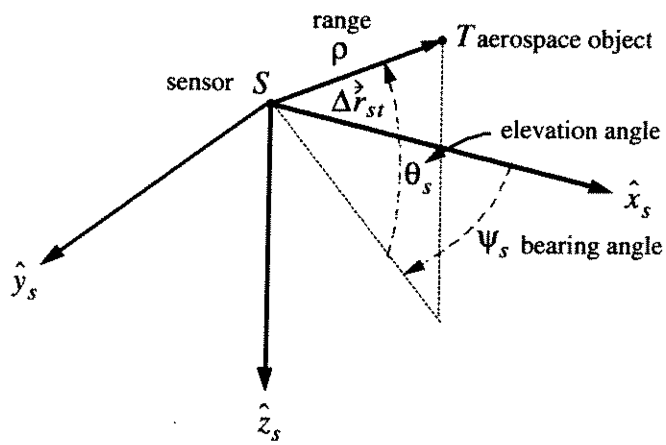


Figure 2-5 Sensor Measurement Items: Range, Bearing, and Elevation

Then

$$(\Delta\vec{r}_{st})_s = \rho\begin{bmatrix} C_{\theta_s}C_{\psi_s} \\ C_{\theta_s}S_{\psi_s} \\ -S_{\theta_s} \end{bmatrix}_s \tag{2.5-1}$$

Of course, these sensor parameters are measured with errors, so, as usual here

$$\tilde{\rho} = \rho + \delta\rho, \quad \tilde{\psi}_s = \psi_s + \delta\psi_s, \quad \tilde{\theta}_s = \theta_s + \delta\theta_s \tag{2.5-2}$$

which produce errors in the sensor-object displacement vector denoted as

$$(\Delta\tilde{\vec{r}}_{st})_s = (\Delta\vec{r}_{st})_s + (\delta\vec{r}_{st})_s \tag{2.5-3}$$

and given by

$$(\delta\vec{r}_{st})_s = \delta\rho\begin{bmatrix} C_{\theta_s}C_{\psi_s} \\ C_{\theta_s}S_{\psi_s} \\ -S_{\theta_s} \end{bmatrix}_s - \rho\delta\theta_s\begin{bmatrix} S_{\theta_s}C_{\psi_s} \\ S_{\theta_s}S_{\psi_s} \\ C_{\theta_s} \end{bmatrix}_s + \rho\delta\psi_s\begin{bmatrix} -C_{\theta_s}S_{\psi_s} \\ C_{\theta_s}C_{\psi_s} \\ 0 \end{bmatrix}_s \quad (2.5\text{-}4)$$

The sensor-object displacement vector is transformed from sensor coordinates into body coordinates as

$$(\Delta\vec{r}_{st})_b = R_{bs}(\Delta\vec{r}_{st})_s \quad (2.5\text{-}5)$$

Because there are errors in both the measured sensor-object displacement vector in sensor coordinates and in the transformation from sensor coordinates to body coordinates, there are conceptually two (groups of three) sources of errors in the measured sensor-object displacement vector in body coordinates. That is

$$(\Delta\tilde{\vec{r}}_{st})_b = (\Delta\vec{r}_{st})_b + (\delta\vec{r}_{st})_b \quad (2.5\text{-}6)$$

where

$$(\delta\vec{r}_{st})_b = R_{bs}(\delta\vec{r}_{st})_s + \delta R_{bs}(\Delta\vec{r}_{st})_s \quad (2.5\text{-}7)$$

Next, in the chain of object vector processing, the lever arm from the body (or INS) center to the sensor system center is added to the sensor-object displacement vector to create the body-object displacement vector, which goes from the point P to the point T. In symbols

$$(\Delta\vec{r}_{bt})_b = (\Delta\vec{r}_{bs})_b + (\Delta\vec{r}_{st})_b \quad (2.5\text{-}8)$$

That is, $(\Delta\vec{r}_{bt})_b$ is the newly defined body-object displacement vector, expressed in body coordinates. There are three (group) sources of error in the body-object displacement vector: the two groups that contribute to the errors in the sensor-object displacement vector, and the errors in the knowledge of the lever arm. The lever arm errors are described directly by

$$(\Delta\tilde{\vec{r}}_{bs})_b = (\Delta\vec{r}_{bs})_b + (\delta\vec{r}_{bs})_b \quad (2.5\text{-}9)$$

where, of course, the errors themselves are represented by $(\delta\vec{r}_{bs})_b$. In symbols,

$$(\Delta\tilde{\vec{r}}_{bt})_b = (\Delta\vec{r}_{bt})_b + (\delta\vec{r}_{bt})_b \quad (2.5\text{-}10)$$

where the total error in the body-object displacement vector is

$$(\delta\vec{r}_{bt})_b = (\delta\vec{r}_{bs})_b + (\delta\vec{r}_{st})_b \quad (2.5\text{-}11)$$

The next step in object vector processing is to transform the body-object displacement vector from body coordinates to local level coordinates, as

$$(\Delta\vec{r}_{bt})_l = R_{bl}^T(\Delta\vec{r}_{bt})_b \quad (2.5\text{-}12)$$

Errors in measured body attitude (roll, pitch, and yaw) cause errors in the transformation matrix from local level to body coordinates, and thus add errors to the body-object displacement vector in local level coordinates. In symbols

$$(\Delta\tilde{\vec{r}}_{bt})_l = (\Delta\vec{r}_{bt})_l + (\delta\vec{r}_{bt})_l \qquad (2.5\text{-}13)$$

where

$$(\delta\vec{r}_{bt})_l = R_{bl}^T(\delta\vec{r}_{bt})_b + \delta R_{bl}^T(\Delta\vec{r}_{bt})_b \qquad (2.5\text{-}14)$$

The next step is to transform the body-object displacement vector from local level coordinates to Earth coordinates, as

$$(\Delta\vec{r}_{bt})_e = R_{le}^T(\Delta\vec{r}_{bt})_l \qquad (2.5\text{-}15)$$

Errors in body position (latitude and longitude) cause errors in this transformation and thus add errors to the body-object displacement vector in Earth coordinates.

$$(\Delta\tilde{\vec{r}}_{bt})_e = (\Delta\vec{r}_{bt})_e + (\delta\vec{r}_{bt})_e \qquad (2.5\text{-}16)$$

where

$$(\delta\vec{r}_{bt})_e = R_{le}^T(\delta\vec{r}_{bt})_l + \delta R_{le}^T(\Delta\vec{r}_{bt})_l \qquad (2.5\text{-}17)$$

The last step is to add the body position vector to the body-object displacement vector to obtain the object position vector, as

$$(\vec{r}_t)_e = (\vec{r}_b)_e + (\Delta\vec{r}_{bt})_e \qquad (2.5\text{-}18)$$

Errors in body position (including altitude) also contribute at this step, and

$$(\tilde{\vec{r}}_t)_e = (\vec{r}_t)_e + (\delta\vec{r}_t)_e \qquad (2.5\text{-}19)$$

where

$$(\delta\vec{r}_t)_e = (\delta\vec{r}_b)_e + (\delta\vec{r}_{bt})_e \qquad (2.5\text{-}20)$$

2.6 Definitions

All these items are defined above, but they are listed here for reference. A few items are in more than one category.

2.6.1 Points

O is the center of the Earth (the center of the Earth model ellipsoid).

B is center of the body frame, also called the center of the ship's INS.

S is the center of the sensor system.

T is the center of the aerospace object, the point on the object sensed by the sensor.

The "centers" of the body frame, the sensor system, and the object are not geometric centers, but merely points fixed to the items in question.

2.6.2 Displacement vectors

$\vec{r}_b$ is the body coordinate frame position vector that goes from O to B.

$\vec{r}_t$ is the aerospace object position vector that goes from O to T.

$\Delta\vec{r}_{bs}$ is the body-sensor lever arm vector that goes from B to S.

$\Delta\vec{r}_{st}$ is the sensor-object displacement vector that goes from S to T.

$\Delta\vec{r}_{bt}$ is the body-object displacement vector that goes from B to T.

2.6.3 Coordinate systems

e-frame, Earth coordinates

l-frame, local level coordinates (at the body coordinate frame position)

b-frame, body coordinates, fixed to the ship

s-frame, sensor coordinates, fixed to the sensor

2.6.4 Coordinate transformation matrices

R_{le} transforms from Earth to local level (at the INS) coordinates.

R_{bl} transforms from local level (at the INS) to body coordinates.

R_{bs} transforms from sensor to body coordinates.

2.6.5 Measured items

(ϕ, λ, h) are the geodetic coordinates of the INS , which are the geodetic latitude, the longitude, and the altitude of the INS above the ellipsoid.

$(\psi_b, \theta_b, \phi_b)$ are the three Euler angles that describe the body coordinate frame orientation with respect to local level. They are the body heading, the body pitch, and the body roll.

$(\Delta\vec{r}_{bs})_b$ is the body-sensor lever arm vector in body coordinates.

R_{bs} is the transformation matrix from sensor to body coordinates.

(ρ, ψ_s, θ_s) are the sensor measurements of range, azimuth (or bearing), and elevation.

In all, there are fifteen scalar error sources associated with the fifteen measured quantities. Of course, beyond these fundamental measured items, there are many derived items. Associated with each of the fifteen measured quantities is an error. These errors are:

- Latitude
- Longitude
- Height
- Three navigation tilts
- Three lever arm errors
- Sensor range
- Sensor bearing
- Sensor elevation
- Three sensor alignments

2.7 The Aerospace Object Vector Processing Chain

While the entire chain of aerospace object vector processing has been discussed above, with error considerations, it may be useful to briefly summarize that chain from a strictly nominal (error free) point of view. All the symbols here are defined above, and all the equations have been previously presented. The starting point is the sensor measurements of the object, and the ending point is the object position vector, which is the vector from the center of the Earth to the aerospace object, expressed in Earth coordinates, Figure 2-6.

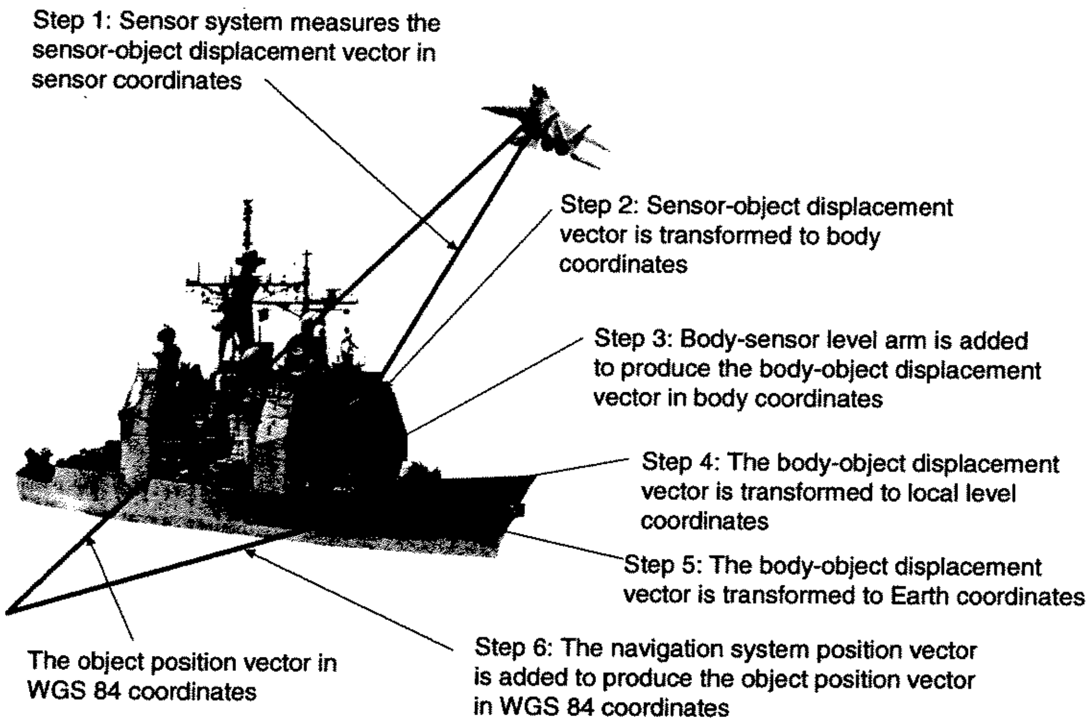


Figure 2-6 The Aerospace Object Vector Processing Chain

Step 1: The sensor system measures the sensor-object displacement vector in sensor coordinates

$$(\Delta\vec{r}_{st})_s = \rho \begin{bmatrix} C_{\theta_s} C_{\psi_s} \\ C_{\theta_s} S_{\psi_s} \\ -S_{\theta_s} \end{bmatrix}_s \tag{2.7-1}$$

Step 2: The sensor-object displacement vector is transformed to body coordinates

$$(\Delta\vec{r}_{st})_b = R_{bs}(\Delta\vec{r}_{st})_s \tag{2.7-2}$$

Step 3: The body-sensor lever arm is added to produce the body-object displacement vector in body coordinates

$$(\Delta\vec{r}_{bt})_b = (\Delta\vec{r}_{bs})_b + (\Delta\vec{r}_{st})_b \tag{2.7-3}$$

Step 4: The body-object displacement vector is transformed to local level coordinates

$$(\Delta\vec{r}_{bt})_l = R_{bl}^T(\Delta\vec{r}_{bt})_b \tag{2.7-4}$$

Step 5: The body-object displacement vector is transformed to Earth coordinates

$$(\Delta\vec{r}_{bt})_e = R_{le}^T(\Delta\vec{r}_{bt})_l \tag{2.7-5}$$

Step 6: The navigation system position vector is added to produce the object position vector in WGS 84 coordinates

$$(\vec{r}_t)_e = (\vec{r}_b)_e + (\Delta\vec{r}_{bt})_e \tag{2.7-6}$$

These same steps are used to derive measured items from the fundamental error free items.

The chain of errors is provided by the following summary of previously developed formulas.

Step 1: Error in the sensor-object displacement vector in sensor coordinates

$$(\Delta\tilde{\vec{r}}_{st})_s = (\Delta\vec{r}_{st})_s + (\delta\vec{r}_{st})_s \tag{2.7-7}$$

$$(\delta\vec{r}_{st})_s = \delta\rho\begin{bmatrix} C_{\theta_s}C_{\psi_s} \\ C_{\theta_s}S_{\psi_s} \\ -S_{\theta_s} \end{bmatrix}_s - \rho\delta\theta_s\begin{bmatrix} S_{\theta_s}C_{\psi_s} \\ S_{\theta_s}S_{\psi_s} \\ C_{\theta_s} \end{bmatrix}_s + \rho\delta\psi_s\begin{bmatrix} -C_{\theta_s}S_{\psi_s} \\ C_{\theta_s}C_{\psi_s} \\ 0 \end{bmatrix}_s \tag{2.7-8}$$

Step 2: Error in the sensor-object displacement vector in body coordinates

$$(\Delta\tilde{\vec{r}}_{st})_b = (\Delta\vec{r}_{st})_b + (\delta\vec{r}_{st})_b \tag{2.7-9}$$

$$(\delta\vec{r}_{st})_b = R_{bs}(\delta\vec{r}_{st})_s + \delta R_{bs}(\Delta\vec{r}_{st})_s \tag{2.7-10}$$

$$\delta R_{bs} = \delta\Omega_{bs}\tilde{R}_{bs} \tag{2.7-11}$$

$$\begin{bmatrix} 0 & \delta\psi_3 & -\delta\psi_2 \\ -\delta\psi_3 & 0 & \delta\psi_1 \\ \delta\psi_2 & -\delta\psi_1 & 0 \end{bmatrix} = \delta\Omega_{bs} \Leftrightarrow (\delta\vec{\psi})_b = \begin{bmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_3 \end{bmatrix}_b \tag{2.7-12}$$

Step 3: Error in the body-object displacement vector in body coordinates

$$(\Delta\tilde{\vec{r}}_{bt})_b = (\Delta\vec{r}_{bt})_b + (\delta\vec{r}_{bt})_b \tag{2.7-13}$$

$$(\delta\vec{r}_{bt})_b = (\delta\vec{r}_{bs})_b + (\delta\vec{r}_{st})_b \tag{2.7-14}$$

Step 4: Error in the body-object displacement vector in local level coordinates

$$(\Delta\tilde{\vec{r}}_{bt})_l = (\Delta\vec{r}_{bt})_l + (\delta\vec{r}_{bt})_l \tag{2.7-15}$$

$$(\delta\vec{r}_{bt})_l = R_{bl}^T(\delta\vec{r}_{bt})_b + \delta R_{bl}^T(\Delta\vec{r}_{bt})_b \tag{2.7-16}$$

$$\delta R_{bl} = \delta\Omega_{bl}\tilde{R}_{bl} \tag{2.7-17}$$

$$\delta\Omega_{bl} = \begin{bmatrix} 0 & \delta\theta_D & -\delta\theta_E \\ -\delta\theta_D & 0 & \delta\theta_N \\ \delta\theta_E & -\delta\theta_N & 0 \end{bmatrix} \tag{2.7-18}$$

$$(\delta\vec{\theta})_b = \begin{bmatrix} \delta\theta_N \\ \delta\theta_E \\ \delta\theta_D \end{bmatrix}_b = \delta\phi_b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_b + \delta\theta_b \begin{bmatrix} 0 \\ C_{\phi_b} \\ -S_{\phi_b} \end{bmatrix}_b + \delta\psi_b \begin{bmatrix} -S_{\theta_b} \\ C_{\theta_b}S_{\phi_b} \\ C_{\theta_b}C_{\phi_b} \end{bmatrix}_b \tag{2.7-19}$$

Step 5: Error in the body-object displacement vector in Earth coordinates

$$(\Delta\tilde{\vec{r}}_{bt})_e = (\Delta\vec{r}_{bt})_e + (\delta\vec{r}_{bt})_e \tag{2.7-20}$$

$$(\delta\vec{r}_{bt})_e = R_{le}^T(\delta\vec{r}_{bt})_l + \delta R_{le}^T(\Delta\vec{r}_{bt})_l \tag{2.7-21}$$

$$\delta R_{le} = \delta\Omega_{le}\tilde{R}_{le} \tag{2.7-22}$$

$$\delta\Omega_{le} = \begin{bmatrix} 0 & \delta\phi_3 & -\delta\phi_2 \\ -\delta\phi_3 & 0 & \delta\phi_1 \\ \delta\phi_2 & -\delta\phi_1 & 0 \end{bmatrix} \tag{2.7-23}$$

$$(\delta\vec{\phi})_l = \begin{bmatrix} \delta\phi_1 \\ \delta\phi_2 \\ \delta\phi_3 \end{bmatrix}_l = -\delta\phi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_l + \delta\lambda \begin{bmatrix} C_\phi \\ 0 \\ -S_\phi \end{bmatrix}_l \qquad (2.7\text{-}24)$$

Step 6: Error in the aerospace object position vector in Earth coordinates

$$(\tilde{\bar{r}}_t)_e = (\bar{r}_t)_e + (\delta\bar{r}_t)_e \qquad (2.7\text{-}25)$$

$$(\delta\bar{r}_t)_e = (\delta\bar{r}_b)_e + (\delta\bar{r}_{bt})_e \qquad (2.7\text{-}26)$$

$$(\delta\bar{r}_b)_e = (r_{ns} + h)\delta\phi \begin{bmatrix} -S_\phi C_\lambda \\ -S_\phi S_\lambda \\ C_\phi \end{bmatrix}_e + (r_{ew} + h)C_\phi \delta\lambda \begin{bmatrix} -S_\lambda \\ C_\lambda \\ 0 \end{bmatrix}_e + \delta h \begin{bmatrix} C_\phi C_\lambda \\ C_\phi S_\lambda \\ S_\phi \end{bmatrix}_e \qquad (2.7\text{-}27)$$

2.8 Time Error

In the development thus far, time has not entered the problem. The point of view that was taken is that there is a single, instantaneous, measurement of the aerospace object by the sensor, accompanied by supporting information—at that same instant of time—about the INS position and attitude, the body-sensor lever arm, and the relationship of the body coordinate frame to the sensor coordinate frame. That single measurement set is then converted into a measured object position with respect to the Earth. However, in reality there are peer-to-peer and peer-to-reference time errors.

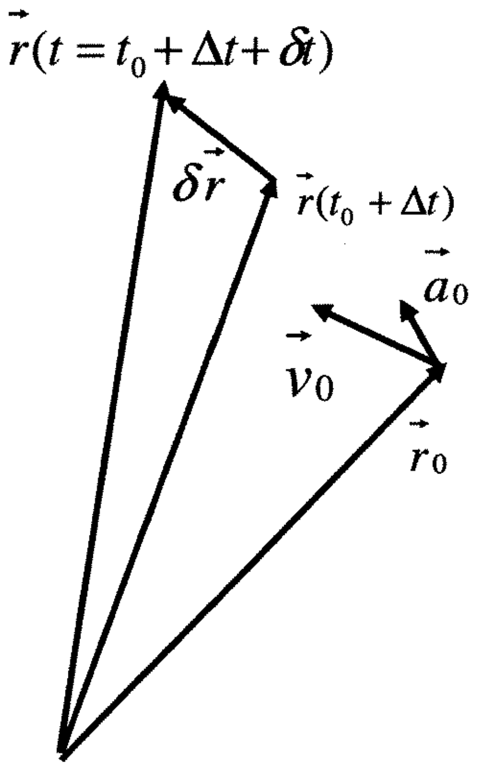
A better conceptual model for what typically happens is that a sequence of such sensor measurements (each accompanied by the time-dependent supporting information) is made. Each of these measurements results in a computed object position in Earth coordinates. The sequence of object positions is then fit to some object trajectory model, such as a constant velocity model or a constant acceleration model, as appropriate, to estimate the trajectory as a function of time. Instead of resulting in just a single object position, the result is an object position vector at some specified time, along with a object velocity vector at that same time, and possibly a object acceleration vector at that time, as well (see Figure 2-7). These three vectors are accompanied by a sensor time tag indicating the time at which they are valid.

The errors in any one estimate of object position have been discussed in the previous sections. The way those errors combine to produce errors in the trajectory estimate is not considered here. That process depends on the details of the trajectory model used, the fitting process used, and especially on the time correlation properties of the errors in the individual measurements and their supporting data.

The object trajectory information is then usually reported to some other participant, possibly remote, to use in dealing with the object. Let us call that other participant the user. Typically, the user is not interested in the object position at the time it was sensed or reported, but is interested in the object position (and perhaps velocity) at some later, extrapolated, time called the time of use. The object position derived by the user has errors for three reasons.

1. Even if all the information about the object trajectory from the sensor is absolutely correct, there will still be errors in extrapolating that information to the time of use. These errors are called extrapolation errors.
2. The object position, velocity, and acceleration data provided by the sensor is in error, and these errors will cause errors in the user-extrapolated items at a later time. These errors, both the original data errors of the sensor and the resulting errors by the user, will be referred to as trajectory data errors.

3. The time tag provided by the sensor is in error (with respect to some idealized absolute time scale), and the user clock also has errors. The difference of these errors is a time error, which will cause errors in the extrapolated trajectory items by the user. These errors in the user items will be called time synchronization errors here. All three types of errors combine to produce what are lumped here as "timing errors".



t_0 = Time aerospace object is observed by the radar

$\vec{r}_0$ = Object position at t_0

$\vec{v}_0$ = Object velocity at time t_0

$\vec{a}_0$ = Object acceleration at time t_0

t = Time object track data is used

Δt = Nominal time latency or nominal time past t_0 target track data is used

$\delta t = t - (t_0 + \Delta t)$ = Time error

$\vec{r}(t_0 + \Delta t)$ = Object position at error-free time $t_0 + \Delta t$

$\delta\vec{r} = \vec{r}(t_0 + \Delta t + \delta t) - \vec{r}(t_0 + \Delta t)$ = Track error

Figure 2-7 Track Error Due To Time Error

Trying to consider all three types of errors at once can be confusing, since they interact. Additional confusion occurs because time is represented as the independent variable in the error equations, so time errors enter in a different way than do the other errors. The goal here is a linearized, first-order error analysis, meaning that the results are only valid to first order in the (presumed small) error quantities. Interactions among various errors may be ignored, and each error may be treated as if the others did not exist, and the results of the three treatments simply summed.

The vector notation of the preceding development is appropriate for the situation when several coordinate systems are involved and everything happens at the same time. It is not so appropriate for this situation, where all vectors are expressed in one coordinate system—Earth coordinates—and time is an issue. Here, it is important to denote the time dependence and not important to denote the coordinate system dependence. The focus here is always on the aerospace object, so it is not necessary to subscript trajectory items to indicate the object. The notation will change to reflect these differences in analysis.

The object position (and velocity and acceleration) are reported in an absolute positional reference system. The same is true of time. That is, there is some absolute time scale—such as universal coordinated time (UTC), or GPS time—used to report time. With respect to this scale, both the sensor and the user clocks are in error.

2.8.1 Extrapolation errors

The (true) time at which the sensor reports of the trajectory items are valid will be denoted by t_0, known as time zero. The true position, the true Earth-relative velocity, and the true Earth-relative acceleration of the aerospace object (all expressed in Earth coordinates), as functions of the (true) time will be denoted as $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$. For convenience, the values of these vectors at time zero will be denoted as $\vec{r}_0$, $\vec{v}_0$, and $\vec{a}_0$.

The basic relationships among these quantities are that velocity is the time derivative of position, and acceleration is the time derivative of velocity. These relationships are more usefully expressed in integral terms as

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(\tau)d\tau \qquad (2.8\text{-}1)$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(\tau)d\tau \qquad (2.8\text{-}2)$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0(t-t_0) + \int_0^t \int_0^\tau \vec{a}(\lambda)d\lambda d\tau \qquad (2.8\text{-}3)$$

If the object acceleration were constant (in general it is not), then it would be true that

$$\vec{v}(t) = \vec{v}_0 + \vec{a}_0(t-t_0) \qquad (2.8\text{-}4)$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0(t-t_0) + \frac{1}{2}\vec{a}_0(t-t_0)^2 \qquad (2.8\text{-}5)$$

Since equations of this form are assumed to be used to extrapolate the object velocity and position, let us define the "constant acceleration" items

$$\vec{v}_a(t) = \vec{v}_0 + \vec{a}_0(t-t_0) \qquad (2.8\text{-}6)$$

$$\vec{r}_a(t) = \vec{r}_0 + \vec{v}_0(t-t_0) + \frac{1}{2}\vec{a}_0(t-t_0)^2 \qquad (2.8\text{-}7)$$

Then we may write

$$\vec{v}(t) = \vec{v}_a(t) + \int_0 [\vec{a}(\tau) - \vec{a}_0]d\tau \tag{2.8-8}$$

$$\vec{r}(t) = \vec{r}_a(t) + \int_0 \int_0 [\vec{a}(\lambda) - \vec{a}_0]d\lambda d\tau \tag{2.8-9}$$

and denote the differences between the actual items and the constant acceleration items by

$$\delta\vec{v}_x(t) = \vec{v}(t) - \vec{v}_a(t) = \int_0 [\vec{a}(\tau) - \vec{a}_0]d\tau \tag{2.8-10}$$

$$\delta\vec{r}_x(t) = \vec{r}(t) - \vec{r}_a(t) = \int_0 \int_0 [\vec{a}(\lambda) - \vec{a}_0]d\lambda d\tau \tag{2.8-11}$$

If all the trajectory information provided by the sensor, including the time tag, were absolutely correct, and if the user clock were also error free, the values of the two differences above (evaluated at the time of use) would be the errors in the extrapolated velocity and position. Thus the extrapolation errors are

$$\delta\vec{v}_x = \delta\vec{v}_x(t_u) = \vec{v}(t_u) - \vec{v}_a(t_u) = \int_0^{t_u} [\vec{a}(\tau) - \vec{a}_0]d\tau \tag{2.8-12}$$

$$\delta\vec{r}_x = \delta\vec{r}_x(t_u) = \vec{r}(t_u) - \vec{r}_a(t_u) = \int_0^{t_u} \int_0 [\vec{a}(\lambda) - \vec{a}_0]d\lambda d\tau \tag{2.8-13}$$

2.8.2 Trajectory data errors

The three vector trajectory items provided by the sensor, with errors, are (defining new notation)

$$\tilde{\vec{r}}_0 = \vec{r}_0 + \delta\vec{r}_0 = \tilde{\vec{r}}(t_0) = \vec{r}(t_0) + \delta\vec{r}(t_0) \tag{2.8-14}$$

$$\tilde{\vec{v}}_0 = \vec{v}_0 + \delta\vec{v}_0 = \tilde{\vec{v}}(t_0) = \vec{v}(t_0) + \delta\vec{v}(t_0) \tag{2.8-15}$$

$$\tilde{\vec{a}}_0 = \vec{a}_0 + \delta\vec{a}_0 = \tilde{\vec{a}}(t_0) = \vec{a}(t_0) + \delta\vec{a}(t_0) \tag{2.8-16}$$

Deviations of the object trajectory from the constant acceleration model cause the extrapolation errors. In treating the trajectory data errors we assume that the constant acceleration model is correct. The error is described by equations 2.8-14 to 2.8-16.

Let the extrapolation time difference or network latency be denoted as $\Delta t = t_u - t_0$. Using equations 2.8-1 to 2.8-3 with constant acceleration leads to the resultant trajectory data errors given by

$$\delta\vec{v}_d = \tilde{\vec{v}}_a(t_u) - \vec{v}_a(t_u) = \delta\vec{v}_0 + \delta\vec{a}_0\Delta t \tag{2.8-17}$$

$$\delta\vec{r}_d = \tilde{\vec{r}}_a(t_u) - \vec{r}_a(t_u) = \delta\vec{r}_0 + \delta\vec{v}_0\Delta t + \frac{1}{2}\delta\vec{a}_0\Delta t^2 \quad (2.8\text{-}18)$$

2.8.3 Time synchronization errors

The time tag reported by the sensor will be denoted as $\tilde{t}_0$, where $\tilde{t}_0 = t_0 + \delta t_s$, with δt_s being the time error of the sensor, see Figure 2-8. The time of use, on the user's clock, is measured as $\tilde{t}_u$, where $\tilde{t}_u = t_u + \delta t_u$, and δt_u is the user time error.

The focus here is on the errors made in extrapolating the trajectory estimate to a position estimate at some future time point, and on the errors caused by time synchronization errors between the sensor clock and the user clock. There may be other time synchronization errors. The vehicle navigation system might have its own clock that differs from the sensor system clock, even though both clocks are nearby on the same vehicle. These other possible timing errors are not considered here.

The "measured" time for extrapolation is

$$\Delta\tilde{t} = \tilde{t}_u - \tilde{t}_0 = t_u + \delta t_u - t_0 - \delta t_s = \Delta t + \delta t \quad (2.8\text{-}19)$$

where the net time synchronization error is $\delta t = \delta t_u - \delta t_s$. That is, $\Delta\tilde{t}$ is the time actually used by the user to extrapolate the sensor trajectory data, instead of using the true network latency interval Δt. The errors in the extrapolated trajectory items produced by the time synchronization error are

$$\delta\vec{v}_t = \vec{a}_0\delta t \quad (2.8\text{-}20)$$

$$\delta\vec{r}_t = \vec{v}_0\delta t + \vec{a}_0\Delta t\delta t \quad (2.8\text{-}21)$$

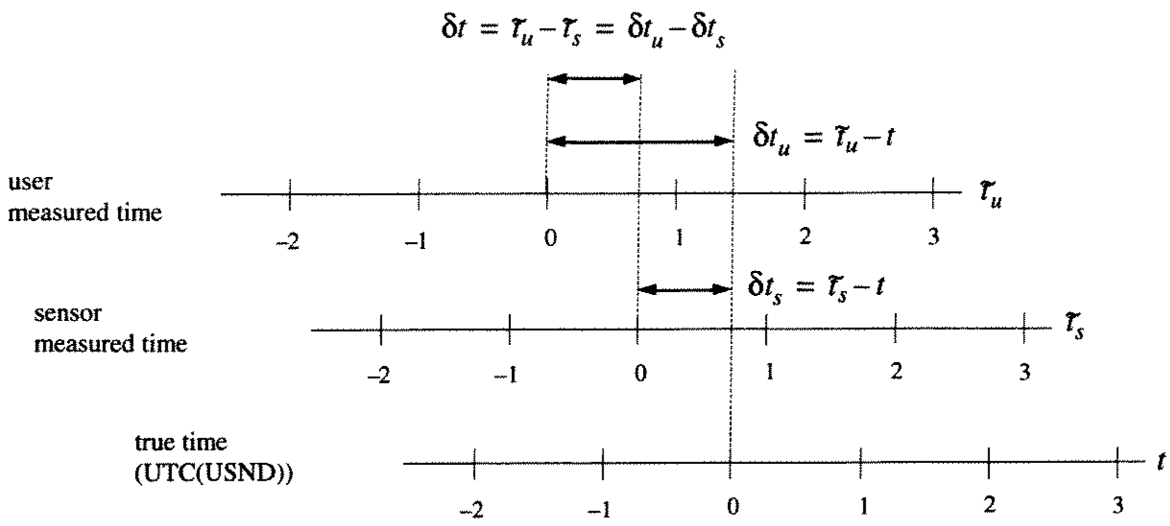


Figure 2-8 Time Synchronization Error

2.8.4 Combined errors

Putting the three results together—by simply adding them—yields the combined error in the trajectory estimates by the user as

$$\delta\vec{v}_c = \delta\vec{v}_x + \delta\vec{v}_d + \delta\vec{v}_t = \int_0^u [\vec{a}(\tau) - \vec{a}_0] d\tau + \delta\vec{v}_0 + \delta\vec{a}_0 \Delta t + \vec{a}_0 \delta t \tag{2.8-22}$$

$$\delta\vec{r}_c = \delta\vec{r}_x + \delta\vec{r}_d + \delta\vec{r}_t \tag{2.8-23}$$

$$\delta\vec{r}_c = \int_0^u \int_0^\tau [\vec{a}(\lambda) - \vec{a}_0] d\lambda d\tau + \delta\vec{r}_0 + \delta\vec{v}_0 \Delta t + \frac{1}{2}\delta\vec{a}_0 \Delta t^2 + \vec{v}_0 \delta t + \vec{a}_0 \Delta t \delta t \tag{2.8-24}$$

Recapping, we have the following interpretation for the terms on the right side of equation 2.8-24:

- $\int_0^u \int_0^\tau [\vec{a}(\lambda) - \vec{a}_0] d\lambda d\tau$ is the error due to using an average acceleration in place of the true acceleration
- $\delta\vec{r}_0$ is the error in the aerospace object position vector in Earth coordinates given by equation 2.7-26

- $\delta\tilde{v}_0 \Delta t + \frac{1}{2}\delta\tilde{a}_0 \Delta t^2$ represents the error in the velocity and acceleration, respectively, used in the tracking algorithm to extrapolate the object's position
- $\vec{v}_0 \delta t + \vec{a}_0 \Delta t \delta t$ is the contribution to track error due to the total time error.

2.8.5 The processing sequence

At the risk of belaboring the issue, the sequence of processing steps by the user will be outlined. The input data to this sequence are the trajectory items and the accompaning time tag provided by the sensor. That is, the inputs are $\left(\tilde{\vec{r}}_0, \tilde{\vec{v}}_0, \tilde{\vec{a}}_0, \tilde{t}_0\right)$. One additional input, provided by the user, is the measured time at which the trajectory estimate is to be used, or $\tilde{t}_u$. The user computes the measured extrapolation time difference as $\Delta\tilde{t} = \tilde{t}_u - \tilde{t}_0$, and then extrapolates the trajectory items using the constant acceleration model

$$\tilde{\vec{v}}(\tilde{t}_u) = \tilde{\vec{v}}_0 + \tilde{\vec{a}}_0 \Delta\tilde{t} \quad (2.8\text{-}25)$$

$$\tilde{\vec{r}}(\tilde{t}_u) = \tilde{\vec{r}}_0 + \tilde{\vec{v}}_0 \Delta\tilde{t} + \frac{1}{2}\tilde{\vec{a}}_0 \Delta\tilde{t}^2 \quad (2.8\text{-}26)$$

The errors in this process are given by

$$\tilde{\vec{v}}(\tilde{t}_u) - \vec{v}(t_u) = \delta\vec{v}_c \quad (2.8\text{-}27)$$

$$\tilde{\vec{r}}(\tilde{t}_u) - \vec{r}(t_u) = \delta\vec{r}_c \quad (2.8\text{-}29)$$

where the combined errors are delineated above.

2.9 The Sensitivities

The analysis presented in the previous sections is characterized as a linear error analysis since only error terms up to first order are retained. As such the results can be expressed in the compact form $\left(\delta\vec{r}_t\right)_e = S\vec{e}$, where S is the sensitivity matrix and $\vec{e}$ is the error vector defined by

$$\vec{e} = \begin{bmatrix} \textit{North Position Error} \\ \textit{East Position Error} \\ \textit{Height Error} \\ \textit{Navigation Tilt, N} \\ \textit{NavigationTilt, E} \\ \textit{Navigation Tilt, D} \\ \textit{Lever Arm Error}, b_1 \\ \textit{Lever Arm Error}, b_2 \\ \textit{Lever Arm Error}, b_3 \\ \textit{Sensor Range Error} \\ \textit{Sensor Bearing Error} \\ \textit{Sensor Elevation Error} \\ \textit{Sensor Misalignment}, \psi_1 \\ \textit{Sensor Misalignment}, \psi_2 \\ \textit{Sensor Misalignment}, \psi_3 \\ \textit{Time Error} \end{bmatrix} \quad (2.9\text{-}1)$$

This form is advantageous since the covariance of track error can be directly obtained from the expression $\text{cov}(\delta\vec{r}) = S\,\text{cov}(\vec{e})\,S^T$. Similarly, the covariance of the track error for groups of error contributors can be calculated by considering the appropriate submatrices of S.

The elements of S provide information on how fast the components of track error change as the individual error terms change. The three rows of S correspond to the x, y, and z components of the WGS 84 coordinate system and each column of S corresponds to an element of the error vector. For example, we will denote *S(*,Sensor Range Error)* as the three sensitivities (x, y, and z directions, respectively) corresponding to the sensor range error.

The elements of the sensitivity matrix are obtained by rearranging the expressions provided in Section 2.6 and making the following two identifications.

First, latitude error is equivalent to north position error and second, east position error is obtained by dividing longitude sensitivity by the cosine of latitude. In what follows, expressions of the form Δr_{α_β} represent the i^th/ element of the displacement vector from *a*, where *a* represents the sensor or body, to the object, expressed in the β-frame. The columns of S are given by the following:

$$S(*, North\ Position\ Error) = (r_{ns} + h)\begin{bmatrix} -S_\phi C_\lambda \\ -S_\phi S_\lambda \\ C_\phi \end{bmatrix}_e + R_{le}^T \begin{bmatrix} -\Delta r_{bt_{l3}} \\ 0 \\ \Delta r_{bt_{l1}} \end{bmatrix}_l \quad (2.9\text{-}2)$$

$$S(*, East\ Position\ Error) = (r_{ew} + h)\begin{bmatrix} -S_\lambda \\ C_\lambda \\ 0 \end{bmatrix}_e + (R_{le}^T / C_\phi)\begin{bmatrix} S_\phi \Delta r_{bt_{l2}} \\ -S_\phi \Delta r_{bt_{l1}} - C_\phi \Delta r_{bt_{l3}} \\ C_\phi \Delta r_{bt_{l2}} \end{bmatrix}_l \quad (2.9\text{-}3)$$

$$S(*, Height\ Error) = \begin{bmatrix} C_\phi C_\lambda \\ C_\phi S_\lambda \\ S_\phi \end{bmatrix}_e \quad (2.9\text{-}4)$$

$$S(*, Navigation\ Tilt, N) = R_{le}^T R_{bl}^T \begin{bmatrix} 0 \\ -\Delta r_{bt_{b3}} \\ \Delta r_{bt_{b2}} \end{bmatrix}_b \quad (2.9\text{-}5)$$

$$S(*, Navigation\ Tilt, E) = R_{le}^T R_{bl}^T \begin{bmatrix} \Delta r_{bt_{b3}} \\ 0 \\ -\Delta r_{bt_{b1}} \end{bmatrix}_b \quad (2.9\text{-}6)$$

$$S(*, Navigation\ Tilt, D) = R_{le}^T R_{bl}^T \begin{bmatrix} -\Delta r_{bt_{b2}} \\ \Delta r_{bt_{b1}} \\ 0 \end{bmatrix}_b \quad (2.9\text{-}7)$$

$$S(*, Lever\ Arm\ Error, b_1) = R_{le}^T R_{bl}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_b \quad (2.9\text{-}8)$$

$$S(*, Lever\ Arm\ Error, b_2) = R_{le}^T R_{bl}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_b \quad (2.9\text{-}9)$$

$$S(*,Lever\ Arm\ Error, b_3) = R_{le}^T R_{bl}^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_b \quad (2.9\text{-}10)$$

$$S(*,Sensor\ Range\ Error) = R_{le}^T R_{bl}^T R_{bs} \begin{bmatrix} C_{\theta_s} C_{\psi_s} \\ C_{\theta_s} S_{\psi_s} \\ -S_{\theta_s} \end{bmatrix}_s \quad (2.9\text{-}11)$$

$$S(*,Sensor\ Bearing\ Error) = R_{le}^T R_{bl}^T R_{bs} \rho \begin{bmatrix} -C_{\theta_s} S_{\psi_s} \\ C_{\theta_s} C_{\psi_s} \\ 0 \end{bmatrix}_s \quad (2.9\text{-}12)$$

$$S(*,Sensor\ Elevation\ Error) = R_{le}^T R_{bl}^T R_{bs} \rho \begin{bmatrix} -S_{\theta_s} C_{\psi_s} \\ -S_{\theta_s} S_{\psi_s} \\ -C_{\theta_s} \end{bmatrix}_s \quad (2.9\text{-}13)$$

$$S(*,Sensor\ Misalignment, \psi_1) = R_{le}^T R_{bl}^T \begin{bmatrix} 0 \\ \Delta r_{st_{b3}} \\ -\Delta r_{st_{b2}} \end{bmatrix}_b \quad (2.9\text{-}14)$$

$$S(*,Sensor\ Misalignment, \psi_2) = R_{le}^T R_{bl}^T \begin{bmatrix} -\Delta r_{st_{b3}} \\ 0 \\ \Delta r_{st_{b1}} \end{bmatrix}_b \quad (2.9\text{-}15)$$

$$S(*,Sensor\ Misalignment, \psi_3) = R_{le}^T R_{bl}^T \begin{bmatrix} -\Delta r_{st_{b2}} \\ \Delta r_{st_{b1}} \\ 0 \end{bmatrix}_b \quad (2.9\text{-}16)$$

$$S(*,Time\ Error) = \begin{bmatrix} v_{0_x} \\ v_{0_y} \\ v_{0_z} \end{bmatrix}_e + \Delta t \begin{bmatrix} a_{0_x} \\ a_{0_y} \\ a_{0_z} \end{bmatrix}_e \quad (2.9\text{-}17)$$

3. RESULTS

3.1 Overview

It should come as no surprise that the analysis results depend strongly on the particular scenario, the operational conditions, and the nominal error budget. The Common Reference Scenarios (CRS) are being developed to provide a uniform set of threat and operational scenarios for analysis purposes. When these CRS are completed much of the scenario-dependent variability will be eliminated. However, at the time of this analysis the CRS were not finalized and we have chosen parameter values that represent realistic conditions for a sea-based system in order to identify the important sensitivities and to quantify the relative importance of the error contributors.

We will use the numerical values of the sensitivity matrix to illustrate how the various errors are mapped into track error and provide insight into the role the geometry of the problem plays. When error budgets are discussed we will measure track accuracy in terms of Spherical Error Probable (SEP), which represents the radius of a sphere centered at the nominal location of the aerospace object that encompasses 50% of the error distribution.

3.2 Nominal Scenario and Operational Parameter Values

The parameter values for the operational scenario are provided in Table 3-1. These parameters place the ship in the Atlantic Ocean sensing an object low in elevation at a range of 100 nm. The aerospace object represents the capabilities of a high-speed aircraft. The roll, pitch, and yaw of the ship are set to zero for convenience, without loss of generality.

Table 3-1 Nominal Operational Parameter Values

Item	Parameter	Units	Value
Ship	Latitude	deg	45
	Longitude	deg	-70
	Height	m	0
	Roll	deg	0
	Pitch	deg	0
	Yaw	deg	0
	Lever arm, b_1	m	-30
	Lever arm, b_2	m	2
	Lever arm, b_3	m	8
Sensor	Range to target	nm	100
	Bearing to target	deg	45
	Elevation to target	deg	2
Network	Latency	sec	5
Aerospace Object	Velocity, x	kt	500
	Velocity, y	kt	0
	Velocity, z	kt	0
	Acceleration, x	g	9
	Acceleration, y	g	0
	Acceleration, z	g	0

3.3 Sensitivity Matrix

The transpose of the sensitivity matrix is provided in Table 3-2. The numerical entries in the table represent the amount of track error in each of the three axes of the WGS 84 coordinate frame that are generated by a unit error term. For example, if x is the first dimension of track accuracy and east position is the second error, then a value of 0.911 for the (1,2) element of the sensitivity matrix means that every meter of east position error causes 0.911 m of track error in the positive direction of x. If the east position error is 1,000 meters then its contribution to track error is 911 m in the positive x direction.

Table 3-2 Sensitivity Matrix (Transposed)

Item	Error Term	Unit	Sensitivities (m/unit) WGS-84 Earth Fixed Axes x	y	z
Ship	North Position	m	-0.24706	0.67879	0.69329
	East Position	m	0.91149	0.36258	0
	Height	m	0.24184	-0.66446	0.70711
	Navigation tilt, N	mrad	-25.58618	89.17173	-92.545
	Navigation tilt, E	mrad	33.20572	-91.23196	87.95773
	Navigation tilt, D	mrad	154.60765	-42.21175	-92.545
	Lever arm error, p_1	m	-0.24184	0.66446	0.70711
	Lever arm error, p_2	m	0.93969	0.34202	0
	Lever arm error, p_3	m	-0.24184	0.66446	-0.70711
Sensor	Range to target	m	0.50159	0.68807	0.52437
	Bearing to target	mrad	154.63536	-42.20016	-92.54359
	Elevation to target	mrad	41.57299	127.58352	127.64471
	Sensor misalignment, ψ_1	mrad	25.57818	-89.17314	92.54359
	Sensor misalignment, ψ_2	mrad	-33.21491	91.25721	-87.97329
	Sensor misalignment, ψ_3	mrad	154.63536	42.20016	92.54359
Network	Time error*	sec	699.15174	0	0

Table 3-3 Unit Radial Track Error

Item	Error Term	Unit	Sensitivities (m/unit) WGS-84 Earth Fixed Axes x	y	z	Unit Radial Track Error
Ship	North Position	m	-0.247	0.679	0.693	1.0
	East Position	m	0.911	0.363	0.000	1.0
	Height	m	0.242	-0.664	0.707	1.0
	Navigation tilt, N	mrad	-26	89	-93	131.0
	Navigation tilt, E	mrad	33	-91	88	131.0
	Navigation tilt, D	mrad	155	-42	-93	185.1
	Lever arm error, p_1	m	-0.242	0.664	0.707	1.0
	Lever arm error, p_2	m	0.940	0.342	0.000	1.0
	Lever arm error, p_3	m	-0.242	0.664	-0.707	1.0
Sensor	Range to target	m	0.502	0.688	0.524	1.0
	Bearing to target	mrad	155	-42	-93	185.1
	Elevation to target	mrad	42	128	128	185.2
	Sensor misalignment, ψ_1	mrad	26	-89	93	131.0
	Sensor misalignment, ψ_2	mrad	-33	91	-88	131.0
	Sensor misalignment, ψ_3	mrad	155	42	93	185.1
Network	Time error*	sec	699	0	0	699.2

*Represents the sensitivity of any time error such as time synchronization, timestamp, etc.

Insight into how individual errors contribute to track error can be obtained by examining the root-sum-square (RSS) of the elements of the sensitivity matrix. These values are provided in the column titled "Unit Radial Track Error" in Table 3-3.

The values of unity for Unit Radial Track Error corresponding to the three rows of navigation position error imply that a meter of position error in north or east, or height, translates one-for-one into a meter of track error. This mapping is independent of operational conditions and scenario, e.g., aerospace object conditions. The same holds for the contribution of lever arm errors, and sensor range error.

If the RSS of the three navigation tilt Unit Radial Track Error values is compared to the RSS of the three sensor misalignment Unit Radial Track Error values, one finds that the difference is insignificant. This implies that the contributions of these two misalignment contributors are nearly identical. The reason for this is that the distance separating the navigation center from the sensor center, 31 m, is small compared to the distance from the sensor to the aerospace object, 100 nm. The magnitude of the contribution of a navigation or sensor misalignment error is proportional to the distance from the sensor to the object.

The contribution of sensor bearing error to track accuracy is proportional to range to the object times the cosine of the object's elevation. The sensor elevation error contribution to track accuracy is proportional to range to the object. The numerical values in Table 3-3 are almost equal because the cosine of 2 degrees elevation is close to unity.

The sensitivity of the time contribution is dependent on the aerospace object's velocity and acceleration and the network latency. If any of these operational or scenario conditions increase (decrease) then the sensitivity will also increase (decrease).

3.4 Nominal Error Values

The nominal error budget for the navigation, sensor, and time errors is presented in Table 3-4. The errors are all assumed to be normally distributed with zero mean. For the present analysis all errors are assumed to be independent. However, nothing in the error analysis of Section 2 requires this to be true. The uncorrelated model was chosen for convenience. More detailed analysis of the individual subsystems is required to develop the more general correlated error model.

Table 3-4 Nominal Error Budget

Item	Error Parameter	Unit	One-Sigma Value
Ship	North position	m	300
	East position	m	300
	Height	m	150
	Navigation tilt, N	mrad	0.1
	Navigation tilt, E	mrad	0.1
	Navigation tilt, D	mrad	0.1
	Lever arm error, b_1	m	0.1
	Lever arm error, b_2	m	0.1
	Lever arm error, b_3	m	0.1
Sensor	Range to object	m	300
	Bearing to object	mrad	1
	Elevation to object	mrad	1
	Sensor misalignment, ψ_1	mrad	1
	Sensor misalignment, ψ_2	mrad	1
	Sensor misalignment, ψ_3	mrad	1
Network	Time error*	sec	0.5

*Represents the total effect of all time errors such as time synchronization, timestamp, etc.

The error budget for each system also provides the statistical description needed to introduce the proper level of bias into each system associated with a platform specific version of the Integrated Architecture Behavior Model (IABM). A random value can be drawn from the distribution of each error each time the complete IABM is executed either during testing or during runs within the Joint Distributed Engineering Plant (JDEP) technical framework.

3.5 Nominal Track Accuracy Error Budget

Combining the sensitivity matrix, Table 3-2, with the nominal error budget, Table 3-4, leads to the nominal track accuracy error budget displayed in Figure 3-1. For each of the error terms listed on the left of the bar chart, the length of the bars shows the contribution to track error, expressed as Spherical Error Probable (SEP), if the error under consideration is the only error in the system. For example, if north position error were the only error, then SEP for the system would be about 250 m.

The bar chart also illustrates the cumulative effect of several errors. The individual yellow bars roll up to the blue bars, the individual blue bars roll up to the red bars and the three red bars roll up to the total error bar color-coded purple. For example, the sensor range, sensor bearing, and sensor elevation error contributors combine to give a total sensing error contribution of approximately 350 m SEP. The sensor misalignment contribution and the sensor measurement contribution combine to yield an approximate contribution of 425 m SEP for the sensor subsystem. The three major error groups, navigation, sensor, and time, combine to yield a total system error of approximately 650 m SEP.

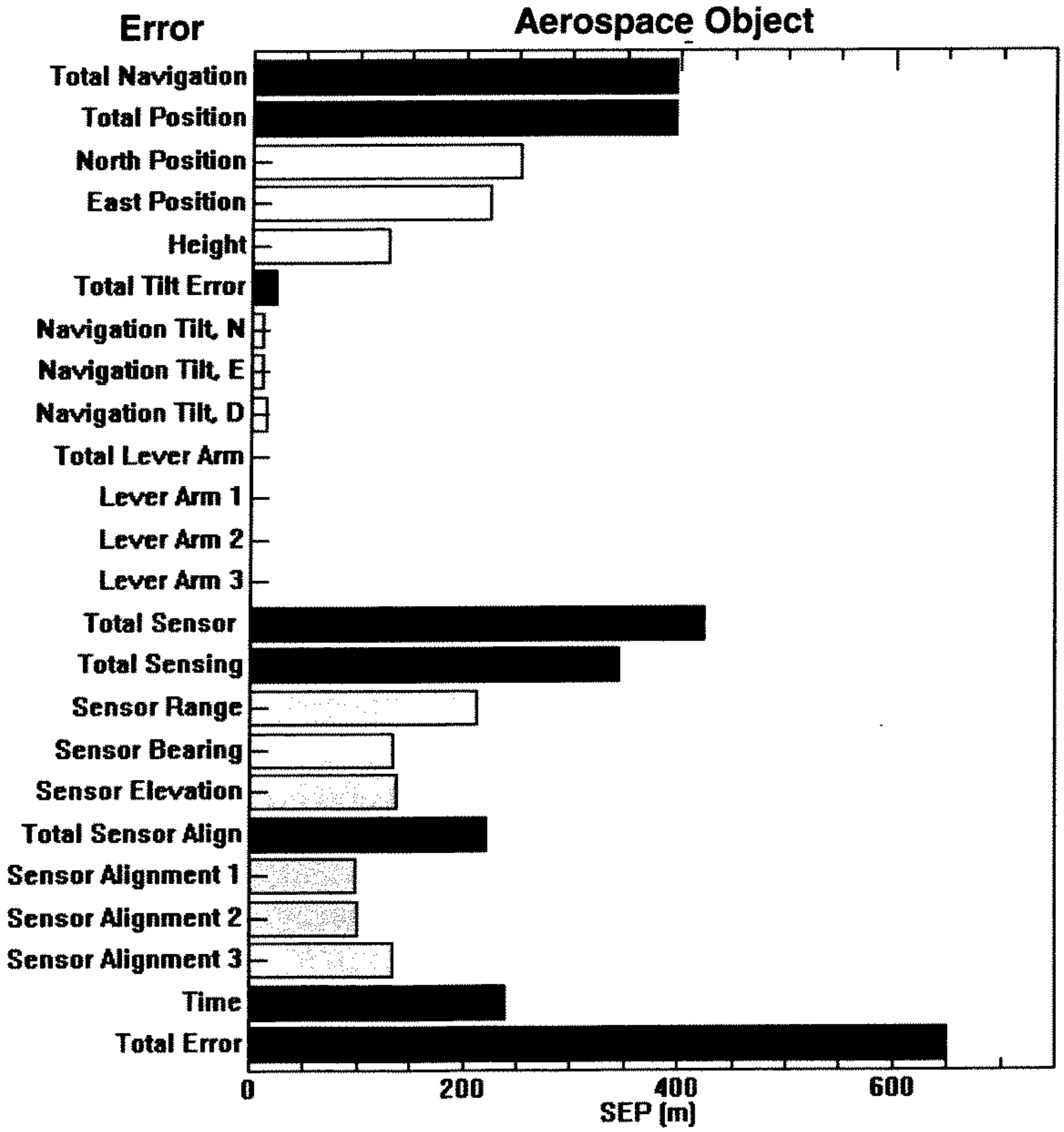


Figure 3-1 Nominal Track Accuracy Error Budget

The error budget also provides a quick way to rank the importance of the various error contributors to track accuracy. For this case, the short length of the bars representing the navigation tilt and lever arm error contributions implies the insignificance of these error terms. That is, resources would be wasted on reducing these error terms since their overall contribution to the total system accuracy is so small. Any further reduction in these errors would cause a minimal reduction on total system error.

The magnitude of the contributions also suggests that when testing the IABM the error terms with small contributions should be included first. Otherwise the larger error terms will dominate and it will be difficult to determine if the small terms are being implemented correctly.

The nominal system also represents a balanced system with respect to navigation and total sensor error. The contributions from these two error categories are nearly equal. This implies that either would be a good candidate to consider for making improvements to track accuracy. However, which is the easiest to correct is another matter to consider. The time error would rank last among the three major error groups as a candidate for improving the total system error because its contribution is relatively small.

3.6 Sensitivity of Track Accuracy to Individual Errors

This section presents a number of charts displaying the sensitivity of track accuracy to changes in the nominal one-sigma error parameters. These charts provide important information on how track accuracy changes if the nominal values change. For comparison purposes the nominal values presented in Figure 3-1 correspond to where the light gray vertical line intersects the sensitivity curves in Figures 3-2 to 3-12. This sensitivity information is relevant if there is some uncertainty in the confidence of the one-sigma values or if there is a need to make predictions of what the total system accuracy will be if a change is made in the one-sigma value. This section is divided into three subsections, each devoted to one of the three major error groups, navigation, sensor, and time. The displays for the sensitivities of track accuracy to time are presented in a different format to show the effect of time latency on system accuracy and to cover two ways in which time error can be mapped to system level track error.

3.6.1 Sensitivity to navigation errors

Figures 3-2 to 3-5 display the sensitivity of track accuracy to changes in the one-sigma values for north position error, east position error, height error, and navigation misalignments, respectively. In all the figures, the curve representing the sensitivity of the position contribution

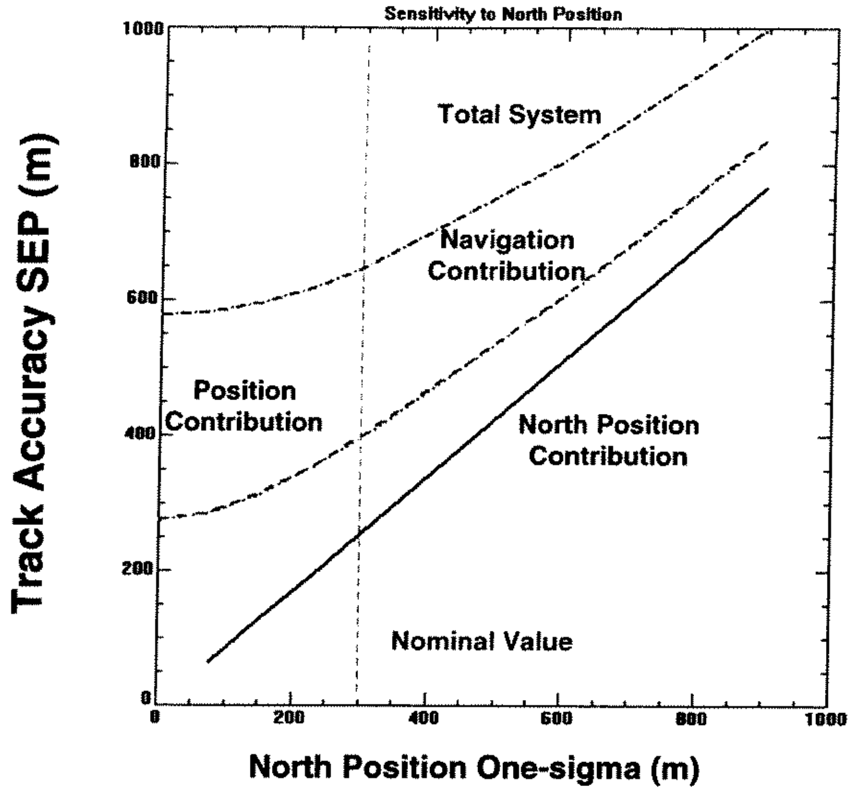


Figure 3-2 Sensitivity to North Position Error

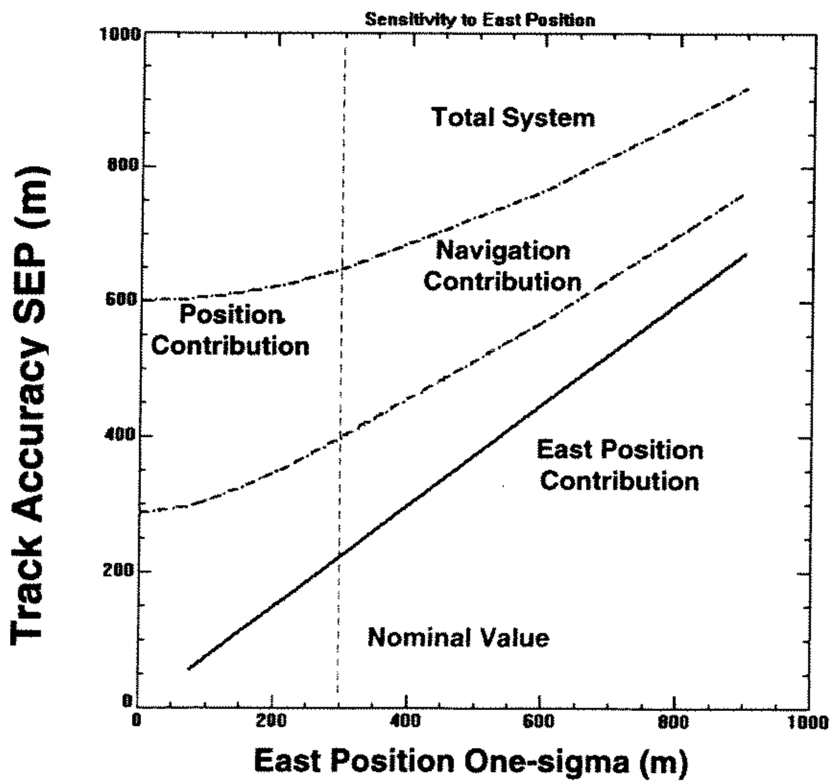


Figure 3-3 Sensitivity to East Position Error

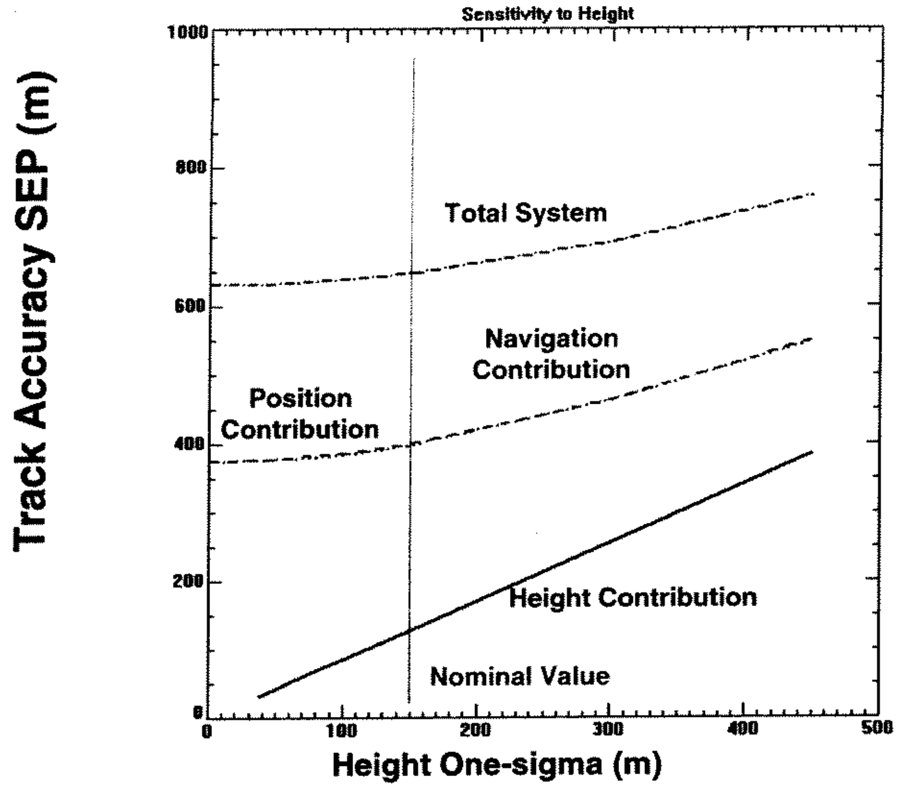


Figure 3-4 Sensitivity to Height Error

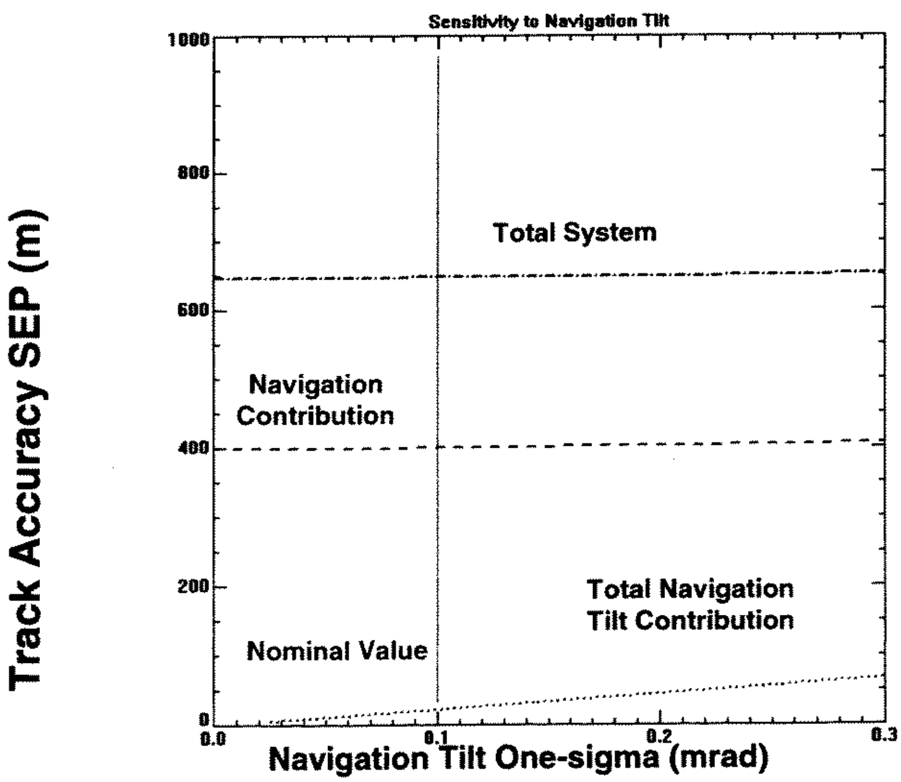


Figure 3-5 Sensitivity to Navigation Tilt

coincides with the curve representing the contribution of navigation because the misalignment contribution is so small. Figures 3-2 and 3-3 show that the sensitivities of total system track error to north position and east position error are very nearly equal. The sensitivity is less pronounced for the height error, Figure 3-4, and is very small for the total navigation misalignment or tilt, Figure 3-5.

3.6.2 Sensitivity to sensor error

The sensitivities of track accuracy to the sensor errors are displayed in Figures 3-6 to 3-11. Unlike the navigation subsystem where the position error group dominated the navigation contribution, both the sensing errors (range, bearing and elevation) and the sensor misalignments about each axis are important contributors to track error. Of all the sensor errors, the sensor range error has the largest influence on track accuracy. This is followed by the sensitivities due to bearing error, ψ_3-axis misalignment, elevation error, and the other two sensor misalignments. Because the system is balanced in terms of the contributions to track error from the navigation and sensor subsystems, decreases in error from the nominal value have a small effect on reducing the total system error. Substantial reductions in track error can only be made by reducing both the navigation position and the sensor errors.

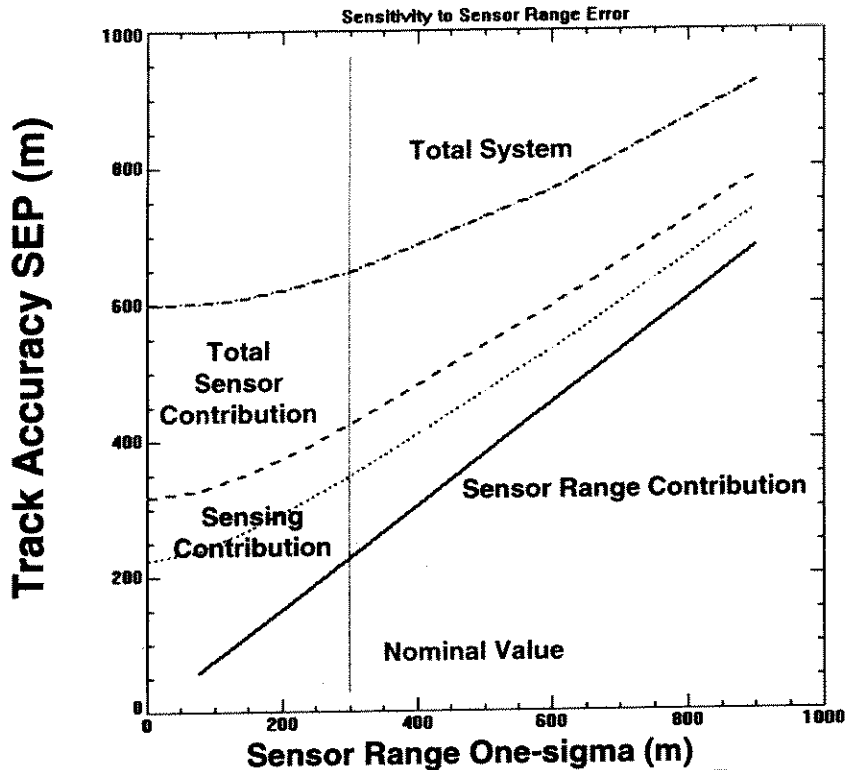


Figure 3-6 Sensitivity to Sensor Range Error

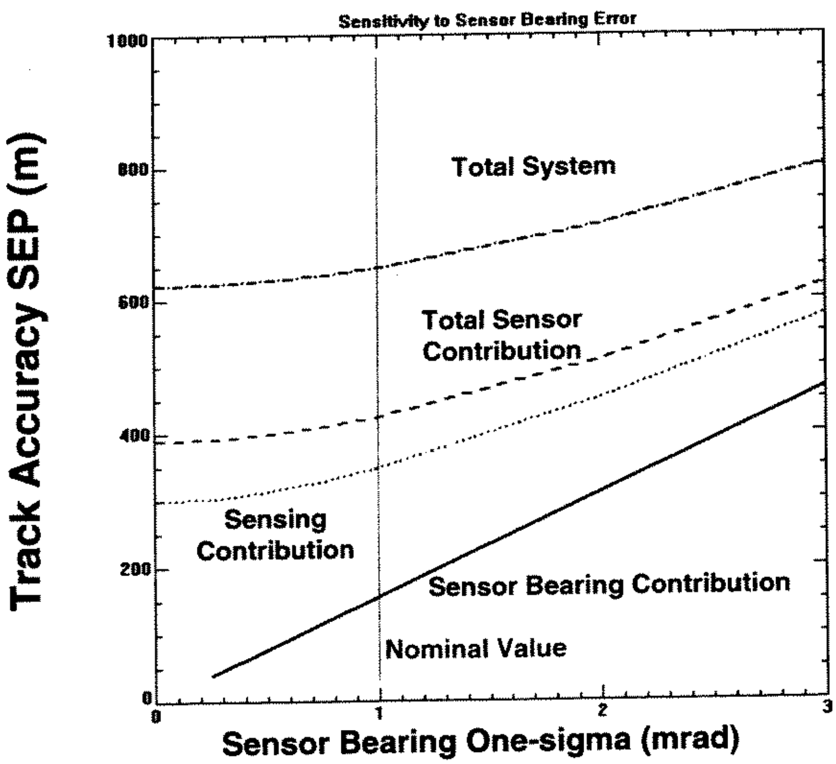


Figure 3-7 Sensitivity to Sensor Bearing Error

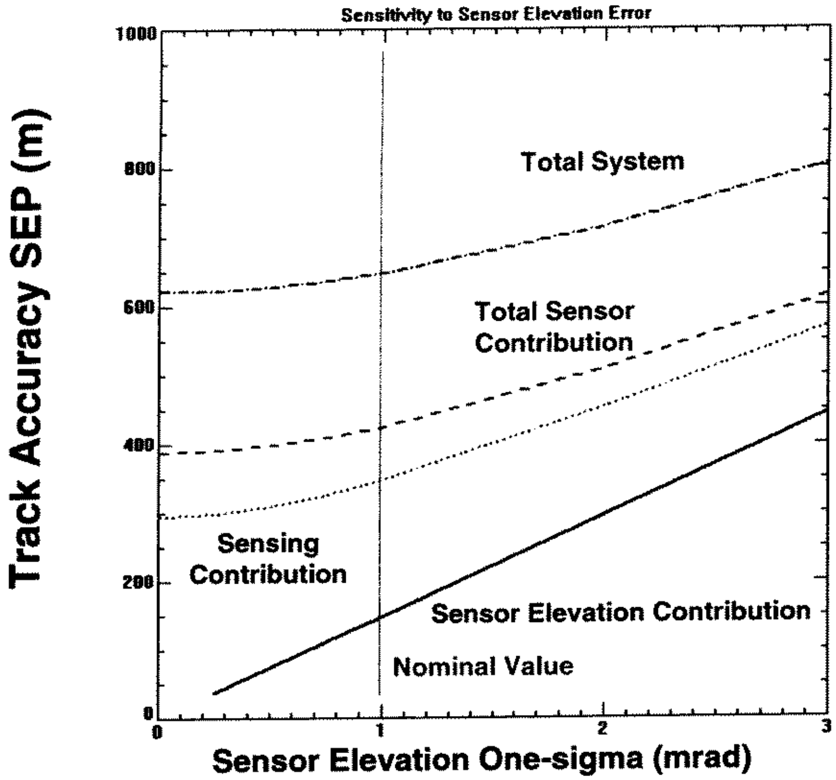


Figure 3-8 Sensitivity to Sensor Elevation Error

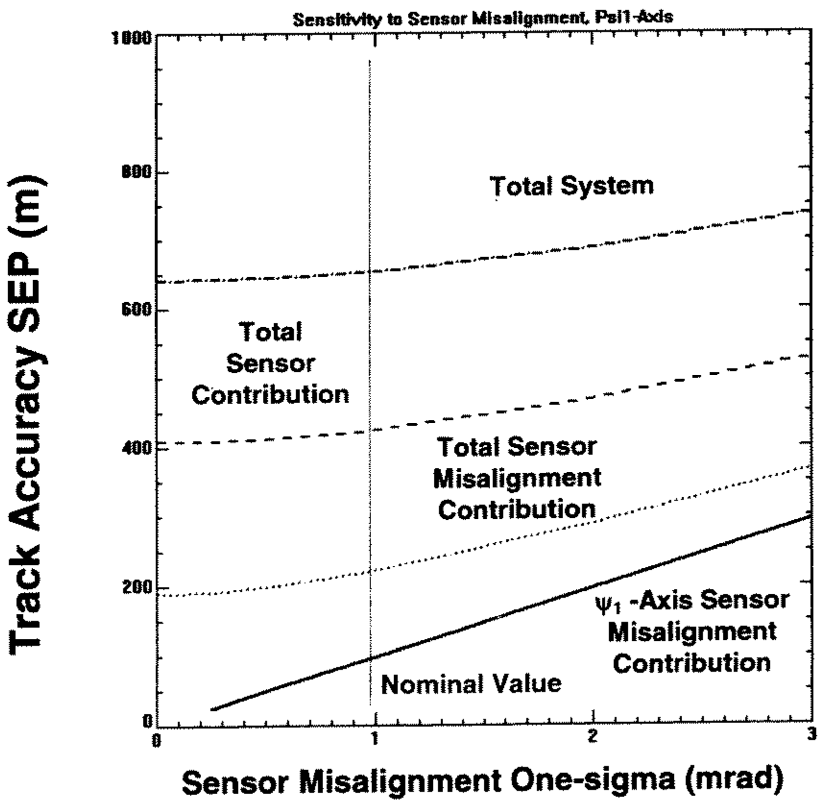


Figure 3-9 Sensitivity to Sensor Misalignment, ψ_1-Axis

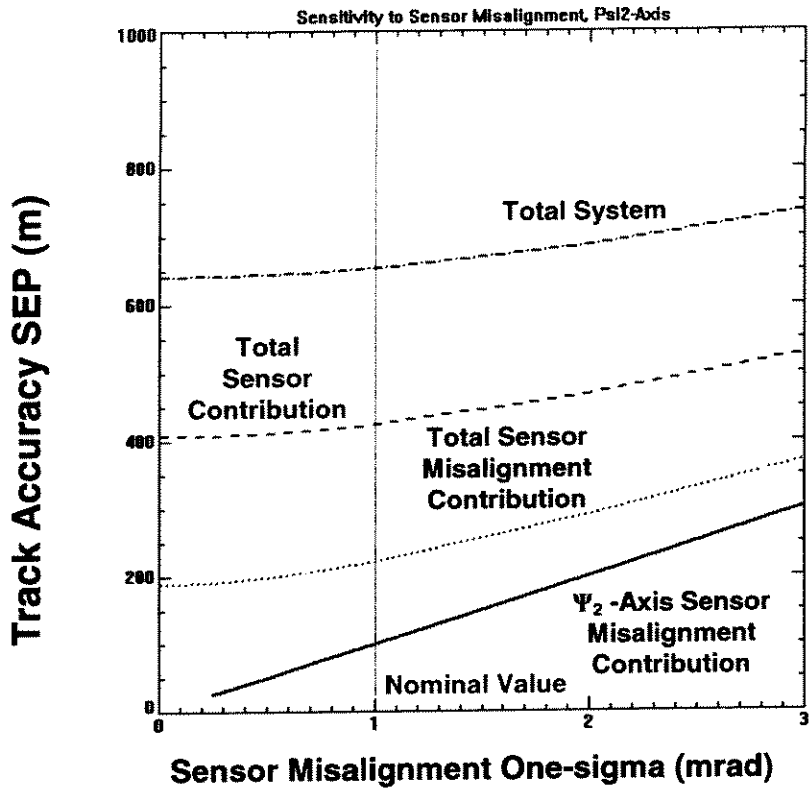


Figure 3-10 Sensitivity to Sensor Misalignment, ψ_2-Axis

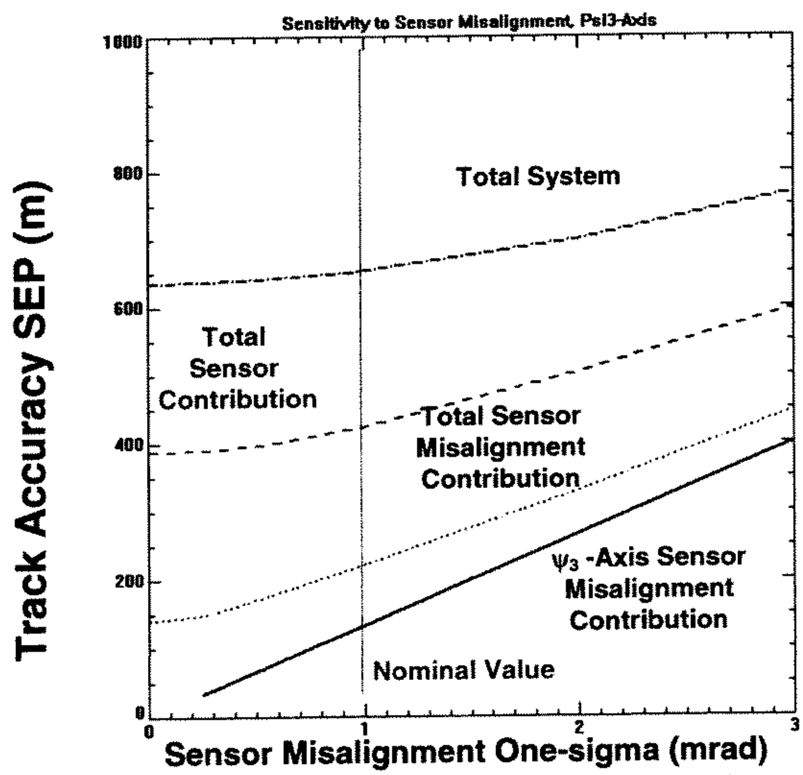


Figure 3-11 Sensitivity to Sensor Misalignment, ψ_3-Axis

3.6.3 Sensitivity to time error

The sensitivity of track accuracy to time error is displayed in Figure 3-12. The curve shows that reducing the time error to zero reduces track error by only a little more than 50 m SEP. This is because the navigation errors and sensor errors dominate the error budget. Doubling the time error budget so it is balanced with navigation and sensor will increase total track SEP by approximately 100 m.

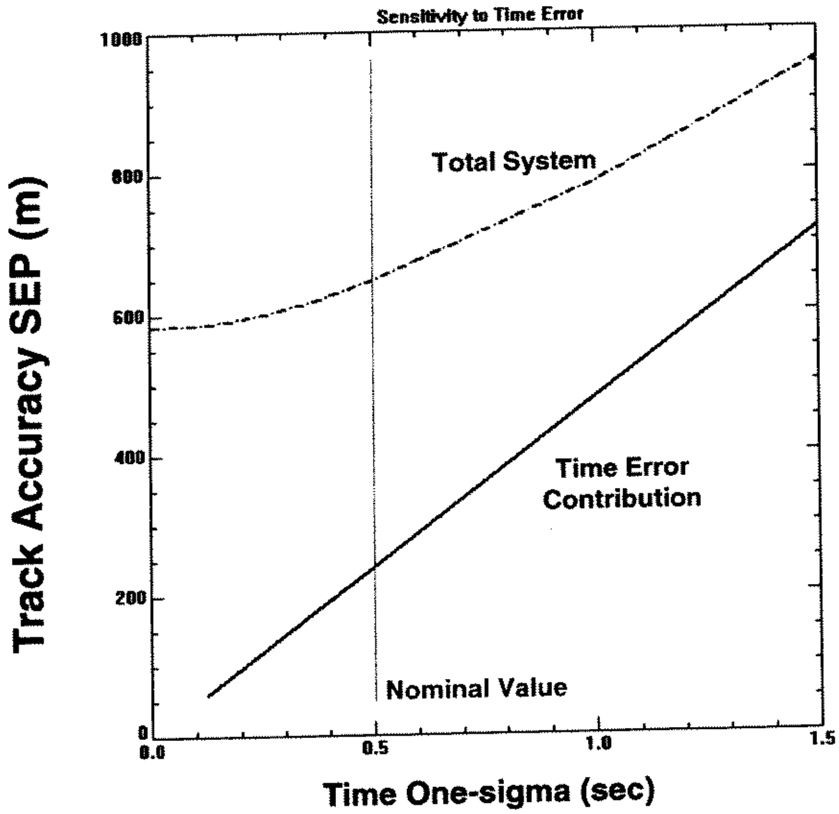


Figure 3-12 Sensitivity to Time Error

Additional insight into the sensitivity of track accuracy to time error is obtained by examining the sensitivities for an expanded set of threat objects and network latency times. The velocity and acceleration for a set of aerospace objects is provided in Table 3-5. These parameter values represent maximum capabilities and provide worst case sensitivities of track accuracy error to time error.

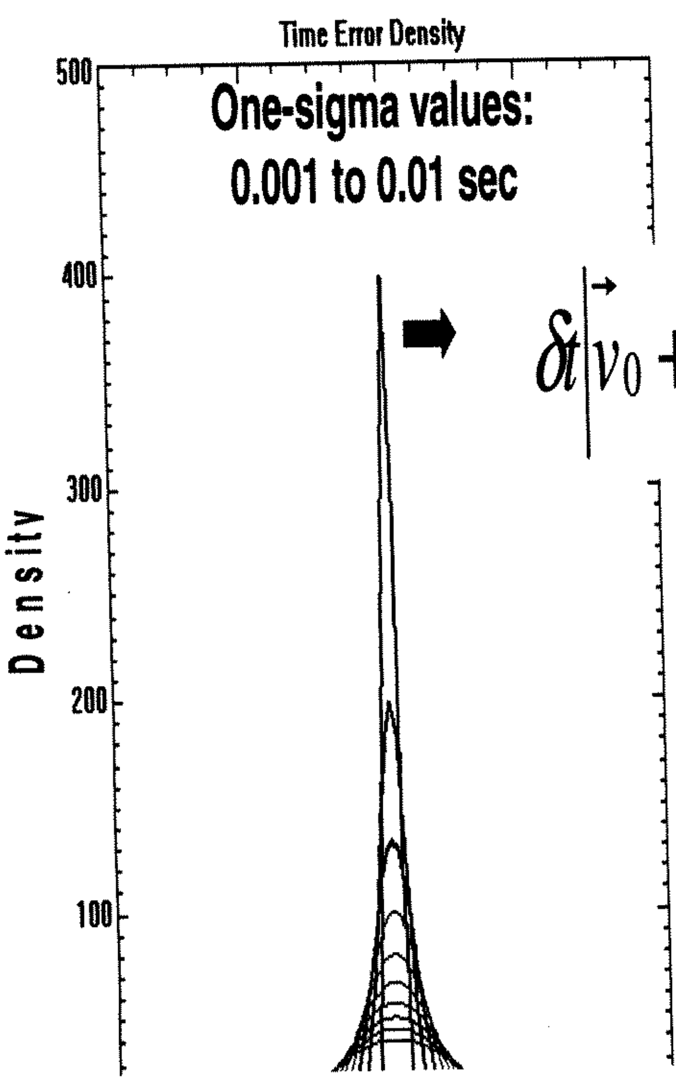
Table 3-5 Velocity and Acceleration Capabilities for Aerospace Objects

Aerospace Object	Velocity Magnitude (kt)	Acceleration Magnitude (g)
Helicopter	200	3.5
Aircraft 1	300	5
Cruise Missile	500	0.5
Aircraft 2	800	9
Aircraft 3	1000	9
Aircraft 4	1200	9
Aircraft 5	1600	9

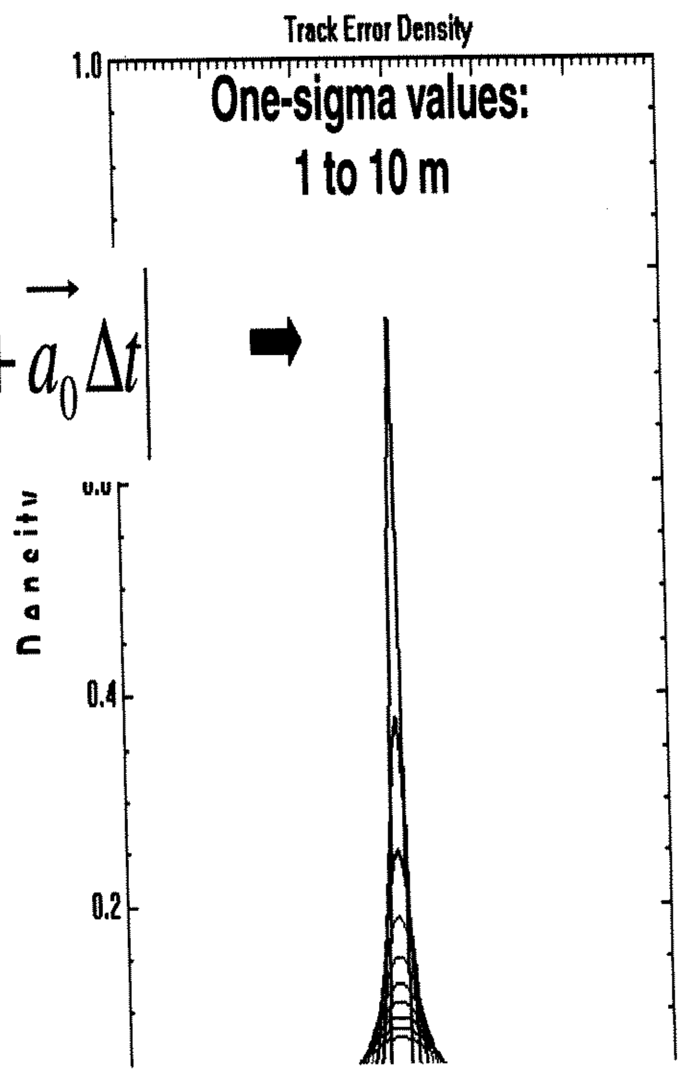
We will first consider how the time errors are mapped into track error. As discussed in section 2, the sensitivity of track error to time error is given by the expression $\left|\vec{v}_0 + \vec{a}_0 \Delta t\right|$ where $\vec{v}_0$ is the object velocity at the time the object is sensed, t_0, and $\vec{a}_0$ is the average acceleration over the nominal latency time, Δt, of the network. We will first consider the case of the directions of velocity and acceleration for the object as being parallel and show the relationship between the normal densities for the time and track errors. Figures 3-13 to 3-15 illustrate this by showing the mapping of the one-sigma values for time error into the one-sigma values for track error for the case of Aircraft 3 and a network latency of 5 sec for three ranges of time error. For the case of Aircraft 3 and a network latency of 5 sec, the sensitivity is 956 m of track error for every sec of time error. We see that time errors in the range of 0.001 to 0.01 sec lead to track errors in the range of 1 to 10 m, time errors between 0.01 and 0.1 sec give rise to track errors between 10 to 96 m, and for the third case of time errors between 0.1 and 1 sec, the track errors are between 96 and 956 m.

Velocity magnitude = 1,000 kt   Acceleration magnitude = 9 g

Time latency = 5 sec

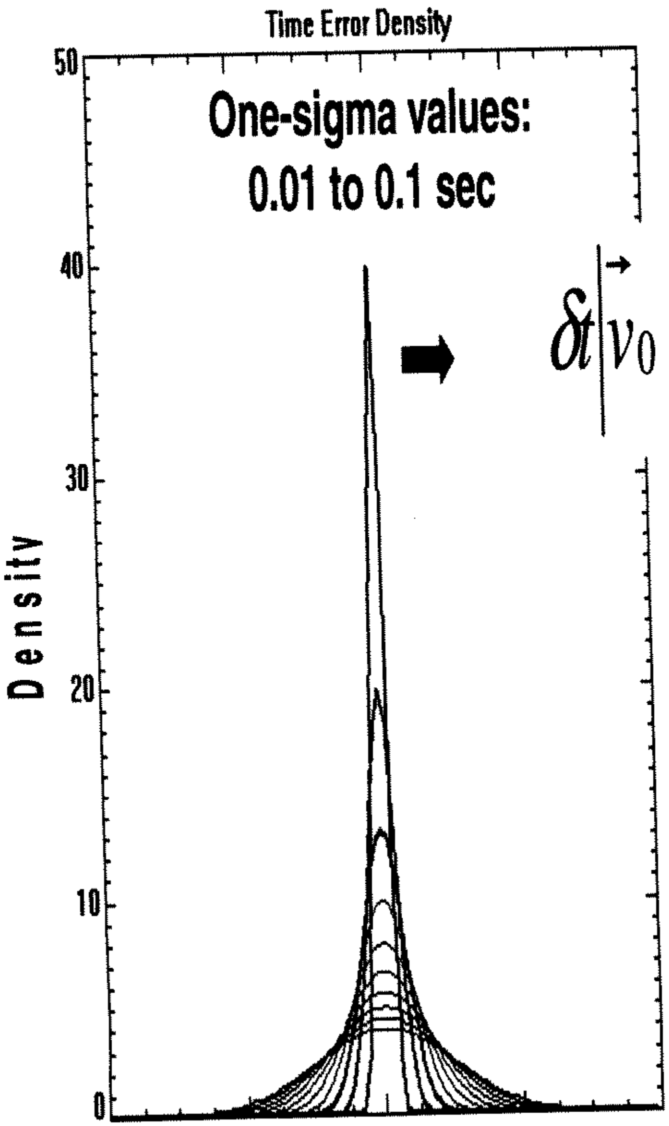


$$\delta t \left| \vec{v}_0 + \vec{a}_0 \Delta t \right|$$

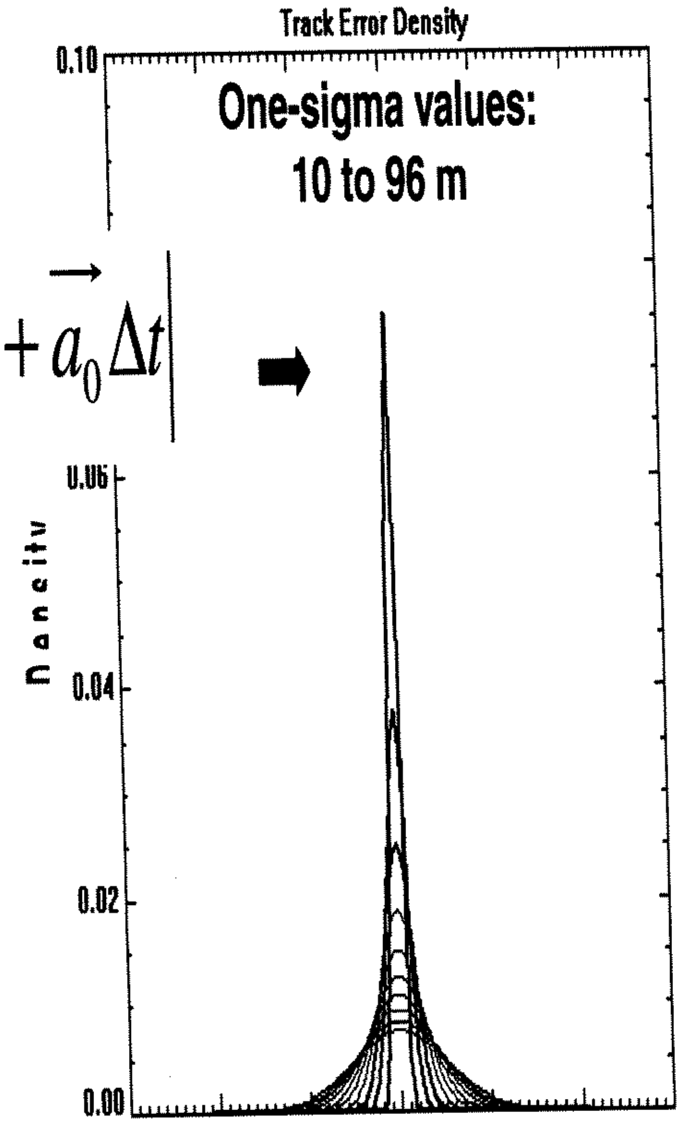


Velocity magnitude = 1,000 kt   Acceleration magnitude = 9 g

Time latency = 5 sec

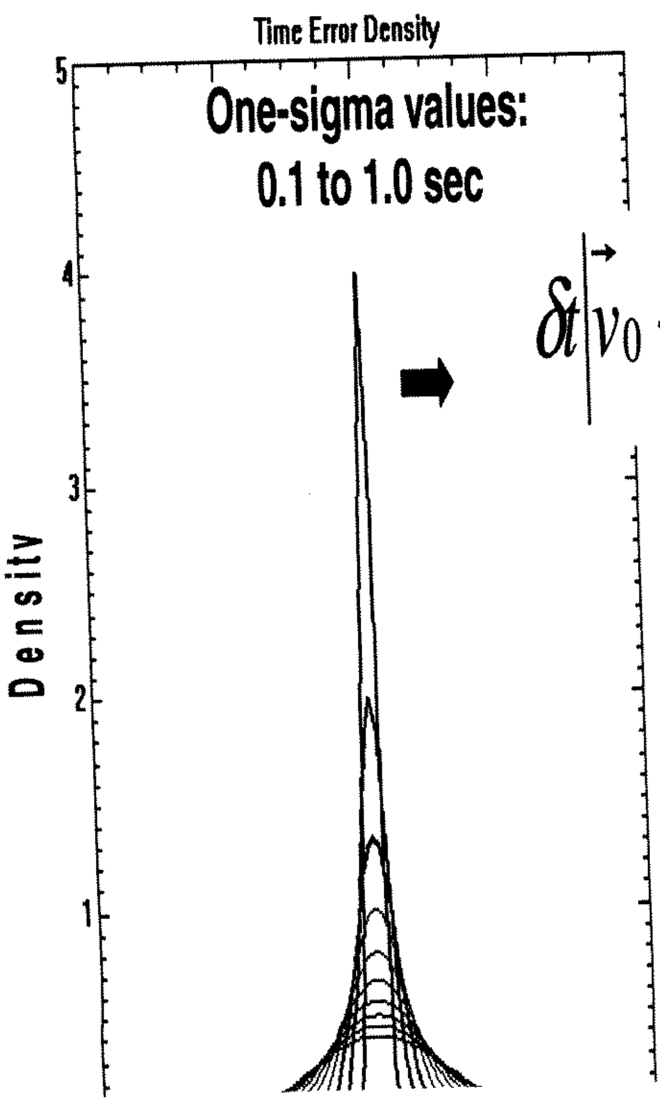


$$\delta t\left|\vec{v}_0 + \vec{a}_0 \Delta t\right|$$

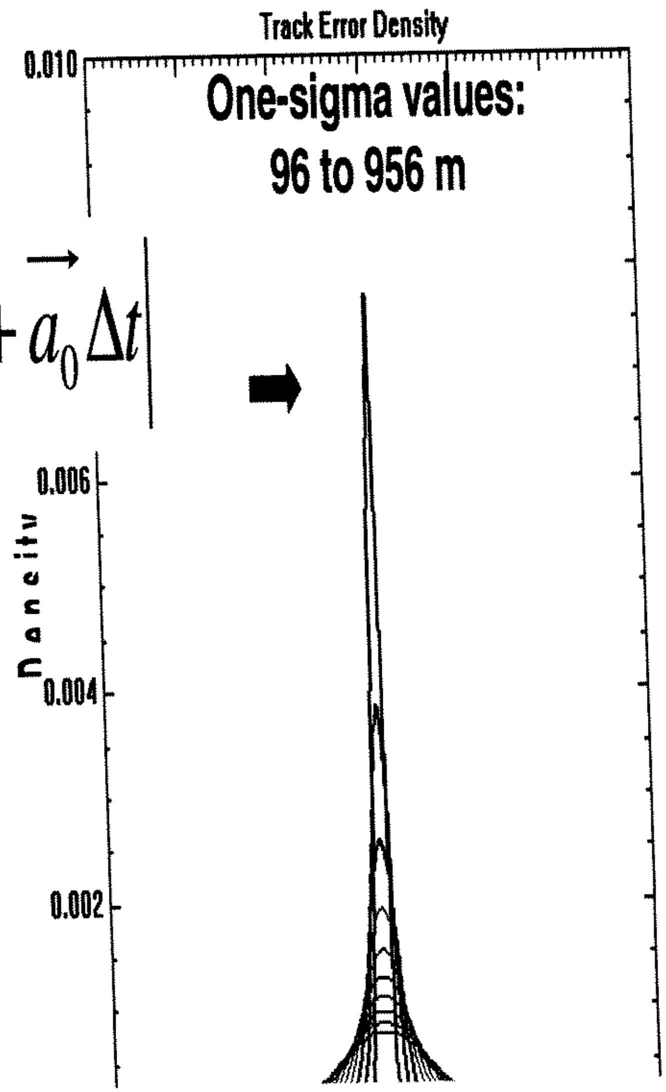


Velocity magnitude = 1,000 kt Acceleration magnitude = 9 g

Time latency = 5 sec



$$\delta t\left|\vec{v}_0 + \vec{a}_0 \Delta t\right|$$



We will next show the sensitivities of track accuracy to time errors for each of the aerospace objects listed in Table 3-5. So far we have assumed that the object velocity and acceleration are known and that these vector quantities are in the same direction. Since the time error is one-dimensional and the directions of the velocity and acceleration are assumed to be known, the density of track error will be a degenerate trivariate normal. That is, the density will be one dimensional along the direction of the vector quantities. We will relax the assumption of known directions for the velocity and acceleration by considering a second case where we let the directions for velocity and acceleration be independent and uniformly distributed in three-space. This assumption will give rise to a truly three-dimensional track error and more realistically represent a true warfighting scenario. It is noted that we will still consider the magnitudes of the velocity and acceleration of the object to be known. Experts in the field of tracking have noted that prior knowledge of the object's velocity and acceleration may not be known exactly and this lack of knowledge often leads to tracking errors. We are presently investigating how to model this situation and include it as a tracking error in future error budget discussions.

To complete the present discussion we will examine the following two cases parametrically.

- Case 1: Assume direction and velocity and acceleration is fixed and known
- Case 2: Assume random direction of velocity and acceleration but magnitudes are known.

We will parametrically vary

- Magnitude of velocity and acceleration by considering the aerospace objects presented in Table 3-5
- Nominal network time latency, Δt
- One-sigma value of the time error.

Track accuracy will be measured by

- Standard deviation (one-sigma) representing 69% coverage for Case 1
- Spherical Error Probable (SEP) representing 50% coverage for Case 2.

The full set of sensitivity curves are presented in Figures 3-16 to 3-23.

Figures 3-16 and 3-17 show the sensitivities of track accuracy for the same helicopter but for different ranges of time error. Figure 3-16 provides a clear indication of the magnitude of the track accuracy ranging from a minimum of approximately 20 m one sigma or SEP to approximately 1,150 m one-sigma and 825 m SEP for time errors between 0.2 and 1 sec and for all cases of network time latency. Figure 3-17 narrows in on the 0.02 to 0.1 sec range for time error and shows the largest track accuracy is approximately 115 m one-sigma for Case 1 and 80 m SEP for Case 2.

Based on the analytic representation of the sensitivity and the family of sensitivity curves we immediately see that track error

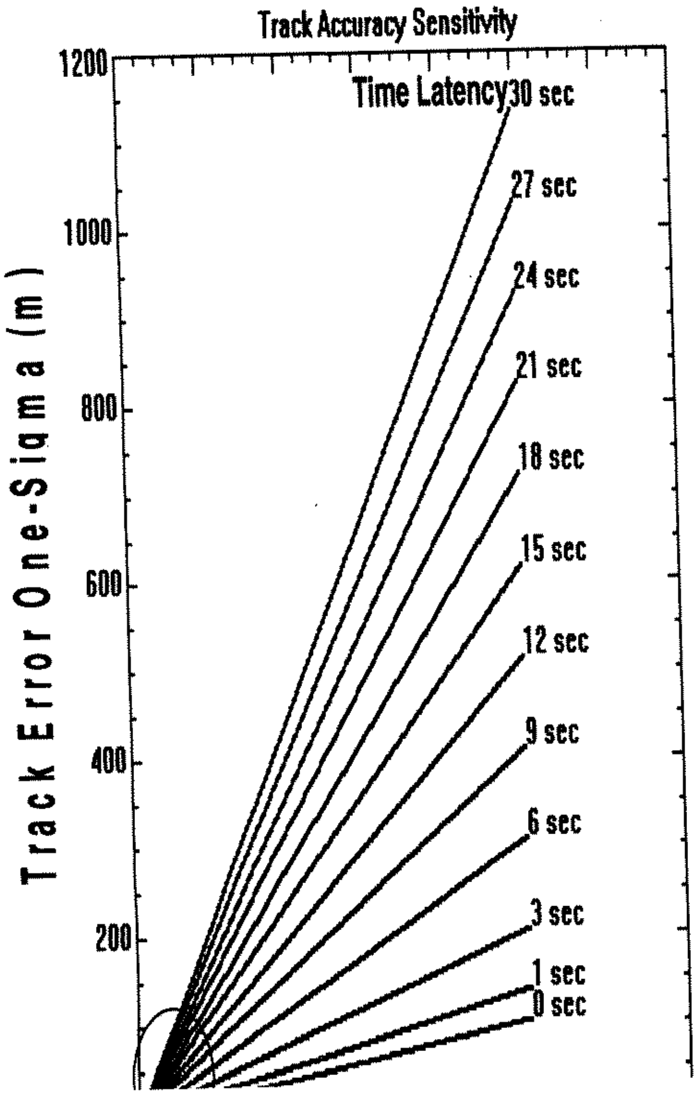
- Increases as the velocity of the object increases
- Increases as the acceleration of the object increases
- Increases as the network time latency increases.

Comparing Figure 3-19 with any of the others shows the combined effect of acceleration and network latency on track accuracy. The narrow spread between successive sensitivity curves displayed in Figure 3-19 is due to the small acceleration magnitude, 0.5 g. The remainder of the figures shows that track error increases when larger accelerations are combined with increased network time latency – a multiplicative effect is taking place.

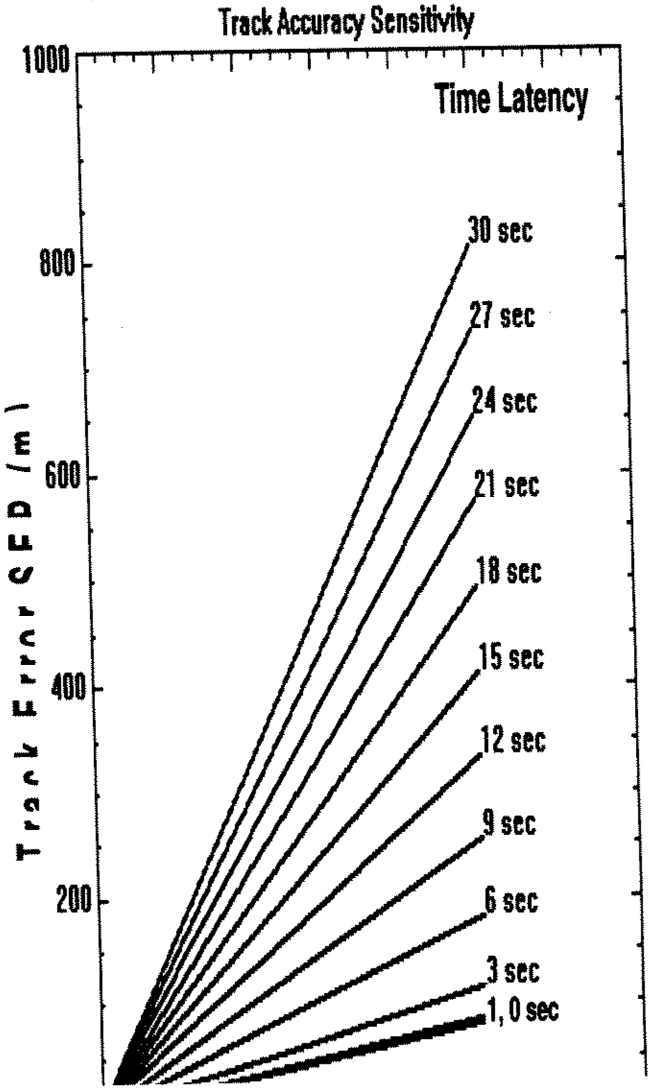
Figure 3-23 shows that under worst case conditions, the track accuracy can be as large as 3,500 m one-sigma and 2,300 m SEP for time errors on the order of 1 sec one-sigma. Discussions with systems experts suggest that time errors will be on the order of 0.3 millisec. Under these conditions the worst case outcome is on the order of 1 m one-sigma for Case 1 and 0.8 m SEP for Case 2. Clearly a time error on the order of less than a millisec will have no appreciable effect on track accuracy.

Velocity magnitude = 200 kt Acceleration magnitude = 3.5 g

Case 1: One-dimensional

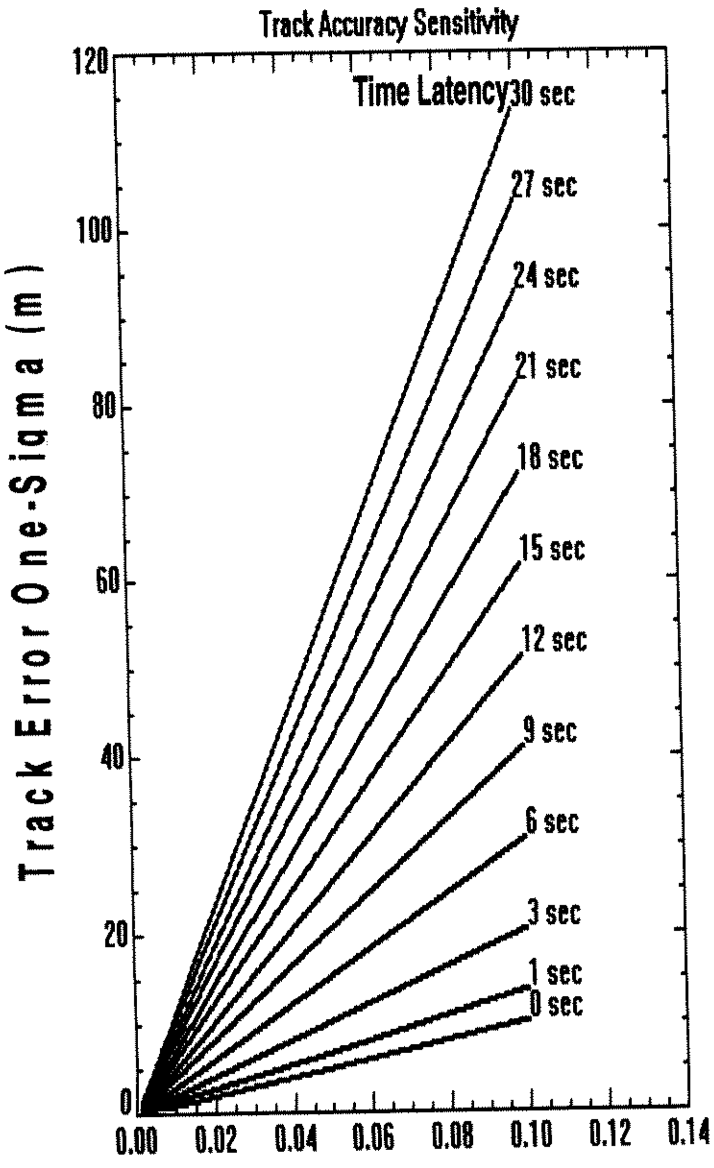


Case 2: Three-dimensional

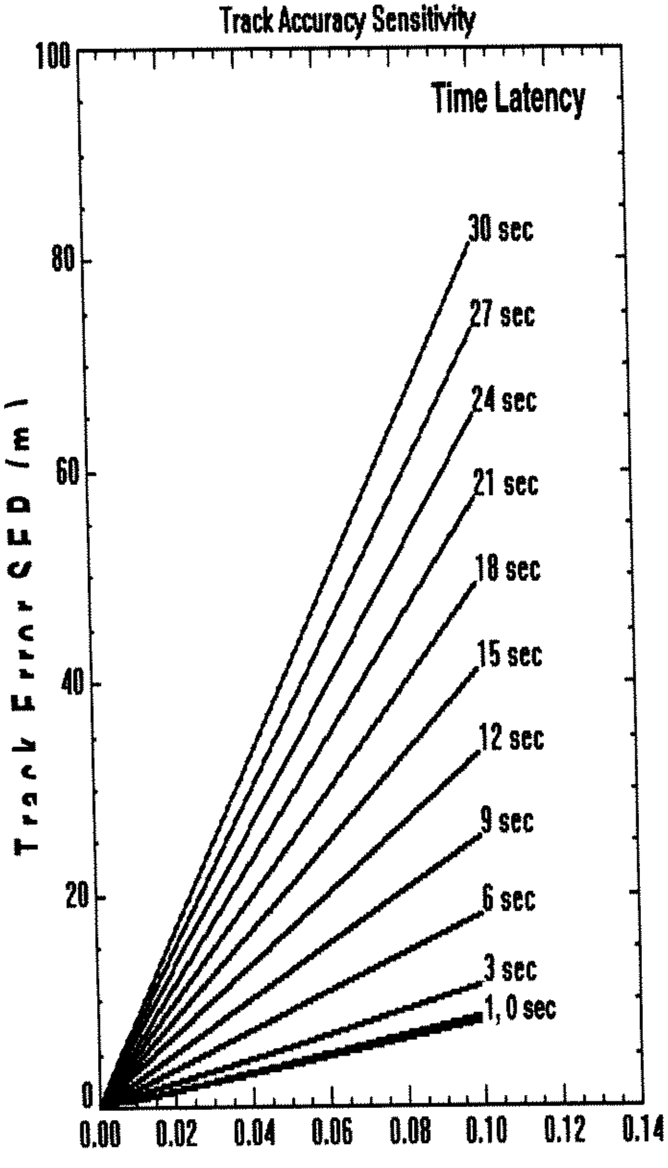


Velocity magnitude = 200 kt   Acceleration magnitude = 3.5 g

Case 1: One-dimensional

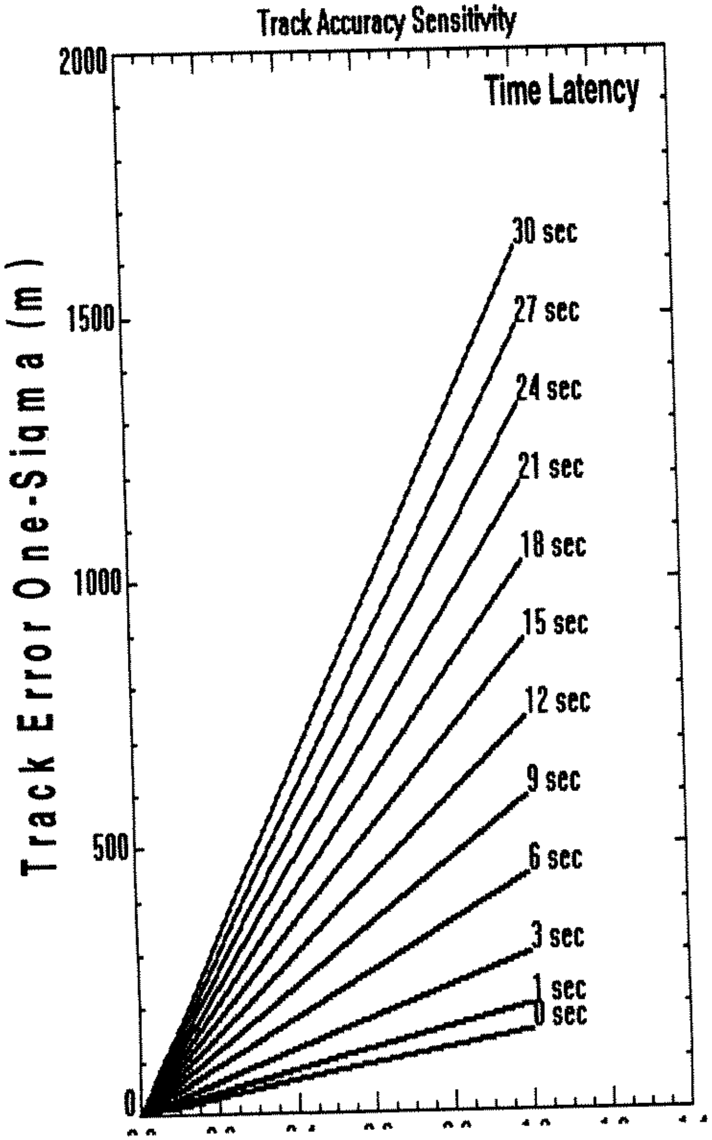


Case 2: Three-dimensional

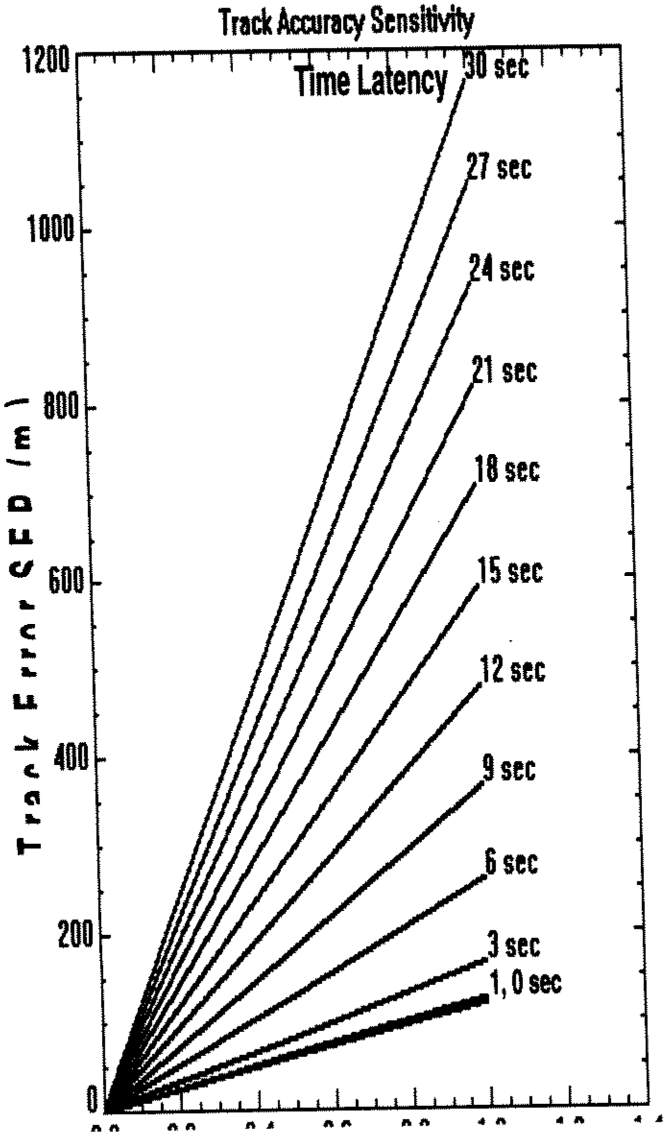


Velocity magnitude = 300 kt   Acceleration magnitude = 5 g

Case 1: One-dimensional

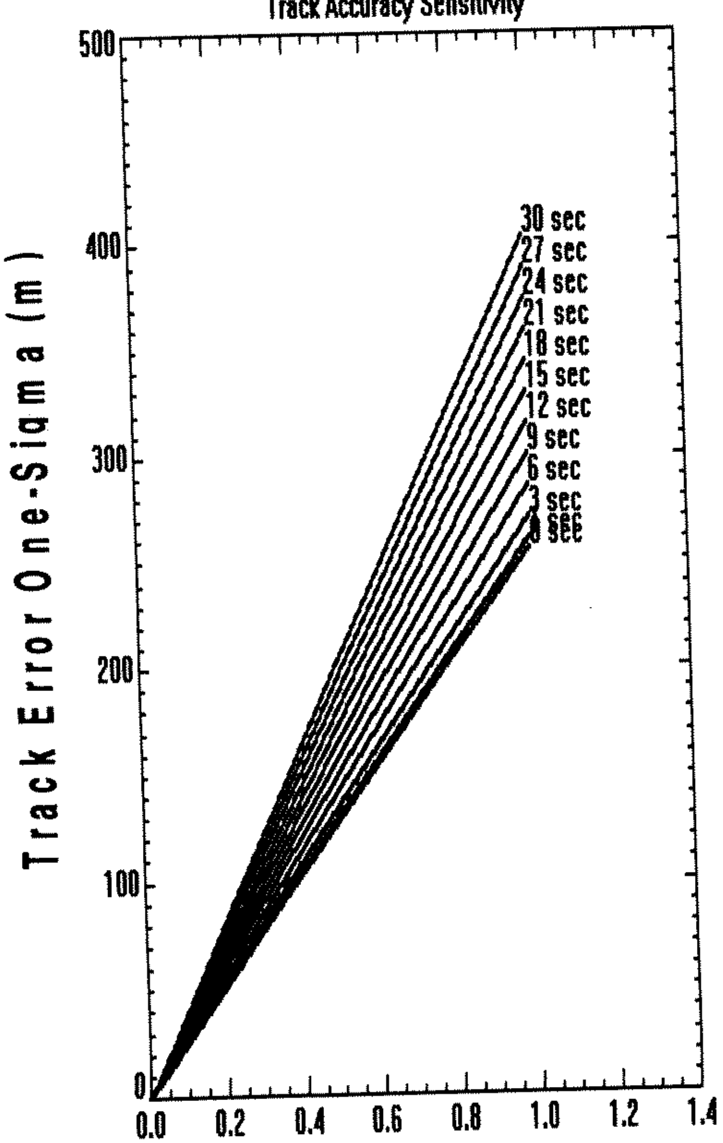


Case 2: Three-dimensional

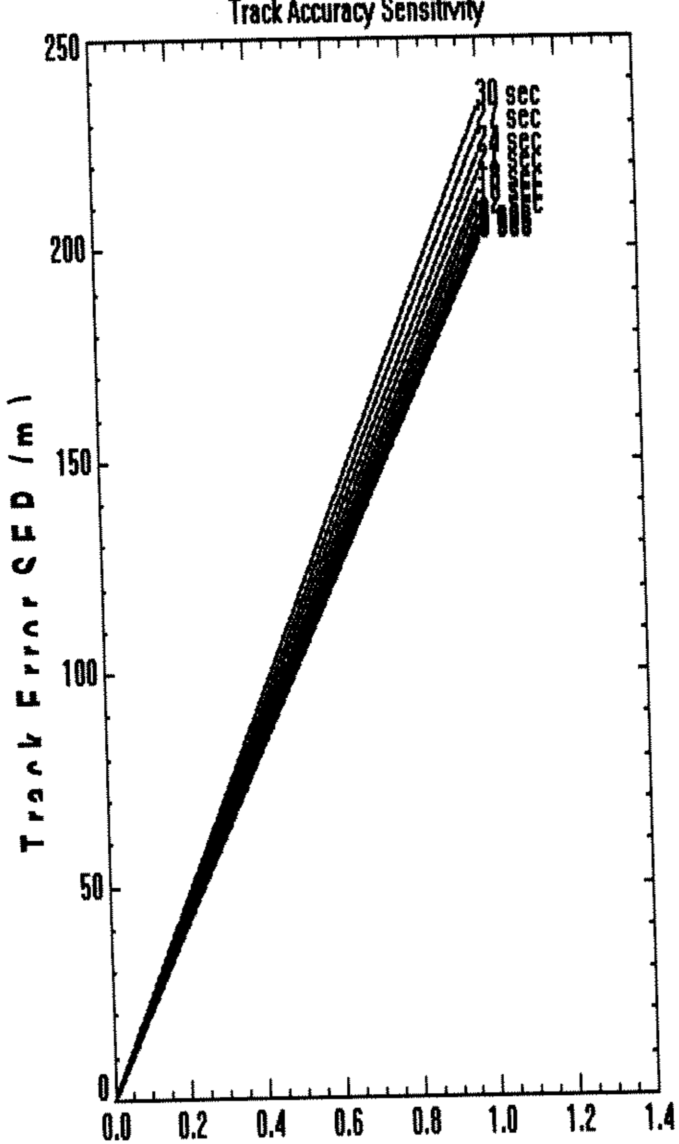


Velocity magnitude = 500 kt Acceleration magnitude = 0.5 g

Case 1: One-dimensional

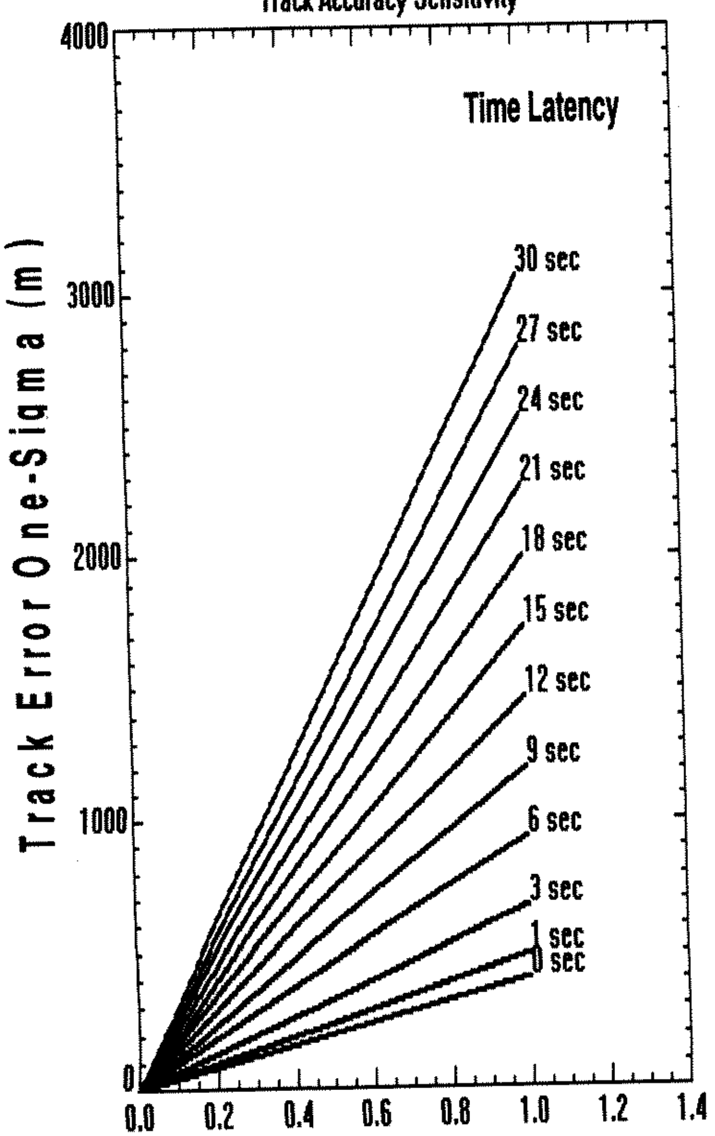


Case 2: Three-dimensional

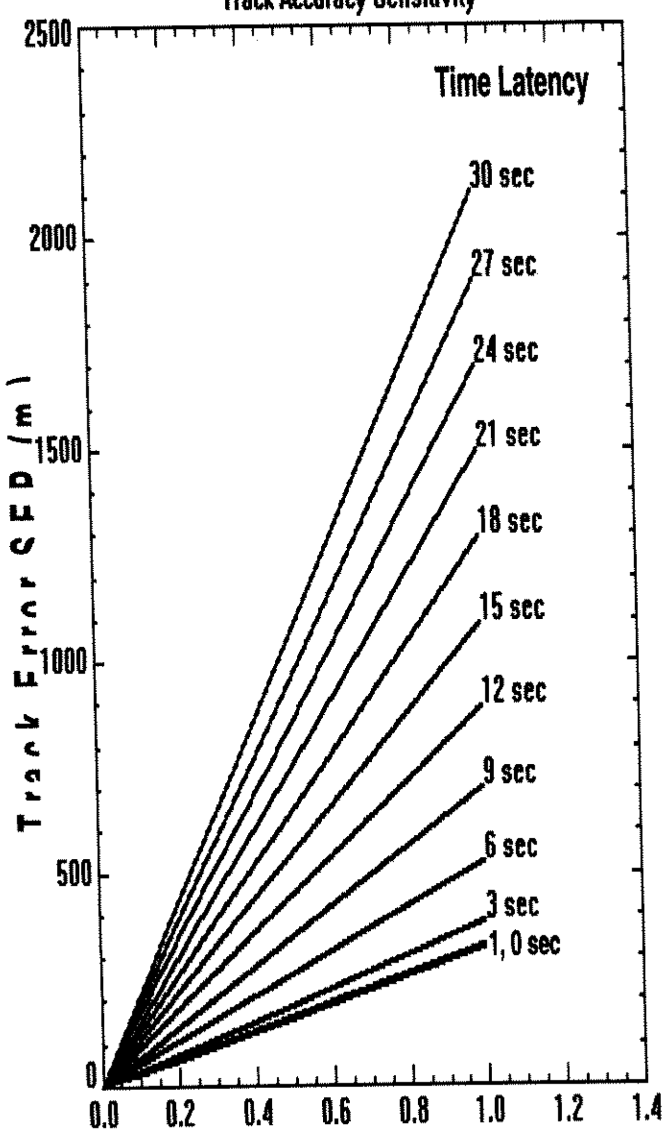


Velocity magnitude = 800 kt　　Acceleration magnitude = 9 g

Case 1: One-dimensional

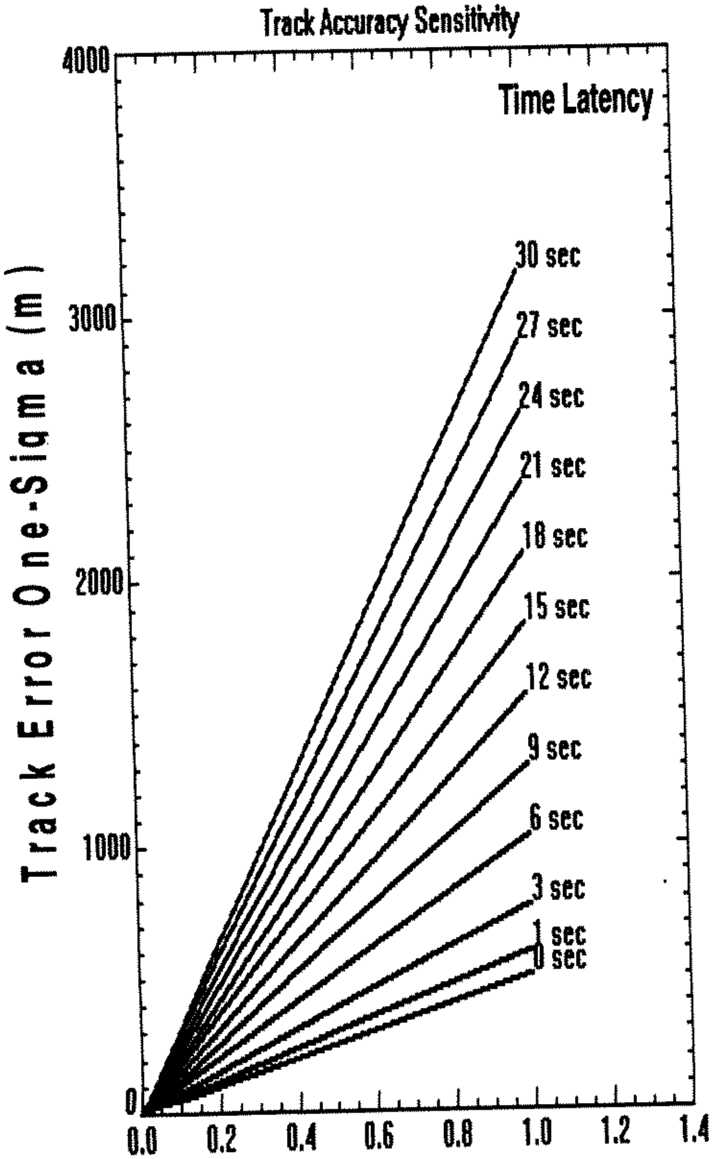


Case 2: Three-dimensional

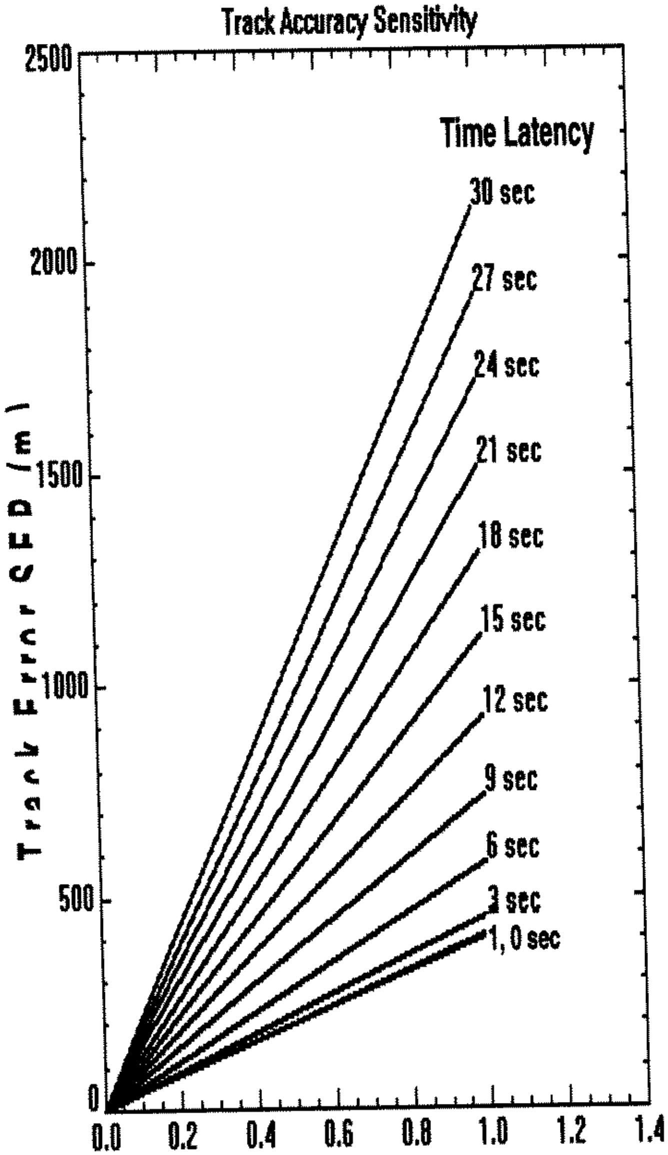


Velocity magnitude = 1,000 kt   Acceleration magnitude = 9 g

Case 1: One-dimensional

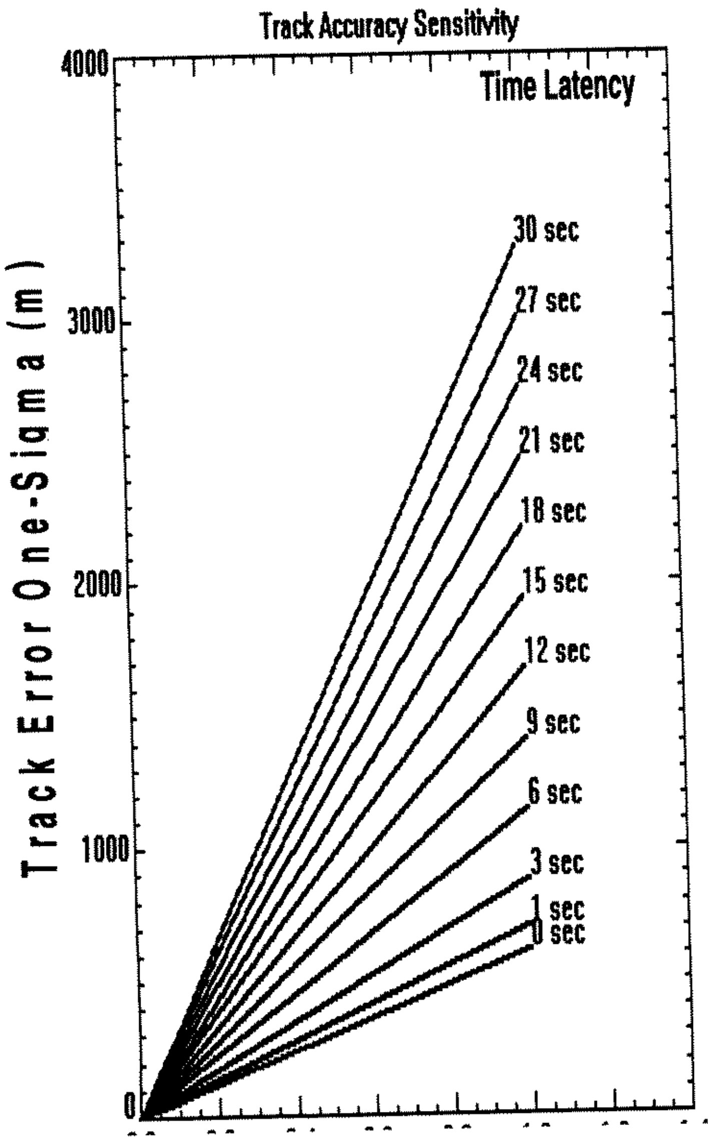


Case 2: Three-dimensional

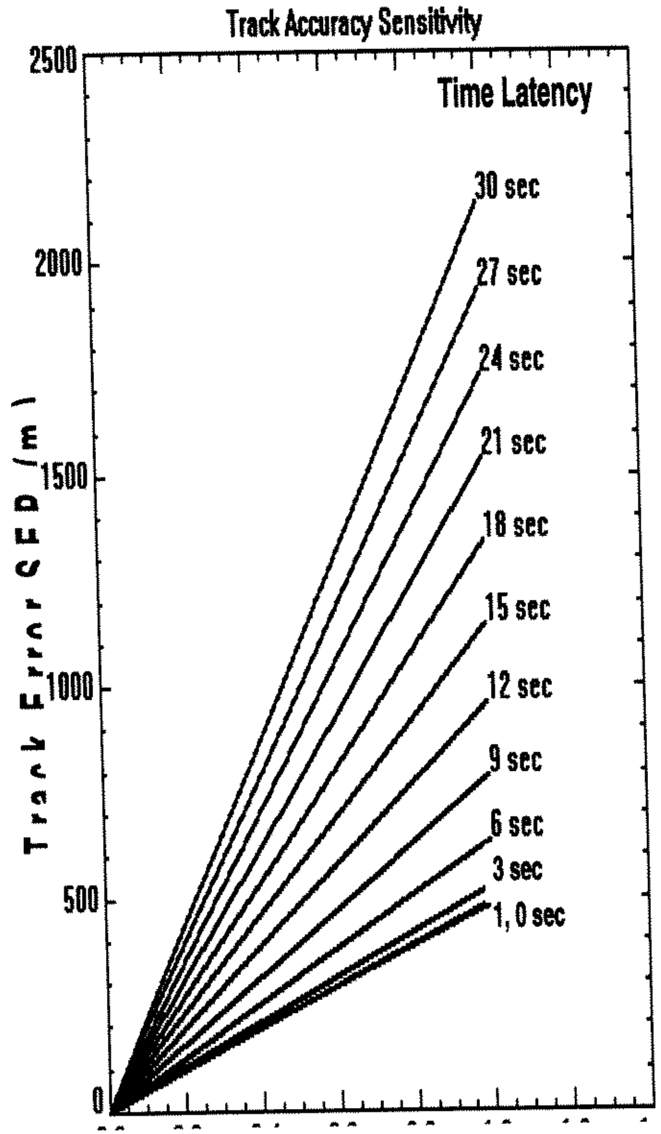


Velocity magnitude = 1,200 kt   Acceleration magnitude = 9 g

Case 1: One-dimensional

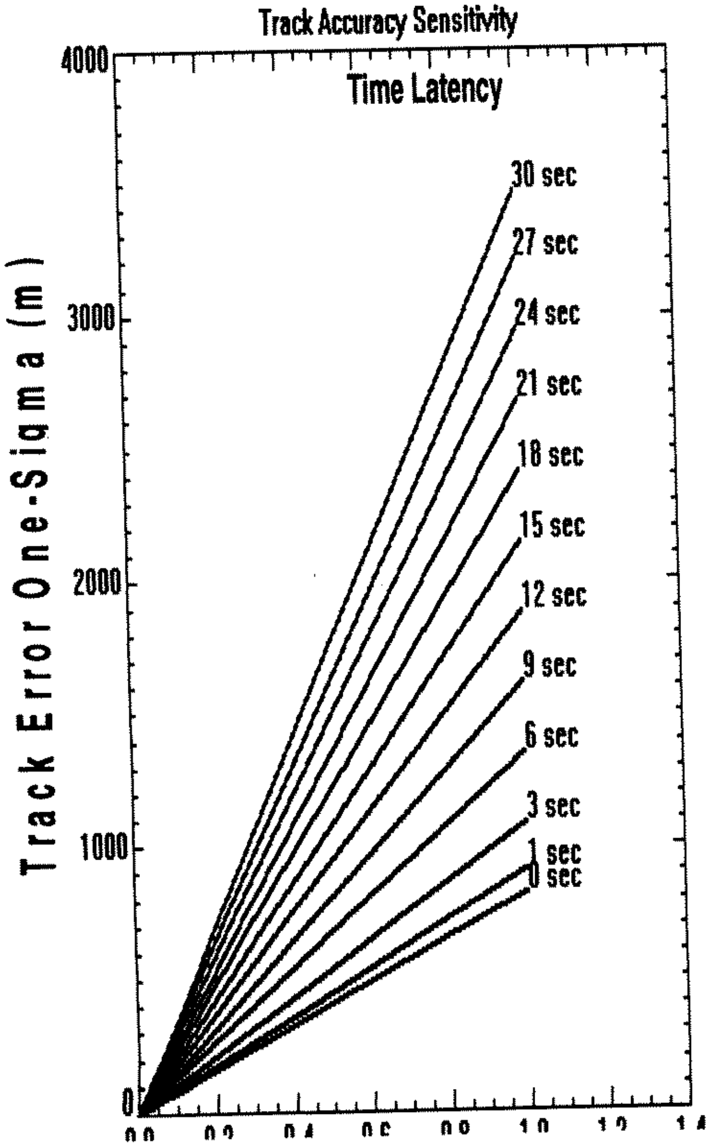


Case 2: Three-dimensional

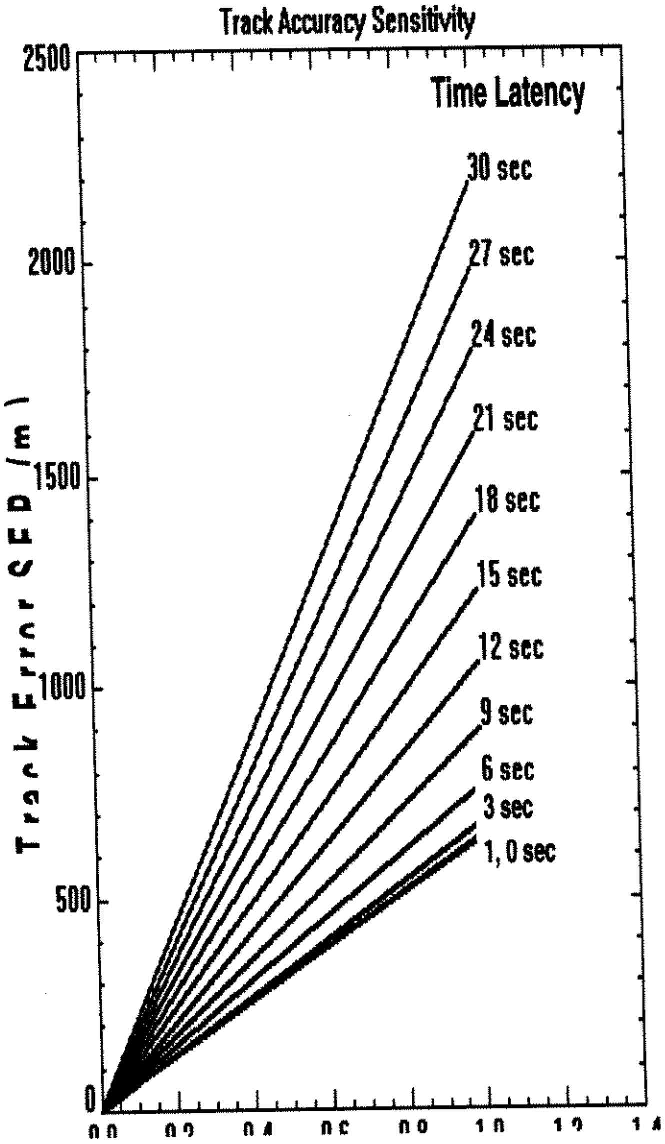


Velocity magnitude = 1,600 kt Acceleration magnitude = 9 g

Case 1: One-dimensional



Case 2: Three-dimensional



4. CONCLUSIONS

4.1 Findings

This report provides the details and the results of an analysis technique to understand and quantify how errors in navigation, sensor, and time affect the SIAP. The analysis approach derives the equation for the location of an aerospace object in the WGS 84 coordinate frame and determines how errors in measurements and alignments affect the estimate of track position. This is necessary since meaningful information can be exchanged between the ship and other units in a distributed system only if the location and orientation of the various coordinate frames are known relative to each other.

An examination of the track equation identifies three groups of error contributors:

- Navigation position and misalignments
- Sensor measurement and misalignments
- Time

After the error mechanisms are identified, a first-order linear perturbation analysis is employed to develop the relationship between track error and the errors identified above. The result of the analysis is the sensitivities of track accuracy to the navigation, sensor, and time errors. Once the sensitivities are calculated, the error budget and parametric sensitivities are determined.

A nominal error budget for the navigation, sensor, and time errors is presented. From this the track accuracy error budget is developed. This budget shows a balance in error contribution between the navigation and sensor subsystems. The time error is a relatively small contributor, so that reductions in time error from the nominal value will result in small decreases in track error. An examination of the navigation subsystem shows that position errors dominate. The contributions from the navigation misalignments and the lever arm errors are very small. The sensor subsystem error budget is almost equally divided between the sensing errors and the sensor misalignment errors.

The sensitivity of track error to time error depends on the aerospace object velocity and acceleration. The sensitivity is found to equal the vector sum of the velocity and the product of the acceleration and the known time latency of the system. For a high-speed combat aircraft and a network latency of 5 sec, the time error is magnified by a factor of 956 m/sec. Thus, a time error of 1 sec is translated into a track error of 956 m. The known time latency of the system has the effect of increasing or decreasing the contribution of the object's acceleration to the sensitivity. For example, reducing the time latency by a factor of two can reduce the track error to 735 m.

The parametric analysis shows that worst case conditions of 1 sec time error and 30 sec network latency lead to track errors of approximately 3,000 m for high-speed aircraft. Time errors on the order of only a millisec lead to track errors of less than 4 m for these worst case conditions.

4.2 Future Work

Although the analysis assumes a shipborne sensor system, the analysis is general and can be applied to all systems that contribute to the SIAP. We are currently developing error budgets for land and airborne systems. In addition, we are expanding the list of error contributors. For example, the analysis of the time error in section 2.8 develops error expressions for extrapolation and trajectory data errors. We are investigating how these errors can be related to tracking errors.

One objective of the error budget analysis is to determine how the individual errors affect the SIAP attributes. We are investigating how to incorporate the error analysis approach into the correlation/decorrelation algorithms to determine sensitivities of the attributes to the individual error contributors.

We also plan to incorporate the Common Reference Scenarios (CRS) into this analysis so that comparisons can be made to other simulation results based on the CRS. Using the CRS will also provide insight into how track accuracy changes as the threat evolves in time.

We are also using the error model and the results of the analysis runs to develop test cases to assist with the IABM verification and validation effort. The error models provide the mechanism to introduce the correct level of bias into each system, and the results of the error model analysis provide examples of track accuracy that the IABM can be checked against.

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